On the quiver and relations of the Borel Schur algebras

Thesis submitted for the degree of

Doctor of Philosophy

at the University of Leicester

by

Degang Liang
(Leicester)

Department of Mathematics

University of Leicester

2007

ProQuest LLC
789 East Eisenhower Parkway
P.O. Box 1346
Ann Arbor, MI 48106-1346

Contents

Acknowledgements

I would like to thank my supervisors Alexander Baranov and Anne Henke for all their guidance and supports. I would also like to thank Prof. Steffen Koenig for numerous suggestions and constant support. I would like to thank my supervisors in China – Prof. Fengwen Niu and Bin Zhu – for their supports. I thank my family and my friend Lina Gu for their helps to my life. I also thank Libin Li, Ming Fang, Ronghua Tan, Dong Yang, Qunhua Liu and Junchang Wang for making my life more easier in Leicester.

Finally, I am grateful to The European Community who provided core funding under the AsiaLink Scheme for my study in Leicester and grateful to the Engineering and Physical Sciences Research Council (EPSRC,UK) who also fund my work.

Abstract

Let K be an infinite field of characteristic $p \geqslant 0$ and let n, r be positive integers. Let $S^+(n,r)$ be the Borel Schur algebra over K, which is a subalgebra of the Schur algebra $S(n,r)$. We aim to give a description of the Borel Schur algebra $S^+(n,r)$ by finding its quiver and relations. We give a complete description of the quiver and relations for $S^+(2,r)$. We also construct a family of embeddings from $S^+(2,r)$ to $S^+(n,r+s)$ which induce embeddings of the corresponding quivers. This gives us some relations for $S^+(n,r)$ for $n > 2$.

We describe the quiver of $S^+(n,r)$ for both $p = 0$ and $p > 0$. We also describe some relations of special type for $p > 0$ and find all relations for $p = 0$.

Introduction

1. Motivation

One wide open problem in the area of representation theory is to understand the representation theory of general linear groups, or equivalently, that of symmetric groups. Here basic representation theoretical questions are still open. For example, it is unknown in general what the dimension of simple modules is, or what the decomposition matrices are. The Schur algebras describe the polynomial representations of general linear groups.

This thesis is concerned with Borel Schur algebras, which are certain subalgebras of Schur algebras. Borel Schur algebras can be used to study Schur algebras. Their representation theory is better understood and their combinatorics is possibly easier than that of Schur algebras.

A Borel Schur algebra is a basic algebra, so we can completely describe it by finding its quiver and relations. This data then allows to calculate (in a relatively easy way) lots of representation theoretical data of this algebra.

2. Descriptions of results

Let K be an infinite field of characteristic $p \geqslant 0$ and let n, r, s be positive integers. Let $S^+(n,r)$ be the Borel Schur algebra over a field K. Our aim is

to give a complete description of the Borel Schur algebra $S^+(n,r)$ by finding its quiver and relations.

Let $\Gamma = (\Gamma_0, \Gamma_1)$ be the quiver of $S^+(n,r)$ where Γ_0 and Γ_1 are the sets of vertices and arrows respectively. Let $\Lambda(n,r)$ be the set of compositions of r with at most n parts. Then the vertex set Γ_0 is equal to $\Lambda(n,r)$ (Sections 1.3 and 1.4).

In general, the quiver for $S^+(n,r)$ is essentially known (see e.g.[13]). Using the results from [16], in Section 4.1 we give a shorter and more elementary proof to describe the arrow set Γ_1 for $S^+(n,r)$.

We find all relations for $S^+(n,r)$ for the case $n = 2$ or $p = 0$ (Sections 2.3 and 5.6). We construct a family of embeddings from $S^+(2,r)$ to $S^+(n,r+s)$ and prove that these embeddings induce embeddings of the corresponding quivers. In this way we find some relations for $S^+(n,r)$ for $n > 2$. We prove a multiplication formula for $S^+(2,r)$ in Section 2.5 and use it to find all $1 \times m$ and $m \times 1$ relations in Γ (Section 5.4).

We also describe some other special relations in Γ. Unfortunately, we have not found all relations for $S^+(n,r)$ for the case $n > 2$ and $p > 0$, which seems to be a difficult problem.

3. Structure of this thesis

In Chapter 1 we give the definitions and background needed to understand the main problem as well as the methods used in the other chapters. We define quivers and list some results about Borel Schur algebras. For more details, see [2], [10], [11] and [16].

In Chapter 2 we describe the quiver and relations for $S^+(2,r)$. We also obtain a multiplication formula for $S^+(2,r)$ in Sections 2.1 and 2.5 and get all relations for $S^+(2,r)$ (Theorem 2.3.1), which we call the p-adic relations.

In Chapter 3 we construct embeddings from $S^+(2,r)$ to $S^+(n,r+s)$. We prove that these embeddings induce embeddings of the corresponding quivers (Section 3.3). In this way we get p-adic relations for $S^+(n,r)$ for $n > 2$. In Section 3.2 we calculate the Cartan invariant $\widehat{c}_{\lambda,\alpha}$ which is the dimension of the vector space $\mathrm{Hom}_A(A\xi_\alpha, A\xi_\lambda)$ where $A = S^+(n,r)$.

In Chapter 4 we give an elementary proof to describe the quiver of $S^+(n,r)$ using some results in [16].

In Chapter 5 we describe *all* relations in the case of characteristic 0 (Section 5.6) using the results from [6] and provide *some* relations in other cases. We consider some special subgraphs of the quiver of $S^+(n,r)$. We obtain a product formula in a rectangle in Section 5.3, to get all relations for the $1 \times m$ and $m \times 1$ rectangles in Section 5.4.

Chapter 1

General background

In this chapter we introduce some notations on the Schur algebras and the Borel Schur algebras, which will be used in the other chapters. We give another proof to find the radical of a Borel Schur algebra in Section 1.2 (first obtained in [11] (Sections 3 and 6)) and show that the Borel Schur algebras are elementary (see Section 1.3) and so is basic. We also introduce the quiver of an algebra in Section 1.4 (for more detail see [2]) and apply this to the Borel Schur algebra $S^+(n,r)$.

Throughout this paper: K is an infinite field of characteristic $p \geqslant 0$; n and r are positive integers.

1.1 The Schur algebra and the Borel Schur algebra

In this section we introduce Schur algebras $S(n,r)$ and Borel Schur algebras $S^+(n,r)$ and describe their elementary properties. For more details see [10]. We also introduce a multiplication formula from [11] and will apply this to the Borel Schur algebra $S^+(n,r)$ in Chapters 2 and 5.

We denote $\underline{n}$ as the set $\{1, \ldots, n\}$. Let $\mathbf{I}(n,r)$ be the set of multi-indices $i = (i_1, \ldots, i_r)$ with $i_\rho \in \underline{n}$ for all $\rho \in \underline{r}$. The symmetric group Σ_r acts on $\mathbf{I}(n,r)$ on the right by place permutations, i.e., $i\pi = (i_{\pi(1)}, \ldots, i_{\pi(r)})$ for $i \in \mathbf{I}(n,r)$ and $\pi \in \Sigma_r$. For example, if $i = (2,1,3,1) \in \mathbf{I}(3,4)$ and $\pi = (132)(4) \in \Sigma_4$, then π acts on i on the right: $i\pi = (3,2,1,1)$.

If we view i as the function $i : \underline{r} \to \underline{n},\ j \mapsto i_j$, then $i\pi$ is just the composition of functions $i \circ \pi$.

Let $i, j \in \mathbf{I}(n,r)$. We define a relation $\sim$ on $\mathbf{I}(n,r)$ by $i \sim j$ if i, j are in the same Σ_r-orbit, that is, $j = i\pi$ for some $\pi \in \Sigma_r$. For example: $(2,1,3,1) \sim (3,2,1,1)$.

The group Σ_r acts on $\mathbf{I}(n,r) \times \mathbf{I}(n,r)$ by place permutations. Let $i, j \in \mathbf{I}(n,r)$ and $\pi \in \Sigma_r$, we can write π acts on (i,j) on the right as follows:

$$(i,j)\pi = (i\pi, j\pi).$$

Similarly, we write $(i,j) \sim (k,l)$ if there exists some $\pi \in \Sigma_r$ such that $(k,l) = (i,j)\pi$, that is, $k = i\pi$ and $l = j\pi$. Note that this yields an equivalence relation on $\mathbf{I}(n,r) \times \mathbf{I}(n,r)$. Let $\Omega(n,r)$ be a set of representatives of equivalence classes of $\mathbf{I}(n,r) \times \mathbf{I}(n,r)$ under the relation $\sim$.

Definition 1.1.1. The *Schur algebra* $S(n,r)$ is an algebra over K with the basis $\{\xi_{i,j} \mid (i,j) \in \Omega(n,r)\}$. The multiplication rule for $S(n,r)$ is given by

$$\xi_{i,j}\xi_{k,l} = \sum_{\substack{(p,q)\in\Omega(n,r) \\ p\sim i,\, q\sim l}} [Z(i,j,k,l,p,q) \cdot 1]\xi_{p,q}, \tag{1.1.1}$$

where $Z(i,j,k,l,p,q) = |\{s \in \mathbf{I}(n,r) \mid (i,j) \sim (p,s) \text{ and } (k,l) \sim (s,q)\}|$.

We call $\{\xi_{i,j} \mid (i,j) \in \Omega(n,r)\}$ the *standard basis* of $S(n,r)$.

Note that $\xi_{i,j} = \xi_{k,l}$ if and only if $(i,j) \sim (k,l)$. $\xi_{i,j}\xi_{k,l} = 0$ unless $j \sim k$.

By Definition 1.1.1, we have the following lemma which will be used later.

Lemma 1.1.2. *([10]) The multiplication rule for $S(n,r)$ given by Equation (1.1.1), can also be written as follows:*

$$\xi_{i,j}\xi_{k,l} = \sum_{\substack{(p',l)\in\Omega(n,r)\\ p'\sim i}} [Z(i,j,k,l,p',l)\cdot 1]\xi_{p',l}, \tag{1.1.2}$$

where $Z(i,j,k,l,p',l) = |\{s' \in \mathbf{I}(n,r) \mid (i,j) \sim (p',s') \text{ and } (k,l) \sim (s',l)\}|$.

Similarly,

$$\xi_{i,j}\xi_{k,l} = \sum_{\substack{(i,q')\in\Omega(n,r)\\ q'\sim l}} [Z(i,j,k,l,i,q')\cdot 1]\xi_{i,q'}, \tag{1.1.3}$$

where $Z(i,j,k,l,i,q') = |\{s' \in \mathbf{I}(n,r) \mid (i,j) \sim (i,s') \text{ and } (k,l) \sim (s',q')\}|$.

Next we introduce a multiplication formula for the Schur algebra $S(n,r)$ in [11] which will be used for the Borel Schur algebra $S^+(n,r)$ later.

Definition 1.1.3. For any $j \in \mathbf{I}(n,r)$, we define

$$P_j := \{\pi \in \Sigma_r \mid j\pi = j\},$$

the stabilizer of j.

Then $P_{h,l} = P_h \cap P_l$ is the stabilizer of the element $(h,l) \in \mathbf{I}(n,r) \times \mathbf{I}(n,r)$. Similarly $P_{h,j,l} = P_h \cap P_j \cap P_l$ is the stabilizer of the element $(h,j,l) \in \mathbf{I}(n,r) \times \mathbf{I}(n,r) \times \mathbf{I}(n,r)$, where the action $\pi \in \Sigma_r$ on $h,j,l \in \mathbf{I}(n,r)$ is as follows:

$$(h,j,l)\pi = (h\pi, j\pi, l\pi).$$

The P_j-obits on $\underline{r}$ are

$$R_a(j) = \{\rho \in \underline{r} \mid j_\rho = a\}, \quad a \in \underline{n}.$$

These sets form a partition of $\underline{r}$, and

$$P_j = \prod_{a \in \underline{n}} P(R_a(j)) \quad \text{(internal direct product)},$$

where $P(R_a(j))$ is the symmetric group on the set $R_a(j)$. In the same way, $P_{h,l}$ has the following orbits on $\underline{r} \times \underline{r}$,

$$R_{a,b}(h,l) = \{\rho \in \underline{r} \mid h_\rho = a,\ l_\rho = b\}, \quad a, b \in \underline{n}.$$

$P_{h,l}$ is the product of the subgroup $P(R_{a,b}(h,l))$ for all $a, b \in \underline{n}$. Similarly

$$R_{a,d,b}(h,j,l) = \{\rho \in \underline{r} \mid h_\rho = a,\ j_\rho = d,\ l_\rho = b\}, \quad a, d, b \in \underline{n}.$$

Theorem 1.1.4. *([11])For any $i, j, l \in \mathbf{I}(n,r)$ there holds*

$$\xi_{i,j}\xi_{j,l} = \sum_h [P_{h,l} : P_{h,j,l}]\xi_{h,l}, \tag{1.1.4}$$

where the sum is over a transversal $\{h\}$ of the $P_{j,l}$-orbits in the set iP_j. The index $[P_{h,l} : P_{h,j,l}]$ appearing in (3.2.5) can be computed from the formula

$$[P_{h,l} : P_{h,j,l}] = \prod_{a,b \in \underline{n}} \frac{r_{a,b}!}{r_{a,1,b}! \cdots r_{a,n,b}!},$$

where, for all $a, d, b \in \underline{n}$, $r_{a,b} = |R_{a,b}(h,l)|$ and $r_{a,d,b} = |R_{a,d,b}(h,j,l)|$.

Remark. 1. We shall always assume that the transversal set contains i.

2. Each integer $z = [P_{h,l} : P_{h,j,l}]$ which appears in (1.1.4) must be interpreted as the element $z \cdot 1_K$ of K. Thus if K has finite characteristic p, these integers are to be taken mod p.

We introduce the following notations. We denote $\Lambda(n,r)$ as the set of compositions of r with at most n parts, i.e. the set of $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\sum_{t=1}^{n} \lambda_t = r$, where λ_t's are nonnegative integers for all $i \in \underline{n}$.

We use the notation

$$\underline{n}^{\lambda} := (1^{\lambda_1}, 2^{\lambda_2}, \ldots, n^{\lambda_n}),$$

where t^{λ_t} is the multi-index $(t, t, \ldots, t)$ with λ_t many t's, for all $t \in \underline{n}$. If $i = \underline{n}^{\lambda}$, we introduce the notation $\xi_{\lambda} := \xi_{i,i}$. It is easy to check that ξ_{λ} is an idempotent in $S(n,r)$, that is, $\xi_{\lambda}^2 = \xi_{\lambda}$.

Definition 1.1.5. Let λ and μ be in $\Lambda(n,r)$. We define

$$\xi_{\lambda,\mu} := \xi_{\underline{n}^{\lambda}, \underline{n}^{\mu}}.$$

Lemma 1.1.6. *([G80], § 2.1) (1)* $\dim S(n,r) = \binom{n^2+r-1}{r}$.
(2) $\{\xi_{\lambda} \mid \lambda \in \Lambda(n,r)\}$ *is a set of orthogonal idempotents, moreover*

$$\varepsilon = \sum_{\lambda \in \Lambda(n,r)} \xi_{\lambda},$$

where ε *is the identity element in* $S(n,r)$.

Let $i, j \in \mathbf{I}(n,r)$ and define $i \leqslant j$ if $i_{\rho} \leqslant j_{\rho}$, for all $\rho = 1, \ldots, r$. We write $i < j$ to mean that $i \leqslant j$ and $i \neq j$. Let

$$\Omega^{+}(n,r) := \{(i,j) \in \Omega(n,r) \mid i \leqslant j\}.$$

We now define the Borel Schur algebra.

Definition 1.1.7. The *Borel Schur algebra* $S^{+}(n,r)$ is the subalgebra of the Schur algebra $S(n,r)$ with basis $\{\xi_{i,j} \mid (i,j) \in \Omega^{+}(n,r)\}$.

Using the multiplication rule for the Schur algebra $S(n,r)$, we can check that $S^+(n,r)$ is indeed a subalgebra of $S(n,r)$.

Example

The dimension of $S(2,2)$ is $\binom{2^2+2-1}{2} = 10$. $S(2,2)$ has the basis:

$$\{\xi_{11,11}, \xi_{11,12}, \xi_{11,22}, \xi_{12,11}, \xi_{12,12}, \xi_{12,21}, \xi_{12,22}, \xi_{22,11}, \xi_{22,12}, \xi_{22,22}\}.$$

The Borel Schur algebra $S^+(2,r)$ has the basis

$$\{\xi(b,a) \mid 0 \leqslant a \leqslant b \leqslant r\},$$

where

$$\xi(b,a) = \xi_{1^b 2^{r-b}, 1^a 2^{r-a}}, \quad \forall\ 0 \leqslant a \leqslant b \leqslant r. \tag{1.1.5}$$

Thus

$$\dim S^+(2,r) = \sum_{b=0}^{r}(b+1) = \binom{r+2}{2}.$$

Then, the dimension of $S^+(2,2)$ is $\binom{2+2}{2} = 6$. $S^+(2,2)$ has the basis:

$$\{\xi_{11,11}, \xi_{11,12}, \xi_{11,22}, \xi_{12,12}, \xi_{12,22}, \xi_{22,22}\},$$

that is,

$$\{\xi(2,2), \xi(2,1), \xi(2,0), \xi(1,1), \xi(1,0), \xi(0,0)\}.$$

We will describe the quiver and relations of $S^+(2,r)$ in Chapter 2.

1.2 The radical of a Borel Schur algebra

From now on, we consider the Borel Schur algebras in more detail. First we find the radical of the Borel Schur algebra $S^+(n,r)$. This was first described

in [11]. Here we give another proof in terms of the distances between two multi-indices in $\mathbf{I}(n,r)$. Related results can be found in [11] and [16].

Let A be a finite dimensional algebra. Recall that the radical of A, denoted by $\operatorname{rad} A$, is the maximal nilpotent ideal of A, or equivalently, the smallest ideal with semi-simple quotient, also equivalently, a nilpotent ideal R of A such that A/R is semi-simple.

Definition 1.2.1. Let $i = (i_1, \ldots, i_r)$ and $j = (j_1, \ldots, j_r)$ be in $\mathbf{I}(n,r)$. We define the *distance* from i to j:

$$\operatorname{dist}(i,j) = \sum_{\rho=1}^{r}(j_\rho - i_\rho) = (\sum_{\rho=1}^{r} j_\rho) - (\sum_{\rho=1}^{r} i_\rho).$$

Let $\xi_{i,j}$ be in $S^+(n,r)$. Then $i \leqslant j$, that is, $i_\rho \leqslant j_\rho$ for all $\rho \in \underline{r}$, so $\operatorname{dist}(i,j) \geqslant 0$.

Lemma 1.2.2. *Let i,j,k,l be in $\mathbf{I}(n,r)$.*

(i) If $i \sim k$ and $j \sim l$, then $\operatorname{dist}(i,j) = \operatorname{dist}(k,l)$.

(ii) $\operatorname{dist}(i,k) = \operatorname{dist}(i,j) + \operatorname{dist}(j,k)$.

(iii) The maximum distance in $\mathbf{I}(n,r)$ is

$$\max_{i,j \in \mathbf{I}(n,r)} \operatorname{dist}(i,j) = (n-1)r. \tag{1.2.1}$$

(iv) If $i < j$, then $\operatorname{dist}(i,j) \geqslant 1$.

Proof. The proof is trivial. ∎

Now we are ready to describe the radical of $S^+(n,r)$.

Proposition 1.2.3. *([11], Sections 3 and 6) The radical of $S^+(n,r)$ has basis $\{\xi_{i,j} \mid i < j,\ (i,j) \in \Omega^+(n,r)\}$.*

Proof. Let T be the vector space of $S^+(n,r)$ spanned by $\{\xi_{i,j} \mid i < j\}$ and let R be the radical of $S^+(n,r)$. By the multiplication rule for $S(n,r)$, we know that T is a two-sided ideal of $S^+(n,r)$. We need to prove $T = R$. Since R is the smallest ideal of $S^+(n,r)$ with semi-simple quotient and $S^+(n,r)$ satisfies

$$S^+(n,r)/T \cong \bigoplus_{\lambda \in \Lambda(n,r)} K \cdot \xi_\lambda,$$

we have $T \supseteq R$.

We claim T is a nilpotent ideal. That implies $T \subseteq R$, so $T = R$. We claim

$$\xi_{i^{(1)},j^{(1)}}\xi_{i^{(2)},j^{(2)}} \cdots \xi_{i^{(L)},j^{(L)}} = 0, \tag{1.2.2}$$

where $\xi_{i^{(t)},j^{(t)}} \in T$ for all $t \in \underline{L}$ and $L = (n-1)r + 1$. So T is nilpotent.

If $j^{(t)} \nsim i^{(t+1)}$ for some t with $t \in \underline{L-1}$, then the product is 0. Thus we can assume $j^{(1)} \sim i^{(2)}$, i.e., $i^{(2)}\pi = j^{(1)}$ for some $\pi \in \Sigma_r$. Then

$$\xi_{i^{(2)},j^{(2)}} = \xi_{i^{(2)}\pi,j^{(2)}\pi} = \xi_{j^{(1)},j^{(2)}\pi}.$$

Hence we can assume that $j^{(t)} = i^{(t+1)}$ for $t \in \{1, 2, \ldots, L-1\}$. Since $\xi_{i^{(2)},j^{(2)}} \in T$, we have $i^{(t)} < j^{(t)}$. By Lemma 1.2.2 (iv), $\mathrm{dist}(i^{(t)}, j^{(t)}) \geqslant 1$ for all $t \in \underline{L}$.

Since $S^+(n,r)$ is a subalgebra, we let

$$\xi_{i^{(1)},j^{(1)}} \cdot \xi_{i^{(2)},j^{(2)}} \cdots \xi_{i^{(L)},j^{(L)}} = \sum_{\substack{i \leqslant j \\ i \sim i^{(1)}, j \sim j^{(L)}}} s_{ij}\xi_{ij},$$

where $s_{i,j} \in K$. Suppose that the above product is not 0. Then there exists some nonzero $s_{ij} \neq 0$ with $i \sim i^{(1)}$ and $j \sim j^{(L)}$. By Lemma 1.2.2 (i), we have $\mathrm{dist}(i,j) = \mathrm{dist}(i^{(1)}, j^{(L)})$. By 1.2.2 (ii), we get

$$\mathrm{dist}(i,j) = \mathrm{dist}(i^{(1)}, j^{(L)}) = \sum_{t=1}^{L} \mathrm{dist}(i^{(t)}, j^{(t)}) \geqslant L > (n-1)r,$$

which contradicts to Lemma 1.2.2 (iii). Hence Equation (1.2.2) holds. Therefore T is a nilpotent ideal. ∎

Definition 1.2.4. Let i be in $\mathbf{I}(n,r)$. We denote the *weight* of i by $wt(i)$, where $wt(i) = (a_1, a_2, \ldots, a_n)$ is in $\Lambda(n,r)$ and a_ρ is the number of $\rho \in \underline{n}$ appearing in i; more formally, if we view i as a function from $\underline{r}$ to $\underline{n}$, $\lambda_\rho = |i^{-1}(\rho)|$, that is,

$$a_\rho = |\{t \mid i_t = \rho, t \in \underline{r}\}|, \quad \forall\, \rho \in \underline{n}.$$

Definition 1.2.5. Let $\alpha = (a_1, a_2, \ldots, a_n)$ and $\beta = (b_1, b_2, \ldots, b_n)$ be in $\mathbf{I}(n,r)$. We denote $\alpha \trianglerighteq \beta$ for the *dominance ordering* if

$$\sum_{s=1}^{t} a_s \geqslant \sum_{s=1}^{t} b_s, \quad \forall\, t \in \underline{n}.$$

1.3 The algebra $S^+(n,r)$ is elementary

In this section we prove that the Borel Schur algebras $S^+(n,r)$ are elementary and so are basic.

Let A be a finite dimensional algebra with an identity over a field K. We start with several well-known general facts on ring theory.

Theorem 1.3.1. *(a) Suppose that*

$$A = P_1 \oplus \ldots \oplus P_n,$$

where P_i's are indecomposable A-modules. Let e_i in P_i for $i = 1, \ldots, n$ be such that

$$1 = e_1 + \ldots + e_n.$$

Then $\{e_1, \ldots, e_n\}$ is a set of nonzero primitive orthogonal idempotents with the property $Ae_i = P_i$ for $i = 1, \ldots, n$.

(b) Suppose that $\{e_1, \ldots, e_n\}$ *is a set of nonzero primitive orthogonal idempotents such that*

$$1 = e_1 + \ldots + e_n.$$

Then Ae_i *is an indecomposable submodule of* A *for all* $i = 1, \ldots, n$ *and*

$$A = Ae_1 \oplus \ldots \oplus Ae_n.$$

Let e be an idempotent. We say e is *primitive*, if e can not be written as a sum of idempotents e_1 and e_2 with $e_1e_2 = 0$.

Lemma 1.3.2. *Let* e *be a nonzero idempotent of* A. *Then* e *is primitive if and only if* Ae *is indecomposable.*

Lemma 1.3.3. *A nonzero* A*-module* M *is indecomposable if and only if the ring* $\operatorname{End} M$ *contains no idempotent except* $0, 1$.

Lemma 1.3.4. *Let* M *be a left* A*-module. Then* $\operatorname{Hom}_A(A, M)$ *has a left* A*-module structure. Moreover, as left* A*-modules,* $M \simeq \operatorname{Hom}_A(A, M)$.

Proof. For all $x \in A$ and $\varphi \in \operatorname{Hom}_A(A, M)$, we define

$$x \cdot \varphi(a) := \varphi(ax),$$

for $a \in A$. We claim that $\operatorname{Hom}_A(A, M)$ is a left A-module with respect to the above action. For all $x_1, x_2 \in A$, we show $(x_1x_2)\varphi = x_1(x_2\varphi)$. For all $a \in A$, we have

$$\begin{aligned}((x_1x_2)\varphi)(a) &= \varphi(a(x_1x_2)) = \varphi((ax_1)x_2) \\ &= (x_2\varphi)(ax_1) = (x_1(x_2\varphi))(a).\end{aligned}$$

We define a map as follows:

$$\begin{array}{ccc} \varphi : M & \longrightarrow & \operatorname{Hom}_A(A, M) \\ m & \longmapsto & \left(\begin{array}{ccc} \varphi(m) : & A & \longrightarrow M \\ & a & \longmapsto am \end{array} \right) \end{array}$$

for all $a \in A$ and $m \in M$. In the following we show that φ is a left A-module isomorphism.

For all $x, a \in A$ and $m \in M$, we have

$$(x\varphi(m))(a) = \varphi(m)(ax) = (ax)m = a(xm) = \varphi(xm)(a).$$

Then $x\varphi(m) = \varphi(xm)$. Hence φ is a left A-module homomorphism. Suppose that $\varphi(m)(a) = 0$, for all $a \in A$ and some $m \in M$. That is $am = 0$ for all $a \in A$ and some $m \in M$. Let $a = 1$, then $m = 0$. This means that $\ker \varphi = 0$, and hence φ is injective. For any $\psi \in \mathrm{Hom}_A(A, M)$, let $\psi(1) = m$, thus $\psi(a \cdot 1) = a\psi(1) = am$ as ψ is a homomorphism. Thus $\psi = \varphi(m)$, and hence φ is surjective. ∎

Lemma 1.3.5. *Let $e, f \in A$ be idempotents. Then $\mathrm{Hom}_A(Ae, Af) \cong eAf$ and $\mathrm{Hom}_A(eA, fA) \cong fAe$. In particular, $\mathrm{Hom}_A(A, A)$ and A are isomorphic.*

Proof. We only prove $\mathrm{Hom}_A(Ae, Af) \cong eAf$. We define a map φ as follows:

$$\begin{array}{rcl} \varphi : eAf & \longrightarrow & \mathrm{Hom}_A(Ae, Af) \\ eaf & \longmapsto & \left(\begin{array}{lll} \varphi(eaf): & Ae & \longrightarrow Af \\ & be & \longmapsto (be)(eaf) \end{array} \right) \end{array}$$

for all $a, b \in A$. It is obvious that φ is a homomorphism. We show that φ is isomorphism.

Suppose that $\varphi(eaf) = 0$ for some $a \in A$. That is, $\varphi(eaf)(be) = 0$, for all $b \in A$. We let $b = e$, then we have $beeaf = eaf = 0$. Hence φ is injective. For any $\psi \in \mathrm{Hom}_A(Ae, Af)$, then $\psi(e) = af$ there exists some $a \in A$. Then $\psi(a'e) = a'e\psi(e) = a'eaf = (a'e)(eaf) = \varphi(eaf)(a'e)$. Thus we have $\psi = \varphi(eaf)$. That means that φ is surjective. ∎

Next we describe primitive orthogonal idempotents of $S^+(n,r)$.

Proposition 1.3.6. *([16], (2.2) Proposition) The set* $\{\xi_\lambda \mid \lambda \in \Lambda(n,r)\}$ *is a complete set of primitive orthogonal idempotents of* $S^+(n,r)$.

Proof. By Lemma 1.1.6 (2), $\{\xi_\lambda \mid \lambda \in \Lambda(n,r)\}$ is a set of orthogonal idempotents, moreover we have

$$\varepsilon = \sum_{\lambda \in \Lambda(n,r)} \xi_\lambda,$$

where ε is the identity element in $S(n,r)$ (so in $S^+(n,r)$). Let $B = S^+(n,r)$. By Lemma 1.3.2, ξ_λ is primitive if and only if $B\xi_\lambda$ is indecomposable. By Lemma 1.3.3, this is equivalent to proving that $\mathrm{End}_B(B\xi_\lambda)$ contains no idempotent except 0 and ξ_λ. By Lemma 1.3.5, we have $\mathrm{End}_B(B\xi_\lambda) \cong \xi_\lambda B \xi_\lambda$. It is obvious that $\xi_\lambda B \xi_\lambda$ has a basis $E := \{\xi_{i,j} \mid \mathrm{wt}(i) = \mathrm{wt}(j) = \lambda,\ (i,j) \in \Omega^+(n,r)\}$. Let $\xi_{i,j} \in E$. Since $\xi_{i,j} \in B$, then $i \leqslant j$; but $\mathrm{wt}(i) = \mathrm{wt}(j) = \lambda$, we have $j = i\pi$ for some $\pi \in \Sigma_r$. Then $i = j$, hence $\xi_{i,j} = \xi_\lambda$. Thus $\xi_\lambda B \xi_\lambda$ is one-dimensional spanned by ξ_λ and contains no idempotents except 0 and ξ_λ. Hence the idempotent ξ_λ is primitive. ∎

Next we introduce a class of algebras which plays a fundamental role in the theory of finite dimensional algebras.

Definition 1.3.7. Let A be a finite dimensional algebra over a field K, and $R = \mathrm{rad}\, A$. Then A is said to be *basic* if A/R is isomorphic to a product of division algebras. Moreover if all these division algebras are isomorphic to K, A is called *elementary*.

Now we prove the main theorem of this section.

Theorem 1.3.8. *The Borel Schur algebra $S^+(n,r)$ is elementary.*

Proof. In Proposition 1.2.3 we determined the radical R of the Borel Schur algebra $S^+(n,r)$, which implies that $S^+(n,r)/R$ as a K-vector space has the basis $\{\xi_\lambda \mid \lambda \in \Lambda(n,r)\}$, i.e.

$$S^+(n,r)/R \cong \bigoplus_{\lambda \in \Lambda(n,r)} K \cdot \xi_\lambda. \tag{1.3.1}$$

Hence by Definition 1.4.5, $S^+(n,r)$ is elementary. ∎

1.4 The quiver of an algebra

In this section we introduce the quiver of an algebra (see III.1 in [2]) and apply this to the Borel Schur algebra $S^+(n,r)$.

A *quiver* $\Gamma = (\Gamma_0, \Gamma_1)$ is a directed graph with a set Γ_0 of vertices and a set Γ_1 of arrows. We define two functions s and e from the set of arrows to the set of vertices: If $\gamma : i \to j$ is an arrow, we denote by $s(\gamma)$ the starting vertex i of the arrow γ and by $e(\gamma)$ the terminating vertex j. A *path* w in the quiver Γ is a word $w = \gamma_h \dots \gamma_2\gamma_1$ with $e(\gamma_i) = s(\gamma_{i+1})$ for $1 \leq i \leq h-1$ and $h \in \mathbb{N}$. We define the function l from the set of paths into the natural numbers by $l(w) := h$, which is just the length of the path w. For every vertex i we define a path of length zero, which we call e_i. For a given quiver Γ we define the *path algebra* $K\Gamma$ as follows: $K\Gamma$ is the K-vector space whose basis consists of all paths in Γ. The algebra multiplication of $K\Gamma$ is given by linear extension of the following multiplication of the basis elements:

$$(\gamma_1 \dots \gamma_h)\cdot(\beta_1 \dots \beta_k) = \begin{cases} 0, & \text{if } s(\gamma_h) \neq e(\beta_1), \\ \gamma_1 \dots \gamma_h\beta_1 \dots \beta_k, & \text{otherwise (path composition)}, \end{cases}$$

and

$$e_i\gamma = \begin{cases} \gamma, & \text{if } e(\gamma) = i, \\ 0, & \text{otherwise,} \end{cases} \qquad \gamma e_j = \begin{cases} \gamma, & \text{if } s(\gamma) = j, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 1.4.1. Let Σ be a ring and V a Σ-bi-module. We write the n-fold tensor product $V \otimes_\Sigma V \otimes_\Sigma \ldots \otimes_\Sigma V$ as V^n. The *tensor ring* $T(\Sigma, V)$ is defined as

$$T(\Sigma, V) = \Sigma \coprod V \coprod V^2 \coprod \ldots \coprod V^i \coprod \ldots,$$

as an abelian group (Here this is direct product, not direct sum). Writing $V^0 = \Sigma$, multiplication is induced by the natural Σ-bilinear maps $V^i \times V^j \to V^{i+j}$ for all $i \geqslant 0$ and $j \geqslant 0$.

The following definition gives the quiver of the tensor ring $T(\Sigma, V)$.

Definition 1.4.2. Let K be a field. For each positive integer n we denote by $\prod_n(K)$ the K-algebra which as a ring is $K \times \ldots \times K$, the product of K with itself n times, and has the K-algebra structure given by the ring morphism $\varphi : K \to \prod_n(K)$ where $\varphi(x) = (x, \ldots, x)$ for all x in K. Let $\Sigma = \prod_n(K)$ and let V be a Σ-bi-module where K acts centrally, that is $av = va$ for $a \in K$ and $v \in V$, and assume that V is finite dimensional over K. Then the tensor ring $T(\Sigma, V)$ is a K-algebra, and we can associate with $T(\Sigma, V)$ a quiver $\Gamma = (\Gamma_0, \Gamma_1)$ in the following way. The set of vertices Γ_0 is $\{1, \ldots, n\}$. Let ϵ_i for $i = 1, \ldots, n$ be the idempotent of Σ with the ith coordinate equal to 1 and the other coordinates zero. Then $\epsilon_j V \epsilon_i$ is a K-subspace of V and there will be $\dim_K(\epsilon_j V \epsilon_i)$ arrows from i to j in Γ. The quiver $\Gamma = (\Gamma_0, \Gamma_1)$ constructed in this way is called the *quiver of* $T(\Sigma, V)$.

We can label the arrows from i to j by the elements of the basis for $\epsilon_j V \epsilon_i$.

Definition 1.4.3. A *relation* σ on a quiver Γ over a field K is a K-linear combination of paths $\sigma = a_1p_1 + \ldots + a_np_n$ with $a_i \in K$ and $e(p_1) = \ldots = e(p_n)$ and $s(p_1) = \ldots = s(p_n)$. We here assume that the length $l(p_i)$ of each p_i, that is the number of arrows in each path, is at least 2. If $\rho = \{\sigma_t\}_{t\in T}$ is a set of relations on Γ over K, the pair (Γ, ρ) is called a *quiver with relations* over K. Associated with (Γ, ρ) is the K-algebra $K(\Gamma, \rho) = K\Gamma/\langle\rho\rangle$, where $\langle\rho\rangle$ denotes the ideal in $K\Gamma$ generated by the set of relations ρ. We have by assumption $\langle\rho\rangle \subset J^2$, where J is the ideal of $K\Gamma$ generated by all the arrows in Γ.

We have the following connection between tensor algebras and path algebras.

Proposition 1.4.4. *([2], Page 53, Proposition 1.3) Let K be a field, and $\Sigma = \prod_n(K)$. Let V be a Σ-bi-module where K acts centrally and which is finite dimensional over K. If Γ is the quiver of the tensor algebra $T(\Sigma, V)$, then there is a K-algebra isomorphism $\phi : T(\Sigma, V) \to K\Gamma$ such that $\phi(\coprod_{j\geqslant t} V^j) = J^t$.*

We now introduce the quiver of an elementary algebra.

Definition 1.4.5. Let Λ be an elementary K-algebra, i.e.

$$\Lambda/R \simeq \prod_n(K), \quad \text{for some } n,$$

where $R = \operatorname{rad}\Lambda$. Then the quiver of the tensor algebra $T(\Lambda/R, R/R^2)$ is called the *quiver of* Λ.

This definition is justified by the following important theorem.

Theorem 1.4.6. *Let Λ be a finite dimensional elementary K-algebra.*

(a) Let $\{e_1, \ldots, e_n\}$ be a complete set of primitive orthogonal idempotents in Λ, and $\{r_1, \ldots, r_t\}$ a set of elements in $R = \operatorname{rad}\Lambda$ such that the images $\overline{r}_1, \ldots, \overline{r}_t$ in R/R^2 generate R/R^2 as a Λ/R-module. Then $\{e_1, \ldots, e_n, r_1, \ldots, r_t\}$ generate Λ as a K-algebra.

(b) There is a surjective ring homomorphism $\widetilde{f} : T(\Lambda/R, R/R^2) \to \Lambda$ with

$$\coprod_{j \geqslant rl(\Lambda)} (R/R^2)^j \subset Ker(\widetilde{f}) \subset \coprod_{j \geqslant 2} (R/R^2)^j,$$

where $rl(\Lambda)$ is the Loewy length of Λ.

(c) $\Lambda \simeq K(\Gamma, \rho)$ with $J^s \subset < \rho > \subset J^2$ for some s, where Γ is the quiver of Λ and ρ is a set of relations of Γ over K, and J is the ideal of $K\Gamma$ generated by all the arrows in Γ.

(d) If $\Lambda \simeq K(\Gamma, \rho)$ with $J^t \subset < \rho > \subset J^2$ for some t, then Γ is the quiver of Λ.

Proposition 1.4.7. *Let Λ be an elementary finite dimensional algebra over K and $1 = \epsilon_1 + \ldots + \epsilon_n$ a decomposition of 1 into a sum of primitive orthogonal idempotents. Let $P_i = \Lambda\epsilon_i$ and $S_i = P_i/RP_i$ for $i = 1, \ldots, n$. Then for a given pair of numbers i, j in $\{1, \ldots, n\}$ the following numbers are the same.*

(a) $\dim_K(\epsilon_j(R/R^2)\epsilon_i)$.

(b) The multiplicity of the simple module S_j in RP_i/R^2P_i.

(c) $\dim_K Ext^1_\Lambda(S_i, S_j)$.

In view of Proposition 1.4.7, the number of arrows from i to j is equal to $\dim_K Ext^1_\Lambda(S_i, S_j)$, the quiver of Λ is sometimes called *Ext-quiver* of Λ.

Our aim is to describe the quiver and relations of the Borel Schur algebra $S^+(n, r)$. Since $S^+(n, r)$ is elementary (see Theorem 1.3.8), this is equivalent

to describing Γ and ρ as in Theorem 1.4.6. Thus we use Definitions 1.4.2 and 1.4.5 to describe Γ, the quiver of $S^+(n,r)$. Let $R = \operatorname{rad} S^+(n,r)$. Since

$$S^+(n,r)/R \cong \bigoplus_{\lambda \in \Lambda(n,r)} K \cdot \xi_\lambda,$$

by Definition 1.4.2 the vertices of Γ are labeled by the primitive orthogonal idempotents ξ_λ, or simply by λ. Then $\xi_\mu(R/R^2)\xi_\lambda$ is a subspace of R/R^2, and by Definition 1.4.2 there will be $\dim \xi_\mu(R/R^2)\xi_\lambda$ arrows from λ to μ in Γ. We label the arrows by the basis elements of $\xi_\mu(R/R^2)\xi_\lambda$.

The relations can be described as follows. By Theorem 1.4.6 (b), there is a surjective ring homomorphism $\widetilde{f} : T(\Sigma, V) \to S^+(n,r)$. By Proposition 1.4.4, the tensor algebra $T(\Sigma, V)$ is isomorphic to the path algebra $K\Gamma$. Then the map $\widetilde{f}$ can be regarded as the map from the path algebra $K\Gamma$ to the algebra $S^+(n,r)$. The relations are all the linear combinations of paths which are in $\ker \widetilde{f}$.

Chapter 2

The quiver and relations of the Borel Schur algebra $S^+(2,r)$

In this chapter we describe the quiver and relations for $S^+(2,r)$ of positive characteristic in Sections 2.2 and 2.3 and for characteristic 0 case in Section 2.4. We consider the product of basis elements in $S^+(2,r)$ and obtain a formulae for $S^+(2,r)$ in Sections 2.1 and 2.5, which will be used in Section 5.4. As for the multiplication formula for $S^+(2,r)$ (first obtained in [16], Section 4), we provide an elementary proof and another proof using the formula in [11].

By the example at the end of Section 1.1, $S^+(2,r)$ has the basis

$$\{\xi(b,a) \mid 0 \leqslant a \leqslant b \leqslant r\},$$

where

$$\xi(b,a) = \xi_{1^b 2^{r-b}, 1^a 2^{r-a}}, \quad \forall\ 0 \leqslant a \leqslant b \leqslant r. \tag{2.0.1}$$

Thus

$$\dim S^+(2,r) = \sum_{b=0}^{r}(b+1) = \binom{r+2}{2}.$$

2.1 The multiplication formula for $S^+(2,r)$

Before we determine the quiver of $S^+(2,r)$, we rewrite the multiplication formula for $S^+(2,r)$. Actually, we give a specific multiplication formula for the Borel Schur algebra $S^+(n,r)$, which will be used in Section 3.1. At the end of this section, we give a formula for the product of elements in $S^+(2,r)$, which will be used in Section 2.3. First, we introduce a notation, which will be used several times in this thesis.

Definition 2.1.1. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be in $\Lambda(n,r)$. For a positive integer $t \in \{1,2,\dots,n-1\}$ and a nonnegative integer m such that $m \leqslant \lambda_{t+1}$, we define $\lambda(t,m) \in \Lambda(n,r)$ by

$$\lambda(t,m) = (\lambda_1, \dots, \lambda_t + m, \lambda_{t+1} - m, \dots \lambda_n).$$

Recall that we denote $\xi_{\underline{n}^\lambda,\underline{n}^\mu}$ by $\xi_{\lambda,\mu}$ (see Definition 1.1.5).

We now have the following multiplication formula:

Lemma 2.1.2. *([16]) Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda(n,r)$. Let $t \in \{1,2,\dots,n-1\}$ and let l and m be nonnegative integers such that $l \leqslant m \leqslant \lambda_{t+1}$. We have*

$$\xi_{\lambda(t,m),\lambda(t,l)}\xi_{\lambda(t,l),\lambda} = \binom{m}{l}\xi_{\lambda(t,m),\lambda}.$$

Proof. Following the above notation, we denote the multi-indices as follows:

$$\begin{aligned} i &= \underline{n}^{\lambda(t,m)} = 1^{\lambda_1} \dots t^{\lambda_t+m}(t+1)^{\lambda_{t+1}-m} \dots n^{\lambda_n}, \\ j &= \underline{n}^{\lambda(t,l)} = 1^{\lambda_1} \dots t^{\lambda_t+l}(t+1)^{\lambda_{t+1}-l} \dots n^{\lambda_n}, \\ k &= \underline{n}^{\lambda} = 1^{\lambda_1} \dots t^{\lambda_t}(t+1)^{\lambda_{t+1}} \dots n^{\lambda_n}. \end{aligned}$$

We need to calculate the product $\xi_{i,j}\xi_{j,k}$. By Lemma 1.1.2,

$$\xi_{i,j}\xi_{j,k} = \sum_{\substack{(i,q)\in\Omega(n,r)\\ q\sim k}} [Z(i,j,j,k,i,q)\cdot 1]\xi_{i,q},$$

where $Z = Z(i,j,j,k,i,k) = |\{s \in \mathbf{I}(n,r) \mid (i,j) \sim (i,s) \text{ and } (j,k) \sim (s,q)\}|$. We now claim that:

(i) $\xi_{i,q} = \xi_{i,k}$, that is, there is only one summand $\xi_{i,q}$ in the above sum. Since $(i,j) \sim (i,s)$ and $i \leqslant j$, Σ_r acts on (i,j) simultaneously, thus we have $i \leqslant s$. In fact, $(i,s) = (i,j)\pi$ for some $\pi \in \Sigma_r$. Then for all $1 \leqslant u \leqslant r$, we have

$$i_u = i_{\pi(u)} \leqslant j_{\pi(u)} = s_u.$$

Thus we get $i \leqslant s$. Similarly $(j,k) \sim (s,q)$ and $j \leqslant k$, then $s \leqslant q$. As $i \leqslant s$ and $s \leqslant q$, we have $i \leqslant q$. Since

$$\begin{aligned} i &= 1^{\lambda_1} \ldots t^{\lambda_t+m}(t+1)^{\lambda_{t+1}-m} \ldots n^{\lambda_n} \\ &\leqslant q \sim k = 1^{\lambda_1} \ldots t^{\lambda_t}(t+1)^{\lambda_{t+1}} \ldots n^{\lambda_n}. \end{aligned}$$

We claim that the first λ_1 entries of q are 1's, that is, $q_1 = \ldots = q_{\lambda_1} = 1$. Otherwise, since $q \sim k$, there exists an integer $l > \lambda_1$ such that $q_l = 1$. But the l-th entry of i, $i_l > 1$. Thus $i_l > 1 = q_l$, which contradicts the assumption $i \leqslant q$. Similarly we get the entries of q as follows:

$$\begin{aligned} &q_{\lambda_1+1} = \ldots = q_{\lambda_1+\lambda_2} = 2, \\ &\ldots \\ &q_{\lambda_1+\ldots+\lambda_{t-2}+1} = \ldots = q_{\lambda_1+\ldots+\lambda_{t-2}+\lambda_{t-1}} = t-1. \end{aligned}$$

Moreover, the entries of q from the $(\lambda_1 + \ldots + \lambda_{t-2} + \lambda_{t-1} + 1)$-th to the $(\lambda_1 + \ldots + \lambda_t + \lambda_{t+1})$-th places will be t or $t+1$. Otherwise there exists a certain integral entry that is $\geqslant t+2$. Since $q \sim k$ and the respective entries of k are $t^{\lambda_t}(t+1)^{\lambda_{t+1}}$, there exists an integer $v \geqslant$

$\lambda_1 + \ldots + \lambda_{t-2} + \lambda_{t+1} + 1$ such that the v-th entry of q is t or $(t+1)$. We know that $i_v \geqslant t+2 > q_v$, which contradicts the assumption $i \leqslant q$. Hence by the same argument as above, we get the entries of q as follows:

$$\begin{aligned} &q_{\lambda_1+\ldots+\lambda_{t+1}+1} = \cdots = q_{\lambda_1+\ldots+\lambda_{t+1}+\lambda_{t+2}} = t+2, \\ &\cdots \\ &q_{\lambda_1+\ldots+\lambda_{n-1}+1} = \cdots = q_{\lambda_1+\ldots+\lambda_{n-1}+\lambda_n} = n. \end{aligned}$$

Furthermore, since $i \leqslant q$ and $q \sim k$, we get that the entries of q from the $(\lambda_1 + \ldots + \lambda_{t-1} + 1)$-th to the $(\lambda_1 + \ldots + \lambda_t + \lambda_{t+1})$-th places are $t^{\lambda_t}(t+1)^{\lambda_{t+1}}$. Hence $(i,q) = (i,k)$.

(ii) $Z = \binom{m}{l}$. By (i), we need to calculate the number of $s \in \mathbf{I}(n,r)$ satisfying the conditions: $(i,j) \sim (i,s)$ and $(j,k) \sim (s,k)$. Since $i \leqslant j \leqslant k$, then we have $i \leqslant s \leqslant k$ and $s \sim j$. Since the entries of i and q are equal, $i_u = k_u$ for all $u = 1, \ldots, \lambda_1 + \ldots + \lambda_{t-1}$, and $u = \lambda_1 + \ldots + \lambda_{t+1} + 1, \ldots, n$. Thus these entries of s, are equal to the ones of i and q. Furthermore, since $i \leqslant s \leqslant k$, we get the remaining entries of s, which we denote as

$$s' := s_{\lambda_1+\ldots+\lambda_{t-1}+1} \cdots s_{\lambda_1+\ldots+\lambda_{t+1}},$$

satisfying the condition:

$$t^{\lambda_t+m}(t+1)^{\lambda_{t+1}-m} \leqslant s' \leqslant t^{\lambda_t}(t+1)^{\lambda_{t+1}},$$

where $s' \sim t^{\lambda_t+l}(t+1)^{\lambda_{t+1}-l}$ since $s \sim j$. From the above condition for the s', we get the entries of s':

$$s'_1 = \ldots = s'_{\lambda_t} = t, \quad s'_{\lambda_t+m+1} = \ldots = s'_{\lambda_t+\lambda_{t+1}} = t+1.$$

Hence the only unknown entries of the s' are $s'_{\lambda_t+1} \ldots s'_{\lambda_t+m}$, which are a permutation of the $t^l(t+1)^{m-l}$ since $s \sim j$. Therefore the number of

s or the number of s', is equal to the combination number $\binom{m}{l}$.

∎

We will give a second proof for Lemma 2.1.2 at the end of Section 3.2, using the multiplication formula in Theorem 1.1.4.

Let $n = 2$ in Lemma 2.1.2. Then $t = 1$ and we obtain the following formula for $S^+(2, r)$:

Corollary 2.1.3. *Let $\lambda = (b, r-b) \in \Lambda(2, r)$ and l, m nonnegative integers such that $l \leqslant m \leqslant r - b$. We have*

$$\xi(b+m, b+l))\xi(b+l, b) = \binom{m}{l}\xi(b+m, b).$$

Finally, we calculate certain products of basis elements in $S^+(2, r)$, which will be used later.

Proposition 2.1.4. *Let t be a positive integer. Let a, $m_0, m_1, \ldots, m_{t-1}$ be nonnegative integers such that*

$$a + m_0 + m_1 + \ldots + m_{t-1} \leqslant r.$$

Then

$$\begin{aligned} &\xi(a+m_0+m_1+\ldots+m_{t-1}, a+m_0+m_1+\ldots+m_{t-2}) \\ &\qquad \xi(a+m_0+m_1+\ldots+m_{t-2}, a+m_0+m_1+\ldots+m_{t-3}) \\ &\qquad\qquad \ldots\ \xi(a+m_0+m_1, a+m_0)\,\xi(a+m_0, a) \\ = &\prod_{i=0}^{t-1} \binom{m_0+m_1+\ldots+m_i}{m_i} \xi(a+\sum_{i=0}^{t-1} m_i, a). \end{aligned}$$

In particular, for $m_0 = m_1 = \ldots = m_{t-1} = l$, we get

$$\begin{aligned} &\xi(a+tl, a+(t-1)l)\xi(a+(t-1)l, a+(t-2)l)\ \ldots\ \xi(a+l, a) \\ = &\prod_{i=0}^{t-1} \binom{(i+1)l}{l} \xi(a+tl, a). \end{aligned}$$

Moreover, there are two special cases:

(i). If $l = 1$, we have

$$\xi(a+t, a+t-1)\xi(a+t-1, a+t-2)\ldots\xi(a+1, a) = t!\xi(a+t, a).$$

(ii). If $l = p^d$ for some nonnegative integer d, we get

$$\begin{aligned} & \xi(a+tp^d, a+(t-1)p^d)\xi(a+(t-1)p^d, a+(t-2)p^d)\ldots\xi(a+p^d, a) \\ = & \prod_{i=0}^{t-1}\binom{(i+1)p^d}{p^d}\xi(a+tp^d, a). \end{aligned}$$

Furthermore, if $t < p$, we have

$$\begin{aligned} & \xi(a+tp^d, a+(t-1)p^d)\xi(a+(t-1)p^d, a+(t-2)p^d)\ldots\xi(a+p^d, a) \\ \equiv & \ t!\xi(a+tp^d, a) \pmod p; \end{aligned}$$

if $t = p$, we have

$$\xi(a+tp^d, a+(t-1)p^d)\xi(a+(t-1)p^d, a+(t-2)p^d)\ldots\xi(a+p^d, a) \equiv 0 \pmod p.$$

Proof. By Corollary 2.1.3, we can calculate the product

$$\xi(a+\sum_{i=0}^{t-1}m_i, a+\sum_{i=0}^{t-2}m_i)\xi(a+\sum_{i=0}^{t-2}m_i, a+\sum_{i=0}^{t-3}m_i)\ldots\xi(a+m_0, a)$$

with the multiplication one by one from $\xi(a+m_0, a)$:

$$\begin{aligned} & \xi(a+\sum_{i=0}^{t-1}m_i, a+\sum_{i=0}^{t-2}m_i)\xi(a+\sum_{i=0}^{t-2}m_i, a+\sum_{i=0}^{t-3}m_i)\ldots\xi(a+m_0, a) \\ = & \binom{m_0+m_1}{m_1}\xi(a+\sum_{i=0}^{t-1}m_i, a+\sum_{i=0}^{t-2}m_i)\ldots\xi(a+m_0+m_1, a) \\ = & \binom{m_0+m_1}{m_1}\binom{m_0+m_1+m_2}{m_2}\xi(a+\sum_{i=0}^{t-1}m_i, a+\sum_{i=0}^{t-2}m_i) \\ & \qquad \ldots\xi(a+m_0+m_1+m_2, a) \\ = & \prod_{i=0}^{t-1}\binom{m_0+m_1+\ldots+m_i}{m_i}\xi(a+\sum_{i=0}^{t-1}m_i, a). \end{aligned}$$

Hence we get the desired multiplication formulae.

(i). This is trivial by straight calculations.

(ii). If $l = p^d$, the product becomes:

$$\begin{aligned} & \xi(a+tp^d, a+(t-1)p^d)\xi(a+(t-1)p^d, a+(t-2)p^d) \ldots \xi(a+p^d, a) \\ = & \prod_{i=0}^{t-1} \binom{(i+1)p^d}{p^d} \xi(a+tp^d, a). \end{aligned}$$

If $t < p$, we have

$$i+1 \leqslant t-1+1 < p, \quad \text{for all } i = 0, 1, \ldots, t-1.$$

By Lemma 2.2.2, we have

$$\binom{(i+1)p^d}{p^d} \equiv \binom{i+1}{1} \equiv i+1 \pmod{p},$$

for any $i = 0, 1, \ldots, t-1$.

Hence we have

$$\prod_{i=0}^{t-1} \binom{(i+1)p^d}{p^d} \xi(a+tp^d, a) \equiv \prod_{i=0}^{t-1} (i+1)\xi(a+tp^d, a) \equiv t!\xi(a+tp^d, a) \pmod{p}.$$

If $t = p$, by Lemma 2.2.2, we get

$$\binom{((t-1)+1)p^d}{p^d} = \binom{p^{d+1}}{p^d} \equiv \binom{1}{0}\binom{0}{1} = 0 \pmod{p}.$$

Hence we have

$$\prod_{i=0}^{t-1} \binom{(i+1)p^d}{p^d} \xi(a+tp^d, a) \equiv 0 \pmod{p}.$$

∎

2.2 The quiver of $S^+(2,r)$ with $p > 0$

By Proposition 1.3.6 $S^+(2,r)$ has primitive orthogonal idempotents $\xi(a,a)$, where $a = 0, 1, \ldots, r$; with $\epsilon = \sum_{a=0}^{r} \xi(a,a)$. By the argument in the end of Section 1.4, the set of vertices Γ_0 of the quiver Γ of $S^+(2,r)$ is the set of primitive orthogonal idempotents, or bijectively the set of the compositions $\Lambda(2,r)$, or simply the following set

$$\Gamma_0 := \{a \mid 0 \leqslant a \leqslant r\}.$$

The vertex set Γ_0 is independent of the characteristic of K.

We now calculate the set of arrows for the quiver of $S^+(2,r)$ for positive characteristic $p > 0$, in terms of the radical and radical square. The quiver of $S^+(2,r)$ for the case $p = 0$ will be discussed in Section 2.4. We also list some examples for the quivers of $S^+(2,r)$ with $p > 0$. First we introduce the following notation.

Definition 2.2.1. Let a be a positive integer and p a prime number, then there is a unique p-adic decomposition of a as follows:

$$a = \sum_{i=0}^{\infty} a_i p^i,$$

where $0 \leqslant a_i \leqslant p-1$, for all i. We introduce the following notation

$$[a]_p = \sum_{i=0}^{\infty} a_i.$$

We will need the following useful lemma. Note that by convention $\binom{a}{b} = 0$ if $a < b$.

Lemma 2.2.2. *([5], P.271) Let n, k be two positive integers, p a prime number. We write n and k p-adically, that is*

$$n = \sum_{i=0}^{t} n_i p^i, \ k = \sum_{i=0}^{t} k_i p^i,$$

where $t \geq 0$, and $0 \leqslant n_i, k_i \leqslant p-1$ for all $0 \leqslant i \leqslant t$. Then

$$\binom{n}{k} \equiv \prod_{i=0}^{t} \binom{n_i}{k_i} \pmod{p}.$$

Lemma 2.2.3. *Let $n \geqslant 2$ be a positive integer, p a prime number. Then we have $\binom{n}{k} \not\equiv 0 \pmod{p}$ for some $1 \leqslant k \leqslant n-1$ if and only if $[n]_p \geqslant 2$.*

Proof. (i) Suppose that $[n]_p \geqslant 2$. Let $n_t p^t$ be the leading term of the p-adic decomposition of n, for some nonnegative integer t. We let $k = p^t$. Since $[n]_p \geqslant 2$, then $n \neq p^t = k$, thus $n > k$. Thus $1 \leqslant k \leqslant n-1$. By Lemma 2.2.2,

$$\binom{n}{k} \equiv \prod_{i=0}^{t} \binom{n_i}{k_i} \equiv \binom{n_t}{1} \equiv n_t \not\equiv 0 \pmod{p}.$$

Such k satisfies the desired condition.

(ii) Conversely, suppose $\binom{n}{k} \not\equiv 0 \pmod{p}$ for some $1 \leqslant k \leqslant n-1$. We need to prove that $[n]_p \geqslant 2$. Otherwise $[n]_p = 1$ since n is a positive integer that $[n]_p > 0$. Then we have $n = p^t$ for some $t \geq 1$ since $n \geqslant 2$. Let $k_s p^s$ be the leading term of the p-adic decomposition of k for some nonnegative integer s. By our assumption for k, we have

$$k \leqslant n - 1 = p^t - 1 = \sum_{i=0}^{t-1} (p-1)p^i.$$

Hence $s < t$. Since $n_s = 0$ and $k_s \neq 0$, we have $n_s < k_s$. By convention $\binom{n_s}{k_s} = 0$. Thus by Lemma 2.2.2, we have

$$\binom{n}{k} \equiv \prod_{i=0}^{t} \binom{n_i}{k_i} \equiv \binom{n_s}{k_s} N \equiv 0 \pmod{p},$$

which is a contradiction to our assumption that $\binom{n}{k} \not\equiv 0 \pmod{p}$. ∎

We are now ready to describe the radical square of $S^+(2,r)$:

Lemma 2.2.4. *Let R be the radical of $S^+(2,r)$. Then R^2 has a basis*

$$\{\xi(b,a) \mid [b-a]_p \geqslant 2, r \geqslant b > a \geqslant 0\}.$$

Hence R/R^2 has a basis

$$\{\xi(b,a) + R^2 \mid [b-a]_p = 1, r \geqslant b > a \geqslant 0\}.$$

Proof. Let I be a subspace of $S^+(2,r)$ with a basis

$$\{\xi(b,a) \mid [b-a]_p \geqslant 2, r \geqslant b > a \geqslant 0\}.$$

We need to prove $R^2 = I$. First we prove that $I \subseteq R^2$. Suppose that $\xi(b,a) \in I$, that is, b and a satisfy the condition: $[b-a]_p \geqslant 2$ for some a and b, where $r \geqslant b > a \geqslant 0$. We need to prove that $\xi(b,a) \in R^2$.

Since $[b-a]_p \geqslant 2$, by Lemma 2.2.3, there exists an integer k with the condition $1 \leqslant k \leqslant b-a-1$ such that

$$\binom{b-a}{k} \not\equiv 0 \pmod{p}. \tag{2.2.1}$$

We now calculate the product $\xi(b,k+a)\xi(k+a,a)$. By Corollary 2.1.3, we have

$$\xi(b,k+a)\xi(k+a,a) = \binom{b-a}{k}\xi(b,a).$$

Then by Equation (2.2.1), the combination number $\binom{b-a}{k}$ is nonzero in the field K of a prime characteristic p. Thus, $\binom{b-a}{k}$ has an inverse $\binom{b-a}{k}^{-1}$. Hence we have

$$\xi(b,a) = \binom{b-a}{k}^{-1}\xi(b,k+a)\xi(k+a,a).$$

By Proposition 1.2.3, the basis elements $\xi(b, k+a)$ and $\xi(k+a, a)$ are in R. Hence we have $\xi(b,a) \in R^2$. Therefore, we get $I \subseteq R^2$.

Next, we prove that $R^2 \subseteq I$. By Proposition 1.2.3, R has a basis

$$\{\xi(b,a) \mid r \geqslant b > a \geqslant 0\}.$$

Let $\xi(c,b)\xi(b',a) \in R^2$, where $r \geqslant c > b \geqslant 0$ and $r \geqslant b' > a \geqslant 0$. We need to prove that $\xi(c,b)\xi(b',a) \in I$. If $b \neq b'$, then $\xi(c,b)\xi(b',a) = 0 \in I$, so we can assume $b = b'$. By Corollary 2.1.3 we have

$$\xi(c,b)\xi(b,a) = \binom{c-a}{b-a}\xi(c,a).$$

If $\binom{c-a}{b-a} = 0$, then we are done. So we can assume $\binom{c-a}{b-a} \not\equiv 0 \pmod{p}$, i.e., there exists an integer $k = b - a$ such that $\binom{c-a}{k} \not\equiv 0 \pmod{p}$ with the condition that $c - a - 1 \geqslant k \geqslant 1$. By Lemma 2.2.3, this is equivalent to the condition that $[c-a]_p \geqslant 2$. Thus we have that $\xi(c,a) \in I$. Thus $\xi(c,b)\xi(b,a) \in I$. Hence we have $R^2 \subseteq I$. Therefore $R^2 = I$.

Since R has a basis

$$\{\xi(b,a) \mid r \geqslant b > a \geqslant 0\}$$

and R^2 has a basis

$$\{\xi(b,a) \mid [b-a]_p \geqslant 2, r \geqslant b > a \geqslant 0\},$$

we get that R/R^2 has a basis

$$\{\xi(b,a) + R^2 \mid [b-a]_p = 1, r \geqslant b > a \geqslant 0\}.$$

∎

We now have the following theorem for the quiver of $S^+(2,r)$ with $p > 0$:

Theorem 2.2.5. *The quiver Γ of the Borel Schur algebra $S^+(2,r)$ over a field K of a prime characteristic p is given as follows:*
The set of vertices is $\Gamma_0 = \{0, 1, \ldots, r\}$; the number of arrows from vertex a to vertex b where $0 \leqslant a < b \leqslant r$, is equal to 1 or 0; this number is equal to 1 if and only if $[b-a]_p = 1$.

Proof. We only need to calculate the arrows. As it is shown at the end of Section 1.4, the number of arrows from vertex a to vertex b for $0 \leqslant a < b \leqslant r$, is the dimension of $\xi(b,b)(R/R^2)\xi(a,a)$. By Lemma 2.2.4, R/R^2 has a basis

$$\{\xi(d,c) + R^2 \mid [d-c]_p = 1,\ r \geqslant d > c \geqslant 0\}.$$

Since $S^+(2,r)$ has the basis

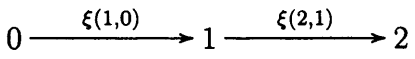
$$\{\xi(d,c) \mid r \geqslant d \geqslant c \geqslant 0\},$$

by the multiplication rule for the Schur algebra, $\xi(b,b)(R/R^2)\xi(a,a)$ is spanned by the $\xi(b,a)$ with the condition that $[b-a]_p = 1$, otherwise $\xi(b,b)(R/R^2)\xi(a,a) = 0$. In other words, the number of the arrows from a to b where $r \geqslant b > a \geqslant 0$, or the dimension of $\xi(b,b)(R/R^2)\xi(a,a)$, is equal to 1 or 0; this number is equal to 1 if and only if $[b-a]_p = 1$. ∎

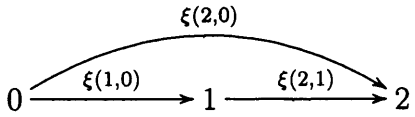
Note that by the definition of the quiver, we may label the arrows from i to j by the elements of the basis for $\epsilon_j V \epsilon_i$. We will identify the arrow from vertex a to vertex b for $[b-a]_p = 1$, with the basis element $\xi(b,a)$ in $\xi(b,b)(R/R^2)\xi(a,a)$ for the algebra $S^+(2,r)$.

Example 2.2.6. *We list some quivers by drawing their graphs (labeling the arrows by the elements in the algebras).*

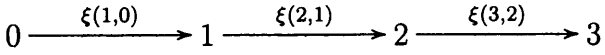
The quiver of $S^+(2,2)$ *over the field* K *of characteristic* $p \neq 2$*:*



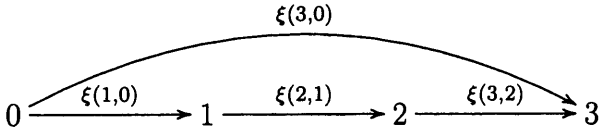
The quiver of $S^+(2,2)$ *for* $p = 2$*:*



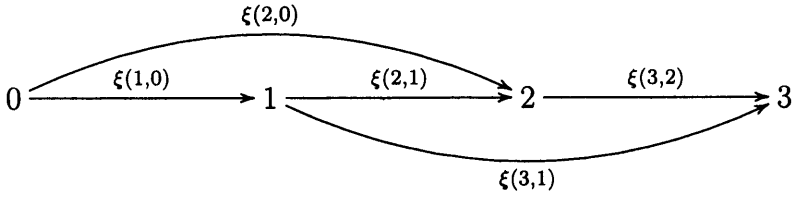
The quiver of $S^+(2,3)$ *for* $p > 3$*:*



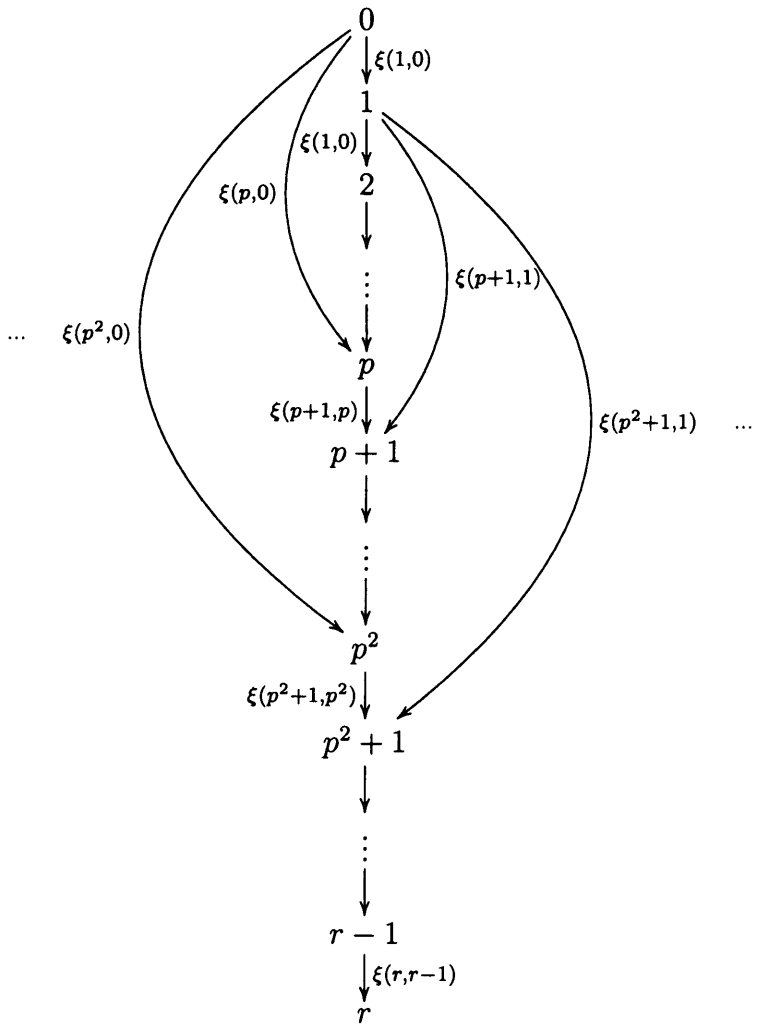
The quiver of $S^+(2,3)$ *for* $p = 3$*:*



The quiver of $S^+(2,3)$ *for* $p = 2$*:*



The quiver of $S^+(2,r)$ over K of characteristic p for arbitrary r and $p > 0$:



Corollary 2.2.7. *Let $f_p(r)$ be the number of arrows of the quiver of $S^+(2,r)$ over a field K of characteristic $p > 0$. We have*

$$f_p(r) = (r+1)(M+1) - (p^{M+1} - 1)/(p-1),$$

where the nonnegative integer M is determined by $p^M \leqslant r < p^{M+1}$.

Proof. We compute $f_p(r)$ using Theorem 2.2.5. So $f_p(r)$ is the number of pairs (a_1, a_2) such that $(a_1, a_2) = (a_1, a_1 + p^t)$ for some $t \geq 0$, and $0 \leqslant a_1 \leqslant r - p^t$.

Thus the number of pairs $(a_1, a_1 + p^t)$ for some nonnegative integers a_1 and t, is $r + 1 - p^t$. Let M be the nonnegative integer such that $p^M \leqslant r < p^{M+1}$. Hence the number of such pairs or the number of arrows, is

$$f_p(r) = \sum_{t=0}^{M}(r + 1 - p^t) = (r + 1)(M + 1) - (p^{M+1} - 1)/(p - 1).$$

∎

2.3 The relations for the quiver of $S^+(2, r)$ with $p > 0$

In this section we describe the relations for the quiver Γ of $S^+(2, r)$ over a field K of a prime characteristic p.

Recall that we identify the arrow from vertex a to vertex b for $[b - a]_p = 1$ in the quiver Γ of $S^+(2, r)$, with the base element $\xi(b, a)$ in $\xi(b, b)(R/R^2)\xi(a, a)$ for the algebra $S^+(2, r)$. We define the *length* of this arrow as $b - a$.

As shown at the end of Section 1.4, the relations in the quiver Γ of $S^+(2, r)$ can be described as follows. There is a surjective ring homomorphism $\widetilde{f} : K\Gamma \to S^+(2, r)$, where $\widetilde{f}$ maps the "product" of the labels of arrows as the paths in the path algebra $K\Gamma$, to the product of labels of arrows in the Borel Schur algebra $S^+(2, r)$. The relations for the quiver Γ of $S^+(2, r)$, are all the linear combinations of paths, or the "products" of the labels of arrows which are in $\ker \widetilde{f}$.

Thus, we will write relations for the quiver Γ of $S^+(2, r)$ as linear combinations of the "products" of $\xi(b, a)$'s where $[b - a]_p = 1$ in the path algebra $K\Gamma$. This shouldn't be confused with products in $S^+(2, r)$.

Let J be the ideal of $K\Gamma$ generated by all the arrows in Γ.

The following theorem describes the relations of the quiver of $S^+(2,r)$.

Theorem 2.3.1. *Let Γ be the quiver of the Borel Schur algebra $S^+(2,r)$ over a field K of a positive characteristic p. Let R_1 and R_2 be the following sets of the products of arrows in the path algebra $K\Gamma$:*

$$R_1 = \{\xi(t+p^{a+1}, t+(p-1)p^a)\xi(t+(p-1)p^a, t+(p-2)p^a)\ldots\xi(t+p^a,t) \mid 0 \leqslant t+p^a \leqslant r,\ t,a \geqslant 0\},$$
$$R_2 = \{\xi(t+p^a+p^b, t+p^a)\xi(t+p^a,t) - \xi(t+p^a+p^b, t+p^b)\xi(t+p^b,t) \mid 0 \leqslant t+p^a+p^b \leqslant r,\ t,a \neq b \geqslant 0\}.$$

Let I be the ideal of the path algebra $K\Gamma$ generated by R_1 and R_2. Then I is the set of all relations for the quiver Γ of $S^+(2,r)$. More precisely, $I \subseteq J^2$ and

$$k\Gamma/I \cong S^+(2,r).$$

Proof. There is a surjective ring homomorphism $\widetilde{f} : K\Gamma \to S^+(2,r)$, where $\widetilde{f}$ maps the product of the arrows in the path algebra $K\Gamma$, to the product of labels of arrows in $S^+(2,r)$. The relations for the quiver Γ, are the paths in $\ker \widetilde{f}$. We need to show that $I = \ker \widetilde{f}$.

By Proposition 2.1.4 (ii), we have

$$\prod_{i=0}^{p-1} \xi(t+(p-i)p^a, t+(p-1-i)p^a) \equiv 0, \pmod p.$$

By Corollary 2.1.3, we have

$$\xi(t+p^a+p^b, t+p^a)\xi(t+p^a,t) = \binom{p^a+p^b}{p^a}\xi(t+p^a+p^b,t)$$
$$\xi(t+p^a+p^b, t+p^b)\xi(t+p^b,t) = \binom{p^a+p^b}{p^a}\xi(t+p^a+p^b,t),$$

So

$$\xi(t+p^a+p^b, t+p^a)\xi(t+p^a, t) = \xi(t+p^a+p^b, t+p^b)\xi(t+p^b, t).$$

Hence $R_1, R_2 \subseteq \ker \widetilde{f}$, so $I \subseteq \ker \widetilde{f}$.

Since $I \subseteq \ker \widetilde{f}$, the map $\widetilde{f}$ induces a surjection:

$$\overline{f} : K\Gamma/I \to S^+(2,r).$$

We need to show that $\ker \overline{f} = 0$, i.e., that $\overline{f}$ is an isomorphism. Since $\overline{f}$ is surjective, it is enough to show that

$$\dim K\Gamma/I \leqslant \dim S^+(2,r),$$

which implies that $\overline{f}$ is injective.

First, fix any $0 \leqslant i \leqslant j \leqslant r$, let $j - i = \sum_s \lambda_s p^s$ be the unique p-adic decomposition of $j-i$. Denote by $P_{j,i}$ the unique path from vertex i to vertex j in the path algebra $K\Gamma$ such that the lengths of the arrows increase and there are exactly λ_s arrows of length p^s for all s.

Let P be a path from i to j in the path algebra $K\Gamma$, written as a product of labels of arrows, where the lengths of arrows must be powers of p by Theorem 2.2.5. Each arrow is labeled by $\xi(b,a)$ where $[b-a]_p = 1$. Thus we can write P in the form

$$P = \xi(a_t, a_{t-1})\xi(a_{t-1}, a_{t-2}) \ldots \xi(a_1, a_0),$$

where $a_t = j$ and $a_0 = i$ and $[a_m - a_{m-1}]_p = 1$ with $t \geqslant m \geqslant 1$. Recall that I is generated by R_1 and R_2. Using the relations in R_2, we reorder the arrows by their lengths, such that the lengths of arrows $\xi(a_m, a_{m-1})$ increase. Next by

the relations in R_1, the number of arrows of the same length does not exceed $p-1$, otherwise $P+I$ will be zero. Hence P is the path, for which the lengths of arrows increase and there are exactly λ_s arrows of length p^s for all s, that is,

$$P + I = P_{j,i} + I.$$

Thus $K\Gamma/I$ is spanned by

$$\{P_{j,i} + I \mid r \geqslant j \geqslant i \geqslant 0\}.$$

Hence $\dim k\Gamma/I$ does not exceed the number of pairs (j,i) with $r \geqslant j \geqslant i \geqslant 0$, which is the dimension of $S^+(2,r)$. ∎

We call the relations of I in the path algebra $K\Gamma$ in Theorem 2.3.1, the *p-adic relations.*

2.4 $S^+(2,r)$ over a field K of characteristic 0

Let K be a field of characteristic 0. In this section, we give the quiver and relations for $S^+(2,r)$ over K. Let $T_{r+1}(K)$ be the algebra of lower triangular matrices of degree $r+1$ over K. We are going to show that $S^+(2,r)$ is isomorphic to $T_{r+1}(K)$.

The following lemma and theorem are proved in the same way as their counterparts in positive characteristic (Lemma 2.2.4 and Theorem 2.2.5).

Lemma 2.4.1. *Let R be the radical of $S^+(2,r)$ over K of characteristic 0, then R^2 has K-basis*

$$\{\xi(b,a) \mid b-a \geqslant 2,\ r \geqslant b > a \geqslant 0\}.$$

Hence R/R^2 has K-basis

$$\{\xi(a+1,a)+R^2 \mid r-1 \geqslant a \geqslant 0\}.$$

Theorem 2.4.2. *The quiver Γ of the Borel Schur algebra $S^+(2,r)$ over a field K of characteristic 0 is given as follows:*
The set of vertices is $\Gamma_0 = \{0,1,\ldots,r\}$; the number of arrows from vertex a to vertex b where $r \geqslant b > a \geqslant 0$, is equal to 1 or 0; this number is equal to 1 if and only if $b-a=1$.

Note that the lengths of arrows in the above quiver Γ of $S^+(2,r)$ over K, are 1. Hence we can give the quiver Γ of $S^+(2,r)$ over K as follows:

$$0 \longrightarrow 1 \longrightarrow \cdots \longrightarrow r-1 \longrightarrow r$$

Recall that $T_{r+1}(K)$ is the algebra of $(r+1)\times(r+1)$ lower triangular matrices over K. Note that the path algebra of this quiver Γ is isomorphic to $T_{r+1}(K)$. We are going to prove that $S^+(2,r) \simeq T_{r+1}(K)$. This implies that the quiver of $S^+(2,r)$ in zero characteristic doesn't have any relation.

Let $E_{i,j}$ be the $(r+1)\times(r+1)$ matrix, whose entry of (i,j) position is 1 and the other ones are zero, with $0 \leqslant i,j \leqslant r$. We define a linear map f from $S^+(2,r)$ (over K) to $T_{r+1}(K)$ as follows:

$$f(\xi(b,a)) = ((b-a)!)^{-1}E_{b,a},$$

where $0 \leqslant a \leqslant b \leqslant r$. Here 0! is defined to be 1. Since f maps a basis of $S^+(2,r)$ to a basis of $T_{r+1}(K)$, f is a bijection. Moreover, we have the following proposition.

Proposition 2.4.3. *The map f is an algebra isomorphism, that is*

$$S^+(2,r) \simeq T_{r+1}(K).$$

Proof. We only need to prove that f preserves the multiplication. By Corollary 2.1.3, for $0 \leqslant a_1 \leqslant b \leqslant a_2 \leqslant r$, we have

$$\begin{aligned} f(\xi(a_2,b)\xi(b,a_1)) &= \binom{a_2-a_1}{b-a_1} f(\xi(a_2,a_1)) \\ &= \binom{a_2-a_1}{b-a_1} ((a_2-a_1)!)^{-1} E_{a_2,a_1} \\ &= ((b-a_1)!)^{-1}((a_2-b)!)^{-1} E_{a_2,a_1}. \end{aligned}$$

On the other hand, we get

$$\begin{aligned} f(\xi(a_2,b))f(\xi(b,a_1)) &= ((a_2-b)!)^{-1} E_{a_2,b}((b-a_1)!)^{-1} E_{b,a_1} \\ &= ((a_2-b)!)^{-1}((b-a_1)!)^{-1} E_{a_2,a_1}. \end{aligned}$$

Hence,

$$f(\xi(a_2,b)\xi(b,a_1)) = f(\xi(a_2,b))f(\xi(b,a_1)),$$

where $0 \leqslant a_1 \leqslant b \leqslant a_2 \leqslant r$. Therefore, f is an algebra isomorphism. ∎

2.5 More about the multiplication formula for $S^+(2,r)$

Let K be a field of characteristic $p > 0$. Let Γ be the quiver of $S^+(2,r)$ over K. Recall the surjective ring homomorphism $\widetilde{f} : K\Gamma \to S^+(2,r)$, where $\widetilde{f}$ maps the product of the labels of arrows as the paths in the path algebra $K\Gamma$, to the product of labels of arrows in the Borel Schur algebra $S^+(2,r)$ (see Sections 1.4 and 2.3).

In this section we will calculate the dimension of the vector space spanned by the paths from one vertex a to another vertex b in the path algebra $K\Gamma$

where $r \geqslant b > a \geqslant 0$. We also calculate the product of labels of arrows as the basis elements in $S^+(2,r)$, for a path from a to b. We get a formula for that product of labels, which will be used in Section 5.4 to get certain relations in the quiver of $S^+(n,r)$.

Lemma 2.5.1. *Let Γ be the quiver of $S^+(2,r)$ over K. Let b and a be nonnegative integers such that $r \geqslant b > a \geqslant 0$. Let $b-a = \sum_{i=0}^{t} m_i p^i$ be the p-adic decomposition of $b-a$. Then the dimension of the vector space T spanned by the paths from a to b in the path algebra $K\Gamma$, is $\binom{m_0+m_1+\ldots+m_t}{m_0,m_1,\ldots,m_t}$.*

Proof. By Theorem 2.2.5, the lengths of the arrows in Γ are powers of p. So the lengths of arrows in the product of labels from a to b in $K\Gamma$, are of the form $(p^0)^{m_0}(p^1)^{m_1}\ldots(p^t)^{m_t}$. Hence the dimension of T spanned by the paths from a to b in $K\Gamma$, is the number of permutations of

$$\underbrace{p^0,\ldots,p^0}_{m_0}\ \underbrace{p^1,\ldots,p^1}_{m_1}\ \cdots\ \underbrace{p^t,\ldots,p^t}_{m_t},$$

which is the combination number $\binom{m_0+m_1+\ldots+m_t}{m_0,m_1,\ldots,m_t}$. ∎

Next we will calculate the product of labels of arrows as the basis elements in the Borel Schur algebra $S^+(2,r)$. By Proposition 2.1.4, this product of labels of the lengths of the powers of p from a to b where $r \geqslant b > a \geqslant 0$, or the product of the basis elements in $S^+(2,r)$, will be a scalar of the basis element $\xi(b,a)$, that is, $M\xi(b,a)$. Our aim is to calculate the coefficient M. Before we calculate such M, we introduce the following definition.

Definition 2.5.2. Let m be a positive integer and let $m = \sum_{i=0}^{t} m_i p^i$ be the p-adic decomposition of m, where t is a nonnegative integer and $0 \leqslant m_i \leqslant p-1$ for all $i = 0, 1 \ldots, t$. We define m_+ and m^+ as follows:

$$m_+ := \prod_{i=1}^{t} (p^i!)^{m_i}, \qquad m^+ := m!/m_+.$$

The following proposition shows that the coefficient M above is the number $(b-a)^+$. In particular, m^+ is an integer and $m^+ \not\equiv 0 \pmod{p}$ for any positive integer m.

Theorem 2.5.3. *Let $0 \leqslant a < b \leqslant r$ and let $m = b - a$. Let $m = \sum_{i=0}^{t} m_i p^i$ be the p-adic decomposition of m. Let Π be the following product of the basis elements in $S^+(2,r)$:*

$$\begin{aligned}\Pi \;=\; & \xi(a+m, a+m-p^t) \ldots \xi(a+m-m_t p^t + p^t, a+m-m_t p^t) \\ & \ldots \quad \xi(a+m_0, a+m_0-1) \ldots \xi(a+1, a).\end{aligned}$$

Then

$$\Pi = M\xi(a+m, a).$$

where

$$M = \prod_{i=0}^{t} \binom{p^i}{p^i}\binom{2p^i}{p^i} \cdots \binom{m_i p^i}{p^i}\binom{m_0 + m_1 p^1 + \ldots + m_i p^i}{m_i p^i}.$$

Moreover,

$$M = m^+, \quad \textit{and} \qquad M \equiv \prod_{i=0}^{t} m_i! \pmod{p}.$$

In particular, $m^+ \not\equiv 0 \pmod{p}$.

Proof. We denote the product of labels of the lengths p^i in the product Π as Π_i, that is, we write Π_i as follows:

$$\begin{aligned}\Pi_i := \xi(a + \textstyle\sum_{j=0}^{i-1} m_j p^j + m_i p^i, a + \sum_{j=0}^{i-1} m_j p^j + (m_i - 1)p^i) \\ \ldots \xi(a + \textstyle\sum_{j=0}^{i-1} m_j p^j + p^i, a + \sum_{j=0}^{i-1} m_j p^j),\end{aligned}$$

where $0 \leqslant i \leqslant t$. By Proposition 2.1.4, we get the product Π_i:

$$\Pi_i = \binom{p^i}{p^i}\binom{2p^i}{p^i} \cdots \binom{m_i p^i}{p^i} \xi(a + \sum_{j=0}^{i-1} m_j p^j + m_i p^i, a + \sum_{j=0}^{i-1} m_j p^j),$$

where $0 \leqslant i \leqslant t$ and $m_i > 0$. If $m_i = 0$, let $\Pi_i = 1$. Thus we have

$$\begin{aligned}\Pi &= \Pi_0\Pi_1 \dots \Pi_t \\ &= \prod_{i=0}^{t} \binom{p^i}{p^i}\binom{2p^i}{p^i} \dots \binom{m_ip^i}{p^i}\xi(a + \sum_{j=0}^{i-1} m_jp^j + m_ip^i, a + \sum_{j=0}^{i-1} m_jp^j).\end{aligned}$$

By Proposition 2.1.4, we calculate the following product

$$\begin{aligned}&\prod_{i=0}^{t} \xi(a + \sum_{j=0}^{i-1} m_jp^j + m_ip^i, a + \sum_{j=0}^{i-1} m_jp^j) \\ = &\prod_{i=0}^{t} \binom{m_0 + m_1p^1 + \dots + m_ip^i}{m_ip^i}\xi(a + m, a).\end{aligned}$$

Hence we have the formula

$$\begin{aligned}\Pi &= \prod_{i=0}^{t} \binom{p^i}{p^i}\binom{2p^i}{p^i} \dots \binom{m_ip^i}{p^i}\xi(a + \sum_{j=0}^{i-1} m_jp^j + m_ip^i, a + \sum_{j=0}^{i-1} m_jp^j) \\ &= \prod_{i=0}^{t} \binom{p^i}{p^i}\binom{2p^i}{p^i} \dots \binom{m_ip^i}{p^i}\binom{m_0 + m_1p^1 + \dots + m_ip^i}{m_ip^i}\xi(a + m, a).\end{aligned}$$

Therefore, we obtain the formula for M:

$$M = \prod_{i=0}^{t} \binom{p^i}{p^i}\binom{2p^i}{p^i} \dots \binom{m_ip^i}{p^i}\binom{m_0 + m_1p^1 + \dots + m_ip^i}{m_ip^i}.$$

By Lemma 2.2.2,

$$\binom{vp^i}{p^i} \equiv v \pmod{p},$$

for any $1 \leqslant v \leqslant p - 1$, and

$$\binom{m_0 + m_1p^1 + \dots + m_ip^i}{m_ip^i} \equiv \binom{m_i}{m_i}\prod_{s=0}^{i-1}\binom{m_s}{0} \equiv 1 \pmod{p},$$

for any $0 \leqslant i \leqslant t$. Hence we have

$$M \equiv \prod_{i=0}^{t} m_i! \not\equiv 0 \pmod{p}.$$

We next prove that $M = m^+$. By Proposition 2.1.4 (i), the product of labels of the lengths 1 can be written as follows:

$$\xi(a + m, a + m - 1) \dots \xi(a + 2, a + 1)\xi(a + 1, a) = m!\xi(a + m, a).$$

By Proposition 2.1.4 (i), furthermore, we can calculate the above product, in terms of the lengths of the product of the powers of p:

$$\begin{aligned}
& \xi(a+m, a+m-1) \ldots \xi(a+2, a+1)\xi(a+1, a) \\
= \; & p^t!\xi(a+m, a+m-p^t) \ldots p^t!\xi(a+m-m_tp^t+p^t, a+m-m_tp^t) \\
& \qquad p^{t-1}!\xi(a+m-m_tp^t, a+m-m_tp^t-p^{t-1}) \ldots \\
& \qquad\qquad \xi(a+m_0, a+m_0-1) \ldots \xi(a+m_0, a) \\
= \; & \prod_{i=0}^{t} (p^i!)^{m_i} (\xi(a+\sum_{j=0}^{i-1} m_jp^j + m_ip^i, a+\sum_{j=0}^{i-1} m_jp^j + (m_i-1)p^i) \\
& \qquad\qquad \ldots \quad \xi(a+\sum_{j=0}^{i-1} m_jp^j + p^i, a+\sum_{j=0}^{i-1} m_jp^j)) \\
= \; & \prod_{i=0}^{t} (p^i!)^{m_i} \Pi_i \\
= \; & \prod_{i=0}^{t} (p^i!)^{m_i} \prod_{i=0}^{t} \Pi_i \\
= \; & m_+\Pi.
\end{aligned}$$

Hence we obtain

$$\Pi = m!/m_+\xi(a+m, a) = m^+\xi(a+m, a).$$

Thus $M = m^+$. ∎

Let $0 \leqslant a < b \leqslant r$. We consider vector space T spanned by the paths from a to b in $K\Gamma$ modulo its relations in Theorem 2.3.1 where Γ is the quiver of $S^+(2, r)$. By the proof in Theorem 2.3.1, T is one-dimensional, and spanned by the path

$$\begin{aligned}
P(b, a) \; = \; & \xi(a+m, a+m-p^t) \ldots \xi(a+m-m_tp^t+p^t, a+m-m_tp^t) \\
& \qquad \ldots \quad \xi(a+m_0, a+m_0-1) \ldots \xi(a+1, a).
\end{aligned}$$

where $m = b - a$ and $m = \sum_{i=0}^{t} m_ip^i$ is the p-adic decomposition of m as in Theorem 2.5.3, but the above multiplication is the multiplication for the path algebra $K\Gamma$. We also say the above path $P(b, a)$ is the path from a to b.

Chapter 3

Embeddings from $S^+(2,r)$ to $S^+(n,r+s)$

We obtained the quiver and relations for the Borel Schur algebra $S^+(2,r)$ in Chapter 2. In this chapter we consider a special type of embedding from the Borel Schur algebras $S^+(2,r)$ to $S^+(n,r+s)$ where $n \geqslant 2$ and $s \geqslant 0$. This embedding embeds the quiver of $S^+(2,r)$ into the quiver of $S^+(n,r+s)$.

In Section 3.1 we construct this embedding φ_t^α from $S^+(2,r)$ to $S^+(n,r+s)$ where $1 \leqslant t \leqslant n-1$ and $\alpha \in \Lambda(n,s)$.

In Section 3.2 we calculate the dimension of the Hom Space $H(\lambda,\alpha) := \mathrm{Hom}_A(A\xi_\alpha, A\xi_\lambda)$ (or the Cartan invariant $\widehat{c}_{\lambda,\alpha}$) where $A = S^+(n,r)$. For more detail see [15]. This will be used in Chapter 5 to find some relations for $S^+(n,r)$. We split Section 3.2 into four parts: in **Part I** we introduce the row semi-standard(RSS) λ-tableau and the number $c_{\lambda,\alpha}$; in **Part II** we introduce the Cartan invariant $\widehat{c}_{\lambda,\alpha}$, which is the dimension of $H(\lambda,\alpha)$ and show that $\widehat{c}_{\lambda,\alpha} = c_{\lambda,\alpha}$; in **Part III** we calculate the dimension of the Borel Schur algebra $S^+(n,r)$ and consider a matrix multiplication for $S^+(2,r)$; in

Part IV, we calculate the Cartan invariant $\widehat{c}_{\lambda,\alpha}$ for some special cases, which will be used in Chapter 5.

In Section 3.3 we consider the properties of our embedding φ_t^α from $S^+(2,r)$ to $S^+(n,r+s)$ and prove that φ_t^α embeds the quiver of $S^+(2,r)$ into the quiver of $S^+(n,r)$.

3.1 Embeddings from $S^+(2,r)$ to $S^+(n,r+s)$

In this section we will define a map from $S^+(2,r)$ to $S^+(n,r+s)$. We prove that this map is indeed an embedding from $S^+(2,r)$ to $S^+(n,r+s)$ where $n \geqslant 2$ and $s \geqslant 0$. Recall that $\{\xi(b,a) \mid r \geqslant b \geqslant a \geqslant 0\}$ is a basis of $S^+(2,r)$.

Definition 3.1.1. Let $t \in \{1,2,\ldots,n-1\}$. Let $\alpha = (a_1,a_2,\ldots,a_n) \in \Lambda(n,s)$ be in $\Lambda(n,r)$. We define a linear map

$$\varphi_t^\alpha : S^+(2,r) \longrightarrow S^+(n,r+s)$$

by $\varphi_t^\alpha(\xi(b,a)) = \xi_{j,i}$ where the multi-indices i and j are obtained as follows:

$$\begin{aligned} j &= 1^{a_1} \cdots (t-1)^{a_{t-1}} t^{a_t+b} (t+1)^{a_{t+1}+r-b} (t+2)^{a_{t+2}} \ldots n^{a_n}, \\ i &= 1^{a_1} \cdots (t-1)^{a_{t-1}} t^{a_t+a} (t+1)^{a_{t+1}+r-a} (t+2)^{a_{t+2}} \ldots n^{a_n}. \end{aligned}$$

For example, for $n=3$, we have the maps

$$\begin{aligned} \varphi_1^{(s)} : \quad & S^+(2,r) \longrightarrow S^+(3,r+s) \\ & \xi_{1^b 2^{r-b}, 1^a 2^{r-a}} \longmapsto \xi_{1^b 2^{r-b} 3^s, 1^a 2^{r-a} 3^s}, \\ \varphi_2^{(s)} : \quad & S^+(2,r) \longrightarrow S^+(3,r+s) \\ & \xi_{1^b 2^{r-b}, 1^a 2^{r-a}} \longmapsto \xi_{1^s 2^b 3^{r-b}, 1^s 2^a 3^{r-a}}. \end{aligned}$$

Remark: Let E_{t+1} be the $1 \times n$ vector in which the $(t+1)$-th entry is 1 and the other entries are 0. Let $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ be the simple roots of type A_{n-1}.

Put $\lambda = \alpha + rE_{t+1}$. Recall our notation in Definition 2.1.1:

$$\begin{aligned}\lambda(t,b) &= \lambda + b\alpha_t \\ &= (\alpha_1, \ldots, \alpha_{t-1}, \alpha_t + b, \alpha_{t+1} + r - b, \alpha_{t+1}, \ldots, \alpha_n); \\ \lambda(t,a) &= \lambda + a\alpha_t \\ &= (\alpha_1, \ldots, \alpha_{t-1}, \alpha_t + a, \alpha_{t+1} + r - a, \alpha_{t+1}, \ldots, \alpha_n).\end{aligned}$$

By Definition 2.1.1, we can write j and i as follows:

$$j = \underline{n}^{\lambda + b\alpha_t} = \underline{n}^{\lambda(t,b)}, \quad i = \underline{n}^{\lambda + a\alpha_t} = \underline{n}^{\lambda(t,a)}.$$

Hence we can rewrite the map φ_t^α as follows:

$$\varphi_t^\alpha(\xi(b,a)) = \xi_{\lambda(t,b),\lambda(t,a)},$$

for all b and a with $r \geqslant b \geqslant a \geqslant 0$.

By Definition 3.1.1, the map φ_t^α from $S^+(2,r)$ to $S^+(n, r+s)$ is injective. Actually, the map φ_t^α is an embedding from $S^+(2,r)$ to $S^+(n, r+s)$.

Proposition 3.1.2. *The linear map φ_t^α is an algebra homomorphism from $S^+(2,r)$ to $S^+(n, r+s)$. Hence φ_t^α is an embedding.*

Proof. Since $\{\xi(b,a) \mid r \geqslant b \geqslant a \geqslant 0\}$ is a basis for $S^+(2,r)$, by Definition 3.1.1, the map φ_t^α is an injective linear map from $S^+(2,r)$ to $S^+(n, r+s)$. It remains to prove that the map φ_t^α preserves the multiplication, i.e., for $0 \leqslant a_1 < b < a_2 \leqslant r$,

$$\varphi_t^\alpha(\xi(a_2,b)\xi(b,a_1)) = \varphi_t^\alpha(\xi(a_2,b))\varphi_t^\alpha(\xi(b,a_1)). \tag{3.1.1}$$

By the multiplication formula in Corollary 2.1.3, that is

$$\varphi_t^\alpha(\xi(a_2,b)\xi(b,a_1)) = \binom{a_2 - a_1}{b - a_1}\varphi_t^\alpha(\xi(a_2,a_1)).$$

By Definition 3.1.1 and Remark, this is equivalent to

$$\xi(\lambda(t,a_2),\lambda(t,b))\xi(\lambda(t,b),\lambda(t,a_1)) = \binom{a_2 - a_1}{b - a_1}\xi(\lambda(t,a_2),\lambda(t,a_1)),$$

where $\lambda = \alpha + rE_{t+1}$. The last equality holds by Lemma 2.1.2. ∎

Remark: The algebra homomorphism φ_t^α in Proposition 3.1.2 does not preserve the identity element 1. Actually,

$$\varphi_t^\alpha(1) = \varphi_t^\alpha(\sum_{a=0}^{r} \xi(a,a)) = \sum_{a=0}^{r} \xi_{\lambda(t,a),\lambda(t,a)},$$

where $\lambda = \alpha + rE_{t+1}$.

3.2 The Hom space $\mathrm{Hom}_A(A\xi_\lambda, A\xi_\mu)$ where $A = S^+(n,r)$

Let $\Lambda = \Lambda(n,r)$. Let $H(\lambda,\alpha)$ be the Hom space $\mathrm{Hom}_A(A\xi_\alpha, A\xi_\lambda)$ where $A = S^+(n,r)$ and $\{\xi_\lambda \mid \lambda \in \Lambda\}$ is a set of primitive orthogonal idempotents of A. In this section we review some results from [15] and [16] and do some calculations for the dimension of $H(\lambda,\alpha)$, or the Cartan invariants, which will be used in Chapter 5.

We split this section into four parts (see above).

Part I: Row Semi-Standard λ-tableau and $c_{\lambda,\alpha}$

In this part we introduce row semi-standard λ-tableau and the numbers $c_{\lambda,\alpha}$ where λ, $\alpha \in \Lambda$.

Definition 3.2.1. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be in Λ. The *diagram* of λ is the set

$$[\lambda] = \{(\mu,\nu) \in \underline{n} \times \underline{r} \mid 1 \leqslant \nu \leqslant \lambda_\mu\}.$$

Any map T^λ from $[\lambda]$ to $\underline{r}$ is called a λ-*tableau* (with values in $\underline{r}$). If $T^\lambda(\mu,\nu) = t_{\mu,\nu}$, we write

$$T^\lambda = \begin{array}{llll} t_{11} & t_{12} & \ldots & t_{1\lambda_1} \\ t_{21} & t_{22} & \ldots & t_{2\lambda_2} \\ \vdots & & & \\ t_{n1} & t_{n2} & \ldots & t_{n\lambda_n}. \end{array}$$

The λ-tableau T^λ is said to be *row semi-standard (RSS)* if the entries in each row of T^λ are weakly increasing from left to right. The *weight* of T^λ is the element α of Λ defined by

$$\alpha_\rho = |\{(\mu, \nu) \in [\lambda] \mid t_{\mu,\nu} = \rho\}|, \quad \forall\, \rho \in \underline{r}.$$

Definition 3.2.2. For each pair λ, α of elements in Λ, we define $c_{\lambda,\alpha}$ as the number of RSS λ-tableau T^λ with weight α such that the entries $t_{\mu,\nu}$ in row μ of T^λ are not greater than μ ($\mu \in \underline{n}$), i.e., $c_{\lambda,\alpha}$ is the number of RSS λ-tableau of the form

$$\begin{array}{l} 1 \ldots\ldots\ldots 1 \\ 1 \ldots \ldots 1 \quad 2 \ldots 2 \\ \vdots \\ 1 \ldots 1 \quad 2 \ldots 2 \quad 3 \ldots 3 \ldots \mu \ldots \mu \\ \vdots \\ 1 \ldots \ldots 1 \quad \ldots \quad n \ldots n. \end{array}$$

Note that, given $\lambda, \alpha \in \Lambda$, a RSS λ-tableau T^λ of weight α as described in the definition above is completely specified by the $n \times n$ matrix $(a_{\mu\nu})$, where for any $\mu, \nu \in \underline{n}$, $a_{\mu\nu}$ is the number of entries ν in row μ of T^λ. Hence $c_{\lambda,\alpha}$ is the number of nonnegative integer lower triangular $n \times n$ matrices whose vector of row sums is $(\lambda_1, \ldots, \lambda_n)$ and whose vector of column sums is $(\alpha_1, \ldots, \alpha_n)$.

Example 3.2.3. *A RSS λ-tableau and its associated matrix are*

$$\begin{array}{l} 1\,1\,1\,1\,1\,1 \\ 1\,2 \\ \\ 2\,2\,4 \\ 3\,4\,4\,5 \end{array} \quad \textit{and} \quad \begin{bmatrix} 6\,0\,0\,0\,0 \\ 1\,1\,0\,0\,0 \\ 0\,0\,0\,0\,0 \\ 0\,2\,0\,1\,0 \\ 0\,0\,1\,2\,1 \end{bmatrix}.$$

In this case $\lambda = (6, 2, 0, 3, 4)$, and the weight of T^λ is $(7, 3, 1, 3, 1)$.

Recall the dominance ordering in $\Lambda(n, r)$ in Definition 1.2.5.

Proposition 3.2.4. *For* $\lambda = (\lambda_1, \ldots, \lambda_n), \alpha = (\alpha_1, \ldots, \alpha_n) \in \Lambda(n, r)$, $\lambda \trianglelefteq \alpha$ *if and only if*

$$\alpha = (\lambda_1 + m_1, \lambda_2 + m_2 - m_1, \lambda_3 + m_3 - m_2, \ldots, \lambda_{n-1} + m_{n-1} - m_{n-2}, \lambda_n - m_{n-1})$$

for some nonnegative integers $m_1, \ldots, m_{n-1}$.

Proof. (The if part): Let $m_\mu = (\alpha_1 + \ldots + \alpha_\mu) - (\lambda_1 + \ldots + \lambda_\mu)$, for all $\mu \in \underline{n}$. Since $\alpha \trianglerighteq \lambda$, $m_\mu \geqslant 0$. Also

$$m_{\mu+1} - m_\mu = \alpha_{\mu+1} - \lambda_{\mu+1},$$

i.e.,

$$\alpha_{\mu+1} = \lambda_{\mu+1} + m_{\mu+1} - m_\mu,$$

where $\mu = 0, 1, \ldots, n-1$, and $m_0 = 0$.

(The only if part): Since $m_\mu \geqslant 0$, we calculate

$$\alpha_1 + \ldots + \alpha_\mu = \lambda_1 + \ldots + \lambda_\mu + m_\mu \geqslant \lambda_1 + \ldots + \lambda_\mu,$$

where $\mu = 1, \ldots, n-1$. Hence $\alpha \trianglerighteq \lambda$. ∎

Proposition 3.2.5. *Let* $\lambda, \alpha \in \Lambda$. *Then*

(i) $c_{\lambda,\alpha} \neq 0$ *if and only if* $\lambda \trianglelefteq \alpha$,

(ii) $c_{\lambda,\lambda} = 1$.

Proof. (i) Assume that $c_{\lambda,\alpha} \neq 0$. Then there exists a nonnegative integer lower triangular $n \times n$ matrix $(a_{\mu,\nu})_{\mu,\nu \in \underline{n}}$ whose vector of row sums is $(\lambda_1, \ldots, \lambda_n)$ and

whose vector of column sums is $(\alpha_1, \ldots, \alpha_n)$. For any $t \in \underline{n}$, since $(a_{\mu,\nu})_{\mu,\nu\in\underline{n}}$ is lower triangular, $a_{\mu,\nu} = 0$ if $\mu < \nu$. Thus we have

$$\begin{aligned}
& (\alpha_1 + \ldots + \alpha_t) - (\lambda_1 + \ldots + \lambda_t) \\
= \; & \textstyle\sum_{\nu=1}^{t} \sum_{\mu=1}^{n} a_{\mu,\nu} - \sum_{\mu=1}^{t} \sum_{\nu=1}^{n} a_{\mu,\nu} \\
= \; & \textstyle\sum_{\nu=1}^{t} \sum_{\mu=1}^{t} a_{\mu,\nu} + \sum_{\nu=1}^{t} \sum_{\mu=t+1}^{n} a_{\mu,\nu} \\
& \qquad - \textstyle\sum_{\mu=1}^{t} \sum_{\nu=1}^{t} a_{\mu,\nu} - \sum_{\mu=1}^{t} \sum_{\nu=t+1}^{n} a_{\mu,\nu} \\
= \; & \textstyle\sum_{\nu=1}^{t} \sum_{\mu=t+1}^{n} a_{\mu,\nu} - \sum_{\mu=1}^{t} \sum_{\nu=t+1}^{n} a_{\mu,\nu} \\
= \; & \textstyle\sum_{\nu=1}^{t} \sum_{\mu=t+1}^{n} a_{\mu,\nu} \\
\geqslant \; & 0,
\end{aligned}$$

i.e., $\lambda \trianglelefteq \alpha$.

Assume that $\lambda \trianglelefteq \alpha$. By Proposition 3.2.4,

$$\alpha = (\lambda_1 + m_1, \lambda_2 + m_2 - m_1, \lambda_3 + m_3 - m_2, \ldots, \lambda_{n-1} + m_{n-1} - m_{n-2}, \lambda_n - m_{n-1})$$

for some nonnegative integers $m_1, \ldots, m_{n-1}$. Now we are going to construct a row semi-standard λ-tableaux T^λ of the form as in Definition 3.2.2.

The first row of T^λ is λ_1's 1. If $m_1 > \lambda_2$, the second row of T^λ is λ_2's 2; If $m_1 \leqslant \lambda_2$, the second row is m_1's 1 and $(\lambda_2 - m_1)$'s 2. For the third row, when $m_1 > \lambda_2$ (the second row of T^λ is λ_2's 2), if $m_1 - \lambda_2 > \lambda_3$, the third row of T^λ is λ_3's 1; if $m_1 - \lambda_2 \leqslant \lambda_3$, then consider $\lambda_2 + m_2$ and $\lambda_3 - (m_1 - \lambda_2)$. If $\lambda_2 + m_2 \leqslant \lambda_3 - (m_1 - \lambda_2)$, then the third row of T^λ is $(m_1 - \lambda_2 \leqslant \lambda_3)$'s 1, $(\lambda_2 + m_2)$'s 2, and $(\lambda_3 - (m_1 - \lambda_2))$'s 3. Otherwise the third row of T^λ is $(m_1 - \lambda_2 \leqslant \lambda_3)$'s 1 and $(\lambda_3 - (m_1 - \lambda_2))$'s 2. And so on, we have a row semi-standard λ-tableaux T^λ with the weight

$$(\lambda_1 + m_1, \lambda_2 + m_2 - m_1, \lambda_3 + m_3 - m_2, \ldots, \lambda_{n-1} + m_{n-1} - m_{n-2}, \lambda_n - m_{n-1}),$$

that is α. Hence $c_{\lambda,\alpha} \neq 0$.

(ii) Since $\lambda \trianglelefteq \lambda$, we have $c_{\lambda,\lambda} \neq 0$, thus there exists a nonnegative integer lower triangular $n \times n$ matrix $(a_{\mu,\nu})_{\mu,\nu \in \underline{n}}$ whose vector of row sums is $(\lambda_1, \ldots, \lambda_n)$ and whose vector of column sums is $(\lambda_1, \ldots, \lambda_n)$. It is easy to check that

$$a_{\mu,\mu} = \lambda_\mu, \quad a_{\mu,\nu} = 0,$$

where $\mu \neq \nu$. So the number of such matrices is 1, i.e., $c_{\lambda,\lambda} = 1$. ∎

Part II: The Cartan invariant $\widehat{c}_{\lambda,\alpha}$

In this part we introduce the Cartan invariant $\widehat{c}_{\lambda,\alpha}$ and give the connection with $c_{\lambda,\alpha}$. First, we will introduce some results on Borel Schur algebras $S^+(n,r)$ in [16].

Let Λ be the set $\Lambda(n,r)$. Let $\lambda \in \Lambda$. Recall that $A = S^+(n,r)$. Denote left A-modules V_λ and k_λ by

$$V_\lambda = A\xi_\lambda \text{ and } k_\lambda = V_\lambda / \operatorname{rad} V_\lambda.$$

In [16], $\{V_\lambda \mid \lambda \in \Lambda\}$ is a full set of pairwise non-isomorphic principal indecomposable A-modules, and $A = \oplus_{\lambda \in \Lambda} V_\lambda$. Also $\{k_\lambda \mid \lambda \in \Lambda\}$ is a full set of pairwise non-isomorphic irreducible A-modules. Note that k_λ is one-dimensional, since $A/\operatorname{rad} A \cong \oplus_{\lambda \in \Lambda(n,r)} K\xi_\lambda$.

Definition 3.2.6. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be in Λ. Let T^λ be a λ-tableau with values in $\underline{r}$, i.e., any map from the diagram $[\lambda]$ to $\underline{r}$.

If $i \in \mathbf{I}(n,r)$, we denote by T_i^λ the λ-tableau

$$T_i^\lambda = \begin{array}{llll} i_{t_{11}} & i_{t_{12}} & \cdots & i_{t_{1\lambda_1}} \\ i_{t_{21}} & i_{t_{22}} & \cdots & i_{t_{2\lambda_2}} \\ \vdots & & & \\ i_{t_{n1}} & i_{t_{n2}} & \cdots & i_{t_{n\lambda_n}}. \end{array}$$

Assume now that T^λ is *standard*, i.e., T^λ is bijective. Let $l(\lambda) = (1^{\lambda_1}, 2^{\lambda_2}, \ldots, n^{\lambda_n})$. Then we define $l(\lambda) \in \mathbf{I}(n,r)$ by the λ-tableau

$$T^\lambda_{l(\lambda)} = \begin{array}{cccc} 1 & 1 & \ldots & 1 \\ 2 & 2 & \ldots & 2 \\ \vdots & & & \\ n & n & \ldots & n. \end{array}$$

Theorem 3.2.7. *([16],§2). The module V_λ has a basis $\{\xi_{i,l(\lambda)} \mid i \in I(n,r), i \leqslant l(\lambda), T^\lambda_i$ a RSS λ-tableau$\}$.*

Definition 3.2.8. Let $\lambda, \alpha \in \Lambda$. The *Cartan invariant* $\widehat{c}_{\lambda,\alpha}$, of the Borel Schur algebra $S^+(n,r)$ is the multiplicity of k_α as a composition factor in V_λ. The *Cartan matrix* of $S^+(n,r)$ is $C = (\widehat{c}_{\lambda,\alpha})_{\lambda,\alpha\in\Lambda}$.

Thus (see [4],(54.16)), we get

$$\widehat{c}_{\lambda,\alpha} = \dim_K \mathrm{Hom}_{S^+(n,r)}(V_\alpha, V_\lambda) = \dim_K \xi_\alpha V_\lambda.$$

Using Theorem 3.2.7 and the fact that $\xi_\alpha \xi_{i,j} = \xi_{i,j}$ or 0 according to whether $\mathrm{wt}(i) = \alpha$ or not, it is easy to see that $\xi_\alpha V_\lambda$ has K-basis $\{\xi_{i,l(\lambda)} \mid i \in I(n,r), i \leqslant l(\lambda), T^\lambda_i$ a RSS λ-tableau$\}$. Thus $\dim_k \xi_\alpha V_\lambda = c_{\lambda,\alpha}$, and we get the following.

Theorem 3.2.9. *For each pair λ, α of elements of Λ, the Cartan invariant $\widehat{c}_{\lambda,\alpha}$ satisfies*

$$\widehat{c}_{\lambda,\alpha} = c_{\lambda,\alpha}.$$

Notice that, since $c_{\lambda,\alpha}$ depends only on λ and α, we have

Corollary 3.2.10. *The Cartan invariants $\widehat{c}_{\lambda,\alpha}$ of $S(B^+)$ are independent of the field K; in particular, they do not depend on its characteristic.*

Part III: The dimension of $S^+(n,r)$ and the matrix multiplication for $S^+(2,r)$

In this part we calculate the dimension of $S^+(n,r)$ in terms of nonnegative integer lower triangular matrices. We also consider the matrix multiplication for $S^+(2,r)$ and get a formula for a product of basis elements.

Theorem 3.2.11. *The dimension of the Borel Schur algebra $S^+(n,r)$ is* $\binom{r+\binom{n+1}{2}-1}{r}$.

Proof. By Definition 3.2.8,

$$\widehat{c}_{\lambda,\alpha} = \dim_K \xi_\alpha V_\lambda.$$

Recall that $\xi_\alpha V_\lambda$ is spanned by the set

$$\{\xi_{i,j} \mid (i,j) \in \Omega^+(n,r),\ \mathrm{wt}(i) = \alpha,\ \mathrm{wt}(j) = \lambda\}.$$

By Theorem 3.2.9,

$$\widehat{c}_{\lambda,\alpha} = c_{\lambda,\alpha}.$$

By Definition 3.2.2, $c_{\lambda,\alpha}$ is the number of nonnegative integer lower triangular $n \times n$ matrices whose vector of row sums is λ and whose vector of column sums is α. Thus the number of elements of the set

$$\{\xi_{i,j} \mid (i,j) \in \Omega^+(n,r),\ \mathrm{wt}(i) = \alpha,\ \mathrm{wt}(j) = \lambda\}$$

is the number of nonnegative integer lower triangular $n \times n$ matrices whose vector of row sums is λ and whose vector of column sums is α. Hence the dimension of the Borel Schur algebra $S^+(n,r)$ is the number of nonnegative integer lower triangular $n \times n$ matrices whose sum of entries is the sum of λ, is

equal to r. Actually, the number of such nonnegative integer lower triangular $n \times n$ matrices, is the number of nonnegative integer solutions of an equation with $1 + 2 + \ldots + n = \binom{n+1}{2}$ indeterminates,

$$x_{11} + x_{21} + x_{22} + \ldots + x_{n1} + \ldots + x_{nn} = r,$$

which is the combination number $\binom{r + \binom{n+1}{2} - 1}{r}$. ∎

Let $n = 2$. By Theorem 3.2.11, the dimension of $S^+(2, r)$ is the combination number $\binom{r+2}{2}$, which has been seen in the example after Definition 1.1.7.

Next, we consider a matrix multiplication for the Borel Schur algebra $S^+(2, r)$.

Let $M(r)$ denote the set of 2×2 matrices with nonnegative integer entries summing to r. Given $i, j \in \mathbf{I}(2, r)$, we define m_{uv} to be the number of $x \in \{1, \ldots, r\}$ such that $i_x = u$, $j_x = v$ for $u, v = 1, 2$. We then define a function $f : \mathbf{I}(2, r) \times \mathbf{I}(2, r) \to M(r)$ be sending (i, j) to the matrix with entries m_{uv}. Now $f((i, j)) = f((k, l))$ if and only if $(i, j) \sim (k, l)$, and so we may index our basis of $S^+(2, r)$ by $M(r)$. In fact we let $M(r)$ be a basis for $S^+(2, r)$ by identifying $\xi_{i,j}$ with $f((i, j))$. We are now going to obtain the multiplication rule for $S^+(2, r)$ in terms of the matrices in $M(r)$; we shall write this as $A \circ B$ to avoid any confusion with ordinary matrix multiplication. In the following, A and B denote matrices.

For $A \in M(r)$, denote by $r_1(A)$, $r_2(A)$ the first and second row sums of A, and by $c_1(A)$, $c_2(A)$ the first and second column sums of A. Now for A, $B \in M(r)$, define $N(A, B)$ to be the set of matrices $C \in M(r)$ with $r_1(C) = r_1(A)$ and $c_1(C) = c_1(B)$. In addition, if $c_1(A) = r_1(B)$, define $R(A, B)$ to be the set of

2×2 matrices D with (possibly negative) integer entries such that $r_u(D) = a_{u1}$, $c_v(D) = b_{1v}$ for $u, v = 1, 2$.

For any 2×2 matrices C, D with integer coefficients (nonnegative in C), we now define

$$\binom{C}{D} = \prod_{u,v=1,2} \binom{c_{uv}}{d_{uv}}.$$

Proposition 3.2.12. *([8], Proposition 2.1) The multiplication rule for the Schur algebra $S(2,r)$ is given in terms of the basis elements $A \in M(r)$ by*

$$A \circ B = \begin{cases} 0 & (c_1(A) \neq r_1(B)) \\ \sum_{C \in N(A,B)} (\sum_{D \in R(A,B)} \binom{C}{D}) 1_K \cdot C & (c_1(A) = r_1(B)) \end{cases}$$

For the Borel Schur algebra $S^+(2,r)$, given $r \geqslant b \geqslant a \geqslant 0$ and $\xi(b,a)$, we have the matrix $M = (m_{uv})$ as above. Since $\xi(b,a) = \xi_{1^b 2^{r-b}, 1^a 2^{r-a}}$, we get $M(b,a) := \begin{pmatrix} a & b-a \\ 0 & r-b \end{pmatrix}$. Let $M^+(r)$ be the subset of $M(r)$ consisting of those upper triangular matrices $M(b,a)$. So $M^+(r)$ is a basis for $S^+(2,r)$ by identifying $\xi(b,a)$ with $M(b,a)$.

Proposition 3.2.13. *The multiplication rule for the Borel Schur algebra $S^+(2,r)$ is given in terms of the basis elements $M(b,a) \in M^+(r)$ by*

$$M(d,c) \circ M(b,a) = \delta_{b,c} \binom{d-a}{b-a} M(d,a).$$

Proof. Since $(c_1(M(d,c)) = c$ and $r_1(M(b,a))) = b$, by Proposition 3.2.12, we have

$$M(d,c) \circ M(b,a) = \begin{cases} 0 & b \neq c \\ \sum_{C \in N^+(M(d,c),M(b,a))} (\sum_{D \in R^+(M(d,c),M(b,a))} \binom{C}{D}) 1 \cdot C & b = c \end{cases}$$

where $N^+(A,B)$ and $R^+(A,B)$ are the subsets of $N(A,B)$ and $R(A,B)$ respectively, consisting of those upper triangular matrices with nonnegative integer entries. So if $b \neq c$, we have $M(d,c) \circ M(b,a) = 0$.

In the following, we let $b = c$. Suppose that $C = \begin{pmatrix} x & y \\ 0 & r-x-y \end{pmatrix}$ is a matrix in $N^+(M(d,c), M(b,a))$. We get, $x+y = d$, $x = a$. So we have $C = \begin{pmatrix} a & d-a \\ 0 & r-d \end{pmatrix} = M(d,a)$. Let $D := \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$ be a matrix with nonnegative integer entries in $R^+(M(d,c), M(b,a))$. We have

$$x_1 + x_2 = c,\ y_1 + y_2 = 0,\ x_1 + y_1 = a.$$

Since $y_1 + y_2 = 0$, so $y_1 = y_2 = 0$. We have $x_1 = a$ and $x_2 = c - a = b - a$. We get $D = \begin{pmatrix} a & b-a \\ 0 & 0 \end{pmatrix}$. Thus

$$\binom{C}{D} = \binom{a}{a}\binom{d-a}{b-a}\begin{pmatrix} r-d \\ 0 \end{pmatrix} = \binom{d-a}{b-a}.$$

Hence

$$M(d,b) \circ M(b,a) = \binom{d-a}{b-a} M(d,a).$$

∎

Part IV: $c_{\lambda,\alpha}$ for some special cases

In this part we do some calculations for $c_{\lambda,\alpha}$ by computing the number of nonnegative integer lower triangular matrices, which will be used in Chapter 5.

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be in $\Lambda(n,r)$. Recall that

$$\lambda(\nu, m) = (\lambda_1, \ldots, \lambda_\nu + m, \lambda_{\nu+1} - m, \ldots \lambda_n),$$

where $\nu \in \{1, 2, \ldots, n-1\}$ and $0 \leqslant m \leqslant \lambda_{\nu+1}$.

Let $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ be the simple roots of the root system A_{n-1}, i.e.,

$$\alpha_i = (0, \ldots, 0, 1, -1, 0, \ldots, 0),$$

where $1 \leqslant i \leqslant n-1$.

Proposition 3.2.14. *(i)* $c_{\lambda,\alpha} = 1$ *for all* $\lambda, \alpha \in \Lambda(2,r)$ *with* $\lambda \trianglelefteq \alpha$.

(ii) Let $\lambda = (a_1, a_2, a_3)$, $\alpha = (b_1, b_2, b_3) \in \Lambda(3,r)$. *Assume* $\lambda \trianglelefteq \alpha$. *Then*

$$c_{\lambda,\alpha} = \min\{a_2, b_2, b_1 - a_1, a_3 - b_3\} + 1.$$

(iii) Let u *and* v *be two positive integers. Let* λ *and* α *be in* $\Lambda(n,r)$ *such that*

$$\alpha - \lambda = u\alpha_i + v\alpha_j,$$

where $1 \leqslant i, j \leqslant n-1$ *and* $|i-j| \geqslant 2$. *Then*

$$c_{\lambda,\alpha} = 1.$$

(iv) Let $\lambda \in \Lambda(n,r)$. *For a positive integer* $\nu \in \{1, 2, \ldots, n-1\}$, *let* a, b *be integers such that* $\lambda_{\nu+1} \geqslant b \geqslant a \geqslant 0$. *Then*

$$c_{\lambda(\nu,a),\lambda(\nu,b)} = 1.$$

Proof. By Definition 3.2.2, $c_{\lambda,\alpha}$ is the number of nonnegative integer lower triangular $n \times n$ matrices whose vector of row sums is $(\lambda_1, \ldots, \lambda_n)$ and whose vector of column sums is $(\alpha_1, \ldots, \alpha_n)$. By Proposition 3.2.5, $c_{\lambda,\alpha} \neq 0$ if and only if $\lambda \trianglelefteq \alpha$. It is easy to check that the compositions λ and α in the above (i)-(iv) satisfy the condition $\lambda \trianglelefteq \alpha$, thus we have $c_{\lambda,\alpha} > 0$. Hence, there exists at least one matrix satisfying the above conditions in (i)-(iv). Next, we will calculate the number of nonnegative integer lower triangular matrices for the cases (i)-(iv).

To prove (i), let

$$\begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix}$$

be a nonnegative integer lower triangular 2×2 matrix whose vector of row sums is $(a, r-a) = \lambda$ and whose vector of column sums is $(b, r-b) = \alpha$. Then we have: $a_{11} = a$, $a_{21} = b - a$, and $a_{22} = r - b$. Hence $c_{\lambda,\alpha} = 1$ in this case.

To prove (ii), let

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

be a nonnegative integer lower triangular 3×3 matrix whose vector of row sums is $(a_1, a_2, a_3) = \lambda$ and whose vector of column sums is $(b_1, b_2, b_3) = \alpha$. We have

$$\begin{aligned} a_{21} &= a_2 - a_{22}, \\ a_{31} &= a_3 - b_3 - b_2 + a_{22} = b_1 - a_1 - a_2 + a_{22}, \\ a_{32} &= b_2 - a_{22}. \end{aligned}$$

So:

$$d =: \max\{0, b_2 + b_3 - a_3\} \leqslant a_{22} \leqslant \min\{a_2, b_2\}.$$

Hence

$$0 \leqslant a_{22} - d \leqslant \min\{a_2, b_2\} - d = \min\{a_2, b_2, b_1 - a_1, a_3 - b_3\}.$$

Thus the number of nonnegative integer lower triangular matrices is the number of a_{22} satisfying the above condition. Hence

$$c_{\lambda,\alpha} = \min\{a_2, b_2, b_1 - a_1, a_3 - b_3\} + 1.$$

To prove (iii). Since

$$\alpha - \lambda = u\alpha_i + v\alpha_j,$$

then we have $\lambda \trianglelefteq \alpha$. Let $(a_{\mu,\nu})_{n\times n}$ be a nonnegative integer lower triangular $n \times n$ matrix whose vector of row sums is $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and whose

vector of column sums is α. Since $1 \leqslant i, j \leqslant n-1$ and $|j-i| \geqslant 2$, we can rewrite α as (assume that $i < j$):

$$\begin{aligned}\alpha = &\ \lambda + u\alpha_i + v\alpha_j \\ = &\ (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i + u, \lambda_{i+1} - u, \lambda_{i+2}, \ldots, \lambda_{j-1}, \\ &\qquad \lambda_j + v, \lambda_{j+1} - v, \lambda_{j+2}, \ldots, \lambda_n).\end{aligned}$$

By the row sums and column sums of $(a_{\mu,\nu})_{n\times n}$, we can calculate that

$$\begin{aligned}&a_{11} = \lambda_1, \\ &a_{21} = 0,\ a_{22} = \lambda_2, \\ &\cdots, \\ &a_{i-1,1} = 0,\ \ldots,\ a_{i-1,i-2} = 0,\ a_{i-1,i-1} = \lambda_{i-1}, \\ &a_{i,1} = 0,\ \ldots,\ a_{i,i-2} = 0,\ a_{i,i-1} = 0,\ a_{i,i} = \lambda_i, \\ &a_{i+1,1} = 0,\ \ldots,\ a_{i+1,i} = u,\ a_{i+1,i+1} = \lambda_{i+1} - u, \\ &\cdots, \\ &a_{j-1,1} = 0,\ \ldots,\ a_{j-1,j-2} = 0,\ a_{j-1,j-1} = \lambda_{j-1}, \\ &a_{j,1} = 0,\ \ldots,\ a_{j,j-2} = 0,\ a_{j,j-1} = 0,\ a_{j,j} = \lambda_j, \\ &a_{j+1,1} = 0,\ \ldots,\ a_{j+1,j} = v,\ a_{j+1,j+1} = \lambda_{j+1} - v, \\ &\cdots, \\ &a_{n,1} = 0,\ \ldots,\ a_{n,n-1} = 0,\ a_{n,n} = \lambda_n.\end{aligned}$$

Hence there is only one matrix $(a_{\mu,\nu})_{n\times n}$ satisfying the condition, that is, $c_{\lambda,\alpha} = 1$.

To prove (iv). Since

$$\lambda(\nu, b) - \lambda(\nu, a) = (b-a)\alpha_\nu,$$

we have $\lambda(\nu, a) \trianglelefteq \lambda(\nu, b)$. Thus $c_{\lambda(\nu,a),\lambda(\nu,b)} \geqslant 1$. Let $(a_{ij})_{n\times n}$ be a nonnegative integer lower triangular $n \times n$ matrix, whose vector of row sums is

$$\lambda(\nu, a) = (\lambda_1, \ldots, \lambda_\nu + a, \lambda_{\nu+1} - a, \ldots, \lambda_n),$$

and whose vector of column sums is

$$\lambda(\nu, b) = (\lambda_1, \ldots, \lambda_\nu + b, \lambda_{\nu+1} - b, \ldots, \lambda_n).$$

Then we can calculate each row and the corresponding column of $(a_{ij})_{n\times n}$:

Row 1: $a_{11} = \lambda_1$;

Column 1: $a_{11} + \ldots + a_{n1} = \lambda_1$, so $a_{21} = \ldots = a_{n1} = 0$;

Row 2: $a_{22} = \lambda_2$;

Column 2: $a_{22} + \ldots + a_{n2} = \lambda_2$, so $a_{32} = \ldots = a_{n2} = 0$;

$\vdots$

Row ν: $a_{\nu\nu} = \lambda_\nu + a$;

Column ν: $a_{\nu\nu} + \ldots + a_{n\nu} = \lambda_\nu + b$, so

$$a_{\nu+1,\nu} + \ldots + a_{n,\nu} = b - a; \tag{3.2.1}$$

Row $\nu + 1$:

$$a_{\nu+1,\nu} + a_{\nu+1,\nu+1} = \lambda_{\nu+1} - a; \tag{3.2.2}$$

Column $\nu + 1$:

$$a_{\nu+1,\nu+1} + \ldots + a_{n,\nu+1} = \lambda_{\nu+1} - b; \tag{3.2.3}$$

We then consider (3.2.1)+(3.2.3)-(3.2.2):

$$a_{\nu+2,\nu} + \ldots + a_{n,\nu} + a_{\nu+2,\nu+1} + \ldots + a_{n,\nu+1} = 0. \tag{3.2.4}$$

Since all entries a_{ij} are nonnegative integers, this implies

$$a_{\nu+2,\nu} = \ldots = a_{n,\nu} = 0 = a_{\nu+2,\nu+1} = \ldots = a_{n,\nu+1}.$$

Now from equations (3.2.1), (3.2.2), and (3.2.3), we get

$$a_{\nu+1,\nu} = b - a, \quad a_{\nu+1,\nu+1} = \lambda_{\nu+1} - b.$$

Continuing the above process, we have:

$$\begin{aligned} &a_{\nu+2,\nu+2} = \lambda_{\nu+2}; \\ &a_{\nu+3,\nu+2} = \ldots = a_{n,\nu+2} = 0; \\ &\ldots \\ &a_{n,1} = \ldots = a_{n,n-1} = 0, \quad a_{n,n} = \lambda_n. \end{aligned}$$

Thus there is only one such a matrix $(a_{ij})_{n\times n}$, hence $c_{\lambda(\nu,a),\lambda(\nu,b)} = 1$. ∎

Proposition 3.2.15. *Let $A = S^+(3,r)$ and let α_1 and α_2 be simple roots of type A_2. Let $\lambda = (a_1, a_2, a_3)$ and α be in $\Lambda(3,r)$ such that*

$$\alpha = \lambda + m\alpha_1 + n\alpha_2,$$

where m and n are nonnegative integers. We assume that $a_2 \geqslant m$. Let H be the vector space

$$H := \mathrm{Hom}_A(A\xi_\alpha, A\xi_\lambda).$$

Let $d = \min\{m,n\}$. Then the dimension of H is $(d+1)$. $H \simeq \xi_\alpha A \xi_\lambda$ is spanned by

$$\{X_t \mid X_t = \xi_{i_t,\underline{3}^\lambda}, \quad i_t = 1^{a_1+m-t}2^{a_2+n-m}1^t3^{a_3-n}, t = 0,1,\ldots,d\}.$$

Moreover, $X_t = \xi_{\underline{3}^\alpha,q_t}$, where $q_t = 1^{a_1}3^t2^{a_2}3^{a_3-t}$ and $0 \leqslant t \leqslant d$.

Proof. Note that $a_2 \geqslant m$. By Proposition 3.2.14 (ii), the dimension of H is the Cartan invariant $\widehat{c}_{\alpha^{00},\alpha^{nm}}$:

$$\dim H = \min\{m, a_2, a_2 - m + n, n\} + 1 = \min\{m,n\} + 1 = d + 1.$$

By Lemma 1.3.5, $H \simeq \xi_\alpha A \xi_\lambda$ as K-vector spaces. If $\xi_{i,j}$ is an element of $\xi_\alpha A \xi_\lambda$. Then $i \leqslant j$ and the weights of i and j are

$$wt(i) = \alpha, \ wt(j) = \lambda.$$

Let $j = \underline{3}^\lambda$. Since $i \leqslant j$ and $i \sim \underline{3}^\alpha$, so it is easy to check that the above set $\{X_t \mid t = 0,1,\ldots,d\}$ form a basis of H.

Next we prove that $\xi_{i_t,\underline{3}^\lambda} = \xi_{\underline{3}^\alpha,q_t}$. We need to prove the multi-indices $(i_t, \underline{3}^\lambda) \sim (\underline{3}^\alpha, q_t)$, that is

$$(1^{a_1+m-t}2^{a_2+n-m}1^t3^{a_3-n}, 1^{a_1}2^{a_2}3^{a_3}) \sim (1^{a_1+m}2^{a_2+n-m}3^{a_3-n}, 1^{a_1}3^t2^{a_2}3^{a_3-t}).$$

We consider the number of the pair $(1,3)$ on both sides. Then the numbers of $(1,3)$ on both sides are both t. So the above relation for the multi-indices holds. ∎

We will use Proposition 3.2.15 in Sections 5.3, 5.4 and 5.5.

Finally, we give a second proof for Lemma 2.1.2:
Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda(n,r)$. Let $t \in \{1, 2, \ldots, n-1\}$ and let l and m be nonnegative integers such that $l \leqslant m \leqslant \lambda_{t+1}$. We have

$$\xi_{\lambda(t,m),\lambda(t,l)}\xi_{\lambda(t,l),\lambda} = \binom{m}{l}\xi_{\lambda(t,m),\lambda}.$$

Second Proof of Lemma 2.1.2: By Proposition 3.2.14 (iv), $c_{\lambda,\lambda(t,m)} = 1$. So $\xi_{\lambda(t,m)}S^+(n,r)\xi_\lambda$ is one-dimensional and spanned by $\xi_{\lambda(t,m),\lambda}$.

We let

$$\begin{aligned} i &= \underline{n}^{\lambda(t,m)} = 1^{\lambda_1} \ldots t^{\lambda_t+m}(t+1)^{\lambda_{t+1}-m} \ldots n^{\lambda_n}, \\ j &= \underline{n}^{\lambda(t,l)} = 1^{\lambda_1} \ldots t^{\lambda_t+l}(t+1)^{\lambda_{t+1}-l} \ldots n^{\lambda_n}, \\ l &= \underline{n}^{\lambda} = 1^{\lambda_1} \ldots t^{\lambda_t}(t+1)^{\lambda_{t+1}} \ldots n^{\lambda_n}. \end{aligned}$$

By Theorem 1.1.4,

$$\xi_{i,j}\xi_{j,l} = [P_{i,l} : P_{i,j,l}]\xi_{i,l}. \tag{3.2.5}$$

where $[P_{i,l} : P_{i,j,l}]$ can be computed from the formula

$$[P_{i,l} : P_{i,j,l}] = \prod_{a,b\in\underline{n}} \frac{r_{a,b}!}{r_{a,1,b}! \cdots r_{a,n,b}!},$$

where, for all $a, d, b \in \underline{n}$, $r_{a,b} = |R_{a,b}(i,l)|$ and $r_{a,d,b} = |R_{a,d,b}(i,j,l)|$.

Now we calculate $r_{a,b}$ and $r_{a,d,b}$, where

$$r_{a,b} = |R_{a,b}(i,l)| = |\{\rho \in \underline{r} \mid i_\rho = a,\ l_\rho = b\}|.$$

We have

$$\begin{aligned} &r_{a,a} = \lambda_a, \ a \neq t, t+1, \\ &r_{t,t} = \lambda_t, \quad r_{t,t+1} = m, \quad r_{t+1,t+1} = \lambda_{t+1} - m, \\ &r_{a,b} = 0, \text{ otherwise.} \end{aligned}$$

and

$$\begin{aligned} &r_{a,a,a} = \lambda_a, \ a \neq t, t+1, \\ &r_{t,t,t} = \lambda_t, \quad r_{t,t,t+1} = l, \quad r_{t,t+1,t+1} = m - l, \quad r_{t+1,t+1,t+1} = \lambda_{t+1} - m, \\ &r_{a,d,b} = 0, \text{ otherwise.} \end{aligned}$$

Hence

$$[P_{i,l} : P_{i,j,l}] = \frac{m!}{l!(m-l)!} = \binom{m}{l}.$$

3.3 Properties of Embeddings

In this section, by calculating the radical and radical square, we show that the embedding $\varphi_t^\alpha : S^+(2,r) \to S^+(n, r+s)$ embeds the quiver of $S^+(2,r)$ into the quiver of $S^+(n, r+s)$.

Throughout this section, let $A = \varphi_t^\alpha(S^+(2,r))$ and $B = S^+(n, r+s)$. Since φ_t^α is an embedding where $t \in \underline{n-1}$ and $\alpha \in \Lambda(n,s)$, A is a subalgebra of B. First, let us discuss the relationship of radical and radical square with the algebras A and B.

Proposition 3.3.1.

$$\operatorname{rad} A = A \cap \operatorname{rad} B.$$

Proof. Since $\operatorname{rad} A$ is the largest nilpotent ideal of A, and $A \cap \operatorname{rad} B$ is nilpotent in A, then $A \cap \operatorname{rad} B \subseteq \operatorname{rad} A$. On the other hand, by Proposition 1.2.3, $\operatorname{rad} A \subset \operatorname{rad} B$. Thus $\operatorname{rad} A \subseteq A \cap \operatorname{rad} B$. ∎

Recall that $S^+(2,r)$ has a basis $\{\xi(b,a) \mid 0 \leqslant a \leqslant b \leqslant r\}$.

Proposition 3.3.2. *Let $\xi \in S^+(2,r)$. Write ξ as a linear combination of basis elements, $\xi = \sum_{b \geqslant a} k_{b,a}\xi(b,a)$, where $k_{b,a} \in K$ for all $r \geqslant b \geqslant a \geqslant 0$. Then we have $\varphi_t^\alpha(\xi) \in A \cap (\operatorname{rad} B)^2$ if and only if $\varphi_t^\alpha(\xi(b,a)) \in A \cap (\operatorname{rad} B)^2$ for all nonzero $k_{b,a}$.*

Proof. (The if part): This is trivial.

(The only if part): By the multiplication rule for the Schur algebra,

$$\xi(b,b)\xi\xi(a,a) = k_{b,a}\xi(b,a).$$

If $k_{b,a} \neq 0$, then

$$\xi(b,a) = {k_{b,a}}^{-1}\xi(b,b)\xi\xi(a,a).$$

Since φ_t^α is an algebra homomorphism, we have

$$\begin{aligned} \varphi_t^\alpha(\xi(b,a)) &= \varphi_t^\alpha({k_{b,a}}^{-1}\xi(b,b)\xi\xi(a,a)) \\ &= {k_{b,a}}^{-1}\varphi_t^\alpha(\xi(b,b))\varphi_t^\alpha(\xi)\varphi_t^\alpha(\xi(a,a)). \end{aligned}$$

Since $\varphi_t^\alpha(\xi) \in A \cap (\operatorname{rad} B)^2$ and $(\operatorname{rad} B)^2$ is an ideal of B, then $\varphi_t^\alpha(\xi(b,a)) \in (\operatorname{rad} B)^2$, i.e. $\varphi_t^\alpha(\xi(b,a)) \in A \cap (\operatorname{rad} B)^2$. ∎

Proposition 3.3.3.

$$(\operatorname{rad} A)^2 = A \cap (\operatorname{rad} B)^2.$$

Proof. By Proposition 3.3.1 $\operatorname{rad} A \subseteq \operatorname{rad} B$, so $(\operatorname{rad} A)^2 \subseteq (\operatorname{rad} B)^2$, thus

$$(\operatorname{rad} A)^2 \subseteq A \cap (\operatorname{rad} B)^2.$$

We need to prove that

$$A \cap (\operatorname{rad} B)^2 \subseteq (\operatorname{rad} A)^2.$$

By Proposition 3.3.2 and Proposition 1.2.3, it is enough to prove that for any $0 \leqslant a < b \leqslant r$ and $\varphi_t^\alpha(\xi(b,a)) \in A \cap (\operatorname{rad} B)^2$, then

$$\varphi_t^\alpha(\xi(b,a)) \in (\operatorname{rad} A)^2.$$

Since $\varphi_t^\alpha(\xi(b,a)) \in A \cap (\operatorname{rad} B)^2$, so $\varphi_t^\alpha(\xi(b,a)) \in (\operatorname{rad} B)^2$. Let

$$\varphi_t^\alpha(\xi(b,a)) = \sum_u k_u f_u g_u,$$

where f_u and g_u are in $\operatorname{rad} B$, and $k_u \in K$. Using the multiplication rule for Schur algebra, we can assume that $f_u = \xi_{i,j}$ and $g_u = \xi_{j,l}$.

Let $\alpha = (a_1, \ldots, a_n) \in \Lambda(n,s)$, then we can write $\varphi_t^\alpha(\xi(a,a))$ as:

$$\varphi_t^\alpha(\xi(a,a)) = \xi_{\alpha^1},$$

where $\alpha^1 = (a_1, \ldots, a_{t-1}, a_t + a, a_{t+1} + r - a, a_{t+2}, \ldots, a_n)$.

Similarly, we can write $\varphi_t^\alpha(\xi(b,b))$ as:

$$\varphi_t^\alpha(\xi(b,b)) = \xi_{\alpha^2},$$

where $\alpha^2 = (a_1, \ldots, a_{t-1}, a_t + b, a_{t+1} + r - b, a_{t+2}, \ldots, a_n)$.

Thus we can rewrite $\varphi_t^\alpha(\xi(b,a))$ as follows:

$$\begin{aligned} \varphi_t^\alpha(\xi(b,a)) &= \varphi_t^\alpha(\xi(b,b)\xi(b,a)\xi(a,a)) \\ &= \xi_{\alpha^2}\varphi_t^\alpha(\xi(b,a))\xi_{\alpha^1} \\ &= \textstyle\sum_u \xi_{\alpha^2} k_u f_u g_u \xi_{\alpha^1} \\ &= \textstyle\sum_u \overline{f}_u \overline{g}_u, \end{aligned}$$

where

$$\begin{aligned} \overline{f}_u &= k_u \xi_{\alpha^2} f_u \in \operatorname{rad} B, \\ \overline{g}_u &= g_u \xi_{\alpha^1} \in \operatorname{rad} B. \end{aligned}$$

Fix any u with $\overline{f}_u \overline{g}_u \neq 0$. We will show that $\overline{f}_u, \overline{g}_u \in \operatorname{rad} A$, so $\varphi_t^\alpha(\xi(b,a)) \in (\operatorname{rad} A)^2$ as required.

Recall that $f_u = \xi_{i,j}$ and $g_u = \xi_{j,l}$ for some i, j, l. Let the shape of j be $\beta = (b_1, \ldots, b_n) \in \Lambda(n, r+s)$, we can rewrite $\overline{f}_u$ and $\overline{g}_u$ as follows:

$$\overline{f}_u = \xi_{\alpha^2} k_u f_u \xi_\beta \in \xi_{\alpha^2} B \xi_\beta,$$
$$\overline{g}_u = \xi_\beta g_u \xi_{\alpha^1} \in \xi_\beta B \xi_{\alpha^1}.$$

By Definition 3.2.8, the Cartan invariant $\widehat{c}_{\beta,\alpha^2}$ is the dimension of $\xi_{\alpha^2} B \xi_\beta$. Here $\overline{f}_u \neq 0$, which implies that $\widehat{c}_{\beta,\alpha^2} \geqslant 1$. By Theorem 3.2.9, we have $\widehat{c}_{\beta,\alpha^2} = c_{\beta,\alpha^2}$. By Proposition 3.2.5, $c_{\beta,\alpha^2} \geqslant 1$ if and only if $\beta \trianglelefteq \alpha^2$. Similarly by $\overline{g}_u \neq 0$, we have $\beta \trianglerighteq \alpha^1$. Hence

$$\alpha^2 \trianglerighteq \beta \trianglerighteq \alpha^1.$$

By the definition of the dominance ordering in Definition 1.2.5, we have

$$a_1 + \ldots + a_k \geqslant b_1 + \ldots + b_k \geqslant a_1 + \ldots + a_k, \quad \text{for all } k \in \{1, \ldots, t-1\}.$$

Thus $b_k = a_k$, for all $k \in \{1, \ldots, t-1\}$.

Similarly, we have $b_k = a_k$, for all $k \in \{t+2, \ldots, n\}$. Hence for the t-th and $(t+1)$-th entries of α^1, α^2 and β, we have the following condition:

$$a_t + b \geqslant b_t = a_t + c \geqslant a_t + a, \quad b_{t+1} = a_{t1} + r - c,$$

then we have

$$b \geqslant c \geqslant a.$$

This means that $\xi_\beta = \varphi_t^\alpha(\xi(c, c))$. Since f_u and g_u are in the radical of B, then $\overline{f}_u$ and $\overline{g}_u$ are in the radical of B. Since idempotents are not contained in the radical, we have the condition:

$$b > c > a.$$

Now $\overline{g}_u$ and $\overline{f}_u$ are just scalar multiples of $\varphi_t^\alpha(\xi(c, a))$ and $\varphi_t^\alpha(\xi(b, c))$, respectively. Thus $\overline{f}_u$ and $\overline{g}_u$ are in the radical of A. Hence $\varphi_t^\alpha(\xi(b, a)) \in (\operatorname{rad} A)^2$.

∎

Let $\lambda \in \Lambda(n,r)$. Recall that

$$\lambda(\nu,m) = (\lambda_1, \ldots, \lambda_\nu + m, \lambda_{\nu+1} - m, \ldots \lambda_n).$$

Theorem 3.3.4. *Let K be a field of characteristic $p \geqslant 0$. Let $\lambda \in \Lambda(n,r)$. Let m be a positive integer such that $m \leqslant \lambda_{\nu+1}$ for some $\nu \in \{1,2,\ldots,n-1\}$. Consider the full sub-quiver Γ' of the quiver Γ of $S^+(n,r)$, generated by all vertices corresponding to the idempotents $\xi_\lambda, \xi_{\lambda(\nu,1)}, \ldots, \xi_{\lambda(\nu,m)}$. Then Γ' is the quiver of $S^+(2,m)$.*

Proof. Let $e_i = (0,\ldots,0,1,0,\ldots,0)$ be a $1\times n$ vector of whose the i-th entry is 1 and the other ones are 0. We let $\alpha = \lambda - me_{\nu+1}$, where $\nu \in \{1,2,\ldots,n-1\}$ and $1 \leqslant m \leqslant \lambda_{\nu+1}$. Thus we have $\alpha \in \Lambda(n, r-m)$. We consider the embedding $\varphi_\nu^\alpha : S^+(2,m) \to S^+(n,r)$.

For the basis element $\xi(b,a)$ where $m \geqslant b \geqslant a \geqslant 0$, by Definition 3.1.1, we have

$$\varphi_\nu^\alpha(\xi(b,a)) = \xi_{\alpha(\nu,b),\alpha(\nu,a)},$$

where

$$\alpha(\nu,b) = (\lambda_1, \ldots, \lambda_\nu + b, \lambda_{\nu+1} - m - b, \ldots, \lambda_n),$$

$$\alpha(\nu,a) = (\lambda_1, \ldots, \lambda_\nu + a, \lambda_{\nu+1} - m - a, \ldots, \lambda_n),$$

and

$$\xi_{\alpha(\nu,b),\alpha(\nu,a)} = \xi_{\underline{n}^{\alpha(\nu,b)},\underline{n}^{\alpha(\nu,a)}}.$$

Let Γ'' be the quiver of $S^+(2,m)$. Note that φ_ν^α maps the basis element $\xi(a,a)$ of $S^+(2,m)$ where $0 \leqslant a \leqslant m$, to the basis element $\varphi_\nu^\alpha(\xi(a,a)) = \xi_{\alpha(\nu,a)}$. This implies that φ_ν^α maps the vertex a of Γ'' to the vertex $\alpha(\nu,a)$ of Γ.

Since φ_ν^α is an embedding from $S^+(2,m)$ to $S^+(n,r)$, $S^+(2,m)$ is isomorphic to $\varphi_\nu^\alpha(S^+(2,m))$. Thus, the radicals of $S^+(2,m)$ and $\varphi_\nu^\alpha(S^+(2,m))$ are isomorphic.

Let $R_1 = \operatorname{rad} S^+(2,m)$, $R = \operatorname{rad} \varphi_\nu^\alpha(S^+(2,m))$ and $R_2 = \operatorname{rad} S^+(n,r)$. Then $R_1 \simeq R$. By Definition 1.4.2, the number of arrows from a to b in Γ'' is

$$c(b,a) := \dim \xi(b,b)(R_1/R_1^2)\xi(a,a),$$

where $0 \leqslant a < b \leqslant r$, and the number of arrows from $\xi_{\alpha(\nu,a)}$ to $\xi_{\alpha(\nu,b)}$ is

$$c'(b,a) := \dim \xi_{\alpha(\nu,b)}(R_2/R_2^2)\xi_{\alpha(\nu,a)}.$$

We need to prove that $c(b,a) = c'(b,a)$ for all $0 \leqslant a < b \leqslant r$.

Recall that p is the characteristic of K. By Theorems 2.2.5 and 2.4.2, $c(b,a) = 1$ if and only if $[b-a]_p = 1$ (if $p = 0$, we have $[b-a]_p = b - a$), otherwise $c(b,a) = 0$.

Note that $\xi_{\alpha(\nu,b)}(R_2/R_2^2)\xi_{\alpha(\nu,a)} \simeq (\xi_{\alpha(\nu,b)}R_2\xi_{\alpha(\nu,a)})/(R_2^2 \bigcap \xi_{\alpha(\nu,b)}R_2\xi_{\alpha(\nu,a)})$. By Proposition 1.2.3, R_2 is spanned by $\{\xi_{ij} \mid i < j, \ (i,j) \in \Omega^+(n,r)\}$. Then $\xi_{\alpha(\nu,b)}R_2\xi_{\alpha(\nu,a)}$, as a vector space, is isomorphic to the vector space spanned by $\{\xi_{ij} \mid i \sim \underline{n}^{\alpha(\nu,b)}, \ j \sim \underline{n}^{\alpha(\nu,a)}\}$, which is also isomorphic to $\xi_{\alpha(\nu,b)}S^+(n,r)\xi_{\alpha(\nu,a)}$. Then by the definition of the Cartan invariant, $\dim \xi_{\alpha(\nu,b)}R_2\xi_{\alpha(\nu,a)} = \dim \xi_{\alpha(\nu,b)}S^+(n,r)\xi_{\alpha(\nu,a)} = c_{\alpha(\nu,a),\alpha(\nu,b)}$. By Proposition 3.2.14 (iv), $c_{\alpha(\nu,a),\alpha(\nu,b)} = 1$ if $b \geqslant a$, otherwise it will be 0. And when $c_{\alpha(\nu,a),\alpha(\nu,b)} = 1$, the vector space $\xi_{\alpha(\nu,b)}R_2\xi_{\alpha(\nu,a)}$ is spanned by the element $\xi_{\alpha(\nu,b),\alpha(\nu,a)}$. This means $c'(b,a) = 0$ or 1.

Therefore if $c(b,a) = 0$, i.e. $\xi(b,a) \in R_1^2$, then $\xi_{\alpha(\nu,b),\alpha(\nu,a)} \in R_2^2$, that is, $c'(b,a) = 0$.

The remaining case is that $c(b,a) = 1$. We will show that $c'(b,a) \geqslant c(b,a)$. Since $c'(b,a) \leqslant 1$, we have $c'(b,a) = 1$.

By Propositions 3.3.1 and 3.3.3, we have

$$R = \varphi_\nu^\alpha(S^+(2,m)) \bigcap R_2, \quad R^2 = \varphi_\nu^\alpha(S^+(2,m)) \bigcap {R_2}^2.$$

Thus the induced map $\varphi_\nu^\alpha : \xi(b,b)(R_1/R_1^2)\xi(a,a) \to \xi_{\alpha(\nu,b)}(R_2/R_2^2)\xi_{\alpha(\nu,a)}$ is well-defined. Actually, we note that

$$\varphi_\nu^\alpha(\xi(b',a') + R_1^2) = \varphi_\nu^\alpha(\xi(b',a')) + R_2^2 = \xi_{\alpha(\nu,b'),\alpha(\nu,a')} + R_2^2,$$

where $0 \leqslant a' \leqslant b' \leqslant m$. If the basis element $\xi(b',a') \in R_1^2$, we have $\varphi_\nu^\alpha(\xi(b',a')) = \xi_{\alpha(\nu,b'),\alpha(\nu,a')} \in R^2$. Since $R^2 = \varphi_\nu^\alpha(S^+(2,m)) \bigcap R_2^2$ by Proposition 3.3.3, we get $\xi_{\alpha(\nu,b'),\alpha(\nu,a')} \in R_2^2$. Hence φ_ν^α is well-defined.

Moreover, if $0 = \xi_{\alpha(\nu,b'),\alpha(\nu,a')} + R_2^2 \in \xi_{\alpha(\nu,b)}(R_2/R_2^2)\xi_{\alpha(\nu,a)}$, that is, $\xi_{\alpha(\nu,b'),\alpha(\nu,a')} \in R_2^2$. Since $R^2 = \varphi_\nu^\alpha(S^+(2,m)) \bigcap R_2^2$ and $\xi_{\alpha(\nu,b'),\alpha(\nu,a')} = \varphi_\nu^\alpha(\xi(b',a')) \in \varphi_\nu^\alpha(S^+(2,m))$, we have $\xi_{\alpha(\nu,b'),\alpha(\nu,a')} \in R^2$. As we know that φ_ν^α is an embedding from $S^+(2,m)$ to $S^+(n,r)$, we have $\xi(b',a') \in R_1^2$. That means φ_ν^α is an embedding from $\xi(b,b)(R_1/R_1^2)\xi(a,a)$ to $\xi_{\alpha(\nu,b)}(R_2/R_2^2)\xi_{\alpha(\nu,a)}$. Hence we have $c'(b,a) \geqslant c(b,a)$. Therefore $c(b,a) = c'(b,a)$.

Hence the quiver Γ'' of $S^+(2,m)$ is a full sub-quiver of the quiver Γ of $S^+(n,r)$.

∎

Recall that we call the relations in Theorem 2.3.1, the p-adic relations. Now by Theorem 3.3.4 these p-adic relations of $S^+(2,m)$ can be embedded in the relations of $S^+(n,r)$. We also call the relations of $S^+(n,r)$ which are embedded

by the p-adic relations of $S^+(2,m)$ in Theorem 2.3.1, the p-adic relations of $S^+(n,r)$.

Let $0 \leqslant a < b \leqslant r$. By the end of Chapter 2, there is a path $P(b,a)$ from a to b. By our embedding, then there will be a path

$$P(\lambda(t,b),\lambda(t,a)) = \varphi_t^{\alpha}(P(b,a))$$

from $\lambda(t,a)$ to $\lambda(t,b)$ where $\lambda = \alpha + rE_{t+1}$. We call $P(\lambda(t,b),\lambda(t,a))$ the path from $\lambda(t,a)$ to $\lambda(t,b)$.

Chapter 4

The quiver of the Borel Schur algebra $S^+(n,r)$

Let K be a field of characteristic $p \geqslant 0$, A the Borel Schur algebra $S^+(n,r)$ over K, and Γ the quiver of A. We know that the vertex set of the quiver Γ is $\Lambda(n,r)$ (see the end of Section 1.4). Using the results from [16], in Section 4.1 we describe the arrow set of Γ. In Section 4.2 we list some quivers for $S^+(3,r)$.

4.1 The quiver of $S^+(n,r)$

In this section we describe the quiver Γ of A.

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be in $\Lambda(n,r)$. We know that ξ_λ is a primitive idempotent of $S^+(n,r)$ by Proposition 1.3.6. Let V_λ be a left ideal

$$V_\lambda := S^+(n,r)\xi_\lambda.$$

Recall the multi-index $l = l(\lambda) = 1^{\lambda_1} \ldots n^{\lambda_n}$ and recall that

$$\lambda(\nu, m) = (\lambda_1, \ldots, \lambda_\nu + m, \lambda_{\nu+1} - m, \ldots \lambda_n).$$

Let $l(\nu, m) = l(\lambda(\nu, m))$. In [16], A. Santana proved the following theorem:

Theorem 4.1.1. *[(4.5)Theorem, [16]] The sets*

(i) $\{\xi_{l(\nu,1),l} \mid \nu \in \{1,2,\ldots,n-1\}\}$, *if* $charK = 0$,
(ii) $\{\xi_{l(\nu,p^{d_\nu}),l} \mid \nu \in \{1,2,\ldots,n-1\}, 1 \leqslant p^{d_\nu} \leqslant \lambda_{\nu+1}\}$, *if* $charK = p$,

are minimal sets of $S^+(n,r)$*-generators of* $\operatorname{rad} V_\lambda$.

The following lemma shows that this minimal set of generators of A-module A is bijective to a K-basis of $(\operatorname{rad} A)/(\operatorname{rad} A)^2$ where $A = S^+(n,r)$.

Lemma 4.1.2. *Let B be an elementary finite dimensional algebra over K, and $R = \operatorname{rad} B$. Then any minimal set of generators of B-module R is bijective to a minimal set of generators of B/R-module R/R^2, or a K-basis of R/R^2. In particular, let $B = S^+(n,r)$. Then the minimal set of generators of $S^+(n,r)$-module R is bijective to a K-basis of R/R^2.*

Proof. By Theorem 1.4.6(a), a minimal set of generators of B-module R is bijective to a minimal set of generators of B/R-module R/R^2. Since B/R is semi-simple, R/R^2 is completely reducible, i.e., R/R^2 is a direct sum of simple B/R-modules. Since B is elementary, i.e., B/R is a product of copies of K, the minimal set of generators of B/R-module R/R^2 is a K-basis of R/R^2.

In particular, let $B = S^+(n,r)$. By Theorem 1.3.8, the Borel Schur algebra $S^+(n,r)$ is elementary over K. Hence the the minimal set of generators of $S^+(n,r)$-module R is bijective to a K-basis of R/R^2. ∎

We now are ready to describe the quiver Γ of the Borel Schur algebra $S^+(n,r)$. The following two theorems give the description for the quiver Γ of $S^+(n,r)$.

Theorem 4.1.3. *Let char $K = p > 0$, and let $\lambda, \mu \in \Lambda(n,r)$ be two vertices of the quiver of $S^+(n,r)$ over K. Then the number of arrows from λ to μ is at most one. Moreover, there is an arrow from λ to μ if and only if $\mu = \lambda(\nu, m)$, where the integers ν and m satisfy the conditions: $1 \leqslant \nu \leqslant n-1$, $1 \leqslant m \leqslant \lambda_{v+1}$ and $m = p^t$ for some nonnegative integer t.*

Proof. Let R be the radical of $S^+(n,r)$. By Definition 1.4.2, the number of arrows from λ to μ in the quiver Γ of $S^+(n,r)$ is the dimension of $\xi_\mu(R/R^2)\xi_\lambda$. By Lemma 4.1.2 and Theorem 4.1.1, R/R^2 has a basis

$$\Delta := \{\xi_{l(\nu,p^{d_\nu}),l} \mid \lambda \in \Lambda(n,r),\ \nu \in \{1,2,\ldots,n-1\}, 1 \leqslant p^{d_\nu} \leqslant \lambda_{\nu+1}\}.$$

We now need to calculate the dimension of

$$\xi_\mu(R/R^2)\xi_\lambda \cong \xi_\mu\Big(\bigoplus_{\xi_{l(\nu,p^{d_\nu}),l} \in \Delta} K \cdot \xi_{l(\nu,p^{d_\nu}),l}\Big)\xi_\lambda =: H(\lambda,\mu).$$

We denote the above vector space as $H(\lambda,\mu)$. Let $c(\lambda,\mu) = \dim H(\lambda,\mu)$. So we only need to prove that $c(\lambda,\mu) = 1$ if and only if $\mu = \lambda(\nu, p^{d_\nu})$ for some nonnegative integers $1 \leqslant \nu \leqslant n-1$ and d_ν; otherwise $c(\lambda,\mu) = 0$.

First we assume that $\mu = \lambda(\nu, p^{d_\nu})$ for some nonnegative integers $1 \leqslant \nu \leqslant n-1$ and d_ν. We will prove that $c(\lambda,\mu) = 1$ in this case. Let $l = l(\lambda)$. Then $\xi_{\mu,\lambda} = \xi_{l(\nu,p^{d_\nu}),l} \in \Delta$. Thus we have $\xi_{\mu,\lambda} \in H(\lambda,\mu)$, so

$$K \cdot \xi_{\mu,\lambda} \subseteq H(\lambda,\mu).$$

By Definition 3.2.8, the Cartan invariant $c_{\lambda,\mu}$ is the dimension of $\xi_\mu S^+(n,r)\xi_\lambda$. By Proposition 3.2.14 (iv), $c_{\lambda,\mu} = c_{\lambda,\lambda(\nu,p^{d_\nu})} = 1$. Thus we have that $\xi_\mu S^+(n,r)\xi_\lambda$ is one-dimensional, spanned by $\xi_{\mu,\lambda}$, that is,

$$\xi_\mu S^+(n,r)\xi_\lambda = K \cdot \xi_{\mu,\lambda}.$$

By the definition of $H(\lambda, \mu)$, we know that

$$H(\lambda, \mu) \subseteq \xi_\mu S^+(n, r)\xi_\lambda.$$

Hence $H(\lambda, \mu)$ satisfies the following condition:

$$K \cdot \xi_{\mu,\lambda} \subseteq H(\lambda, \mu) \subseteq \xi_\mu S^+(n, r)\xi_\lambda = K \cdot \xi_{\mu,\lambda},$$

Therefore, $H(\lambda, \mu) = K \cdot \xi_{\mu,\lambda}$. Thus we have $c(\lambda, \mu) = \dim H(\lambda, \mu) = 1$.

Next, we suppose $c(\lambda, \mu) \neq 0$. We want to show that $c(\lambda, \mu) = 1$. Let h be a non-zero element in $H(\lambda, \mu)$. Note that the set Δ spans the vector space H. We can write h as a linear combination of the elements in Δ:

$$h = \sum_{\xi \in \Delta} a_\xi \xi,$$

where $a_\xi \in K$. Since $h \neq 0$, there exists some $a_\xi \neq 0$ for some $\xi \in \Delta$. Note that $h = \xi_\mu h \xi_\lambda$ since $h \in H(\lambda, \mu)$. We know that $\xi_\mu \eta \xi_\lambda = \eta$ or 0 for all $\eta \in \Delta$. Since ξ occurs in the sum of h and $\xi \in \Delta$, we have

$$\xi = \xi_\mu \xi \xi_\lambda.$$

Let $\xi = \xi_{\alpha(\nu, p^{d_\nu}), \alpha}$ for some $\alpha \in \Lambda(n, r)$ and nonnegative integers $1 \leqslant \nu \leqslant n-1$ and d_ν. Since $\xi = \xi_\mu \xi \xi_\lambda$, by the multiplication rule for the Schur algebra $S(n, r)$, we have

$$\underline{n}^\lambda \sim \underline{n}^\alpha, \quad \underline{n}^{\alpha(\nu, p^{d_\nu})} \sim \underline{n}^\mu.$$

Thus $\alpha = \lambda$ and $\mu = \alpha(\nu, p^{d_\nu})$. Hence $\mu = \lambda(\nu, p^{d_\nu})$. For such λ and μ, we have proved that $H(\lambda, \mu) = K \cdot \xi_{\mu,\lambda}$. Therefore $c(\lambda, \mu) = 1$. ∎

Similarly, by Theorem 4.1.1, we can get the quiver of $S^+(n, r)$ over a field K of characteristic 0.

Theorem 4.1.4. *Let char $K = 0$, and let $\lambda, \mu \in \Lambda(n, r)$ be two vertices of the quiver of $S^+(n, r)$ over K. Then the number of arrows from λ to μ is at most one. Moreover, there is an arrow from λ to μ if and only if $\mu = \lambda(\nu, 1)$, where the integers ν satisfy the condition: $1 \leqslant \nu \leqslant n-1$.*

By Theorems 4.1.3 and 4.1.4, there is precisely one arrow from λ to μ if and only if $\mu = \lambda(\nu, m)$, where the integers ν and m satisfy the conditions: $1 \leqslant \nu \leqslant n-1$, $1 \leqslant m \leqslant \lambda_{v+1}$ and $m = p^t$ for some nonnegative integer t. (When the characteristic of K is 0, we let $m = 1$). We label this arrow as $\xi_{\mu,\lambda}$, where $\xi_{\mu,\lambda}$ sometimes denote an arrow in the quiver of $S^+(n, r)$ instead of an element of $S^+(n, r)$.

Definition 4.1.5. Let λ and μ be two compositions of r with at most n parts. Let λ and μ satisfy the conditions in Theorems 4.1.3 or 4.1.4, i.e.,

$$\mu - \lambda = p^t \alpha_\nu,$$

for some $1 \leqslant \nu \leqslant n-1$ and $t \geqslant 0$ ($p^t = 1$ if $p = 0$). Then there is precisely one arrow from λ to μ in the quiver of $S^+(n, r)$, labeled by $\xi_{\mu,\lambda}$. We say that this arrow *belongs to* the simple root α_ν, and has *length* p^t.

Hence all arrows in the quiver of $S^+(n, r)$ over K of characteristic 0, have length 1.

By Definition 4.1.5, every arrow in the quiver Γ of $S^+(n, r)$ belongs to certain simple root of type A_{n-1}: $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$.

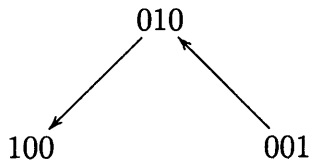
Furthermore, by Definition 3.1.1 and Definition 4.1.5, it is easy to check that the embedding φ_t^α has the following property.

Proposition 4.1.6. *Let $s \leqslant r$ be a positive integer and $\xi(b,a)$ a label for an arrow in the quiver of $S^+(2,s)$ over a field K of characteristic p, that is, $b-a = p^u$ for some $u \geqslant 0$. Let φ_t^α be an embedding as in Definition 3.1.1, from $S^+(2,s)$ to $S^+(n,r)$ over K, where $1 \leqslant t \leqslant n-1$ and $\alpha \in \Lambda(n, r-s)$. Then the arrow labeled by the element $\varphi_t^\alpha(\xi(b,a))$ belongs to the simple root α_t and has the length p^u.*

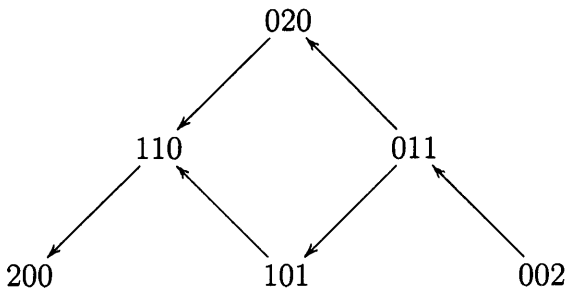
4.2 The quiver of $S^+(3,r)$

In this section we list some quivers of $S^+(3,r)$ over a field K of characteristic p. For simplicity, we denote a composition (a,b,c) by abc, for example 010 denotes the composition $(0,1,0)$.

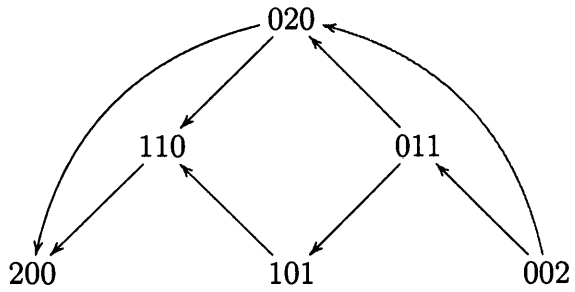
The quiver of $S^+(3,1)$ for any p:



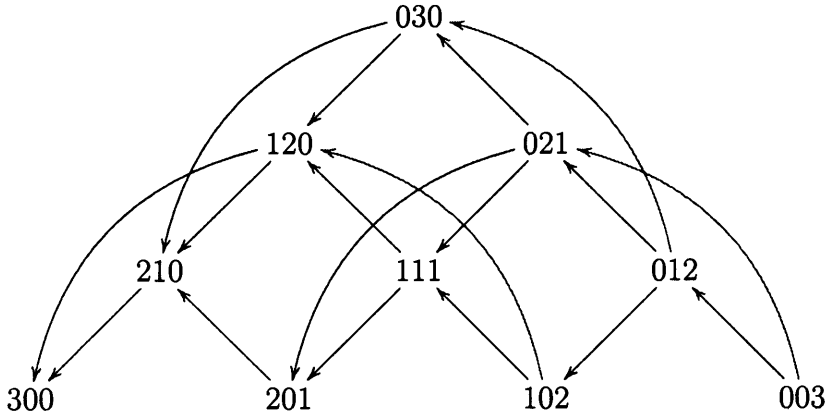
The quiver of $S^+(3,2)$ for $p = 0$:



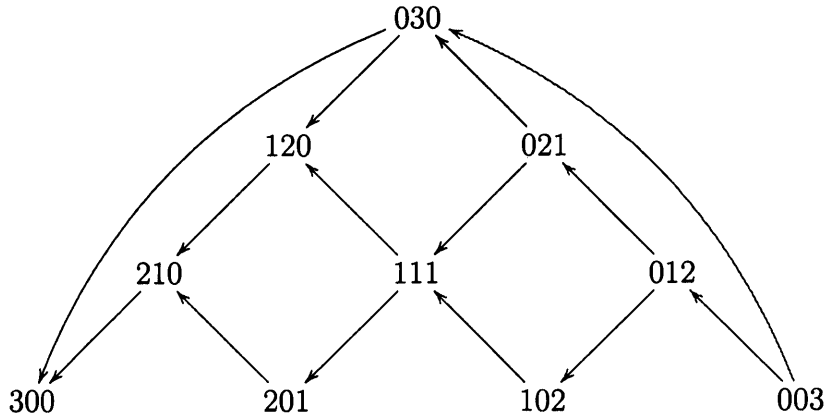
The quiver of $S^+(3,2)$ for $p=2$:



The quiver of $S^+(3,3)$ for $p=2$:



The quiver of $S^+(3,3)$ for $p=3$:



Chapter 5

Special types of relations for the quiver of $S^+(n,r)$

In the previous chapter we described the quiver Γ of the Borel Schur algebra $S^+(n,r)$ over a field K (Theorems 4.1.3 and 4.1.4). We now consider the relations of the quiver Γ. Recall that there is a surjective homomorphism $\widetilde{f}$: $K\Gamma \to S^+(n,r)$ where $\widetilde{f}$ maps the labels of the paths in the path algebra $K\Gamma$ to the basis elements in the Borel Schur algebra $S^+(n,r)$ by Definition 4.1.5, and the relations of the quiver Γ is the kernel of $\widetilde{f}$.

We describe *all* relations in the case of characteristic 0 (Section 5.6) and we provide *some* relations in the case of positive characteristic for some special subgraphs of the quiver Γ.

In Section 5.1 the subgraph in Γ is a rectangle, which we call a diamond. We obtain a relation in which the arrows belong to the simple roots α_i and α_j with $|i-j| \geqslant 2$.

In Section 5.2 we describe the relations for $S^+(3,r)$ for the paths from $(0,0,r)$ to $(r,0,0)$ in Γ.

In Section 5.3 we consider the $n \times m$ and $m \times n$ rectangles in Γ of $S^+(3,r)$. We obtain a formula for a special path in these rectangles in Theorems 5.3.5 and 5.3.10.

In Section 5.4 we consider the $1 \times m$ and $m \times 1$ rectangles in Γ. Using the formula for the corresponding paths in Section 5.3, we describe the relations for these rectangles in Theorems 5.4.5 and 5.4.12. In this case, we obtain all relations (see Theorems 5.4.7 and 5.4.12).

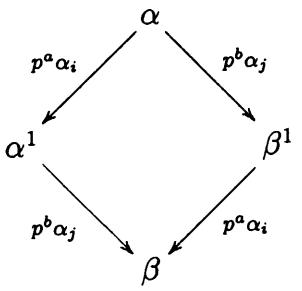
In Section 5.5 we get the relations for the $n \times m$ rectangle in Γ of $S^+(3,r)$ in zero characteristic, in terms of the words of R's and D's.

In Section 5.6 we use the results from [6], to describe all relations in Γ of $S^+(n,r)$ over a field of characteristic 0.

Our approach is as follows. We consider some special subgraphs in the quiver Γ of $S^+(n,r)$ over a field K. We obtained already some relations (which we call p-adic) for the quiver Γ in Theorems 3.3.4 and 2.3.1. We will study the relations modulo these known p-adic relations. Our method is to calculate the dimension of the vector space spanned by the paths from one vertex λ to another vertex μ in the subgraph in the path algebra $K\Gamma$, the dimension of the corresponding Hom space $\mathrm{Hom}_A(A\xi_\mu, A\xi_\lambda)$ with $A = S^+(n,r)$ which is the Cartan invariant $\widehat{c}_{\lambda,\mu}$ by Definition 3.2.8, and the dimension of the quotient of $K\Gamma$ modulo the known relations. We then consider the relationship between these dimensions to check whether we found all the relations.

5.1 The diamond relation

Let $n \geqslant 4$. Let D be the following subgraph of the quiver Γ of $S^+(n,r)$ over K, (we also call D a *diamond*),



where α, α^1, β, and β^1 are compositions in $\Lambda(n,r)$; the integer p is the characteristic of the field K; α_i and α_j are two simple roots of type A_{n-1} such that $1 \leqslant i,j \leqslant n-1$ and $|i-j| \geqslant 2$; a,b are nonnegative integers, satisfying the following conditions:

$$\begin{aligned} \alpha^1 &= \alpha + p^a\alpha_i, \\ \beta^1 &= \alpha + p^b\alpha_j, \\ \beta &= \alpha + p^a\alpha_i + p^b\alpha_j, \end{aligned}$$

where $p^a = p^b = 1$ if $p = 0$.

Proposition 5.1.1. *Let $n \geqslant 4$. Let D be the above subgraph in the quiver Γ satisfying the above conditions. Then for the $\widetilde{f}$-images of the paths $\xi_{\beta,\alpha^1}\xi_{\alpha^1,\alpha}$ and $\xi_{\beta,\beta^1}\xi_{\beta^1,\alpha}$ in the path algebra $K\Gamma$, we have*

$$\xi_{\beta,\alpha^1}\xi_{\alpha^1,\alpha} = \xi_{\beta,\beta^1}\xi_{\beta^1,\alpha}, \tag{5.1.1}$$

in the algebra $S^+(n,r)$. Hence we have a relation

$$\xi_{\beta,\alpha^1}\xi_{\alpha^1,\alpha} - \xi_{\beta,\beta^1}\xi_{\beta^1,\alpha},$$

for the quiver Γ of $S^+(n,r)$.

Proof. Let $A = S^+(n,r)$. By Proposition 3.2.14 (iii), the Cartan invariant $c_{\alpha,\beta} = 1$. By Definition 3.2.8, $c_{\alpha,\beta}$ is the dimension of $H(\alpha,\beta) := \mathrm{Hom}_A(A\xi_\beta, A\xi_\alpha)$. Since $\xi_{\beta,\alpha}$ is an element of $H(\alpha,\beta)$, we have $H(\alpha,\beta) = K\xi_{\beta,\alpha}$. By the multiplication for the Schur algebra in Definition 1.1.1, the LHS and RHS of Equation (5.1.1) are both scalar multiples of $\xi_{\beta,\alpha}$.

By the multiplication formula for Schur algebras in Definition 1.1.1

$$LHS = \xi_{\beta,\alpha^1}\xi_{\alpha^1,\alpha} = Z\xi_{\beta,\alpha},$$

where Z is the number of multi-indices s of $\mathbf{I}(n,r)$ satisfying the condition:

$$(\underline{n}^\beta, \underline{n}^{\alpha^1}) \sim (\underline{n}^\beta, s), \quad (\underline{n}^{\alpha^1}, \underline{n}^\alpha) \sim (s, \underline{n}^\alpha). \tag{5.1.2}$$

Since $|i-j| \geqslant 2$, by the definition of the relation $\sim$, we can assume that $i = 1$ and $j = 3$. Let $\alpha = (a_1, \ldots, a_n)$ be a composition of r with at most n parts. Then

$$\alpha^1 = \alpha + p^a\alpha_1 = (a_1 + p^a, a_2 - p^a, a_3, \ldots, a_n),$$
$$\beta = \alpha + p^a\alpha_1 + p^b\alpha_3 = (a_1 + p^a, a_2 - p^a, a_3 + p^b, a_4 - p^b, a_5, \ldots, a_n).$$

By the definition of the relation $\sim$ in Section 1.1, and by the second relation in Equation 5.1.2, the only difference between s and $\underline{n}^\alpha$ is in the first $(a_1 + a_2)$ places. By the first relation in Equation 5.1.2, the first $(a_1 + a_2)$ places of $\underline{n}^\beta$ and s are the same. Thus s must be the multi-index $\underline{n}^{\alpha^1}$, so $Z = 1$. Hence

$$LHS = \xi_{\beta,\alpha^1}\xi_{\alpha^1,\alpha} = \xi_{\beta,\alpha}.$$

Similarly, we can calculate that

$$RHS = \xi_{\beta,\beta^1}\xi_{\beta^1,\alpha} = \xi_{\beta,\alpha}.$$

Hence we have

$$\xi_{\beta,\alpha^1}\xi_{\alpha^1,\alpha} = \xi_{\beta,\beta^1}\xi_{\beta^1,\alpha}.$$

∎

Remark. We can use Theorem 1.1.4 to calculate the coefficient Z in LHS.

$$Z = \prod_{a,b \in \underline{n}} \frac{r_{a,b}!}{r_{a,1,b}! \cdots r_{a,n,b}!},$$

where $i = \underline{n}^{\beta}$, $j = \underline{n}^{\alpha^1}$ and $l = \underline{n}^{\alpha}$, for all $a, d, b \in \underline{n}$, $r_{a,b} = |R_{a,b}(i,l)|$ and $r_{a,d,b} = |R_{a,d,b}(i,j,l)|$.

Recall that

$$r_{a,b} = |R_{a,b}(i,l)| = |\{\rho \in \underline{r} \mid i_\rho = a,\ l_\rho = b\}|.$$

Let $\alpha = (a_1, \ldots, a_n)$ be in $\Lambda(n,r)$. We have

$$\begin{array}{l} r_{t,t} = a_t,\ t \neq i, i+1, j, j+1, \\ r_{i,i} = a_i, \quad r_{i,i+1} = p^a, \quad r_{i+1,i+1} = a_{i+1} - p^a, \\ r_{j,j} = a_j, \quad r_{j,j+1} = p^b, \quad r_{j+1,j+1} = a_{i+1} - p^b, \\ r_{a,b} = 0, \text{ otherwise.} \end{array}$$

and

$$\begin{array}{l} r_{t,t,t} = a_t,\ t \neq i, i+1, j, j+1, \\ r_{i,i,i} = a_t, \quad r_{i,i,i+1} = p^a, \quad r_{i+1,i+1,i+1} = a_{i+1} - p^a, \\ r_{j,j,j} = a_j \quad r_{j,j,j+1} = p^b, \quad r_{j+1,j+1,j+1} = a_{j+1} - p^b, \\ r_{a,d,b} = 0, \text{ otherwise.} \end{array}$$

Hence $Z = 1$.

We call the relation in Proposition 5.1.1 a *diamond relation.*

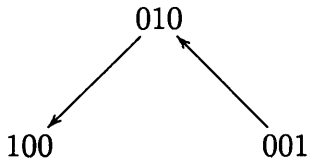
By Proposition 5.1.1, if $n \geqslant 4$, we have the diamond relations for the quiver Γ of $S^+(n,r)$ where the arrows belong to the simple roots α_i and α_j such that $|i-j| \geqslant 2$. We then consider the relations for the quiver Γ where the relative arrows belong to two simple roots α_i and α_{i+1} for $1 \leqslant i \leqslant n-2$. This can be studied for the quiver of $S^+(3,r)$, where the simple roots are α_1 and α_2. Hence from now on, we mainly consider the relations for the quiver of $S^+(3,r)$ except for Section 5.6.

5.2 Some relations for the quiver of $S^+(3,r)$

In this section we consider the paths from $(0,0,r)$ to $(r,0,0)$ in the path algebra $K\Gamma$ where Γ is the quiver of $S^+(3,r)$ of zero characteristic.

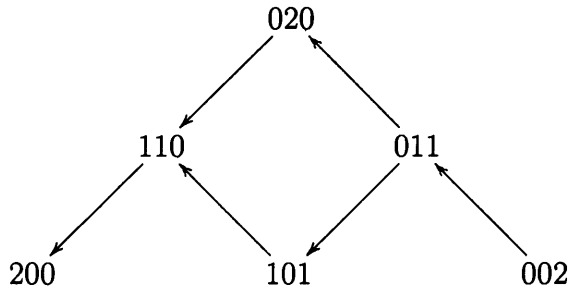
Let T be the vector space spanned by the paths from $(0,0,r)$ to $(r,0,0)$ in the path algebra $K\Gamma$. Let $D(r)$ be the dimension of T. For $r = 1,2,3$, we calculate the corresponding $D(r)$ as follows (for the quiver Γ, see Section 4.2):

The quiver of $S^+(3,1)$ for $p = 0$:



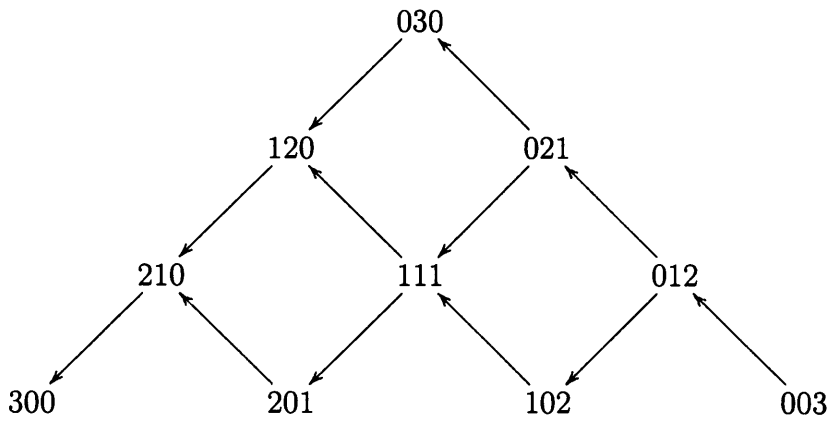
$D(1) = 1$.

The quiver of $S^+(3,2)$ for $p = 0$:



$D(2) = 2$.

The quiver of $S^+(3,3)$ for $p = 0$:



$D(3) = 5$.

We now introduce the following definition.

Definition 5.2.1. ([17], Page 221) The n-th *Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the number of Dyck paths from $(0,0)$ to $(2n,0)$, i.e., lattice paths with steps $(1,1)$ and $(1,-1)$, never falling below the x-axis.

Proposition 5.2.2. $D(r) = C_r$.

Proof. By Definition 5.2.1, C_r is the number of the lattice paths from $(0,0)$ to $(2n,0)$, with steps $(1,1)$ and $(1,-1)$, never falling below the x-axis, which is the $D(r)$. ∎

Proposition 5.2.3. *There are* $(C_r - 1)$ *linearly independent relations in* T.

Proof. By Proposition 5.2.2, the dimension of T is $D(r) = C_r$. Let $\lambda := (0,0,r)$ and $\alpha := (r,0,0)$. By Proposition 3.2.14 (ii), the Cartan invariant $c_{\lambda,\alpha} = 1$. By Definition 3.2.8, this implies the Hom space $\mathrm{Hom}_A(A\xi_\alpha, A\xi_\lambda)$ is one-dimensional and generated by $\xi_{\alpha,\lambda}$ where $A = S^+(3,r)$. So the paths from

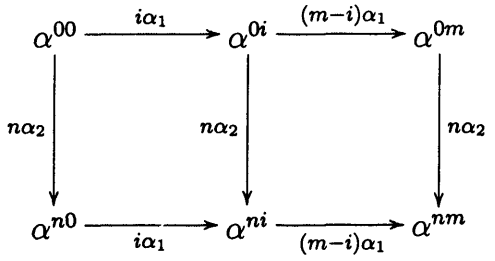
λ to α in the path algebra $K\Gamma$ is mapped by $\widetilde{f}$ to a scalar multiple of $\xi_{\alpha,\lambda}$ in $S^+(3,r)$. Hence the $\widetilde{f}$-images of these C_r paths in T are all multiples of $\xi_{\alpha,\lambda}$. In other words, we obtain $(C_r - 1)$ linearly independent relations in T. ∎

5.3 The $n \times m$ and $m \times n$ rectangles

Let K be a field of characteristic $p \geqslant 0$. Let A be the Borel Schur algebra $S^+(3,r)$ over K. Let Γ be the quiver of A. Let α_1 and α_2 be simple roots of type A_2.

In this section we consider the $n \times m$ and $m \times n$ rectangle in the quiver Γ where n and m are positive integers. We calculate a special path in these two rectangles.

Let i be an integer with $0 \leqslant i \leqslant m$. We first consider the following $n \times m$ rectangle:



We assume that $\alpha^{00} = (a_1, a_2, a_3)$ with the conditions $a_2 \geqslant m$ and $a_3 \geqslant n$. Then we write α^{00} as follows:

$$\alpha^{00} = (u, m+v, n+w),$$

where u, v, w are nonnegative integers. Thus, the compositions α^{0i} and α^{ni} are as follows:

$$\alpha^{0i} := \alpha^{00} + i\alpha_1 = (u+i, m-i+v, n+w),$$
$$\alpha^{ni} := \alpha^{0i} + n\alpha_2 = (u+i, m-i+n+v, w).$$

We have

$$\alpha^{0m} = (u+m, v, n+w), \quad \alpha^{n0} = (u, m+n+v, w), \quad \alpha^{nm} = (m+u, n+v, w).$$

Recall that we consider all paths in the path algebra $K\Gamma$ modulo the p-adic relations (i.e. those given by $S^+(2,s)$ sub-quivers (see after Theorem 3.3.4)). In particular, this implies that there is only one path from α^{0k} to α^{0l} (or from α^{nk} to α^{nl}) where $0 \leqslant k < l \leqslant m$.

Denote by P_i the path in the path algebra $K\Gamma$ modulo the p-adic relations, from α^{00} to α^{0i}, to α^{ni} and to α^{nm}.

Recall the map $\widetilde{f}$ from $K\Gamma$ to $S^+(n,r)$ (see Section 1.4).

Proposition 5.3.1.

$$\widetilde{f}(P_i) = i^+(m-i)^+ n^+ \xi_{\alpha^{nm},\alpha^{ni}} \xi_{\alpha^{ni},\alpha^{0i}} \xi_{\alpha^{0i},\alpha^{00}}. \tag{5.3.1}$$

Proof. The path P_i can be written as follows:

$$P_i = P(\alpha^{nm}, \alpha^{ni}) P(\alpha^{ni}, \alpha^{0i}) P(\alpha^{0i}, \alpha^{00}),$$

where $P(\alpha^{nm}, \alpha^{ni})$ is the unique path from α^{ni} to α^{nm}, $P(\alpha^{ni}, \alpha^{0i})$ is the unique path from α^{0i} to α^{ni} and $P(\alpha^{0i}, \alpha^{00})$ is the unique path from α^{00} to α^{0i}. By Theorem 2.5.3, the $\widetilde{f}$-images of the paths

$$\begin{aligned}
&\widetilde{f}(P(\alpha^{nm}, \alpha^{ni})) = (m-i)^+ \xi_{\alpha^{nm},\alpha^{ni}}, \\
&\widetilde{f}(P(\alpha^{ni}, \alpha^{0i})) = n^+ \xi_{\alpha^{ni},\alpha^{0i}}, \\
&\widetilde{f}(P(\alpha^{0i}, \alpha^{00})) = i^+ \xi_{\alpha^{0i},\alpha^{00}}.
\end{aligned}$$

Hence we obtain Equation (5.3.1). ∎

By Proposition 3.2.15, we have the following proposition.

Proposition 5.3.2. *Let* $d := \min\{m, n\}$. *Let* H *be* $\mathrm{Hom}_A(A\xi_{\alpha^{nm}}, A\xi_{\alpha^{00}})$. *Then the dimension of* H *is* $(d+1)$. *Moreover,* $H \simeq \xi_{\alpha^{nm}} A \xi_{\alpha^{00}}$ *is spanned by*

$$\{X_t \mid X_t = \xi_{i_t, \underline{3}^{\alpha^{00}}}, \quad i_t = 1^{u+m-t}2^{n+v}1^t3^w, t = 0, 1, \ldots, d\}.$$

Moreover, $X_t = \xi_{\underline{3}^{\alpha^{nm}}, q_t}$, *where* $q_t = 1^u3^t2^{m+v}3^{n+w-t}$ *and* $0 \leqslant t \leqslant d$.

Our aim is to calculate $\widetilde{f}(P_i)$.

Lemma 5.3.3.

$$\xi_{\alpha^{ni}, \alpha^{0i}} \xi_{\alpha^{0i}, \alpha^{00}} = \xi_{\alpha^{ni}, \alpha^{00}}.$$

Proof. Let $d' = \min\{n, i\}$. By Proposition 5.3.2, $\xi_{\alpha^{ni}} A \xi_{\alpha^{00}}$ has a basis

$$\{\xi_{i'_t, \underline{3}^{\alpha^{00}}} \mid i'_t = 1^{u+i-t}2^{m-i+n+v}1^t3^w, t = 0, 1, \ldots, d'\}.$$

By Lemma 1.1.2, we have

$$\xi_{\alpha^{ni}, \alpha^{0i}} \xi_{\alpha^{0i}, \alpha^{00}} = \sum_{t=0}^{d'} Z_t \xi_{i'_t, \underline{3}^{\alpha^{00}}},$$

where Z_t is the number of multi-indices $s = (s_1, \ldots, s_r)$ in $\mathbf{I}(3, r)$ satisfying the following conditions:

$$(\underline{3}^{\alpha^{ni}}, \underline{3}^{\alpha^{0i}}) \sim (i'_t, s), \qquad (\underline{3}^{\alpha^{0i}}, \underline{3}^{\alpha^{00}}) \sim (s, \underline{3}^{\alpha^{00}}).$$

That is,

$$(1^{u+i}2^{m-i+n+v}3^w, 1^{u+i}2^{m-i+v}3^{n+w}) \sim (1^{u+i-t}2^{m-i+n+v}1^t3^w, s), \quad (5.3.2)$$
$$(1^{u+i}2^{m-i+v}3^{n+w}, 1^u2^{m+v}3^{n+w}) \sim (s, 1^u2^{m+v}3^{n+w}). \quad (5.3.3)$$

By (5.3.3), the last $(n+w)$ places of s are 3's. By (5.3.2), the first $(u+i-t)$ places and the t places from the last $(w+t)$-th to the last $(w+1)$-th ones of s are 1's.

We claim that $t = 0$. Otherwise $t \geqslant 1$. Then the t places from the last $(w+t)$-th to the last $(w+1)$-th places of s are 1's. This is a contradiction with that the last $(n+w)$ places of s are 3's where $n \geqslant 1$. Hence $t = 0$. By (5.3.2), the first $(u+i)$ places of s are 1's. Since $s \sim 1^{u+i}2^{m-i+v}3^{n+w}$, $s = 1^{u+i}2^{m-i+v}3^{n+w}$. Thus $Z_0 = 1$. Note that $i_0' = \underline{3}^{\alpha^{ni}}$. Hence we obtain our formula. ∎

Lemma 5.3.4. *Let* $d_i := \min\{m-i, n\}$. *Then*

$$\xi_{\alpha^{nm},\alpha^{ni}}\xi_{\alpha^{ni},\alpha^{00}} = \sum_{t=0}^{d_i} \binom{m-t}{i} X_t.$$

where $0 \leqslant i \leqslant m$. *In particular,*

$$\xi_{\alpha^{nm},\alpha^{n0}}\xi_{\alpha^{n0},\alpha^{00}} = \sum_{t=0}^{d} X_t.$$

Proof. By Lemma 1.1.2 and Proposition 5.3.2, we have

$$\xi_{\alpha^{nm},\alpha^{ni}}\xi_{\alpha^{ni},\alpha^{00}} = \sum_{t=0}^{d} Z_t \xi_{\underline{3}^{\alpha^{nm}},q_t},$$

where $q_t = 1^u3^t2^{m+v}3^{n+w-t}$ and Z_t is the number of multi-indices $s = (s_1, \ldots, s_r)$ in $\mathbf{I}(3,r)$ satisfying the condition:

$$(1^{u+m}2^{n+v}3^w, 1^{u+i}2^{m-i+n+v}3^w) \sim (1^{u+m}2^{n+v}3^w, s), \tag{5.3.4}$$
$$(1^{u+i}2^{m-i+n+v}3^w, 1^u2^{m+v}3^{n+w}) \sim (s, 1^u3^t2^{m+v}3^{n+w-t}), \tag{5.3.5}$$

By (5.3.5), the first u places of s are 1's. By (5.3.4), the last w places of s are 3's. Since $s \sim 1^{u+i}2^{m-i+n+v}3^w$, it remains which places of s will be the i copies of 1's.

By (5.3.4), the first $(u+m)$ places of s must be 1's or 2's. Thus, these i copies of 1's of s are in the m places from the first $(u+1)$-th to the first $(u+m)$-th ones.

By (5.3.5), the t places of s from the first $(u+1)$-th to the first $(u+t)$-th place are 2's. Hence it remains to choose i places from $(m-t)$ places from the first $(u+t+1)$-th to the first $(u+t+m)$-th places of s, to put i copies of 1's. So the number of such s is $\binom{m-t}{i}$. That is,

$$Z_t = \binom{m-t}{i}.$$

We claim $t \leqslant m-i$. Otherwise $t > m-i$. Then $m-t < i$. Note that we need to put i 1's in the $(m-t)$ places from the first $(u+t+1)$-th to the first $(u+m)$-th places. This is impossible. Hence $t \leqslant m-i$. By Proposition 5.3.2, $t \leqslant d\min\{m,n\}$. Let $d_i := \min\{m-i,n\}$, then $0 \leqslant t \leqslant d_i$. Hence

$$\xi_{\alpha^{nm},\alpha^{ni}}\xi_{\alpha^{ni},\alpha^{00}} = \sum_{t=0}^{d_i}\binom{m-t}{i}X_t.$$

Note that $d_0 = d$. So we obtain the special formula when $i = 0$. ∎

We now calculate $\widetilde{f}(P_i)$ where $i = 0, 1, \ldots, m$.

Theorem 5.3.5. *Let* $d_i := \min\{m-i,n\}$. *Then*

$$\widetilde{f}(P_i) = i^+(m-i)^+n^+\sum_{t=0}^{d_i}\binom{m-t}{i}X_t,$$

where $0 \leqslant i \leqslant m$. *In particular,*

$$\widetilde{f}(P_0) = m^+n^+\sum_{t=0}^{d}X_t, \quad \widetilde{f}(P_m) = m^+n^+X_0.$$

Proof. By Equation (5.3.1),

$$\widetilde{f}(P_i) = i^+(m-i)^+n^+\xi_{\alpha^{nm},\alpha^{ni}}\xi_{\alpha^{ni},\alpha^{0i}}\xi_{\alpha^{0i},\alpha^{00}}.$$

By Lemma 5.3.3, we have

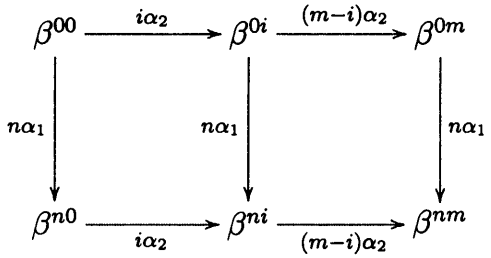
$$\xi_{\alpha^{ni},\alpha^{0i}}\xi_{\alpha^{0i},\alpha^{10}} = \xi_{\alpha^{ni},\alpha^{00}}.$$

So, by Lemma 5.3.4,

$$\widetilde{f}(P_i) = i^+(m-i)^+n^+\xi_{\alpha^{nm},\alpha^{ni}}\xi_{\alpha^{ni},\alpha^{00}} = i^+(m-i)^+n^+\sum_{t=0}^{d_i}\binom{m-t}{i}X_t.$$

∎

Finally, we consider the related $m \times n$ rectangle in the quiver Γ. This subgraph can also be obtained by interchanging the simple roots α_1 and α_2 in the above $n \times m$ rectangle:



where β^{00} is the following composition

$$\beta^{00} = (u, n+v, m+w).$$

Thus, the compositions β^{0i} and β^{ni} are as follows:

$$\begin{aligned}\beta^{0i} &:= \beta^{00} + i\alpha_2 = (u, v+n+i, w+m-i),\\ \beta^{ni} &:= \beta^{0i} + n\alpha_1 = (u+n, v+i, w+m-i).\end{aligned}$$

Denote by Q_i the path in the path algebra $K\Gamma$ modulo the p-adic relations, from β^{00} to β^{0i}, to β^{ni} and to β^{nm}.

We simply list the corresponding propositions and theorems.

Proposition 5.3.6.

$$\widetilde{f}(Q_i) = i^+(m-i)^+n^+\xi_{\beta^{nm},\beta^{ni}}\xi_{\beta^{ni},\beta^{0i}}\xi_{\beta^{0i},\beta^{10}}. \tag{5.3.2}$$

Proposition 5.3.7. *Let* $d := \min\{m,n\}$. *Let* H' *be the* $\mathrm{Hom}_A(A\xi_{\beta^{nm}}, A\xi_{\beta^{00}})$. *Then the dimension of* H' *is* $(d+1)$. *Moreover,* $H \simeq \xi_{\beta^{nm}}A\xi_{\beta^{00}}$ *is spanned by*

$$\{X'_t \mid X'_t = \xi_{i'_t,\underline{3}^{\beta^{00}}}, \quad i'_t = 1^{u+n-t}2^{m+v}1^t3^w, t = 0,1,\ldots,d\}.$$

Moreover, $X'_t = \xi_{\underline{3}^{\beta^{nm}},q_t}$, *where* $q'_t = 1^u3^t2^{n+v}3^{m+w-t}$ *and* $0 \leqslant t \leqslant d$.

Lemma 5.3.8.

$$\xi_{\beta^{nm},\beta^{ni}}\xi_{\beta^{ni},\beta^{0i}} = \xi_{\beta^{nm},\beta^{0i}}.$$

Lemma 5.3.9. *Let* $d'_i := \min\{i,n\}$. *Then*

$$\xi_{\beta^{nm},\beta^{0i}}\xi_{\beta^{0i},\beta^{00}} = \sum_{t=0}^{d'_i}\binom{m-t}{m-i}X'_t.$$

where $0 \leqslant i \leqslant m$. *In particular,*

$$\xi_{\beta^{nm},\beta^{0m}}\xi_{\beta^{0m},\beta^{00}} = \sum_{t=0}^{d}X'_t.$$

Theorem 5.3.10. *Let* $d'_i := \min\{i,n\}$. *Then*

$$\widetilde{f}(Q_i) = i^+(m-i)^+n^+\sum_{t=0}^{d'_i}\binom{m-t}{m-i}X'_t,$$

where $0 \leqslant i \leqslant m$. *In particular,*

$$\widetilde{f}(Q_0) = m^+n^+X'_0, \quad \widetilde{f}(Q_m) = m^+n^+\sum_{t=0}^{d}X'_t.$$

5.4 The $1 \times m$ and $m \times 1$ relations

In this section we consider the special case $n = 1$ for the $n \times m$ and $m \times n$ rectangles in Section 5.3. In this special case we obtain the $1 \times m$ and $m \times 1$ relations.

Let K be a field of characteristic $p \geqslant 0$. Let A be $S^+(3, r)$ over K. Let α_1 and α_2 be simple roots of type A_2.

First we consider the following $1 \times m$ subgraph:

$$\begin{array}{ccccccccc}
\alpha^{00} & \xrightarrow{\alpha_1} & \alpha^{01} & \xrightarrow{\alpha_1} & \cdots & \xrightarrow{\alpha_1} & \alpha^{0,m-1} & \xrightarrow{\alpha_1} & \alpha^{0m} \\
\downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_2} & & & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_2} \\
\alpha^{10} & \xrightarrow{\alpha_1} & \alpha^{11} & \xrightarrow{\alpha_1} & \cdots & \xrightarrow{\alpha_1} & \alpha^{1,m-1} & \xrightarrow{\alpha_1} & \alpha^{1m}
\end{array}$$

We assume that $\alpha^{00} = (a_1, a_2, a_3)$ with the conditions $a_1 \geqslant m$ and $a_3 \geqslant 1$. We write α^{0i} and α^{1i} $(0 \leqslant i \leqslant m)$ as the following compositions in $\Lambda(3, r)$:

$$\begin{aligned}
\alpha^{0i} &:= (a_1 + i, a_2 - i, a_3) = \alpha^{00} + i\alpha_1, \\
\alpha^{1i} &:= (a_1 + i, a_2 - i + 1, a_3 - 1) = \alpha^{0i} + \alpha_2.
\end{aligned}$$

Let T be the vector space spanned by the paths from α^{00} to α^{1m} in the path algebra $K\Gamma$ (modulo the p-adic relations!). Next we consider the relations in T.

By the end of Chapter 3, we have the path $P(\alpha^{0i}, \alpha^{00})$ from α^{00} to α^{0i} in T. Define by P_i the path in T as in the proof of Proposition 5.3.1:

$$P_i = P(\alpha^{1m}, \alpha^{1i})P(\alpha^{1i}, \alpha^{0i})P(\alpha^{0i}, \alpha^{00})).$$

where $0 \leqslant i \leqslant m$.

Proposition 5.4.1. *The set* $\{P_i \mid 0 \leqslant i \leqslant m\}$ *is a basis of* T. *Hence the dimension of* T *is* $(m+1)$.

Proof. Since the simple roots in type A_2 are α_1 and α_2, there is no path from α^{00} to α^{1m} which goes partly outside the rectangle in $K\Gamma$. Thus the paths from α^{00} to α^{1m} in T are linear combinations of the products of the paths P_i for $0 \leqslant i \leqslant m$. In other words, $\{P_i \mid 0 \leqslant i \leqslant m\}$ spans T.

Next we prove that $\{P_i \mid 0 \leqslant i \leqslant m\}$ is linearly independent in T. In the path algebra $K\Gamma$, we know that $\{P_i \mid 0 \leqslant i \leqslant m\}$ is linearly independent. Since our p-adic relations at the end of Chapter 3, are for the paths which have their different parts only in the first row (related to the simple root α_1), or only in the second row (related to the simple root α_2). In other words, there is no p-adic relation for the paths whose different parts are of part in the first row and of part in the second row. Thus there is no p-adic relation for the set $\{P_i \mid 0 \leqslant i \leqslant m\}$. Hence $\{P_i \mid 0 \leqslant i \leqslant m\}$ are linearly independent in $K\Gamma$ modulo our p-adic relations, that is, are linearly independent in T.

Therefore $\{P_i \mid 0 \leqslant i \leqslant m\}$ is a basis of T. ∎

We list the corresponding results for $n = 1$ in the $n \times m$ rectangle in Section 5.3.

Proposition 5.4.2.

$$\widetilde{f}(P_i) = i^{+}(m-i)^{+}\xi_{\alpha^{1m},\alpha^{1i}}\xi_{\alpha^{1i},\alpha^{0i}}\xi_{\alpha^{0i},\alpha^{00}}. \tag{5.4.1}$$

By Proposition 5.3.2, we have:

Proposition 5.4.3. *Let $A = S^+(3,r)$ and $H = \mathrm{Hom}_A(A\xi_{\alpha^{1m}}, A\xi_{\alpha^{00}})$. Then the dimension of H is 2. Moreover, $H \simeq \xi_{\alpha^{1m}} A \xi_{\alpha^{00}}$ is spanned by $\{X_0, X_1\}$, where*

$$X_0 := \xi_{\underline{3}\alpha^{1m},\underline{3}\alpha^{00}}, \ X_1 := \xi_{\underline{3}\alpha^{1m},1^{a_1}32^{a_2}3^{a_3-1}}.$$

By Theorem 5.3.5, we have:

Proposition 5.4.4.

$$\widetilde{f}(P_i) = i^+(m-i)^+ \left(\binom{m}{i} X_0 + \binom{m-1}{i} X_1 \right).$$

In particular,

$$\widetilde{f}(P_0) = m^+(X_0 + X_1), \quad \widetilde{f}(P_m) = m^+ X_0.$$

We now describe relations in T in terms of P_i's where $i = 0, 1 \ldots, m$.

Theorem 5.4.5.

$$i^+(m-i)^+ \binom{m-1}{i} \widetilde{f}(P_0) - m^+ \widetilde{f}(P_i) + i^+(m-i)^+ \binom{m-1}{i-1} \widetilde{f}(P_m) = 0,$$

where $1 \leqslant i \leqslant m-1$. That is, we have relations in the quiver Γ:

$$i^+(m-i)^+ \binom{m-1}{i} P_0 - m^+ P_i + i^+(m-i)^+ \binom{m-1}{i-1} P_m,$$

where $1 \leqslant i \leqslant m-1$.

Proof. By Proposition 5.4.4,

$$\begin{aligned}
m^+ \widetilde{f}(P_i) &= i^+(m-i)^+ \left(\binom{m}{i} m^+ X_0 + \binom{m-1}{i} m^+ X_1 \right) \\
&= i^+(m-i)^+ \left(\left(\binom{m-1}{i} + \binom{m-1}{i-1} \right) m^+ X_0 + \binom{m-1}{i} m^+ X_1 \right) \\
&= i^+(m-i)^+ \left(\binom{m-1}{i} (X_0 + X_1) + \binom{m-1}{i-1} X_0 \right), \\
&= i^+(m-i)^+ \left(\binom{m-1}{i} \widetilde{f}(P_0) + \binom{m-1}{i-1} \widetilde{f}(P_m) \right),
\end{aligned}$$

as required. ∎

Corollary 5.4.6. *(a). Let a be a nonnegative integer, and let $m = m'p^a$ and $i = i'p^a$ in Theorem 5.4.5, with $p \geqslant m' > i' \geqslant 1$. Let $P_{i'}^a$ be the path $P_{i'p^a}$, then $P_i^0 = P_i$ as above. Thus the relations in Theorem 5.4.5 can be written as*

$$(m' - i')P_0^a - m'P_{i'}^a + i'P_{m'}^a,$$

where $1 \leqslant i' \leqslant m' - 1$. In particular, let $m' = 2$, then $i' = 1$, and we have the relation

$$P_0^a - 2P_1^a + P_2^a.$$

(b). Let K be of characteristic 0, that is, $p = 0$. Then the relations in Theorem 5.4.5 can be written as

$$(m - i)P_0 - mP_i + iP_m,$$

where $1 \leqslant i \leqslant m - 1$.

Proof. (a). By the formula in Theorem 2.5.3

$$(i'p^a)^+ \equiv i'!, \qquad (m'p^a - i'p^a)^+ \equiv (m' - i')! \pmod{p}.$$

By Lemma 2.2.2, since $i' \leqslant m' - 1 \leqslant p - 1$,

$$\binom{m'p^a - 1}{i'p^a} = \binom{(m' - 1)p^a + (p - 1)p^{a-1} + \ldots + (p - 1)}{i'p^a},$$

and

$$\binom{(m' - 1)p^a + (p - 1)p^{a-1} + \ldots + (p - 1)}{i'p^a} \equiv \binom{m' - 1}{i'} \prod_{s=0}^{a-1} \binom{p - 1}{0}.$$

Thus

$$\binom{m'p^a - 1}{i'p^a} \equiv \binom{m' - 1}{i'} \pmod{p}.$$

Similarly, we have

$$\begin{aligned}\binom{mp^a-1}{ip^a-1} &= \binom{(m'-1)p^a+(p-1)p^{a-1}+\ldots+(p-1)}{(i'-1)p^a+(p-1)p^{a-1}+\ldots+(p-1)}\\ &\equiv \binom{m'-1}{i'-1} \pmod{p}.\end{aligned}$$

Hence we get

$$i'!(m'-i')!\binom{m'-1}{i'} = (m'-i')(m'-1)!, \quad i'!(m'-i')!\binom{m'-1}{i'-1} = i'(m'-1)!.$$

Thus the formula in Theorem 5.4.5 can be written as

$$(m'-i')(m'-1)!\widetilde{f}(P_0^a) - m'!\widetilde{f}(P_{i'}^a) + i'(m'-1)!\widetilde{f}(P_{m'}^a) = 0,$$

where $p \geqslant m' > i' \geqslant 1$. Since $m' \leqslant p$, so $(m'-1)! \neq 0 \pmod{p}$, hence we obtain the relations

$$(m'-i')P_0^a - m'P_{i'}^a + i'P_{m'}^a,$$

where $1 \leqslant i' \leqslant m'-1$.

(b). Since $m^+ = m!$ for K of characteristic 0, the formula can be written as

$$i!(m-i)!\binom{m-1}{i}\widetilde{f}(P_0) - m!\widetilde{f}(P_i) + i!(m-i)!\binom{m-1}{i-1}\widetilde{f}P_m) = 0.$$

So we have the relations

$$(m-i)P_0 - mP_i + iP_m,$$

where $1 \leqslant i \leqslant m-1$. ∎

Theorem 5.4.7. *All the relations in T are generated by the relations in Theorem 5.4.5:*

$$i^+(m-i)^+\binom{m-1}{i}P_0 - m^+P_i + i^+(m-i)^+\binom{m-1}{i-1}P_m,$$

where $1 \leqslant i \leqslant m-1$.

Proof. By Propositions 5.4.1 and 5.4.3, we need to find $(m-1)$ linearly independent relations in T. So it remains to prove that those $(m-1)$ relations in Theorem 5.4.5 are linearly independent.

Let

$$x_i = i^+(m-i)^+\binom{m-1}{i}, \qquad y_i = i^+(m-i)^+\binom{m-1}{i-1}.$$

We write the formula in Theorem 5.4.5 as follows:

$$D\widetilde{P} = 0_{1\times(m+1)},$$

where the matrix D is the following $(m-1)\times(m+1)$ matrix,

$$D = \begin{pmatrix} x_1 & -m^+ & 0 & \dots & 0 & y_1 \\ x_2 & 0 & -m^+ & \ddots & \vdots & y_2 \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ x_{m-1} & 0 & \dots & 0 & -m^+ & y_{m-1} \end{pmatrix},$$

and $\widetilde{P}$ is the following $(m+1)\times 1$ vector,

$$\widetilde{P} = (\widetilde{f}(P_0), \widetilde{f}(P_1), \dots, \widetilde{f}(P_m))'.$$

The matrix D has a sub-matrix

$$\begin{pmatrix} -m^+ & 0 & \dots & 0 \\ 0 & -m^+ & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -m^+ \end{pmatrix}$$

which is nonsingular since $m^+ \not\equiv 0 \pmod{p}$ for any positive integer m by Theorem 2.5.3. Thus the matrix D has rank $(m-1)$. Hence these relations are linearly independent. Therefore, we obtain all relations in T. ∎

We call the relations in Theorem 5.4.5, the $1\times m$ *relations.*

We can rewrite the relations in Theorem 5.4.5 as follows.

Theorem 5.4.8.

$$\widetilde{f}(P_i) - (m^+)^{-1}i^+(m-i)^+\binom{m-1}{i}\widetilde{f}(P_0) \\ -(m^+)^{-1}i^+(m-i)^+\binom{m-1}{i-1}\widetilde{f}(P_m) = 0,$$

where $1 \leqslant i \leqslant m-1$. *That is, we have the following relations in the quiver* Γ*:*

$$P_i - (m^+)^{-1}i^+(m-i)^+\binom{m-1}{i}P_0 - (m^+)^{-1}i^+(m-i)^+\binom{m-1}{i-1}P_m,$$

where $1 \leqslant i \leqslant m-1$. *Hence these relations are in one-to-one correspondence with the paths* P_i *where* $1 \leqslant i \leqslant m-1$.

Proof. By Theorem 2.5.3, $m^+ \not\equiv 0 \pmod{p}$ for any positive integer m. It remains to multiply by $-(m^+)^{-1}$ the equations in Theorem 5.4.5. ∎

Finally, we consider the related $m \times 1$ subgraph in the quiver Γ. This subgraph can also be obtained by interchanging the simple roots α_1 and α_2 in the above $1 \times m$ subgraph(or this is the special case $n = 1$ for the $m \times n$ rectangle in Section 5.3):

$$\begin{array}{ccccccccc} \beta^{00} & \xrightarrow{\alpha_2} & \beta^{01} & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_2} & \beta^{0,m-1} & \xrightarrow{\alpha_2} & \beta^{0m} \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_1} & & & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_1} \\ \beta^{10} & \xrightarrow{\alpha_2} & \beta^{11} & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_2} & \beta^{1,m-1} & \xrightarrow{\alpha_2} & \beta^{1m} \end{array}$$

where $\beta^{00} = (b_1, b_2, b_3)$ is in $\Lambda(3, r)$ and for $i = 0, 1, \ldots, m$,

$$\beta^{0i} = \beta^{00} + i\alpha_2 = (b_1, b_2 + i, b_3 - ip^a),$$

$$\beta^{1i} = \beta^{0i} + \alpha_1 = (b_1 + 1, b_2 - 1 + ip^a, b_3 - ip^a).$$

We simply list the corresponding propositions and theorems.

Proposition 5.4.9. *Let $A = S^+(3,r)$. Then*

$$H' := \mathrm{Hom}_A(A\xi_{\beta^{1m}}, A\xi_{\beta^{00}})$$

is 2-dimensional and spanned by

$$Y_0 := \xi_{\underline{3}\beta^{1m},\underline{3}\beta^{00}},\ Y_1 := \xi_{\underline{3}\beta^{1m},1^{b_1}32^{b_2}3^{b_3-1}}.$$

Let T' be the vector space spanned by the paths from β^{00} to β^{1m} modulo the p-adic relations in Theorem 3.3.4 in the path $K\Gamma$. Let Q_i be the following path in T' from β^{00} to β^{0i}, to β^{1i} and to β^{1m}. Then Q_i can be written as

$$\widetilde{f}(Q_i) = i^+(m-i)^+\xi_{\beta^{1m},\beta^{1i}}\xi_{\beta^{1i},\beta^{0i}}\xi_{\beta^{0i},\beta^{00}}, \tag{5.4.2}$$

where $i = 0, 1, \ldots, m$.

Lemma 5.4.10. *We have*

$$\xi_{\beta^{1m},\beta^{0i}}\xi_{\beta^{0i},\beta^{00}} = \binom{m}{i}Y_0 + \binom{m-1}{i}Y_1,$$

where $0 \leqslant i \leqslant m$. In particular,

$$\xi_{\beta^{2m},\beta^{10}}\xi_{\beta^{10},\beta^{10}} = Y_0 + Y_1,$$

$$\xi_{\beta^{2m},\beta^{1m}}\xi_{\beta^{1m},\beta^{10}} = Y_0.$$

Proposition 5.4.11. *We have*

$$\widetilde{f}(Q_i) = i^+(m-i)^+\left(\binom{m}{i}Y_1 + \binom{m-1}{i-1}Y_2\right).$$

In particular,

$$\widetilde{f}(Q_0) = m^+Y_0, \quad \widetilde{f}(Q_m) = m^+(Y_0 + Y_1).$$

Theorem 5.4.12.

$$i^+(m-i)^+\binom{m-1}{i}\widetilde{f}(Q_0) - m^+\widetilde{f}(Q_i) + i^+(m-i)^+\binom{m-1}{i-1}\widetilde{f}(Q_m) = 0,$$

where $0 \leqslant i \leqslant m$. *Then we have all relations in* T'*:*

$$i^+(m-i)^+\binom{m-1}{i}Q_0 - m^+Q_i + i^+(m-i)^+\binom{m-1}{i-1}Q_m,$$

where $0 \leqslant i \leqslant m$.

Corollary 5.4.13. *(a). Let* a *be a nonnegative integer, and let* $m = m'p^a$ *and* $i = i'p^a$ *in Theorem 5.4.12, with* $p \geqslant m' > i' \geqslant 1$. *Let* $Q^a_{i'}$ *be the path* $Q_{i'p^a}$, *then* $Q^0_i = Q_i$. *Thus the relations in Theorem 5.4.5 can be written as*

$$(m'-i')Q^a_0 - m'Q^a_{i'} + i'Q^a_{m'},$$

where $1 \leqslant i' \leqslant m'-1$. *In particular, let* $m' = 2$, *then* $i' = 1$, *and we have the relation*

$$Q^a_0 - 2Q^a_1 + Q^a_2.$$

(b). Let K *be of characteristic* 0, *that is,* $p = 0$. *Then the relations in Theorem 5.4.12 can be written as*

$$(m-i)Q_0 - mQ_i + iQ_m,$$

where $1 \leqslant i \leqslant m-1$.

Theorem 5.4.14.

$$\widetilde{f}(Q_i) - (m^+)^{-1}i^+(m-i)^+\binom{m-1}{i}\widetilde{f}(Q_0) \\ -(m^+)^{-1}i^+(m-i)^+\binom{m-1}{i-1}\widetilde{f}(Q_m) = 0,$$

where $1 \leqslant i \leqslant m-1$. *That is, we have all relations in* T'*:*

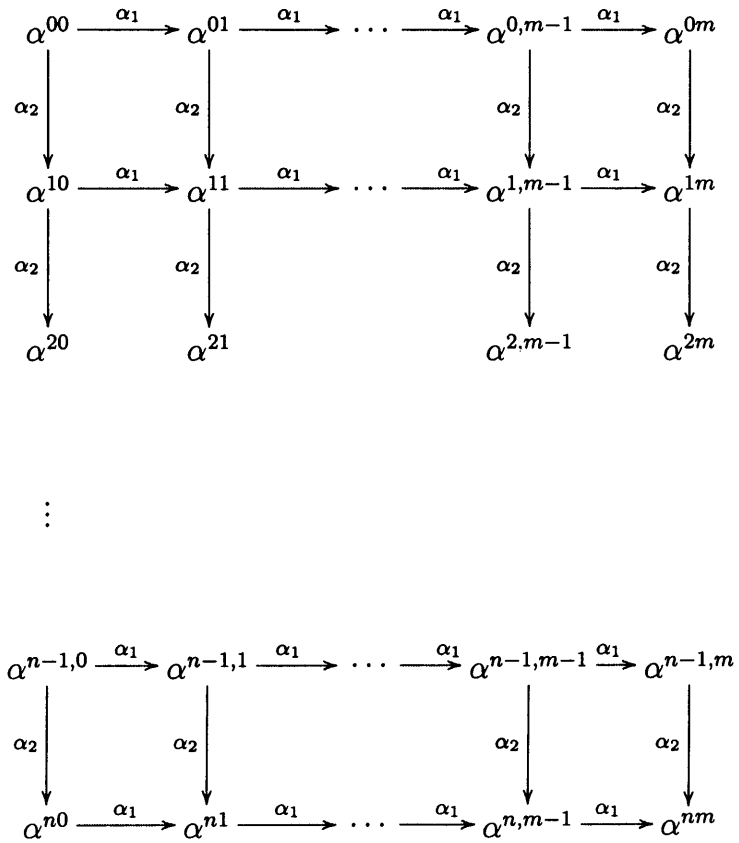
$$Q_i - (m^+)^{-1}i^+(m-i)^+\binom{m-1}{i}Q_0 - (m^+)^{-1}i^+(m-i)^+\binom{m-1}{i-1}Q_m,$$

where $1 \leqslant i \leqslant m-1$. *Hence these relations are in one-to-one correspondence to the paths* Q_i *where* $1 \leqslant i \leqslant m-1$.

We call the relations in Theorem 5.4.12, the $m \times 1$ *relations.*

5.5 The $n \times m$ rectangle for characteristic 0

Let A be the Borel Schur algebra $S^+(3,r)$ over K of characteristic 0. Let Γ be the quiver Γ of A. By Theorem 4.1.4, the lengths of the arrows in Γ are 1. Let m, n be positive integers. We consider the following $n \times m$ rectangle in the quiver Γ:



where $\alpha^{00} = (a_1, a_2, a_3)$ and the compositions α^{ji} $(0 \leqslant i \leqslant m$ and $0 \leqslant j \leqslant n)$ are as follows:

$$\alpha^{ji} = \alpha^{00} + i\alpha_1 + j\alpha_2 = (a_1 + i, a_2 - i + j, a_3 - j),$$

with the conditions that $a_2 \geqslant m$ and $a_3 \geqslant n$.

By Proposition 3.2.15, we have the following lemma.

Lemma 5.5.1. *Let $d := \min\{m, n\}$. Let $H = \mathrm{Hom}_A(A\xi_{\alpha^{nm}}, A\xi_{\alpha^{00}})$. Then the dimension of H is equal to $(d+1)$.*

Let T be the vector space spanned by the paths from α^{00} to α^{nm} in the $n \times m$ rectangle in the path algebra $K\Gamma$. Let $\Psi(m, n)$ be the set of all words consisting of m letters R (right) and n letters D (down).

Lemma 5.5.2. *The set $\Psi(m, n)$ can be regarded as a basis of T. Hence the dimension of T is equal to $\binom{m+n}{m}$.*

Proof. Since there is only one path from α^{ji} to α^{ki} (or from α^{ji} to α^{jl}) where $0 \leqslant i \leqslant l \leqslant j < k \leqslant m$, the paths from α^{00} to α^{nm} in the path algebra $K\Gamma$ can be labeled by words from $\Psi(m, n)$. The set $\Psi(m, n)$ is a basis of T. The number of such words is $\binom{m+n}{m}$. Hence the dimension of T is $\binom{m+n}{m}$. ∎

Hence we can rewrite the relations in Corollary 5.4.6 as follows:

Lemma 5.5.3. *Let i and m' be integers with $1 \leqslant i \leqslant m' - 1 \leqslant m - 1$. Then*

$$(m' - i)W_0(DR^{m'})W_1 - m'W_0R^i(DR^{m'-i})W_1 + iW_0(R^{m'}D)W_1,$$

where W_0 and W_1 are words of R's and D's such that $W_0W_1 \in \Psi(m-1, n-m')$. Similarly, we have

$$(m' - i)V_0(D^{m'}R)V_1 - mV_0(D^iRD^{m'-i})V_1 + iV_0(RD^{m'})V_1,$$

where V_0 and V_1 are words of R's and D's such that $V_0V_1 \in \Psi(m-m', n-1)$. In particular, let $m' = 2$, then $i = 1$, we have

$$W_0(DR^2)W_1 - 2W_0(RDR)W_1 + W_0(R^2D)W_1, \tag{5.5.1}$$

$$V_0(D^2R)V_1 - 2V_0(DRD)V_1 + V_0(RD^2)V_1, \tag{5.5.2}$$

where W_0,W_1, V_0 and V_1 are words of R's and D's such that $W_0W_1 \in \Psi(m-1, n-2)$ and $V_0V_1 \in \Psi(m-2, n-1)$. These relations are generated by the following relations:

$$DR^2 - 2RDR + R^2D, \qquad D^2R - 2DRD + RD^2. \tag{5.5.3}$$

Definition 5.5.4. We call the relations in Equation (5.5.3), the 1×2 *relation* and 2×1 *relation* respectively.

Our aim is to prove that all relations in T are generated by 1×2 *relation* or 2×1 *relation.*

Define the lexicographic ordering on the words in $\Psi(m, n)$ by setting $D \succ R$.

Example 5.5.5. *The 1×2 relation:*

$$DRR - 2RDR + RRD, \qquad \text{where } DRR \succ RDR \succ RRD.$$

The 2×1 relation:

$$DDR - 2DRD + RDD, \qquad \text{where } DDR \succ DRD \succ RDD.$$

We can list all words of R's and D's in $\Psi(m, n)$ by this lexicographic ordering.

Theorem 5.5.6. *The relations from α^{00} to α^{nm} in T are generated by the 1×2 or 2×1 relations.*

Proof. By Lemma 5.5.2, $\dim T = \binom{m+n}{m}$. By Lemma 5.5.1, the Hom space $H := \mathrm{Hom}_A(A\xi_{\alpha^{nm}}, A\xi_{\alpha^{00}})$ is $(d+1)$ dimensional. Recall that the surjective map $\widetilde{f}$ maps T to the Hom space H. The relations in T are the kernel of $\widetilde{f}$. So we only need to find $\binom{m+n}{m} - d - 1$ linearly independent relations in T generated by the 1×2 or 2×1 relations in T.

There are $\binom{m+n}{m}$ words of R and D in $\Psi(m,n)$.

The words in $\Psi(m,n)$, which don't contain a sub-word DDR or DRR, are of the following forms

$$R^{n-i}(DR)^i D^{m-i},$$

where $0 \leqslant i \leqslant d$. There are $(d+1)$ such words. Hence, the number of the words containing a sub-word DDR or DRR is equal to $\binom{m+n}{m} - d - 1$.

We assume that $P = W(DDR)V$ is in $\Psi(m,n)$ where $WV \in \Psi(m-2, n-1)$. Then the words $W(DRD)V$ and $W(RDD)V$ are in $\Psi(m,n)$. Hence we have the relation

$$W(DDR)V - 2W(DRD)V + W(RDD)V, \qquad (W(DDR)V \succ W(DRD)V \succ W(RDD)V),$$

which is generated by the 2×1 relation. Similarly, if $P = W'(DRR)V'$ is in $\Psi(m,n)$ where $W'V' \in \Psi(m-1, n-2)$. Then we have the relation

$$W(DRR)V - 2W(RDR)V + W(RRD)V, \qquad (W(DRR)V \succ W(RDR)V \succ W(RRD)V),$$

which is generated by the 1×2 relation. Note that the word P is the leading term in the above relations in the lexicographic ordering. Thus these relations in T are linearly independent. Hence we found $\binom{m+n}{m} - d - 1$ linearly independent relations in T. ∎

5.6 Relations for $S^+(n,r)$ in characteristic 0

In this section we use the results from [6] to describe all relations for $S^+(n,r)$ over a field of characteristic 0. Without loss of generality we can assume that the ground field is the field of rational numbers $\mathbb{Q}$.

Recall that $\varepsilon_1, \ldots, \varepsilon_n$ is the standard orthogonal basis of the euclidean space $\mathbb{R}^n$. Let $(\ ,\ )$ denote the inner product on this space and define $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. $\{\alpha_1, \ldots, \alpha_{n-1}\}$ is a base of simple roots of the root system of type A_{n-1}.

Doty and Qiaquinto in [6] found the following presentation of the Schur algebra by generators and relations.

Theorem 5.6.1. *([6]) The $\mathbb{Q}$-algebra $S(n,r)$ is the associative algebra (with 1) given by generators 1_λ $(\lambda \in \Lambda(n,r))$, e_i, f_i $(1 \leqslant i \leqslant n-1)$ subject to the relations*

$$1_\lambda 1_\mu = \delta_{\lambda\mu} 1_\lambda, \quad \textstyle\sum_{\lambda\in\Lambda(n,r)} 1_\lambda = 1$$

$$e_i 1_\lambda = \begin{cases} 1_{\lambda+\alpha_i} e_i & if \lambda + \alpha_i \in \Lambda(n,r) \\ 0 & otherwise \end{cases}$$

$$f_i 1_\lambda = \begin{cases} 1_{\lambda-\alpha_i} f_i & if \lambda - \alpha_i \in \Lambda(n,r) \\ 0 & otherwise \end{cases}$$

$$1_\lambda e_i = \begin{cases} e_i 1_{\lambda-\alpha_i} & if \lambda - \alpha_i \in \Lambda(n,r) \\ 0 & otherwise \end{cases}$$

$$1_\lambda f_i = \begin{cases} f_i 1_{\lambda+\alpha_i} & if \lambda + \alpha_i \in \Lambda(n,r) \\ 0 & otherwise \end{cases}$$

$$e_i f_j - f_j e_i = \delta_{ij} \sum_{\lambda\in\Lambda(n,r)} (\lambda_j - \lambda_{j+1}) 1_\lambda$$

$$e_i^2 e_j - 2 e_i e_j e_i + e_j e_i^2 = 0 \quad (|i-j| = 1)$$

$$e_i e_j - e_j e_i = 0 \quad (|i-j| > 1)$$

$$f_i^2 f_j - 2 f_i f_j f_i + f_j f_i^2 = 0 \quad (|i-j| = 1)$$

$$f_i f_j - f_j f_i = 0 \quad (|i-j| > 1).$$

It follows from the results in [6] that the Borel Schur algebra $S^+(n,r)$ over $\mathbb{Q}$ has the following presentation by generators and relations.

Theorem 5.6.2. *([6]) The $\mathbb{Q}$-algebra $S^+(n,r)$ is the associative algebra A (with 1) given by generators 1_λ ($\lambda \in \Lambda(n,r)$), e_i ($1 \leqslant i \leqslant n-1$) subject to the relations*

$$(\mathcal{R}_1) \qquad 1_\lambda 1_\mu = \delta_{\lambda\mu} 1_\lambda, \quad \textstyle\sum_{\lambda\in\Lambda(n,r)} 1_\lambda = 1$$

$$(\mathcal{R}_2) \qquad \begin{aligned} e_i 1_\lambda &= \begin{cases} 1_{\lambda+\alpha_i} e_i & if \lambda+\alpha_i \in \Lambda(n,r) \\ 0 & otherwise \end{cases} \\ 1_\lambda e_i &= \begin{cases} e_i 1_{\lambda-\alpha_i} & if \lambda-\alpha_i \in \Lambda(n,r) \\ 0 & otherwise \end{cases} \end{aligned}$$

$$(\mathcal{R}_3) \qquad \begin{aligned} e_i^2 e_j - 2e_i e_j e_i + e_j e_i^2 = 0 \quad (|i-j|=1) \\ e_i e_j - e_j e_i = 0 \quad (|i-j|>1). \end{aligned}$$

In the following, we let A be the associative algebra in Theorem 5.6.2.

Set $E_i^\lambda = e_i 1_\lambda$ for $1 \leqslant i \leqslant n-1$ and $\lambda \in \Lambda(n,r)$. Note that by the relations $(\mathcal{R}_2)$ in Theorem 5.6.2, $E_i^\lambda = 0$ unless $\lambda + \alpha_i \in \Lambda(n,r)$.

Lemma 5.6.3. *Let $\nu, \lambda, \mu \in \Lambda(n,r)$. Then*

$$1_\mu E_i^\nu 1_\lambda = \delta_{\mu,\nu+\alpha_i}\delta_{\nu,\lambda} E_i^\nu.$$

Moreover, $E_i^\mu E_j^\lambda = 0$ unless $\mu - \alpha_j = \lambda$, and

$$E_i^{\lambda+\alpha_j} E_j^\lambda = e_i e_j 1_\lambda.$$

Proof. By the definition of E_i^ν and the relations $(\mathcal{R}_1)$ in Theorem 5.6.2, we have

$$1_\mu E_i^\nu = 1_\mu e_i 1_\nu = 1_\mu 1_{\nu+\alpha_i} e_i = \delta_{\mu,\nu+\alpha_i} 1_{\nu+\alpha_i} e_i = \delta_{\mu,\nu+\alpha_i} e_i 1_\nu.$$

Thus

$$1_\mu E_i^\nu 1_\lambda = e_i 1_\nu 1_\lambda = \delta_{\nu,\lambda} e_i 1_\nu = \delta_{\nu,\lambda} E_i^\nu.$$

Moreover, by the above formula,

$$E_i^\mu E_j^\lambda = (E_i^\mu 1_\mu)(1_{\lambda+\alpha_j} E_j^\lambda) = \delta_{\mu,\lambda+\alpha_j} E_i^\mu E_j^\lambda,$$

and

$$E_i^{\lambda+\alpha_j} E_j^{\lambda} = e_i 1_{\lambda+\alpha_j} e_j 1_{\lambda} = e_i e_j 1_{\lambda}.$$

Note that if $\lambda + \alpha_j$ is not in $\Lambda(n,r)$, i.e. $E_i^{\lambda+\alpha_j} = 0$, the above equation still holds. ∎

Recall that rad A, the radical of A, is a nilpotent ideal R of A such that A/R is semi-simple (see Section 1.2). Next we will calculate the radical of A.

By Lemma 5.6.3, $E_i^{\mu} E_j^{\lambda} = 0$ unless $\mu - \alpha_j = \lambda$, which gives the multiplication formula for E_i^{λ}'s in A. Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda(n,r)$. Suppose that there exists some i with $1 \leqslant i \leqslant n-1$ such that $\lambda_i \geqslant 1$, then $\lambda - \alpha_i \in \Lambda(n,r)$. This means for the diagram of λ we can move 1 box from the i-th line to the $(i+1)$-th line. For each λ_i with $1 \leqslant i \leqslant n-1$, we can follow this procedure for the diagram of λ, moving the boxes of line i to the bottom line n. We can always repeat this procedure, until the remaining diagram will have the only line on the bottom, i.e.$(0, \ldots, 0, r)$. More precisely, we have

$$\lambda - \Sigma_{i=1}^{n-1} \lambda_i (\alpha_i + \ldots + \alpha_{n-1}) = (0, \ldots, 0, r),$$

where $\lambda - \lambda_i(\alpha_i + \ldots + \alpha_{n-1})$ means moving the i-th line of the diagram of λ to the n-th line, this implies we need to move $n-i$ times for each of the λ_i boxes of line i. Thus the total number of moves for this procedure will be

$$n_\lambda := (n-1)\lambda_1 + (n-2)\lambda_2 + \ldots + \lambda_{n-1}.$$

Note that for $(0, \ldots, 0, r)$, we can not run this procedure, since for any i, $(0, \ldots, 0, r) - \alpha_i$ is not in $\Lambda(n,r)$. Thus, the maximal number of moves we can make to to remain in $\Lambda(n,r)$ is n_λ. Note that

$$\max_{\lambda} n_\lambda = (n-1)r, \text{ which is attained for } \lambda = (r, 0, \ldots, 0).$$

Hence, for any $\lambda \in \Lambda(n,r)$,

$$\lambda - \sum_i t_i\alpha_i, \text{ with the condition } \sum_i t_i \geqslant (n-1)r+1,$$

then $\lambda - \sum_i t_i\alpha_i$ will be not in $\Lambda(n,r)$.

Proposition 5.6.4. *Let $R = \operatorname{rad} A$. Then R is generated by all E_i^λ as an algebra. Moreover, the vector space R/R^2 has a basis*

$$\{E_i^\lambda + R^2 \mid 1 \leqslant i \leqslant n-1,\ \lambda \in \Lambda(n,r),\ \lambda + \alpha_i \in \Lambda(n,r)\}.$$

Proof. Let E be the subalgebra of A generated by all E_i^λ's. We need to prove that $E = R$. Note that for any i with $1 \leqslant i \leqslant n-1$,

$$e_i = e_i 1 = e_i \sum_{\lambda\in\Lambda(n,r)} 1_\lambda = \sum_{\lambda\in\Lambda(n,r)} E_i^\lambda.$$

Since $E_i^\lambda \in E$, we have $e_i \in E$. By Lemma 5.6.3, E is stable under multiplications by 1_λ's, so E is an ideal of A. It is easy to see that

$$A/E \cong \bigoplus_{\lambda\in\Lambda(n,r)} K1_\lambda,$$

so A/E is semi-simple.

Let $N = (n-1)r+1$. Note that $N > \max\limits_\lambda n_\lambda$. Consider E^{N+1}. Let $x \in E^{N+1}$, then x is a linear combination of the products $E_{i_0}^{\lambda^0} E_{i_1}^{\lambda^1} \ldots E_{i_M}^{\lambda^M}$, where $M \geqslant N$.

Since $E_i^\mu E_j^\lambda = 0$ unless $\mu - \alpha_j = \lambda$, the above product $E_{i_0}^{\lambda^0} E_{i_1}^{\lambda^1} \ldots E_{i_M}^{\lambda^M} = 0$ unless

$$\lambda^0 - \sum_{k=1}^{M} \alpha_{i_k} \in \Lambda(n,r).$$

However, this is impossible since $M \geqslant N > \max\limits_\lambda n_\lambda$. Thus any product in the linear combination of x will be 0. Hence $x = 0$. That is $E^{N+1} = 0$, which implies that E is nilpotent. Therefore E is the radical of A, i.e. $E = R$.

Now it is clear that R/R^2 is spanned by all nonzero $E_i^\lambda + R^2$, where $1 \leqslant i \leqslant n-1, \lambda \in \Lambda(n,r)$. Next we prove that these nonzero $E_i^\lambda + R^2$ are linearly independent. Suppose that there exist rational numbers $a_{i,\lambda}$, such that

$$\sum_{i,\lambda} a_{i,\lambda}(E_i^\lambda + R^2) = 0,$$

we need to prove that all $a_{i,\lambda} = 0$. Assume there exists $a_{i,\lambda} \neq 0$, for some i and λ. Since

$$\sum_{i,\lambda} a_{i,\lambda}(E_i^\lambda + R^2) = \sum_{i,\lambda} a_{i,\lambda}(E_i^\lambda) + R^2 = 0,$$

we multiply $1_{\lambda+\alpha_i}$ on the left hand side, and multiply 1_λ on the right hand side of the above equation, then by Lemma 5.6.3, only E_i^λ in the sum survives, we get:

$$1_{\lambda+\alpha_i}(\sum_{i,\lambda} a_{i,\lambda}(E_i^\lambda) + R^2)1_\lambda = a_{i,\lambda}E_i^\lambda + R^2 = 0.$$

Thus $a_{i,\lambda}E_i^\lambda \in R^2$. Since $a_{i,\lambda} \neq 0$, we have $E_i^\lambda \in R^2$, which is a contradiction. Hence all $E_i^\lambda + R^2$ will be linear independent and form a basis of R/R^2. ∎

Now we describe the quiver Γ of A.

Theorem 5.6.5. *The quiver Γ of A over $\mathbb{Q}$ is given as follows. The set of vertices is $\Gamma_0 = \Lambda(n,r)$; the number of arrows from vertex λ to vertex μ for $\lambda, \mu \in \Lambda(n,r)$, is equal to 1 or 0; this number is equal to 1 if and only if $\mu - \alpha_i = \lambda$ for some $1 \leqslant i \leqslant n-1$.*

Proof. Let $R = \operatorname{rad} A$. By Proposition 5.6.4,

$$A/R \cong \bigoplus_{\lambda \in \Lambda(n,r)} K1_\lambda.$$

By Definition 1.4.5, A is elementary. Thus, by Definition 1.4.2, the vertex set Γ_0 is $\Lambda(n,r)$.

Next we consider the arrows. Note that the number of arrows from vertex λ to vertex μ for $\lambda, \mu \in \Lambda(n,r)$, is the dimension of $1_\mu(R/R^2)1_\lambda$. And by Proposition 5.6.4, R/R^2 has a basis

$$\{E_i^\nu + R^2 \mid 1 \leqslant i \leqslant n-1,\ \nu \in \Lambda(n,r),\ \nu + \alpha_i \in \Lambda(n,r)\}.$$

Hence we need to calculate the dimension of

$$1_\mu(R/R^2)1_\lambda \cong 1_\mu(\bigoplus_{\nu,i} K \cdot E_i^\nu + R^2)1_\lambda =: H(\mu,\lambda).$$

By Lemma 5.6.3, $H(\mu,\lambda) \neq 0$, if and only if there exists $1 \leqslant i \leqslant n-1$ and $\nu \in \Lambda(n,r)$ such that $\lambda = \nu$ and $\mu - \alpha_i = \nu$. Moreover, in this case $H(\mu,\lambda)$ is one-dimensional and spanned by $E_i^\nu + R^2$, where $\lambda = \nu$ and $\mu - \alpha_i = \nu$. ∎

Note that in Theorem 5.6.5, there is precisely one arrow from λ to $\mu = \lambda + \alpha_i$ for some $1 \leqslant i \leqslant n-1$, corresponding to E_i^λ and labeled by E_i^λ.

Let $\mathbb{Q}\Gamma$ be the path algebra of Γ over $\mathbb{Q}$. Then by Theorem 1.4.6, there is a surjective ring homomorphism $\varphi : \mathbb{Q}\Gamma \longrightarrow A$, defined as follows: $\varphi(1_\lambda) = 1_\lambda$ and $\varphi(E_i^\lambda) = E_i^\lambda = e_i 1_\lambda$ for all $1 \leqslant i \leqslant n-1$ and $\lambda \in \Lambda(n,r)$. By Theorem 1.4.6, $\ker\varphi$ contains exactly all relations of the quiver Γ.

Definition 5.6.6. We define the following relations for the path algebra $\mathbb{Q}\Gamma$:

$$(\mathcal{R}) \qquad \begin{aligned} E_i^{\lambda+\alpha_i+\alpha_j} E_i^{\lambda+\alpha_j} E_j^\lambda - 2E_i^{\lambda+\alpha_i+\alpha_j} E_j^{\lambda+\alpha_i} E_i^\lambda \\ + E_j^{\lambda+2\alpha_i} E_i^{\lambda+\alpha_i} E_i^\lambda \quad (|i-j|=1) \\ E_i^{\lambda+\alpha_j} E_j^\lambda - E_j^{\lambda+\alpha_i} E_i^\lambda \quad (|i-j|>1), \end{aligned}$$

where E_k^ν is treated as zero if either $\nu \notin \Lambda(n,r)$ or $\nu + \alpha_k \notin \Lambda(n,r)$ (i.e. there is no such arrow E_k^ν).

Let T be the ideal of $\mathbb{Q}\Gamma$ generated by the above relations $(\mathcal{R})$.

Lemma 5.6.7. $T \subseteq \ker \varphi$.

Proof. We need to prove that $\varphi(\mathcal{R}) = 0$. By the definition of φ and the relations $(\mathcal{R}_2)$ in Theorem 5.6.2 (or see Lemma 5.6.3),

$$\begin{aligned} & \varphi(E_i^{\lambda+\alpha_i+\alpha_j} E_i^{\lambda+\alpha_j} E_j^{\lambda}) \\ = \; & \varphi(E_i^{\lambda+\alpha_i+\alpha_j})\varphi(E_i^{\lambda+\alpha_j})\varphi(E_j^{\lambda}) \\ = \; & e_i 1_{\lambda+\alpha_i+\alpha_j} e_i 1_{\lambda+\alpha_j} e_j 1_{\lambda} \\ = \; & e_i^2 1_{\lambda+\alpha_j} e_j 1_{\lambda} \\ = \; & e_i^2 e_j 1_{\lambda}. \end{aligned}$$

Similarly

$$\begin{aligned} \varphi(E_i^{\lambda+\alpha_i+\alpha_j} E_j^{\lambda+\alpha_i} E_i^{\lambda}) &= e_i e_j e_i 1_{\lambda}, \\ \varphi(E_j^{\lambda+2\alpha_i} E_i^{\lambda+\alpha_i} E_i^{\lambda}) &= e_j e_i^2 1_{\lambda}. \end{aligned}$$

By Theorem 5.6.2, A satisfies the relations $(\mathcal{R}_3)$, hence

$$\begin{aligned} & \varphi(E_i^{\lambda+\alpha_i+\alpha_j} E_i^{\lambda+\alpha_j} E_j^{\lambda} - 2E_i^{\lambda+\alpha_i+\alpha_j} E_j^{\lambda+\alpha_i} E_i^{\lambda} + E_j^{\lambda+2\alpha_i} E_i^{\lambda+\alpha_i} E_i^{\lambda}) \\ = \; & (e_i^2 e_j - 2e_i e_j e_i + e_j e_i^2) 1_{\lambda} = 0, \end{aligned}$$

where $|i-j| = 1$. And by Lemma 5.6.3,

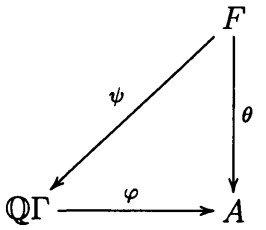
$$\varphi(E_i^{\lambda+\alpha_j} E_j^{\lambda} - E_j^{\lambda+\alpha_i} E_i^{\lambda}) = (e_i e_j - e_j e_i) 1_{\lambda} = 0$$

where $|i-j| > 1$. Thus $T \subseteq \ker \varphi$. ∎

Let F be the free associative algebra with unit and free generators $e_i, 1_\lambda$, where $1 \leqslant i \leqslant n-1$ and $\lambda \in \Lambda(n,r)$. Let θ and ψ be the canonical homomorphisms from F to A and $\mathbb{Q}\Gamma$, defined as follows: $\theta(1_\lambda) = 1_\lambda$, $\theta(e_i) = e_i$; $\psi(1_\lambda) = 1_\lambda$, $\psi(e_i) = \sum_{\lambda \in \Lambda(n,r)} E_i^{\lambda}$. Note that θ is surjective. The map ψ is also surjective, since

$$E_i^{\lambda} = (\sum_{\lambda \in \Lambda(n,r)} E_i^{\lambda}) 1_{\lambda} = \psi(e_i)\psi(1_{\lambda}) = \psi(e_i 1_{\lambda}),$$

for all $1 \leqslant i \leqslant n-1$ and $\lambda \in \Lambda(n,r)$. Thus we have the following diagram:



Note that

$$\varphi\psi(1_\lambda) = \varphi(1_\lambda) = 1_\lambda = \theta(1_\lambda),$$

and

$$\varphi\psi(e_i) = \varphi(\sum_{\lambda\in\Lambda(n,r)} E_i^\lambda) = \sum_{\lambda\in\Lambda(n,r)} e_i 1_\lambda = e_i \cdot 1 = e_i = \theta(e_i),$$

for all $1 \leqslant i \leqslant n-1$ and $\lambda \in \Lambda(n,r)$. Hence $\theta = \varphi\psi$, i.e.the above diagram commutes.

Lemma 5.6.8. *For the above diagram,* $\ker\varphi = \psi(\ker\theta)$.

Proof. Let $x \in \psi(\ker\theta)$. Then $x = \psi(y)$ for some $y \in \ker\theta$. Thus $\varphi(x) = \varphi\psi(y) = \theta(y) = 0$, i.e. $x \in \ker\varphi$. Thus $\psi(\ker\theta) \subseteq \ker\varphi$.

Now let $a \in \ker\varphi$. Since ψ is surjective, then there exists $b \in F$ such that $a = \psi(b)$. Then $\theta(b) = \varphi\psi(b) = \varphi(a) = 0$, i.e. $b \in \ker\theta$, thus $a = \psi(b) \in \psi(\ker\theta)$. Hence $\ker\varphi = \psi(\ker\theta)$. ∎

Theorem 5.6.9. *The ideal T of A contains exactly all relations of A, i.e.*

$$\ker\varphi = T.$$

Proof. By Lemma 5.6.7, we only need to prove that $\ker\varphi \subseteq T$. By Lemma 5.6.8, we need to prove that $\psi(\ker\theta) \subseteq T$. By Theorem 5.6.2, $\ker\theta$ is

generated by the relations $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$ as an ideal. Thus it is enough to prove that ψ maps each relation in $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$ into T.

($\mathcal{R}_1$): Note that path algebra $\mathbb{Q}\Gamma$ satisfies all relations in $\mathcal{R}_1$, so the image of all these relations in $\mathbb{Q}\Gamma$ is zero.

($\mathcal{R}_2$): Assume $\lambda \in \Lambda(n,r)$, and $\lambda + \alpha_i \in \Lambda(n,r)$. Then one has

$$\psi(e_i 1_\lambda) = E_i^\lambda,$$

and

$$\psi(1_{\lambda+\alpha_i} e_i) = \psi(1_{\lambda+\alpha_i})\psi(e_i) = 1_{\lambda+\alpha_i}(\sum_{\nu\in\Lambda(n,r)} E_i^\nu) = E_i^\lambda = \psi(e_i 1_\lambda),$$

thus

$$\psi(e_i 1_\lambda - 1_{\lambda+\alpha_i} e_i) = 0.$$

Assume now $\lambda \in \Lambda(n,r)$ and $\lambda + \alpha_i \notin \Lambda(n,r)$. Since $\psi(e_i 1_\lambda) = E_i^\lambda$, and $\lambda + \alpha_i \notin \Lambda(n,r)$, $E_i^\lambda = 0$. Thus $\psi(e_i 1_\lambda) = 0$. And the other case in the relations $\mathcal{R}_2$ is considered similarly. Thus $\psi(\mathcal{R}_2) = 0$.

($\mathcal{R}_3$): Let $1 \leqslant i, j \leqslant n-1$. Since $\psi(e_i) = \sum_{\lambda\in\Lambda(n,r)} E_i^\lambda$, and $E_i^\mu E_j^\lambda = 0$ unless $\mu - \alpha_j = \lambda$ by Lemma 5.6.3,

$$\begin{aligned}\psi(e_i e_j) &= \psi(e_i)\psi(e_j)\\ &= (\textstyle\sum_{\mu\in\Lambda(n,r)} E_i^\mu)(\sum_{\lambda\in\Lambda(n,r)} E_j^\lambda)\\ &= \textstyle\sum_{\lambda,\mu\in\Lambda(n,r)} E_i^\mu E_j^\lambda\\ &= \textstyle\sum_{\lambda\in\Lambda(n,r)} E_i^{\lambda+\alpha_j} E_j^\lambda.\end{aligned}$$

Thus

$$\psi(e_i^2 e_j) = \sum_{\lambda\in\Lambda(n,r)} E_i^{\lambda+\alpha_i+\alpha_j} E_i^{\lambda+\alpha_j} E_j^\lambda.$$

Similarly

$$\psi(e_i e_j e_i) = \sum_{\lambda\in\Lambda(n,r)} E_i^{\lambda+\alpha_i+\alpha_j} E_j^{\lambda+\alpha_i} E_i^\lambda.$$

and

$$\psi(e_j e_i^2) = \sum_{\lambda \in \Lambda(n,r)} E_j^{\lambda+2\alpha_i} E_i^{\lambda+\alpha_i} E_i^{\lambda}.$$

Hence

$$\begin{aligned}\psi(e_i^2 e_j - 2e_i e_j e_i + e_j e_i^2) &= \psi(e_i^2 e_j) - 2\psi(e_i e_j e_2) + \psi(e_j e_i e_i) \\ &= \sum_{\lambda \in \Lambda(n,r)} (E_i^{\lambda+\alpha_i+\alpha_j} E_i^{\lambda+\alpha_j} E_j^{\lambda} \\ &\quad -2E_i^{\lambda+\alpha_i+\alpha_j} E_j^{\lambda+\alpha_i} E_i^{\lambda} + E_j^{\lambda+2\alpha_i} E_i^{\lambda+\alpha_i} E_i^{\lambda}) \in T,\end{aligned}$$

if $|i-j| = 1$. Similarly we have

$$\psi(e_i e_j - e_j e_i) = \psi(e_i e_j) - \psi(e_j e_i) = \sum_{\lambda \in \Lambda(n,r)} (E_i^{\lambda+\alpha_j} E_j^{\lambda} - E_j^{\lambda+\alpha_i} E_i^{\lambda}) \in T,$$

if $|i-j| > 1$. Thus $\psi(\mathcal{R}_3) \subseteq T$. Therefore $\psi(\ker\theta) \subseteq T$. Hence $\ker\varphi = \psi(\ker\theta) = T$. ∎

Bibliography

[1] J. Alperin. Local representation theory, Cambridge studies in advanced mathematics, vol 11, Cambridge University Press, 1986.

[2] M. Auslander, I. Reiten and S. O. Smalø, Representation Theory of Artin Algebras, Cambridge University Press, 1995.

[3] D. J. Benson, Representations and Cohomology, Cambridge University Press, 1995.

[4] C. W. Curtis and I. Reiner, Representation Theory of Finite Groups and Associative Algebras, JOHN WILEY & SONS, INC, 1962.

[5] L. E. Dickson, History of the Theory of Numbers, vol. 1, Washington, 1919.

[6] S. Doty and A. Qiaquinto, Presenting Schur algebras, 2002.

[7] Y. A. Drozd, V. V. Kirichenko, Finite Dimensional Algebras, Springer-Verlag Berlin Heidelberg, 1994.

[8] M. Fayers and S. Martin, Schur subalgebras, Journal of Algebra **240**, 859-73(2001).

[9] P. Gabriel, Auslander-Reiten sequences and representation-finite algebras, Representation theory I, Ottawa 1979, Lecture Notes in Mathematics 831, Springer-Verlag, Berlin/New York 1980.

[10] J. A. Green, Polynomial Representations of GL_n, Lecture Notes in Mathematics, vol. 830, Springer-Verlag, Berlin, 1980.

[11] J. A. Green, On certain subalgebras of the Schur algebra, Journal of Algebra **131**, 265-280(1990).

[12] N. Jacobson, Basic Algebra II, W. H. Freeman and Company, 1980.

[13] J. C. Jantzen, Representation of Algebraic Groups, Second Edition, *Mathematical Surveys and Monographs*, **107**, Amer. Math. Soc. 2003.

[14] M. E. Keating, A First Course in Module Theory, Imperial College Press, 1998.

[15] A. Kovačec and A. P. Santana, A cominatorial theorem on the Cartan invariants of the Schur algebra $S(B^+)$, Linear Algebra and Its Applications 197, 198: 79-91(1994).

[16] A. P. Santana, The Schur algebra $S(B^+)$ and projective resolutions of Weyl modules, Journal of Algebra **161**, 480-504(1993).

[17] R. P. Stanley, Enumerative Combinatorics (Volume 2), Cambridge University Press, 1999.