The Fundamental Groupoid and the Geometry of Monoids

Thesis submitted for the degree of
Doctor of Philosophy
at the University of Leicester

by

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February 2016

Declaration

This thesis, submitted to the University of Leicester, is my original research work, unless cited otherwise. Wherever the results of others are used, every effort is made to indicate this clearly.

I further declare that, with the exception of Chapter 2, Subsection 2.3.1, no part of my results have been submitted for any other degree at any other University.

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Abstract

This thesis is divided in two equal parts. We start the first part by studying the Kato-spectrum of a commutative monoid M, denoted by $\mathsf{KSpec}(M)$. We show that the functor $M \mapsto \mathsf{KSpec}(M)$ is representable and discuss a few consequences of this fact. In particular, when M is additionally finitely generated, we give an efficient way of calculating it explicitly.

We then move on to study the cohomology theory of monoid schemes in general and apply it to vector- and particularly, line bundles. The isomorphism class of the latter is the Picard group. We show that under some assumptions on our monoid scheme X, if k is an integral domain (resp. PID), then the induced map

$$\mathsf{Pic}(X) \to \mathsf{Pic}(X_k)$$

from X to its realisation is a monomorphism (resp. isomorphism).

We then focus on the Pic functor and show that it respects finite products. Finally, we generalise several important constructions and notions such as cancellative monoids, smoothness and Cartier divisors, and prove important results for them.

The main results of the second part can be summed up in fewer words. We prove that for good topological spaces X the assignment $U \mapsto \Pi_1(U)$ is the terminal object of the 2-category of costacks. Here U is an open subset of X and $\Pi_1(U)$ denotes the fundamental groupoid of U. This result translates to the étale fundamental groupoid as well, though the proof there is completely different and involves studying and generalising Galois categories.

Acknowledgements

I would like to thank my supervisor Dr. Frank Neumann, for his friendly and patient teaching, guiding and help during my studies.

Further, I would like to thank my entire family for their help and support.

Contents

Introduction

This thesis is made up of two independent parts. Outside of Section 5.2, which gives a brief connection, there will be basically no intersection and dependence between these parts. The first part, consisting of Chapters 2 and 3, focuses on the study of commutative monoids and monoid schemes. In the second part, we move toward the so called 2-categories and give a new characterisation of the fundamental groupoid, both in the classical (topological), as well as the étale (algebraic) case. It should be noted that while the main statement in both cases will be identical, the proofs given, are vastly different.

Introduction to 'The Geometry of Monoids'

The formal study of monoids and related objects (for example semigroups) goes back over a hundred years. But as the term 'semigroup' already suggests, these were considered to be generalisations of groups and as such they were studied from much the same perspective.

It was mainly the paper by Kato [26] in 1994 that led to a new perspective on commutative monoids, as it demonstrated the link between toric varieties and monoid schemes. In 2008 Deitmar built on this result and showed that the realisation of irreducible, connected, integral (cancellative in our terminology) monoid schemes of finite type over the complex numbers were (complex) toric varieties [14, Theorem 4.1]. These two papers alone give validity to the study of monoid schemes, as they can be thought of as a generalisation of the classical field of toric varieties without a fixed base field.

There are however many other areas that use monoid schemes. These include tropical geometry [37], logarithmic geometry [1][27] and the geometry over $\mathbb{F}_1$, both the version of Connes [10][11] and of Deitmar [13].

The fundamental idea of monoid schemes is to treat monoids as if they were rings, but disregard addition everywhere. For example, an ideal I of a monoid M would simply be a subset, satisfying $a \in I$, $m \in M \Rightarrow am \in I$.

The first chapter of this thesis focuses on the set of prime ideals of a commutative monoid M. This set, which is denoted by $\mathsf{KSpec}(M)$, after K. Kato, is endowed with a natural topology, called the Zariski topology, which is defined in much the same way as for rings. Unlike for ring however, here, the union of prime ideals is again a prime ideal. This induces a monoid structure on the Kato-spectrum. One of the main results we prove in the second chapter, claims that for any commutative monoid M, one has a natural isomorphism of topological monoids

$$\mathsf{KSpec}(M) \cong \mathsf{Hom}(M, \mathbb{I}), \tag{1}$$

where Hom is taken in the category of commutative monoids and $\mathbb{I} = \{0, 1\}$ is the monoid with the obvious multiplication. For the topology involved in this isomorphism see Section 2.3. From isomorphism (1) one easily deduces the 'reduction isomorphism'

$$\mathsf{KSpec}(M) \cong \mathsf{KSpec}(M^{sl}), \tag{2}$$

where M^{sl} is the monoid obtained by quotiening out M with the congruence generated by $m^2 \sim m$. That is to say, it is the canonical semilattice associated to M.

Another result says that for any commutative monoid M there is an injective, order preserving map

$$\alpha_M : M^{sl} \to \mathsf{KSpec}(M),$$

which is an isomorphism, provided M is finitely generated. In particular, these results give an effective way of computing $\mathsf{KSpec}(M)$ for an arbitrary, finitely generated, commutative monoid M. Note that M^{sl} is considered as a poset where $v \leq u$ if and only if $uv = v$, $u, v \in M^{sl}$.

Lastly, we focus on localisation and show that every localisation of M is isomorphic to a localisation by a single element. We further show that two elements will define the same localisation if and only if they map to the same element with the canonical map $M \to M^{sl}$. This result was previously already obtained in [12, Lemmas 1.1 and 1.3].

Having obtained these results, we then move on to study monoid schemes in the next chapter. Let X denote a monoid scheme. We will show that the category of sheaves on the underlying topological space of X is equivalent to the category of functors on its underlying poset. Part ii) of Proposition 3.1.9 shows that for separated monoid schemes, we can use the simpler Ĉech cohomology to calculate the (Grothendieck) cohomology. These two results greatly simplify the calculation of sheaf cohomologies of monoid schemes.

In [13, Propositin 4.3], Deitmar showed that the Picard group (also called the group of line bundles) $\mathsf{Pic}(\mathbb{P}_1)$ of the projective line (in the monoid world) is $\mathbb{Z}$, which agrees with the Picard group of the complex projective line $\mathbb{P}_1(\mathbb{C})$. This raises the natural question whether this result generalises and whether we can use the cohomology of monoid schemes to calculate the cohomology of their realisations. In Section 3.2 we show that the vector bundles over a monoid scheme (being defined as locally free M-sets) can be calculated using cohomology, exactly as in the classical case. We then proceed to prove that over any separable noetherian monoid scheme X, every vector bundle is a coproduct of line bundles.

This already clearly shows that getting a strong relation between the vector bundles over a monoid scheme and its realisation is unlikely. For line bundles however, the situation is much more interesting and hopeful. Indeed, as we will show in Section 3.3, Corollary 3.3.3, under some assumptions on X, there is an isomorphism

$$\mathsf{Pic}(X) \to \mathsf{Pic}(X_k),$$

when k is a PID. This result shows the importance of $\mathsf{Pic}(X)$ for a separated monoid scheme X and the rest of the chapter is devoted to studying it in more detail.

We first prove that the functor Pic is additive in the monoid world. In other words, the canonical map

$$\mathsf{Pic}(X) \oplus \mathsf{Pic}(Y) \cong \mathsf{Pic}(X \times Y)$$

is an isomorphism.

In Section 3.5 we move on to study the Picard group in more detail. For this we restrict our class of monoid schemes even more. We introduce the notion of s-cancellative monoids and monoid schemes as well as s-regular elements. These generalise the notions of cancellative monoids and regular elements respectively.

This is important because as we will show, it is the biggest class of monoid schemes where the maps induced by localisations are injective on the invertible elements. This allows us to embed the sheaf $(X, \mathcal{O}_X^*)$ in a (in general) bigger constant sheaf than the sheaf of meromorphic functions. We show that for s-cancellative monoid schemes, there is an isomorphism

$$\mathsf{Pic}(X) \cong s\mathsf{Cl}(X),$$

where s-Cl is the analogue of CaCl, the classes of Cartier divisors, in our setting.

Lastly we generalise the notion of smooth monoid schemes with s-smooth monoid schemes and show that for such schemes, the higher cohomologies vanish. We then give a few examples of s-smooth monoids which are not smooth, as well as a small conjecture.

Introduction to 'Stacks, Costacks and the Fundamental Groupoid'

The second part of this thesis focuses on the theory of 2-categories. Its foundations can be traced back to Grothendieck and his school, more precisely, the gluing of the categories of schemes, also known as the descent data. While this theory was not formally known as 2-categories back then, it carried much of the essence. The main idea here is that a 2-category is essentially a category enriched in categories and as such two objects can now be either equal, isomorphic or equivalent. Every categorical construction has a natural, 2-categorical analogue. For example, stacks are the analogue of sheaves.

In this thesis we will focus on the dual notion of stacks, namely costacks. We will show that while slightly overlooked, they are an important class of 2-functors. They will allow us to give an axiomatic description of the fundamental groupoid, both in the classical, as well as the algebraic case.

More precisely, we will show that the assignment $U \mapsto \Pi_1(U)$ defines the 2-terminal costack over X. Here X can be a topological space with $U \subset X$ an open subset, or the site of étale coverings over a noetherian scheme, with U an object in the said site.

While it might seem like an artificial property for a (strict) 2-functor $\mathfrak{F}$ to be a costack, it is in essence only saying that $\mathfrak{F}$ satisfies (a slightly reformulated version) of the Seifert-van Kampen theorem. Hence Theorems 5.1.4 and 8.0.9 can be reinterpreted as saying that the Seifert-van Kampen theorem is in fact the defining property of the fundamental groupoid.

We give separate proofs for the topological and algebraic case, both of which are of intrinsic interest.

The proof for the topological case gives an effective way to calculate the fundamental groupoid explicitly, whenever we are given a so called discrete covering (Definition 5.1.3). In Section 5.2 we give a big class of such spaces and demonstrate its application.

The algebraic case mainly involves the study of Galois categories from a 2-categorical viewpoint and generalising them to the finitely connected case. We then proceed to prove that the 2-category of finitely connected, profinite groupoids (see Definition 4.1.12) is 2-equivalent to the 2-category of finitely connected Galois categories.

Though we do not touch on this subject in this thesis, it is likely that this proof can be modified to prove that the Nori-fundamental groupoid scheme (adequately defined) will be the terminal costack with values in the 2-category of groupoid schemes. To do this, we would have to replace the Galois categories with Tannakian categories.

Chapter 1

Preliminaries from Category Theory and Homological Algebra

In this chapter we will give some basic results regarding sheaf cohomology and areas related to that. Everything here is of course well known and is only stated to fix notation and for the convenience of the reader. In more detail, we will start by defining what a Grothendieck topology, also called a site, is and give a few examples. This will be used in our discussion of the étale fundamental groupoid.

We will then define the Ĉech cohomology of a presheaf and the Grothendieck cohomology of a sheaf, for which we will state an array of important results in Theorem 1.4.1. Finally we mention a few words about non-abelian cohomology, until restricting ourselves to the cohomology theory of topological spaces.

1.1 Grothendieck Topology

Following [2], a *Grothendieck topology* $\mathbf{T}$, which is also called a *site*, consists of a small category $\mathsf{Cat}\ \mathbf{T}$ and a set $\mathsf{Cov}\ \mathbf{T}$ of families of morphisms $\{U_i \xrightarrow{\phi_i} U\}_{i \in I}$ in $\mathsf{Cat}\ \mathbf{T}$ called *coverings*, satisfying the following:

i) If ϕ is an isomorphism then $\{\phi\} \in \mathsf{Cov}\ \mathbf{T}$;

ii) If $\{U_i \to U\}_{i \in I} \in \mathsf{Cov}\ \mathbf{T}$ and $\{V_{ij} \to U_i\}_{j \in J_i} \in \mathsf{Cov}\ \mathbf{T}$ then the family

$$\{V_{ij} \to U\}_{i \in I, j \in J_i} \in \mathsf{Cov}\ \mathbf{T};$$

iii) If $\{U_i \to U\}_{i\in I} \in \mathsf{Cov}\ \mathbf{T}$ and $V \to U \in \mathsf{Cat}\ \mathbf{T}$ is arbitrary then $U_i \times_U V$ exists and $\{U_i \times_U V \to V\}_{i\in I} \in \mathsf{Cov}\ \mathbf{T}$.

By abuse of notation we will call $\mathbf{T}$ a Grothendieck topology.

Definition 1.1.1. *Let $\{U_i \to U\}_{i\in I}$ and $\{V_s \to U\}_{s\in S}$ be two coverings of U. A morphism of coverings*

$$\{U_i \to U\}_{i\in I} \to \{V_s \to U\}_{s\in S}$$

is given by a map $\epsilon : I \to S$, and for every $i \in I$ a morphism $f_i : U_i \to V_{\epsilon(i)}$, such that the diagram

$$\begin{tikzcd} U_i \arrow[rr, "f_i"] \arrow[dr] & & V_{\epsilon(i)} \arrow[dl] \\ & U & \end{tikzcd}$$

commutes. This is also called a refinement *of $\{U_i \to U\}_{i\in I}$.*

Remark 1: In more modern literature this is known as a *Grothendieck pre-topology*. Since however a pre-topology defines a topology in a unique way, we can use this significantly simpler definition for the purposes of this thesis.

Example 1: Let X be a topological space. One can associate to it the following Grothendieck topology. Define $\mathfrak{Off}(X)$ to be the category corresponding to the poset of open subsets of X. That is, objects of $\mathfrak{Off}(X)$ are open subsets of X, while $Hom_{\mathfrak{Off}(X)}(V, U)$ has one element if $V \subset U$ and is empty otherwise. This category, with the usual coverings, is a Grothendieck topology.

Example 2: Let X be a noetherian scheme. We define the *faithfully flat topology* $\mathsf{FF}(X)$ as follows:

- $\mathsf{Cat}\,\mathsf{FF}(X)$ is the category of faithfully flat schemes of finite presentation over X.

- $\mathsf{Cov}\,\mathsf{FF}(X)$ are finite surjective families of maps.

Example 3: Let X be a noetherian scheme. We define the *fpqc topology* $\mathsf{Fpqc}(X)$ by declaring:

- $\mathsf{Cat}\,\mathsf{Fpqc}(X)$ to be the category of schemes over X which are faithfully flat, of finite presentation and quasi-compact.
- $\mathsf{Cov}\,\mathsf{Fpqc}(X)$ to be finite surjective families of maps.

Example 4: Let X be a noetherian scheme and define $\mathsf{FEC}(X)$ as follows:

- $\mathsf{Cat}\,\mathsf{FEC}(X)$ is the category of étale schemes over X. Note that by our definition, see Section 8, this includes finiteness.
- $\mathsf{Cov}\,\mathsf{FEC}(X)$ are finite surjective families of maps.

It is well known (see [2, Example (0.7)]) that for any scheme Z over X the functor $\mathsf{Hom}_{\mathsf{Sch}/X}(-, Z)$ is a sheaf in the faithfully flat topology, where Sch/X denotes the category of schemes over X.

Let $\mathbf{T}$ be a Grothendieck topology. A *presheaf* of sets is a contravariant functor from $\mathbf{T}$ to the category of sets. A presheaf F is called a *sheaf* if for any covering $\{U_i \to U\} \in \mathsf{Cov}\ \mathbf{T}$, the diagram

$$F(U) \to \prod_{i \in I} F(U_i) \rightrightarrows \prod_{ij} F(U_i \times_U U_j)$$

is exact. Recall that exactness means the following: If both parallel arrows map an element $(a_i) \in \prod_{i \in I} F(U_i)$ to the same element in $\prod_{ij} F(U_i \times_U U_j)$, then there exists a unique element $a \in F(U)$ such that $a \mapsto (a_i)$ via the first map. An other way of saying that is that $F(U)$ is the kernel, or limit, of the diagram $F(U_i) \rightrightarrows \prod_{ij} F(U_i \times_U U_j)$. One can also talk about sheaves with values in groups, rings etc.

For a sheaf F and an object U of $\mathsf{Cat}\ \mathbf{T}$ elements of the set $F(U)$ are sometimes called *sections of F over U*. In the case of a topological space X with $U = X$, we will simply say section or global section of F.

1.2 Čech Cohomology

Let $\mathbf{T}$ be a Grothendieck topology. For a given covering $\{U_i \to U\}_{i\in I}$ one constructs iterated fibre products $U_{i_0} \times_U U_{i_1} \times_U \cdots \times_U U_{i_n}$. Then for any presheaf of abelian groups F one can form a cochain complex (see [2, Section 3])

$$\prod_i F(U_i) \to \prod_{ij} F(U_i \times_U U_j) \to \prod_{ijk} F(U_i \times_U U_j \times_U U_k) \to \cdots.$$

The n-th cohomology of this cochain complex is denoted by $H^n(\{U_i \to U\}, F)$ and is called the n-th *Čech cohomology of the covering* $\{U_i \to U\}$ *with coefficients in* F. From the morphism of coverings of a site one obtains a cochain map

$$\begin{tikzcd}
\prod_s F(V_s) \arrow[r] \arrow[d, "f^*"'] & \prod_{st} F(V_s \times_U V_t) \arrow[r] \arrow[d, "f^*"] & \prod_{ijk} F(V_s \times_U V_t \times_U V_r) \arrow[r] \arrow[d, "f^*"] & \cdots \\
\prod_i F(U_i) \arrow[r] & \prod_{ij} F(U_i \times_U U_j) \arrow[r] & \prod_{ijk} F(U_i \times_U U_j \times_U U_k) \arrow[r] & \cdots.
\end{tikzcd}$$

It is well known that any two morphisms $\{U_i \to U\}_{i\in I} \to \{V_s \to U\}_{s\in S}$ yield homotopic chain maps [2, Proposition 3.4] and hence the induced homomorphism in cohomology $H^n(\{V_s \to U\}, F) \to H^n(\{U_i \to U\}, F)$ is independent from our choice of morphism of coverings.

For a given object U let us consider the following poset: Elements are coverings $\{U_i \to U\}_{i\in I}$. One says that $\{U_i \to U\} \geq \{V_s \to U\}$ if there is a morphism of coverings $\{U_i \to U\} \to \{V_s \to U\}$. It follows that the assignment

$$\{U_i \to U\}_{i\in I} \mapsto H^n(\{U_i \to U\}, F)$$

gives rise to a functor on that poset. We let $\check{H}^n(U, F)$ be the colimit of this functor. That is, we define:

$$\check{H}^n(U, F) := \mathsf{colim}_{\{U_i \to U\}} H^n(\{U_i \to U\}, F).$$

These groups are known as the *Čech cohomology* of U with coefficients in a presheaf F.

Observe that the category $\mathcal{P}$ of presheaves on $\mathbf{T}$ with values in the category of abelian groups is an abelian category. A sequence of presheaves

$$0 \to F_1 \to F \to F_2 \to 0$$

is short exact if and only if for any object U, the sequence of abelian groups

$$0 \to F_1(U) \to F(U) \to F_2(U) \to 0$$

is exact. If this is the case, one has a long exact sequence of abelian groups

$$0 \to \check{H}^0(U, F_1) \to \check{H}^0(U, F) \to \check{H}^0(U, F_2) \to \check{H}^1(U, F_1) \to \check{H}^1(U, F) \to \check{H}^1(U, F_2) \to \cdots.$$

1.3 Sheafification

For any abelian presheaf F, the group $\check{H}^0(\{U_i \to U\}_{i\in I}, F)$ coincides with the set of all collections $(a_i)_{i\in I}$, $a_i \in F(U_i)$, such that the image of a_i under the map $F(U_i) \to F(U_i \times_U U_j)$ agrees with the image of a_j under the map $F(U_j) \to F(U_i \times_U U_j)$.

We observe that this definition makes sense even for presheaves with values in sets. Since the ordered set of all coverings is filtered [2, Remark on p.25] the same is true for $\check{H}^0(U, F)$.

A presheaf F is called *separated* if for any $\{U_i \to U\} \in \mathsf{Cov}\ \mathbf{T}$ the natural map $F(U) \to \prod_i F(U_i)$ is injective.

It is clear that a presheaf is a sheaf if and only if for any $\{U_i \to U\} \in \mathsf{Cov}\ \mathbf{T}$ the natural map $F \to \check{H}^0(\{U_i \to U\}_{i\in I}, F)$ is an isomorphism.

For a presheaf (of sets) F, one defines a presheaf F^+ by

$$F^+(U) = \check{H}^0(U, F).$$

We have the following result (see [2, Lemma 1.4]):

Proposition 1.3.1. *i) For any presheaf F, the presheaf F^+ is separated.*

ii) For any separated presheaf F, the presheaf F^+ is a sheaf. Further, the natural map $F \to F^+$ is injective.

It follows that for any presheaf F, the presheaf $\hat{F} = (F^+)^+$ is a sheaf. This is called the *sheafification* of F. The natural morphism $i : F \to F^+$ yields a morphism $i : F \to \hat{F}$, which has the following universal property: For any sheaf G and any morphism of presheaves $f : F \to G$ there is a unique morphism $g : \hat{F} \to G$ such that $f = gi$.

1.4 Sheaf Cohomology

Denote by $\mathcal{S}$ the category of sheaves on a site $\mathbf{T}$ with values in abelian groups. There is an inclusion functor $i : \mathcal{S} \to \mathcal{P}$. It is well known that both categories are abelian categories. The kernel of a morphism of sheaves can be computed in the category of presheaves. Hence for any exact sequence of sheaves

$$0 \to F_1 \to F \to F_2$$

one has an exact sequence

$$0 \to F_1(U) \to F(U) \to F_2(U).$$

However, if $f : F_1 \to F$ is a morphism of sheaves, then the presheaf

$$U \mapsto \mathsf{Coker}(F_1(U) \to F(U))$$

is not a sheaf in general. The sheafification of this presheaf is the cokernel of f in the category $\mathcal{S}$.

The sheafification gives rise to a functor $\hat{} : \mathcal{P} \to \mathcal{S}$, which is exact (see [2, Theorem 2.14]).

A sheaf I is called *injective* if the functor $Hom_{\mathcal{S}}(-, I) : \mathcal{S} \to Ab$ is exact. It is well known [2] that for any sheaf F there is an injective sheaf I and a monomorphism $0 \to F \to I$. It follows that any $F \in \mathcal{S}$ has an *injective resolution*, that is an exact sequence of sheaves

$$0 \to F \to I_0 \to I_1 \to \cdots$$

such that all I_n, $n \geq 0$ are injective sheaves. For any object U one can consider the cochain complex

$$0 \to I_0(U) \to I_1(U) \to \cdots.$$

The n-th cohomology of this complex is denoted by $H^n(U, F)$ and is called the n-th *Grothendieck cohomology* or *cohomology of U with coefficients in a sheaf F*. We have the following well-known facts ([2], [19, Section II.5.4],[44]):

Theorem 1.4.1. *i) For all $n \geq 0$, we have a functor $H^n(U, -) : \mathcal{S} \to \mathsf{Ab}$, given by the assignment $F \mapsto H^n(U, F)$.*

ii) If I is an injective sheaf, then $H^n(U, I) = 0$, for $n > 0$.

iii) For any sheaf F one has

$$F(U) = H^0(U, F).$$

iv) For any short exact sequence of sheaves

$$0 \to F_1 \to F \to F_2 \to 0$$

one has a long exact sequence of abelian groups

$$0 \to H^0(U, F_1) \to H^0(U, F) \to H^0(U, F_2) \to H^1(U, F_1) \to H^1(U, F) \to H^1(U, F_2) \to \cdots.$$

v) For any sheaf F and any $\{U_i \to U\}_{i \in I} \in \mathsf{Cov}\ \mathbf{T}$ there is a spectral sequence

$$E_2^{pq} = H^p(\{U_i \to U\}_{i \in I}; \mathsf{H}^q(F)) \Rightarrow H^{p+q}(U, F).$$

Here H^q is the presheaf given by $V \mapsto H^q(V, F)$. Further, we have the five-term exact sequence:

$$0 \to E_2^{10} \to H^1(U, F) \to E_2^{01} \to E_2^{20} \to H^2(U, F).$$

vi) For any sheaf F there are natural homomorphisms

$$\eta^n : \check{H}^n(U,F) \to H^n(U,F), n \geq 0.$$

The homomorphisms η^0 and η^1 are isomorphisms, while η^2 is a monomorphism.

1.5 Non-abelian Cohomology

Following [25], we can also talk about non-abelian cohomology. Assume G is a presheaf on $\mathbf{T}$ with values in the category of groups. In this case one can still define the cohomology with coefficients in G, but only in dimensions 0 and 1. In fact we already defined the 0-th cohomology. Recall that $H^0(\{U_i \to U\}, G)$ consists of families (g_i) such that $g_i \in G(U_i)$ and for any i and j both elements g_i and g_j have the same image in $G(U_i \times_U U_j)$. One observes that this set is in fact a group. It follows that $\check{H}^0(U,G)$ is also a group.

To define the first cohomology $H^1(\{U_i \to U\}, G)$, one needs to consider families of the form (g_{ij}), where $g_{ij} \in G(U_i \times_U U_j)$, $i,j \in I$. Such a family is called a 1-*cocycle* provided for any $i,j,k \in I$ the following equation

$$g_{ik} = g_{ij} g_{jk}$$

holds in the group $G(U_i \times_U U_j \times_U U_k)$. By putting $i = j = k$ one obtains that $g_{ii} = 1$, and hence from $k = i$ we get $g_{ji} = g_{ij}^{-1}$. Denote the collection of all 1-cocycles by $Z^1(\{U_i \to U\}, G)$. Two 1-cocycles (g_{ij}) and (f_{ij}) are called *cohomologous* provided that there are elements $h_i \in G(U_i)$ such that the equation

$$f_{ij} = h_i g_{ij} h_j^{-1}$$

holds in the group $G(U_i \times_U U_j)$. One checks that this is an equivalence relation and the set of equivalence classes is denoted by $H^1(\{U_i \to U\}, G)$ (see [25, Section 5.1]). The first non-abelian cohomology has in general no group structure, though it still has a special element, which is the class of the trivial family $g_{ij} = 1$ for all i and j.

As in the abelian case one passes to colimits with respect to all coverings to obtain the group $\check{H}^1(U,G)$. In the case where G is a sheaf of groups we write $H^1(U,G)$ for $\check{H}^1(U,G)$. By (vi) of Theorem 1.4.1, this is consistent with the abelian case. Let

$$1 \to G_1 \xrightarrow{i} G \to G_2 \to 1$$

be a short exact sequence of sheaves of groups. Thus for any object U the map $i(U) : G_1(U) \to G(U)$ is injective. Moreover the subgroup $\mathsf{Im}(i(U))$ is a normal subgroup of $G(U)$ and G_2 is the sheafification of the presheaf $U \mapsto \mathsf{Coker}(i(U))$.

In this generality the group $G_2(U)$ acts on the set $H^1(U,G_1)$, as follows from [25, p.75]. Take $x \in G_2(U)$ and take a covering $\{U_i \to U\}_{i\in I}$ such that there are elements $g_i \in G(U_i)$ with the following property: the image of g_i in $G_2(U_i)$ is the same as the image of x in $G_2(U_i)$. Such a covering exists because $G \to G_2$ is an epimorphism of sheaves. Next, take an element $y \in H^1(U,G_1)$, without loss of generality we can assume that y is represented by a 1-cocycle $(y_{ij}) \in Z^1(\{U_i \to U\},G_1)$. Then one defines $x \star y$ to be the class in $H^1(U,G_1)$ represented by the 1-cocycle

$$(g_i y_{ij} g_j^{-1}) \in Z^1(\{U_i \to U\},G_1).$$

It can be checked that this action is well defined. We have the following result [25, Proposition 5.3.1]:

Proposition 1.5.1. *Let*

$$1 \to G_1 \xrightarrow{i} G \to G_2 \to 1$$

be a short exact sequence of sheaves of groups. Then one has an exact sequence of pointed sets

$$1 \to G_1(U) \to G(U) \to G_2(U) \xrightarrow{\delta} H^1(U,G_1) \xrightarrow{i_1} H^1(U,G) \to H^1(U,G_2).$$

Moreover the first two non-trivial homomorphisms are group homomorphisms and for elements $y,z \in H^1(U,G_1)$, one has $i_1(y) = i_2(y)$ if and only if $z = x \star y$ for some element $x \in G_2(U)$.

From this we immediately obtain the following:

Proposition 1.5.2. *Assume*

$$1 \to G_1 \xrightarrow{\alpha} G \xrightarrow{\beta} G_2 \to 1$$

is a split short exact sequence of sheaves of groups. We have a short exact sequence of pointed sets

$$1 \to H^1(U, G_1)_{G_2(U)} \xrightarrow{\alpha_*} H^1(U, G) \to H^1(U, G_2) \to 1$$

Here $H^1(U, G_1)_{G_2(U)}$ is the orbit space of $H^1(U, G_1)$ under the action of the group $G_2(U)$.

1.6 The Case of Topological Spaces

In the case of topological spaces, one can check exactness of a sequence of sheaves 'pointwise'. For this we need to introduce the notion of a stalk of a sheaf over a point.

Let X be a topological space. For a presheaf F and a point $x \in X$ one defines the *stalk* F_x of F over x by

$$F_x := \mathsf{colim}_{x \in U} F(U).$$

The sequence of sheaves

$$F' \to F \to F''$$

is exact in the category $\mathcal{S}$, if and only if the sequence of abelian groups

$$F'_x \to F_x \to F''_x$$

is exact for all $x \in X$. Elements of $F(X)$ are known as *global sections* of F.

There is a big class of sheaves for which the higher cohomology vanishes: A sheaf F on a topological space X is called *flasque* provided for any open subset $U \subset X$, the restriction map $F(X) \to F(U)$ is an epimorphism. We have the following well-known result [19, Theorem 4.4.3]:

Lemma 1.6.1. *If a sheaf of abelian groups F is flasque, then $H^n(U,F) = 0$ for all $n > 0$ and any open subset U.*

We also need a vanishing result due to Grothendieck [19, Theorem 4.15.2]. Recall that a topological space X is called *noetherian* (Zariski space in the terminology of [24]) provided for any closed subsets

$$Y_1 \supseteq Y_2 \supseteq \cdots$$

there exist a natural number n such that $Y_m = Y_{m+1}$ for all $m \geq n$. For such spaces, one writes $\mathsf{dim}(X) \leq n$, if any strictly descending sequence of irreducible subsets contains at most $n+1$ members. Here, a closed subset A is called *irreducible*, if for closed subsets B and C the equality $A = B \cup C$ implies $A = B$ or $A = C$. The following result is well known [24, Theorem 3.6.5].

Theorem 1.6.2. *If X is a noetherian space such that $\mathsf{dim}(X) \leq n$, then for any sheaf F one has $H^i(X,F) = 0$ provided $i > n$.*

1.7 Constant Sheaves

Let A be an abelian group. Denote by P_A the presheaf given by $P_A(U) = A$ for all objects $U \in \mathbf{T}$, where for any morphism $V \to U$, the induced map

$$A = P_A(V) \to P_A(U) = A$$

is the identity. The sheafification $\hat{P}_A$ of P_A is known as the *constant sheaf* associated to the abelian group A. A constant sheaf considered as a presheaf is in general not constant.

In this section we mainly restrict ourselves to topological spaces. In this case, for any constant sheaf $\hat{P}_A$ and an open set U, one has:

$$\hat{P}_A(U) = \{\text{locally constant functions U} \to \text{A}\}.$$

If $\mathcal{U} = \{U_i \subset U\}_{i \in I}$ is an open covering of X, then the *nerve* $N\mathcal{U}$ of the covering $\mathcal{U}$ is a simplicial complex, whose vertices are elements of I. A finite subset J of I forms a simplex of the nerve, provided $\bigcap_{j \in J} U_j \neq \varnothing$. We have the following result:

Lemma 1.7.1. *Let A be an abelian group and $\mathcal{U} = \{U_i \subset U\}_{i \in I}$ any covering of non-empty open sets such that the intersections $\bigcap U_i$ are empty or connected. One has an isomorphism*

$$H^*(\{U_i \to U\}, \hat{P}_A) = H^*(N\mathcal{U}, A)$$

where the cohomology on the right hand side is the cohomology of simplicial complexes.

Proof. Since $\hat{P}_A(V) = A$ if V is a non-empty, connected open subset and $\hat{P}_A(\varnothing) = 0$, it follows that we have the following:

$$\begin{tikzcd}
\prod \hat{P}_A(U_i) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, equal] & \prod \hat{P}_A(U_{ij}) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, equal] & \prod \hat{P}_A(U_{ijk}) \cdots \arrow[d, equal] \\
\prod\limits_{(N\mathcal{U})_0} A \arrow[r, shift left] \arrow[r, shift right] & \prod\limits_{(N\mathcal{U})_1} A \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \prod\limits_{(N\mathcal{U})_2} A \cdots
\end{tikzcd} .$$

Thus the cochain complexes $C^*(\{U_i \to U\}, \hat{P}_A)$ and $C^*(N\mathcal{U}, A)$ are the same and hence so are their cohomologies. $\mathcal{Q.E.D}$

We also need the following well-known result due to Grothendieck (see for example [35, Theorem 1.1] or [19]).

Theorem 1.7.2. *If X is irreducible and F is a constant sheaf, then $H^i(X, A) = 0$ for all $i > 0$.*

Part I

The Geometry of Monoids

Chapter 2

The Kato-Spectrum

In the last years there has been considerable interest in the theory of monoid schemes [10],[11],[13],[14],[26]. It is believed that these objects play a central role in the theory of schemes over 'the field with one element'. See also [30] for more about geometry over the field with one element. As such, the prime ideal spectrum of commutative monoids, as well as localisations, are of importance.

The aim of this chapter is to prove several useful results, about the spectrum of commutative monoids as well as about localisations. Thereby, we essentially define affine monoid schemes. This chapter is organised as follows:

In the first section we will give a brief overview of what a monoid is. In Section 2.2 we will discuss posets and in particular semilattices. Our interest in them is, as we will show in the following section, that the set of prime ideals of a monoid form a natural semilattice.

In Subsection 2.3.1 we will give several ways of explicitly calculating the spectrum of a commutative monoid, especially when it is finitely generated. Lastly we will discuss localisation.

2.1 Introduction

A set M is said to be a *semigroup* if we are given a binary, associative operation $\times : M \times M \to M$. Of course instead of writing $\times(a,b)$ we will write ab. If additionally we have a distinguished element 1_M, such that $1_M m = m 1_M = m$, then we say that M is a *monoid*. We immediately see that a monoid is essentially a group without necessarily having inverses. It is further said to be *commutative*, if $ab = ba$. Throughout this thesis, we will only work with commutative monoids.

Let M and N be two monoids. We say that a map $f : M \to N$ is a *monoid homomorphism* (henceforth referred to as a homomorphism unless there is ambiguity) if $f(1_M) = 1_N$ and $f(ab) = f(a)f(b)$. For simplicity, we will use 1 instead of 1_M from now on. It is obvious that if M and N are monoids, then the set of homomorphisms $\mathsf{Hom}(M, N)$ has a natural monoid structure where $fg(a) = f(a)g(a)$.

The free commutative monoid with one generator is denoted by $\mathbb{N}$ and is isomorphic to $\langle x \rangle := \{1, x, x^2, \cdots, x^n, \cdots\}$, with the obvious operation. An important fact is that every monoid is a quotient of a free monoid, with possibly infinitely many generators. Hence, every monoid can be written as $\langle x_1, \cdots, x_n, \cdots \rangle / K$, where K is a congruence relation. That is, an equivalence relation respecting the monoid structure. It is clear that for every relation R there exists a unique associated congruence relation K_R, which is the smallest congruence relation containing R. A typical way of defining monoids is in terms of generators and relations.

If M and N are commutative monoids, their *tensor product* $M \otimes N$ is defined to be the commutative monoid generated by the elements $m \otimes n$ with $m \in M, n \in N$, modulo the relations:

- $(m_1 m_2) \otimes n = (m_1 \otimes n)(m_2 \otimes n)$
- $m \otimes (n_1 n_2) = (m \otimes n_1)(m \otimes n_2)$
- $1 \otimes n = m \otimes 1 = 1 \otimes 1$.

Note that the last relation does not follow from the first two, as it does in the case of abelian groups. We immediately see that the exponential property

$$\mathsf{Hom}(M \otimes N, S) \cong \mathsf{Hom}(M, \mathsf{Hom}(N, S))$$

holds, where M, N and S are commutative monoids.

Most of the categorical definitions, like limits and colimits, that exist in abelian groups, also exist for monoids and are constructed in the same way. The connection of monoids with groups is of course long known. But it has been observed relatively recently that they also have many similarities with commutative rings. For example, one can consider the analogue of prime ideals for monoids as well, and define $\mathsf{KSpec}(M)$, the Kato-Spectrum of M, with its own Zariski topology and structure sheaf. Using this, we can also define gluing and extend the category of monoids to monoid schemes. This has much intrinsic interest but one of the main uses of it is the fact that we can generalise toric varieties using monoid schemes.

2.2 Posets, Semilattices and Lattices

The results of this section are well known. See for example [20]. The notations we use however, do not follow it.

As already mentioned, abelian groups form a full subcategory of monoids. Now we will discuss the 'opposite' of groups, i.e. a full subcategory where for every element $m \in M$, $m^2 = m$. It is the opposite in the sense that if we were to invert an element m of M, we would get $m = 1$. In other words $M_m \cong M/m$, where M_m is the localisation of M (see Subsection 2.3.2) with respect to m and M/m is the quotient of M by $m \sim 1$. In particular, $M^{Gr} = M_M \cong 1$, where M^{Gr} is the Grothendieck group of M. That is, the localisation of M with respect to all of M. As we will see, this category, called the category of semilattices, is very important to do 'geometry' over monoids.

Definition 2.2.1. *Let P be a partially ordered set, or poset for short. We call an element $a \vee b$ the* join *of a and b if $a, b \leq a \vee b$ and for every element $m \in M$ such that $a, b \leq m$, we have $a \vee b \leq m$. A poset P is said to be a* join semilattice *if for every pair of elements (a, b) the join $a \vee b$ exists.*

Dually we define the *meet* $a \wedge b$ of a and b to be an element satisfying $a, b \geq a \wedge b$, and for every element $m \in M$ such that $a \geq m, b \geq m$, $a \wedge b \geq m$. Hence we can define a *meet semilattice* as well.

Definition 2.2.2. *Let S and S' be two join semilattices. A map $f : S \to S'$, is a morphism of join semilattices if $f(a \vee b) = f(a) \vee f(b)$.*

Similarly we can define the morphisms of meet semilattices as well. It is obvious that these two categories are equivalent, and the equivalence is given by reversing the ordering. A semilattice (meet or join) is called *bounded* if additionally we have a unit. In other words an element 1 such that $1 \wedge a = a$ (or $0 \vee a = a$) for all $a \in M$.

If the semilattice is a join semilattice, then the identity is called the *least element*. In the case of meet semilattices, it is referred to as the *greatest element*.

From now on, whenever we use the term semilattice, we mean bounded. Furthermore unless we specifically care about the orientation of the ordering, we will simply say semilattice instead of join or meet semilattice and use the notations of a join semilattice.

It is immediately obvious that the category of semilattice is equivalent to the category of *idempotent monoids*, i.e. where $m^2 = m$ for every $m \in M$. The operation of the monoid is induced by the join (or meet) of the semilattice. Conversely, the ordering is given by $x \leq y$ if $xy = y$. We have a functor

$$\mathsf{sl} : \mathsf{Monoids} \to \mathsf{Semilattices} \tag{2.1}$$

given by $M \mapsto M/a^2 \sim a$ which is the left adjoint of the inclusion functor. Explicitly, this relation is given as follows: [20, Theorem 1.2 of Chapter III]. We have $a \sim b$ provided there exist natural numbers $m, n \geq 1$ and elements $u, v \in M$, such that $a^m = ub$ and $b^n = va$.

We denote the image of M under sl by M^{sl} and the class of an element $a \in M$ in M^{sl} by $[a]$.

Lemma 2.2.3. *1. The relation $\sim$ is a congruence relation on M.*

2. If X is a join semi-lattice and $f : M \to X$ is a monoid homomorphism, then f can be uniquely decomposed as:

$$\begin{tikzcd} M \arrow[r] \arrow[dr, "f"'] & M^{sl} \arrow[d] \\ & X. \end{tikzcd}$$

Remark 2: Let M be a monoid. It is obvious that the ordering on M^{sl} is induced by the monoid structure on M. In other words $[a] \leq [b] \Leftrightarrow [b] = [ab]$.

Corollary 2.2.4. *Let $f : M \to N$ be a monoid homomorphism. Then the induced map $f_* : M^{sl} \to N^{sl}$ respects ordering.*

Definition 2.2.5. *A poset L is said to be a lattice if it is both a join and a meet semilattice.*

Lemma 2.2.6. *If L is a finite join semi-lattice, then L is a lattice.*

Proof. First, we show that L possesses a greatest element. In fact, if $L = \{x_1, \cdots, x_n\}$, then $x_1 \vee \cdots \vee x_n$ is the greatest element. Now for any $a, b \in L$ we consider the subset

$$Q_{a,b} := \{x \in L | x \leq a, \text{ and } x \leq b\}.$$

It is clear that the least element belongs to $Q_{a,b}$, so it is non-empty. It is also clear that $Q_{a,b}$ is a join subsemilattice of L. Thus, $Q_{a,b}$ has the greatest element $a \wedge b$ and we are done. $\mathcal{Q.E.D}$

Let $f : L \to L'$ be a morphism of finite join semilattices. Then, as we have seen, L and L' are lattices. However, in general, f is not a morphism of lattices.

Definition 2.2.7. *Let P be a poset. Define a topology on P by declaring $U \subset P$ to be open if $x \in U, y \leq x$ implies $y \in U$. This is called a* poset topological space *and we denote it by X_P.*

Note that we can also defined a topology on a poset P by declaring U to be open if $x \in U, y \geq x$ implies $y \in U$. However, the categories obtained, are clearly equivalent. For more on poset topologies, see Subsection 3.1.1.

2.3 Prime Ideals of Commutative Monoids

In this section we will focus on the set of prime ideals of a commutative monoid M. This set, together with its natural topology, is denoted by $\mathsf{KSpec}(M)$ after K.Kato, who introduced it in [26]. It plays the same role in the theory of monoid schemes, as the classical set of prime ideals $\mathsf{Spec}(R)$, of a commutative ring R, does in the theory of (ring) schemes. We will first show that $\mathsf{KSpec}(M)$ has an additional structure compared to its classical counterpart. Namely, the union of (prime) ideals is again a (prime) ideal. We will prove several important results for $\mathsf{KSpec}(M)$, most notably Theorems 2.3.5 and 2.3.10.

Having given an effective way of calculating $\mathsf{KSpec}(M)$ for finitely generated monoids, we will move on to localisations. We will show in Theorem 2.3.18 that for a finitely generated monoid, every localisation can be assumed to be a localisation by the multiplicative subset generated by an element, or equivalently, the localisation at a prime ideal. This result first appears in [12, Lemmas 1.1 and 1.3]. After this, we will prove that if we have two elements f and g, with $f \cong g$ under the image $\mathsf{sl} : M \to M^{sl}$, they will define the same localisation.

Definition 2.3.1. *A subset $\mathfrak{a} \subset M$ of a monoid M is called an* ideal*, provided for any $a \in \mathfrak{a}$ and $x \in M$ one has $ax \in \mathfrak{a}$. For an element $a \in M$, we let (a) be the* principal ideal *aM.*

Definition 2.3.2. *An ideal $\mathfrak{p}$ is called* prime*, provided $\mathfrak{p} \neq M$ and the complement of $\mathfrak{p}$ in M is a submonoid.*

Thus, an ideal $\mathfrak{p}$ is prime, if and only if $1 \notin \mathfrak{p}$, and if $xy \in \mathfrak{p}$, either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. We let $\mathsf{KSpec}(M)$ be the set of all prime ideals of M. It is equipped with a topology, where the sets

$$D(a) = \{\mathfrak{p} \in \mathsf{KSpec}(M) | a \notin \mathfrak{p}\}$$

are the base of open sets. Here a is an arbitrary element of M, see [13, p.92],[26],[30]. Observe that if $f : M_1 \to M_2$ is a homomorphism of monoids, for any prime ideal $\mathfrak{p} \in \mathsf{KSpec}(M_2)$, the pre-image $f^{-1}(\mathfrak{p})$ is a prime ideal of M_1. Hence, any monoid homomorphism $f : M_1 \to M_2$ gives rise to a continuous map

$$f^{-1} : \mathsf{KSpec}(M_2) \to \mathsf{KSpec}(M_1); \quad \mathfrak{p} \mapsto f^{-1}(\mathfrak{p}).$$

Unlike in the case of rings, we have the following easy, but important facts.

Lemma 2.3.3. *Let M be a commutative monoid, then:*

1. *The union of ideals is an ideal.*
2. *The union of prime ideals is a prime ideal.*
3. *The empty set $\varnothing$ is a prime ideal.*
4. *The maximal ideal $\mathfrak{m} = \bigcup_{\mathfrak{q} \in \mathsf{KSpec}(M)} \mathfrak{q} = M \setminus M^*$, satisfies $\mathfrak{m} \cup \mathfrak{p} = \mathfrak{m}$ for all $\mathfrak{p}$,*

5. *If $\mathfrak{a}$ is an ideal, the set of prime ideals contained in $\mathfrak{a}$ has a greatest element, denoted $\mathfrak{m}_{\mathfrak{a}}$.*

6. $\mathsf{KSpec}(M)$ *forms a lattice.*

Proof. 1. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals. Take $u \in \mathfrak{a} \cup \mathfrak{b}$ and $m \in M$. Since $u \in \mathfrak{a} \cup \mathfrak{b}$, without loss of generality $u \in \mathfrak{a}$, hence $mu \in \mathfrak{a} \subset \mathfrak{a} \cup \mathfrak{b}$. So $\mathfrak{a} \cup \mathfrak{b}$ is again an ideal.

2. Assume $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals. To see that their union is again prime, consider $xy \in \mathfrak{p} \cup \mathfrak{q}, x \notin \mathfrak{p} \cup \mathfrak{q}$. Without loss of generality $xy \in \mathfrak{p}$. Since $x \notin \mathfrak{p} \cup \mathfrak{q} \Rightarrow x \notin \mathfrak{p}$. Hence we have $xy \in \mathfrak{p}, x \notin \mathfrak{p} \Rightarrow y \in \mathfrak{p} \subset \mathfrak{p} \cup \mathfrak{q}$, proving the assertion.

3,4. Obvious.

5. It is easily seen that $\mathfrak{m}_{\mathfrak{a}} = \bigcup_{\mathfrak{p} \subset \mathfrak{a}} p$, $\mathfrak{p} \in \mathsf{KSpec}(M)$ does the trick.

6. This is a result of the above statements, with the two operations being

$$\mathfrak{p} \vee \mathfrak{q} \mapsto \mathfrak{p} \cup \mathfrak{q}, \ \mathfrak{p} \wedge \mathfrak{q} \mapsto \mathfrak{m}_{\mathfrak{p} \cap \mathfrak{q}}.$$

$\mathcal{Q}.\mathcal{E}.\mathcal{D}$

It should be pointed out that for a monoid homomorphism $f : M_1 \to M_2$ the induced map

$$f^{-1} : \mathsf{KSpec}(M_2) \to \mathsf{KSpec}(M_1)$$

is only a homomorphism of join semilattices (i.e. the operation induced by union), even when both M_1 and M_2 are finitely generated. Hence we will still refer to the associated lattice of a monoid as semilattice, to avoid confusion.

Lemma 2.3.4. $\mathsf{KSpec}(M)$ *is a topological monoid.*

Proof. Take a principal open subset $D(f)$ of $\mathsf{KSpec}(M)$ and consider its preimage via the operation map $\cup : \mathsf{KSpec}(M) \times \mathsf{KSpec}(M) \to \mathsf{KSpec}(M)$. Clearly

$$\cup^{-1}(D(f)) = (D(f), D(f))$$

which is open in the product topology. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Example 5: Here we give a small example of how the Kato-spectrum of a commutative monoid looks like, with its lattice structure.

$$
\begin{tikzcd}
 & & & & (x,y) & \\
 & (x) & & (x) \arrow[ur] & & (y) \arrow[ul] \\
\varnothing & \varnothing \arrow[u] & & & \varnothing \arrow[ul] \arrow[ur] & \\
\mathsf{KSpec}(\mathbb{Z}) & \mathsf{KSpec}(\mathbb{N}) & & & \mathsf{KSpec}(\mathbb{N}^2). &
\end{tikzcd}
$$

2.3.1 Reduction Theorem

Recall that for any monoid M there is a semilattice M^{sl} and the quotient homomorphism $f : M \to M^{sl} = M/\sim$ (see lemma 2.2.3), universal among all homomorphism into semilattices. Denote by $\mathbb{I}$ the topological monoid with the two elements $\{0,1\}$ and obvious multiplication. The open sets are $\{\varnothing, 1, \mathbb{I}\}$.

Theorem 2.3.5. *Let M be a commutative monoid. Then:*

1. *The canonical homomorphism $f : M \to M^{sl}$ yields the isomorphism of topological monoids*

$$f^{-1} : \mathsf{KSpec}(M^{sl}) \xrightarrow{\sim} \mathsf{KSpec}(M).$$

2. *There is an isomorphism of topological monoids*

$$\mathsf{Hom}(M, \mathbb{I}) \cong \mathsf{KSpec}(M).$$

The topology on the LHS is induced by the product topology on $\prod_{m \in M} \mathbb{I}$.

3. *There is an isomorphism of topological monoids*

$$M \otimes \mathbb{I} \cong M^{sl},$$

where the topology in M^{sl} is induced by the ordering (see Definition 2.2.7).

Proof. 1. Surjectivity of f implies that the induced morphism $\mathsf{KSpec}(M^{sl}) \to \mathsf{KSpec}(M)$ is injective. To see that it is also surjective, take a prime ideal $\mathfrak{p}$ of M. Since f is surjective, $f(\mathfrak{p}) = \mathfrak{q}$ is an ideal of M^{sl}.

Our first claim is that $1 \notin \mathfrak{q}$. For this, assume $1 \in \mathfrak{q}$. Hence, there exists $a \in \mathfrak{p}$ such that $f(a) = 1$. That is, $1 \sim a$ where $\sim$ is the congruence given in lemma 2.2.3. We get that a must be invertible, which contradicts the condition that $\mathfrak{p}$ is prime.

To see that $\mathfrak{q}$ is a prime ideal of M^{sl}, suppose $q(x)q(y) \in \mathfrak{q}$. Hence, there exists an element $a \in \mathfrak{p}$, such that $xy \sim a$. So $(xy)^n = au$ for some $n \geq 1$. It follows that $x^n y^n \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime, we see that $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. This means that $q(x)$ or $q(y)$ belongs to $\mathfrak{q}$, implying that $\mathfrak{q}$ is a prime ideal. It remains to show that

$$\mathfrak{p} = f^{-1}(\mathfrak{q}) = f^{-1}(f(\mathfrak{p})).$$

Take an element $x \in f^{-1}(f(\mathfrak{p}))$. There is an element $b \in \mathfrak{p}$ such that $f(x) = f(b)$. Thus $x \sim b$ i.e. $x^n = bv \in \mathfrak{p}$, which implies that $x \in \mathfrak{p}$. So we have proven that $\mathfrak{p} \supset f^{-1}(f(\mathfrak{p}))$. Since $\mathfrak{p} \subset f^{-1}(f(\mathfrak{p}))$ is obvious, the result follows.

For every $a \in M$, $f(D(a)) = D(f(a))$, f^{-1} is continuous. Conversely,

$$f^{-1}(D([a])) = \bigcup_{f(b) \in [a]} D(b),$$

and the result follows. Indeed as we will show in 2.3.15, we don't need to take the union over all such b. It suffices to just take any one.

2. Since $\mathbb{I}$ is a monoid we can talk about $\mathsf{KSpec}(\mathbb{I})$. It is obvious that the only two prime ideals of $\mathbb{I}$ are $\{\varnothing, 0\}$. Since the preimage of a prime ideal is prime, we have a homomorphism $u : \mathsf{Hom}(M, \mathbb{I}) \to \mathsf{KSpec}(M)$, given by $u(f) = f^{-1}(0)$. On the other hand, we have the map $v : \mathsf{KSpec}(M) \to \mathsf{Hom}(M, \mathbb{I})$, given by $v(\mathfrak{p}) = f_v$ where $f_v(x) = \begin{cases} 0, & x \in \mathfrak{p} \\ 1, & x \notin \mathfrak{p} \end{cases}$. Since $M \setminus \mathfrak{p}$ is a submonoid of M, this

map is a monoid homomorphism as well. It is straight forward to see that they are inverse to each other.

By the definition of the product topology, subsets of the form

$$P(a) = \{f : M \to \mathbb{I} \,|\, f(a) = 1\}, \quad a \in M$$

form a prebase of the topology on $\prod_{m \in M} \mathbb{I}$. Here f is a map of sets. Hence, a pre-base of the induced topology on $\mathsf{Hom}(M, \mathbb{I})$ is given by the subsets

$$U(a) = \{f \in \mathsf{Hom}(M, \mathbb{I}) \,|\, f(a) = 1\}.$$

As $U(a) \cap U(b) = U(ab)$, these subsets form a basis. Since $u(U(a)) = D(a)$, the result follows.

3. Let S be a semilattice. We have

$$\mathsf{Hom}(M \otimes \mathbb{I}, S) \cong \mathsf{Hom}(M, \mathsf{Hom}(\mathbb{I}, S)) \cong \mathsf{Hom}(M, S) \cong \mathsf{Hom}(M^{sl}, S).$$

The first isomorphism comes from the fact that the tensor product is the adjoint of the Hom-functor. For the second isomorphism we use the fact that $\mathbb{I}$ is the free object with one generator in the category of semilattices. Now Yoneda's lemma gives us the desired result.

$\mathcal{Q.E.D}$

Corollary 2.3.6. *Let $f : M \to N$ be a morphism of monoids.*

- *If f is surjective, then the associated map $f^{-1} : \mathsf{KSpec}(N) \to \mathsf{KSpec}(M)$ is injective.*
- *If f is surjective, then the associated map $f^{sl} : M^{sl} \to N^{sl}$ is surjective.*

Proof. This follows from Theorem 2.3.5, part 2 and 3 respectively, and the fact that $\mathsf{Hom}(-, \mathbb{I})$ maps epimorphisms to monomorphisms and $- \otimes \mathbb{I}$ is right exact. $\mathcal{Q.E.D}$

Corollary 2.3.7. *Let I be a small category and $M : I \to \{\text{comutative monoids}\}$ a functor. Then we have a natural isomorphism of topological monoids:*

$$\mathsf{KSpec}(\operatorname{colim}_i M_i) \cong \lim_i \mathsf{KSpec}(M_i).$$

In particular,

$$\mathsf{KSpec}(M \times N) \cong \mathsf{KSpec}(M) \times \mathsf{KSpec}(N).$$

Proof. This follows straight from part 2 of Theorem 2.3.5, and the fact that $\mathsf{Hom}(-, A)$ sends colimits to limits. The second assertion is due to the fact that for commutative monoids, finite products and coproducts agree. $\mathcal{Q.E.D}$

We also obtain the following fact, which sharpens Lemma 4.2 in [14].

Corollary 2.3.8. *Let B be a submonoid of A and assume for any element $a \in A$ there exist a natural number n such that $a^n \in B$. Then*

$$\mathsf{KSpec}(A) \to \mathsf{KSpec}(B)$$

is a bijection.

Proof. It suffice to show that $B^{sl} \to A^{sl}$ is an isomorphism. Take any element $a \in A$. Since $a \sim a^n$ for all n we see that the map in question is surjective. Now take two elements b_1, b_2 in B and assume $b_1 \sim b_2$ in A. Then there are $u, v \in A$ such that $b_1^k = ub_2$ and $b_2^m = vb_1$. Take r such that $u_1 = u^r \in B$ and $v_1 = v^r \in B$. Then $b_1^{kr} = u_1 b_2^r$ and $b_2^r = v_1 b_1^r$. Thus $b_1 \sim b_2$ in B and we are done. $\mathcal{Q.E.D}$

Lemma 2.3.9. *Let M be a finitely generated monoid. Denote by $L(M, \mathfrak{p}) = \{\mathfrak{q} | \mathfrak{q} \subset \mathfrak{p}\}$ the sublattice of $\mathsf{KSpec}(M)$ consisting of sub prime ideals of $\mathfrak{p}$. Then principal open sets are in a one-to-one correspondence with open sets of the form $L(M, \mathfrak{p})$.*

Proof. First observe that since there is an isomorphism of topological monoids $\mathsf{KSpec}(M) \to \mathsf{KSpec}(M^{sl})$, it suffices to prove this assertion when M is a finitely generated lattice. That is, when M is a finite lattice. One side of this correspondence is straight forward. If $D(f)$ is a principal open set, $L(M, \mathfrak{p}_f)$ does the trick. Here where $\mathfrak{p}_f = \bigcup_{\mathfrak{p} \in D(f)} \mathfrak{p}$. The other side requires the finiteness condition.

Let $\mathfrak{p}$ be a prime ideal of M and consider $M \setminus \mathfrak{p}$. By the definition of prime ideals, this is a submonoid of M, and as such, a finite lattice. Take its maximal element $m_{\mathfrak{p}} = \prod_{m_i \notin \mathfrak{p}} m_i$. We define

$$L(M, \mathfrak{p}) \mapsto D(m_{\mathfrak{p}}) \subset \mathsf{KSpec}(M).$$

Clearly $L(M, \mathfrak{p}) \subset D(m_{\mathfrak{p}})$. For the other side, take $\mathfrak{q} \in D(m_{\mathfrak{p}})$. Then

$$m_{\mathfrak{p}} = \prod_{m_i \notin \mathfrak{p}} m_i \notin \mathfrak{q} \Leftrightarrow m_i \notin \mathfrak{q}$$

for every i. Hence, for every element $m_i \in M \setminus \mathfrak{p}$, we have

$$m_i \in M \setminus \mathfrak{q} \Rightarrow M \setminus \mathfrak{q} \supset M \setminus \mathfrak{p} \Leftrightarrow \mathfrak{q} \subset \mathfrak{p} \Leftrightarrow \mathfrak{q} \in L(M, \mathfrak{p}).$$

$\mathcal{Q.E.D}$

Theorem 2.3.10. *Let M be a finitely generated monoid. There exists a (non-functorial) isomorphism of topological monoids*

$$\alpha_{sl} : \mathsf{KSpec}(M) \cong M^{sl},$$

where the topology in M^{sl} is induced by the ordering.

Proof. First recall the classical result [20, Chapter III, Lemma 1.1] that $D(f) = D(g)$ if and only if there are $n, m \in \mathbb{N}$ and $x, y \in M$ such that $f^n = gy$ and $fx = g^m$. This is the same as the congruence given on page 22, in order to obtain M^{sl}. Hence, there is a one-to-one correspondence between principal open sets and elements of M^{sl}. We have already shown in Lemma 2.3.9, that prime ideals themselves are in a one-to-one relation with principal open sets. This implies the result. The topology is respected since it's induced by the ordering in both sides, and all the isomorphisms given above, respect ordering. $\mathcal{Q.E.D}$

Remark 3: For an alternative proof see [38, Corollary 3.9]. There we also construct the above map more explicitly, and show that it is given as follows : To a prime ideal of M, we associate the maximal element of M^{sl} whose intersection with $\mathfrak{p}$ is empty. Conversely, to an element $[f]$ of M^{sl} we associate the biggest prime ideal $\mathfrak{p}$ not containing $[f]$. That is, the union of all prime ideals not containing $[f]$.

As we have shown in Lemma 2.3.3, the Kato-spectrum of a monoid is a lattice and as such again a monoid. We can iterate the functor KSpec and we denote by $\mathsf{KSpec}^2(M)$ the 2-fold composite of this functor. We have the following result:

Corollary 2.3.11. *Let M be a finitely generated monoid. There is a (non-functorial) isomorphism*

$$\mathsf{KSpec}^2(M) \cong \mathsf{KSpec}(M).$$

Proof. This follows straight from the above theorem and the fact that the functor $\mathsf{sl} : \mathsf{Monoids} \to \mathsf{Semilattices}$ (given in formula 2.1 on page 22) is stable under iteration. $\mathcal{Q.E.D}$

Corollary 2.3.12. *Let M be a finitely generated monoid with n generators. The number of elements of $\mathsf{KSpec}(M)$ is bounded by 2^n.*

Proof. As shown, $\mathsf{KSpec}(M)$ is isomorphic to M^{sl}, which is itself a quotient of M. Thus, it has at most n generators. Since $m^2 = m$ in M^{sl}, every element is just a distinct combination of generators, and so their number is bounded by 2^n. $\mathcal{Q.E.D}$

Example 6: Let $M = \langle a, b, c, d \rangle / \{ab = b,\ cd = d,\ bc = ad\}$ be a commutative monoid. By Theorem 2.3.10 we have know that M^{sl} is M, modulo the relation $m^2 = m$. Hence $M^{sl} = \{[1], [a], [b], [c], [d], [ac], [ad]\}$. The only nontrivial operation here is that $[ac][ad] = [ad]$. The semilattice structure is given below, with the arrows indicating the ordering.

$$M^{sl}$$

$$\begin{tikzcd}
 & {[ad]} & \\
{[b]} \arrow[ur] & {[ac]} \arrow[u] & {[d]} \arrow[ul] \\
{[a]} \arrow[u] \arrow[ur] & & {[c]} \arrow[ul] \arrow[u] \\
 & {[1]} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

$$\mathsf{KSpec}(M)$$

$$\begin{tikzcd}
 & (a,b,c,d) & \\
(a,b,d) \arrow[ur] & & (b,c,d) \arrow[ul] \\
(a) \arrow[u] & (b,d) \arrow[ul] \arrow[ur] & (c) \arrow[u] \\
 & \varnothing \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Here $\alpha_L(ad) = \varnothing, \alpha_L(b) = (c)$, etc. We also see an obvious, but important fact. While $\mathsf{KSpec}(M)$ is just M^{sl} with the inverted ordering, they are very different in terms of generators and relations. We have $M^{sl} = \langle a,b,c,d\rangle / \{ab = b, cd = d, bc = ad\}$ and $\mathsf{KSpec}(M) = \langle x,y,z\rangle / \{xz = xyz\}$, in the category of semilattices.

2.3.2 Localisation of Monoids

It is an obvious fact that we have the inclusion functor

$$\mathsf{Monoids} \to \mathsf{Abelian\ Groups}.$$

This functor has both left and right adjoints. The right adjoint maps a monoid to its invertible elements, which is clearly an abelian group. The left adjoint is constructed via localisation. Let M be a monoid and $N \subset M$ a submonoid. We define the *localisation* of M by N, denoted M_N as follows: Elements of M_N are equivalence classes of pairs (m,n), $m \in M$, $n \in N$, where $(m,n) \sim (m',n')$ if and only if there exists an element $r \in N$ such that $rmn' = rm'n$. In the case when $N = M$, M_N is a group, called the Grothendieck Group of M. It is universal in the following sense: We have a homomorphism $g : M \to M^{Gr}$, such that for any other homomorphism $f : M \to G$, where G is an abelian group, we have a homomorphism $h : M^{Gr} \to G$ with $f = h \circ g$. An other way of saying that, is the functor

$$Gr : \textsf{Monoids} \to \textsf{Abelian Groups}$$

is the left adjoint of the forgetful functor.

An important case is the localisation with an element $f \in M$. For this, take an element $f \in M$, and consider the submonoid $\langle f \rangle \subset M$ generated by f. The localisation of M by the submonoid $\langle f \rangle$, is denoted by M_f rather then $M_{\langle f \rangle}$.

Proposition 2.3.13. *Let $f \in M$. Then $M_f \cong M_{[f]}$.*

Proof. First recall that $[f] = [g]$ if there exists $n, m \in \mathbb{N}, u, v \in M$ such that $f^n = gu$ and $g^m = fv$. We clearly have $\langle f \rangle \subset [f]$ and hence we can define the natural map $M_f \to M_{[f]}$ given by $(r, f^k) \mapsto (r, f^k)$, $r \in M$, $k \in \mathbb{N}$.
Injectivity: Say $(r, f^k) \cong_{M_{[f]}} (r', f^{k'})$. This is to say there exists $g \in [f]$ such that $grf^{k'} = gr'f^k$. But since $g \in [f]$, $[g] = [f]$ and so we have $gu = f^n$. From this we get:

$$\begin{aligned} gurf^{k'} &= gur'f^k \\ \Leftrightarrow f^n r f^{k'} &= f^n r' f^k \\ \Leftrightarrow (r, f^k) &\sim_{M_f} (r', f^{k'}). \end{aligned}$$

Surjectivity: Take a general element $(r, g) \in M_{[f]}$. Since $rug = rf^n$, we immediately see that $(ru, f^n) \sim_{M_{[f]}} (r, g)$, where (ru, f^n) is clearly in the image. $\mathcal{Q.E.D}$

Corollary 2.3.14. *Let $f, g \in M$, such that they define the same class in M^{sl}, i.e. $[f] = [g]$. Then $M_f \cong M_g$.*

Proof. This follows straight from Proposition 2.3.13 since $M_f \cong M_{[f]} = M_{[g]} \cong M_g$. $\mathcal{Q.E.D}$

Lemma 2.3.15. *Let M be a monoid and $N \subset M$ be a multiplicative subset, such that $N \subset [f]$, for some $f \in M$. Then $M_N \cong M_f$.*

Proof. By Corollary 2.3.14 we can assume that $f \in N$ and as such $\langle f \rangle \subset N \subset [f]$. Hence we have the following canonical homomorphisms

$$M_f \to M_N \to M_{[f]}.$$

By Proposition 2.3.13, we know that the composition is an isomorphism. This immediately implies that $M_f \to M_N$ is injective. To show surjectivity, it suffices to show that $M_N \to M_{[f]}$ is injective as well. So let $(r,s) \sim_{M_{[f]}} (r',s'), r \in M, s \in N$. That is to say, we have an element $g \in [f]$ such that $grs' = gr's$. Multiplying both sides by u yields $f^n rs' = f^n r's$ and since $\langle f \rangle \subset N$, we have $(r,s) \sim_{M_N} (r',s')$. $\mathcal{Q.E.D}$

Lemma 2.3.16. *For a subset S of M, we denote by $\langle S \rangle$ the associated multiplicative subset. Let S_1 and S_2 be subsets of M. Then*

$$(M_{\langle S_1 \rangle})_{\langle S_2 \rangle} \cong M_{\langle S_1 \cup S_2 \rangle}.$$

Proof. The proof of this is straightforward. We will just mention that the map is given by $((r,s_1),s_2) \mapsto (r, s_1 s_2)$. $\mathcal{Q.E.D}$

Just like in the classical case, we can also localise at a prime ideal. For this, take the complement $M \setminus \mathfrak{p}$ of $\mathfrak{p}$. It is clearly a submonoid and so we can localise M with respect to it.

Lemma 2.3.17. *Let M be a monoid and assume $[f] \leq [g]$. Then $M_{[g]} \cong M_{[f] \cup [g]}$.*

Proof. We have

$$M_{[f]\cup[g]} = M_{\langle [f]\cup[g] \rangle} \cong (M_{[f]})_{[g]} \cong M_{fg} \cong M_{[fg]} \cong M_{[g]}.$$

$\mathcal{Q.E.D}$

We note that the following theorem is already in [12] as Lemmas 1.1 and 1.3.

Theorem 2.3.18. *Let M be a finitely generated monoid.*

i) For every multiplicative subset $N \subset M$, there exists an element f_N, such that $M_{f_N} \cong M_N$.

ii) For every prime ideal $\mathfrak{p} \subset M$, we have an isomorphism $M_{\mathfrak{p}} \cong M_{\alpha_{sl}(\mathfrak{p})}$. Here α_{sl} is the isomorphism defined in Theorem. 2.3.10.

Proof. i) Since $[1] \leq [f]$ for all $f \in M$, by Lemma 2.3.17, we can assume that N is a submonoid. We have the composition $\Phi : N \subset M \xrightarrow{sl} M^{sl}$. Since M^{sl} is finite, so is the sublattice $\Phi(N) \subset M^{\mathsf{sl}}$. Denote the preimage of the elements of $\Phi(N)$ under sl by $[f_i]$ and the preimage of its maximal element by $[f_N] = [f_1 \times \cdots \times f_n]$. Further let $N_i = N \cap [f_i]$. Clearly $N \subset \bigcup [f_i]$ and so $N = \bigcup N_i$. We have

$$M_N = M_{\bigcup N_i} \cong ((M_{N_1})_{N_2})_{\cdots N_n} \cong ((M_{f_1})_{f_2})_{\cdots f_n} \cong M_{f_N}.$$

Here the second equality is coming from Lemma 2.3.16 and the third from Lemma 2.3.15.

ii) This essentially follows from Remark 3 on page 31 and Proposition 2.3.16, since localising with respect to the submonoid $M \setminus \mathfrak{p}$ is canonically isomorphic to localisation by the maximal element of $(M)^{sl}$, whose intersection with $\mathfrak{p}$ is empty.

$\mathcal{Q.E.D}$

Note that the second statement also holds the other way. Namely, that for every element $[f]$ of M^{sl} (and hence for every element of M), we can find a prime ideal $\mathfrak{p}_f$ such that $M_{[f]} \cong M_{\mathfrak{p}_f}$. This is due to Theorem 2.3.10.

Chapter 3

Cohomology of monoid schemes

The aim of this chapter is to study the cohomology theory of monoid schemes in general and apply it to vector- and line bundles. We will prove that any vector bundle over any separated monoid scheme is a coproduct of line bundles. This fact generalises the main result of [5, Theorem 2.6], where they proved this for the special case when $X = \mathbb{P}^n$. It should be said that our methods are different from the ones used in the above. We use cohomological machinery, which could have other applications beyond monoid schemes.

This result shows that vector bundles, being $H^i(X, \mathcal{O}_X^*)$, are not very interesting in the case when $i \geq 2$. In this case when $i = 1$ however, things are very different. As in the classical case, $H^1(X, \mathcal{O}_X^*)$ forms an abelian group under the tensor product. It is called the Picard group and is denoted by $\mathsf{Pic}(X)$. We investigate the relationship between line bundles over a monoid scheme X and line bundles over its geometric realisation $k[X]$, where k is a commutative ring. We prove that if k is an integral domain (resp. principal ideal domain) and X is a cancellative and torsion free (resp. seminormal and torsion-free) monoid scheme, then the induced map $\mathsf{Pic}(X) \to \mathsf{Pic}(k[X])$ is a monomorphism (resp. isomorphism).

The assignment $X \mapsto \mathsf{Pic}(\mathsf{X})$ gives rise to a contravariant functor from the category of monoid schemes to the category of abelian groups. We will prove that Pic respects finite products, in other words the natural map

$$\mathsf{Pic}(X) \oplus \mathsf{Pic}(Y) \xrightarrow{\sim} \mathsf{Pic}(X \times Y),$$

is an isomorphism for separated monoid schemes X and Y.

Next we will introduce the notion of s-cancellative monoids. They are monoids for which the equality $ax = ay$ implies that $(xy)^n x = (xy)^n y$ for some natural number n. By taking $n = 0$, we see that every cancellative monoid is s-cancellative. A monoid scheme is s-cancellative if it is obtained by gluing s-cancellative monoids. This class is important since it is the biggest class of monoids where the subgroup of invertible elements M^* of M map injectively in its group of fractions, denoted by M^{Gr}. As we will see in Section 3.5, this will enable us to embed $\mathcal{O}_X^*$ in a constant sheaf. We develop the theory of s-divisors and we prove that for an s-cancellative monoid scheme X, the group $\mathsf{Pic}(X)$ can be described in terms of s-divisors. For cancellative monoid schemes, s-divisors agree with the Cartier divisors.

Lastly, we introduce a class of monoid schemes, called s-smooth monoid schemes, which includes the class of all smooth monoid schemes, and we will prove that for them $H^i(X, \mathcal{O}_X^*) = 0$ for all $i \geq 2$.

This chapter is organised as follows: In the first section we will discuss posets and the topological spaces that they define. These are important, as the underlying topological spaces of monoid schemes of finite type (see Definition 3.1.6) are merely finite posets, and then proceed to clarify the relationship between sheaf cohomology and poset cohomology.

In Section 3.2, using a cohomological argument, we will prove that over separated monoid schemes, any vector bundle is a coproduct of line bundles. This already implies that vector bundles are less interesting for monoid schemes and as such we focus our attention on line bundles. Section 3.3 considers the relationship between line bundles over monoid schemes and its realisations, and shows that in many cases, we actually have an isomorphism.

Recall that the isomorphism classes of line bundles forms an abelian group, which is isomorphic to the Picard group. We show in the next section that the functor Pic is additive in the monoid world.

In Section 3.5, we translate the classical theory of Cartier divisors for monoid schemes and we show that for cancellative monoid schemes line bundles, up to isomorphisms, can be described using Cartier divisors. Afterwards we generalise these results and prove that for s-cancellative monoid schemes line bundles can be classified using s-divisors.

In the last two sections we define and study s-smooth monoids and monoid schemes and we will prove that the cohomology $H^i(X, \mathcal{O}_X^*)$ vanishes for them. Finally we give several examples of s-smooth monoids which are not smooth.

3.1 Basic Facts about Monoid Schemes

In this section we will define monoid schemes and show that instead of working with sheaves over a topological space, we can work with functors from a poset, see Lemma 3.1.7. This greatly simplifies the cohomology of monoid schemes. Indeed from this we will obtain, amongst other things, that the Grothendieck cohomology is equivalent to the significantly simpler Čech cohomology.

3.1.1 Monoid Schemes and the Poset Topology

The results of this subsection, other then Lemma 3.1.7, are well known with citations given below.

Let M be a commutative monoid. As we have shown in Section 2.3, there is a functorial way of assigning a topological space $\mathsf{KSpec}(M)$ to M. But more is true. Recalling the results obtained in [13, Section 2.1] [12, p.4], one can also define a sheaf of monoids on $\mathsf{KSpec}(M)$. Just like in classical algebraic geometry, we assign the localisations M_f to the principal open subsets $D(f) \subset \mathsf{KSpec}(M)$ and extend this in the natural way. It is called an *affine scheme*.

Definition 3.1.1. *A* monoid scheme *is a pair* $(X, \mathcal{O}_X)$*, where* X *is a topological space, and* $\mathcal{O}_X$ *is a sheaf of monoids on* X *that is locally affine. In other words, it is locally isomorphic to* $(\mathsf{KSpec}(M), \mathcal{O}_{\mathsf{KSpec}(M)})$ *for a monoid* M.

For simplicity, when there is no ambiguity, we will write X instead of $(X, \mathcal{O}_X)$. If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are two monoid schemes, then $f : X \to Y$ is a *morphism of monoid schemes*, if it is a morphism of sheaves that is additionally *local*. In other words, for every point $x \in X$, the induced morphism $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$, maps non-invertible elements of $\mathcal{O}_{Y,f(x)}$, to non-invertible elements of $\mathcal{O}_{X,x}$. We have the bijection

$$\mathsf{Hom}(M, N) \cong \mathsf{Hom}_{\mathsf{MSchm}}(\mathsf{KSpec}(N), \mathsf{KSpec}(M)),$$

where $\mathsf{KSpec}(M)$ denotes the affine scheme (i.e. the space together with the structure sheaf) of M.

Theorem 3.1.2. *Products exist in the category of monoid schemes. Furthermore:*

i) The underlying topological space of the product, is the product of the underlying topological spaces;

ii) $\Gamma(X \times Y, \mathcal{O}_{X\times Y}) \cong \Gamma(X, \mathcal{O}_X) \times \Gamma(Y, \mathcal{O}_Y)$.

Here $(X, \mathcal{O}_X)$ *and* $(Y, \mathcal{O}_Y)$ *are monoid schemes, and* Γ *is the global section functor.*

Proof. The first assertion, that products exist, is shown in [13, p.10]. Part i) is shown in [12, Proposition 3.1].

To see assertion ii) we cover X and Y with affine open subsets $X = \bigcup U_\alpha$ and $Y = \bigcup V_\beta$. By Part i) we have $X \times Y = \bigcup_{\alpha,\beta} U_\alpha \times U_\beta$. Hence $\Gamma(X \times Y, \mathcal{O}_{X\times Y}) =$

$$\mathcal{O}_{X\times Y}(X \times Y) \cong \ker[\prod_{\alpha,\beta} \mathcal{O}_{X\times Y}(U_\alpha \times U_\beta) \rightrightarrows \prod_{\alpha,\beta,\alpha',\beta'} \mathcal{O}_{X\times Y}(U_\alpha \times U_\beta) \cap (U_{\alpha'} \times U_{\beta'})].$$

The fact that the global sections commute with products in the affine case is checked in [13, p.11]. Hence the above is equivalent to:

$$\ker[\prod_{\alpha,\beta} \mathcal{O}_X(U_\alpha) \times \mathcal{O}_Y(U_\beta) \rightrightarrows \prod_{\alpha,\beta,\alpha',\beta'} \mathcal{O}_X(U_\alpha \cap U'_\alpha) \times \mathcal{O}_X(U_\beta \cap U'_\beta)] \cong$$

$$\ker[\prod_{\alpha} \mathcal{O}_X(U_\alpha) \rightrightarrows \prod_{\alpha,\alpha'} \mathcal{O}_X(U_\alpha \cap U'_\alpha)] \times \ker[\prod_{\beta} \mathcal{O}_X(U_\beta) \rightrightarrows \prod_{\beta,\beta'} \mathcal{O}_X(U_\beta \cap U'_\beta)] \cong$$

$$\mathcal{O}_X(X) \times \mathcal{O}_Y(Y) \cong \Gamma(X \times \mathcal{O}_X) \times \Gamma(Y \mathcal{O}_Y).$$

This proves the assertion. $\mathcal{Q.E.D}$

The aim of this subsection is to show that monoid schemes can be described in a simpler way.

Definition 3.1.3. *A* T_0 space *is a topological space, satisfying the following condition: for any two distinct points* $x, y \in X$*, there is an open set* U*, such that, either* $x \in U$ *and* $y \notin U$ *or* $y \in U$ *and* $x \notin U$*. A* T_0*-topological space is called a* $\mathfrak{P}$-topological space*, provided any intersection of open sets is again open.*

Recall the topology defined on a poset in Definition 2.2.7. The link between $\mathfrak{P}$-topological spaces and poset topological spaces is well known, see [40] and [34]. We state and prove the following facts (which are discussed in the above citation), for the convenience of the reader:

Proposition 3.1.4. *1) Let P be a poset and X_P its associated topological space. For any element $x \in X_P$, the smallest open subset of X_P containing x is $L(P,x) = \{x' | x' \leq x\}$. Further, the open subsets of the type $L(P,x)$ form a basis of the topology.*

2) Let X_P and X_Q be the two topological spaces associated to P and Q respectively. Then we have a bijection

$$\mathsf{C}(X_P, X_Q) \cong \mathsf{MHom}(P, Q),$$

where MHom denotes the set of monotonic maps from P to Q, and C denotes the set of continuous maps from X_P to X_Q.

3) The category of $\mathfrak{P}$-topological spaces with continues maps is equivalent to the category of posets with monotonic maps.

Proof. 1) This follows straight from the definition.

2) Let $f : X_P \to X_Q$ be a continuous map and let $x \leq y \in X_P$. Take

$$L(Q, f(y)) \subset X_Q.$$

Unless $f(x) \leq f(y)$, in other words $f(x) \in L(Q, f(y))$, x is not an element of

$$f^{-1}(L(P, f(y))) = L(P, y).$$

But then we get a contradiction, since $x \leq y$.

Now let $f : P \to Q$ be a monotonic map and $U \subset X_Q$ open. Take $y \in f^{-1}(U)$ and $x \leq y$. Since f is monotonic, $f(x) \leq f(y)$. Hence $f(x) \in U$ implies that

$f(x)$ is also an element of U, which implies that $x \in f^{-1}(U)$.

3) Take two distinct points $x, y \in X_P$. Without loss of generality, assume either $x \leq y$, or they are not comparable. Either way, for the open subset $L(P,x)$, we have $x \in L(P,x)$ but $y \notin L(P,x)$. Hence X_p is a T_0 space.

Take a collection of open subsets $\{U_i\}$ and let $U = \bigcap U_i$. Pick $y \in U$ and $x \leq y$. Since y is in the intersection, for every i, we have $y \in U_i$. This implies that $x \in U_i$, so $x \in U$. Hence U is again open and so X_P is a $\mathfrak{P}$-topological space.

To see the reverse, take a $\mathfrak{P}$-topological space X, and construct a poset P_X as follows: Elements of P_X are points of X, and we say $x \leq y$ if and only if $U_x = (\bigcap U_i | x \in U_i) \subset U_y = (\bigcap U_j | y \in U_j)$. It is straightforward to see that this defines a poset and that these two constructions are inverse to each other.

$\mathcal{Q.E.D}$

Using this proposition, we will identify posets equipped with the ordering topology given above, with $\mathfrak{P}$-topological spaces. It is a well known fact that a sheaf $\mathfrak{F}$ over a topological space X is uniquely defined by its values on the bases of open sets. Hence, using part 1 of the above Proposition, to define a sheaf on a $\mathfrak{P}$-topological space, it is sufficient to define it only on the open subsets of the form U_x. That is, it suffices to only define it on the stalks of the underlying poset P. If $x \leq y \Rightarrow U_x \subset U_y$, which yields a homomorphism $\mathfrak{F}_y \to \mathfrak{F}_x$.

Lemma 3.1.5. *Let X_P be a $\mathfrak{P}$-topological space and $\mathfrak{F}$ a sheaf of sets on X_P. Then the map $\mathfrak{F} \mapsto \mathfrak{F}_x$ gives rise to an equivalence between the category of sheaves on X, and the category of contravariant functors on the poset P.*

To see this result, it suffices to say that the corresponding sheaf $\mathfrak{F}$ on X_P is defined by $\mathfrak{F}(U) = lim_{x \in U} \mathfrak{F}_x$.

Definition 3.1.6. *A monoid scheme X is called of* finite type*, provided there is a finite open cover by affine schemes $U_i = \mathsf{KSpec}(M_i)$, such that all M_i's are finitely generated monoids.*

Lemma 3.1.7. *Let $(X, \mathcal{O}_X)$ be a monoid scheme of finite type. The underlying topological space X, is a finite $\mathfrak{P}$-topological space and equivalent to X_P, where P satisfies the property that for all $x \in P$, $L(P, x)$ is a finite lattice. We say that such a poset is* locally a finite lattice.

Proof. Since openness of a subset is a local condition, it suffices to consider the case when X is affine. But this follows straight from the results of the previous chapter. $\mathcal{Q.E.D}$

Thus monoids schemes can and will be described using pairs $(X, \mathcal{O}_X)$, where X is a poset that is locally a finite lattice, and $\mathcal{O}_X$ is a contravariant functor defined on it.

As we said in the introduction, a (commutative) monoid is a set M with an operation $\times$, such that $\times$ is commutative and associative. Additionally, we have a unique element 1 such that $1 \times m = 1$. There is however a slightly different theory developed in [11][10][12], which is based on pointed monoids. Recall that a pointed monoid is a monoid with distinguished element 0. In this theory, all homomorphisms and all ideals are pointed. If M is a usual monoid, we let M_+ denote the pointed monoid obtained from M by adding a new element 0.

Observe that the space of prime ideals of M and the space of pointed prime ideals of M_+ are essentially the same. Since the functor $M \mapsto M_+$ is compatible with localisation, it has a unique extension to monoid schemes $(X, \mathcal{O}_M) \mapsto (X, \mathcal{O}_{M_+})$. The resulting functor from the category of monoid schemes in the sense of [13] to the category of monoid schemes in the sense of [11][10][12], is faithful and preserves finite products. Because of this, the concepts and results from [12] can be applied to monoid schemes in the sense of [13]. For instance, we can talk about *separated* monoid schemes [12, Definition 3.3], which are formally defined using the product as for classical schemes.

We have the following fact, as proven in [12, Corollary 3.8].

Lemma 3.1.8. *If X is a separated monoid scheme, the intersection of two affine open subsets is again affine.*

It should be noted that from here on onwards, monoids and monoid schemes are again assumed to be unpointed. Furthermore, the above lemma states the only property about separated monoid schemes that we will care about in this thesis.

3.1.2 Cohomology of Posets

Recall that if X is a topological space and F is a presheaf of groups on X, then for any open cover $\mathcal{U} = (U_i \hookrightarrow X)_{i\in I}$, $\bigcup_i U_i = X$, the zeroth cohomology group $H^0(\mathcal{U}, F)$ and the first cohomology pointed set $H^1(\mathcal{U}, F)$ are defined. If F has values in abelian groups, then there are well defined cohomology groups $H^n(\mathcal{U}, F)$, $n \geq 0$, and these groups are abelian. If $\mathcal{F}$ is a sheaf, we also have the sheaf (or Grothendieck) cohomology $H^n(X, \mathcal{F})$ defined. It is well known that

$$H^i(X, \mathcal{F}) = colim_{\mathcal{U}} H^i(\mathcal{U}, \mathcal{F}) = \check{H}^i(X, \mathcal{F}),\ i = 0, 1,$$

where the colimit is taken with respect to all open covers.

Proposition 3.1.9. *Let X be a monoid scheme and F a sheaf of groups.*

i) If X is affine, then

$$H^1(X, F) = 0.$$

Moreover, if F is a sheaf of abelian groups, then $H^i(X, F) = 0$ for all $i \geq 1$.

ii) Let X be a separated monoid scheme and $\mathfrak{U}$ an affine cover of X. If F is a sheaf of abelian groups, then $H^*(X, F) = H^*(\mathfrak{U}, F)$.

Proof. i) Let M be a monoid and denote $X = \mathsf{KSpec}(M)$. Let $\mathfrak{m}$ be the subset of non-invertible elements of M. If $\mathfrak{m} \in D(f)$ for an element $f \in M$, then f is invertible, and thus $D(f) = X$. Hence, the only open subset of X which contains $\mathfrak{m}$, is X itself. Thus, for any sheaf F one has

$$F(X) = F_{\mathfrak{m}}.$$

Since $F \mapsto F_{\mathfrak{m}}$ is an exact functor in the category of sheaves, we see that the global section functor is an exact functor. This already proves the result for abelian sheaves, because $H^*(X, -)$, being derived functors of an exact functor, vanish in positive dimensions. If F is not necessary abelian, the proof is essentially the same: Any open cover of X must contain X as a member.

A cover consisting of a single element X is cofinal among all covers. The cosimplicial object corresponding to this cover is constant, and so the result follows.

ii) Since X is separated, the intersection of any two open, affine, monoid subschemes is again affine (see Lemma 3.1.8). Hence, ii) is a formal consequence of i) and Leray's theorem (see for example [19, Corollary of Theorem II.5.4.1]).

In more detail, let $\mathfrak{U} = \{U_i\}s_{i\in I}$ be a covering of X, where every U_i is the affine scheme associated to the monoid M_i. Using the spectral sequence associated to said covering (Theorem 1.4.1, v)), we know that $H^p(\mathfrak{U}, \mathsf{H}^q(F))$ abuts to $H^{p+q}(X, F)$. Here H^q denotes the presheaf given by $V \mapsto H^q(V, F)$. From i) of this proposition, we know that $H^q(U_i, F) = \begin{cases} F(U_i), & q = 0 \\ 0, & q \geq 1 \end{cases}$.

So $E_2^{pq} = \begin{cases} H^p(\mathfrak{U}, F), & q = 0 \\ 0, & q \geq 0 \end{cases}$, and hence the second page of the Leray-Serre spectral sequence looks as follows:

$$\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & \cdots \\ \\ H^0(\mathfrak{U}, F) & H^1(\mathfrak{U}, F) & H^2(\mathfrak{U}, F) & H^3(\mathfrak{U}, F) & H^4(\mathfrak{U}, F) & \cdots . \end{array}$$

As this is clearly already the stable page, the result follows.

$\mathcal{Q.E.D}$

Recall that for a monoid scheme X of finite type, P_X is finite. It follows that if X is additionally separated, then $L(X, x) \cap L(X, y)$ is either an empty set or again of the type $L(X, z)$, where $z = x \wedge y$. Thus, if X is a connected separated monoid scheme of finite type, then X is a meet semi-lattice with smallest element. In particular, the nerve of such a poset is contractible.

Lemma 3.1.10. *Let X be a separated monoid scheme of finite type.*

i) *For any constant sheaf G with values in groups, one has*

$$H^1(X, G) = 0.$$

If G is also abelian, then $H^n(X, G) = 0$ for all $n \geq 1$.

ii) *Assume X is connected and $m_1, \ldots, m_k$ are maximal elements of X. For any sheaf F on X the cohomology $H^*(X, F)$ can be computed using the cochain complex*

$$\prod_{i=1}^{k} F_{m_i} \to \prod_{i,j} F_{m_i \wedge m_j} \to \prod_{i,j,k} F_{m_i \wedge m_j \wedge m_k} \to \cdots \to F_{m_1 \wedge \cdots \wedge m_k} \to 0 \to \cdots.$$

In particular, $H^n(X, F) = 0$, for all $n > k$, where n is the minimal natural number such that X has an open cover by n affine monoid schemes.

Proof. i) We can assume that X is connected. In this case the nerve of the poset X is contractible, as stated just above the lemma, and hence by Lemma 1.7.1 the result follows.

ii) This follows from part ii) of the Lemma 3.1.9, applied to the open covering $(U_i)_{i=1}^k$, where $U_i = L(X, m_i)$. In fact, since $m_{j_1} \wedge \cdots \wedge m_{j_k}$ is the greatest element in $U_{i_{j_1}} \cap \cdots \cap U_{i_{j_k}}$, one has $F(U_{i_{j_1}} \cap \cdots \cap U_{i_{j_k}}) = F_{m_{j_1} \wedge \cdots \wedge m_{j_k}}$.

$\mathcal{Q.E.D}$

Let us recall that for a point $x \in X$, the *height of x* is the supremum of natural numbers k, for which there is a sequence $x_0 < \cdots < x_k = x$.

Definition 3.1.11. *Let X be a monoid scheme of finite type. The* Krull dimension *$dim(X)$ of X is defined as*

$$dim(X) = \mathsf{sup}_{x \in X} ht(x).$$

Similarly to the last assertion of Lemma 3.1.10, we have $H^i(X, F) = 0$, for all $k > dim(X)$. This is a consequence of the well-known result of Grothendieck on noetherian spaces (Theorem 1.6.2).

Corollary 3.1.12. *Assume X is a non-affine monoid scheme of finite type. Then there exists a sheaf F of abelian groups such that $H^1(X,F)$ is non-trivial.*

Proof. It immediately follows from the assumptions that X has at least 2 maximal elements. Call them p and q. Then by Proposition 3.1.5, for any abelian group G, there is a sheaf F for which

$$F_x = \begin{cases} 0, & \text{if } x \neq p \wedge q \\ G, & \text{if } x = p \wedge q. \end{cases}$$

It follows that $H^1(X,F) = G$. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

3.2 Vector Bundles over Monoid Schemes

The main result of this section claims that any vector bundle over any separated monoid scheme is a coproduct of line bundles. We start by recalling the notion of a vector bundle over a monoid scheme [5, Definition 2.4, Remark 2.5], [14]. Then we use a cohomological description of isomorphism classes of n-dimensional vector bundles to prove the main result.

Definition 3.2.1. *Let M be a monoid and A a left M-set (also called M-act). Then A is* free of rank n*, if it is isomorphic to an M-set of the form $A = M \times X$. Here X is a set of cardinality n and M acts on A by $n(m,x) = (nm,x)$, $x \in X$ and $m,n \in M$.*

The disjoint union $\coprod$ of the underlying sets induces the coproduct in the category of M-sets. If S and T are left M-sets, the tensor product $S \otimes_M T$ is an M-set defined to be the quotient of $S \times T$ by the equivalence relation generated by

$$(ms,t) \sim (s,mt) =: m(s,t),$$

where $s \in S$, $t \in T$ and $m \in M$. One easily sees that if S and T are free of rank m and n respectively, $S \coprod T$ and $S \otimes_M T$ are free of rank $n+m$ and nm respectively. Hence, a free M-set of rank n is isomorphic to the coproduct of n-copies of free modules of rank 1. Because of this we can write $M^{\coprod n}$ for a free M-set of rank n.

Denote by $GL(n, M)$ the group of automorphisms of $M^{\amalg n}$. Then we have the following result, as shown in [13, p.15]:

Proposition 3.2.2. *The group $GL(n, M)$ is isomorphic to the group of all $n \times n$ matrices (with the obvious multiplication) such that:*

- *there is exactly one non-zero entry in each row and column;*
- *every entry is an invertible element of M.*

Proposition 3.2.3. *The automorphism group $GL(n, M)$ of $M^{\amalg n}$ is isomorphic to the semidirect product $(M^*)^n \rtimes \Sigma_n$, where Σ_n is the symmetric group and M^* is the subgroup of invertible elements. The action of Σ_n on $(M^*)^n$ is given by permuting the factors.*

Proof. A free M-set is $M^{\amalg n} = M \times S$, where $S = \{1, \cdots, n\}$, with the action given by $m(m', s) = (mm', s')$. Hence $f : M \times S \to M \times S$ is an M-isomorphism if f is a bijection and $(m(m', s)) = mf(m', s)$. We have $f(1, s) = (x(s), \varphi(s))$, where $x(s) \in M$ and $\varphi(s) \in S$. Since they only depend on s, we actually have 2 functions $x : S \to M$ and $\varphi : S \to S$. By definition, we have $(m, s) = m(1, s)$, and so

$$f(m, s) = mf(1, s) = m(x(s), \varphi(s)) = (mx(s), \varphi(s)).$$

As f is a bijection, there exists $g : M \times S \to M \times S$ such that $g \circ f = \mathsf{Id}$. We will fix notations and say that $g(m, s) = (my(s), \phi(s))$. Hence we have:

$$(1, s) = \mathsf{Id}(1, s) = g \circ f(1, s) = g(x(s), \varphi(s)) = (x(s)y(\varphi(s)), \phi(\varphi(s))) \tag{3.1}$$

$$(1, s) = \mathsf{Id}(1, s) = f \circ g(1, s) = f(y(s), \phi(s)) = (y(s)x(\phi(s)), \varphi(\phi(s))) \tag{3.2}$$

The first component of 3.1 implies $x(s)y(\varphi(s)) = 1$, so $x(S) \subset M^*$. To see that $\varphi \in \Sigma_n$, i.e. that $\varphi : S \to S$ is a bijection, take the second components of 3.1 and 3.2. This shows that $\varphi \circ \phi = \phi \circ \varphi = \mathsf{Id}$. These two equations also show that we have a semidirect product $(M^*)^n \rtimes \Sigma_n$. $\mathcal{Q.E.D}$

By the above proposition, we have a split short exact sequence of groups

$$1 \to (M^*)^n \xrightarrow{i_n} GL(n, M) \to \Sigma_n \to 1. \tag{3.3}$$

The coproduct induces an obvious homomorphism of groups

$$c = c_{k,n} : GL(k, M) \times GL(n, M) \to GL(k + n, M),$$

which fits in the following commutative diagram with exact rows:

$$\begin{tikzcd}
1 \arrow[r] & (M^*)^k \times (M^*)^n \arrow[r] \arrow[d, "c'"', "\cong"] & GL(k, M) \times GL(n, M) \arrow[r] \arrow[d, "c"'] & \Sigma_k \times \Sigma_n \arrow[r] \arrow[d, "\tilde{c}"] & 1 \\
1 \arrow[r] & (M^*)^{k+n} \arrow[r] & GL(k + n, M) \arrow[r] & \Sigma_{k+n} \arrow[r] & 1.
\end{tikzcd}$$

Here $c'((x_1, \cdots, x_k), (y_1, \cdots, y_n)) = (x_1, \cdots, x_k, y_1, \cdots, y_n)$, and for $\sigma_1 \in \Sigma_k, \sigma_2 \in \Sigma_n$, the permutation $\tilde{c}(\sigma_1, \sigma_2) \in \Sigma_{k+n}$ is given by:

$$\tilde{c}(\sigma_1, \sigma_2)(i) = \begin{cases} \sigma_1(i) & 1 \le i \le k, \\ \sigma_2(i - k) & k \le i \le k + n. \end{cases}$$

Definition 3.2.4. *A* vector bundle of rank n *on a monoid scheme* X *is a sheaf* $\mathcal{V}$ *of sets on* X*, together with an action of* $\mathcal{O}_X$*, such that locally,* $\mathcal{V}$ *is isomorphic to* $\mathcal{O}_X^{\amalg n}$*. We let* $\mathbf{Vect_n(X)}$ *be the category of vector bundles of rank n on* X*. The set of their isomorphism classes of rank n on* X *is denoted by* $\mathsf{Vect}_n(X)$*. In the special case when the rank is one, we use the term* line bundle *and write* $\mathsf{Pic}(X)$*.*

For a given monoid scheme X, we denote the group of automorphism of $X^{\amalg n}$ by $GL(n, X)$. Since a monoid homomorphism maps invertible elements to invertible elements, the description of $GL(n, M)$, as given in Proposition 3.2.2, gives rise to a functor. We have the following result (see [13, p.16, Proposition 5.2]):

Proposition 3.2.5. *The functor* $GL(n, M)$ *is representable as a monoid scheme.*

This immediately implies that the construction given in Proposition 3.2.2, extends to a sheaf, and it is straightforward to see that it agrees with $GL(n, X)$.

The coproduct and tensor product of M-sets yield corresponding operations on vector bundles:

$$\coprod : \mathbf{Vect}_m(X) \times \mathbf{Vect}_n(X) \to \mathbf{Vect}_{n+m}(X)$$

and

$$\otimes_{\mathcal{O}_X} : \mathbf{Vect}_m(X) \times \mathbf{Vect}_n(X) \to \mathbf{Vec}t_{mn}(X).$$

Proposition 3.2.6. *The tensor product yields an abelian group structure on* $\mathsf{Pic}(X)$.

Proof. The fact that it is a semigroup is trivial. The identity is clearly $\mathcal{O}_X$, making it a monoid. To see that it is a group, we need to give an inverse for a line bundle $\mathfrak{L}$. Just like in the classical case, $\mathsf{Hom}_{\mathcal{O}_X}(\mathfrak{L}, \mathcal{O}_X)$ does the trick since $\mathfrak{L} \otimes \mathsf{Hom}_{\mathcal{O}_X}(\mathfrak{L}, \mathcal{O}_X) \cong \mathcal{O}_X$. The isomorphism is given by $x \otimes f \mapsto f(x)$. $\mathcal{Q.E.D}$

Proposition 3.2.7. *There is a natural bijection*

$$\mathsf{Vect}_n(X) \cong H^1(X, GL(n, \mathcal{O}_X))$$

and isomorphism of groups

$$\mathsf{Pic}(X) = H^1(X, \mathcal{O}_X^*).$$

Moreover one has a commutative diagram

$$\begin{tikzcd}
\mathsf{Vect}_k(X) \times \mathsf{Vect}_n(X) \arrow[r] \arrow[d, "\sim"'] & \mathsf{Vect}_{k+n}(X) \arrow[d, "\sim"] \\
H^1(X, GL(k, \mathcal{O}_X)) \times H^1(X, GL(n, \mathcal{O}_X)) \arrow[r] & H^1(X, GL(k+n, \mathcal{O}_X)).
\end{tikzcd}$$

Proof. This is standard. Assume we are given an open cover $\mathcal{U} = (U_i \hookrightarrow X)_{i \in I}$, $\bigcup_i U_i = X$ and a 1-cocycle $(f_{ij} \in GL(X, \mathcal{O}_X))$. As in the classical case, the associated vector bundle is obtained from the trivial vector bundles on U_i, by gluing on $U_i \bigcap U_j$ via f_{ij}. One easily checks that this construction yields a bijection. $\mathcal{Q.E.D}$

Corollary 3.2.8. *Any vector bundle over an affine monoid scheme is trivial.*

For line bundles this fact first appears in [15, Lemma 5.2].

Proof. This follows from the fact that cohomology vanishes for affine monoid schemes. See Proposition 3.1.9. $\mathcal{Q.E.D}$

Theorem 3.2.9. *Let X be a connected, separated monoid scheme of finite type. Then any vector bundle of rank n is a coproduct of n copies of line bundles. Moreover, this decomposition is unique up to permuting summands.*

Proof. The coproduct induces the natural map

$$\mathsf{Pic}(X)^n \to \mathsf{Vect}_n(X). \tag{3.4}$$

Since the operation induced by the coproduct is commutative, this factors through the orbit space $\mathsf{Pic}(X)^n_{\Sigma_n} \to \mathsf{Vect}_n(X)$. We need to show that this map is a bijection. By the commutativity of the diagram in Proposition 3.2.7, we see that the map in (3.4) is the same (up to isomorphism) as the map

$$H^1(X, (\mathcal{O}_X^*)^n) \xrightarrow{i_n^*} H^1(X, GL(n, \mathcal{O}_X)).$$

Here i_n is the sheaf homomorphism, which fits in the following split short exact sequence:

$$0 \to (\mathcal{O}_X^*)^n \xrightarrow{i_n} GL(n, \mathcal{O}_X) \to \Sigma_n \to 0.$$

The last term Σ_n, is considered as a constant sheaf. Apply Proposition 1.5.2 to get the short exact sequence of pointed sets

$$0 \to \left(H^1(X, \mathcal{O}_X^*)\right)^n_{\Sigma_n} \xrightarrow{i_n^*} H^1(X, GL(k, \mathcal{O}_X)) \to H^1(X, \Sigma_n) \to 1.$$

By Lemma 3.1.10 i), the last term vanishes. Hence i_n yields the isomorphism

$$(\mathsf{Pic}(X))^n_{\Sigma_n} \cong \left(H^1(X, \mathcal{O}_X^*)\right)^n_{\Sigma_n} \cong H^1(X, GL(k, \mathcal{O}_X)) \cong \mathsf{Vect}_n(X),$$

and the result follows. $\mathcal{Q.E.D}$

For the special case when $X = \mathbb{P}^n$ (in the monoid world), this theorem was first proven in [5, Theorem 2.6], by completely different means.

3.3 Comparison between Line Bundles over a Monoid Scheme and its Realisation

Fix a commutative ring k. If M is a monoid, we can define the *monoid ring* $k[M]$, by considering the free module $k^{|M|}$ of rank $\#|M|$ over k. We define the multiplication on the basis according to the monoid structure. As described in [12, Section 5], or [13], we can extend this construction to monoid schemes, and as such, we have a functorial way of assigning a k-scheme $k[X]$ to a monoid scheme X. We call $k[X]$ the *realisation* of X over k.

If S is a free M-set, then $k[S]$ is a free $k[M]$-module. After gluing, one obtains a functor $\mathbf{Vect}_n(X) \to \mathbf{Vect}_n(k[X])$. This induces a homomorphism $\mathsf{Pic}(X) \to \mathsf{Pic}(k[X])$, or more generally a map $\mathsf{Vect}_n(X) \to \mathsf{Vect}_n(k[X])$, $n > 0$.

For example, if k is a field, $X = \mathbb{P}^1$, then by the classical result of Grothendieck [23, Theorem 2.1] any vector bundle over $\mathbb{P}^1_k$ is a direct sum of line bundles. By Theorem 3.2.9 and the computation of $\mathsf{Pic}(\mathbb{P}^1)$ made in [13, Proposition 4.3], it follows that

$$\mathsf{Vect}_n(\mathbb{P}^1) \to \mathsf{Vect}_n(\mathbb{P}^1_k)$$

is a bijection for all $n \geq 1$. We will see soon that this fact is a particular case of a more general result. See Corollary 3.3.3 below.

Let X be a separated monoid scheme. To analyse the natural homomorphism $\mathsf{Pic}(X) \to \mathsf{Pic}(k[X])$ we will use the low-dimensional exact cohomological sequence associated to a spectral sequence of a cover, see Theorem 1.4.1 v). In our case it takes the following form. Let $\mathcal{U} = (U_i \hookrightarrow X)_{i \in I}$ be an open cover of a separated monoid scheme X, using affine monoid schemes $U_i = \mathsf{KSpec}(M_i)$, where M_i are monoids. By our assumption on X and Lemma 3.1.8, we have that for any i, j we have $U_{ij} = U_i \cap U_j = \mathsf{KSpec}(M_{ij})$ for some monoid M_{ij}. Then the cochain complex of the covering $\mathcal{U}_k$ of $k[X]$ and the sheaf of invertible elements of $k[\mathcal{O}_X] = \mathcal{O}_{k[X]}$ looks as follows:

$$\prod_{i \in I} (k[M_i])^* \to \prod_{i,j} (k[M_{ij}])^* \to \cdots.$$

The p dimensional cohomology of this cochain complex is denoted by E_2^{p0}. We also have

$$E_2^{01} = \mathsf{Ker}\left(\prod_{i\in I}\mathsf{Pic}(k[M_i]) \to \prod_{i,j}\mathsf{Pic}(k[M_{ij}])\right).$$

Then we have an exact sequence (Theorem 1.4.1, v)):

$$0 \to E_2^{10} \to \mathsf{Pic}(k[X]) \to E_2^{01} \to E_2^{20}. \tag{3.5}$$

Recall that a monoid is *torsion-free* if and only if $x^n = y^n$ for some $n > 0$ implies $x = y$. We will say that a monoid scheme X is *torsion-free* provided for any affine open monoid subscheme $\mathsf{KSpec}(M)$, the monoid M is torsion-free. One easily sees that X is torsion-free if and only if $\mathcal{O}_{X,x}$ is torsion-free for any $x \in X$.

Proposition 3.3.1. *Assume k is an integral domain and X is a torsion-free cancellative monoid scheme. Then $E_2^{10} = \mathsf{Pic}(X)$ and hence the natural map*

$$\mathsf{Pic}(X) \to \mathsf{Pic}(k[X])$$

is a monomorphism.

Proof. By Theorem 11.1 in [18], if k is an integral domain and M is a torsion-free cancellative monoid, one has $(k[M])^* = k^* \times M^*$. Thus, in this case, the cochain complex computing E_2^{*0}-terms is a direct sum of two subcomplexes, corresponding to the ring and monoid factors. Hence

$$E_2^{*0} \cong H^*(\mathcal{U}, \mathcal{O}_X{}^*) \oplus H^*(\mathcal{U}, k^*),$$

where k^* is considered as a constant sheaf on X. The homology of the second summand vanishes in positive dimensions and hence the result follows. $\mathcal{Q.E.D}$

A finitely generated cancellative monoid M is called *seminormal* if for any $x \in M^{gr}$ with $x^2, x^3 \in M$, it follows that $x \in M$. Here M^{gr} is the group of fractions of M. A monoid scheme X is *seminormal* provided for any affine open monoid subscheme $\mathsf{KSpec}(M)$, the monoid M is seminormal.

Proposition 3.3.2. *Assume k is a PID and X is a seminormal monoid scheme. Then the natural map*

$$\mathsf{Pic}(X) \to \mathsf{Pic}(k[X])$$

is an epimorphism.

Proof. It is well known that if M is seminormal, then $\mathsf{Pic}(k[M]) = 0$ (see [7, Theorem 8.4]. It follows that $E_2^{01} = 0$. Hence the result follows from the exact sequence (3.5) and Proposition (3.3.1). $\mathcal{Q.E.D}$

Corollary 3.3.3. *Assume k is a PID, and X is a seminormal and torsion-free monoid scheme. Then the natural map*

$$\mathsf{Pic}(X) \to \mathsf{Pic}(k[X])$$

is an isomorphism.

For $X = \mathbb{P}^n$, this result was first proven by direct computation in [13, Proposition 4.3] (if $n = 1$) and it follows trivially from [5, Theorem 2.6] (if $n \geq 2$).

3.4 Additivity of the Functor Pic

We now continue to study the Picard functor in more detail. In this section we will prove that the natural map $\mathsf{Pic}(X) \oplus \mathsf{Pic}(Y) \xrightarrow{\sim} \mathsf{Pic}(X \times Y)$ is an isomorphism provided X and Y are separated monoid schemes.

Let T be a contravariant functor defined on the category of separated monoid schemes, with values in the category of abelian groups. We will say that the functor T is *additive for the pair* (X, Y), if the natural morphism $T(X) \oplus T(Y) \to T(X \times Y)$ is an isomorphism. Moreover, T is called *additive*, if it is additive for all pairs (X, Y), where X and Y are separated monoid schemes. For instance, the functor

$$X \mapsto \mathcal{O}_X^*(X) = H^0(X, \mathcal{O}_X^*)$$

is additive, as shown in Theorem 3.1.2.

Lemma 3.4.1. *Let X, Y be separated monoid schemes and $U, V \subset X$ open, such that $X = U \cup V$ and $W = U \cap V$. Assume the functors $X \mapsto H^i(X, \mathcal{O}_X^*)$, $i \in \{n-1, n\}$ are additive for the pairs $(U, Y), (V, Y)$ and (W, Y). Then $X \mapsto H^n(X, \mathcal{O}_X^*)$ will also be additive for the pair (X, Y).*

Proof. The proof follows from the Mayer-Vietoris sequences for the coverings $X = U \cup V$ and $Y = Y \cup Y$, and using the five-lemma to compare their direct sum with the Mayer-Vietoris sequence of the covering $X \times Y = (U \times Y) \cup (V \times Y)$. In more detail, since $X = U \cup V$, we get the Mayer-Vietoris sequence:

$$\begin{aligned} \cdots \to H^{n-1}(X) \to H^{n-1}(U) \oplus H^{n-1}(V) \to H^{n-1}(W) \to \\ \to H^n(X) \to H^n(U) \oplus H^n(V) \to H^n(W) \to \cdots. \end{aligned}$$

Here we denote $H^n(A, \mathcal{O}_A^*)$ by $H^n(A)$ for simplicity. Likewise, by looking at Y as the union with itself, i.e. $Y = Y \cup Y$, we get

$$\begin{aligned} \cdots \to H^{n-1}(Y) \to H^{n-1}(Y) \oplus H^{n-1}(Y) \to H^{n-1}(Y) \to \\ \to H^n(Y) \to H^n(Y) \oplus H^n(Y) \to H^n(Y) \to \cdots. \end{aligned}$$

Putting these two together yields:

$$\begin{aligned} \cdots \to H^{n-1}(X) \oplus H^{n-1}(Y) \to H^{n-1}(U) \oplus H^{n-1}(V) \oplus H^{n-1}(Y) \oplus H^{n-1}(Y) \to \\ \to H^{n-1}(W) \oplus H^{n-1}(Y) \to H^n(X) \oplus H^n(Y) \to \\ \to H^n(U) \oplus H^n(V) \oplus H^n(Y) \oplus H^n(Y) \to H^n(W) \oplus H^n(Y) \to \cdots. \end{aligned}$$

From the assumption now, we have the following:

$$\begin{aligned} \cdots \to H^{n-1}(X) \oplus H^{n-1}(Y) \to H^{n-1}(U \times Y) \oplus H^{n-1}(V \times Y) \to H^{n-1}(W \times Y) \to \\ \to H^n(X) \oplus H^n(Y) \to H^n(U \times Y) \oplus H^n(V \times Y) \to H^n(W \times Y) \to \cdots \end{aligned} \tag{3.6}$$

The Mayer-Vietoris sequence for $X \times Y$, gives us:

$$\begin{aligned} \cdots \to H^{n-1}(X \times Y) \to H^{n-1}(U \times Y) \oplus H^{n-1}(V \times Y) \to H^{n-1}(W \times Y) \to \\ \to H^{n}(X \times Y) \to H^{n}(U \times Y) \oplus H^{n}(V \times Y) \to H^{n}(W \times Y) \to \cdots. \end{aligned} \tag{3.7}$$

Now comparing (3.6) and (3.7) and using the five-lemma, we get the desired result.

$\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Theorem 3.4.2. *Let X and Y be separated monoid schemes with finite affine coverings. For all $i \geq 0$, one has $H^i(X \times Y, \mathcal{O}^*_{X\times Y}) \cong H^i(X, \mathcal{O}^*_X) \oplus H^i(Y, \mathcal{O}^*_Y)$. In particular*

$$\mathsf{Pic}(X \times Y) \cong \mathsf{Pic}(X) \oplus \mathsf{Pic}(Y).$$

Proof. First recall that for $i = 0$, this result holds by Theorem 3.1.2. Let $i \geq 1$. We will prove this by induction on the number of affine coverings of X and Y. The base step holds since the cohomologies vanish for affine monoid schemes, as shown in Corollary 3.2.8. Now assume additivity holds for all pairs with n and m or less affine coverings. We will then prove that it also holds for $n+1$ and m affine coverings. By symmetry, it will hold for n and $m+1$ as well, and hence for any pair of finite affine coverings.

Let X be a monoid scheme with $n+1$ and Y with m affine covers. Then X can be written as $X = X' \cup U$, where U is affine and X' can be covered by n affine components. Since $X' \cap U$ can be covered by n affine components, by our assumption of separability, the condition of the above Lemma is satisfied. Hence additivity has been proven for X and Y, which implies the theorem. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

For the special case $Y = \mathsf{KSpec}(\mathbb{N})$, this was first proven in [15] by completely different means, for the functor Pic.

3.5 Divisors and Line Bundles

We start by defining an analogue of the Cartier divisors for monoid schemes. To do so, we will closely follow the classical construction of the Cartier divisors for the usual schemes (see for example [4, p. 434–444]).

Definition 3.5.1. *Let M be a monoid, we call an element $m \in M$* regular*, provided for every elements x, y such that $mx = my$, we have $x = y$.*

Denote by $R(M)$ the submonoid of all regular elements of M and by $M_{R(M)}$ the localisation of M with respect to $R(M)$. The canonical map $M \to M_{R(M)}$ is injective.

It is clear that $U \mapsto \mathcal{O}_X(U)_{\mathcal{R}_X(U)}$ is a presheaf of monoids. The sheaf associated to it is denoted by $\mathcal{M}_X$ and referred to as the *meromorphic functions* on X. The same argument as for the classical case shows that the presheaf $U \mapsto \mathcal{O}_X(U)_{\mathcal{R}_X(U)}$ is separated and hence by Proposition 1.3.1 is a subpresheaf of its sheafification $\mathcal{M}_X$. Hence $\mathcal{O}_X \to \mathcal{M}_X$ is injective. Consider now the short exact sequence of sheaves of abelian groups:

$$1 \to \mathcal{O}_X^* \to \mathcal{M}_X^* \to \mathcal{M}_X^*/\mathcal{O}_X^* \to 1. \tag{3.8}$$

As in the classical case, the global sections of the sheaf $\mathcal{M}_X^*/\mathcal{O}_X^*$ are called the *Cartier divisors*. A Cartier divisor is called *principal*, provided it corresponds to the image of an element of $\mathcal{M}_X^*(X)$.

The quotient of the group of Cartier divisors by the principal divisors is denoted by $\mathsf{CaCl}(X)$.

Recall that a monoid is called *cancellative* if for every $x, y, z \in M$, $xz = yz$ implies that $x = y$. A separated monoid scheme X is called *cancellative* if for any affine open monoid subscheme $\mathsf{KSpec}(M)$, the monoid M is cancellative. One easily sees that X is cancellative if and only if $\mathcal{O}_{X,x}$ is cancellative for any $x \in X$.

Proposition 3.5.2. *One has a monomorphism $\mathsf{CaCl}(X) \to \mathsf{Pic}(X)$, which is an isomorphism if X is cancellative.*

Proof. Take the short exact sequence in (3.8). We apply iv) from Theorem 1.4.1 to get the exact sequence

$$\mathcal{M}_X^*(X) \to \mathcal{M}_X^*/\mathcal{O}_X^*(X) \to \mathsf{Pic}(X) \to H^1(X, \mathcal{M}_X^*).$$

Since $\mathcal{M}_X$ is a constant sheaf, so is $\mathcal{M}_X^*$ and hence its first cohomology is zero. The result now follows from the definition of $\mathsf{CaCl}(X)$. $\mathcal{Q.E.D}$

This very proof shows us that we actually don't need for $\mathcal{O}_X$ to map injectively to $\mathcal{M}_X$. We only need for $\mathcal{O}_X^*$ to map injectively to $\mathcal{M}_X^*$. The rest of this section will be devoted to generalising the notions of cancellative monoids, regular elements and the Cartier divisors.

Definition 3.5.3. *A monoid M is called s-*cancellative *provided for any elements a, x, y with $ax = ay$ we have $(xy)^n x = (xy)^n y$ for some $n \in \mathbb{N}$.*

It is clear that any cancellative monoid is s-cancellative. Our next aim is to give several equivalent conditions for a monoid to be s-cancellative, but first a notation. For an element $c \in M$ we set

$$\mathfrak{p}_c = \bigcup_{c \notin \mathfrak{p} \in \mathsf{KSpec}(M)} \mathfrak{p}.$$

Since the union of prime ideals in a monoid is a prime ideal, we see that $\mathfrak{p}_c$ is the maximal prime ideal which does not contain c.

Lemma 3.5.4. *For elements b, c of a monoid M, one has $b \notin \mathfrak{p}_c$ if and only if there is a natural number n and $t \in M$, such that*

$$c^n = bt.$$

Proof. Clearly $b \notin \mathfrak{p}_c$ if and only if any prime ideal which contains b contains c as well. Thus the result follows from [20, Lemma III.1.1]. $\mathcal{Q.E.D}$

Theorem 3.5.5. *Let M be a finitely generated monoid. Then the following conditions are equivalent:*

1. *M is s-cancellative.*
2. *For any elements x, y, a with $xa = ya$ and any prime ideal $\mathfrak{p}$ such that $x, y \notin \mathfrak{p}$, there exists an element $b \notin \mathfrak{p}$ such that $xb = yb$.*
3. *For any prime ideals $\mathfrak{q} \subset \mathfrak{p}$, the induced map on invertible elements*

$$(M_{\mathfrak{p}})^* \to (M_{\mathfrak{q}})^*$$

is injective.

4. For any elements x, y, a with $xa = ya$ there exists an element $b \notin \mathfrak{p}_{xy}$ such that $xb = yb$.

Proof. 1) $\Longrightarrow$ 2). It suffices to take $b = (xy)^n$.

2) $\Longrightarrow$ 3). Without loss of generality we may assume that $\mathfrak{q} = \varnothing$. In this case $M_{\mathfrak{q}}$ is a group, which is obtained by localising M with respect to all of M. This group is denoted by G. Take an element

$$\frac{x}{y} \in Ker\left((M_{\mathfrak{p}})^* \to G\right), \quad x \in M, \ y \notin \mathfrak{p}.$$

Thus there exists an element $a \in M$ such that $xa = ya$. By assumption this implies that $xb = yb$, where $b \notin \mathfrak{p}$ and therefore $\frac{x}{y} = 1$ in $M_{\mathfrak{p}}$.

3) $\Longrightarrow$ 4). By assumption the map $(M_{\mathfrak{p}})^* \to G$ is injective for any prime ideal $\mathfrak{p}$, where G is the same as in the previous case. Suppose $a, x, y \in M$ are such elements that $xa = ya$. Since $x, y \notin \mathfrak{p}_{xy}$, we have

$$z = \frac{x}{y} \in \left(M_{\mathfrak{p}_{xy}}\right)^*.$$

The condition $xa = ya$ implies that the image of z in G is 1. Hence $z = 1$ by assumption. Therefore $xb = yb$ for some $b \notin \mathfrak{p}_{xy}$.

4) $\Longrightarrow$ 1). Assume $xa = ya$. Then by assumption $xb = yb$, where $b \notin \mathfrak{p}_{xy}$. By Lemma 3.5.4 there is a natural number n and an element $t \in M$ such that $(xy)^n = bt$. So

$$(xy)^n x = btx = bty = (xy)^n y.$$

$\mathcal{Q.E.D}$

Definition 3.5.6. *A monoid scheme X is called* s-cancellative *provided for any $\mathfrak{q} \leq \mathfrak{p}$ the induced map $\mathcal{O}^*_{X,\mathfrak{p}} \to \mathcal{O}^*_{X,\mathfrak{q}}$ is injective.*

By Theorem 3.5.5 an affine monoid scheme $\mathsf{KSpec}(M)$ is s-cancellative if and only if M is s-cancellative.

Proposition 3.5.7. *If X and Y are s-cancellative monoid schemes, then $X \times Y$ is again s-cancellative.*

Proof. The proof of this is actually trivial. Since the stalks respect the product, i.e. on the point $(\mathfrak{p}, \mathfrak{q})$ we have $M_\mathfrak{p} \times M_\mathfrak{q}$, where $M_\mathfrak{p}$ and $M_\mathfrak{q}$ are the monoids standing on the stalks $\mathfrak{p} \in X$ and $\mathfrak{q} \in Y$ respectively, it is clear that if every morphism $M_\mathfrak{p} \to M_{\mathfrak{p}'}$ and $M_\mathfrak{q} \to M_{\mathfrak{q}'}$ coming from inclusions $\mathfrak{p}' \subset \mathfrak{p} \in X$ and $\mathfrak{q}' \subset \mathfrak{q} \in Y$ are injections, then the morphism $M_\mathfrak{p} \times M_\mathfrak{q} \to M_{\mathfrak{p}'} \times M_{\mathfrak{q}'}$ is an injection as well. With the above statement, the proposition is proven. $\mathcal{Q.E.D}$

Next we will generalise the notion of regular elements. We call an element $a \in M$ *s-regular*, provided for some elements a, u, v with $a^m u = a^m v$ for any $m \in \mathbb{N}$, we have $u(uv)^n = v(uv)^n$ for some $n \in \mathbb{N}$. Denote the set of s-regular elements by $\mathsf{S}(M)$.

Lemma 3.5.8. *For any element $f \in M$ the localisation homomorphism $M \to M_f$ sends s-regular elements to s-regular elements. In particular, if M is a finitely generated monoid, then the same is also true for the localisation homomorphism $M \to M_\mathfrak{p}$, for any prime ideal $\mathfrak{p}$.*

Proof. Let $a \in S(M)$ be an s-regular element of M. Assume we have

$$a^n u' = a^n v'$$

in M_f. If $u' = \frac{u}{f^i}$ and $v' = \frac{v}{f^j}$, $u, v \in M$, then we can rewrite

$$a^n \frac{u}{f^i} = a^n \frac{v}{f^j}.$$

Hence there exists a natural number $k \in \mathbb{N}$ such that $f^k a^n u f^j = f^k a^n v f^i$ in M. Since a is semi-regular, there exists a natural number $m \in \mathbb{N}$ such that

$$(f^{2k+j+i} uv)^m f^k u f^j = (f^{2k+j+i} uv)^m f^k v f^i.$$

We obtain

$$f^{k(2m+1)} (uv)^m u f^{m(i+j)} f^j = f^{k(2m+1)} (uv)^m v f^{m(i+j)} f^i.$$

Hence

$$\frac{(uv)^m u}{f^{m(i+j)} f^i} = \frac{(uv)^m v}{f^{m(i+j)} f^j}.$$

Or, equivalently $(u'v')^m u' = (u'v')^m v'$. Thus a is semi-regular in M_f. To see the second part, observe that if M is finitely generated, then $X = \mathsf{KSpec}(M)$ is a finite T_0-space. Hence any point $\mathfrak{p} \in \mathsf{KSpec}(M)$ has a smallest open neighbourhood, say $D(f)$. Then

$$M_{\mathfrak{p}} = \mathcal{O}_{X,\mathfrak{p}} = \mathcal{O}_X(D(f)) = M_f$$

and the result follows. $\mathcal{Q.E.D}$

Proposition 3.5.9. *The collection of all s-regular elements $\mathsf{S}(M) \subset M$ is a multiplicative subset containing* 1, *and the localisation map $M^* \to (M_{\mathsf{S}(M)})^*$ is injective.*

Proof. Let $a, b \in S(M)$. Assume that $(ab)^m u = (ab)^m v$ for some $m \in \mathbb{N}$ and $u, v \in M$. Since $a \in S(M)$, we have $a^m(b^m u) = a^m(b^m v)$ imply that

$$\begin{aligned} (b^m u b^m v)^n (b^m u) &= (b^m u b^m v)^n (b^m v) \\ b^{m(2n+1)}(uv)^n u &= b^{m(2n+1)}(uv)^n v. \end{aligned}$$

Since $b \in S(M)$, we have

$$\begin{aligned} ((uv)^n u (uv)^n v)^{n'} (uv)^n u &= ((uv)^n u (uv)^n v)^{n'} (uv)^n v \\ (u^{(2n+1)n'} u^n)(v^{(2n+1)n'} v^n) u &= (u^{(2n+1)n'} u^n)(v^{(2n+1)n'} v^n) v \\ (uv)^{(2n+1)n'+n} u &= (uv)^{(2n+1)n'+n} v \end{aligned}$$

Hence $ab \in S(M)$. To see that localising with the s-regular elements of a monoid induces an injection on the invertible elements, we only have to prove that $\frac{m}{1} = \frac{m'}{1}$ in $M_{S(M)}$ if and only if $m = m'$ in M^*. But this is easy to see, since $\frac{m}{1} = \frac{m'}{1}$ is equivalent to saying that there exists an element $a \in \mathsf{S}(M)$ such that $ma = m'a$. Since a is s-regular, we have $(mm')^n m = (mm')^n m'$. But since $m, m' \in M^*$, it implies that $mm' \in M^*$, hence $m = m'$, as required. $\mathcal{Q.E.D}$

It follows that one can take the localisation $M_{\mathsf{S}(M)}$. Moreover, the assignment $\mathfrak{p} \mapsto (M_\mathfrak{p})_{\mathsf{S}(M_p)}$ gives rise to a contravariant functor on the poset of $\mathsf{KSpec}(M)$. The associated sheaf is denoted by $s\mathcal{M}_X$ and is called the sheaf of *s-meromorphic* functions on X. Even though $\mathcal{O}_X \to s\mathcal{M}_X$ is no longer injective in general, we still have that $\mathcal{O}_X^* \to s\mathcal{M}_X^*$ is injective. Hence we get the short exact sequence of abelian groups:

$$1 \to \mathcal{O}_X^* \to s\mathcal{M}_X^* \to s\mathcal{M}_X^*/\mathcal{O}_X^* \to 1. \tag{3.9}$$

We call the sections of the sheaves $s\mathcal{M}_X^*/\mathcal{O}_X^*$ and $s\mathcal{M}_X^*$ the *s-divisors* and the *principal s-divisors* respectively. The quotient of the group of the s-divisors by the principal s-divisors is denoted by $s\mathsf{Cl}(X)$. If M is s-cancellative any element in M is s-regular since for every elements a, u, v with $a^m u = a^m v$ we have $(uv)^n u = (uv)^n v$ as $a^m \equiv b \in M$. Hence $s\mathcal{M}_X^*$ is the constant sheaf G, where G is the Grothendieck group of M. Clearly this is also true for any s-cancellative monoid scheme X. By Theorem 3.5.5 the sheaf $\mathcal{O}_X^*$ is a subsheaf of $s\mathcal{M}_X^*$. For cancellative schemes one has $s\mathcal{M}_X^* = \mathcal{M}_X^*$.

Hence by applying the long cohomological exact sequence to the short exact sequence (3.9) we obtain the following fact.

Proposition 3.5.10. *One has a monomorphism* $s\mathsf{Cl}(X) \to \mathsf{Pic(X)}$*, which is an isomorphism if X is s-cancellative.*

Remark 4: Note that neither in the definition of s-regular elements, nor in the Propositions 3.5.8 and 3.5.9 did we use the fact that we were working with monoids and not with rings. Indeed this generalisation as well as that of the Cartier divisors holds for rings as well.

3.6 Vanishing of $H^i(X, \mathcal{O}_X^*)$, $i \geq 2$

A monoid scheme of finite type is called *smooth*, provided there exist an open covering with affine monoid schemes of the type $\mathsf{KSpec}(M)$, where $M = \mathbb{N}^r \times \mathbb{Z}^s$. It is well known that $H^i(X, \mathcal{O}_X^*)$, $i \geq 2$ vanishes for smooth (ring)-schemes. The aim of this section is to prove that the analogue holds for monoid schemes. Actually, we will prove a more general result, see Theorem 3.6.6 below. We will generalise smoothness by introducing the notion of s-smoothness and we prove that the vanishing result is true for s-smooth monoid schemes.

3.6.1 S-Flasque Sheaves and Functors

A sheaf F of abelian groups on a topological space X is called *s-flasque* provided for any open subset U the restriction map $F(X) \to F(U)$ has a section. By Lemma 3.1.5 in our circumstance sheaves can be replaced by functors. So we will work with functors instead of sheaves.

Let P be a poset. Recall that a subset $X \subset P$ is called *open*, provided for any $x \in X$ and $y \leq x$, it follows that $y \in X$. Let F be a contravariant functor on P with values in the category of abelian groups. F is called *s-flasque*, provided for all open subsets $X \subset Y$ the induced map

$$\Gamma(Y, F) = \lim_{y \in Y} F_y \to \lim_{x \in X} F_x = \Gamma(X, F)$$

is a split epimorphism. It is obvious that such functors correspond exactly to the s-flasque sheaves under the equivalence constructed in Lemma 3.1.5.

Let $A^e = (A^e_x)_{x \in P}$ be a collection of abelian groups indexed by a poset P. A functor *generated by the collection* A^e is the contravariant functor A on P defined by

$$A_x = \prod_{y \leq x} A^e_y,$$

and for $x \leq y$, the map $A_y \to A_x$ is the natural projection.

Lemma 3.6.1. *Let X be a $\mathfrak{P}$-topological space and P the poset associated to X as in Proposition 3.1.4, part 3. A contravariant functor F defined on P is s-flasque, if and only if F is isomorphic to a functor generated by a collection of abelian groups $A^e = (A^e_x)_{x \in P}$.*

Proof. Assume F is a functor generated by a collection A^e. Then for any open subset $U \subset X$, one has

$$\lim_{u \in U} F_u \cong \prod_{x \in U} A^e_x, \tag{3.10}$$

and hence F is s-flasque.

The converse will be proven by induction on the Krull dimension $dim(X)$ of X. The case $dim(X) = 0$ being trivial. Assume F is s-flasque. Let us define the collection A^e by

$$A_x^e = \ker(F_x \xrightarrow{f_x} \lim_{y<x} F_y).$$

We claim there is an isomorphism $\theta : F \to A^e$. To construct θ, we choose at once sections s_x of the maps $f_x : F_x \to \lim_{y<x} F_y$ for all $x \in X$. Next, we define

$$\theta_x : F_x \to A_x$$

by induction on the hight of x. If the hight of x is zero, then x is a minimal element of X, thus in this case $F_x = A_x^e = A_x$, and we can take θ_x to be the identity map. Assume θ_y is defined for all y, for which $ht(y) < ht(x)$ and let $Y = \{y \in X | y < x\}$. The Krull dimension of Y is strictly smaller than the Krull dimension of X. Thus by the induction assumption, the functor F restricted on Y is isomorphic to a functor generated by a collection of abelian groups $(A_y^e)_{y<x}$, and hence by the isomorphism (3.10) one has an isomorphism:

$$\lim_{y<x} F_y \to \prod_{y<x} A_y^e.$$

Using s_x, we now can define θ_x as the composite of isomorphisms

$$F_x \to A_x^e \times \lim_{y<x} F_y \to A_x^e \times \prod_{y<x} F_y = \prod_{y \leq x} F_y.$$

$\mathcal{Q.E.D}$

This result has the following immediate consequence.

Proposition 3.6.2. *Let F be a sheaf on a $\mathfrak{P}$-topological space X. Then F is s-flasque if and only if F is locally s-flasque.*

Recall that if P and Q are posets, then $P \times Q$ is a poset, with $(p_1, q_1) \leq (p_2, q_2)$ if and only if $p_1 \leq p_2$ and $q_1 \leq q_2$.

Lemma 3.6.3. *Let P (resp. Q) be a poset with the least element denoted by e (resp. f). Let A (resp. B) be a contravariant functor defined on P (resp. on Q). Define $A \times B$ to be a contravariant functor defined on $P \times Q$ by*

$$(A \times B)_{(p,q)} = A_p \times B_q.$$

If A and B are s-flasque, then $A \times B$ is also s-flasque.

Proof. By assumption A is generated by $(A_x^e)_{x \in P}$ and B is generated by $(B_y^e)_{y \in Q}$. Define

$$C_{(x,y)}^e = \begin{cases} A_x^e \times B_y^e & x = e, y = f \\ A_x^e & x \neq e, y = f \\ B_y^e & x = e, y \neq f \\ 0 & x \neq e, y \neq f. \end{cases}$$

One sees that

$$(A \times B)_{(p,q)} = (\prod_{x \leq p} A_x^e) \times (\prod_{y \leq q} B_y^e) = \prod_{(x,y) \leq (p,q)} C_{x,y}^e$$

and the result follows. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

3.6.2 S-Smooth Monoid Schemes

All monoids in this subsection are assumed to be of finite type.

Definition 3.6.4. *An s-cancellative monoid scheme is called* s-smooth *provided the sheaf $s\mathcal{M}_X^*/\mathcal{O}_X^*$ is s-flasque. An s-cancellative monoid M is s-smooth provided $X = \mathsf{KSpec}(M)$ is s-smooth.*

Proposition 3.6.5. *i) For a monoid scheme to be s-smooth is a local property.*

ii) If X and Y are s-smooth monoid schemes, then $X \times Y$ is also s-smooth.

iii) Any smooth monoid scheme is s-smooth.

Proof. i) This is a direct consequence of Theorems 3.6.2 and 3.5.7.

ii) According to [12] $X \times Y$ locally looks like $\mathsf{KSpec}(M_1 \times M_2)$ for monoids M_1 and M_2. So by (i) we need to consider only the affine case. In this case the statement follows from Lemma 3.6.3.

iii) By previous results, we only need to show that $\mathbb{Z}$ and $\mathbb{N}$ are s-smooth monoids. Both are trivial to prove.

$\mathcal{Q.E.D}$

Proposition 3.6.6. *If X is an s-smooth monoid scheme, then for all $i \geq 2$ one has*

$$H^i(X, \mathcal{O}_X^*) = 0.$$

Proof. By the same argument as in the proof of Proposition 3.5.2, the sheaf $s\mathcal{M}_X^*$ is constant, provided X is s-cancellative. Thus for separated and s-cancellative monoid schemes one has

$$H^i(X, \mathcal{O}_X^*) = H^{i-1}(X, s\mathcal{M}_X^*/\mathcal{O}_X^*), \quad i \geq 2.$$

If additionally X is s-smooth, the last group vanishes, because any s-flasque sheaf is flasque and hence has zero cohomology in all positive dimensions, as mentioned in Lemma 1.6.1. $\mathcal{Q.E.D}$

As a corollary we obtain that $H^i(X, \mathcal{O}_X^*) = 0$, for all $i \geq 2$, provided X is smooth. This finishes the classical analogue and while the proof might have been longer, it is also more general. Now we give examples of monoids, which are s-smooth, but not smooth.

Example 7: Let

$$M = \langle a_1, \cdots, a_n, e_1, \cdots, e_m \rangle / a_1 \cdots a_n = a_1 \cdots a_n \cdot e_1^{i_1} \cdots e_n^{i_n}, \quad e_j^{k_j} = e_j^{k_j + i_j}.$$

Then M is s-smooth.

Example 8: Another example of an s-smooth monoid is $M = \langle u, a, b \rangle / u^2 = ab = u^3$.

Conjecture. *While these examples are distinct from each other, neither are cancellative. Indeed I have a suspicion that if M is a finitely generated, cancellative monoid and $\mathcal{M}_X^*/\mathcal{O}_X^*$ is flasque, then $M/M^* \cong \mathbb{N}^r$.*

Since we are not giving proofs, the next example, which is a special case of Ex.7, is written out in detail to convince the reader.

Example 9: Let $S = \langle a, b, e \rangle / ab = abe, \quad e^2 = e$. Then S is s-smooth.

Proof. First observe that $e^2 = e$ is not going to affect the associated semilattice S^{sl} of S. Hence the Kato-spectrum of this monoid is going to be dual to the monoid M given in Example 6 on page 31. That is $\mathsf{KSpec}(S)$ and M^{sl} will have the same ordering. This immediately tells us how the semilattice $\mathsf{KSpec}(S)$ is going to look, including the fact that it has 7 elements. In more detail, we have

$$X = \mathsf{KSpec}(S) = \{\varnothing, (a), (b), (a,b), (a,e), (b,e), (a,b,e)\}.$$

Hence $\mathcal{O}_X^*$ and $s\mathcal{M}_X^*/\mathcal{O}_X^*$ look as follows, in the category of abelian groups (in other words $\langle a \rangle$ denotes $\mathbb{Z}$, rather then $\mathbb{N}$, etc.):

$\mathcal{O}_X^*$ $\qquad\qquad\qquad\qquad$ $s\mathcal{M}_X^*/\mathcal{O}_X^*$

$$\begin{tikzcd}
 & 1 \arrow[dl] \arrow[d] \arrow[dr] & \\
\langle a \rangle \arrow[d] & 1 \arrow[dl] \arrow[dr] & \langle b \rangle \arrow[d] \\
\langle a \rangle \arrow[dr] & & \langle b \rangle \arrow[dl] \\
 & \langle a, b \rangle &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & \langle a, b \rangle \arrow[dl] \arrow[d] \arrow[dr] & \\
\langle b \rangle \arrow[d] & \langle a, b \rangle \arrow[dl] \arrow[dr] & \langle a \rangle \arrow[d] \\
\langle b \rangle \arrow[dr] & & \langle a \rangle \arrow[dl] \\
 & 1 &
\end{tikzcd}$$

Since every morphism is the canonical morphism, i.e. $a \mapsto a$, it is straightforward to see that $s\mathcal{M}_X^*/\mathcal{O}_X^*$ is indeed s-flasque. It is also clear that M is not smooth. $\mathcal{Q.E.D}$

Lemma 3.6.7. *Let M be a finitely generated s-cancellative monoid and $X = \mathsf{KSpec}(M)$. Then*

$$s\mathcal{M}_X^*/\mathcal{O}_X^* = s\mathcal{M}_{X/X^*}^*/\mathcal{O}_{X/X^*}^*$$

where X/X^* denotes the Spectrum of M/M^*.

Part II

Stacks, Costacks And The Fundamental Groupoid

Chapter 4

2-Mathematics

The roots of '2-mathematics' go back to Grothendieck and possibly even earlier. However, it was only relatively recently that it started to gain popularity in mainstream mathematics. Today however, 2-categories, or more generally n- or even ∞-categories are seen everywhere in algebraic topology, algebraic geometry and of course category theory.

Purely from its definition, a (strict) 2-category is just a category enriched in categories, i.e. where for every two objects A and B, $\mathsf{Hom}(A, B)$ is a category. As such, the categorical philosophy here is that we generalise sets to categories. Furthermore, just as we were generally interested in objects up to isomorphisms in category theory, here we are interested in objects up to equivalences. This already introduces notions like 2-limits and 2-colimits in a natural way.

The topological philosophy however, is slightly different. Here the analogue of a set, which can be seen as a topological space (up to homotopy) with only π_0, is not a category, but a groupoid. Formally, a groupoid is just a category where every morphism is an isomorphism. But they can be thought of as a topological space (up to homotopy) that only has π_0 and π_1.

This viewpoint enables for a more conceptual approach to many aspects of topology. One of its many uses, is the fact that it enables us to get rid of the basepoint when dealing with the fundamental group. We will build on this advantage and show in Theorems 5.1.2 and 8.0.7 that the fundamental groupoid has a universal property. More precisely, we will given an axiomatic definition using costacks, the dual notion of stacks. To do so however, we need to be familiar with 2-limits and 2-colimits.

In the first section, we will first give some basic definitions and results on 2-categories, including groupoids. In Section 4.2 we will talk about the various types of limits that arise in 2-categories and compare them with each other. We will also give explicit constructions for these definitions, except for the general case of a 2-colimit. As we will prove in Subsection 4.2.5, we can reduce the 2-colimit of a diagram, to the colimit, by 'deforming' our categories. This result has many calculatory uses as we will show in Section 5.2.

After this, we will talk a little about the 2-mathematical analogue of sheaves, namely stacks. Instead of defining them using the descent data however, we will use the more categorical approach and define them using 2-limits. This will give us a direct way of defining their dual notion, the less well-known costacks. While we do not know if costackification exists, in Subsection 4.3.2.3 we give a simplification for checking whether a 2-functor $\mathfrak{F}$ is a costack or not.

4.1 Introduction to 2-Category Theory

4.1.1 Definition and Basic Results on 2-Categories

This subsection, as well as 4.3.1, follows the second chapter (Lecture) of [36] very closely, even on notations.

Definition 4.1.1. *A strict 2-category $\mathfrak{C}$ consists of the following:*

1. *A class of objects, denoted by* $\mathsf{obj}(\mathfrak{C})$*;*
2. *A family of categories* $\mathsf{Hom}_{\mathfrak{C}}(X,Y)$*, indexed by* $\mathsf{obj}(\mathfrak{C}) \times \mathsf{obj}(\mathfrak{C})$*;*

    objects of $\mathsf{Hom}_{\mathfrak{C}}(X,Y)$ *are called morphisms of* $\mathfrak{C}$*, or sometimes 1-cells,*

    morphisms of $\mathsf{Hom}_{\mathfrak{C}}(X,Y)$ *are called 2-morphisms of* $\mathfrak{C}$*, or 2-cells;*
3. *A family of functors*

$$\mu_{x,y,z} : \mathsf{Hom}_{\mathfrak{C}}(X,Y) \times \mathsf{Hom}_{\mathfrak{C}}(Y,Z) \to \mathsf{Hom}_{\mathfrak{C}}(X,Z)$$

indexed by $\mathsf{obj}(\mathfrak{C}) \times \mathsf{obj}(\mathfrak{C}) \times \mathsf{obj}(\mathfrak{C})$, *called composition functors, such that the following diagram*

$$\begin{tikzcd}
\mathsf{Hom}_{\mathfrak{C}}(X,Y) \times \mathsf{Hom}_{\mathfrak{C}}(Y,Z) \times \mathsf{Hom}_{\mathfrak{C}}(Z,T) \arrow[r, "\mathsf{Id}\times\mu_{Y,Z,T}"] \arrow[d, "\mu_{X,Y,Z}\times\mathsf{Id}"'] & \mathsf{Hom}_{\mathfrak{C}}(X,Y) \times \mathsf{Hom}_{\mathfrak{C}}(Y,T) \arrow[d, "\mu_{X,Y,T}"] \\
\mathsf{Hom}_{\mathfrak{C}}(X,Z) \times \mathsf{Hom}_{\mathfrak{C}}(Z,T) \arrow[r, "\mu_{X,Z,T}"'] & \mathsf{Hom}_{\mathfrak{C}}(X,T)
\end{tikzcd}$$

is commutative for all 4-tuples (X,Y,Z,T) *of objects of* $\mathfrak{C}$;

4. *For each object* X *of* $\mathfrak{C}$, *there is an object* i_X *of* $\mathsf{Hom}_{\mathfrak{C}}(X,X)$ *such that the following diagrams*

$$\begin{tikzcd}
\mathbf{1} \times \mathsf{Hom}_{\mathfrak{C}}(\mathbf{X},\mathbf{Y}) \arrow[rr, "i_X\times\mathsf{Id}"] \arrow[dr, "="'] & & \mathsf{Hom}_{\mathfrak{C}}(X,X) \times \mathsf{Hom}_{\mathfrak{C}}(X,Y) \arrow[dl, "\mu_{X,X,Y}"] \\
& \mathsf{Hom}_{\mathfrak{C}}(X,Y) &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathsf{Hom}_{\mathfrak{C}}(Y,X) \times \mathbf{1} \arrow[rr, "\mathsf{Id}\times i_X"] \arrow[dr, "="'] & & \mathsf{Hom}_{\mathfrak{C}}(Y,X) \times \mathsf{Hom}_{\mathfrak{C}}(X,X) \arrow[dl, "\mu_{Y,X,X}"] \\
& \mathsf{Hom}_{\mathfrak{C}}(Y,X) &
\end{tikzcd}$$

commute. Here $\mathbf{1}$ *denotes the category with one object and only the identity morphism.*

Whenever one of our arrows are invertible, we add the prefix iso. For example, an invertible morphism (i.e. object of $\mathsf{Hom}_{\mathfrak{C}}(X,Y)$)is called an *isomorphism*, an invertible 2-morphism a *2-isomorphism*, and so on.

Definition 4.1.2. *A morphism* $F : A \to B$ *in a 2-category is called an* equivalence*, if we are given the following:*

- *A morphism* $G : B \to A$;
- *Two 2-isomorphisms* $f : F \circ G \to \mathsf{Id}_B$ *and* $g : G \circ F \to \mathsf{Id}_A$.

Note that there exists a more general version of this, where composition need not be associative, but only associative up to a 2-isomorphism which satisfies a coherence condition. But since we will only deal with strict 2-categories, we will not define the more general version. Indeed we will usually try to define the least general cases that we need.

Definition 4.1.3. *Let $\mathfrak{C}$ and $\mathfrak{D}$ be two 2-categories. We say $\mathfrak{F} : \mathfrak{C} \to \mathfrak{D}$ is a (covariant)* 2-functor *if*

- *$\mathfrak{F}$ takes objects to objects, morphisms to morphisms and 2–morphisms to 2-morphisms.*
- *For all chains $A \xrightarrow{i} B \xrightarrow{j} C$ in $\mathfrak{C}$, we have a 2-isomorphism $\tau_{i,j} : j_* i_* \Rightarrow (ji)_*$, where i_* denotes $\mathfrak{F}(i)$, such that for all triple compositions $A \xrightarrow{i} B \xrightarrow{j} C \xrightarrow{k} D$, the diagram*

$$\begin{tikzcd}
(kji)_* \arrow[r, Rightarrow, "\tau_{k,ji}"] \arrow[d, Rightarrow, "\tau_{i,kj}"] & k_*(ji)_* \arrow[d, Rightarrow, "k_*\tau_{i,j}"] \\
(kj)_* i_* \arrow[r, Rightarrow, "\tau_{j,k} i^*"] & k^* j^* i^*
\end{tikzcd}$$

  commutes. This is called the coherent compatibility condition*.*
- *On 2-morphisms we require that $\mathfrak{F}$ respects the composition.*

If the above 2-functor is invertible, that is, if we have $\mathfrak{G} : \mathfrak{D} \to \mathfrak{C}$ with $\mathfrak{F} \circ \mathfrak{G} = \mathsf{Id}$ and $\mathfrak{G} \circ \mathfrak{F} = \mathsf{Id}$, then $\mathfrak{F}$ is called a 2-*isomorphism.* In this case, we say that $\mathfrak{C}$ and $\mathfrak{D}$ are 2-*isomorphic.*

Remark 5: It should be pointed out, that we use the term 2-isomorphism for two different concepts. However, since there can be no ambiguity from the context, we will keep this terminology for simplicity.

Definition 4.1.4. *A 2-functor $\mathfrak{F} : \mathfrak{C} \to \mathfrak{D}$ is said to be* strict*, if for all $i, j \in I$, the isomorphism $\tau_{i,j}$ is the identity.*

Definition 4.1.5. *Let $\mathfrak{F} : \mathfrak{C} \to \mathfrak{D}$ and $\mathfrak{G} : \mathfrak{C} \to \mathfrak{D}$ be two 2-functors. A* natural transformation *$\phi : \mathfrak{F} \Rightarrow \mathfrak{G}$ of 2-functors is the following data:*

- *A morphism* $\phi_U : \mathfrak{F}(U) \to \mathfrak{G}(U)$ *for each* $U \in \mathsf{obj}(\mathfrak{C})$,
- *A 2-isomorphism* $\alpha_i : i_*\phi_V \Rightarrow \phi_U i_*$ *for each* $i : V \to U$, *i.e.*

$$\begin{tikzcd} \mathfrak{F}(V) \arrow[r, "\phi_V"] \arrow[d, "i_*"'] \arrow[dr, phantom, "\alpha_i \Nearrow"] & \mathfrak{G}(V) \arrow[d, "i_*"] \\ \mathfrak{F}(U) \arrow[r, "\phi_U"'] & \mathfrak{G}(U) \end{tikzcd}$$

such that for each composition $W \xrightarrow{j} V \xrightarrow{i} U$, *the diagram*

$$\begin{tikzcd} & (ij)_*\phi_W \arrow[rr, Rightarrow, "\alpha_{ij}"] \arrow[dl, Rightarrow, "\tau_{j,i}\phi_W"] & & \phi_U(ij)_* \arrow[dr, Rightarrow, "\phi_U\tau_{j,i}"] & \\ i_*j_*\phi_W \arrow[rr, Rightarrow, "i_*\alpha_j"] & & i_*\phi_V j_* \arrow[rr, Rightarrow, "\alpha_i j_*"] & & \phi_U i_* j_* \end{tikzcd}$$

commutes.

An invertible natural transformation is called a *natural isomorphism.*

Definition 4.1.6. *Let* $\phi, \psi : \mathfrak{F} \Rightarrow \mathfrak{G}$ *be two natural transformations of two 2-functors. A* natural 2-transformation $\mu : \phi \to \psi$, *also known as a fibred transformation or modification, consists of 2-morphisms*

$$\mu_U : \phi_U \Rightarrow \psi_U,$$

for each $U \in \mathsf{obj}(\mathfrak{C})$, *such that for every* $i : V \to U$, *the diagram*

$$\begin{tikzcd} i_*\phi_V \arrow[r, Rightarrow, "\alpha_i"] \arrow[d, Rightarrow, "i_*\mu_V"'] & \phi_U i_* \arrow[d, Rightarrow, "\mu_U i_*"] \\ i_*\psi_V \arrow[r, Rightarrow, "\beta_i"'] & \psi_U i_* \end{tikzcd}$$

commutes.

Following the notations, if μ is invertible, we will call it a *natural 2-isomorphism.*

Definition 4.1.7. *Let $\mathfrak{C}$ and $\mathfrak{D}$ be two 2-categories and $\mathfrak{F} : \mathfrak{C} \to \mathfrak{D}$ a 2-functor between them. We say that $\mathfrak{F}$ is a 2-*equivalence*, if there exists a 2-functor $\mathfrak{G} : \mathfrak{D} \to \mathfrak{C}$, and two natural isomorphisms $\phi : \mathfrak{F} \circ \mathfrak{G} \Rightarrow \mathsf{Id}$, $\varphi : \mathfrak{G} \circ \mathfrak{F} \Rightarrow \mathsf{Id}$. If that is the case, $\mathfrak{C}$ and $\mathfrak{D}$ are said to be 2-*equivalent*.*

We have the following well-known (see for example in [29, Prop. 1.5.13]) result.

Proposition 4.1.8. *The 2-functor $\mathfrak{F} : \mathcal{A} \to \mathcal{B}$ is a 2-equivalence of 2-categories if and only if the following two conditions hold:*

- *For every $A, A' \in \mathsf{obj}(\mathcal{A})$ we have an equivalence of categories*

$$\mathsf{Hom}_{\mathcal{A}}(A, A') \to \mathsf{Hom}_{\mathcal{B}}(\mathfrak{F}(A), \mathfrak{F}(A'));$$

- *For every object $B \in \mathcal{B}$, there exists an object $B' \in \mathcal{B}$, such that B' is equivalent to B and B' is in the image of $\mathfrak{F}$.*

When $\mathfrak{F}$ satisfies the first condition, we say that $\mathfrak{F}$ is *full and faithful*. When it satisfies the second, we call $\mathfrak{F}$ *essentially surjective*.

4.1.2 Groupoids

Definition 4.1.9. *Let $\mathfrak{G}$ be a category. We say that $\mathfrak{G}$ is a *groupoid*, if every morphism in $\mathfrak{G}$ is an isomorphism.*

Note that we can look at a group as a groupoid by defining its objects to be a single object, and its endomorphisms to be the elements of the group. Composition is given by the group law. As such, groupoids generalise groups in a very natural way. Indeed, one can think of a group as a connected groupoid, due to the following simple, and well-known lemma.

Lemma 4.1.10. *Let $\mathfrak{G}$ be a groupoid such that for every pair of objects $x, y \in \mathfrak{G}$, there exists a morphism $\phi : x \to y$. Then $\mathfrak{G}$ is equivalent to a groupoid coming from a group.*

A groupoid satisfying the condition of the lemma is called a *connected groupoid*. A connected groupoid such that $\mathsf{Aut}(x)$ is trivial, is called a *trivial* or *simply connected* groupoid. It is equivalent to the groupoid with one object and only the identity morphism, and is denoted by $\mathbf{1}$.

But in general, a groupoid is not connected. The set of connected components of a groupoid is denoted by $\pi_0(\mathfrak{G})$. Likewise, $\mathsf{Aut}(x)$ is sometimes written as $\pi_1(\mathfrak{G}, x)$. We say that a groupoid is *discrete*, if for every object x of our groupoid, $\mathsf{Aut}(x)$ only has the identity.

Proposition 4.1.11. *Let $\mathfrak{G}$ be a connected groupoid and let $x \in \mathfrak{G}$. We have an equivalence of categories*

$$\mathsf{Hom}_{\mathfrak{Cat}}(\mathfrak{G}, \mathsf{Sets}) \cong \mathsf{Aut}(x)\text{-}\mathsf{Sets}$$

where $\mathsf{Aut}(x)$-Sets *denotes the category of* $\mathsf{Aut}(x)$*-Sets.*

Proof. We know that equipping a set S with a G-set structure is equivalent to saying that we have a map $G \to \mathsf{Aut}(S)$. Since $\mathfrak{G}$ is connected, we can assume that it only has one object x. Take $\mathcal{F} \in \mathsf{Hom}_{\mathfrak{Cat}}(\mathfrak{G}, \mathsf{Sets})$ and let $\mathcal{F}(x) = S$, where S is a set. Since a morphism in the 2-category of categories is a functor, $\mathcal{F}$ induces a map $\mathsf{Aut}(x) \to \mathsf{Aut}(S)$, hence defines an $\mathsf{Aut}(x)$-set. It can be easily checked that natural transformations between two such functors $\mathcal{F}$ and $\mathcal{G}$ are equivalent to equivariant maps. $\mathcal{Q.E.D}$

If we changed Sets with FSets, the category of finite sets, this result would still hold. This result enables us to generalise group actions to groupoid actions, which we will use in chapter 6

Definition 4.1.12. *Let $\mathfrak{G}$ be a groupoid. We say that $\mathfrak{G}$ is a* finitely connected, profinite groupoid*, if it is equivalent to the 2-coproduct, in the 2-category of groupoids (see Subsection 4.2.4), of finitely many, connected groupoids coming from profinite groups.*

Note that this is not the best way to define a profinite groupoid in the general case, as it does not deal with the topology on its connected components. One should define it as simply the filtered 2-limit (see Subsection 4.2.2) of finite groupoids endowed with the profinite topology. But for the purposes of this thesis however, this simpler definition is adequate.

4.2 2-Limits and 2-Colimits

Just like the limit and colimit of a functor are of fundamental importance in category theory, so too are the 2-limit and 2-colimit of a 2-functor in 2-category theory. If the 2-functor is strict however, we can also talk about its limit and colimit.

Analogous to how a set S can be seen as a category $\mathcal{S}$, whose objects are the elements of S and the only morphisms are identities, a category can be 'lifted' to a 2-category. Here the objects and morphisms remain the same and the 2-morphisms are just taken to be the identities. Hence, we can talk about 2-functors between categories and 2-categories.

The aim of this section is to give the definitions and constructions of the various limits of 2-categories. Note that, just like in the last subsection, we will not define the most general cases, but only up to the generality we will actually need. Most notably, this means that the source of our functors will be categories, rather then 2-categories.

We will give the definitions, as well as constructions, of the limit/2-limit and the colimit of 2-functors. However, we will not give the general construction for the 2-colimit. The reason is that it is not an easy construction, and as we will show in Subsection 4.2.5, we will not actually need to for the purposes of this thesis.

We will now fix the notations, that will be used throughout this section. Let $\mathfrak{F} : I \to \mathfrak{Cat}$ be a covariant 2-functor from the category I to the 2-category $\mathfrak{Cat}$ of categories. For an element $i \in I$, let $\mathfrak{F}_i$ be the value of $\mathfrak{F}$ at i. For a morphism $\psi : i \to j$, let $\psi_* : \mathfrak{F}_i \to \mathfrak{F}_j$ be the induced functor. For any chain $i \xrightarrow{\psi} j \xrightarrow{\nu} k$, one has the natural transformation $\mu_{\psi,\nu} : \nu_*\psi_* \Rightarrow (\nu\psi)_*$, satisfying the coherent condition.

4.2.1 Limits of Categories

Definition 4.2.1. *Let $\mathfrak{F} : I \to \mathfrak{Cat}$ be a strict 2-functor. The limit of $\mathfrak{F}$ is a category $\lim\limits_i \mathfrak{F}_i$, together with a collection of morphisms $f_i : \lim\limits_i \mathfrak{F}_i \to \mathfrak{F}_i$, such that $\psi_* \circ f_i = f_j$.*

This data has to satisfy the following property:

- *For any category* $\mathfrak{G}$ *with* $g_i : \mathfrak{G} \to \mathfrak{F}_i$ *satisfying* $\psi_* \circ g_i = g_j$, *there is a unique morphism* $g : \mathfrak{G} \to \lim\limits_i \mathfrak{F}_i$, *such that for all* i, *the diagram*

$$\begin{tikzcd} \mathfrak{G} \arrow[d, "g"'] \arrow[dr, "g_i"] & \\ \lim\limits_i \mathfrak{F}_i \arrow[r, "f_i"'] & \mathfrak{F}_i \end{tikzcd}$$

  commutes.

- *If* $g, h : \mathfrak{G} \to \lim\limits_i \mathfrak{F}_i$ *are two functors, then for any collection of natural transformations*

$$\alpha_i : f_i \circ g \Rightarrow f_i \circ h$$

  for which $\psi_*(\alpha_i) = \alpha_j$, *there exists a unique natural transformation* $\alpha : g \Rightarrow h$, *such that* $\alpha_i = f_i \alpha$

Proposition 4.2.2. *Limits exist, and objects of the category* $\lim\limits_i \mathfrak{F}_i$ *are families* (x_i), *where* x_i *is an object of the category* $\mathfrak{F}_i$, *such that all maps* $\psi : i \to j$, *one has* $\psi_*(x_i) = x_j$. *A morphism* $(x_i) \to (y_i)$ *is a family* (f_i), *where* $f_i : x_i \to y_i$ *is a morphism of* $\mathfrak{F}_i$, *such that for any* $\psi : i \to j$, *one has* $\psi_*(f_i) = f_j$.

This is well known. See for example [17, p.5]. To define limits for other 2-categories, such as the 2-category of groupoids, one only needs to consider the uniqueness condition in said 2-category. Alternatively, one can define limits as follows:

Definition 4.2.3. *Let* $\mathfrak{F} : I \to \mathcal{A}$ *be a strict covariant 2-functor in a general 2-category* $\mathcal{A}$. *The limit of* $\mathfrak{F}$ *is the following:*

- *An object* $\lim\limits_i \mathfrak{F}_i$ *in* $\mathcal{A}$, *together with morphisms* $f_i : \lim\limits_i \mathfrak{F}_i \to \mathfrak{F}_i$, *satisfying* $f_i \circ \psi_* = f_j$;

- *For all objects $A \in \mathcal{A}$, the functor*

$$\mathsf{Hom}_{\mathcal{A}}(A, \lim_i \mathfrak{F}_i) \to \lim_i \mathsf{Hom}_{\mathcal{A}}(A, \mathfrak{F}_i)$$

given by $\kappa : \chi \mapsto (\chi \circ f_i)$, where $\chi \in \mathsf{Hom}_{\mathcal{A}}(A, \lim_i \mathfrak{F}_i)$, is an isomorphism.

It is straightforward to see that when $\mathcal{A}$ is the 2-category of categories, these two definitions coincide.

4.2.2 2-Limits of Categories

Definition 4.2.4. *The 2-limit of $\mathfrak{F} : I \to \mathfrak{Cat}$ is a category $2\text{-}\lim_i \mathfrak{F}_i$, together with a family of functors $(f_i : 2\text{-}\lim_i \mathfrak{F}_i \to \mathfrak{F}_i)_{i \in \mathsf{obj}(I)}$ and natural transformations*

$$(\zeta_\psi : \psi_* \circ f_i \Rightarrow f_j)_{\psi \in \mathsf{Hom}_I(i,j)},$$

satisfying the coherent condition for any composable maps $i \xrightarrow{\psi} j \xrightarrow{\nu} k$ in I. In other words, $\zeta_\nu \zeta_\psi \mu_{\psi,\nu} = \zeta_{\nu\psi}$. These data must satisfy the following properties:

- *For any category $\mathfrak{G}$ with $g_i : \mathfrak{G} \to \mathfrak{F}_i$ and compatible $\eta_\psi : \psi_* \circ g_i \to g_j$, we have a functor $a : \mathfrak{G} \to 2\text{-}\lim_i \mathfrak{F}_i$ and a natural isomorphism $a_i : g_i \Rightarrow f_i \circ a$, such that the diagram*

$$\begin{tikzcd} \psi_* \circ g_i \arrow[r, Rightarrow, "\eta_\psi"] \arrow[d, Rightarrow, "\psi \circ a_i"'] & g_j \arrow[d, Rightarrow, "a_j"] \\ \psi_* \circ f_i \circ a \arrow[r, Rightarrow, "\zeta_\psi \circ a"'] & f_j \circ a \end{tikzcd}$$

commutes.

- *If there is given a category $\mathfrak{G}$ and two functors $a, b : \mathfrak{G} \to 2\text{-}\lim_i \mathfrak{F}_i$, then for any collection of natural transformations*

$$\alpha_i : f_i \circ a \Rightarrow f_i \circ b,$$

for which the diagram

$$\begin{tikzcd} \psi_* \circ f_i \circ a \arrow[r, Rightarrow, "\psi_*(\alpha_i)"] \arrow[d, Rightarrow, "\zeta_\psi \circ a"'] & \psi_* \circ f_i \circ b \arrow[d, Rightarrow, "\zeta_\psi \circ b"] \\ f_j \circ a \arrow[r, Rightarrow, "\alpha_j"'] & f_j \circ b \end{tikzcd}$$

commutes, there exists a unique natural transformation $\alpha : a \Rightarrow b$*, such that* $\alpha_i = f_i\alpha$

Proposition 4.2.5. *2-Limits exist and:*

- *Objects of the category* $2\text{-}\lim_i \mathfrak{F}_i$ *are collections* (x_i, ξ_ψ)*, where* x_i *is an object of* $\mathfrak{F}_i$*, while* $\xi_\psi : \psi_*(x_i) \to x_j$ *for* $\psi : i \to j$ *is an isomorphism of the category* $\mathfrak{F}_j$*, satisfying the* 1-cocycle condition*. That is, for any* $i \xrightarrow{\psi} j \xrightarrow{\nu} k$*, the diagram:*

$$\begin{tikzcd} \nu_*(\psi_*(x_i)) \arrow[r, "\nu_*(\xi_\psi)"] \arrow[d, "\mu_{\psi,\nu}"] & \nu_*(x_j) \arrow[d, "\xi_\nu"] \\ (\nu\psi)_*(x_i) \arrow[r, "\xi_{\psi\nu}"'] & x_k \end{tikzcd}$$

commutes

- *Morphisms from* (x_i, ξ_ψ) *to* (y_i, η_ψ) *are collections of morphisms* $(f_i : x_i \to y_i)$*, such that for any* $\psi : i \to j$*, the following is a commutative diagram:*

$$\begin{tikzcd} \psi_*(x_i) \arrow[r, "\xi_\psi"] \arrow[d, "\psi_*(f_i)"'] & x_j \arrow[d, "f_j"] \\ \psi_*(y_i) \arrow[r, "\eta_\psi"'] & y_j. \end{tikzcd}$$

This result is well known and is a particular case of a far more general construction given in [42]. Comparing this construction to the one given for limits shows us immediately that the following holds:

Lemma 4.2.6. *Let $\mathfrak{F} : I \to \mathfrak{Cat}$ be a strict 2-functor. Then the functor*

$$\gamma : \lim \mathfrak{F} \to 2\text{-}\mathsf{lim}\, \mathfrak{F},$$

given by

$$\gamma(x_i) = (x_i, \mathsf{Id}),$$

is full and faithful.

Remark 6: Let $f : A \to B$ and $g : B \to C$ be two morphisms in the 2-limit of $\mathfrak{F} : I \to \mathfrak{Cat}$. Then composition is defined componentwise. To see that consider the following commutative diagram:

$$\begin{tikzcd}
\psi_*(A_i) \arrow[r, "\alpha_\psi"] \arrow[d, "\psi_*(f_i)"'] & A_j \arrow[d, "f_j"] \\
\psi_*(B_i) \arrow[r, "\beta_\psi"] \arrow[d, "\psi_*(g_i)"'] & B_j \arrow[d, "g_j"] \\
\psi_*(C_i) \arrow[r, "\gamma_\psi"] & C_j .
\end{tikzcd}$$

Since ψ_*, the induced functor of $\psi : i \to j$, is by definition a functor, we have that $\psi_*(f_i)\psi_*(g_i) = \psi_*(f_i g_i)$. Hence $(f_i g_i)$ defines a morphism in the 2-limit of $\mathfrak{F}$, which we denote by fg.

Remark 7: Let (x_i, ξ_ψ) and (y_i, ν_ψ) be objects in the 2-limit of $\mathfrak{F} : I \to \mathfrak{Cat}$. From Proposition 4.2.5 we immediately see that we have the following exact sequence of sets:

$$\mathsf{Hom}_{2\text{-}\underset{i}{\mathsf{lim}}\, \mathfrak{F}_i}(a, b) \longrightarrow \prod_i \mathsf{Hom}_{\mathfrak{F}_i}(a_i, b_i) \rightrightarrows \prod_{\psi : i \to j} \mathsf{Hom}_{\mathfrak{F}_j}(\psi_*(a_i), b_j).$$

The alternative definition of 2-limits is as follows:

Definition 4.2.7. *Let $\mathfrak{F} : I \to \mathcal{A}$ be a covariant 2-functor in a general 2-category $\mathcal{A}$. The 2-limit of $\mathfrak{F}$ is the following data:*

- *An object 2-*$\lim_i \mathfrak{F}_i$ *with morphisms* $f_i : 2\text{-}\lim_i \mathfrak{F}_i \to \mathfrak{F}_i$ *and 2-morphisms* $\zeta_\psi : f_i \circ \psi_* \Rightarrow f_j$*, satisfying the coherent condition, meaning* $\zeta_\nu \zeta_\psi \mu_{\psi,\nu} = \zeta_{\nu\psi}$*;*

- *Additionally we require that for all objects* $A \in \mathcal{A}$*, the functor*

$$\mathsf{Hom}_{\mathcal{A}}(A, 2 - \lim_i \mathfrak{F}_i) \to 2 - \lim_i \mathsf{Hom}_{\mathcal{A}}(A, \mathfrak{F}_i)$$

  given by $\kappa : \chi \to (\chi \circ f_i, \chi \circ \zeta_\phi)$*, where* $\chi \in \mathsf{Hom}_{\mathcal{A}}(A, 2 - \lim_i \mathfrak{F}_i)$*, is an equivalence of categories .*

It can be shown that this definition agrees with the above definition in the case when $\mathcal{A}$ is the 2-category of categories. We have the following fact which is straightforward to check:

Proposition 4.2.8. *The assignment*

$$\mathfrak{L} : \{\text{2-Functors over } I \text{ with values in categories}\} \to \mathfrak{Cat}$$

defined in the following way:

i) $\{\mathfrak{F} : I \to \mathfrak{Cat}\} \mapsto 2\text{-}\lim \mathfrak{F}$ *on objects;*

ii) $\{\varphi : \mathfrak{F} \to \mathfrak{G}\} \mapsto \{\varphi' : 2\text{-}\lim \mathfrak{F} \to 2\text{-}\lim \mathfrak{G}\}$ *given by*

$$x = (x_i, \xi) \mapsto (\varphi(x_i), \varphi(\xi)), \qquad \{f : x \to y\} = (f_i) \mapsto (\varphi(f_i))$$

on morphisms, and

iii) $\{\lambda : \varphi \Rightarrow \phi\} \mapsto \{\lambda' : \varphi' \Rightarrow \phi'\}$ *given by* $\lambda'(\varphi'(x)) = \lambda(\varphi(x_i), \varphi(\xi))$ *on 2-morphisms,*

defines a 2-functor.

4.2.3 Colimits of Categories

Definition 4.2.9. *Let $\mathfrak{F} : I \to \mathfrak{Cat}$ be a strict 2-functor. The colimit of $\mathfrak{F}$ is a category $\operatorname{colim}_i \mathfrak{F}_i$, together with morphisms $f_i : \mathfrak{F}_i \to \operatorname{colim}_i \mathfrak{F}_i$ such that $f_i = f_j \circ \psi_*$. This data must satisfy the following property:*

- *For any other category $\mathfrak{G}$ with $g_i : \mathfrak{F}_i \to \mathfrak{G}$ satisfying $g_i = g_j \circ \psi_*$, we have a unique morphism $g : \operatorname{colim}_i \mathfrak{F}_i \to \mathfrak{G}$, such that for all i, the diagram*

$$\begin{tikzcd} \mathfrak{F}_i \arrow[d, "f_i"'] \arrow[dr, "g_i"] & \\ \operatorname{colim}_i \mathfrak{F}_i \arrow[r, "g"'] & \mathfrak{G} \end{tikzcd}$$

  commutes.

- *If $g, h : \operatorname{colim}_i \mathfrak{F}_i \to \mathfrak{G}$ are two functors, then for any collection of natural transformations*

$$\alpha_i : g \circ f_i \Rightarrow h \circ f_i$$

  for which $\psi_*(\alpha_i) = \alpha_j$, there exists a unique natural transformation $\alpha : g \Rightarrow h$, such that $\alpha_i = \alpha f_i$

Alternatively we can define the colimit as follows:

Definition 4.2.10. *Let $\mathfrak{F} : I \to \mathcal{A}$ be a strict 2-functor and $\mathcal{A}$ a general 2-category. Then the colimit of $\mathfrak{F}$ is:*

- *An object $\operatorname{colim}_i \mathfrak{F}_i$ of $\mathcal{A}$, together with a collection of functors*

$$f_i : \mathfrak{F}_i \to \operatorname{colim}_i \mathfrak{F}_i$$

  such that $f_i = f_j \circ \psi_*$.

- *Additionally, one requires that for any object G of $\mathcal{A}$, the canonical functor*

$$c : \mathsf{Hom}_{\mathcal{A}}(\operatorname{colim}_i \mathfrak{F}_i, G) \to \lim_i \mathsf{Hom}_{\mathcal{A}}(\mathfrak{F}_i, G),$$

given by $c(\chi) = (\chi \circ f_i)$*, is an isomorphism of categories. Here,* $\chi : \operatorname*{colim}_i \mathfrak{F}_i \to G$ *is is a morphism in* $\mathcal{A}$*,*

It is well known that colimits exist in the 2-categories of groupoids and categories [17, p. 36].

4.2.3.1 Construction in the General Case

The construction of the colimit of a diagram of small categories, is quite a bit harder then the limit. First we will need to define what a graph is, then take the colimit of graphs, make a free category out of that obtained graph and finally quotient the obtained category in such a way, that compositions are respected. The following construction is of course well known. See for example [17, p.4].

Definition 4.2.11. *Let* A^0 *and* A^1 *be sets and let*

$$A^1 \underset{d^1}{\overset{d^0}{\rightrightarrows}} A^0 \xrightarrow{i} A^1$$

be a diagram. We say that it is a graph *if* $d^0 \circ i = d^1 \circ i = \mathsf{Id}_{A^1}$*, and denote it with* A^* *for short.*

Elements of A_0 are called objects, while elements of A_1 are called arrows. For an arrow $f \in A_1$, $d^0(f)$ and $d^1(f)$ are called the domain and range of f. For an object $a \in A_0$, the arrow $i(a)$ is called the identity arrow of a.

Definition 4.2.12. *Let* A^* *and* B^* *be graphs. A morphism of graphs* $f^* : A^* \to B^*$ *is defined to be a pair of maps* $f^0 : A^0 \to B^0$ *and* $f^1 : A^1 \to B^1$*, such that*

- $d^i \circ f^0 = f^1 \circ d^i$ *for* $i = 0, 1$ *and*
- $i \circ f^1 = f^0 \circ i$*.*

Hence we can talk about the category of graphs, which we denote by Graphs. There is an obvious forgetful functor $U : \mathfrak{Cat} \to \mathsf{Graphs}$, which assigns to a small category C the sets C^0, C^1 of objects and arrows. The forgetful functor has a left adjoint functor $F : \mathsf{Graphs} \to \mathfrak{Cat}$ given by assigning to a graph A^* the free category generated by A^*.

Objects of $F(A^*)$ are the elements of A^0. For two objects $a, b \in A^0$, a morphism is defined to be a sequence $(f^1, \cdots, f^k)$ where $f^1, \cdots, f^k \in A^1, d^0(f^k) = a, d^1(f^1) = b^1$, and $d^1(f^i) = d^0(f^{i+1}), 0 \le i \le k$. Moreover, if $k > 1$, then $f^i \in A^1 \setminus i(A^0), i = 1, \cdots, k$. Composition of $(f^1, \cdots, f^k)$ and $(g^1, \cdots, g^n)$ is defined in the obvious way if $k, n \ge 2$. If $k = 1$ and $f^1 = i(a)$ or $n = 1$ and $g^1 = i(b)$, then the composition is $(g^1, \cdots, g^n)$ or $(f^1, \cdots, f^k)$ respectively.

Let $\mathfrak{G} : I \to \mathsf{Graphs}$ be a functor. For any $i \in I$, one has a graph $\mathfrak{G}_i$ with set of objects $\mathfrak{G}_i^0$ and set of arrows $\mathfrak{G}_i^1$. In this way, one obtains two functors

$$\mathfrak{G}^0, \mathfrak{G}^1 : I \to \mathsf{Sets}$$

and natural transformations

$$\begin{tikzcd} \mathfrak{G}^1 \arrow[r, shift left, "d^0"] \arrow[r, shift right, "d^1"'] & \mathfrak{G}^0 \arrow[r, "i"] & \mathfrak{G}^1 \end{tikzcd} .$$

Now passing to the colimits, we obtain a diagram of sets

$$\begin{tikzcd} \operatorname{colim} \mathfrak{G}^1 \arrow[r, shift left, "d^0"] \arrow[r, shift right, "d^1"'] & \operatorname{colim} \mathfrak{G}^0 \arrow[r, "i"] & \operatorname{colim} \mathfrak{G}^1 \end{tikzcd} ,$$

which is a graph, and it is the colimit in the category Graphs. Assume $\mathfrak{F} : I \to \mathfrak{Cat}$ is a functor. To take the colimit of $\mathfrak{F}$, we first take the colimit of $\mathfrak{G} :\equiv U \circ \mathfrak{G} : I \to \mathsf{Graphs}$ and obtain

$$\begin{tikzcd} \mathfrak{G}^1 \arrow[r, shift left, "d^0"] \arrow[r, shift right, "d^1"'] & \mathfrak{G}^0 \arrow[r, "i"] & \mathfrak{G}^1 \end{tikzcd} .$$

Denote the graph by B. We can form a free category $F(B)$. For any i, we have a morphism of graphs $\alpha_i : U(\mathfrak{F}_i) \to F(B)$. The colimit of $\mathfrak{F} : I \to \mathfrak{Cat}$ is the quotient of $F(B)$ by the minimal congruence on $F(B)$ under which $\alpha_i(g) \circ \alpha_i(f) \sim \alpha_i(g \circ f)$, for all $i \in I$ and all composable morphisms $a \xrightarrow{f} b \xrightarrow{g} c$ in $\mathfrak{F}_i$.

4.2.3.2 The Filtered Case

Let I be a filtered category and let $\mathfrak{F} : I \to \mathfrak{Cat}$ a strict 2-functor. For any $i \in Obj(I)$, any object $A \in Obj(\mathfrak{F}_i)$ defines an object $[A]$ in $\operatorname{colim} \mathfrak{F}_i$. For $B \in Obj(\mathfrak{F}_i)$, one has $[A] = [B]$ if and only if there are morphisms $i \xrightarrow{\alpha} k \xleftarrow{\beta} j$, such that $\alpha_*(A) = \beta_*(B)$

in $\mathfrak{F}_k$. Any morphism $f : A \to A'$ of the category $\mathfrak{F}_i$ defines the morphism

$$[f] : [A] \to [A'].$$

For a morphism $g : B \to B'$ of the category F_j, one has $[f] = [g]$, if and only if there are morphism $i \xrightarrow{\alpha} k \xleftarrow{\beta} j$, such that

$$\alpha_*(A) = \beta_*(B), \alpha_*(A') = \beta_*(B'), \alpha_*(f) = \beta_*(g).$$

If $f : A \to A'$ is a morphism in $\mathfrak{F}_i$ and $g : B \to B'$ is a morphism in F_j such that $[A'] = [B']$, then the composite $[g] \circ [f] : [A] \to [B']$ is defined as follows: Choose morphisms $i \xrightarrow{\alpha} k \xleftarrow{\beta} j$ such that $\alpha_*(A') = \beta_*(B)$. Then

$$[g] \circ [f] := [\beta_* g \circ \alpha_* f].$$

4.2.4 2-Colimits of Categories

Definition 4.2.13. *Let $\mathfrak{F} : I \to \mathfrak{Cat}$ be a 2-functor. The 2-colimit of $\mathfrak{F}$ is the following:*

- *A category 2-$\operatorname{colim}_i \mathfrak{F}_i$, together with a family of functors*

$$\alpha_i : \mathfrak{F}_i \to 2 - \operatorname{colim}_i \mathfrak{F}_i$$

and natural transformations $\lambda_\psi : \alpha_j \psi_* \Rightarrow \alpha_i$, satisfying the following condition: For any $i \xrightarrow{\psi} j \xrightarrow{\nu} k$, the following diagram

$$\begin{tikzcd}
\alpha_k(\nu\psi)_* \arrow[r, Rightarrow, "\lambda_{\nu\psi}"] & \alpha_i \\
\alpha_k \nu_* \psi_* \arrow[u, Rightarrow, "\alpha_k \circ \mu_{\psi,\nu}"] \arrow[r, Rightarrow, "\lambda_{\nu_*} \circ \psi_*"'] & \alpha_j \psi_* \arrow[u, Rightarrow, "\lambda_\psi"']
\end{tikzcd}$$

commutes.

- *For any category C, the canonical functor*

$$\kappa : \mathsf{Hom}_{\mathfrak{Cat}}(2-\operatorname*{\mathsf{colim}}_i \mathfrak{F}_i, C) \to 2-\lim_i \mathsf{Hom}_{\mathfrak{Cat}}(\mathfrak{F}_i, C)$$

is an equivalence of categories. Here κ is given by $\kappa(\chi) = (\chi \circ \alpha_i, \chi_i \circ \lambda_\psi)$.

It is well known that the 2-colimit exists and is unique up to an equivalence of categories, see, [8, pp. 192-193]. The analogous statement for groupoids holds as well [22, Exposé VI, Section 6].

4.2.4.1 The Filtered Case

Let I be a filtered category and $\mathfrak{F} : I \to \mathfrak{Cat}$ a 2-functor in the 2-category of small categories. The set of objects of the category 2-$\mathsf{colim}\,\mathfrak{F}$ is the disjoint union of the objects of $\mathfrak{F}_i$. Thus, any object $A \in \mathfrak{F}_i$ is also an object of 2-$\mathsf{colim}\,\mathfrak{F}$. Take $A \in \mathfrak{F}_i$ and $B \in \mathfrak{F}_j$. Any diagram of the form

$$i \xrightarrow{\alpha} k \xleftarrow{\beta} j$$

and any morphism $\alpha_*(A) \xrightarrow{h} \beta_*(B)$ in $\mathfrak{F}_k$ determines a morphism $A \to B$ in the category 2-$\mathsf{colim}\,\mathfrak{F}$. This morphism will be denoted by $[\alpha, \beta, h]$. If we take an other morphism $[\alpha', \beta', h']$ with $i \xrightarrow{\alpha'} k' \xleftarrow{\beta'} j$ and $h' \in \mathsf{Hom}_{\mathfrak{F}_k}(\alpha'_*(A), \beta'_*(B))$, then

$$[\alpha, \beta, h] = [\alpha', \beta', h']$$

if and only if there exists a diagram $k \xrightarrow{\gamma} l \xleftarrow{\gamma'} k'$, such that $\gamma\alpha = \gamma'\alpha', \gamma\beta = \gamma'\beta'$ and that the following is a commutative diagram:

$$\begin{tikzcd}
(\gamma\alpha)_*(A) \arrow[r] \arrow[d, "\mathsf{Id}"'] & \gamma_*\alpha_*(A) \arrow[r, "\gamma_* h"] & \gamma_*\beta_*(B) \arrow[r] & (\gamma\beta)_*(B) \arrow[d, "\mathsf{Id}"] \\
(\gamma'\alpha')_*(A) \arrow[r] & \gamma'_*\alpha'_*(A) \arrow[r, "\gamma'_* h'"'] & \gamma'_*\beta'_*(B) \arrow[r] & (\gamma'\beta')_*(B).
\end{tikzcd}$$

It follows from the definition, that if $[\alpha,\beta,h] : A \to B$ is a morphism in 2-$\mathsf{colim}\,\mathfrak{F}$, then for every morphism $\nu : k \to l$, one has

$$[\alpha,\beta,h] = [\nu\alpha,\nu\beta,\nu_*(h)].$$

To define the composition in 2-$\mathsf{colim}\,\mathfrak{F}$, let $[\alpha,\beta,h] : A \to B$ and $[\gamma,\delta,g] : B \to C$ be morphism in 2-$\mathsf{colim}\,\mathfrak{F}$, where $h : \alpha_*(A) \to \beta_*(A)$ and $g : \gamma_*(B) \to \delta_*(C)$ are morphism in $\mathfrak{F}_k$ and $\mathfrak{F}_m$ respectively. Here, $i \xrightarrow{\alpha} k \xleftarrow{\beta} j$ and $j \xrightarrow{\gamma} m \xleftarrow{\gamma} l$ are diagrams in I. Since I is directed, there are morphisms $k \xrightarrow{\mu} n \xleftarrow{\zeta} m$ such that $\mu\beta = \zeta\gamma$. Then one puts

$$[\gamma,\delta,g] \circ [\alpha,\beta,h] = [\mu\alpha,\mu\beta = \zeta\gamma,\zeta_* g \circ \mu_* h].$$

Proposition 4.2.14. *Let I be a filtered category and $\mathfrak{F} : I \to \mathfrak{Cat}$ be a 2-functor. Take $\mathcal{A} \in \mathfrak{F}_i$ and $\alpha : i \to j$, a morphism in I. Then, A and $\alpha_*(A)$ are isomorphic in the 2-colimit of $\mathfrak{F}$.*

Proof. The diagram $i \xrightarrow{\alpha} j \xleftarrow{\mathsf{Id}} j$ in I, induces the map $[\alpha,\mathsf{Id}_j,\mathsf{Id}_{\alpha_*(A)}] : A \to \alpha_*(A)$. Its inverse is given by $[\mathsf{Id}_j,\alpha,\mathsf{Id}_{\alpha_*(A)}] : \alpha_*(A) \to A$, which is coming from the diagram $j \xrightarrow{\mathsf{Id}} k \xleftarrow{\alpha} i$. $\mathcal{Q.E.D}$

By the very definition of morphisms in the 2-colimit, we have the following result:

Proposition 4.2.15. *Let $\mathfrak{F} : I \to \mathfrak{Cat}$ be a filtered 2-colimit. Take $a \in \mathfrak{F}_i$ and $b \in \mathfrak{F}_j$ and denote by $I_{i,j} := \{i \xrightarrow{\alpha} k, j \xrightarrow{\beta} k\}$. Then we have a bijection*

$$\mathsf{Hom}_{2\text{-}\underset{i}{\mathsf{colim}}\,\mathfrak{F}_i}(a,b) = \underset{k\in I_{i,j}}{\mathsf{colim}}\,\mathsf{Hom}_{\mathfrak{F}_k}(\alpha_*(a),\beta_*(b)).$$

We have the following straight forward to check fact:

Proposition 4.2.16. *The assignment*

$$\mathfrak{L} : \{\text{2-Functors over } I \text{ with values in categories}\} \to \mathfrak{Cat}$$

defined in the following way:

i) $\{\mathfrak{F} : I \to \mathfrak{Cat}\} \mapsto$ 2-$\mathsf{colim}\,\mathfrak{F}$ *on objects;*

ii) $\{\varphi : \mathfrak{F} \to \mathfrak{G}\} \mapsto \{\varphi' : 2\text{-}\mathsf{colim}\,\mathfrak{F} \to 2\text{-}\mathsf{colim}\,\mathfrak{G}\}$ *given by*

$$x = x_i \mapsto \varphi(x_i), \qquad [\alpha, \beta, f : x_i \to y_j] \mapsto [\varphi(\alpha), \varphi(\beta), \varphi(f)]$$

on morphisms, and

iii) $\{\lambda : \varphi \Rightarrow \phi\} \mapsto \{\lambda' : \varphi' \Rightarrow \phi'\}$ *given by* $\lambda'(\varphi'(x)) = \lambda'(\varphi'(x_i)) = \lambda(\varphi(x_i))$ *on 2-morphisms,*

defines a 2-functor.

4.2.5 Comparison of the Colimit and the 2-Colimit

Proposition 4.2.17. *Let I be a filtered poset and $\mathfrak{F} : I \to \mathfrak{Cat}$ a strict 2-functor. We have an equivalence of categories*

$$\delta : 2\text{-}\operatorname*{\mathsf{colim}}_i \mathfrak{F}_i \to \operatorname*{\mathsf{colim}}_i \mathfrak{F}_i.$$

Proof. We will start by describing the functor δ. Recall that objects of the 2-colimit are the disjoint union of the objects of the $\mathfrak{F}_i$-s (Section 4.2.4.1). Since the objects of the colimit of $\mathfrak{F}$ are a quotient of the disjoint union of $\mathsf{obj}(\mathfrak{F}_i)$ (Section 4.2.3.2), we clearly have δ defined on objects. We also immediately see that δ is essentially surjective (indeed it is surjective).

To define δ on morphisms, take $A \in \mathfrak{F}_i$, $B \in \mathfrak{F}_j$ and

$$\varphi \in \mathsf{Hom}_{2\text{-}\operatorname*{\mathsf{colim}}_i \mathfrak{F}_i}(a, b) \cong \operatorname*{\mathsf{colim}}_{k \in I_{i,j}} \mathsf{Hom}_{\mathfrak{F}_k}(\alpha_*(A), \beta_*(B))$$

(for the result and notation see Proposition 4.2.15). Pick a representative

$$\mathfrak{F}_k \ni \varphi_k : \alpha_*(A) \to \alpha_*(B)$$

of φ. Since every morphism in every $\mathfrak{F}_k$ defines a morphism in the colimit, clearly so does φ_k. It is straightforward to see that this map is well defined.

To show that δ is full and faithful, assume $\delta(\varphi) = \delta(\phi)$ in $\mathsf{Hom}_{\mathsf{colim}_i \mathfrak{F}_i}(\delta(A), \delta(B))$. There exists $k \in I_{i,j}$ such that $\delta(\varphi)_k = \delta(\phi)_k$. This is the exact same condition as for the equality in the 2-colimit. Lastly, to prove surjectivity between the Hom-sets, take $f : A \to B$ in the colimit of $\mathfrak{F}$. Since f is in the colimit, choose a representative f_k of f in some $\mathfrak{F}_k$. Clearly $[\mathsf{Id}, \mathsf{Id}, f_k] \in \mathsf{Hom}_{2\text{-}\mathsf{colim}_i \mathfrak{F}_i}(A, B)$ maps to f. $\mathcal{Q.E.D}$

Lemma 4.2.18 (Deformation Lemma)**.** *Let*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "i_1"] \arrow[d, "i_2"'] \arrow[dr, phantom, "\swarrow\lambda"] & \mathcal{B} \arrow[d, "j_1"] \\ \mathfrak{C} \arrow[r, "j_2"'] & \mathfrak{D} \end{tikzcd}$$

be a 2-diagram of groupoids, that is, $\lambda : j_1 i_1 \Rightarrow j_2 i_2$ is a natural isomorphism. If the functor i_1 is injective on objects, there exist a functor $j_1' : \mathcal{B} \to \mathfrak{D}$ and a natural transformation $\kappa : j_1 \Rightarrow j_1'$, for which the diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "i_1"] \arrow[d, "i_2"'] & \mathcal{B} \arrow[d, "j_1'"] \\ \mathfrak{C} \arrow[r, "j_2"'] & \mathfrak{D} \end{tikzcd}$$

commutes and λ coincides with $\kappa \circ i_1 : j_1 i_1 \to j_1' i_1 = j_2 i_2$.

Proof. Define j_1' on objects by

$$j_1'(b) = \begin{cases} j_1(b), & \text{if } b \notin Im(i_1) \\ j_2 i_2(a), & \text{if } b = i_1(a). \end{cases}$$

To define j_1' on morphisms, we proceed as follows. Let $\beta : b_1 \to b_2$ be a morphism in B. To define

$$j_1'(\beta) : j_1'(b_1) \to j_1'(b_2),$$

we have to consider five different cases.

Case 1. $b_1 = i_1(a_1)$ and $b_2 \notin Im(i_1)$. One defines $j_1'(\beta)$ to be the composite:

$$j_1'(b_1) = j_2 i_2(a_1) \xrightarrow{\lambda^{-1}(a_1)} j_1 i_1(a_1) = j_1(b_1) \xrightarrow{j_1(\beta)} j_1(b_2) = j_1'(b_2).$$

Case 2. $b_1 = i_1(a_1)$, $b_2 = i_1(a_2)$, but $\beta \notin Im(i_1)$. One defines $j_1'(\beta)$ to be the composite:

$$j_1'(b_1) = j_2 i_2(a_1) \xrightarrow{\lambda^{-1}(a_1)} j_1 i_1(a_1) = j_1(b_1) \xrightarrow{j_1(\beta)} j_1(b_2) = j_1 i_1(a_2) \xrightarrow{\lambda(a_2)} j_2 i_2(a_2) = j_1'(b_2).$$

Case 3. $\beta \in Im(j_1)$. We choose $\alpha : a_1 \to a_2$ in A such that $\beta = i_1(\alpha)$. One defines $j_1'(\beta)$ by

$$j_1'(b_1) = j_2 i_2(a_1) \xrightarrow{j_2 i_2(\alpha)} j_2 i_2(a_2) = j_1'(b_2).$$

To check that this is independent of the choice of α, consider $\alpha' : a_1 \to a_2$ with the property $i_1(\alpha') = \beta$. Since λ is a natural transformation, we have a commutative diagram

$$\begin{tikzcd}
j_1 i_1(a) \arrow[rr, "\lambda(a)"] \arrow[d, shift right=2, "j_1 i_1(\alpha)"'] \arrow[d, shift left=2, "j_1 i_1(\alpha')"] \arrow[d, phantom, "="] & & j_2 i_2(a) \arrow[d, shift right=2, "j_2 i_2(\alpha)"'] \arrow[d, shift left=2, "j_2 i_2(\alpha')"] \\
j_1 i_1(a') \arrow[rr, "\lambda(a')"'] & & j_2 i_2(a')
\end{tikzcd}$$

Since the left vertical arrows are equal and the horizontal ones are isomorphisms, it follows from the commutativity, that the right vertical arrows are also equal. Hence, $j_1'(\beta)$ is well-defined in this case.

Case 4. $b_1 \notin Im(i_1)$ and $b_2 = i_1(a_2)$. One defines $j_1'(\beta)$ to be the composite

$$j_1'(b_1) = j_1(b_1) \xrightarrow{j_1(\beta)} j_1(b_2) = j_1 i_i(a_2) \xrightarrow{\lambda(a_2)} j_2 i_2(a_2) = j_2'(b_2).$$

Case 5. $b_1 \notin Im(i_1)$ and $b_2 \notin Im(i_1)$. One defines $j_1'(\beta)$ as

$$j_1'(b_1) = j_1(b_1) \xrightarrow{j_1(\beta)} j_1(b_2) = j_1'(b_2).$$

Checking case by case shows that j_1' is really a functor, with $j_1' i_1 = j_2 i_2$. Define κ by

$$\kappa(b) = \begin{cases} \mathsf{Id}_{j_1(b)}, & \text{if } b \notin Im(i_1) \\ \lambda(a), & \text{if } b = i_1(a). \end{cases}$$

One easily sees that j_1' and κ satisfy the assertions of the Lemma. $\mathcal{Q.E.D}$

Theorem 4.2.19. *Let*

$$\begin{tikzcd} A \arrow[r, "i_1"] \arrow[d, "i_2"'] & B \\ C & \end{tikzcd}$$

be a diagram of groupoids, where i_1 is injective on objects. Then the colimit and the 2-colimit are equivalent.

Proof. Consider the diagram

$$\begin{tikzcd} A \arrow[r, "i_1"] \arrow[d, "i_2"'] & B \arrow[d] \arrow[ddr] & \\ C \arrow[r] \arrow[drr] & P & \\ & & W, \arrow[ul, "\alpha"'] \end{tikzcd}$$

where P is the pushout and W is the 2-pushout (weak pushout) of $C \overset{i_2}{\leftarrow} A \overset{i_1}{\rightarrow} B$. Let E be any groupoid and consider:

$$\begin{tikzcd} \mathsf{Hom}(A,E) & \mathsf{Hom}(B,E) \arrow[l, "i_1^*"'] & \\ \mathsf{Hom}(C,E) \arrow[u, "i_2^*"] & \mathsf{Hom}(P,E) \arrow[l] \arrow[u] \arrow[dr, "\alpha^*"] & \\ & & \mathsf{Hom}(W,E). \arrow[ull] \arrow[uul] \end{tikzcd}$$

Since $\mathsf{Hom}(-,E)$ maps pushouts and 2-pushouts to pullbacks and 2-pullbacks, we have that $\mathsf{Hom}(P,E)$ and $\mathsf{Hom}(W,E)$ are the pullback and 2-pullback of the diagram

$\mathsf{Hom}(A,E) \xleftarrow{i_1^*} \mathsf{Hom}(B,E) \xrightarrow{i_2^*} \mathsf{Hom}(P,E)$ respectively. Using Lemma 4.2.6, we know that α^* is full and faithful. To see that it's an equivalence of categories, we only need to show that it's essentially surjective.

Since as mentioned $\mathsf{Hom}(W,E)$ is the 2-pullback, taking an object there is the same as taking objects in $\mathsf{Hom}(B,E)$ and $\mathsf{Hom}(C,E)$, and an equivalence between these objects in $\mathsf{Hom}(A,E)$. That is to say we, have a 2-commutative diagram:

$$\begin{tikzcd} A \arrow[r, "i_1"] \arrow[d, "i_2"'] \arrow[dr, phantom, "\nearrow"] & B \arrow[d] \\ C \arrow[r] & E. \end{tikzcd}$$

From this, we immediately see that the Deformation Lemma (4.2.18) shows essential surjectivity of α^*. This in turn, using the Yoneda lemma for 2-categories [32, Lemma 2.3], shows that $\alpha : W \to P$ is an equivalence of categories, completing the proof.

$\mathcal{Q.E.D}$

4.3 Stacks and Costacks

In this section we will talk about the 2-categorical analogues of sheaves and its dual notion cosheaves, which are called stacks and costacks respectively. We will define these using the 2-limit and 2-colimit constructed in the previous section.

As we will show, a costack is a very important construction. Indeed, saying that a 2-functor is a costack, is equivalent to saying that it satisfies a slightly reformulated version of the Seifert-van Kampen theorem for every covering. Lastly, we will give a simplified method for checking whether a 2-functor $\mathfrak{F}$ is a costack, by showing that it suffices to check whether $\mathfrak{F}$ satisfies the costack condition for every covering with only 2 objects.

Stacks and costacks will play a major role in this thesis, as it is our aim to prove that costacks enable us to axiomatise the fundamental groupoid. We will use stacks to prove this result in the algebraic case, as we will work with Galois categories, which form a stack over the site of étale coverings.

4.3.1 Stacks

As already mentioned, this subsection follows [36] closely, even on notation. Let X be a site and $\mathfrak{F} : X^{op} \to \mathfrak{Cat}$ a 2-functor, where $\mathfrak{Cat}$ is the 2-category of categories. This is called a *fibred category* over X. If $\mathfrak{F}$ is a strict 2-functor, we would call it a strict fibred category. It should be noted to avoid any confusion, that this is often called a prestack (which we call something else). We will however stick to this terminology. If we have two (strict) fibred categories over X, then a morphism between them is called a *(strict) fibred functor*. The following important fact holds:

Lemma 4.3.1. *Let $\mathfrak{F}$ be a fibred category over X and A, B be objects in $\mathfrak{F}(U)$. The assignment $V \mapsto Hom_{\mathfrak{F}(V)}(i^*(A), i^*(B))$, for any morphism $i : V \to U$, defines a presheaf on $X|_U$. It is denoted by $Hom_{\mathfrak{F}}(A, B)$.*

Definition 4.3.2 (Prestack)**.** *If the presheaf $Hom_{\mathfrak{F}}(A, B)$ is in addition a sheaf, we say that $\mathfrak{F}$ is a* prestack*.*

It is clear that every prestack is a fibred category. Hence, we have the inclusion 2-functor

$$\mathcal{A} : \{\textit{Prestacks over } X\} \to \{\textit{Fibred Categories over } X\}$$

given in the obvious way. This 2-functor has a left adjoint, called *prestackification*, which is given by sheafifying the functors $\mathsf{Hom}_{\mathfrak{F}}(A, B)$ for all A and B. More precisely, to a contravariant 2-functor $\mathfrak{F} : X \to \mathfrak{Cat}$, we assign $\overline{\mathfrak{F}} : X \to \mathfrak{Cat}$. The objects of the category $\overline{\mathfrak{F}}(U)$ are the same as that of $\mathfrak{F}$, and for every pair of objects (A, B), we define $Hom_{\overline{\mathfrak{F}}(U)}(A, B)$ to be the section of the sheafification of $Hom_{\mathfrak{F}}(A, B)$, seen as a contravariant functor.

Let X be a site and $\mathfrak{F} : X^{op} \to \mathfrak{Cat}$ a 2-functor. Let U be an object in X and $\mathfrak{U} = \{U_i \to U\}$ a covering of U. Then we can consider the following diagram:

$$\prod_{i \in I} \mathfrak{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathfrak{F}(U_{ij}) \Rrightarrow \prod_{i,j,k \in I} \mathfrak{F}(U_{ijk}),$$

where $U_{ij} := U_i \times_U U_j$ and $U_{ijk} := U_i \times_U U_j \times_U U_k$. We denote its 2-limit by 2-lim$(\mathfrak{U}, \mathfrak{F})$ and if the above 2-functor is additionally strict, its limit by $\lim(\mathfrak{U}, \mathfrak{F})$. Note that the last part $\prod_{i,j,k \in I} \mathfrak{F}(U_{ijk})$ does not factor in the limit, if it is defined, only the 2-limit. Also note that the 2-limit in this case is usually called the descent data and the following notation $\mathsf{Des}(\mathfrak{U},\mathfrak{F})$ is often used. But since we will work with 2-limits and 2-colimits throughout this thesis, we will keep using the above notation.

Definition 4.3.3 (Sheaf of categories)**.** *A strict fibred category $\mathfrak{F}$ over a site X is called a sheaf, if for all objects U of X and for all coverings $\mathfrak{U}$ of U, the functor $\mathfrak{F}(U) \to \lim(\mathfrak{U}, F)$ is an isomorphism of categories.*

Definition 4.3.4 (Stack)**.** *A fibred category $\mathfrak{F}$ over X is called a* stack, *if for all objects U of X and for all coverings $\mathfrak{U}$ of U, the functor $\mathfrak{F}(U) \to 2\text{-}\lim(\mathfrak{U},\mathfrak{F})$ is an equivalence of categories.*

Let X be a site. From Remark 7 on page 80 we immediately see that every stack over X is also a prestack over X, i.e. we have an inclusion 2-functor:

$$\mathcal{B} : \{Stacks\ over\ X\} \to \{Prestacks\ over\ X\}.$$

Proposition 4.3.5. *[Stackification] The 2-functor $\mathcal{B}$ has a left adjoint, called stackification. If $\mathfrak{F}$ is a prestack, its associated stack is defined by*

$$\hat{\mathfrak{F}}(U) := 2\text{-}\,\mathsf{colim}_{\mathfrak{U}}(2\text{-}\,\mathsf{lim}(\mathfrak{U},\mathfrak{F})).$$

For the proof of the above proposition, see [36, Theorem 2.1]. Just like for sheaves, the associated stack of a 2-functor can also be defined by its uniqueness property.

Definition 4.3.6 (Uniqueness Property)**.** *Let $\mathfrak{F}$ be a fibred category over a site X. Then $\hat{\mathfrak{F}}$ is the associated stack of $\mathfrak{F}$ if we are given a natural transformation $\mathfrak{F} \Rightarrow \hat{\mathfrak{F}}$ of 2-functors, such that for every stack $\mathfrak{G}$ over X, the induced functor*

$$\mathsf{Hom}_{\mathsf{St}}(\hat{\mathfrak{F}},\mathfrak{G}) \xrightarrow{\cong} \mathsf{Hom}_{\mathsf{Fb}}(\mathfrak{F},\mathfrak{G}),$$

is an equivalence of categories. Here St *denotes the 2-category of stacks and* Fb *the 2-category of fibred categories.*

Hence, by simply composing $\mathcal{A}$ and $\mathcal{B}$ and their adjoints, we can associate a stack to every fibred category over X. But there exists another, direct, way of stackifying a 2-functor, as shown in [41, Theorem 3.8]:

Proposition 4.3.7 (Direct Stackification). *Let $\mathfrak{F} : X^{op} \to \mathfrak{Cat}$ be a 2-functor and define $\mathfrak{F}'(U) := 2\text{-}\mathsf{colim}_{\mathfrak{U}}(2\text{-}\lim(\mathfrak{U}, \mathfrak{F}))$. Clearly we can iterate this construction. Then,*

$$\hat{\mathfrak{F}} := \mathfrak{F}'''(U) = 2\text{-}\mathsf{colim}_{\mathfrak{U}}(2\text{-}\lim(\mathfrak{U}, \mathfrak{F}''))$$

is the associated stack of $\mathfrak{F}$.

4.3.2 Costacks

4.3.2.1 Cosheaves

Before we define the dual notion of a stack, we will mention what it means for a functor to be a cosheaf.

Definition 4.3.8. *Let X be a site and $G : X \to \mathbf{A}$ be a functor with values in a category $\mathbf{A}$. We say that G is a* cosheaf *if for any covering $\mathfrak{U} = \{U_i \to U\}$ of any object $U \in X$, the diagram*

$$\coprod_{i,j} G(U_i \cap U_j) \overset{b_0}{\underset{b_1}{\rightrightarrows}} \coprod_i G(U_i) \xrightarrow{a} G(U) \longrightarrow \mathbf{1}$$

is exact, where $\mathbf{1}$ is the terminal object of $\mathbf{A}$. In other words, $G(U)$ is the coequaliser of b_0, b_1. That is to say, G is a cosheaf if and only if for any object S of A the presheaf F defined by $F(U) = \mathsf{Hom}_A(G(U), S)$ is a sheaf of sets [8].

Definition 4.3.9. *Let $G : X \to \mathbf{A}$ be a functor from a site X to a category $\mathbf{A}$. We say that $\hat{G} : X \to \mathbf{A}$ is the associated cosheaf of G, if:*

- *We are given a natural transformation $\epsilon : \hat{G} \Rightarrow G$;*

- *For any cosheaf $G' : X \to \mathbf{A}$ and any natural transformation $\varphi : G' \Rightarrow G$, there* *is a unique natural transformation $\hat{\varphi} : G' \Rightarrow \hat{G}$, such that the diagram*

$$\begin{tikzcd} G' \arrow[r, Rightarrow, "\varphi"] \arrow[dr, Rightarrow, "\hat{\varphi}"'] & G \\ & \hat{G} \arrow[u, Rightarrow, "\epsilon"'] \end{tikzcd}$$

commutes.

4.3.2.2 Costacks

Let X be a site and $\mathfrak{F} : X \to \mathfrak{Cat}$ a 2-functor, where $\mathfrak{Cat}$ is the 2-category of categories. Dually to Subsection 4.3.1, we call this a *cofibred category* over X. Take an object in $U \in X$ with a covering $\mathfrak{U} = \{U_i \to U\}$. Consider the following diagram:

$$\prod_{i \in I} \mathfrak{F}(U_i) \leftleftarrows \prod_{i,j \in I} \mathfrak{F}(U_{ij}) \Lleftarrow \prod_{i,j,k \in I} \mathfrak{F}(U_{ijk}),$$

where $U_{ij} := U_i \times_U U_j$ and $U_{ijk} := U_i \times_U U_j \times_U U_k$. We denote its 2-colimit by 2-$\mathsf{colim}(\mathfrak{U}, \mathfrak{F})$. If $\mathfrak{F}$ is additionally strict, we can also talk about its colimit, which is denoted by $\mathsf{colim}(\mathfrak{U}, \mathfrak{F})$. Note that the last part $\prod_{i,j,k \in I} \mathfrak{F}(U_{ijk})$ does not factor in the colimit, only the 2-colimit.

Definition 4.3.10 (Cosheaf)**.** *A strict cofibred category $\mathfrak{F}$ over X is called a* cosheaf*, provided, for every object U of X and every covering $\mathfrak{U}$ of U, the induced functor $\mathfrak{F}(U) \leftarrow \mathsf{colim}(\mathfrak{U}, \mathfrak{F})$, is an isomorphism of categories.*

Definition 4.3.11 (Costack)**.** *A cofibred category $\mathfrak{F}$ over X is called a* costack*, if for every object U of X and every covering $\mathfrak{U}$ of U, the functor $\mathfrak{F}(U) \leftarrow 2\text{-}\mathsf{colim}(\mathfrak{U}, \mathfrak{F})$ is an equivalence of categories.*

Due to the fact that $\mathsf{Hom}(-, A)$ is left exact, we immediately obtain the following lemma:

Lemma 4.3.12. *Let $\mathfrak{F}$ be a cofibred category over X. Then $\mathfrak{F}$ is a costack, if and only if for every category C, the assignment $U \mapsto \mathsf{Hom}_{\mathfrak{C}}(\mathfrak{F}(U), C)$ is a stack.*

It should be noted that since $\mathsf{Hom}(-,A)$ is not (in general) right exact, the duality between stacks and costacks breaks down here. Namely, it would not be sufficient to check that $U \mapsto \mathsf{Hom}_{\mathfrak{Cat}}(\mathfrak{F}(U),-)$ is a costack for $\mathfrak{F}$ to be a stack.

Definition 4.3.13 (Uniqueness Property)**.** *Let $\mathfrak{F}$ be a cofibred category over a site X. Then $\hat{\mathfrak{F}}$ is the associated costack of $\mathfrak{F}$ if we are given a natural transformation $\mathfrak{F} \Rightarrow \hat{\mathfrak{F}}$, such that for every costack $\mathfrak{G}$ over X, the induced functor*

$$\mathsf{Hom}_{\mathsf{Cofb}}(\mathfrak{G},\mathfrak{F}) \cong \mathsf{Hom}_{\mathsf{Cost}}(\mathfrak{G},\hat{\mathfrak{F}}),$$

is an equivalence of categories. Here Cofb *denotes the 2-category of cofibred categories and* Cost *the 2-category of costacks.*

Note that if our 2-functor $\mathfrak{F}$ took values in groupoids, it would suffice to check it only for costacks with values in groupoids. Unfortunately, unlike for stacks, it is not known whether every cofibred category has an associated costack.

4.3.2.3 From 2-Pushouts to Costacks

We will restrict ourself to topological spaces in this subsection, rather then work with general sites. Although this result should hold for sites as well, there are some additional difficulties, and since we will only need it for spaces, we will lower the generality.

Definition 4.3.14. *Let $n \geq 2$ be an integer. We will say that a strict 2-functor $\mathfrak{F} : \mathfrak{Off}(X) \to \mathfrak{Cat}$ has the property $\mathsf{sh}(n)$ (resp. $\mathsf{st}(n)$), if for any open subset U of X and any open cover $U_1, \cdots, U_n$ of U, the canonical functor*

$$\mathfrak{F}(U) \longrightarrow \lim[\textstyle\prod_i \mathfrak{F}(U_i) \rightrightarrows \prod_{i,j} \mathfrak{F}(U_{ij})]$$

is an isomorphism of categories, (resp.

$$\mathfrak{F}(U) \longrightarrow 2\text{-}\mathsf{lim}[\textstyle\prod_i \mathfrak{F}(U_i) \rightrightarrows \prod_{i,j} \mathfrak{F}(U_{ij}) \rightrightarrows\!\!\!\!\to \displaystyle\prod_{i,j,k} \mathfrak{F}(U_{ijk})]$$

is an equivalence of categories). That is to say, $\mathfrak{F}$ satisfies the sheaf (resp. stack) condition for all coverings having n-members.

In an analogous way we can define $\mathsf{cosh}(n)$ and $\mathsf{cost}(n)$.

Theorem 4.3.15. *Let X be a topological space and $\mathfrak{F} : \mathfrak{Off}(X) \to \mathsf{Grpd}$ a covariant 2-functor that commutates with filtered colimits.*

- *Assume $\mathfrak{F}$ is a strict 2-functor. Then $\mathfrak{F}$ is a cosheaf if and only if $\mathfrak{F}$ has the property $\mathsf{cosh}(2)$.*
- *$\mathfrak{F}$ is a costack if and only if $\mathfrak{F}$ has the property $\mathsf{cost}(2)$.*

To prove this, we first need to prove the following lemma.:

Lemma 4.3.16. *If $\mathfrak{F}$ satisfies the condition $\mathsf{sh}(2)$ (resp. $\mathsf{st}(2)$), then $\mathfrak{F}$ satisfies the condition $\mathsf{sh}(n)$ (resp. $\mathsf{st}(n)$) for any $n \geq 2$.*

Proof. We consider only the case $n = 3$, since the only difference between this and the general case is the notation. By definition, the objects of

$$2\text{-}\mathsf{lim}[\textstyle\prod_i \mathfrak{F}(U_i) \rightrightarrows \prod_{i,j} \mathfrak{F}(U_{ij}) \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \displaystyle\prod_{i,j,k} \mathfrak{F}(U_{ijk})]$$

are

$$((g_1, g_2, g_3), \alpha_{12}, \alpha_{13}, \alpha_{23})$$

where g_i is an object of $\mathfrak{F}(U_i)$, $i = 1, 2, 3$ and $\alpha_{12} : g_1|_{U_{12}} \to g_2|_{U_{12}}$, $\alpha_{13} : g_1|_{U_{13}} \to g_3|_{U_{13}}$, and $\alpha_{23} : g_2|_{U_{23}} \to g_3|_{U_{23}}$ are morphisms in $\mathfrak{F}(U_{12})$, $\mathfrak{F}(U_{13})$ and $\mathfrak{F}(U_{23})$ respectively. One also requires that these data satisfy the 1-cocycle condition. The morphisms are $(h_1 : g_1 \to g_1', h_2 : g_2 \to g_2', h_3 : g_3 \to g_3')$, such that restricted on the intersections, the obvious diagrams commute.

We set $V = U_1 \cup U_2$. Since $\mathfrak{F}$ satisfies the condition $st(2)$, we can assume that the objects of $F(U) = F(V \cup U_3)$ are the same as those of the category

$$2\text{-}\mathsf{lim}[\mathfrak{F}(V) \times \mathfrak{F}(U_3) \rightrightarrows \mathfrak{F}(V \cap U_3)].$$

Hence, we can assume that they are of the form

$$((g_v, g_3), \gamma : g_v|_{V\cap U_3} \to g_3|_{V\cap U_3}),$$

where g_v is an object of $\mathfrak{F}(V)$. Since $V = U_1 \cup U_2$ and $\mathfrak{F}$ satisfies the condition $st(2)$, we may assume that the objects of $\mathfrak{F}(V)$ have the form

$$g_v = ((g_1, g_2), \delta : g_1|_{U_{12}} \to g_2|_{U_{12}}).$$

Thus, the objects of $\mathfrak{F}(U) = \mathfrak{F}(V \cup U_3)$, can be written as

$$((g_1, g_2, g_3), \delta : g_1|_{U_{12}} \to g_2|_{U_{12}}, \gamma_1 : g_1|_{U_{13}} \to g_3|_{U_{13}}, \gamma_2 : g_2|_{U_{23}} \to g_3|_{U_{23}}),$$

such that

$$\begin{tikzcd}
g_1|_{U_{123}} \arrow[r] \arrow[d, "\delta|_{U_{123}}"'] & g_3|_{U_{123}} \arrow[d, "\mathsf{Id}"] \\
g_2|_{U_{123}} \arrow[r] & g_3|_{U_{123}}
\end{tikzcd}$$

commutes. Since the last condition is exactly the 1-cocycle condition, we can see that the categories $\mathfrak{F}(U)$ and

$$\mathsf{2\text{-}lim}[\textstyle\prod_i \mathfrak{F}(U_i) \rightrightarrows \prod_{i,j} \mathfrak{F}(U_i \times_U U_j) \Rrightarrow \displaystyle\prod_{i,j,k} \mathfrak{F}(U_i \times_U U_j \times_U U_k)]$$

have essentially the same objects.

The morphisms of $\mathfrak{F}(U) = \mathfrak{F}(V \cup U_3)$ are $(\varphi_V : g_V \to g'_V, \varphi_{U_3} : g_3 \to g'_3)$, such that the obvious diagram commutes. Those of $\mathfrak{F}(V)$ are $(\lambda_{U_1} : g_1 \to g'_1, \lambda_{U_2} : g_2 \to g'_2)$, such that again, the obvious diagram commutes. Hence, plugging one into the other, we get: $(\varphi_V : g_{U_1\cup U_2} \to g'_{U_1\cup U_2}, \varphi_{U_3} : g_3 \to g'_3)$. That is, the morphisms of $\mathfrak{F}(U)$ are

$$(\varphi_{U_1} : g_1 \to g'_1, \varphi_{U_2} : g_2 \to g'_2, \varphi_{U_3} : g_3 \to g'_3),$$

such that on the pairwise intersections, the restrictions agree. This shows that $\mathfrak{F}(U)$

is equivalent to

$$2\text{-}\mathsf{lim}[\prod_i \mathfrak{F}(U_i) \rightrightarrows \prod_{i,j} \mathfrak{F}(U_i \times_U U_j) \rightrightarrows \prod_{i,j,k} \mathfrak{F}(U_i \times_U U_j \times_U U_k)]\ .$$

Hence $st(3)$ holds.

The fact that $\mathsf{s}h(2)$ implies $\mathsf{s}h(3)$ is a special case of the above proof, by simply taking the α_i-s to be identities. $\mathcal{Q.E.D}$

Proof of Thm 4.3.15. The 'only if' part is obvious. Assume $\mathfrak{F}$ satisfies the condition $\mathsf{c}osh(2)$. This is equivalent to saying that the functor $\mathfrak{F}_{\mathcal{G}}$, defined by

$$\mathfrak{F}_{\mathcal{G}}(U) \coloneqq \mathsf{Hom}_{\mathsf{Grpd}}(\mathfrak{F}(U), \mathcal{G}),$$

satisfying the property $\mathsf{sh}(2)$ for every groupoid $\mathcal{G}$ (see Lemma. 4.3.12). From Lemma 4.3.16, we know that $\mathfrak{F}_{\mathcal{G}}$ satisfies $\mathsf{sh}(n)$ for every $n \in \mathbb{N}$ and every groupoid $\mathcal{G}$. Hence, by using Lemma 4.3.12 again, we know that $\mathfrak{F}$ has the property $\mathsf{cosh}(n)$.

Let U be an open set and $\{U_i\}_{i \in I}$ be a cover of U. Denote by $f(I)$ the set of finite subsets of I. Then $f(I)$ is a filtered system. For a fixed $\lambda \in f(I)$, we denote $U_\lambda = \bigcup_i U_i$, $i \in \lambda$. By assumption, $\mathfrak{F}$ satisfies the cosheaf condition for any finite covering. We have

$$\mathfrak{F}(U) = \mathfrak{F}(\operatorname*{\mathsf{colim}}_{\lambda} U_\lambda) = \operatorname*{\mathsf{colim}}_{\lambda} \mathfrak{F}(U_\lambda) = \operatorname*{\mathsf{colim}}_{\lambda}(\mathsf{colim}[\coprod_{i,j \in \lambda} \mathfrak{F}(U_{ij}) \rightrightarrows \coprod_{i \in \lambda} \mathfrak{F}(U_i)]) =$$

$$\mathsf{colim}[\operatorname*{\mathsf{colim}}_{\lambda} \coprod_{i,j \in \lambda} \mathfrak{F}(U_{ij}) \rightrightarrows \operatorname*{\mathsf{colim}}_{\lambda} \coprod_{i \in \lambda} \mathfrak{F}(U_i)]s = \mathsf{colim}[\coprod_{i,j \in I} \mathfrak{F}(U_{ij}) \rightrightarrows \coprod_{i \in I} \mathfrak{F}(U_i)]$$

as desired. Here the second equality is true by the assumption that $\mathfrak{F}$ commutes with filtered colimits. Recall that by Proposition 4.2.17, we have

$$\mathfrak{F}(U) = \operatorname*{\mathsf{colim}}_{\lambda} \mathfrak{F}(U_\lambda) \cong 2\text{-}\operatorname*{\mathsf{colim}}_{\lambda} \mathfrak{F}(U_\lambda).$$

Thus, we can repeat the above computation, replacing $\operatorname*{colim}_{\lambda}$ with $2\text{-}\operatorname*{colim}_{\lambda}$ and

$$\mathsf{colim}[\coprod_{i,j\in I}\mathfrak{F}(U_{ij}) \rightrightarrows \coprod_{i\in I}\mathfrak{F}(U_i)]$$

with

$$2\text{-}\mathsf{colim}[\coprod_{i,j,k\in\lambda}\mathfrak{F}(U_{ijk}) \Rrightarrow \coprod_{i,j\in I}\mathfrak{F}(U_{ij}) \rightrightarrows \coprod_{i\in I}\mathfrak{F}(U_i)].$$

In this way we, prove the analogous statement for costacks. $\mathcal{Q.E.D}$

Theorem 4.3.17. *Let X be a topological space and $\mathfrak{F} : \mathfrak{Off}(X) \to \mathsf{Grpd}$ a cosheaf satisfying the following properties:*

- *For every inclusion $U \subset V$, the induced functor $\mathfrak{F}(U) \to \mathfrak{F}(V)$ is injective on objects;*
- *$\mathfrak{F}$ commutes with filtered colimits.*

Then $\mathfrak{F}$ is a costack.

Proof. Since $\mathfrak{F}$ is a cosheaf, it clearly satisfies the condition $\mathsf{cosh}(2)$. Using Theorem 4.2.19, we know that $\mathfrak{F}$ satisfies $\mathsf{cost}(2)$ as well. Hence the second statement of Theorem 4.3.15 now implies the desired result. $\mathcal{Q.E.D}$

4.4 Examples of Stacks

4.4.1 Modules and Algebras

Let R be a commutative ring with unit and $R \to S$ a faithfully flat R-algebra. By abuse of notations, we denote $P \otimes_R Q$ by $P \otimes Q$. For any homomorphism of R-modules

$$\psi : P_1 \otimes \cdots \otimes P_n \to Q_1 \otimes \cdots \otimes Q_m,$$

we denote by

$$\psi_i : P_1 \otimes \cdots \otimes P_{i-1} \otimes S \otimes P_i \otimes \cdots \otimes P_n \to Q_i \otimes \cdots \otimes Q_{i-1} \otimes S \otimes Q_i \otimes \cdots \otimes Q_m \tag{4.1}$$

the homomorphism obtained by tensoring with Id_S in the i-th position. Let M_0 be an R-module and $M = M_0 \otimes S$. The canonical automorphism $g : S \otimes M \to M \otimes S$ given by $g(s_1 \otimes x \otimes s_2) = x \otimes s_1 \otimes s_2$ satisfies the the 1-cocycle condition, i.e. $g_2 = g_3 g_1$, where the g_i are defined as in 4.1. More explicitly, the diagram

$$\begin{tikzcd}
S \otimes S \otimes M \arrow[rr, "g_2"] \arrow[dr, "g_1"'] & & M \otimes S \otimes S \\
& S \otimes M \otimes S \arrow[ur, "g_3"'] &
\end{tikzcd}$$

commutes. Such an $S \otimes S$ automorphism g is called a *descent datum* for the S-module M. The collection of all such pairs (M, g) with morphisms $f : (M, g) \to (N, h)$ being the obvious commutative squares, is the *descent datum of the covering* $R \to S$ in the faithfully flat topology. As we already remarked in our discussion with stacks, we prefer to work in terms of 2-limits. Hence, we observe that we have the diagram

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r] & S \arrow[r, shift left] \arrow[r, shift right] & S \otimes_R S \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & S \otimes_R S \otimes_R S
\end{tikzcd}$$

associated to $R \to S$. Acting on this with the functor $\mathcal{M}$, where $\mathcal{M}$ denotes the assignment $R \mapsto \{Modules\ over\ R\}$, gives us the diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{M}(R) \arrow[r] & \mathcal{M}(S) \arrow[r, shift left] \arrow[r, shift right] & \mathcal{M}(S) \otimes_R \mathcal{M}(S) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathcal{M}(S) \otimes_R \mathcal{M}(S) \otimes_R \mathcal{M}(S)
\end{tikzcd}$$

whose 2-limit we denote by 2-lim$(S/R, \mathcal{M})$. This is the descent data we mentioned above and hence, we have defined the functor

$$\mathfrak{L}_S : \mathcal{M}(R) \to 2\text{-}\lim(S/R, \mathcal{M})\}$$

on objects. On morphisms, we send $f : M_0 \to M_0'$ to $f \otimes 1 : M_0 \otimes_R S \to M_0' \otimes_R S$.

Proposition 4.4.1. *The functor $\mathfrak{L}_S$ constructed above is an equivalence of categories.*

For the proof of this, see [28, chapter 3. (p.106) Prop. (1.1.1)]. But indeed more is true. Denote by $\mathcal{M}(Y)$ the category of quasi-coherent sheaves on Y. The following holds:

Proposition 4.4.2. *Let $f : Z \to Y$ be a faithfully flat morphism of schemes and let $\mathfrak{L}_Z : \mathcal{M}(Y) \to 2\text{-}\lim(Z/Y, \mathcal{M})$ be a functor, extending $\mathfrak{L}_S$ in the obvious way. Then, $\mathfrak{L}_Z$ is an equivalence of categories.*

For the proof see [21, p.154, Corollary 1.3]. Let X be a scheme and $\mathcal{FF}_X$ the site of faithfully flat schemes over X. We say that a 2-functor $\mathfrak{F} : \mathcal{FF}_X \to \mathfrak{Cat}$ is additive, if for every $Y, Y' \in \mathcal{FF}_X$, $\mathfrak{F}(Y \coprod Y') = \mathfrak{F}(Y) \times \mathfrak{F}(Y')$.

Lemma 4.4.3. *Let X be a quasi-compact scheme and $\mathfrak{F} : \mathcal{FF}_X \to \mathfrak{Cat}$ be additive. Then $\mathfrak{F}$ is a stack, if the functor $\mathfrak{L}_Z : \mathfrak{F}(Y) \to 2\text{-lim}(Z/Y, \mathfrak{F})$ is an equivalence of categories for every covering $Z \to Y$ in $\mathcal{FF}_X$. Here $\mathfrak{L}_Y$ denotes the canonical inclusion map into the 2-limit.*

Proof. Let $Y \to X$ be an object in $\mathcal{FF}_X$ and $\{Z_i \to Y\}$ a covering of Y. Since every $Z_i \to Y$ is flat, the coproduct $\coprod Z_i \to Y$ is flat, and since the collection is a covering, it is furthermore faithful. Hence, $Z := \coprod Z_i \in \mathcal{FF}_X$ and so we can use the condition of the proposition to say that

$$0 \longrightarrow \mathfrak{F}(Y) \longrightarrow \mathfrak{F}(Z) \rightrightarrows \mathfrak{F}(Z \times_Y Z) \Rrightarrow \mathfrak{F}(Z \times_Y Z \times_Y Z)$$

is 2-exact. By the assumption that $\mathfrak{F}$ is additive, we know that

$$\mathfrak{F}(Z \times_Y Z) \cong \prod \mathfrak{F}(Z_{ij}) \text{ and } \mathfrak{F}(Z \times_Y Z \times_Y Z) \cong \prod \mathfrak{F}(Z_{ijk}),$$

where the Z_{ij}-s and the Z_{ijk}-s are the 'intersections' of the Z_i-s, i.e. $Z_i \times_Y Z_j$. We have the following 2-exact sequence:

$$0 \longrightarrow \mathfrak{F}(Y) \longrightarrow \prod \mathfrak{F}(Z_i) \rightrightarrows \prod \mathfrak{F}(Z_{ij}) \Rrightarrow \prod \mathfrak{F}(Z_{ijk}).$$

This proves the assertion. $\mathcal{Q.E.D}$

Proposition 4.4.4. *Let X be a quasi-coherent scheme. The assignment $Y \mapsto \mathcal{M}_Y$ forms a stack in the faithfully flat topology over X.*

Proof. By Proposition 4.4.2 and Lemma 4.4.3 we immediately see that the only thing to prove is that $\mathcal{M}$ is an additive functor. In other words, that it takes coproducts to products. By the fact, that a module over a scheme is a sheaf of modules on its underlying space, it is clear that we only need to check it for the basis of open sets, which are affine. So we only need to check that for a ring R,

$$\mathcal{M}_R(S_1 \times S_2) \cong \mathcal{M}_R(S_1) \times \mathcal{M}_R(S_2).$$

To see this take an $S_1 \times S_2$ module M, and consider $M_1 := e_1 M$ and $M_2 := e_2 M$ as modules over S_1 and S_2 respectively. Here, e_1 and e_2 are the obvious idempotents of $S_1 \times S_2$. In essence we only need to check that $M \cong e_1 M \times e_2 M$.

Define the map $M \to e_1 M \times e_2 M$ by $m \mapsto (e_1 m, e_2 m)$ and the inverse maps by $(m_1, m_2) \mapsto m_1 + m_2$. Clearly, $m \mapsto (e_1 m, e_2 m) \mapsto e_1 m + e_2 m$ is the identity, since $e_1 m + e_2 m - (1,1)m = 0$. The other side follows from the fact that $e_1 M_2 = 0 = e_2 M_1$, where M is a module on S_1 and M' on S_2. On homomorphisms, we define it by sending $f \in \mathsf{Hom}(M, N)$ to $(e_1 f, e_2 f)$ and $(f_1, f_2) \in \mathsf{Hom}(M_1 \times M_2, N_1 \times N_2)$ to $f_1 \times f_2$. $\mathcal{Q.E.D}$

It is clear that this result also extends to the case of finite modules by essentially just repeating the proof.

Proposition 4.4.5. *Let X be a quasi-coherent scheme. The assignment $Y \mapsto \mathcal{A}_Y$ forms a stack in the fpqc topology over X, where $\mathcal{A}_Y$ denotes the category of affine schemes over Y.*

The proof of this is given in [43, Theorem 4.33]. We also mention that the above is true for quasi-affine schemes as well.

Chapter 5

The Topological Fundamental Groupoid

In this chapter, we will prove one of the main theorem of this thesis. We will show that the topological fundamental groupoid can be defined by the Seifert-van Kampen theorem. More precisely, we will show that for a topological space X, the assignment $U \mapsto \Pi_1(U)$ defines the 2-terminal costack over X.

The methods we use to prove this result however have also independent uses. In Section 5.2, we will give several examples of such applications, including giving a simple way to calculate the fundamental group of real, smooth, toric varieties explicitly, in terms of generators and relations.

5.1 The Seifert-van Kampen Theorem

Our aim in this section is to establish a 2-dimensional analogue of the following easy, but important fact. Recall that for an open subset $U \subset X$, the set of connected components is denoted by $\pi_0(U)$.

Lemma 5.1.1. *Let X be a locally connected topological space. The assignment $U \mapsto \pi_0(U)$ is a cosheaf, which is a terminal object in the category of cosheaves on X. Alternatively, it is the cosheafification of the functor* pt *given by* $\mathsf{pt}(U) = \star$ *where* $\star$ *denotes the singleton.*

Proof. The proof is done by the universal property of the associated cosheaf. For this, we first need to construct a map α from G to $\hat{\mathsf{pt}}$ for, every cosheaf G defined on X. Then, we need to check that it is unique. To define $G(U) \to \hat{\mathsf{pt}}(U)$, consider the cover $\{U_i\}$ of U by its connected components. Since the intersection of connected components is either itself or empty, and π_0 of a connected space is the singleton, it is clear that we have unique maps

$$\begin{tikzcd}
\coprod_{i,j} G(U_i \cap U_j) \arrow[r, shift left, "b_0"] \arrow[r, shift right, "b_1"'] \arrow[d] & \coprod_i G(U_i) \arrow[r, "a"] \arrow[d] & G(U) \arrow[r] \arrow[d, dotted, "\exists!"] & 1 \\
\coprod_{i,j} \pi_0(U_i \cap U_j) \arrow[r, shift left, "b_0"] \arrow[r, shift right, "b_1"'] & \coprod_i \pi_0(U_i) \arrow[r, "a"] & \pi_0(U) \arrow[r] & 1.
\end{tikzcd}$$

It follows from the exactness of the above sequences that there is a unique map $G(U) \to \pi_0(U)$. $\mathcal{Q.E.D}$

Now we turn to costacks with values in groupoids. Recall that a groupoid G is simply connected, provided for any two objects a and b of G, there is exactly one morphism $a \to b$.

Recall that an object t of a 2-category A is 2-terminal, provided that for any object x of A the category $\mathsf{Hom}_A(x,t)$ is a simply connected groupoid, which is to say that $\mathsf{Hom}_A(x,t)$ is equivalent to the groupoid $\mathbf{1}$, with one object and one arrow.

For a topological space X, we let $\Pi_1(X)$ be the *fundamental groupoid* of X, see [6]. Recall that the objects of $\Pi_1(X)$ are the points of X, while morphisms are the homotopy classes of paths. It follows immediately that for every element $x \in \Pi_1(X)$ we have $\mathsf{Aut}(x) = \pi_1(X,x)$. Furthermore, if we denote by $\pi_0(\Pi_1(X))$ the set of connected components of the fundamental groupoid, then $\pi_0(\Pi_1(X)) = \pi_0(X)$. We have the following result:

Theorem 5.1.2 (Seifert-Van Kampen)**.** *Let X be a topological space. The strict 2-functor given by $\Pi_1 : U \mapsto \Pi_1(U)$ defines simultaneously a cosheaf and a costack with values in the 2-category of small groupoids.*

Proof. In [6, p. 226, Statement 6.7.2] R.Brown demonstrated that for $V = U_1 \cup U_2$

and $W = U_1 \cap U_2$, the pushout of the diagram

$$\begin{tikzcd} \Pi_1(U_1) & \\ \Pi_i(W) \arrow[u] \arrow[r] & \Pi_1(U_2) \end{tikzcd}$$

is $\Pi_1(U)$. This essentially says that the strict 2-functor Π_1 satisfies $\mathsf{cosh}(2)$. Since Π_1 commutes with filtered colimits, we can use Theorem 4.3.15 to deduce that Π_1 is a cosheaf.

Since for any inclusion $W \hookrightarrow U_i$, the the functor $\Pi_1(W) \to \Pi_1(U_i)$ is injective on objects, Theorem 4.3.17 gives us the desired result. $\mathcal{Q.E.D}$

It should be noted that [33, Theorem II.7] essentially showed the cosheaf condition, but only for coverings with connected open sets.

This already demonstrates the importance of costacks, but indeed more is true.

Definition 5.1.3. *Let X be a topological space and $U \subset X$ an open subset. We call a covering $\mathfrak{U} = \{U_i\}_{i \in I}$ of U* discrete *(of order 3), if every $\Pi_1(U_i), \Pi_1(U_{ij})$ and $\Pi_1(U_{ijk})$ are discrete groupoids. Recall that we called a groupoid discrete, if for any two objects x, y we had at most one morphism between them.*

Call a topological space X *good*, if any open subset U of X possesses a discrete covering. We can now state the main theorem of this section.

Theorem 5.1.4. *Let X be a good topological space. Consider the constant, strict 2-functor $P : U \mapsto \mathbf{1}$, where $\mathbf{1}$ is the trivial groupoid. Then P has an associated costack, which is the fundamental groupoid, i.e. $\hat{P}(U) = \Pi_1(U)$. Thus the costack given by $U \mapsto \Pi_1(U)$ is a 2-terminal object in the 2-category of all costacks on X.*

Proof. By Theorem 5.1.2, the 2-functor $\Pi_1 : U \mapsto \Pi_1(U)$ is a costack. To prove that it is the costackification of P, we will use the universal property. It is clear that for any 2-functor $Q : \mathfrak{Off}(X) \to \mathsf{Grpd}$ we have a canonical 2-morphism $\mathfrak{q} : Q \to P$ of 2-functors. So we only have to prove that given a costack $Q : \mathfrak{Off}(X) \to \mathsf{Grpd}$, there exists an essentially unique 2-morphism $\alpha : Q \to \Pi_1$ such that $\mathfrak{q} = \mathfrak{p} \circ \alpha$, where $\mathfrak{p} : \Pi_1 \to P$ is the canonical 2-functor.

To define $\alpha(U)$, cover U by a discrete covering (of order 3). We can do this by the assumption on X. Since Q is a costack

$$\coprod_{i,j,k} Q(U_i \cap U_j \cap U_k) \rightrightarrows\!\!\!\!\!\rightarrow \coprod_{i,j} Q(U_i \cap U_j) \rightrightarrows \coprod_i Q(U_i) \longrightarrow Q(U) \longrightarrow 1$$

is 2-exact. Likewise for Π_1. Note that on the empty set Q is empty, just like Π_1. For the U_i-s and their non-empty intersections $U_i \cap U_j$ and $U_i \cap U_j \cap U_k$, Π_1 is a trivial groupoid, since they are simply connected. It follows that there is essentially a unique functor from $Q(U_i), Q(U_i \cap U_j)$ and $Q(U_i \cap U_j \cap U_k)$ to $\Pi_1(U_i), \Pi_1(U_i \cap U_j)$ and $\Pi_1(I_i \cap U_j \cap U_k)$, respectively.

Hence, by 2-exactness, we get a map, which is unique up to a unique natural transformation $\alpha(U) : Q(U) \to \Pi_1(U)$ satisfying the compatibility condition. It is also clear that $\mathfrak{q} = \mathfrak{p} \circ \alpha$ since $\mathfrak{q}$ is unique.

$\mathcal{Q.E.D}$

5.2 Examples

In this section, we will use the obtained results to calculate the fundamental group of some real geometric objects. (By real we mean a subset of $\mathbb{R}^n$). Throughout this part, we will use the following notations for discrete groupoids:

- Denote the groupoid with one object and one morphism by $\bullet$;
- Denote the groupoid with 2 objects and no non-identity morphisms by $\bullet \qquad \bullet$;
- Denote the groupoid with 2 objects x, y and one non-identity isomorphism $\alpha : x \to y$ from x to y by $\bullet \xrightarrow{\alpha} \bullet$; and so on.

Let X be a topological space and $\mathfrak{U} = \{U_i\}_{i \in I}$ a covering of X. Assume that I is a finite set and for all i, $\Pi_1(U_i)$ is a finitely connected, discrete groupoid. The advantage of 2-colimits is that the groupoids involved can be changed by equivalent ones. Hence, using Theorem 5.1.2, we can assume that all functors in the diagram associated to a covering

$$\coprod_{i,j,k} \Pi_1(U_{ijk}) \rightrightarrows\!\!\!\!\rightarrow \coprod_{i,j} \Pi_1(U_{ij},) \rightrightarrows \coprod_{i} \Pi_i(U_i)$$

are injective on objects. Using Theorems 4.2.19 and 4.3.15, we know that the 2-colimit of the above diagram is equivalent to its colimit. That is, the colimit of the following sub diagram:

$$\coprod_{i,j} \Pi_1(U_{ij}) \rightrightarrows \coprod_{i} \Pi_i(U_i).$$

Example 10 (The Sphere): Let us cover S^1 by U_1 and U_2, where both are open subsets of S^1, with a single point removed (of course different). Let V be the intersection of U_1 and U_2. Then we have $\Pi_1(U_1) = \Pi_1(U_2) = \bullet$ and $\Pi_1(V) = \bullet \qquad \bullet$. Hence the diagram of groupoids, associated to the covering, is

$$\begin{tikzcd}
\bullet & & & \bullet \\
& \bullet \arrow[ul] \arrow[urr] & \bullet \arrow[ull] \arrow[ur] & .
\end{tikzcd}$$

This is equivalent to

$$\begin{tikzcd}
\bullet \arrow[r, bend left, "\alpha"] & \bullet & & & \bullet \arrow[r, bend left, "\beta"] & \bullet \\
& & \bullet \arrow[ull] \arrow[urr] & \bullet \arrow[ull] \arrow[urr] & & .
\end{tikzcd}$$

From the construction of the colimit given in Subsection 4.2.3.1, we know that the colimit of the above diagram is the free category of the graph $\begin{tikzcd} \bullet \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & \bullet \arrow[l, bend right=80, "\alpha^{-1}"'] \arrow[l, bend left=80, "\beta^{-1}"] \end{tikzcd}$, modulo the composition quotient. Since we are only interested in the colimit up to an equivalence of categories, we can use any of the arrows to equate the two objects. We choose α and see that the fundamental groupoid of S^1 is connected, and the automorphism group is isomprhic to

$$\left\langle \alpha, \alpha^{-1}, \beta, \beta^{-1} \right\rangle / [\alpha = 1, \alpha \circ \alpha^{-1} = 1, \beta \circ \beta^{-1} = 1].$$

This is clearly $\mathbb{Z}$.

Compare this computation with the computation in [6, pp. 233-234] which uses the more complicated groupoid $\Pi_1 X A$, where A is an additional set.

Since the construction of the colimit was described for categories, we also had to write the inverses for every morphism and then quotient out the fact that $\alpha \circ \alpha^{-1} = 1$. But from now on, since we are restricting ourselves to groupoids, we will omit them from our notation. Hence for example, the above graph would become $\bullet \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \bullet$.

5.2.1 The Fundamental Group of Real Smooth Toric Varieties

The above method makes it essentially straight forward to calculate the fundamental groupoid of an object X, if we are given a discrete covering. Note that while most real geometric objects admit a discrete covering, writing it down explicitly can be tricky. However, there are situations when we can do it, as we will discuss in this subsection.

For a monoid scheme X, we denote by $\mathbb{R}[X]$, the realisation of X over the real numbers. Here we look at $\mathbb{R}[X]$ as a real geometric object, rather then a scheme. For simplicity, we will restrict ourselves to the case when X is noetherian, smooth and connected.

Recall that a monoid scheme is smooth if it is coverable by affine monoid schemes of the form $\mathsf{KSpec}(\mathbb{N}^{r_i} \times \mathbb{Z}^{s_i})$. The assumption that X is connected implies that $r_i + s_i = m$, where m is a constant integer. The localisation homomorphism

$$\mathbb{N}^{r_i} \times \mathbb{Z}^{s_i} \to \mathbb{N}^{r_{i-1}} \times \mathbb{Z}^{s_{i+1}}$$

corresponds to removing the hyperplane $x_j = 0$ (where x_j was inverted) in its realisation.

An affine covering $\{A_i\}$ of X induces a covering of its realisation, which we call an *affine covering* in this subsection. Clearly $\Pi_1(\mathbb{R}[\mathbb{N}^{r_i} \times \mathbb{Z}^{s_i}])$ is equivalent to a discrete groupoid with 2^{s_i} elements and the localisation homomorphisms

$$\Pi_1(\mathbb{R}[\mathbb{N}^{r_i} \times \mathbb{Z}^{s_i}]) \to \Pi_1(\mathbb{R}[\mathbb{N}^{r_{i-1}} \times \mathbb{Z}^{s_{i+1}}])$$

are surjective on objects.

By adding isomorphic objects, we can replace a functor between groupoids by an equivalent one, which is injective on objects. It follows that the groupoids $\Pi_1(\mathbb{R}_{A_i}), \Pi_1(\mathbb{R}_{A_{ij}})$ and $\Pi_1(\mathbb{R}_{A_{ijk}})$ can be assumed to have $2^m = 2^{r_i+s_i}$ objects, and every functor in our affine covering to be bijective on objects. The colimit and 2-colimit of this diagram will now be equivalent.

These objects of the discrete groupoid $\Pi_1(\mathbb{R}[\mathbb{Z}^m])$, correspond to subsets of $\mathbb{R}^m$ of the form

$$\{x_1 > 0, x_2 > 0, \cdots x_m > 0\}, \{x_1 > 0, x_2 > 0, \cdots x_m < 0\}, \cdots, \{x_1 < 0, x_2 < 0, \cdots x_m < 0\}.$$

Two such objects, p and q, are connected in $\Pi_1(\mathbb{R}[\mathbb{N}^{r_i} \times \mathbb{Z}^{s_i}])$, if there exists a generator x_j, such that x_j has the same sign in both p and q, and is not invertible in $\mathbb{N}^{r_i} \times \mathbb{Z}^{s_i}$.

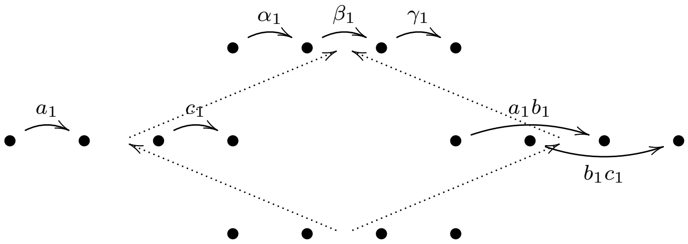
The gluing automorphism will induce an automorphism of these objects, allowing us to draw a diagram of discrete groupoids. In this way, we can calculate the fundamental groupoid of a smooth toric variety over the reals numbers.

We will now give a few demonstrations of this approach. The examples are taken from [16, pp.6].

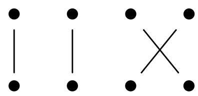
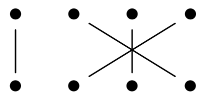
Example 11 (The real projective plane): Lemmas 3.1.5 and 3.1.7 enable us to describe a monoid scheme using a functor from a locally lattice poset. The diagram below, with the obvious maps, represents $\mathbb{P}^2$:

$$\begin{tikzcd}
\langle a,b\rangle \arrow[d] \arrow[dr] & \langle a,a^{-1}b\rangle \arrow[dl] \arrow[dr] & \langle ab^{-1},b^{-1}\rangle \arrow[dl] \arrow[d] \\
\langle a,a^{-1},b\rangle \arrow[dr] & \langle a,b,b^{-1}\rangle \arrow[d] & \langle a,a^{-1}b,b^{-1}\rangle \arrow[dl] \\
& \langle a,a^{-1},b,b^{-1}\rangle . &
\end{tikzcd}$$

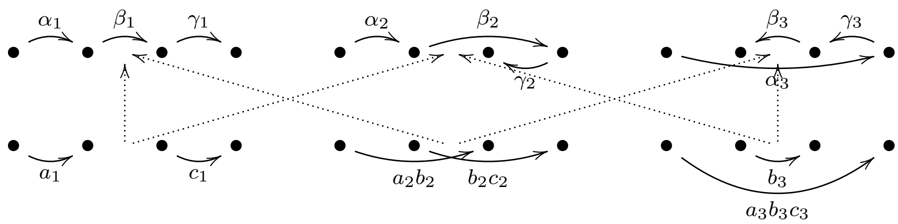
Its realisation is the real projective plane. We observe that this is covered by three copies of $\mathbb{N}^2$, $A_1 = \langle a,b\rangle, A_2 = \langle a,a^{-1}b\rangle$ and $A_3 = \langle ab^{-1},b^{-1}\rangle$. On its own, every one of the affine components will look as follows:



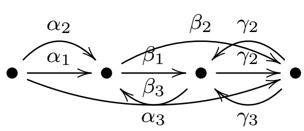
Here the dotted arrows indicate the morphisms between the groupoids. These functors are bijective between objects (by the argument given above) and map a_1 to α_1, a_1b_1 to $\alpha_1 \circ \beta_1$ etc. The gluing isomorphisms induce the following bijections:

$A_1 \to A_2 \Rightarrow$  and $A_1 \to A_3 \Rightarrow$ 

They imply that the diagram $\coprod_{i,j} \Pi_1(\mathbb{R}_{A_{ij}}) \rightrightarrows \coprod_i \Pi_i(\mathbb{R}_{A_i})$ looks as follows:



Here the triple intersection is omitted, as it does not factor in the colimit. Hence, the fundamental groupoid of $\mathbb{P}^2(\mathbb{R})$ is the free groupoid associated to the graph



module the relations induced by the arrows in the second line of the above diagram:

$$a_1 \Rightarrow \alpha_1 = \alpha_2; \qquad a_2b_2 \Rightarrow \alpha_1\beta_1 = \alpha_3\gamma_3^{-1}; \qquad a_3b_3c_3 \Rightarrow \alpha_2\beta_2 = \alpha_3;$$
$$c_1 \Rightarrow \gamma_1 = \gamma_2^{-1}; \qquad b_2c_2 \Rightarrow \beta_1\gamma_1 = \beta_3^{-1}\gamma_3^{-1}; \qquad b_3 \Rightarrow \beta_2\gamma_2^{-1} = \beta_3^{-1}.$$

Finally, since we are only interested in the groupoid up to equivalence, we can use any of the connecting arrows between two objects to equate them. We choose α_1, β_1 and γ_1 and set them to 1, to get the connected groupoid, (i.e. group)

$$\langle \alpha_2, \alpha_3, \beta_2, \beta_3, \gamma_2, \gamma_3 \rangle / [\alpha_2 = 1, \gamma_2^{-1} = 1, \alpha_3\gamma_3^{-1} = 1, \beta_3^{-1}\gamma_3^{-1} = 1, \alpha_2\beta_2 = \alpha_3, \beta_2\gamma_2^{-1} = \beta_3^{-1}].$$

This can be easily seen to be the cyclic group with 2 elements.

This method, which is quite a bit simpler than it looks, works for every finite, discrete covering in exactly the same way.

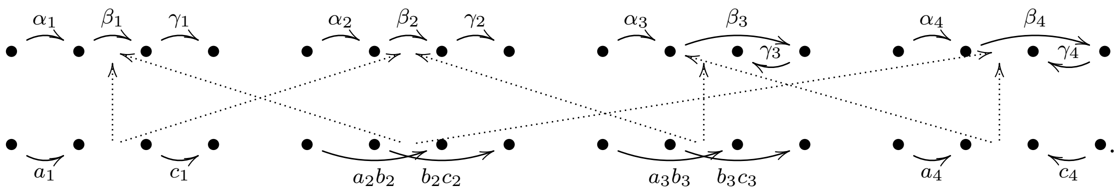
Example 12: Another example, (also from [16]) is the following: Consider the scheme covered by four affine components,

$$A_1 = \langle x, y \rangle, A_2 = \langle x, y^{-1} \rangle, A_3 = \langle x^{-1}, x^n y^{-1} \rangle, A_4 = \langle x^{-1}, x^n y \rangle$$

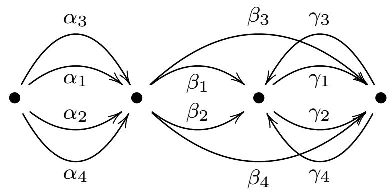
all being isomorphic to $\mathbb{N}^2$. The patching data is given by

$$\begin{tikzcd}
\langle x, y \rangle \arrow[r] \arrow[d] & \langle x^{-1}, x^n y \rangle \arrow[d] \\
\langle x, y^{-1} \rangle \arrow[r] & \langle x^{-1}, x^{-n} y^{-1} \rangle
\end{tikzcd}$$

with the obvious morphisms. Assume $n \geq 1$ and n is odd. The induced diagram of groupoids looks as follow:



Hence the fundamental groupoid is the free groupoid associated to the graph:



modulo the congruence generated by the second row of the above diagram, i.e.

$$a_1 \Rightarrow \alpha_1 = \alpha_2; \quad a_2b_2 \Rightarrow \alpha_1\beta_1 = \alpha_4\beta_4\gamma_4^{-1}; \quad a_3b_3 \Rightarrow \alpha_2\beta_2 = \alpha_3\beta_3\gamma_3^{-1}; \quad a_4 \Rightarrow \alpha_3 = \gamma_4$$

$$c_1 \Rightarrow \gamma_1 = \gamma_2; \qquad b_2c_2 \Rightarrow \beta_1\gamma_1 = \beta_4; \qquad b_3c_3 \Rightarrow \beta_2\gamma_2 = \beta_3; \qquad c_4 \Rightarrow \gamma_3 = \gamma_4.$$

As in the previous example, we impose $\alpha_1 = 1, \beta_1 = 1$ and $\gamma_1 = 1$ to equate our objects and get the abelian group $\mathbb{Z}^2$.

In the case when n is even (including 0), we get the abelian group $\mathbb{Z}^2$ as well, thought the diagram will look a bit different.

Chapter 6

Galois Categories

Our aim for the rest of this thesis is to give an axiomatisation of the étale fundamental groupoid of a finitely connected, noetherian scheme X, in exactly the same way as we did for the topological one. While most parts of the proof given in Chapter 5 would easily translate to this setting, the restriction we had on topological spaces would be too rigid for the algebraic case. More precisely, there are very few schemes that admit a discrete covering, as required in the previous proof.

As such, we will have to try and costackify the association $U \mapsto \mathsf{pt}$, where pt is the trivial groupoid. Unfortunately we don't know anything about costackification, so instead, we will compose our 2-functor with $\mathsf{Hom}_{\mathfrak{Cat}}(-, \mathsf{FSets})$ and work with stackification. The exact nature of this approach will be discussed in the proof of Theorem 8.0.9, but one of the main technical difficulties lies in showing the following:

Assume we have a contravariant 2-functor given by $U \mapsto (\mathfrak{G}_U, \mathsf{FSets})$, with the functors $\mathsf{Hom}(\mathfrak{G}_U, \mathsf{FSets}) \to \mathsf{Hom}(\mathfrak{G}_V, \mathsf{FSets})$ being induced by the functors between the groupoids $\mathfrak{G}_V \to \mathfrak{G}_U$. We want to show that its associated stack is again of this type. There is however a difficulty. While we know that

$$\mathsf{Hom}(2\text{-}\lim G_i, A) \cong 2\text{-}\mathsf{colim}\,\mathsf{Hom}(G_i, A),$$

we have no such formula for $\mathsf{Hom}(2\text{-}\mathsf{colim}\, G_i, A)$, even when the 2-colimit is filtered.

This is why we will use the so called Galois categories. We will show that the classical result by Grothendieck generalises to finitely connected, profinite groupoids. In other words, that $\mathfrak{G}$-FSets, being $\mathsf{Hom}_{\mathfrak{Cat}}(\mathfrak{G}, \mathsf{FSets})$ (see page 124)), is equivalent to a slightly reformulated Galois category, as given in Definition 6.1.1. As we will show in Chapter 7, individually, every one of the axioms is preserved by both the 2-limit as well as the filtered 2-colimit. This will show that the stackification of the 2-functor $U \mapsto \mathsf{Hom}(G_U, \mathsf{FSets})$ is again of the form $U \mapsto (\hat{\mathfrak{G}}_U, \mathsf{FSets})$, even though we do not know what $\hat{\mathfrak{G}}_U$ is in terms of the original groupoids $\mathfrak{G}_U$-s.

Unfortunately, reducing the equivalence between finitely connected Galois categories and $\mathfrak{G}$-FSets, where $\mathfrak{G}$ is a finitely connected, profinite groupoid, to the connected case is not as easy as one might expect. The main part of this chapter will be devoted to doing just that.

In the first section, we will show that the analogue of Grothendieck's classical result holds, and that a finitely connected Galois category is equivalent to the category of $\mathfrak{G}$-FSets, where $\mathfrak{G}$ is a finitely connected, profinite groupoid and FSets denotes the category of finite sets.

In the next section, we will talk about the whole 2-category of such Galois categories, which includes the morphisms and 2-morphisms. We will sharpen the result above, by proving that there is a 2-equivalence between the 2-category of finitely connected categories and the 2-category of finitely connected, profinite groupoids. While there are many generalisations of Galois categories, some of which are no doubt more general than ours, this reformulation of the classical theorem seems to be new.

Indeed, it will follow that an even more general formulation is true, namely Corollary 6.2.8.

6.1 Finitely Connected Galois Categories

In order to give the main definition, recall that a morphism $u : A \to B$ of a category $\mathcal{C}$ is an *epimorphism* (resp. *monomorphism*), if for any object X, the induce map $Hom_{\mathcal{C}}(B, X) \to Hom_{\mathcal{C}}(A, X)$ (resp. $Hom_{\mathcal{C}}(X, A) \to Hom_{\mathcal{C}}(X, B)$) is injective. Moreover, an epimorphism u is called a *strict epimorphism* if the pull-back

$$\begin{tikzcd} A \times_B A \arrow[r, "p_1"] \arrow[d, "p_2"] & A \arrow[d, "u"] \\ A \arrow[r, "u"] & B \end{tikzcd}$$

exists and B is the coequaliser of the diagram

$$A \times_B A \underset{p_2}{\overset{p_1}{\rightrightarrows}} A \xrightarrow{u} B.$$

It should be noted that the following definition of a Galois category differs from the standard definition of a Galois category (see for example [9]).

Definition 6.1.1. *A* (finitely-connected) Galois category *is a category $\mathcal{C}$ together with a set of covariant functors $\{\mathcal{F}_j : \mathcal{C} \to \mathsf{FSets}\}_{j\in J}$, satisfying the following axioms:*

1. *Finite limits exist in $\mathcal{C}$.*

2. *Finite colimits exist in $\mathcal{C}$.*

3. *Any morphism $u : Y \to X$ in $\mathcal{C}$ factors as $Y \xrightarrow{u'} X' \xrightarrow{u''} X$, where u' is a strict epimorphism and u'' is a monomorphism and there is an isomorphism $v : X' \coprod X'' \to X$ such that $u'' = vi_1$, where $i_1 : X' \to X' \coprod X''$ is the standard inclusion.*

4. *Every $\mathcal{F}_j$ is right exact, i.e. $\mathcal{F}_j$ respects finite colimits.*

5. *Every $\mathcal{F}_j$ is left exact, i.e. $\mathcal{F}_j$ respects finite limits.*

6. *Let $\{u : Y \to X\}$ be a morphism in $\mathcal{C}$. Then there exists a finite subset $I \subset J$ such that u is an isomorphism if and only if $\mathcal{F}_i(u)$ is an isomorphism for all $i \in I$.*

If I can be chosen to be a one element set, then $\mathcal{C}$ is called *connected*. This is equivalent to the standard definition of a Galois category. However, from now on, Galois category will refer to Definition 6.1.1. For more on connected Galois categories, see [9]. Several facts (Lemma 2.6 i), ii); Proposition 3.1; Proposition 3.2 (1), (3) i), iii)) proven in [9] have immediate generalisation in our situation. To state these statements, recall that an object X is called *connected*, if $X \neq 0$ and for any decomposition $X = Y \coprod Z$ one has $X = 0$ or $Y = 0$. Here 0 denotes the initial object.

Proposition 6.1.2. *Let $\mathcal{C}$ be a finitely connected Galois category. Then the following properties hold:*

i) *A morphism u is a monomorphism (resp. strong epimorphism), if and only if for all $i \in I$, the map $\mathcal{F}_i(u)$ is injective (resp. surjective). A morphism is an isomorphism, if and only if it is a monomorphism and a strong epimorphism.*

ii) *An object X is initial (resp. terminal), provided for all $i \in I$, the set $\mathcal{F}_i(X) = \varnothing$ (resp. $\mathcal{F}(X) = *$). Here $*$ denotes the singleton.*

iii) *Any object X has a unique decomposition $X = U_1 \coprod \cdots \coprod U_k$, where the U_i-s are connected.*

iv) *If U and V are connected, any morphism $U \to V$ is a strong epimorphism. In particular, any endomorphism $U \to U$ is an automorphism.*

v) *If U is connected, then for any objects $A_1 \cdots A_m$ the natural map*

$$\coprod_{i=1}^{m} \mathsf{Hom}(U, A_i) \to \mathsf{Hom}(U, \coprod_{i=1}^{m} A_i)$$

is a bijection.

The proof is the same as for the connected case. (see [9]).

Lemma 6.1.3. *Let $\mathcal{C}$ be a finitely-connected Galois category and $t = e_1 \coprod \cdots \coprod e_d$ be a decomposition of the terminal object as a coproduct of connected objects. Then for each $1 \leq i \leq d$ one has (after re-indexing) $\mathcal{F}_i(e_i) = *$ and $\mathcal{F}_i(e_j) = \varnothing$, $j \neq i$, $1 \leq i, j \leq d$.*

Proof. Let $1 \leq i \leq d$. Since $0 \to e_i$ is not an isomorphism, there exist at least one $\mathcal{F}$ such that $\mathcal{F}(e_i) \neq \varnothing$. After reindexing we can assume that $\mathcal{F} = \mathcal{F}_i$. Since $\mathcal{F}_i(t) = *$ and $\mathcal{F}_i$ respects coproducts we see that

$$\mathcal{F}_i(e_1) \coprod \cdots \coprod \mathcal{F}_i(e_d) = *$$

So, all terms except $\mathcal{F}_i(e_i)$ are empty sets and the result follows. $\mathcal{Q.E.D}$

Now we are in a position to prove the following result.

Lemma 6.1.4 (Main Lemma). *Let $\mathcal{C}$ be a finitely-connected Galois category and*

$$t = e_1 \coprod \cdots \coprod e_d$$

the decomposition of the terminal object t as a coproduct of connected objects. Then there is an equivalence of categories

$$\mathcal{C} \cong \mathcal{C}_1 \times \cdots \times \mathcal{C}_d$$

where C_i , $1 \le i \le d$ is the following full subcategory of C

$$\mathcal{C}_i = \{X \in \mathcal{C} |\ \mathcal{F}_j(X) = \varnothing, j \neq i, 1 \le j \le d\}.$$

Furthermore the pair $(\mathcal{C}_i, \mathcal{F}_i)$ is a connected Galois category and for any element $k \in J$ the functor $\mathcal{F}_k$ is isomorphic to exactly one of the functors $\mathcal{F}_1, \cdots, \mathcal{F}_d$.

Proof. We proceed by induction on d. Assume $d = 1$. Thus t is connected. In this case $\mathcal{C}_1 = \mathcal{C}$. Take any of $\mathcal{F}_i$ and call it $\mathcal{F}$. First we show that $\mathcal{F}$ reflects isomorphisms, meaning that if v is a morphism, such that $\mathcal{F}(v)$ is an isomorphism, then v is an isomorphism. If U is connected, then $U \to t$ is a strict epimorphism thanks to Proposition 6.1.2 iv). It follows that $\mathcal{F}(U) \to \mathcal{F}(t) = *$ is a strict epimorphism. Thus $\mathcal{F}(U) \neq \varnothing$. Since any object is a coproduct of connected ones and $\mathcal{F}$ respect coproducts, it follows that if A is not an initial object, then $\mathcal{F}(A) \neq \varnothing$. Assume $u : A \to B$ is a monomorphism, such that $\mathcal{F}(u)$ is an isomorphism, then $B \cong A \coprod \mathcal{C}$. Hence $\mathcal{F}(\mathcal{C}) = \varnothing$ so, $\mathcal{C} = 0$ and u is an isomorphism.

Now let $v : A \to B$ be a general morphism, such that $\mathcal{F}(v)$ is an isomorphism.

Consider the following commutative diagram

$$\begin{tikzcd}
A \arrow[dr, "\delta"] \arrow[drr, "\mathsf{Id}"] \arrow[ddr, "\mathsf{Id}"'] & & \\
& A \times_B A \arrow[r] \arrow[d] & A \arrow[d, "v"] \\
& A \arrow[r, "v"] & B
\end{tikzcd}.$$

Here δ is the diagonal map and hence a monomorphism. Apply $\mathcal{F}$ to this diagram and use the fact that $\mathcal{F}$ preserves pullbacks and $\mathcal{F}(v)$ is an isomorphism. We obtain that $\mathcal{F}(\delta)$ is an isomorphism. Thus δ is an isomorphism. It follows that v is a monomorphism (thanks to [9, Lemma 2.4]) and hence an isomorphism. Thus, $\mathcal{F}$ reflects isomorphisms and $(\mathcal{C}, \mathcal{F})$ is a connected Galois category. By [9, Theorem 2.8] any other $\mathcal{F}_i$ is isomorphic to $\mathcal{F}$. This proves the lemma for $d = 1$.

Assume now that $d > 1$. Thanks to Lemma 6.1.3 we have $\mathcal{F}_i(e_j) = \varnothing$ for all $j \neq i$ and $\mathcal{F}_i(e_i) = *$, $1 \le i, j \le d$. One easily sees that for each $1 \le i \le d$ the subcategory $\mathcal{C}_i$ is closed under finite limits and colimits.

Assume $X = \coprod_{j=1}^{k} U_j$ is a decomposition as a coproduct of connected objects.

<u>Claim 1</u>: We have $X \in \mathcal{C}_i$ if and only if $U_1, \cdots, U_k \in \mathcal{C}_i$. In fact if $U_1, \cdots, U_k \in \mathcal{C}_i$, then for any $j \neq i$, $1 \le j \le d$ one has $\mathcal{F}_j(U_1) = \cdots = \mathcal{F}_j(U_k) = \varnothing$. Thus,

$$\mathcal{F}_j(X) = \mathcal{F}_j(U_1) \coprod \cdots \coprod \mathcal{F}_j(U_k) = \varnothing.$$

Hence $X \in \mathcal{C}_i$. Conversely, if $X \in \mathcal{C}_i$, then

$$\varnothing = \mathcal{F}_j(X) = \mathcal{F}_j(U_1) \coprod \cdots \coprod \mathcal{F}_j(U_k).$$

Thus $\mathcal{F}_j(U_1) = \cdots = \mathcal{F}_j(U_k) = \varnothing$ and $U_1, \cdots U_k \in \mathcal{C}_i$.

<u>Claim 2</u>: The object e_i is a terminal object in the category $\mathcal{C}_i$. So, we have to prove that the set $\mathsf{Hom}(X, e_i)$ is a singleton provided $X \in \mathcal{C}_i$. By the first claim it is

enough to assume that X is connected. According to Proposition 6.1.2 v)

$$* = \mathsf{Hom}(X,t) = \mathsf{Hom}(X,e_1)\coprod\cdots\coprod\mathsf{Hom}(X,e_d)$$

So the set $\mathsf{Hom}(X,e_i)$ has at most one element. To show that it has exactly one element, we need to show that $\mathsf{Hom}(X,e_j) = \varnothing$ for $j \neq i$. Assume there is a morphism $X \to e_j$. Since both objects are connected, this map must be a strict epimorphism. This implies that $\varnothing = \mathcal{F}_j(X) \to \mathcal{F}_j(e_j) = *$ is surjective and hence a contradiction. Thus, our second claim is proven. It follows from the case $d = 1$, that the pair $(\mathcal{C}_i, \mathcal{F}_i)$ is a connected Galois category.

Claim 3: Our third claim is that if $i \neq j$, then for any objects $0 \neq X \in \mathcal{C}_i$ and $Y \in \mathcal{C}_j$ one has $\mathsf{Hom}(X,Y) = \varnothing$. In fact, since e_j is terminal in $\mathcal{C}_j$ there exist a unique morphism $Y \to e_j$. Thus it suffice to show that $\mathsf{Hom}(X,e_j) = \varnothing$, but this was shown in the proof of claim 2.

Define the functor

$$\xi : \mathcal{C}_1 \times \cdots \times \mathcal{C}_d \to \mathcal{C}$$

by

$$\xi(X_1,\cdots,X_d) = X_1\coprod\cdots\coprod X_d.$$

We will show that the functor ξ is an equivalence of categories. Take an object $X \in \mathcal{C}$ and consider the pull-back

$$\begin{tikzcd} X_i \arrow[r] \arrow[d] & X \arrow[d] \\ e_i \arrow[r] & t. \end{tikzcd}$$

Claim 4: We want to show that $X_i \in \mathcal{C}_i$. Take any $j \neq i$ and consider the image of the above diagram under $\mathfrak{F}_j$. We obtain a diagram of sets:

$$\begin{tikzcd} \mathcal{F}_j(X_i) \arrow[r] \arrow[d] & \mathcal{F}_j(X) \arrow[d] \\ \varnothing \arrow[r] & \mathcal{F}_j(t). \end{tikzcd}$$

Since the functor $\mathcal{F}_j$ respects pullbacks, it follows that $\mathcal{F}_j(X_i) = \varnothing$, and thus $X_i \in \mathcal{C}_i$. Moreover, the natural morphism $X_1 \coprod \cdots \coprod X_d \to X$ is an isomorphism, because every $\mathcal{F}_i$ takes it to an isomorphism. It follows that the functor ξ is essentially surjective. It remains to show that the functor ξ is full and faithful. Take objects $X_i, Y_i \in \mathcal{C}_i$, $i = 1, \cdots, d$. We have:

$$\mathsf{Hom}(\coprod_{i=1}^{d} X_i, \coprod_{i=1}^{d} Y_i) = \prod_{i=1}^{d} \mathsf{Hom}(X_i, Y_1 \coprod \cdots \coprod Y_d).$$

It remains to show that for any object $Z \in \mathcal{C}_i$, one has

$$\mathsf{Hom}(Z, Y_1 \coprod \cdots \coprod Y_d) = \mathsf{Hom}(Z, Y_i)$$

If Z is connected, this follows from Proposition 6.1.2 v) and claim 3. For the general case, we decompose $Z = Z_1 \coprod \cdots \coprod Z_k$.

We have

$$\begin{aligned} \mathsf{Hom}(Z, Y_1 \coprod \cdots \coprod Y_d) &= \mathsf{Hom}(\coprod_{j=1}^{k} Z_j, Y_1 \coprod \cdots \coprod Y_d) \\ &= \prod_{j=1}^{k} \mathsf{Hom}(Z_j, Y_1 \coprod \cdots \coprod Y_d) \\ &= \prod_{j} \mathsf{Hom}(Z_j, Y_i) \\ &= \mathsf{Hom}(\coprod Z_j, Y_i) \\ &= \mathsf{Hom}(Z, Y_i) \end{aligned}$$

Hence ξ is an equivalence of categories. $\mathcal{Q.E.D}$

Recall that we showed in Proposition 4.1.11, that for a connected groupoid $\mathfrak{G}$, we have an equivalence of categories

$$\mathsf{Hom}_{\mathfrak{Cat}}(\mathfrak{G}, \mathsf{FSets}) \cong \mathsf{Aut}(x)\text{-}\,\mathsf{FSets}\,.$$

Hence, we can now generalise group actions to groupoid actions, and for a groupoid $\mathfrak{G}$, we write $\mathfrak{G}$-FSets or $\mathsf{Hom}_{\mathfrak{Cat}}(\mathfrak{G}, \mathsf{FSets})$. It should be emphasised that we use two different notations for the exact same category. The second notation will mainly be used in calculations. If however, $\mathfrak{G}$ is a profinite groupoid, then we only consider the continuous actions. For simplicity though, we will still refer to it as $\mathsf{Hom}_{\mathfrak{Cat}}(\mathfrak{G}, \mathsf{FSet})$ or $\mathfrak{G}$-FSets.

Corollary 6.1.5. *Let $\{\mathcal{F}_i : \mathcal{C} \to \mathsf{FSets}\}_{i\in I}$ be a finitely connected Galois Category. Then it is equivalent to $\mathfrak{G}$-FSets, where $\mathfrak{G}$ is a finitely connected profinite groupoid.*

Proof. As proven in the Lemma 6.1.4,

$$\{\mathcal{F}_i : \mathcal{C} \to \mathsf{FSets}\}_{i\in I} \cong \{\mathcal{F}_j : \prod_{j'\in J} \mathcal{C}_{j'} \to \mathsf{FSets}\}_{j\in J},$$

such that J is a finite set and $\mathcal{F}_j(C_k) = \varnothing$ for $k \neq j$. This shows that our Galois category is equivalent to $\prod_{j\in J}(\mathcal{F}_j : \mathcal{C}_j \to \mathsf{FSets})$. Again by the above lemma, we know that for each $j \in J$, the functor $\mathcal{F}_j : \mathcal{C}_j \to \mathsf{FSets}$ is a connected Galois category. Using lemma 4.1.11 we know that its equivalent to $\mathfrak{G}$-FSets where $\mathfrak{G}$ is a connected, profinite groupoid. The result now follows from the fact that

$$\prod_{j\in J} \mathsf{Hom}_{\mathfrak{Cat}}(\mathfrak{G}_j, \mathsf{FSets}) \cong \mathsf{Hom}_{\mathfrak{Cat}}(\coprod_{j\in J} \mathfrak{G}_j, \mathsf{FSets}).$$

$\mathcal{Q.E.D}$

6.2 The 2-category of Galois Categories

Definition 6.2.1. *Let $\{\mathcal{F}_i : \mathcal{C} \to \mathsf{FSets}\}_{i\in I}$ and $\{\mathcal{G}_j : \mathcal{D} \to \mathsf{FSets}\}_{j\in J}$ be two Galois categories. A morphism of Galois categories consists of a map $f : J \to I$, a functor*

$\varphi : \mathcal{C} \to \mathcal{D}$ preserving finite limits and finite colimits, and a collection of natural isomorphisms $\lambda_j, j \in J$, as given in the following commutative diagram:

$$\begin{tikzcd}
\mathcal{D} \arrow[dr, "\mathcal{G}_j"'] & & \mathcal{C} \arrow[ll, "\varphi"'] \arrow[dl, "\mathcal{F}_{f(j)}"] \\
& \mathsf{FSets}. \arrow[u, phantom, "\overset{\Lleftarrow}{\lambda_j}"] &
\end{tikzcd}$$

We will refer to it as $\{(\varphi, \lambda_j) : \mathcal{F}_{f(j)} \to \mathcal{G}_j\}_{j \in J}$.

To define composition, we need to define the composition of the λ_j's. Say we have

$$\begin{tikzcd}
\mathcal{E} \arrow[drr, "\mathcal{H}_k"'] & & \mathcal{D} \arrow[ll, "\phi"'] \arrow[d, "\mathcal{G}_{f(k)}"] \arrow[dl, phantom, "\swarrow \lambda_{k,\phi}"] \arrow[dr, phantom, "\lambda_{j,\varphi} \nwarrow"] & & \mathcal{C} \arrow[ll, "\varphi"'] \arrow[dll, "\mathcal{F}_g(j)"] \\
& & \mathsf{FSets} & &
\end{tikzcd}$$

Define $\lambda_{k,\phi} \circ \lambda_{k,\varphi}(x) = \lambda_{k,\phi}(\varphi(x)) \circ \lambda_{k,\varphi}(x)$. In more detail we have

$$\lambda_{k,\phi} \circ \lambda_{k,\varphi}(x) : \mathcal{F}(x) \xrightarrow{\lambda_{k,\varphi}(x)} \mathcal{G}(\varphi(x)) \xrightarrow{\lambda_{k,\phi}(\varphi(x))} \mathcal{E}(\phi \circ \varphi(x)).$$

It is easily verified that the above construction is strictly associative.

Definition 6.2.2. *A 2-morphism between $\{(\varphi, \lambda_{j,\varphi}) : \mathcal{F}_{f(j)} \to \mathcal{G}_j\}$ and $\{(\phi, \lambda_{j,\phi}) : \mathcal{F}_{f(j)} \to \mathcal{G}_j\}$ is a collection of natural transformations*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "\varphi"] \arrow[r, bend right, "\phi"'] \arrow[r, phantom, "\Downarrow \zeta_j"] & \mathcal{D},
\end{tikzcd}$$

such that additionally the following diagram

$$\begin{tikzcd}
\mathcal{F}(x) \arrow[r, "\lambda_{j,\varphi}(x)"] \arrow[dr, "\lambda_{j,\phi}(x)"'] & \mathcal{G}(\varphi(x)) \arrow[d, "\zeta(x)"] \\
& \mathcal{G}(\phi(x))
\end{tikzcd}$$

commutes for all j and all x.

This shows that we can talk about the (strict) 2-category of Galois categories. We will denote it by GCat.

Lemma 6.2.3. *Let $\{\mathcal{F}_i : \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \to \mathsf{FSets}\}_{i \in I}$ be a finitely connected Galois category and $\mathcal{G} : \mathcal{D} \to \mathsf{FSets}$ a connected Galois category.*

Let $\mathcal{A} : \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \to \mathcal{D}$ be a functor between the Galois categories preserving finite limits and finite colimits. Then there exists an $i \in I$ and $\mathcal{A}_i : \mathcal{C}_i \to \mathcal{D}$, such that $\mathcal{A}$ factors through $\mathcal{A}_i$.

Proof. Take $t \in \mathcal{C}$. As in Lemma 6.1.4, $t = \coprod_{i \in I} e_i$, where every e_i is connected. Since $\mathcal{D}$ is connected, its terminal object is connected. Since $\mathcal{A}$ respects finite limits and colimits there exists $i \in I$ such that $\mathcal{A}(e_i) = \star$ and $\mathcal{A}(e_j) = \varnothing$ for all $j \neq i$. Recall the equivalence $\mathcal{C} \cong \mathcal{C}_1 \times \cdots \times \mathcal{C}_n$ as constructed in Lemma 6.1.4. For every $X \in \mathcal{C}$, we have $X \cong \coprod_i^n X_i$ where X_i is the pullback of the diagram:

$$\begin{tikzcd} & X \arrow[d] \\ e_i \arrow[r] & t = \coprod_{i \in I} e_i. \end{tikzcd}$$

$\mathcal{A}$ respects pullbacks, and so for all $j \neq i$, we know that $\mathcal{A}(X_j)$ is the pullback of

$$\begin{tikzcd} & \mathcal{A}(X) \arrow[d] \\ \mathcal{A}(e_j) \arrow[r] & \mathcal{A}(t). \end{tikzcd}$$

Since $Y \to \varnothing$ implies that Y is the empty set, we get that $\mathcal{A}(X_j) = \varnothing$. So

$$\mathcal{A}(X) = \coprod_j \mathcal{A}(X_j) = \mathcal{A}(X_i).$$

Hence $\mathcal{A}$ factors through the projection $\mathcal{C}_1 \times \cdots \times \mathcal{C}_n \to \mathcal{C}_i$. $\mathcal{Q}.\mathcal{E}.\mathcal{D}$

Corollary 6.2.4. *Let $\mathfrak{G}_i$ be a finitely connected, profinite groupoid and $\mathfrak{H}$ be a* *connected, profinite groupoid. Then we have an equivalence of categories:*

$$\mathsf{Hom}_{\mathsf{GCat}}(\mathsf{Hom}_{\mathfrak{Cat}}(\coprod_{i\in I}\mathfrak{G}_i,\mathsf{FSets}),\mathsf{Hom}_{\mathfrak{Cat}}(\mathfrak{H},\mathsf{FSets}))\cong$$

$$\cong\coprod_{i\in I}\mathsf{Hom}_{\mathsf{GCat}}(\mathsf{Hom}_{\mathfrak{Cat}}(\mathfrak{G}_i,\mathsf{FSets}),\mathsf{Hom}_{\mathfrak{Cat}}(\mathfrak{H},\mathsf{FSets})).$$

Proof. This follows from the Lemmas 6.2.3 and 6.1.5, and the fact that by definition, functors in GCat respect finite limits and finite colimits. $\mathcal{Q.E.D}$

Theorem 6.2.5. *The 2-category of finitely connected Galois categories is contravariantly 2-equivalent to the 2-category of profinite, finitely connected, groupoids.*

Proof. This equivalence is given by associating to a profinite groupoid $\mathfrak{G}$, the Galois category $\mathsf{Hom}_{\mathfrak{Cat}}(\mathfrak{G},\mathsf{Sets})$. On functors and natural transformations, the 2-functor is defined in the obvious way by composition. Using Proposition 4.1.8, we only need to show that it's essentially surjective and full and faithful. Both of course in the 2-mathematical sense. Essential surjectivity is proven in Corollary 6.1.5.
Full and Faithful: Let $\mathfrak{G}$ and $\mathfrak{H}$ be profinite and finitely connected groupoids. We have $\mathfrak{G}\cong\coprod_{i\in I}\mathfrak{G}_i$ and $\mathfrak{H}\cong\coprod_{j\in J}\mathfrak{H}_j$, where the $\mathfrak{G}_i$-s and $\mathfrak{H}_j$-s are profinite groupoids with one object. Hence

$$\begin{aligned}\mathsf{Hom}_{\mathfrak{Cat}}(\mathfrak{G},\mathfrak{H}) &\cong \mathsf{Hom}_{\mathfrak{Cat}}(\coprod_{j\in J}\mathfrak{G}_i,\coprod_{j\in J}\mathfrak{H}_j)\\ &\cong \prod_{i\in I}\mathsf{Hom}_{\mathfrak{Cat}}(\mathfrak{G}_i,\coprod_{j\in J}\mathfrak{H}_j)\\ &\cong \prod_{i\in I}\coprod_{j\in J}\mathsf{Hom}_{\mathfrak{Cat}}(\mathfrak{G}_i,\mathfrak{H}_j).\end{aligned}$$

The last equivalence comes from the fact that the $\mathfrak{G}_i$-s and $\mathfrak{H}_j$-s have a single object and so any functor $\mathfrak{G}_i$ can only go to a single $\mathfrak{H}_j$. From [21, Corolarry 6.2. p.111] we get that $\mathsf{Hom}_{\mathfrak{Cat}}(\mathfrak{G}_i,\mathfrak{H}_j)\cong\mathsf{Hom}_{\mathsf{GCat}}(\mathsf{Hom}_{\mathfrak{Cat}}(\mathfrak{H}_j,\mathsf{FSets}),\mathsf{Hom}_{\mathfrak{Cat}}(\mathfrak{G}_i,\mathsf{FSets}))$. On the other hand, we have:

$$\begin{aligned}
& \mathsf{Hom}_{\mathsf{GCat}}(\mathsf{Hom}_{\mathfrak{Cat}}(\mathfrak{H}, \mathsf{FSets}), \mathsf{Hom}_{\mathfrak{Cat}}(\mathfrak{G}, \mathsf{FSets})) \\
\cong\ & \mathsf{Hom}_{\mathsf{GCat}}(\mathsf{Hom}_{\mathfrak{Cat}}(\coprod_{j\in J}\mathfrak{H}_j, \mathsf{FSets}), \mathsf{Hom}_{\mathfrak{Cat}}(\coprod_{i\in I}\mathfrak{G}_i, \mathsf{FSets})) \\
\cong\ & \mathsf{Hom}_{\mathsf{GCat}}(\mathsf{Hom}_{\mathfrak{Cat}}(\coprod_{j\in J}\mathfrak{H}_j, \mathsf{FSets}), \prod_{i\in I}\mathsf{Hom}_{\mathfrak{Cat}}(\mathfrak{G}, \mathsf{FSets})) \\
\cong\ & \prod_{i\in I}\mathsf{Hom}_{\mathsf{GCat}}(\mathsf{Hom}_{\mathfrak{Cat}}(\coprod_{j\in J}\mathfrak{H}_j, \mathsf{FSets}), \mathsf{Hom}_{\mathfrak{Cat}}(\mathfrak{G}, \mathsf{FSets})).
\end{aligned}$$

Using Corollary 6.2.4, we get the desired result. $\mathcal{Q.E.D}$

Corollary 6.2.6. *Let $E : \mathfrak{G} \to \mathfrak{H}$ be a functor between finitely connected, profinite groupoids and $F : \mathsf{Hom}_{\mathfrak{Cat}}(\mathfrak{H}, \mathsf{FSets}) \to \mathsf{Hom}_{\mathfrak{Cat}}(\mathfrak{G}, \mathsf{FSets})$ the induced functor. Then F is an equivalence if and only if E is an equivalence.*

Proof. From Theorem 6.2.5 we know that

$$\mathsf{Hom}_{\mathfrak{Cat}}(\mathfrak{G}, \mathfrak{H}) \cong \mathsf{Hom}\mathfrak{Cat}(\mathfrak{G}\text{-}\,\mathsf{FSets}, \mathfrak{H}\text{-}\,\mathsf{FSets}).$$

This equivalence clearly holds when restricted to $\mathsf{Aut}_{\mathfrak{Cat}}$, being the category of equivalences. $\mathcal{Q.E.D}$

Unfortunately, due to our definition, we can not take general 2-limits with values in GCat. So we can not talk about stacks with values in the 2-category of Galois categories for a general site. However, it follows from Proposition 7.1.10, that if we only consider sites where every covering can be replaced by a finite one, we can avoid that problem.

Definition 6.2.7. *We call a site* finitely coverable*, if for every covering $\{U_i \to U\}_{i\in I}$, there exists a refinement $\{V_j \to U\}_{j\in J}$, such that J is a finite set.*

Let X be a finitely coverable site. We will call a 2-functor $\mathfrak{F} : X^{op} \to \mathsf{GCat}$ a *fibered Galois category* over X. If X was a prestack, we would refer to it as a *Galois prestack*. Similarly, we use the term *Galois Stack*. A natural transformation between two 2-functors $\mathfrak{F}, \mathfrak{G} : X \to \mathsf{GCat}$, that respects the structure, will be called a *Galois transformation*.

Let X be a site and $\mathfrak{F} : X \to \mathsf{Grpd}$ a covariant 2-functor. We denote by $\mathfrak{F}_S$ the contravariant 2-functor given by $U \mapsto \mathsf{Hom}_{\mathfrak{Cat}}(\mathfrak{F}(U), \mathsf{Sets})$. Take two covariant 2-functors $\mathfrak{E}, \mathfrak{F} : X \to \mathsf{Grpd}$ and $F : \mathfrak{E} \Rightarrow \mathfrak{F}$ a natural transformation between them. It is clear that $F_S : \mathfrak{F}_S \Rightarrow \mathfrak{E}_S$ is a Galois transformation. But indeed Theorem 6.2.5 shows that the reverse is also true. Hence, we have the following:

Corollary 6.2.8. *Let X be a site. The 2-category of fibered Galois categories over X is contravariantly equivalent to the 2-category of cofibered profinite groupoids over X.*

Chapter 7

Properties Preserved under Stackification

In Chapter 4 we defined the notion of a stack (Definition 4.3.4) and showed that for every fibered category $\mathfrak{F}$ over X, we had an associated stack $\hat{\mathfrak{F}}$ (Proposition 4.3.5). In this chapter, we will talk a little about some of the structures preserved under this construction. To do so, we will use the direct stackification given in Proposition 4.3.7, which showed that we only have to check that a given property is preserved by both 2-limits and filtered 2-colimits.

Our main interest in them is to study how Galois categories behave under stackification. However, since there is much intersection, we will mention a little about abelian categories as well, although we will not use these results.

7.1 Properties Preserved under 2-Limits

7.1.1 General Properties

Let $\mathfrak{F} : X \to \mathfrak{Cat}$ be a 2-functor and $f : A \to B$, $g : B \to C$ and $h : A \to C$ morphisms in $2\text{-}\lim_i \mathfrak{F}_i$. It follows straight from Remark 6 on page 80, that if the projects of h and $f \circ g$ agree for every i, then $f \circ g = h$. From this, we immediately see that a diagram $\mathfrak{D}$ in the 2-limit of $\mathfrak{F}$ commutes, if and only if its projections commute for every i.

Theorem 7.1.1 (Limits and Colimits). *Let I be a small category, $\mathcal{A} : I \to \mathfrak{Cat}$ a 2-functor and denote by $L := 2\text{-}\lim_{i \in I} \mathcal{A}_i$.*

i) Assume limits exist in every category $\mathcal{A}_i$ and the functors $\mathcal{A}_i \to \mathcal{A}_j$ preserve limits. Then limits exist in L and the canonical functors $L \to \mathcal{A}_i$ respect limits as well.

ii) Assume colimits exist in every category $\mathcal{A}_i$ and the functors $\mathcal{A}_i \to \mathcal{A}_j$ preserve colimits. Then colimits exist in L and the canonical functors $L \to \mathcal{A}_i$ respect colimits as well.

Proof of Thm. 7.1.1. Let C be a small category and $A : C \to L$ a functor. Consider the composite functor $C \xrightarrow{A} L \xrightarrow{f_i} \mathcal{A}_i$ and denote its limit by P_i. Here f_i is the canonical projection. First, we aim to show that the P_i define an object in L. To this end, consider the diagram

$$\begin{tikzcd} C \arrow[r, "A"] & L \arrow[rr, "f_i"] \arrow[drr, "f_j"'] & & \mathcal{A}_i \arrow[d, "\psi_*"] \arrow[dl, phantom, "\zeta_\psi \Nearrow"] \\ & & & \mathcal{A}_j . \end{tikzcd}$$

Since P_i is the limit of $f_i \circ A : C \to \mathcal{A}_i$, we have maps $p_{ic} : P_i \to (f_i \circ A)(c)$. Hence, we have

$$\psi_*(p_{ic}) : \psi_*(P_i) \to \psi_*((f_i \circ A)(c)) = \psi_*(f_i(A(c))) \xrightarrow{\zeta_\psi} f_j(A(c)),$$

where ζ_ψ is the natural isomorphism from the diagram. By the universality of P_j, this defines a map $p_\psi : \psi_*(P_i) \to P_j$.

To see that the collection $(P_i, \mathfrak{P}_\psi)$ is compatible, consider the following diagram:

$$\begin{tikzcd}
\nu_*(\psi_*(P_i)) \arrow[rr, "\nu_*(p_\psi)"] \arrow[dr, "{\nu_*,\psi_*,p_{ic}}"] \arrow[dd, "\mu_{\psi,\nu}"] & & \nu_*(P_j) \arrow[dr, "\nu_* p_{jc}"] \arrow[dd, "p_\nu"] & \\
& \nu_*(\psi_*(f_i(A(c)))) \arrow[rr, "\nu_*\alpha_{c\psi}"] \arrow[dd, "\mu_{\psi,\nu}"] & & \nu_*(f_j(A(c))) \arrow[dd, "\alpha_{c\nu}"] \\
(\nu\psi)_*(P_i) \arrow[rr, "p_{(\nu\psi)}"] \arrow[dr, "(\nu\psi)_* p_{ic}"] & & P_k \arrow[dr, "p_{kc}"] & \\
& (\nu\psi)_*(f_i(A(c))) \arrow[rr, "\alpha_{c(\nu\psi)}"] & & f_k(A(c)).
\end{tikzcd}$$

Here p_{ic}, p_{jc} and p_{kc} are the natural projections of P_i, P_j and P_k onto $f_i(A(c)), f_j(A(c))$ and $f_k(A(c))$ respectively, and $\alpha_{c\psi}$ is the compatible isomorphism. It exists since $f_i(A(c))$ and $f_j(A(c))$ are the natural projections of the objects of L. First look at the 'top' square of our cube:

$$\begin{tikzcd}
\psi_*(P_i) \arrow[rr, "p_\psi"] \arrow[dr, "{\psi_*,p_{ic}}"] & & (P_j) \arrow[dr, "p_{jc}"] & \\
& \psi_*(f_i(A(c))) \arrow[rr, "\alpha_{c\psi}"] & & f_j(A(c)).
\end{tikzcd} \tag{1}$$

We know that it commutes by the construction of the morphism p_ψ (which was the 'gluing' of the $\alpha_{c\psi}$'s) and composing with ν_*, which is a functor, maps commutative diagrams to commutative diagrams. Likewise the 'bottom' square will commute since $p_{(\nu\psi)}$ is the gluing of the $\alpha_{c(\nu\psi)}$'s. The 'left' square commutes since $\mu_{\psi,\nu}$ is a natural isomorphism. The 'right' square will commute since p_ν is the gluing of the $\alpha_{c\nu}$. The 'front' square:

$$\begin{tikzcd}
\nu_*(\psi_*(f_i(A(c)))) \arrow[r, "\nu_*\alpha_{c\psi}"] \arrow[d, "\mu_{\psi,\nu}"'] & \nu_*(f_j(A(c))) \arrow[d, "\alpha_{c\nu}"] \\
(\nu\psi)_*(f_i(A(c))) \arrow[r, "\alpha_{c(\nu\psi)}"] & f_k(A(c))
\end{tikzcd}$$

commutes since $f_i(A(c)), f_j(A(c))$ and $f_k(A(c))$ are objects of the 2-limit. Hence the following diagram:

$$\begin{tikzcd}
\nu_*(\psi_*(P_i)) \arrow[r, "\nu_*(p_\psi)"] \arrow[d, "\mu_{\psi,\nu}"] & \nu_*(P_j) \arrow[d, "p_\nu \circ p_{kc}"] \\
(\nu\psi)_*(P_i) \arrow[r, "\alpha_{(\nu\psi)} \circ \alpha_{c(\nu\psi)}"] & f_k(A(c))
\end{tikzcd}$$

will commute. Since this is true for all $c \in C$, the 'back' square commutes as well. This shows that the (P_i, p_ψ) define an object in the 2-limit L.

To see that P is indeed the limit of $A : C \to L$, we need to show that we have compatible maps $P \to A(c)$ and that P is universal with respect to this property. We know that $P = (P_i, p_\psi : P_i \to P_j)$ and that

$$A(c) = (f_i(A(c)), \alpha_{c\psi} : f_i(A(c)) \to f_j(A(c)).$$

Hence, we define the map

$$p_c : P \to A(c) := (p_{ic} : P_i \to f_i(A(c)).$$

The fact that they are compatible and define a morphism in L is saying that Square (1) commutes, which we already know. The fact that it is compatible is straight forward. To see that P is universal, let $Q \in L$ be an other object with compatible maps $q_c : Q \to A(c)$. Since $Q \in L$, $Q = (Q_i, \mathfrak{q}_\psi : Q_i \to Q_j)$, and so for every $i \in I$, we can define compatible $a_i : Q_i \to P_i$. To see that the (a_i) define a morphism in L, consider the following diagram:

$$\begin{tikzcd}
\psi_*(Q_i) \arrow[rr, "\psi_*(a_i)"] \arrow[dr, "\psi_*(q_{ic})"] \arrow[dd, "q_\psi"] & & \psi_*(P_i) \arrow[dl, "\psi_*(p_{ic})"'] \arrow[dd, "p_\psi"] \\
& \psi_*(f_i(A(c))) \arrow[dd, "f_i(u)"] & \\
Q_j \arrow[rr, "a_j"] \arrow[dr, "q_{ic}"'] & & P_j \arrow[dl, "p_{ic}"] \\
& f_j(A(c)). &
\end{tikzcd}$$

The top and bottom triangle commute since P_i and P_j are limits in $\mathcal{A}_i$ and $\mathcal{A}_j$

respectively, and ψ_* is a functor. The left and right squares will commute since $(Q_i, q_\psi : Q_i \to Q_j)$ and $(P_i, p_\psi : P_i \to P_j)$ define objects in the limit L (see diagram (1)). Hence the whole diagram commutes, which implies that the (a_i) are compatible.

Since for all $i \in I$, the composition triangle commutes in $\mathcal{A}_i$ after composing with f_i, we have $q_c = p_c \circ a$. This proves the assertion for the limit of $A : C \to L$.

Since we didn't use the construction of the limit at any point, just the universality, it is clear that the exact same proof, with the obvious adjustments (i.e. we will not have arrows $p_{ic} : (f_i \circ A)(c) \to P_i$ for example, and the induced arrows will be $p_\psi : P_j \to \psi_*(P_i)$), will also hold for the colimit. $\mathcal{Q.E.D}$

Proposition 7.1.2. *Let I be a category, $\mathfrak{F}, \mathfrak{G} : I \to \mathfrak{Cat}$ be two 2-functors and $\varphi : \mathfrak{F} \Rightarrow \mathfrak{G}$ a natural transformation. Let φ' denote the induced functor between the 2-limits* $\text{2-lim}\,\mathfrak{F}$ *and* $\text{2-lim}\,\mathfrak{G}$ *of $\mathfrak{F}$ and $\mathfrak{G}$ respectively.*

i) If for every i, $\varphi(i)$ preserves limits, then so will φ'.

ii) If for every i, $\varphi(i)$ preserves colimits, then so will φ'.

Proof. Since φ' is defined componentwise and every component respects limits (colimits), it is clear that it will respect limits (colimits) as well. $\mathcal{Q.E.D}$

Remark 8: We can see from the proofs of Theorem 7.1.1 and Proposition 7.1.2, that their analogues would hold for finite limits and colimits as well.

Proposition 7.1.3. *Let $\mathfrak{F} : I \to \mathfrak{Cat}$ be a 2-functor and $u : a \to b$ be a morphism in* $\underset{i}{\text{2-lim}}\,\mathfrak{F}_i$.

i) If for all i, $u_i : a_i \to b_i$ is a monomorphism, then u is a monomorphism.

ii) If for all i, $u_i : a_i \to b_i$ is an epimorphism, then u is an epimorphism.

iii) If for all i, $u_i : a_i \to b_i$ is a strict epimorphism, then u is a strict epimorphism.

iv) If for all i, $u_i : a_i \to b_i$ is an isomorphism, then u is an isomorphism.

Proof. i) Recal that $u : a \to b$ is a monomorphism if for all maps $v, v' : c \to a$,

$$\begin{tikzcd} c \arrow[r, shift left, "v"] \arrow[r, shift right, "v'"'] & a \arrow[r, "u"] & b \end{tikzcd}$$

$vu = v'u \Rightarrow v = v'$. It is clear that $vu = v'u$ implies $v_i u_i = v_i' u_i$, and since by assumption the u_i's were monomorphisms, we have $v_i = v_i'$ for every i, implying that $v = v'$ as desired.

ii) The proof is identical to the above one.

iii) From Theorem 7.1.2 we know that $a \times_b a$ exists, since its a limit of a finite diagram. Further, we know that for all i, $a_i \times_{b_i} a_i$ is the pullback of the projected diagrams onto the $\mathfrak{F}_i$-s. By assumption, the for every i, b_i is the colimit of the diagram

$$\begin{tikzcd} a_i \times_{b_i} a_i \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & a_i. \end{tikzcd}$$

Hence, by the same theorem, we know that b is the colimit of the diagram

$$\begin{tikzcd} a \times_b a \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & a, \end{tikzcd}$$

proving the assertion.

iv) Follows from i) and iii).

$\mathcal{Q.E.D}$

7.1.2 Abelian Categories

Let I be a category and $\mathfrak{F} : I \to \mathfrak{Cat}$ a 2-functor. We say that $\mathfrak{F}$ takes values in abelian categories if:

- For all $i \in I$, $\mathfrak{F}_i$ is an abelian category;
- For all morphisms $j \to i$, the induced functors $\mathfrak{F}_i \to \mathfrak{F}_j$ preserve the structure, i.e. are exact.

The aim of this subsection is to sprove that $\operatorname{2-lim}_i \mathfrak{F}$ is again an abelian category, which we will do piecewise.

7.1.2.1 Additive Categories

Let $\mathfrak{F} : I \to \mathfrak{Cat}$ be a 2-functor with values in pre-additive categories. By that we mean that for all $i \in I$, $\mathfrak{F}_i$ is a pre-additive category, and for every morphism $\psi : i \to j$, the induced functor $\psi_* : \mathfrak{F}_j \to \mathfrak{F}_i$ is additive. That is to say, the map

$$\mathsf{Hom}(a_i, b_i) \to \mathsf{Hom}(\psi_*(a_i), \psi_*(b_i))$$

is a group homomorphism. We want to show that $\operatorname{2-lim}_i \mathfrak{F}$ will be a pre-additive category as well.

Proposition 7.1.4. *Let $a, b \in \operatorname{2-lim}_i \mathfrak{F}$. Then $\mathsf{Hom}(a,b)$ is an abelian group and the composition map $\circ : \mathsf{Hom}(a,b) \times \mathsf{Hom}(b,c) \to \mathsf{Hom}(a,c)$ is bilinear.*

Proof. This follows straight from the formula given in Remark 7 on page 80. $\mathcal{Q.E.D}$

Proposition 7.1.5. *The projection functors $\mathfrak{p}_i : \operatorname{2-lim}_i \mathfrak{F} \to \mathfrak{F}_i$ are additive.*

Proof. The proof of this follows from the very definition of the group structure in the hom-sets of $\operatorname{2-lim}_i \mathfrak{F}_i$. $\mathcal{Q.E.D}$

This shows that $\operatorname{2-lim}_i \mathfrak{F}$ is a pre-additive category. Let us now assume that $\mathfrak{F}$ took values in additive categories. Theorem 7.1.1 then implies the 2-limit is additive.

Remark 9: Let A and B be additive categories and $T : A \to B$ a functor. The following are equivalent:

- T is additive, that is for all $a, a' \in A$, $\mathsf{Hom}_A(a, a') \to \mathsf{Hom}_B(T(a), T(a'))$ is a homomorphism.
- T respects finite direct summands, $T(a) \oplus T(a') \xrightarrow{\cong} T(a \oplus a')$.

See [31, p.77].

Proposition 7.1.6. *Let $\mathfrak{F}, \mathfrak{G} : I \to \mathsf{AdCat}$ be two 2-functors with values in the 2-category of additive categories and additive functors, and $\phi : \mathfrak{F} \to \mathfrak{G}$ be a morphism of 2-functors. By the uniqueness of the 2-limit, we get a functor $\phi' : 2\text{-}\lim_i \mathfrak{F} \to 2\text{-}\lim_i \mathfrak{G}$.*

i) If for every i, $\phi(i)$ is additive, then ϕ' is additive as well.

ii) If for every i, $\phi(i)$ maps finite biproducts to finite biproducts, then so does ϕ'.

Proof. The first assertion follows straight from Proposition 4.2.8 point (ii), and Remark 9 shows that the second statement is equivalent to the first. $\mathcal{Q.E.D}$

7.1.2.2 Preabelian Categories

Next we want to show that the 2-limit of a preabelian category is preabelian. Recall that a pre-abelian category is an additive category that has all kernels and cokernels. Assume that $\mathfrak{F} : I \to \mathsf{AdCat}$ satisfies all the previous conditions. We already know that additivity is preserved under the 2-limit and hence, we have a zero morphism in $2\text{-}\lim_i \mathfrak{F}_i$.

Proposition 7.1.7. *Assume that for all $i \in I$, $\mathfrak{F}_i$ has kernels/cokernels, and for all morphism $\psi : i \to j$, the induced functor $\psi_* : \mathfrak{F}_i \to \mathfrak{F}_j$ maps kernels/cokernels to kernels/cokernels. Then $2\text{-}\lim_i \mathfrak{F}$ has kernels/cokernels and the projection functors $\mathfrak{p}_i : 2\text{-}\lim_i \mathfrak{F} \to \mathfrak{F}_i$ map kernels/cokernels to kernels/cokernels.*

Proof. This follows from the fact that for a morphism $f : A \to B$,

$$\mathsf{Ker}(f) = \lim\Big(\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & B \end{tikzcd} \Big)$$

and

$$\mathsf{Coker}(f) = \mathsf{colim}\Big(\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & B \end{tikzcd} \Big),$$

and Theorem 7.2.1. $\mathcal{Q.E.D}$

Proposition 7.1.8. *Let $\mathfrak{F} : I \to \mathsf{Ab}$ and $\mathfrak{G} : I \to \mathsf{Ab}$ be two 2-functors and $\phi : \mathfrak{F} \Rightarrow \mathfrak{G}$ a natural transformation. Assume that for all i, $\phi(i) : \mathfrak{F} \to \mathfrak{G}$ maps kernels/cokernels* *to kernels/cokernels. Then the induced map* $\psi' : 2\text{-}\lim_i \mathfrak{F}_i \to 2\text{-}\lim_i \mathfrak{F}_i$ *maps kernels/cokernels to kernels/cokernels as well.*

Proof. As we have already mentioned in the previous proposition, the kernel and cokernel are both limits/colimits of some diagram. Hence, this result follows from Proposition 7.1.2.

$\mathcal{Q.E.D}$

7.1.2.3 Abelian Categories

Lastly we want to show that the 2-limit preserves abelian categories.

Proposition 7.1.9. *Let* $\mathfrak{F} : I \to \mathsf{Ab}$ *be a 2-functor with values in abelian categories. Then* $2\text{-}\lim_i \mathfrak{F}_i$ *is abelian as well and the projection maps* $2\text{-}\lim_i \mathfrak{F}_i \to \mathfrak{F}_i$ *respect the structure.*

Proof. Using the results from the previous sections, we already know that the 2-limit is preabelian and the canonical projections are structure preserving. Hence, using [3, Theorem 2.3.2 (p.100)], we know that $2\text{-}\lim_i \mathfrak{F}_i$ is an abelian category if for every morphism

$$\lambda : A :\to B$$

with kernel $(\mathsf{Ker}\lambda, \kappa)$ and cokernel $(\mathsf{Coker}\lambda, \chi)$, the morphism

$$\psi : \mathsf{Coker}\kappa \to \mathsf{Ker}\chi$$

is an isomorphism. But since by assumption, for every i the projection ψ_i of this morphism in $\mathfrak{F}_i$ is an isomorphism, so is ψ. $\mathcal{Q.E.D}$

7.1.3 Galois Categories

Proposition 7.1.10. *Let* I *be a finite category and* $\mathfrak{F} : I \to \mathsf{GCat}$ *a 2-functor from* I *to the 2-category of Galois categories, given by* $i \mapsto \{\mathcal{F}_{ij} : \mathcal{C}_i \to \mathsf{FSets}\}_{j \in J_i}$. *The 2-*

limit of $\mathfrak{F}$ *is again a Galois category and all the canonical projections* $2\text{-}\lim_i \mathfrak{F}_i \to \mathfrak{F}_i$ *preserve the Galois structure, i.e. are exact.*

Proof. We claim that $\{\mathcal{F}_{ij} : \mathcal{C} \to \mathsf{FSets}\}, ij \in J_i$ is the 2-limit, where C the 2-limit of the $\mathcal{C}_i$-s and by abuse of notations $\mathcal{F}_{ij}$ is the composition $\mathcal{C} \longrightarrow \mathcal{C}_i \xrightarrow{\mathcal{F}_{ij}} \mathsf{FSets}$. In order to prove this, we have to check that the axioms are respected by the 2-limit:

1,4. Follows from Theorem 7.1.1 i) (see Remark 8).

2,5. Follows from Theorem 7.1.1 ii) (see Remark 8).

3. Let $u : Y \to X$ be a morphism in $\mathcal{C}$. We know that for all i, it's canonical projection factors as $Y_i \xrightarrow{u'_i} X'_i \xrightarrow{u''_i} X_i$ in C_i. By [9, p. 174, Lemma 2.3], we know that the X'_i-s and X''_i-s define objects X' and X'' in the 2-limit (indeed the ξ_ψ-s are the identities) and u'_i-s and u''_i-s define morphisms u' and u'' in the 2-limit. Since for all i, $u_i = u'_i \circ u''_i$, we have $u = u' \circ u''$. Finally, Theorem 7.1.1 ii) implies that $X \cong X' \coprod X'$ and 'Proposition 7.1.3 i) and iii) implies the rest.

6. By assumption, there exists a finite subset $K_i \subset J_i$ for every i, such that the $\{F_i k\}_{ik \in K_i}$ reflect isomorphisms in $\mathcal{C}_i$. Proposition 7.1.3 iv) shows that the union of the K_i-s (which is finite since I is finite) does the trick.

$\mathcal{Q.E.D}$

7.2 Properties Preserved under Filtered 2-Colimits

7.2.1 General Properties

Theorem 7.2.1 (2-colimits preserve finite limits and colimits)**.** *Let I be a filtered category and $\mathfrak{F} : I \to \mathfrak{Cat}$ a 2-functor.*

- *Assume finite limits exist in every category $\mathfrak{F}_i$ and the maps $\mathfrak{F}_i \to \mathfrak{F}_j$ preserve finite limits. Then finite limits exist in* $2\text{-}\mathsf{colim}_{i \in I} \mathfrak{F}_i$ *and the canonical maps* $\mathfrak{F}_i \to 2\text{-}\mathsf{colim}_{i \in I} \mathfrak{F}_i$ *respect finite limits as well.*

- *Assume finite colimits exist in every category $\mathfrak{F}_i$ and the maps $\mathfrak{F}_i \to \mathfrak{F}_j$ preserve finite colimits. Then finite colimits exist in $2\text{-}\mathsf{colim}_{i\in I}\mathfrak{F}_i$ and the canonical maps $\mathfrak{F}_i \to 2\text{-}\mathsf{colim}_{i\in I}\mathfrak{F}_i$ respect finite colimits as well.*

Proof. Let C be a finite category and $A : C \to 2\text{-}\mathsf{colim}_{i\in I}\mathfrak{F}_i$ a functor. By the definition of the 2-colimit for filtered systems, every A_c can be thought to be in one of the $\mathfrak{F}_i$'s. Since C is finite and I is filtered, Proposition 4.2.14 shows that we can find a single $\mathfrak{F}_j$, such that the whole diagram $A : C \to 2\text{-}\mathsf{colim}_{i\in I}\mathfrak{F}_i$ can be represented in it. We then take the colimit of A in any such $\mathfrak{F}_j$. This gives us an element in L which we denote it by P. Note that it does not depend on our choice of $\mathfrak{F}_j$. To show that it is indeed the colimit in $2\text{-}\mathsf{colim}_{i\in I}\mathfrak{F}_i$, consider an other object $Q \in 2\text{-}\mathsf{colim}_{i\in I}\mathfrak{F}_i$, such that we have compatible maps $Q \to A_c$ for all $c \in C$. Again by the definition of the 2-colimit, Q is in one of the $\mathfrak{F}_i$'s and we can again find a category $\mathfrak{F}_k$, such that the diagram $A : C \to 2\text{-}\mathsf{colim}_{i\in I}\mathfrak{F}_i$, P, Q, as well as all the morphisms, are inside it and hence we will have a map $Q \to P$. Uniqueness follows trivially as well. The proof for the colimit is analogous to the above one. $\mathcal{Q.E.D}$

Proposition 7.2.2. *Let I be a category, $\mathfrak{F}, \mathfrak{G} : I \to \mathfrak{Cat}$ be two 2-functors and $\varphi : \mathfrak{F} \Rightarrow \mathfrak{G}$ a natural transformation between them. Let φ' denote the induced functor between the 2-colimits $L_{\mathfrak{F}}, L_{\mathfrak{G}}$ of $\mathfrak{F}$ and $\mathfrak{G}$ respectively.*

- *If for every i, $\varphi(i)$ preserves finite limits, then so will φ'.*
- *If for every i, $\varphi(i)$ preserves finite colimits, then so will φ'.*

Proof. Consider the following diagram:

$$\begin{tikzcd}
C \arrow[r, "\exists"] \arrow[dr, "A"'] & \mathfrak{F}_i \arrow[r, "\varphi(i)"] \arrow[d] & \mathfrak{G}_i \arrow[d] \\
& 2\text{-}\mathsf{colim}_{i\in I}\mathfrak{F}_i \arrow[r, "\varphi'"'] & 2\text{-}\mathsf{colim}_{i\in I}\mathfrak{G}_i.
\end{tikzcd}$$

Here the square commutes. For the triangle on the left, we know that the limit (colimit) of $A : C \to 2\text{-}\mathsf{colim}_{i\in I}\mathfrak{F}_i$ is isomorphic to the limit (colimit) of the composition

$C \to \mathfrak{F}_i \to \text{2-colim}_{i \in I} \mathfrak{F}_i$. Since we have already shown that $\mathfrak{G}_i \to \text{2-colim}_{i \in I} \mathfrak{G}_i$ respects limits (colimits), as does by assumptions $\varphi(i)$, clearly so does the composite

$$\varphi' = \mathfrak{F}_i \xrightarrow{\varphi(i)} \mathfrak{G}_i \to \text{2- colim}_{i \in I} \mathfrak{G}_i.$$

Q.E.D

Proposition 7.2.3. *Let $\mathfrak{F} : I \to \mathfrak{Cat}$ be a 2-functor, $L_i : \mathfrak{F}_i \to \text{2-colim}\, \mathfrak{F}_i$ the canonical functor and $u_i : a_i \to b_i$ a morphism in $\mathfrak{F}_i$.*

i) If u_i is a monomorphism and for all $\alpha : i \to j$ the induced functor $\alpha_* : \mathfrak{F}_i \to \mathfrak{F}_j$ respects monomorphisms, then $L_i(u_i)$ is a monomorphism.

ii) If u_i is an epimorphism and for all $\alpha : i \to j$ the induced functor $\alpha_* : \mathfrak{F}_i \to \mathfrak{F}_j$ respects epimorphisms, then $L_i(u_i)$ is an epimorphism.

iii) If u_i is a strict epimorphism and for all $\alpha : i \to j$ the induced functor $\alpha_* : \mathfrak{F}_i \to \mathfrak{F}_j$ respects strict epimorphisms, then $L_i(u_i)$ is a strict epimorphism.

iv) If u_i is an isomorphism and for all $\alpha : i \to j$ the induced functor $\alpha_* : \mathfrak{F}_i \to \mathfrak{F}_j$ respects isomorphisms, then $L_i(u_i)$ is an isomorphism.

Proof. i) Recall that $L_i(u_i)$ is a monomorphism if for all maps $v, v' : c \to a_i$ in $\text{2-colim}_{i \in I} \mathfrak{F}_i$,

$$\begin{tikzcd} c \arrow[r, shift left, "v"] \arrow[r, shift right, "v'"'] & a_i \arrow[r, "u_i"] & b_i \end{tikzcd}$$

$vu_i = v'u_i \Rightarrow v = v'$. Since $v : c \to a_i \in \text{2-colim}_{i \in I} \mathfrak{F}_i$, there exists $j \in I$, such that v is coming from $v_j : c_j \to \alpha_*(a_i)$. Similarly there exists a j' for v'. Since I is filtered, we can find $k \in I$ such that we have maps $\alpha_{ik} : i \to k$, $\alpha_{jk} : j \to k$ and $\alpha_{j'k} : j' \to k$. Hence, the whole diagram

$$\begin{tikzcd} c_k \arrow[r, shift left, "v_k"] \arrow[r, shift right, "v'_k"'] & a_k \arrow[r, "u_k"] & b_k \end{tikzcd}$$

can be represented in $\mathfrak{F}_k$, where by abuse of notation we write a_k instead of $\alpha_{ik*}(\alpha_*(a_i))$ etc. While in general $v_k u_k$ does not have to equal $v'_k u_k$, we can find $k' \in I$, such that $v_{k'}u_k = v'_{k'}u_k$ in $\mathfrak{F}_{k'}$. Since all the α_* respected monomorphisms, $u_{k'} : a_{k'} \to b_{k'}$ is a monomorphism and so $v_{k'} = v'_{k'}$. This implies that $v = v'$, proving the assertion.

ii) This is done in exactly the same way.

iii) From by Theorem 7.2.1 we know that $a \times_b a$ exists, since its a limit of a finite diagram. Further we know, that for all i, $a_i \times_{b_i} a_i$ is the pullback of the projected diagrams onto the $\mathfrak{F}_i$-s. By assumption, the for every i, b_i is the colimit of the diagram

$$a_i \times_{b_i} a_i \overset{p_1}{\underset{p_2}{\rightrightarrows}} a_i.$$

By the same theorem, we know that b is the colimit of the diagram

$$a \times_b a \overset{p_1}{\underset{p_2}{\rightrightarrows}} a.$$

This proves the assertion.

iv) Follows from i) and iii).

$\mathcal{Q.E.D}$

7.2.2 Abelian Categories

Let I be a category and $\mathfrak{F} : I \to \mathfrak{Cat}$ a 2-functor. We want to prove that $2\text{-}\mathsf{colim}_i \mathfrak{F}$ is again an abelian category. Again we will prove this piecewise.

7.2.2.1 Additive Categories

Let $\mathfrak{F} : I \to \mathfrak{Cat}$ be a 2-functor with values in pre-additive categories. We want to show that $2\text{-}\mathsf{colim}_i \mathfrak{F}$ will be a pre-additive category as well.

Proposition 7.2.4. *Let $a, b \in 2\text{-}\mathsf{colim}_i \mathfrak{F}$. Then $\mathsf{Hom}(a,b)$ is an abelian group and the composition map $\circ : \mathsf{Hom}(a,b) \times \mathsf{Hom}(b,c) \to \mathsf{Hom}(a,c)$ is bilinear.*

Proof. Since the filtered colimit of abelian groups, considered as sets, is again an abelian group, the first assertion follows from Proposition 4.2.15. Bilinearity of the composition follows from the fact that composition commutes with filtered colimits and the fact that $I_{i,j}$, as defined in the same proposition, is a filtered category. $\mathcal{Q.E.D}$

Proposition 7.2.5. *The canonical functors $L_i : \mathfrak{F}_i \to 2\text{-}\underset{i}{\mathsf{colim}}\,\mathfrak{F}$ are additive.*

Proof. The proof of this follows directly from the definition of the group structure in the Hom-sets of $2\text{-}\underset{i}{\mathsf{colim}}\,\mathfrak{F}_i$. $\mathcal{Q.E.D}$

This shows that $2\text{-}\underset{i}{\mathsf{colim}}\,\mathfrak{F}$ of pre-additive categories is a pre-additive category. Let us now assume that $\mathfrak{F}$ took values in additive categories. We then immediately see from 7.2.1 that the 2-colimit will as well.

Proposition 7.2.6. *Let $\mathfrak{F}, \mathfrak{G} : I \to \mathsf{AdCat}$ be two 2-functors with values in the 2-category of additive categories and $\phi : \mathfrak{F} \Rightarrow \mathfrak{G}$ be a natural transformation of 2-functors. By the uniqueness of the 2-colimit, we get a functor $\phi' : 2\text{-}\underset{i}{\mathsf{colim}}\,\mathfrak{F} \to 2\text{-}\underset{i}{\mathsf{colim}}\,\mathfrak{G}$.*

i) If for every i, $\phi(i)$ is additive, then ϕ' is be additive as well.

ii) If for every i, $\phi(i)$ maps finite biproducts to finite biproducts, then so does ϕ'.

Proof. i) This follows straight from proposition 4.2.16, point (ii).

ii) See (i) Remark 9.

$\mathcal{Q.E.D}$

7.2.2.2 Preabelian Categories

Next we want to show that the 2-colimit of a preabelian category is preabelian. We already know that additivity is preserved under the 2-colimit and hence we have a zero morphism in $2\text{-}\underset{i}{\mathsf{colim}}\,\mathfrak{F}_i$.

Proposition 7.2.7. *Assume for all $i \in I$, $\mathfrak{F}_i$ has kernels/cokernels, and for all morphism $\psi : i \to j$, $\psi_* : \mathfrak{F}_i \to \mathfrak{F}_j$ maps kernels/cokernels to kernels/cokernels. Then $2\text{-}\underset{i}{\mathsf{colim}}\,\mathfrak{F}$ has kernels/cokernels and the canonical maps $\mathfrak{L}_i : \mathfrak{F}_i \to 2\text{-}\underset{i}{\mathsf{colim}}\,\mathfrak{F}$ map kernels/cokernels to kernels/cokernels.*

Proof. This follows from the fact that for a morphism $f : A \to B$, $\mathsf{Ker}(f) = \lim(A \overset{f}{\underset{0}{\rightrightarrows}} B)$ and $\mathsf{Coker}(f) = \mathsf{colim}(A \overset{f}{\underset{0}{\rightrightarrows}} B)$ and Theorem 7.1.1. $\mathcal{Q.E.D}$

Proposition 7.2.8. *Say we have $\mathfrak{F} : I \to \mathfrak{Cat}$ and $\mathfrak{G} : I \to \mathfrak{Cat}$ and a natural transformation $\phi : \mathfrak{F} \Rightarrow \mathfrak{G}$ between them, that maps kernels/cokernels to kernels/cokernels. Then the induced functor $\phi' : 2\text{-}\mathsf{colim}_i \mathfrak{F}_i \to 2\text{-}\mathsf{colim}_i \mathfrak{F}_i$ maps kernels/cokernels to kernels/cokernels as well.*

Proof. Take a map $f : X \to Y$ in $2\text{-}\mathsf{colim}_i \mathfrak{F}_i$ and let K be its kernel. By the definition of the 2-colimit we know that we can choose a representative $f_i : X_i \to Y_i \in \mathfrak{F}_i$ with kernel K_i.

Kernels exist in $2\text{-}\mathsf{colim}_i \mathfrak{G}_i$ by Proposition 7.2.7 and so $\phi(f) : \phi'(X) \to \phi'(Y)$ has a kernel, which we denote by $Ker(\phi(f))$. This means that we only have to show that $Ker(\phi(f))$ is isomorphic to $\phi(K)$. But since they agree on the restrictions by Proposition 7.2.8, by the uniqueness property of the 2-colimit, they are isomorphic. The same argument works for cokernels as well. $\mathcal{Q.E.D}$

7.2.2.3 Abelian Categories

Proposition 7.2.9. *Let I be a filtered system and $\mathfrak{F} : I \to \mathsf{Ab}$ a 2-functor with values in abelian categories. Then $2\text{-}\mathsf{colim}_i \mathfrak{F}_i$ is an abelian category and the projection functors $\mathfrak{F}_i \to 2\text{-}\mathsf{colim}_i \mathfrak{F}_i$ respect the structure.*

Proof. Using the results from the previous sections, we already know that the 2-colimit is preabelian and the canonical projections are structure preserving. Hence, using [3, Theorem 2.3.2 (p.100)], we know that $2\text{-}\mathsf{colim}_i \mathfrak{F}_i$ is an abelian category if for every morphism

$$\lambda : A \to B,$$

with kernel $(\mathsf{Ker}\lambda, \kappa)$ and cokernel $(\mathsf{Coker}\lambda, \chi)$, the morphism

$$\psi : \mathsf{Coker}\kappa \to \mathsf{Ker}\chi$$

is an isomorphism. By the definition of the 2-colimit for filtered systems, we can assume that the whole diagram

$$\begin{tikzcd}
\mathsf{Ker}\lambda \arrow[r, "\kappa"] & A \arrow[rrr, "\lambda"] \arrow[dr] & & & B \arrow[r, "\chi"] & \mathsf{Coker} \\
& & \mathsf{Coker}\kappa \arrow[r, "\psi"] & \mathsf{Ker}\chi \arrow[ur] & &
\end{tikzcd}$$

is in some $\mathfrak{F}_i$. By assumption, ψ is an isomorphism in $\mathfrak{F}_i$ and thus by Proposition 4.2.14, it is an isomorphism in the 2-colimit. $\mathcal{Q.E.D}$

7.3 Properties Preserved under Stackification

As mentioned at the beginning of this chapter, if a property is preserved by both 2-limits and filtered 2-colimits, it is preserved by stackification. This section will basically be a summing up of the results obtained in the previous two sections, for the convenience of the reader.

Theorem 7.3.1. *Let X be a site and $\mathfrak{F} : X \to \mathfrak{C}$ a 2-functor from X to the 2-category $\mathfrak{C}$. The associated stack $\hat{\mathfrak{F}} : X \to \mathfrak{C}$ exists if $\mathfrak{C}$ is any of the following:*

i) The 2-category of (finite) complete categories;

ii) The 2-category of (finite) cocomplete categories;

iii) The 2-category of (finite) bicomplete categories;

iv) The 2-category of pre-additive categories;

v) The 2-category of additive categories;

vi) The 2-category of pre-abelian categories;

vii) The 2-category of abelian categories.

Proof. This follows from Proposition 4.3.7 and:

i) Theorems 7.1.1 i) and 7.2.1 i) (see Remark 8, p.134);

ii) Theorems 7.1.1 ii) and 7.2.1 ii) (see Remark 8, p.134);

iii) The above two statements;

iv) Propositions 7.1.4, 7.1.5, 7.2.4 and 7.2.5;

v) Statement iii) and iv);

vi) Propositions 7.1.7 and 7.2.7;

vii) Propositions 7.1.9 and 7.2.9.

$\mathcal{Q.E.D}$

Lemma 7.3.2. *Let X be a site, $\mathfrak{F} : X \to \mathfrak{Cat}$ a 2-functor and $\mathfrak{G} : X \to \mathfrak{Cat}$ a stack. Let $\psi : \mathfrak{F} \Rightarrow \mathfrak{G}$ be a natural transformation. Denote by $\hat{F}$ the associated stack of $\hat{\mathfrak{F}}$ and let $\hat{\psi} : \hat{\mathfrak{F}} \Rightarrow \mathfrak{G}$ be the associated natural transformation. Then the following results hold:*

i) If ψ respects (finite) limits, so does $\hat{\psi}$;

ii) If ψ respects (finite) colimits, so does $\hat{\psi}$;

iii) If ψ is additive, then so is $\hat{\psi}$.

Proof. This follows from Proposition 4.3.7 and:

i) Propositions 7.1.2 i) and 7.2.2 i) (see Remark 8, p.134);

ii) Propositions 7.1.2 ii) and 7.2.2 ii) (see Remark 8, p.134);

iii) Propositions 7.1.6 and 7.2.6.

$\mathcal{Q.E.D}$

Recall that using Proposition 7.1.10 we can talk about stacks with values in Galois categories (called Galois stacks). As such, using the universal definition, we still have the notion of an associated Galois stack, even if we don't prove that it always exists. We then have the following:

Corollary 7.3.3. *Let $\mathfrak{F} : X \to \mathsf{GCat}$ be a 2-functor, $\mathfrak{G} : X \to \mathsf{GCat}$ a Galois stack and $\psi : \mathfrak{F} \Rightarrow \mathfrak{G}$ a Galois transformation. If $\mathfrak{F}$ has the associated Galois stack $\hat{\mathfrak{F}}$, then $\hat{\psi} : \hat{\mathfrak{F}} \Rightarrow \mathfrak{G}$ is a Galois transformation as well.*

Proof. This follows from i) and ii) of the above lemma and iii) of Theorem 7.3.1.

$\mathcal{Q.E.D}$

Chapter 8

The Étale Fundamental Groupoid

In this chapter we will prove our final main result of this thesis. The original problem, posed by my supervisor, was to prove a version of the van-Kampen theorem for the étale fundamental groupoid. The version we prove is that for any covering $Y \mapsto X$ of a noetherian scheme X, the association $Y \mapsto \Pi_1(Y)$ is a costack, rather then a cosheaf, as initially expected.

But more is true. The étale fundamental groupoid is indeed defined by the Seifert-van Kampen theorem, in complete analogy to the topological case, however, the proof is vastly different.

To prove that it's the 2-terminal costack over any noetherian scheme X is equivalent to saying that it is the associated costack of the constant, covariant 2-functor taking the trivial groupoid as its value. Unfortunately we don't know much about costackification, so instead we will compose it with the functor $\mathsf{Hom}(-, \mathsf{FSets})$, and hence get a Galois category.

Since by the definition of a costack, after composing with $\mathsf{Hom}(-, \mathsf{FSets})$, we will get a stack, we can use stackification to prove this theorem. This is also our main reason for studying properties preserved under 2-limits and filtered 2-colimits in the above chapter.

We follow [35] on the notations. Let k be a field. A polynomial $f \in k[x]$ is *separable* if $f(x)$ is irreducible over k and has no multiple roots. A field extension $k \to L$ is *separable* if the minimal polynomial $m_k(a)$ of every element $a \in L$ over k is separable over k.

A homomorphism $A \to B$ is called *finite*, if B is finitely generated as an A-module. Now let A and B be local rings and $f : A \to B$ a local homomorphisms. We say that f is *unramified* if $A/m_A \to B/f(m_A)B$ is a finite, separable field extension. Let Y be a scheme such that it can be covered by affine subschemes $V_i = \mathsf{Spec}(A_i)$, where the A_i are noetherian. Let $f : X \to Y$ be a morphism of schemes, and assume further that for all i, $f^{-1}(V_i)$ can be covered by finitely many affine subschemes $U_j = \mathsf{Spec}(B_j)$, where the B_j-s are noetherian as well.

We say that $f : X \to Y$ is *of finite type* if the B_j-s are finitely generated A_i-algebras for every i. Note that in this case we require them to be finitely generated as algebras, not as modules, as we did in the case of finite homomorphisms.

A morphism $\varphi : Y \to X$ of schemes is *unramified* if it is of finite type and if $\mathcal{O}_{X,\varphi(y)} \to \mathcal{O}_{Y,y}$ are unramified for all $y \in Y$.

A morphism $A \to B$ of rings is called *flat*, if the functor $M \mapsto B \otimes_A M$ from A-modules to B-modules is exact. A morphism of schemes is *flat*, if the local homomorphisms $\mathcal{O}_{X,\varphi(y)} \to \mathcal{O}_{Y,y}$ are flat for every $y \in Y$.

Let X, Y be schemes and $f : X \to Y$ a morphism between them. We say that f is *étale* if it is flat and unramified. In this case, we also say that X is étale over Y. If additionally f is surjective on the underlying topological spaces of Xs and Y, then f is said to be an étale covering. See also Example 4 on page 8.

Let X be a noetherian scheme. Denote by $\mathsf{FEC}(X)$ the site of finite étale coverings of X. It is a well-known result that the category of finite étale coverings is a Galois category.

Lemma 8.0.1. *Let X and Y be quasi-compact, noetherian schemes. We have $\mathsf{FEC}(X \coprod Y) \cong \mathsf{FEC}(X) \times \mathsf{FEC}(Y)$, where $\mathsf{FEC}(X)$ denotes the category of finite etale coverings over X.*

Proof. From the proof of Proposition 4.4.4 we know that this is true for the underlining modules. Further, we know that it suffices to check it only in the affine case. Let R and S be commutative rings and denote by $\mathsf{Alg}(R)$ the category of algebras over R.

As Algebras: Take an algebra $A \in \mathsf{Alg}(R \times S)$. Since it's also a module over $R \times S$, we define modules $e_1 A$ and $e_2 A$ over R and S respectively. It is clear that the ring structure on A induces a ring structure on $e_1 A$ by saying that $(1,0)a \times (1,0)b = (1,0)ab$.

Similarly for e_2A. For the reverse, take A_1 and A_2 in $\mathsf{Alg}(R)$ and $\mathsf{Alg}(S)$ respectively. By defining $(a_1, a_2)(b_1, b_2) = (a_1b_1, a_2b_2)$ we get a ring structure on $A_1 \times A_2$. This argument trivially extends to morphisms as well. Hence we have $\mathsf{Alg}(R \times S) \cong \mathsf{Alg}(R) \times \mathsf{Alg}(S)$.

Unramifiedness: Recall that if $A \cong A_R \times A_S$, then we have

$$\mathsf{Spec}(A) \cong \mathsf{Spec}(A_R) \coprod \mathsf{Spec}(A_S), \tag{8.1}$$

given by $\mathfrak{p}_R \mapsto (\mathfrak{p}_R, A_S)$ and $\mathfrak{p}_S \mapsto (A_R, \mathfrak{p}_S)$. Let $f : R \times S \to A$ be an $R \times S$-algebra and take $\mathfrak{p} \in \mathsf{Spec}(A)$. We know that there are $f_R : R \to A_R$, $f_S : S \to A_S$ with $A \cong A_R \times A_S$ and $f = f_R \times f_S$. Without loss of generality, we can assume that $\mathfrak{p} = (\mathfrak{p}_R, A_S)$. We have

$$f^{-1}(\mathfrak{p}) = f^{-1}(\mathfrak{p}_R, A_S) = (f_R^{-1}(\mathfrak{p}_R), f_S^{-1}(A_S)).$$

$\mathfrak{p}_R$ is prime, so is its pre-image and $f^{-1}(\mathfrak{p}) = (f_R^{-1}(\mathfrak{p}_R), A_S)$. This implies that

$$(R \times S)_{f^{-1}(\mathfrak{p})} \cong (R \times S)_{f^{-1}(\mathfrak{p}_R), A_S}.$$

Since $(1, A_S) \subset R \times S \setminus (f_R^{-1}(\mathfrak{p}_R), A_S)$, we get $(R \times S)_{f^{-1}(\mathfrak{p})} \cong R_{f_R^{-1}(\mathfrak{p}_R)}$. Similarly $A_{\mathfrak{p}} \cong (A_R)_{\mathfrak{p}_R}$. Hence $f : R \times S \to A$ being unramified is without loss of generality equivalent to

$$f_R : R_{f_R^{-1}(\mathfrak{p}_R)} \to (A_R)_{\mathfrak{p}_R}$$

being unramified.

Flatness: Recall that we have

$$(M_R \times M_S) \otimes_{R \times S} (N_R \times N_S) \cong (M_R \otimes_R N_R) \times (M_S \otimes_S N_S).$$

Assume now that M_R and M_S are flat over R and S respectively. Take any module N over $R \times S$. By Proposition 4.4.4 we know that $N \cong N_R \times N_S$ were N_R is an

R-module and N_S is an S-module. By the above formula and our assumption, we get that N is flat over $R \times S$. Same for the other side. This implies the result.
Surjectivity: Let $f : A \to R \times S$ be a be a ring homomorphisms that is surjective on the set of prime ideals. Formula 8.1 implies that its equivalent to

$$f_R^{-1} \amalg f_S^{-1} : \mathsf{Spec}(R) \coprod \mathsf{Spec}(S) \to \mathsf{Spec}(A_R) \coprod \mathsf{Spec}(A_S)$$

being surjective. This now implies the result. $\mathcal{Q.E.D}$

Given a geometric point $\overline{s} : \mathsf{Spec}(\omega) \to X$ of X, we denote the underlying set associated to the scheme $Y_{\overline{s}} := Y \times_X \mathsf{Spec}(\omega)$ by $X_{\overline{s}}^{\mathsf{Set}}$. Thus, we obtain the functor

$$\mathfrak{F}_{\overline{s}} : \mathsf{FEC}(X) \to \mathsf{FSets},$$

given by

$$\mathfrak{F}_{\overline{s}}(Y) = Y + \overline{s}^{\mathsf{Sets}}.$$

We have the following well-known result:

Theorem 8.0.2. *Let X be a finitely connected, noetherian scheme. The category $\mathsf{FEC}(X)$ of finite etale coverings of X, together with all the functors*

$$\{\mathfrak{F}_{\overline{s_i}} : \mathsf{FEC}(X) \to \mathsf{FSets}\}_{i \in I}$$

induced by every geometric point $\overline{s_i} : \mathsf{Spec}(\omega_i) \to X, i \in I$, is a (finitely connected) Galois category.

Theorem 8.0.3. *Let X be a finitely connected, noetherian scheme and denote by $\mathsf{FEC}(X)$ the site of finite étale coverings of X. The 2-functor ${}_{\mathcal{F}}\mathcal{E}_{\mathcal{C}} : \mathsf{FEC}(X) \to \mathsf{GCat}$, given by*

$$Y \mapsto \{\mathfrak{F}_{\overline{s_i}} : \mathsf{FEC}(Y) \to \mathsf{FSets}\}_{i \in I},$$

is a stack.

Proof. Recall first that for finitely coverable sites, we could talk about stacks with values in Galois categories (Definition 6.2.7). Since for a noetherian scheme X the site of finite étale coverings $\mathsf{FEC}(X)$ satisfies said finiteness condition, the proof follows from [21, p.187, Proposition 4.1], and Lemmas 8.0.1 and 4.4.3. $\mathcal{Q.E.D}$

Definition 8.0.4. *Let* $\{\mathcal{F}_j : \mathcal{C} \to \mathsf{FSets}\}_{j\in J}$ *be a finitely connected Galois category. We define its* fundamental groupoid $\Pi_1(\mathcal{C})$ *as follows:*

- *Objects of* $\Pi_1(\mathcal{C})$ *are the functors* $\{\mathcal{F}_j : \mathcal{C} \to \mathsf{FSets}\}_{j\in J}$*;*
- *For* $\mathcal{F}_i$ *and* $\mathcal{F}_j$ *we define* $\mathsf{Hom}_{\mathsf{Fnct}}(\mathcal{F}_i, \mathcal{F}_j) := \mathsf{Iso}_{\mathsf{Fnct}}(\mathcal{F}_i, \mathcal{F}_j)$ *where* $\mathsf{Iso}_{\mathsf{Fnct}}$ *denotes the set of natural isomorphisms.*

Definition 8.0.5. *Let X be a finitely connected site. Define the* étale fundamental groupoid $\Pi_1(X)$ *of X to be the fundamental groupoid of the Galois category* ${}_{\mathcal{F}}\mathcal{E}_{\mathcal{C}}(X)$*, being* $\{\mathfrak{F}_{\overline{s_i}} : \mathsf{FEC}(X) \to \mathsf{FSets}\}_{i\in I}$*.*

Equivalently, Theorem 8.0.3 can be stated as follows:

Theorem 8.0.6. *The 2-functor* ${}_{\mathcal{F}}\mathcal{E}_{\mathcal{C}} : \mathsf{FEC}(X) \to \mathsf{GCat}$ *given by* $Y \mapsto \Pi_1(Y)\text{-}\mathsf{FSets}$*, where* $\Pi_1(Y)$ *denotes the étale fundamental groupoid of Y, is a stack.*

Proof. In the case of connected schemes this result is a classical result by Grothendieck. We use our proof of the generalised case, when X need not be connected, given in Theorem 6.2.5 to obtain this result. $\mathcal{Q.E.D}$

Theorem 8.0.7 (Seifert-van Kampen Theorem)**.** *Let X be a finitely connected, noetherian scheme. The assignment* $Y \mapsto \Pi_1(Y)$ *defines a costack on the site of finite étale coverings of X.*

Proof. For any covering $Z \in \mathsf{Cov}(Y)$, we have the 2-functor $2\text{-}\mathsf{colim}(Z, \Pi_1) \to \Pi_1(Z)$, where $2\text{-}\mathsf{colim}(Z, \Pi_1)$ denotes the 2-colimit of

$$\Pi_1(Z \times_X Y) \leftleftarrows \Pi_1(Z \times_X Y \times_X Y) \Lleftarrow \Pi_1(Z \times_X Y \times_X Y \times_X Y).$$

Hence we get the associated functor $\Pi_1(Z)\text{-}\mathsf{FSets} \to 2\text{-}\mathsf{colim}(Z, \Pi_1)\text{-}\mathsf{FSets}$, where we denoted by $\mathfrak{G}\text{-}\mathsf{FSets}$ the functor category $\mathsf{Hom}_{\mathfrak{Cat}}(\mathfrak{G}, \mathsf{FSets})$. Since $\mathsf{Hom}_{\mathfrak{Cat}}(-, \mathsf{FSets})$

is left exact, we have

$$2\text{-}\,\mathsf{colim}(Z,\Pi_1)\text{-}\,\mathsf{FSets} \cong 2\text{-}\lim((Z,\Pi_1)\text{-}\,\mathsf{FSets}),$$

where 2-lim$((Z,\Pi_1)$-$\mathsf{FSets})$ denotes the 2-limit of

$$\Pi_1(Z\times_X Y)\text{-}\,\mathsf{FSets} \rightrightarrows \Pi_1(Z\times_X Y\times_X Y)\text{-}\,\mathsf{FSets} \Rrightarrow \Pi_1(Z\times_X Y\times_X Y\times_X Y)\text{-}\,\mathsf{FSets}\,.$$

By Theorem 8.0.6, the functor $\Pi_1(Z)$- $\mathsf{FSets} \to$ 2-lim$((Z,\Pi_1)$-$\mathsf{FSets})$ is an equivalence of categories. Hence from Corollary 6.2.6, 2-$\mathsf{colim}(Z,\Pi_1) \to \Pi_1(Z)$ is an equivalence of categories as well. This proves the assertion. $\mathcal{Q.E.D}$

Definition 8.0.8. *Let $\mathfrak{C}$ be a 2-category. We say that $\mathfrak{T}$ is the* 2-terminal object *of $\mathfrak{C}$, if for any other object $C \in \mathfrak{C}$, $Hom_{\mathfrak{Cat}}(C,\mathfrak{T})$ is equivalent to the 1-point category.*

Theorem 8.0.9. *Let X be a finitely connected, noetherian scheme. The assignment $U \mapsto \Pi_1(U)$, $U \in X$ is the 2-terminal costack of groupoids over the site of étale coverings of X.*

To prove this theorem, we first need a few other results.

Lemma 8.0.10. *Consider the constant covariant 2-functor $\mathfrak{s} : U \mapsto \mathsf{FSets}$. Its associated prestack is given by $\bar{\mathfrak{s}} : U \mapsto CS(U)$, where $CS(U)$ denotes the category of constant sheaves of finite sets on U.*

Proof. Recall our discussion about prestacks in Subsection 4.3.1. We know that the objects remain the same, but the morphisms are replaced by the sections of the sheafification of the presheaves $Hom_U(a,b)$. We claim that this is equivalent to the category of constant sheaves on U. The fact that the objects of these two categories are equivalent is clear. To see that the Hom-sets are isomorphic, first observe that the sheafification of $Hom_U(a,b)$ is $Hom(\pi_0(u), Hom(a,b))$, which is isomorphic to

$(b^a)^{\pi_0(U)} = b^{a\pi_0(U)}$. Denote by $\overline{a}$ the constant sheaf with value a. Then

$$\begin{aligned}\mathsf{Hom}_{Sheaf}(\overline{a},\overline{b}) &= \mathsf{Hom}_{Presheaf}(\overline{a},\overline{b}) \\ &= \mathsf{Hom}_{Presheaf}(a,\overline{b}).\end{aligned}$$

Since $\overline{b}$ is given by $U \mapsto Hom(\pi_0(U), b)$, we have

$$Hom_{Presheaf}(a,\overline{b}) = Hom(a, Hom(U,b)).$$

This is isomorphic to $(b^{\pi_0(U)})^a = b^{\pi_0(U)a}$, proving the assertion. $\mathcal{Q.E.D}$

Corollary 8.0.11. *Consider the constant covariant 2-functor* $\mathfrak{s} : U \mapsto \mathsf{FSets}$*. Its associated stack* $\hat{\mathfrak{s}}$ *is given by* $\hat{\mathfrak{s}}(U) = LCS(U)$*, where* $LCS(U)$ *denotes the category of locally constant sheaves of finite sets on* U*.*

Let A be a covariant 2-functor. Recall that we denoted by A_S the contravariant 2-functor given by $U \mapsto Hom(A(U), \mathsf{FSets})$.

Proof of Thm. 8.0.9. We have already shown that the 2-functor $U \to \Pi_1(U)$ forms a costack. Hence, to prove this theorem, we only have to show that for every costack C of groupoids, we have an essentially unique map $C \Rightarrow \Pi_1$.

Denote by P the constant, covariant assignment $U \mapsto \mathsf{pt}$, where pt is seen as a groupoid. It is clear that we have a natural transformation $C \Rightarrow P$, and as such, a Galois transformation $P_S \Rightarrow C_S$. As shown in Corollary 8.0.11, the stackification of P_S exists and is $U \mapsto LCS(U)$, where $LCS(U)$ denotes the category of locally constant sheaves on U. In the case of noetherian schemes, $LCS(U)$ is equivalent to the Galois category $\Pi_1(U)$-FSets. Since C was a costack, C_S is a Galois stack and hence, $P_S \Rightarrow C_S$ factors through Π_{1S}. Using Corollary 7.3.3, we know that the obtained natural transformation $\Pi_{1S} \Rightarrow C_S$ is a Galois transformation.

By the uniqueness of the associated stack (see Definition 4.3.6) and Corollary 6.2.8 respectively, we have the following equivalences of categories:

$$\mathsf{Hom}_{\mathsf{Gal}}(P_S, C_S) \cong \mathsf{Hom}_{\mathsf{Gal}}(\Pi_{1S}, C_S) \cong \mathsf{Hom}_{\mathfrak{Cat}}(C, \Pi_1).$$

Here $\mathsf{Hom}_{\mathsf{Gal}}$ denotes the category of Galois transformations (i.e. of natural transformations respecting finite limits and colimits). Uniqueness and existence now comes from the fact that we have precisely one exact natural transformation $P_S \Rightarrow C_S$. The last claim is true because P_S is equivalent to FSets and hence a functor respecting finite colimits is defined by its value on the singleton $\star$, which has to map to the terminal object of C_S. $\mathcal{Q.E.D}$

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