Tilings Generated by Non-Parallel Projection Schemes

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by

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Abstract

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This thesis defines and investigates rational and irrational 2:1 X-projection schemes and non-parallel projection schemes with strips at rational gradients.

Both irrational 2:1 X-projection schemes and non-parallel projection schemes with strips at rational gradients are shown to produce tilings with infinitely many prototiles, with the tilings produced by the second of these schemes nonetheless shown to display a property similar to repetitivity.

Rational 2:1 X-projection schemes are shown to produce tilings with a finite number of prototiles, with a subset of these tilings shown to be repetitive. The points in the fundamental domain of our lattice L that correspond to translates of these tilings are also investigated, with these points shown to be either dense in a finite number of lines or dense in the fundamental domain. This also leads to a proof of repetitivity in all rational 2:1 X-projection tilings and aperiodicity in a subset of these tilings. The tiling spaces of such tilings are also investigated.

In addition, the proportions in which the prototiles in a rational 2:1 X-projection tiling appear are also looked at, and a possible explanation of the values observed is provided.

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Contents

1 Introduction

People have been creating tilings for thousands of years, from the wall and floor tilings of ancient civilisations such as the Romans and Persians (see for example [6]) to much more recent work like the drawings of M. C. Escher. These patterns are generally periodic tilings of the plane, meaning that they consist of a finite number of tiles arranged in a certain way within some patch, with this patch then repeated in a regular way throughout the plane.

In this document we will be concerned with tilings that are aperiodic (see definition 2.9), so do not consist of a single patch of tiles that repeats in a regular way. However, the tilings that we are interested in may be repetitive (see definition 2.10), which means that any patch of tiles in the tiling will reappear throughout the tiling and always within some fixed distance (that depends on the patch) of any point in the tiling. This is a property that all periodic tilings have, but that is not necessarily shared by an aperiodic tiling.

1.1 Crystals and Quasicrystals

Aperiodic tilings and tile sets (set of tiles that will only fit together to form aperiodic tilings) of the plane have been studied for decades, with one-dimensional aperiodic tilings even older, however physical analogues to these mathematical objects were not encountered until the 1980s, when quasicrystals were discovered (first reported in [15]).

Crystals are 3-dimensional structures in which the constituent atoms or molecules are arranged in regular repeating pattern, and so are much like 3-dimensional periodic tilings. The structure of a crystal can be determined by looking at its corresponding diffraction pattern, which is produced by shining X-rays through a thin slice of the crystal. The diffraction patterns produced by crystals look like patterns of points with a rotational symmetry of order 2, 3, 4 or 6, with any other order of rotational symmetry impossible (see [16]).

Quasicrystals were first identified by their diffraction patterns, which were pure point (i.e. consisting of distinct bright spots) like those of crystals, but displayed forbidden symmetries such as 5-fold and 10-fold rotational symmetry. The pure point diffraction patterns suggested that these substances were "crystal-like" in the sense that they must have structures that are somewhat regular, but the symmetries of the patterns ruled out the possibility of these structures being periodic.

Identifying the structure of quasicrystals provided some physical motivation for the study of aperiodic tilings.

1.2 Generating Aperiodic Tilings

There are several methods for generating tilings. We will briefly look at three of these methods here. The third of these (the projection method) is the one that we will be most interested in for the remainder of this document.

The first method for generating tilings is to start with a set of tiles and impose matching rules on them so that they can only fit together in certain ways (see for example [13]). For example, to produce a Wang tiling (see [1], [7]) we will start with a set of square tiles with the edges of each coloured in some way and then fit them together so that matching edges have the same colour (see figure 1.1). A similar effect can be achieved by altering the shapes of the edges slightly so that only edges with the same colour can fit together.

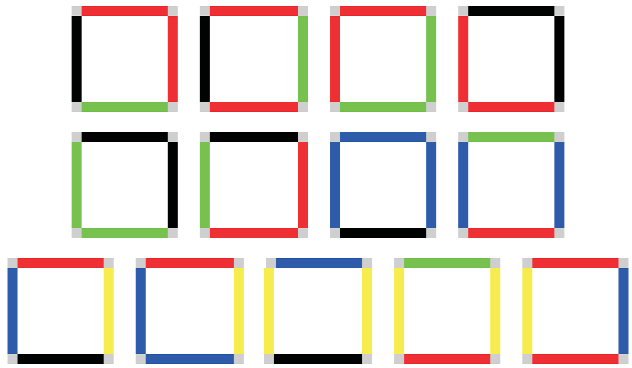

Figure 1.1: An aperiodic set of 13 Wang tiles [3]

The second way to obtain tilings is by substitution (see for example [5]). For this method we take a set of tiles and define a substitution rule for each of them, where a single tile is replaced by a patch of one or more tiles from our set at each iteration. We can thus start with a tile or patch of tiles and perform the substitution to get a larger patch, then substitute again to get an even larger patch, and so on. For example, we can create a Penrose tiling [12] in this way (see figure 1.2). Note that Penrose tilings can also be constructed using matching rules [11].

The final method for generating tilings that we shall mention here is the projection method. For this method we start with some lattice in a higher dimension, typically $\mathbb{Z}^N$, select some subset of the points of this lattice and then project these points onto the space in which we want our tiling. We then form a tiling from these points. This is examined in greater detail in the following chapter.

In this document we will define a modification of the projection setup, where the points selected from the higher dimensional lattice (we will be looking at $\mathbb{Z}^2$) are those contained in two "strips" that have different gradients, and the projection is onto a space that has a gradient independent of those of the strips.

We will see that there are several versions of this setup, giving different

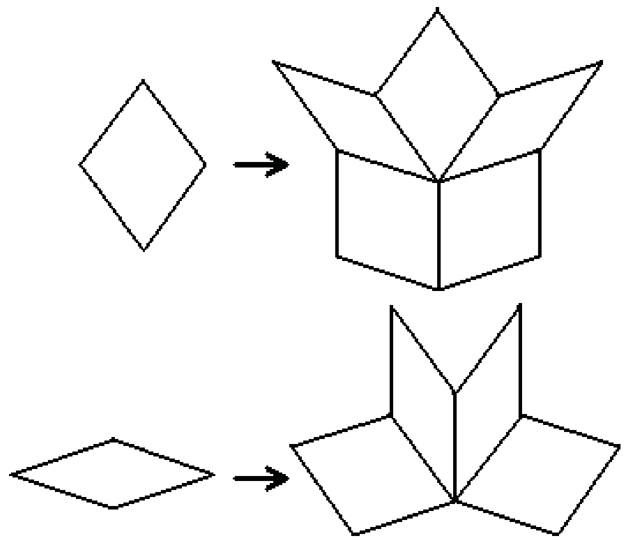

Figure 1.2: Substitution rules for the Penrose Rhombs

classes of tilings. We will then examine the different types, particularly the tilings generated by rational 2:1 X-projection schemes.

1.3 Document Layout

As explained above, this document is largely concerned with projection tilings, with the projection typically being from a 2-dimensional lattice onto a 1-dimensional space.

In chapter 2 we define tiles and tilings and then projection tilings, with particular emphasis on canonical 2:1 projections, and prove some basic results about these tilings.

In chapter 3 we introduce non-parallel projection tilings, which are generated by selecting lattice points from within two non-parallel strips and projecting these onto a line with a gradient independent of either strip. These come in three distinct types, and we provide some basic results about each of these types.

In chapter 4 we look in greater detail at rational 2:1 X-projection tilings (one of the three types of non-parallel projection introduced in chapter 3), examining the positions in which we can place our two strips to obtain translates of our tiling. We also prove that certain rational 2:1 X-projection setups will produce aperiodic tilings.

In chapter 5 we look at the tiling spaces associated to rational 2:1 X-projection tilings. In addition, we see that all rational 2:1 X-projection schemes give repetitive tilings.

In chapter 6 we provide some examples of both rational and irrational 2:1 X-projection tilings and also look at the number of prototiles in a tiling produced by a rational 2:1 X-projection scheme and the proportions in which these prototiles appear.

Finally, the conclusion provides a summary of the results obtained in this document along with some remaining questions.

Figure 1.3 shows a more pictographic overview of the structure of this thesis, with the solid lines indicating the connection between different sections (and the dotted line showing a possible connection).

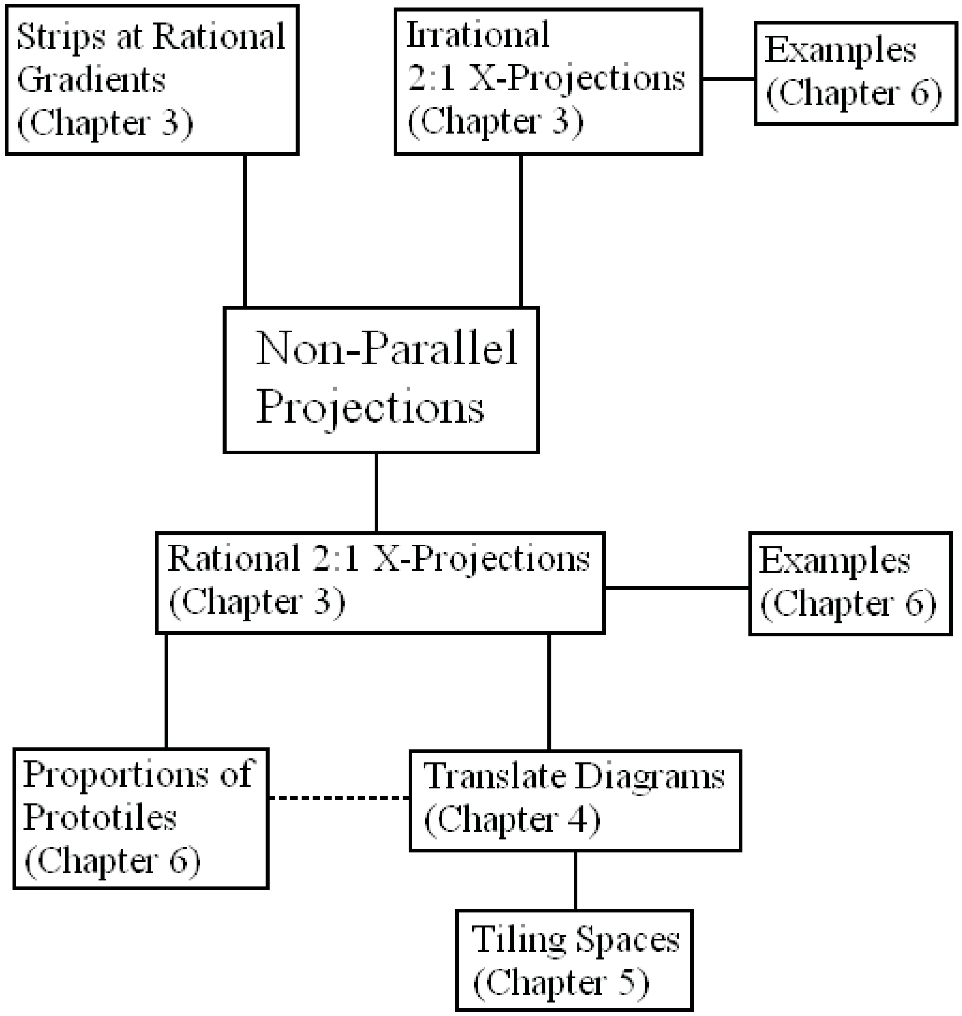

Figure 1.3: A summary of the contents of this document.

2 Tiles, Tilings, Model Sets and Projection Schemes

We begin this chapter with the basic definitions involved in the study of tilings, starting with the definitions of a tile and a tiling that we will be using throughout this document (see for example [14]).

Definition 2.1. A set $t \subseteq \mathbb{R}^n, n \geq 1$, is called a *tile* if it is compact and equal to the closure of its interior. We will also always assume that tiles are homeomorphic to topological balls. So tiles in $\mathbb{R}$ are closed intervals and normally tiles in $\mathbb{R}^2$ will be polygons.

Definition 2.2. A *tiling* T of $\mathbb{R}^n$ is a collection of tiles that,

- *Pack* $\mathbb{R}^n$, meaning that any two tiles have pairwise disjoint interiors.
- *Cover* $\mathbb{R}^n$, i.e., the union of all the tiles is $\mathbb{R}^n$.

Definition 2.3. We say that two tiles t_1, t_2 are *equivalent* if one is a translate of the other.

Definition 2.4. Equivalence class representative of tiles are called *prototiles*.

We will mostly be interested in tilings that have a finite set of prototiles, which means that they are made up of only finitely many "types" of tile, however there will be some occasions when we look at tilings with an infinite set of prototiles.

Definition 2.5. A *patch* of tiles is a finite set of tiles in a tiling whose union is connected.

2.1 Model Sets and Projection Schemes

Model sets, which are also known as cut-and-project sets (first constructed in [2]), are sets of points that are generated by selecting certain points of a higher dimensional lattice (through a *cut-and-project scheme*) and projecting these down onto a space of smaller dimension.

Definition 2.6. A *cut-and-project scheme* (see for example [10]) consists of a lattice, L, in the space $\mathbb{R}^m \times \mathbb{R}^n$ (i.e. a discrete subgroup of $\mathbb{R}^m \times \mathbb{R}^n$ that spans $\mathbb{R}^m \times \mathbb{R}^n$) and projections $\pi_1 : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ and $\pi_2 : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$, where $\pi_1|_L$ is injective and $\pi_2(L)$ is dense in $\mathbb{R}^n$.

$\mathbb{R}^n$ is the space in which the Model set will be generated and hence is known as the *pattern space*, whereas the space $\mathbb{R}^m$ is called the *internal space*.

If we take a subset $K \subset \mathbb{R}^m$ then we denote by $\Lambda(K)$ the point pattern in $\mathbb{R}^n$ given by the projection into $\mathbb{R}^n$ of the points of L that are projected by π_2 into K, i.e.

$$\Lambda(K) = \{\pi_1(x) \in \mathbb{R}^n : x \in L, \pi_2(x) \in K\}.$$

Here we call K the *acceptance domain*.

Definition 2.7. A *model set* [9] (or *cut-and-project set*) is a subset Γ of $\mathbb{R}^n$ satisfying $\Lambda(W^\circ) \subset \Gamma \subset \Lambda(W)$, for some W connected and compact in $\mathbb{R}^n$, $W = \overline{W^\circ} \neq \emptyset$. The model set Γ is *regular* if the boundary $\partial W = W \backslash W^\circ$ of W is of Lebesgue measure 0.

We will largely be concerned with 2:1 projections. That is, projections of points from a 2-dimensional lattice onto a 1-dimensional space. In the case of 2:1 projections we have a lattice L in $\mathbb{R}^2$ with two projections from $\mathbb{R}^2$ onto the axes, satisfying the above conditions, i.e., $\pi_1|_L$ is injective and $\pi_2(L)$ is dense in $\mathbb{R}$.

We will usually refer to the pattern space as E with the internal space being called $E^\perp$.

Example 2.1. *If we take L to be a square lattice then with K being a single closed interval in $\mathbb{R}$ we will get a situation much like that illustrated in figure 2.1.*

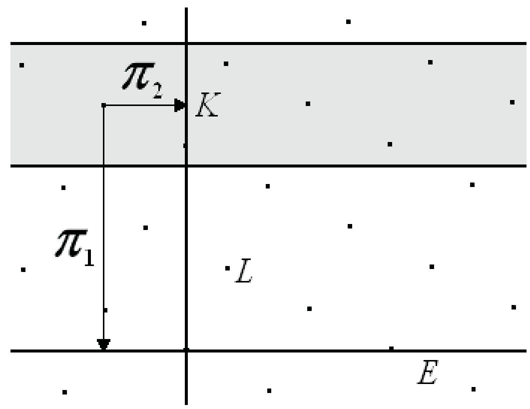

Figure 2.1: A 2:1 projection scheme.

In a similar way we can consider the lattice to be fixed and the pattern space (E) to be at an irrational gradient relative to the square lattice, as shown in figure 2.2.

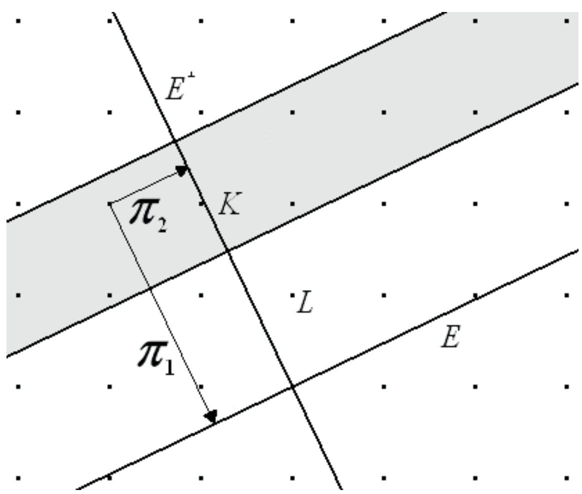

Figure 2.2: An alternative way of viewing the 2:1 projection scheme.

2.2 Canonical 2:1 Projection Tilings

Definition 2.8. A *canonical 2:1 projection scheme* is a cut-and-project scheme as detailed above with lattice $L = \mathbb{Z}^2$ and acceptance domain K being a closed interval, where the width of this interval, and therefore the strip that lattice points are projected from, is taken to be equal to the projection of a unit square onto $E^\perp$. In addition, the acceptance domain K is chosen so that the boundaries of the strip do not intersect any points of L.

So a strip S has *canonical width* if the point (α, β) (for $\alpha, \beta \in \mathbb{R}$) is on the lower boundary of the strip if and only if the point $(\alpha - 1, \beta + 1)$ is on the upper boundary.

Note that a canonical width strip will have width $\sin\theta + \cos\theta$, where θ is the angle of the strip relative to the integer lattice, as can be seen from figure 2.3.

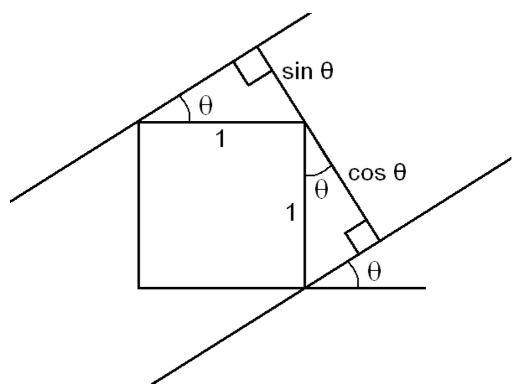

Figure 2.3: Canonical strip width.

Observe also that θ must be an irrational multiple of 2π, i.e., the strip must have an irrational gradient. This is because a projection with rational gradient would not result in $\pi_1|_L$ being injective or $\pi_2(L)$ being dense in $E^\perp$.

Proposition 2.1. *It is possible to choose an acceptance domain K so that the boundaries of the strip $K + E$ do not intersect any points of the lattice.*

Proof. Choosing a strip whose boundaries do not intersect any lattice points is equivalent to choosing a line that does not intersect any lattice points. This is because the width of the strip is chosen so that the upper boundary line is the translation of the lower boundary line by $(-1, 1)$. So there is a lattice point on the lower boundary line if and only if there is a lattice point on the upper boundary line.

So start with a line, L, in $\mathbb{R}^2$. Then any translate of L by a vector not parallel to L, say by a vector parallel to $L^\perp$, will not intersect L and so will not intersect any of the lattice points on L. More generally, two translates by non-equal vectors parallel to $L^\perp$ will not intersect each other and so will not contain any of the same lattice points. However, there are uncountably many translates of L of this type and only countably many lattice points, so therefore there must exist translates of L that do not intersect any lattice points. $\square$

2.2.1 Characteristics of Canonical 2:1 Projection Tilings

If we construct a canonical 2:1 projection tiling as above, then the tiling will have certain attributes, some of which are detailed in this section.

Firstly, the "horizontal" and "vertical" widths of the strip are given by,

$$horizontal\ width = 1 + \frac{\cos\theta}{\sin\theta}$$
$$vertical\ width = 1 + \frac{\sin\theta}{\cos\theta}$$

as can be seen from figure 2.4.

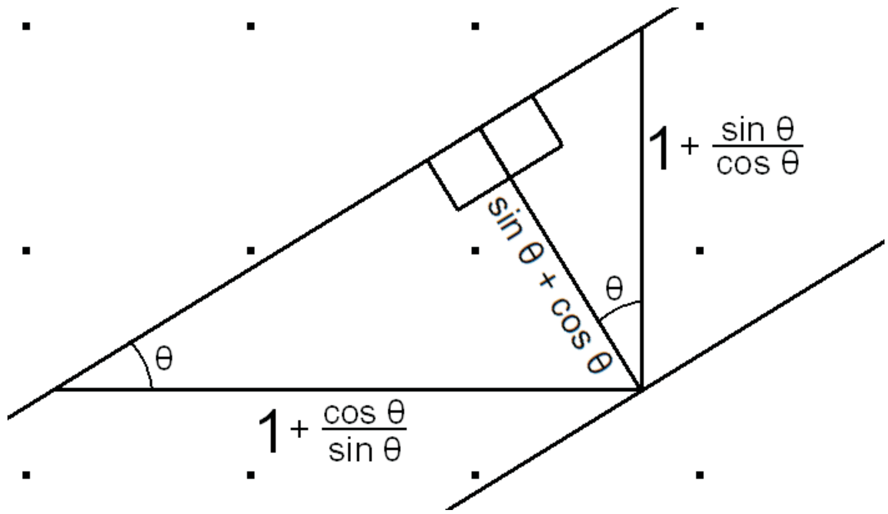

Figure 2.4: The horizontal and vertical widths of the strip.

Proposition 2.2. *There are exactly two types of tile in a canonical 2:1 projection tiling with irrational gradient.*

Proof. If we have a lattice point (x, y) within the strip $K + E$ then exactly one of the points $(x, y + 1)$ and $(x + 1, y)$ is contained in the strip.

If (x, y) is less than distance $\sin\theta$ from the lower boundary of the strip then (x, y) is within "vertical" distance $\frac{\sin\theta}{\cos\theta}$ and within "horizontal" distance 1 of the lower boundary of the strip. Therefore we get that the lattice point $(x, y+1)$ is contained in the strip and the lattice point $(x + 1, y)$ is not contained in the strip.

If (x, y) is greater than distance $\sin\theta$ from the lower boundary of the strip (and therefore less than distance $\cos\theta$ from the upper boundary) then it is within "horizontal" distance $\frac{\cos\theta}{\sin\theta}$ and "vertical" distance 1 of the upper boundary, and thus the lattice point $(x + 1, y)$ is contained in the strip and the lattice point $(x, y + 1)$ is not.

Note that by the choice of position of the strip you can never have a lattice point exactly distance $\sin\theta$ from the lower boundary, as this would imply that the points $(x+1, y)$ and $(x, y+1)$ were on the boundaries of the strip.

So for any lattice point (x, y) within the strip exactly one of the lattice points $(x, y+1)$ and $(x+1, y)$ is also contained in the strip. Similarly, exactly one of $(x-1, y)$ and $(x, y-1)$ is contained within the strip.

We are restricting to the case where $0 < \theta < \frac{\pi}{2}$, so $(x+1, y)$ and $(x, y+1)$ are both projected further along E than the projection of (x, y) and therefore all subsequent lattice points in the strip are projected yet further along. So the projection of any lattice point (x, y) is followed by the projection of either $(x, y+1)$ or $(x+1, y)$, meaning that there are at most two tile lengths.

Note that you will only get one length of tile if $(x, y+1)$ and $(x+1, y)$ are projected to the same point, however this can only happen if $\theta = \frac{\pi}{4}$, which is discounted by the choice of irrational gradient of the strip.

Now, $0 < \theta < \frac{\pi}{2}$, so

$$horizontal\ width = 1 + \frac{\cos\theta}{\sin\theta} > 1$$

and

$$vertical\ width = 1 + \frac{\sin\theta}{\cos\theta} > 1\,.$$

θ is irrational, so there are points arbitrarily close to the boundaries of the strip $K + E$ which are within the strip. Therefore for any irrational θ satisfying $0 < \theta < \frac{\pi}{2}$ there are lattice points within the strip $K + E$ that are within distance $\sin\theta$ of the lower boundary of the strip, and there are lattice points within distance $\cos\theta$ of the upper boundary. Therefore both types of tile appear in any canonical 2:1 projection tiling with irrational gradient.

$\square$

So there are exactly two different lengths of tile in every 2:1 canonical projection tiling. The lengths of these two tiles are $\sin\theta$ and $\cos\theta$ as can be seen from figure 2.5.

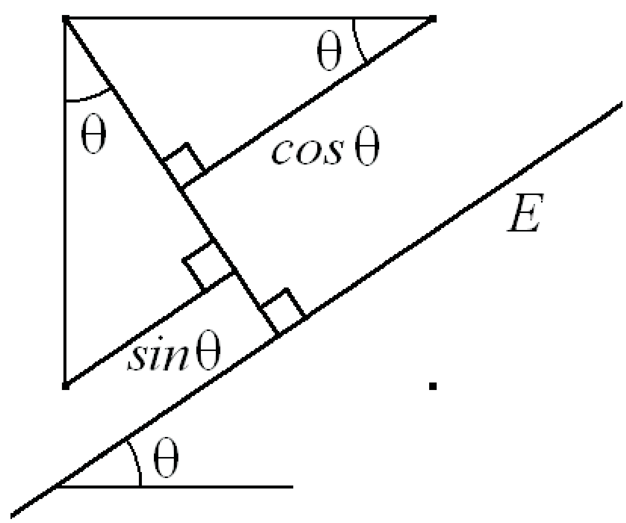

Figure 2.5: The projections of horizontal and vertical steps.

Proposition 2.3. *A canonical 2:1 projection tiling has two prototiles, the shorter of which always appears flanked by two longer tiles.*

Proof. We can relate the "horizontal" and "vertical" widths of the strip to the lengths of the two tiles in the following way:

$$horizontal\ width = 1 + \frac{length\ of\ tile\ 1}{length\ of\ tile\ 2}$$

$$vertical\ width = 1 + \frac{length\ of\ tile\ 2}{length\ of\ tile\ 1}\,.$$

As already stated, the lengths of the two tiles cannot be equal when $\theta \neq \frac{\pi}{4}$, so the "horizontal" and "vertical" widths are not equal and,

$$longer\ width = 1 + \frac{length\ of\ longer\ tile}{length\ of\ shorter\ tile} > 2$$

$$shorter\ width = 1 + \frac{length\ of\ shorter\ tile}{length\ of\ longer\ tile} < 2\,.$$

If $0 < \theta < \frac{\pi}{4}$ then the "vertical" width is shorter and since it is strictly less than two it is not possible to have more than two lattice points arranged in a vertical row within the strip (recall: lattice points are distance 1 apart), i.e., there cannot be more than one tile of length $\sin\theta$ in a row. Similarly, if $\frac{\pi}{4} < \theta < \frac{\pi}{2}$ then the "horizontal" width is shorter and there cannot be more than one tile of length $\cos\theta$ in a row.

But note that for $0 < \theta < \frac{\pi}{4}$ the tile of length $\sin\theta$ is the shorter tile, and for $\frac{\pi}{4} < \theta < \frac{\pi}{2}$ the tile of length $\cos\theta$ is the shorter tile. So in any canonical 2:1 projection tiling you never get two shorter tiles next to each other. $\square$

Proposition 2.4. *In a canonical 2:1 projection tiling the longer of the two prototiles appears in patches, with the number of tiles in each patch equal to* $\left\lfloor \frac{\text{length of longer tile}}{\text{length of shorter tile}} \right\rfloor$ *or* $\left\lfloor \frac{\text{length of longer tile}}{\text{length of shorter tile}} \right\rfloor + 1$.

Proof. We know that the horizontal and vertical widths of the strip are given by,

$$\textit{horizontal width} = 1 + \frac{\cos\theta}{\sin\theta}$$

$$\textit{vertical width} = 1 + \frac{\sin\theta}{\cos\theta}\,.$$

For $0 < \theta < \frac{\pi}{4}$, the tile of length $\cos\theta$ (corresponding to a horizontal step between lattice points) is the longer tile.

In this case, the maximum number of consecutive lattice points in a horizontal row within the strip is $2 + \left\lfloor \frac{\text{length of longer tile}}{\text{length of shorter tile}} \right\rfloor$, because you can fit a horizontal line of length $1 + \left\lfloor \frac{\text{length of longer tile}}{\text{length of shorter tile}} \right\rfloor$ within the strip. Also, the smallest number of lattice points in a horizontal row in the strip must be $1 + \left\lfloor \frac{\text{length of longer tile}}{\text{length of shorter tile}} \right\rfloor$, because it is possible to position a horizontal line of length $\left\lfloor \frac{\text{length of longer tile}}{\text{length of shorter tile}} \right\rfloor$ within the strip with endpoints less than "horizontal" distance 1 from the boundaries of the strip, but this cannot be done with a horizontal line of length $\left\lfloor \frac{\text{length of longer tile}}{\text{length of shorter tile}} \right\rfloor - 1$.

The case where $\frac{\pi}{4} < \theta < \frac{\pi}{2}$ is similar but with the tiles of length $\sin\theta$ (corresponding to vertical steps) being longer and appearing in blocks.

So the longer tiles in a canonical 2:1 projection tiling come in blocks of length $1 + \left\lfloor \frac{length\ of\ longer\ tile}{length\ of\ shorter\ tile} \right\rfloor$ or $\left\lfloor \frac{length\ of\ longer\ tile}{length\ of\ shorter\ tile} \right\rfloor$.

$\square$

Note that $\frac{length\ of\ longer\ tile}{length\ of\ shorter\ tile}$ is not an integer, since $\frac{length\ of\ longer\ tile}{length\ of\ shorter\ tile}$ is equal to $\frac{\sin\theta}{\cos\theta}$ or $\frac{\cos\theta}{\sin\theta}$, and so is equal to the gradient of the strip, or 1 divided by the gradient, which cannot be an integer by the choice of an irrational gradient.

Thus the tiling given by a projection of this kind must have two types of tile and consist of blocks of the longer tiles of length $1 + \left\lfloor \frac{length\ of\ longer\ tile}{length\ of\ shorter\ tile} \right\rfloor$ or $\left\lfloor \frac{length\ of\ longer\ tile}{length\ of\ shorter\ tile} \right\rfloor$ divided by solitary tiles of shorter length.

Now we look at the proportions in which the two prototiles appear in a tiling by considering $\frac{number\ of\ short\ tiles}{number\ of\ long\ tiles}$ for some connected section of the tiling, and the limit of this sequence as the section is lengthened. This will show the gradient, or $\frac{1}{gradient}$, of the line drawn from the lattice point that is projected to the start of this section of tiles to the lattice point that is projected to the end, i.e., you get the gradient of the line shown in figure 2.6.

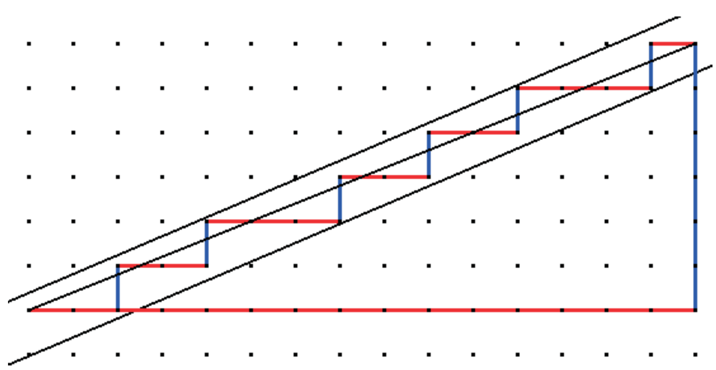

Figure 2.6: Horizontal and vertical steps in a section of the strip.

Note 2.1. If $\frac{\pi}{4} < \theta < \frac{\pi}{2}$ then the shorter tiles will correspond to horizontal “steps”, so you will be measuring $\frac{1}{gradient}$.

As you examine longer sections of the tiling you will be looking at larger sections of the strip, but both the endpoints are contained within the strip, so the gradient of the line drawn between them can only differ slightly from the gradient of the strip.

Proposition 2.5. *In a canonical 2:1 projection tiling we have that,*

$$\lim_{n\to\infty}\left(\frac{number\ of\ shorter\ tiles}{number\ of\ longer\ tiles}\right) = \frac{length\ of\ shorter\ tile}{length\ of\ longer\ tile}.$$

If $0 < \theta < \frac{\pi}{4}$ *then this value is the gradient of the strip.*

If $\frac{\pi}{4} < \theta < \frac{\pi}{2}$ *then this value is one over the gradient of the strip.*

Proof. The situation is shown in figure 2.7, where n is the number of tiles corresponding to horizontal steps in the section that you are examining.

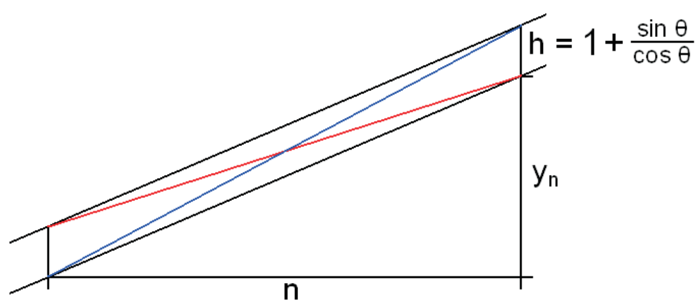

Figure 2.7: The maximum and minimum possible gradients.

The gradient of the strip is given by,

$$gradient = \frac{y_n}{n}.$$

Then the maximum and minimum possible gradients of lines drawn between lattice points in the strip, corresponding to the blue and red lines respectively, are:

$$maximum\ gradient = \frac{y_n + h}{n} = \frac{y_n}{n} + \frac{h}{n} = gradient\ of\ strip\ + \frac{h}{n}$$

$$minimum\ gradient = \frac{y_n - h}{n} = \frac{y_n}{n} - \frac{h}{n} = gradient\ of\ strip\ - \frac{h}{n}\,.$$

The value of h remains constant, so at the limit the gradient of the line must be equal to the gradient of the strip.

Now, if $0 < \theta < \frac{\pi}{4}$ then the shorter tiles correspond to the vertical "steps" and are of length $\sin\theta$ and therefore,

$$\lim_{n\to\infty}\left(\frac{number\ of\ shorter\ tiles}{number\ of\ longer\ tiles}\right) = strip\ gradient$$

$$= \tan\theta$$

$$= \frac{\sin\theta}{\cos\theta}$$

$$= \frac{length\ of\ shorter\ tile}{length\ of\ longer\ tile}\,.$$

If $\frac{\pi}{4} < \theta < \frac{\pi}{2}$ then the shorter tiles correspond to the horizontal "steps" and are of length $\cos\theta$ so,

$$\lim_{n\to\infty}\left(\frac{number\ of\ shorter\ tiles}{number\ of\ longer\ tiles}\right) = \frac{1}{strip\ gradient}$$

$$= \frac{1}{\tan\theta}$$

$$= \frac{\cos\theta}{\sin\theta}$$

$$= \frac{\textit{length of shorter tile}}{\textit{length of longer tile}}\,.$$

Thus, at the limit, the ratio of the number of short tiles to the number of long tiles is equal to the ratio of their lengths.

□

There are two more characteristics of projection tilings that will be shown below. The first of these is the aperiodicity of the tiling.

Definition 2.9. A tiling T of $\mathbb{R}^n$ is said to be *aperiodic* if for any non-zero vector $v \in \mathbb{R}^n$ the tiling $T + v$ (that is, the translate of tiling T by vector v) does not coincide with T.

A tiling is said to be *periodic* if there *is* a non-zero translate of the tiling that coincides with the original tiling.

Note that a tiling of $\mathbb{R}^n$ could be periodic in only some directions and have no periodicity in others. However, we will largely be concerned with 1-dimensional tilings, for which this is not a problem.

Theorem 2.6. *A tiling produced by a canonical 2:1 projection is aperiodic.*

Proof. First we assume that the tiling T is periodic. Then there exists a vector $\underline{v} = (v_1, v_2)$ in $\mathbb{R}^2$ parallel to the pattern space E that maps T to itself.

So the projection of the lattice points after being translated by $\underline{v}$ is the same as the projection of the original lattice points (i.e. both projections give T).

Consider a lattice point $x_0 = (x, y)$ in the strip. Then $x_0 + \underline{v}$ is in the strip and because the gradient is irrational it does not coincide with any other lattice point. But there is a lattice point y_0 in the strip that is projected to the same point as $x_0 + \underline{v}$, so y_0 is some distance ε from $x_0 + \underline{v}$ in a direction parallel to $E^{\perp}$. So we will write $y_0 = x_0 + \underline{v} + \underline{\varepsilon}$, where $\underline{\varepsilon}$ is parallel to $E^{\perp}$ and of length ε.

If x_1 is the next lattice point to be projected onto E after x_0 (i.e. $x_1 = (x+1, y)$ or $(x, y+1)$) then $x_1 + \underline{v}$ must have a corresponding lattice point y_1 which is equal to $y_0 + (1,0)$ if $x_1 = (x+1, y)$ or $y_0 + (0,1)$ if $x_1 = (x, y+1)$ and so is the same distance (ε) in the same direction (along a line parallel to $E^{\perp}$) from $x_1 + \underline{v}$ as y_0 is from $x_0 + \underline{v}$. This is because the projection is onto a line at irrational gradient, so the step between lattice points x_0 and x_1 must be the same as the step between y_0 and y_1 or the tiles would not be the same length.

Thus inductively, all the subsequent lattice points satisfy, $y_i = x_i + \underline{v} + \underline{\varepsilon}$.

However, because of the irrational gradient of the strip there will be lattice points x_j arbitrarily close to the boundaries of the strip, and in particular, within distance ε of each boundary, which implies that some of the y_i are located outside the strip.

Therefore, the tiling T of E must be aperiodic. $\square$

The final attribute of tilings that are produced by canonical 2:1 projections examined in this section is that they are *repetitive*. This term is defined below.

Definition 2.10. A tiling T of $\mathbb{R}^n$ is called *repetitive* if for any patch of tiles P in T there is a number $r > 0$ such that for any point $t \in \mathbb{R}^n$ there is a translate of the patch P belonging to T and contained in the ball $B_r(t)$ (in the 1-dimensional case this is the interval of length $2r$ centred at t).

So in a repetitive tiling any finite set of connected tiles will appear throughout the tiling, never more than some distance r from any point.

Theorem 2.7. *A tiling generated by a canonical 2:1 projection is repetitive.*

Proof. Any patch, P, in T is a finite set of tiles whose union is connected. So the endpoints of the tiles in P are the projections of a finite set of lattice points $\{(x_i, y_i)\}$ within the strip.

Now, because of the choice of strip width (and bearing in mind that we have restricted θ to be between 0 and $\frac{\pi}{2}$), the lattice point $(x_i - 1, y_i + 1)$ is the same distance from the upper boundary of the strip as (x_i, y_i) is from the lower boundary, and $(x_i + 1, y_i - 1)$ is the same distance from the lower boundary as (x_i, y_i) is from the upper boundary. This is because, by the choice of width of the strip, if you moved the strip so that (x_i, y_i) was on the lower boundary then $(x_i - 1, y_i + 1)$ would be on the upper boundary and if (x_i, y_i) was on the upper boundary then $(x_i + 1, y_i - 1)$ would be on the lower boundary. Figure 2.8 illustrates the situation.

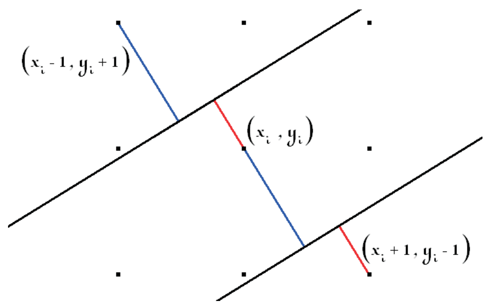

Figure 2.8: Proximity of lattice points to the strip boundaries.

So in particular, each of the points $(x_i + 1, y_i - 1)$ and $(x_i - 1, y_i + 1)$ is at least as far from the strip as the distance from (x_i, y_i) to the nearest boundary.

Now, since there are only a finite number of lattice points in the set $\{(x_i, y_i)\}$ there will be a lattice point (x_j, y_j) of minimal distance from the strip boundaries, i.e., if (x_j, y_j) is distance ε from the closest boundary then no other lattice point is within distance ε of either boundary.

Then every time a lattice point within the strip is closer than distance 2ε to the boundary that is closest to (x_j, y_j) you will get the same configuration of lattice points surrounding that point as are found in $\{(x_i, y_i)\}$, because shifting all of the points in $\{(x_i, y_i)\}$ by up to distance ε in a direction perpendicular to the strip will not translate any of them outside the strip or move any new points into the strip within the confines of the patch, i.e., all points (x_i, y_i) will remain within the strip and therefore all points of the form $(x_i + 1, y_i - 1)$ and $(x_i - 1, y_i + 1)$ will remain outside the strip. So a copy of the patch P will be found around any lattice point that is within distance 2ε of the relevant boundary.

So to show that the tiling is repetitive we have to show that given a line, L, with irrational gradient in an integer lattice, and $\varepsilon > 0$, there exists some $r > 0$ such that for any point t on the line there is a lattice point within the strip of width 2ε and length $2r$ extending out from the line in one direction with t at the centre of one side. The situation is shown in figure 2.9. (Note that the strip could also be below L).

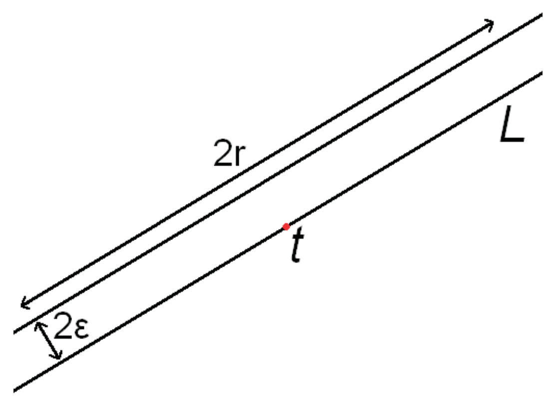

Figure 2.9: The strip of length $2r$ and width 2ε with t at the midpoint of one side.

If x is an integer, then there will be a lattice point with first coordinate x within this strip if the interval $\left(L(x), L(x) + \frac{2\varepsilon}{\cos\theta}\right)$ contains an integer, where $L(x)$ is the y-value of the line L at x (note that $\frac{2\varepsilon}{\cos\theta}$ is the vertical width of this strip and is greater than 2ε because of our choice of θ).

At $x + n$ (for $n \in \mathbb{Z}$) the situation is similar but with the relevant interval being $\left(L(x) + n\tan\theta, L(x) + \frac{2\varepsilon}{\cos\theta} + n\tan\theta\right)$.

Equivalently, we can look at the fractional parts of these intervals within the unit interval I. The gradient, $\tan\theta$, is irrational, so the set of fractional parts of the intervals for all integers n form an open cover of I, and this has a finite subcover by compactness of I.

This finite subcover must have an interval corresponding to $x+m$ with m of maximum modulus, so taking r to correspond to m steps along in each direction from the initial point (i.e. $r = \frac{m}{\cos\theta}$) will guarantee that there is a lattice point less than 2ε from the line and no more than distance r along the line from the initial point.

This will work from any point on L with integer x-coordinate because starting at a different x is like shifting every set in the cover by the same fixed amount, so will still result in a cover. If we take r to correspond to $m+1$ steps then it will work for any point on L.

So therefore for any point on L there is a value $r > 0$ satisfying the relevant conditions, and so the tiling is repetitive.

$\square$

So, to sum up, a tiling generated by a 2:1 canonical projection has the following characteristics:

- The tiling has exactly two lengths of tile, these lengths being $\sin\theta$ and $\cos\theta$.
- The tiling consists of "blocks" of one or more longer tiles divided up by single short tiles.
- The "blocks" of longer tiles contain either $1 + \left\lfloor \frac{length\ of\ longer\ tile}{length\ of\ shorter\ tile} \right\rfloor$ or $\left\lfloor \frac{length\ of\ longer\ tile}{length\ of\ shorter\ tile} \right\rfloor$ tiles.
- The limit of the ratio,

$$number\ of\ short\ tiles : number\ of\ long\ tiles$$

is equal to the ratio

$$length\ of\ short\ tile : length\ of\ long\ tile\,.$$

- The tiling is aperiodic and repetitive.

2.3 N:1 Projections

As with the 2:1 projections discussed above we can also generate 1-dimensional tilings using $N:1$ projection schemes, where we project points from an N-dimensional lattice onto a 1-dimensional pattern space.

In a canonical N:1 projection we have an integer lattice $\mathbb{Z}^N$ and a strip that is defined by translating a unit N-cube parallel to some vector $(a_1, a_2, ..., a_N)$, which we will take to be a unit vector.

Note that here $a_i \neq 0$ for all i, since if we had an a_i being equal to 0 we would effectively be in the $N-1$ case. In fact, as with 2:1 projections, we will restrict to the case where $a_i > 0$ for all i.

If a lattice point within the strip is projected onto the pattern space E then the next lattice point to be projected onto E must be a unit step along from the first point in one of N directions. These N possible steps give (up to) N possible tile lengths in the tiling produced by this projection.

Definition 2.11. A canonical N:1 projection with defining vector $(a_1, a_2, ..., a_N)$ is said to be *degenerate* if we have that $a_i = a_j$ for some pair $i, j \in \{1, 2, ..., N\}$ $(i \neq j)$.

Conversely, we say that an N:1 projection is *non-degenerate* if every a_i is distinct.

Proposition 2.8. *Given a standard N:1 projection setup with strip defined by unit vector $(a_1, a_2, ..., a_N)$ the lengths of the prototiles in the corresponding tiling are a_i for $i \in \{1, ..., N\}$.*

Proof. If we take the line through the origin that is parallel to $(a_1, a_2, ..., a_N)$, i.e., the line $t(a_1, a_2, ..., a_N)$, for $t \in \mathbb{R}$, then the projections of the points $(1, 0, ..., 0)$, $(0, 1, 0, ..., 0)$,..., $(0, 0, ..., 0, 1)$ onto this line give the lengths of the tiles from a standard $N:1$ projection with strip defined by the vector above. Each of these points will be projected to the closest point on the line.

So consider the vector from the point $(1, 0, ..., 0)$ to the point $s(a_1, a_2, ..., a_N)$ on the line. This vector is equal to $(sa_1 - 1, sa_2, ..., sa_N)$.

Now we look at the square of the length of this vector as s varies. This is given by:

$$\begin{aligned}\phi(s) &= (a_1 s - 1)^2 + (a_2 s)^2 + ... + (a_N s)^2 \\ &= (a_1^2 + a_2^2 + ... + a_N^2)s^2 - 2a_1 s + 1 \\ &= s^2 - 2a_1 s + 1\ .\end{aligned}$$

This will be minimised at the turning point of $\phi(s)$,

$$\phi'(s) = 2s - 2a_1\ .$$

So,

$$\phi'(s) = 0 \Leftrightarrow s = a_1\ .$$

So the closest point on the line to the point $(1, 0, ..., 0)$ is the point $a_1(a_1, a_2, ..., a_N)$, which is distance a_1 from the origin (since $(a_1, ..., a_N)$ is a unit vector).

Therefore the length of the tile corresponding to the unit step $(1, 0, ..., 0)$ is a_1, and a similar argument applies to the other steps. □

As with the canonical 2:1 case we will also get the N different possible steps between lattice points in our strip appearing in proportion to the lengths of their projections. However, note that we may get degenerate cases where the terms a_i for $i \in \mathbb{R}$ are not all different, resulting in two or more distinct steps giving tiles of the same length in the tiling. Thus with this sort of setup the tiles may not actually appear in proportion to their lengths in the tiling.

In the canonical 2:1 case, only the step corresponding to the longer tile can appear in multiples of more than one within the strip. This is also true in the (non-degenerate) canonical N:1 case.

Proposition 2.9. *A tiling generated by a non-degenerate canonical $N{:}1$ projection scheme contains only one prototile that can appear next to a copy of itself.*

Proof. Since in a non-degenerate $N:1$ projection we have that the prototiles corresponding to different steps are all of different lengths it suffices to prove that the strip in any such projection scheme has only one step that can appear in multiples of more than one.

We will look at the case where $a_i > 0$ for all i, with all other cases being similar. Here we will look at the strip generated by translating the unit N-cube with vertices $(0,0,...,0),(1,0,...,0),...,(1,1,...,1)$ along the vector $(a_1,a_2,...,a_N)$. This strip has all the points of the unit N-cube on its edges apart from the points $(0,0,...,0)$ and $(1,1,...,1)$, which are in the interior.

A point $(x_1,x_2,...,x_N) \in \mathbb{R}^N$ is contained in this strip if and only if there exists some value $\mu \in \mathbb{R}$ such that $(x_1,...,x_N)+\mu(a_1,...,a_N)$ is contained in the unit N-cube described above.

We will assume that $a_1 > a_i$ for all $i \in \{2,3,...,N\}$. Since $(a_1,...,a_N)$ is a unit vector and all a_i are non-zero we have also that $0 < a_i < 1$ for all i.

Now, if we look at the point $(-1,0,...,0)$ we find that for this to be contained in the strip we must have that there exists some $\mu \in \mathbb{R}$ such that,

$$0 < \mu a_1 - 1 < 1$$

$$0 < \mu a_2 < 1$$

$$...$$

$$0 < \mu a_N < 1\,.$$

I.e., all the values μa_i for $i \in \{2, 3, ..., N\}$ are between 0 and 1 and μa_1 takes a value between 1 and 2. However, since a_1 is the largest of the a_i there will be such a μ. Thus the strip can contain more than one step of the type $(1, 0, ..., 0)$ consecutively.

To discover whether any other step can appear in multiples of two or more we examine whether the points that are two steps of the form $(0, ..., 0, -1, 0, ..., 0)$ along from each vertex of the unit N-cube lie within the strip.

As a starting point we can immediately discount every vertex that has a 1 anywhere other than in position i, since this would give inequality $0 < \mu a_j + 1 < 1$ for some j whilst also having either $0 < \mu a_i - 2 < 1$ or $0 < \mu a_i - 1 < 1$, requiring μ to be both positive and negative.

Thus we look to see whether any points of the form $(0, ..., 0, -1, 0, ..., 0)$ are contained within the strip. For this to be the case we must have that there exists some $\mu \in \mathbb{R}$ that satisfies the following inequalities:

$$0 < \mu a_i - 1 < 1$$

$$0 < \mu a_1 < 1$$

$$...$$

$$0 < \mu a_N < 1\,.$$

However, this cannot be the case, since $a_1 > a_i$ (for $i \neq 1$), so $\mu a_i > 1$ implies that $\mu a_1 > 1$.

Therefore the step $(1, 0, ..., 0)$ is the only one that can appear twice in a row within the strip.

$\square$

Thus a tiling generated by a non-degenerate canonical N:1 projection scheme will have N prototiles appearing in proportion to their lengths with only the longest prototile appearing in patches of more than one at any point in the tiling.

2.4 Model Multi-sets

A variant of the projection tiling setup is the model multi-set (see for example [8]), where multiple model sets are generated from the same cut-and-project scheme and effectively overlaid.

This setup differs from the one that we will be investigating in the following chapters, where the sets that we will be looking at are the overlaying of points generated by separate cut-and-project schemes.

2.5 Tiling Spaces

One way in which we can study a tiling T is by constructing a space Ω_T of tilings and looking at the topology of Ω_T (see [14]).

We start by defining a metric on tilings, where two tilings are considered to be close if they coincide on a large ball around the origin after some small translate.

Definition 2.12. Given two tilings T_1 and T_2 of $\mathbb{R}^n$ we define the distance between these two tilings, $d(T_1, T_2)$, to be equal to,

$$\inf \left\{ \{1\} \bigcup \left\{ \varepsilon : T_1 + s_1 = T_2 + s_2 \text{ on } B_{\frac{1}{\varepsilon}} \text{ with } s_1, s_2 \in \mathbb{R}^n, \|s_1\|, \|s_2\| < \frac{\varepsilon}{2} \right\} \right\}$$

where $B_{\frac{1}{\varepsilon}}$ denotes the ball of radius $\frac{1}{\varepsilon}$ centred at the origin.

Note that here $T + s$ is the tiling obtained by translating tiling T by vector s (or equivalently moving the origin by $-s$).

We can now look at the translates of a tiling and how far these are from the original tiling in the tiling metric.

Definition 2.13. The *orbit* of a tiling T of $\mathbb{R}^n$ is defined to be,

$$\mathcal{O}(T) = \{T + s : s \in \mathbb{R}^n\}.$$

That is, the set of all translates of the tiling T.

Definition 2.14. A *tiling space* Ω is a set of tilings that is closed under translation and complete in the tiling metric. I.e., if $T \in \Omega$ then $\mathcal{O}(T) \subset \Omega$, and every Cauchy sequence of tilings in Ω has a limit in Ω.

Definition 2.15. The *hull* or *orbit closure* Ω_T of a tiling T is the closure of $\mathcal{O}(T)$.

The hull of a tiling T is the set of tilings that locally look like T. A tiling T' is in Ω_T if and only if every patch of T' is found in a translate of T.

3 Non-Parallel Projections

In this chapter we introduce a new type of projection setup involving the projections of lattice points from within two strips that are not parallel to each other (and indeed with neither necessarily being parallel to the pattern space onto which we are projecting).

A 2:1 X-projection tiling is produced by projecting the lattice points contained within two canonical-width strips at different irrational gradients onto a line. A more formal definition is given below.

Definition 3.1. A *strip* S at gradient q in $\mathbb{R}^2$ is defined to be $K \times F_q$ where F_q is a line at gradient q in $\mathbb{R}^2$, and K is a compact, closed and connected subset of $F_q^{\perp}$ (i.e. a closed interval in $F_q^{\perp}$).

Strip S is said to be of *canonical width* if K has length equal to the projection of a unit square onto $F_q^{\perp}$. Equivalently, when $q > 0$, the point (α, β) is on the lower boundary of S (for $\alpha, \beta \in \mathbb{R}$) if and only if the point $(\alpha - 1, \beta + 1)$ is on the upper boundary of S.

Definition 3.2. A 2:1 X-projection scheme consists of:

- The integer lattice L sitting in $\mathbb{R}^2$.
- Two strips S_1 and S_2 of canonical width at gradients q_1 and q_2 respectively, with q_1 and q_2 satisfying,

$$q_1, q_2 \notin \mathbb{Q}$$

$$q_1, q_2 > 0$$

$$q_1 \neq q_2 \,.$$

  In addition we have that the strips are positioned so that $\partial S_i \bigcap L = \emptyset$ for $i = 1, 2$.

- A line E, known as the pattern space, at gradient p with $p > 0$, $p \neq 1$.
- Orthogonal projection $\pi : \mathbb{R}^2 \to E$.

We thus get a pattern of points $P = \{\pi(x) : x \in L \bigcap (S_1 \bigcup S_2)\}$. From this pattern we get a 2:1 X-projection tiling by taking the points to be the endpoints of the tiles.

Definition 3.3. Note that $\pi|_L$ is not assumed to be injective, so the pattern space E could be taken to have rational or irrational gradient. We will call a projection a *rational X-projection* if E has rational gradient and an *irrational X-projection* if E has irrational gradient.

As with normal projections, 2:1 X-projection schemes give sets of points in $\mathbb{R}$. We denote by θ, ϕ_1 and ϕ_2 the angles between the horizontal in the lattice L and the pattern space E and strips S_1 and S_2 respectively. We then restrict θ, ϕ_1 and ϕ_2 to be between 0 and $\frac{\pi}{2}$ as with the standard projection case.

Note that changing the value of θ does not affect the strips, and in particular does not change the "staircase" function of "up" and "across" (left to right) steps within each strip. Restricting θ to be between 0 and $\frac{\pi}{2}$ ensures that the second lattice point in an "up" or "across" step will be projected further along E than the first.

Thus the set of points P_i generated by strip S_i will have two possible distances between consecutive points unless $\theta = \frac{\pi}{4}$ (but note that we specified that E should not be at gradient 1, so this case does not arise), with these distances being the lengths of the projections of a vertical unit interval and a horizontal unit interval onto E, i.e., $\sin\theta$ and $\cos\theta$ (notice that these lengths are independent of the gradient of S_i). The order in which these two different "steps" appear will however be the same as with normal projections because the "staircase" function within the strip is the same.

So the pattern P_i generated by strip S_i with projection onto space E will consist of an infinite set of points with two possible distances between consecutive points and each of these distances corresponds to one of the two distances that you get with a standard projection with strip S_i where the projection is onto a pattern space that is parallel to the strip.

Proposition 3.1. *In the 2:1 X-projection scheme there must be at least one point of the integer lattice contained in* $S_1 \bigcap S_2$*, and only finitely many of such points.*

Proof. S_1 and S_2 both contain "staircase functions", i.e., within each strip is a line consisting of horizontal and vertical steps between lattice points in that strip, and because the strips are non-parallel these lines must cross. Both the staircase functions are subsets of a unit square grid, so their intersection must also be a subset of this grid. If the two lines intersect at a point other than a lattice point then they must both contain the entire unit interval in which that point is located and must therefore intersect on the entire interval and in particular the two lattice points at the ends of the interval.

So the staircase functions associated to the two strips must contain a common lattice point and this lattice point must therefore be in $S_1 \bigcap S_2$.

S_1 and S_2 are non-parallel and have fixed width, so their intersection is a compact parallelogram in $\mathbb{R}^2$ and therefore can contain only finitely many lattice points. □

3.1 Rational 2:1 X-Projections

In this section we will investigate rational 2:1 X-projections. These will be looked at in greater detail in the following chapters. We will denote by P_i the set of points that are the projections of points from the strip S_i, and the tiling associated to P_i will be called T_i. The tiling that is the combination of T_1 and T_2 will be denoted by U.

Definition 3.4. A set of points $P \subset \mathbb{R}^n$ is said to be *uniformly discrete* if there exists a positive real number r such that $\forall x, y \in P, |x - y| \geq 2r$.

Definition 3.5. A set of points P is said to be *relatively dense* in $\mathbb{R}^n$ if there exists a positive real number R such that every sphere of radius greater than R contains at least one point of P in its interior.

Definition 3.6. A set of points $P \subset \mathbb{R}^n$ is a *Delone set* if it is uniformly discrete and relatively dense.

The sets generated by standard cut-and-project schemes are Delone sets. As we shall see, this is also true of the sets generated by rational 2:1 X-projections.

Lemma 3.2. *In a tiling generated by a rational 2:1 X-projection scheme with pattern space E at gradient $\frac{a}{b}$ the lengths, $|t_1|$ and $|t_2|$, of the two prototiles t_1, t_2 in the tilings T_i are rational multiples of each other.*

Furthermore, the longer of these prototiles, which we shall label t_2 has length,

$$|t_2| = \frac{\max\{a,b\}}{\min\{a,b\}}|t_1|\,.$$

Proof. The two prototiles have lengths that are equal to $\sin\theta$ and $\cos\theta$, so depend only on the gradient of the pattern space E. If E has rational gradient then $\frac{\sin\theta}{\cos\theta} = \tan\theta$ is rational.

We label the longer of the prototiles as t_2, thus we get,

$$\theta < \frac{\pi}{4} \Rightarrow \frac{a}{b} < 1 \Rightarrow \frac{|t_1|}{|t_2|} = \frac{a}{b} \Rightarrow |t_2| = \frac{b}{a}|t_1|$$

$$\theta > \frac{\pi}{4} \Rightarrow \frac{a}{b} > 1 \Rightarrow \frac{|t_2|}{|t_1|} = \frac{a}{b} \Rightarrow |t_2| = \frac{a}{b}|t_1|\,.$$

Either way we have that,

$$|t_2| = \frac{\max\{a,b\}}{\min\{a,b\}}|t_1|\,.$$

□

Proposition 3.3. *The set of points generated by a rational 2:1 X-projection with pattern space E at gradient $\frac{a}{b}$ is a Delone set, with the distance between any two points of the set being an integer multiple of $\frac{1}{\min\{a,b\}}\ |t_1|$ (where t_1 is the shorter of the two prototiles from the tilings T_1 and T_2).*

Proof. The set generated by this type of projection is the union of the points in P_1 and P_2, so is the union of two sets which consist of points in E separated by either $|t_1|$ or $|t_2|= \frac{\max\{a,b\}}{\min\{a,b\}}\ |t_1|$. So in each P_i every point is an integer multiple of $\frac{1}{\min\{a,b\}}\ |t_1|$ from every other point, and P_1 and P_2 have at least one point in common, therefore every point in $P_1 \bigcup P_2$ is an integer multiple of $\frac{1}{\min\{a,b\}}\ |t_1|$ from every other point in the union.

So the minimum possible distance between points is $\frac{1}{\min\{a,b\}}\ |t_1|$ and the maximum possible distance between consecutive points is $|t_2|= \frac{\max\{a,b\}}{\min\{a,b\}}\ |t_1|$. Thus the point set is discrete and relatively dense, and therefore a Delone set.

$\square$

Corollary 3.4. *A tiling produced by a rational 2:1 X-projection scheme with pattern space E at gradient $\frac{a}{b}$ has at most* $\max\{a,b\}$ *prototiles.*

Proof. We know that any two consecutive points in the combined set must be separated by a distance that is an integer multiple of $\frac{1}{\min\{a,b\}}|t_1|$ and this distance cannot be more than $|t_2| = \frac{\max\{a,b\}}{\min\{a,b\}}|t_1|$. Thus the tiling given by this setup can have at most $\max\{a,b\}$ prototiles. $\square$

The pattern space E has rational gradient and passes through a lattice point (the origin), so therefore has infinitely many lattice points evenly spaced along its length. Similarly, $E^{\perp}$ contains infinitely many lattice points with the same spacing as those along E.

So if we look at all the lines parallel to $E^{\perp}$ passing through lattice points on E then we get a square sublattice of L, this sublattice will be used extensively in the following chapters, so we define it formally below.

Definition 3.7. Let E be a line with rational gradient $\frac{a}{b}$, with a and b coprime (i.e. the fraction is written in its lowest terms), passing through a point of the lattice $L = \mathbb{Z}^2$ which we will refer to as O.

We define the lattice Λ to be the sublattice of L containing the point O and generated by the vectors (b, a) and $(-a, b)$.

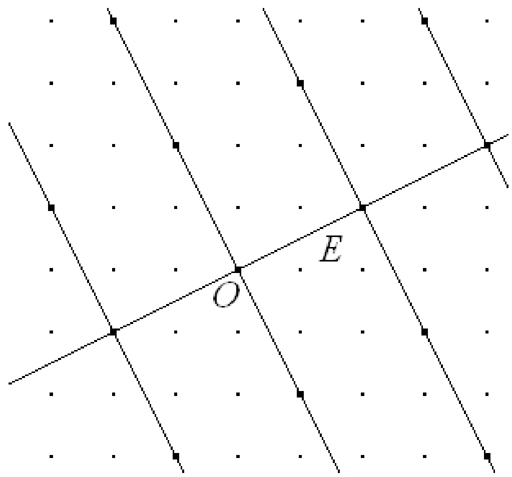

Figure 3.1: The lattice Λ for E at gradient $\frac{1}{2}$.

From this point on both L and Λ coordinates will be used, so we will introduce notation to cover this here.

Definition 3.8. Translations in L coordinates will be denoted by (x, y) as before.

Translations in Λ coordinates (with Λ defined as above) will be denoted by $(x, y)_\sim$, where the relationship between the two types of translation is as follows.

$$(1, 0)_\sim = (b, a)$$

$$(0, 1)_\sim = (-a, b)\,.$$

So the translate $(x,y)_{\sim} = (xb - ya, xa + yb)$.

Lemma 3.5. *A line I has irrational gradient relative to the lattice Λ if and only if it has irrational gradient relative to the lattice L (where L and Λ are as above).*

Proof. Take I to run through a point of both L and Λ, which we will call $(0,0)$. Then the fact that I has irrational gradient relative to L implies that I does not intersect any more points of L, and therefore does not intersect any more points of Λ, since $\Lambda \subset L$.

Thus I must have irrational gradient relative to Λ.

If I has rational gradient relative to L then I passes through the lattice point (m,n) (with at least one of m and n non-zero) and all integer multiples of this point.

The lattice Λ is at rational gradient relative to L and is generated by the vectors (a,b) and $(-b,a)$. But the point $(a^2+b^2)(m,n) = (an-bm)(-b,a) + (am+bn)(a,b)$ is common to both lattices and lies on I, so I must also have rational gradient relative to Λ. $\square$

Recall the definition of repetitive as given in the previous chapter:

Definition 2.10. A tiling T of $\mathbb{R}^n$ is called *repetitive* if for any patch of tiles P in T there is a number $r > 0$ such that for any point $t \in \mathbb{R}^n$ there is a translate of the patch P belonging to T and contained in the ball $B_r(t)$ (in the 1-dimensional case this is the interval of length $2r$ centred at t).

Theorem 3.6. *Rational 2:1 X-projection schemes with strips S_1 and S_2 having gradients that are irrational and rationally related relative to Λ give repetitive tilings.*

Proof. Each T_i has two prototiles t_1 and t_2, with $|t_2|>|t_1|$. The projections of two consecutive lattice points in T_i can never be more than $|t_2|$ apart, therefore any patch in U of diameter greater than $|t_2|$ must contain the projections of lattice points from both strips. Note that there may be smaller patches in U that contain tiles with all endpoints from the same strip, but any one of these patches will appear as part of a larger patch containing projections of points from both strips and will therefore reappear in the tiling whenever this larger patch reappears. Thus it is enough to prove that any patch with tile endpoints coming from the projections of points in both strips will appear throughout the tiling U.

If you have a finite patch of the above type then each strip will contain a finite set of lattice points that are projected into this patch, and thus each will have a lattice point that is closest to one of the boundaries of the strip. We call these points $x_1 \in S_1$ and $x_2 \in S_2$, and their distances from the relevant boundaries ε_1 and ε_2 respectively.

As in the proof of repetitivity of a standard 2:1 projection, whenever there is a lattice point, y_1, within $2\varepsilon_1$ of the relevant boundary of S_1 the patch from T_1 will appear in U and whenever there is a lattice point, y_2, within $2\varepsilon_2$ of the relevant boundary of S_2 the patch from T_2 will appear in U. So for the complete patch to appear in U it is required that we have such lattice points y_1 and y_2 with,

$$\pi(y_1) - \pi(y_2) = \pi(x_1) - \pi(x_2) .$$

If we draw lines I_1 through x_1 parallel to S_1 and I_2 through x_2 parallel to S_2 then for the complete patch to reappear we need lattice points y_1 and y_2 whose projections have the above relation, with y_1 being within ε_1 of S_1 and y_2 within ε_2 of S_2.

We now set $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, and also position the pattern space E so that it runs through x_2, which we shall henceforth refer to as the origin in both the normal lattice L and the sublattice Λ.

In addition we will denote by (γ, δ) the vector between x_1 and x_2. So we have,

$$x_1 = x_2 + (\gamma, \delta)\,.$$

This gives a situation that looks a bit like that shown in figure 3.2.

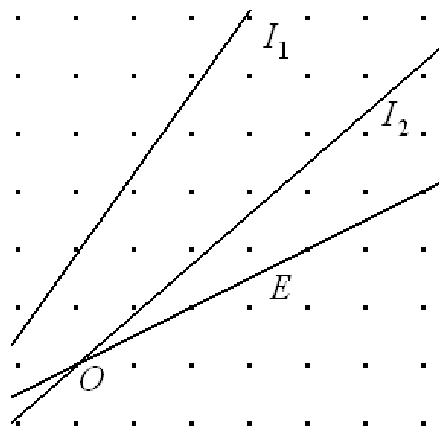

Figure 3.2: The lines I_1 and I_2 in lattice L.

With the original patch reappearing whenever we have lattice points y_1 and y_2 within ε of I_1 and I_2 respectively, satisfying,

$$\pi(y_1) - \pi(y_2) = \pi(x_1) - \pi(x_2) = \pi(\gamma, \delta)\,.$$

Relative to the sublattice Λ the situation looks a bit like that shown in figure 3.3 (at least when the gradients of S_1 and S_2 are greater than that of E).

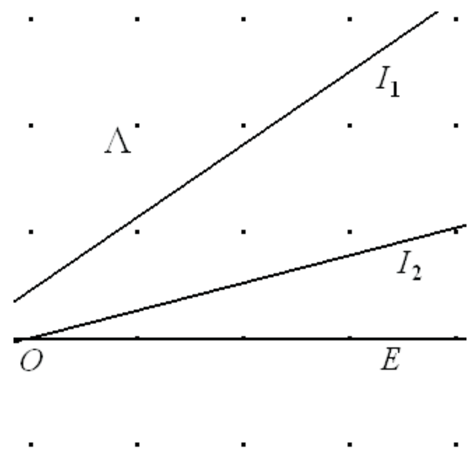

Figure 3.3: The lines I_1 and I_2 in lattice Λ.

Here the line I_1 runs through the point (γ, δ), which is a point in the lattice L, but may not be a point of Λ.

Now, assume that the gradients of S_1 and S_2 are rationally related and irrational relative to Λ, so $gradient(S_1) = \frac{c}{d} gradient(S_2)$ (w.l.o.g. assume $|c| > |d|$). Then if the point $(\alpha, \beta)_\sim \in \Lambda$ is within ε of I_2 the point $(\alpha, \frac{c}{d}\beta)_\sim + (\gamma, \delta)$ will be within $|\frac{c}{d}|\ \varepsilon$ of I_1. However, the point $(\alpha, \frac{c}{d}\beta)_\sim + (\gamma, \delta)$ is only a lattice point if $(\alpha, \frac{c}{d}\beta)_\sim$ is a lattice point, which only happens when $\frac{c}{d}\beta \in \mathbb{Z}$, i.e., when β is a multiple of d.

If we have a point $(\alpha, \beta)_\sim$ within $\frac{\varepsilon}{|c|}$ of I_2 then the point $y_1 = (d\alpha, b\beta)_\sim$ is within $|\frac{d}{c}|\varepsilon$ of I_2, and therefore within distance ε. Also the point $y_2 = (d\alpha, \frac{c}{d} d\beta)_\sim + (\gamma, \delta) = (d\alpha, c\beta)_\sim + (\gamma, \delta)$ is a lattice point within $|\frac{c}{d}|\ .\ |\frac{d}{c}|\ .\varepsilon = \varepsilon$ of I_1.

Of course,

$$\pi(d\alpha, d\beta)_\sim = \pi(d\alpha, c\beta)_\sim$$

so,

$$\pi(y_1) - \pi(y_2) = \pi(x_1) - \pi(x_2)\,.$$

Thus for every point of Λ within $\frac{\varepsilon}{|c|}$ of I_2 there will be lattice points y_1 within ε of I_1 and y_2 within ε of I_2 whose projections have the required relation. Since there will be points of Λ within this distance of I_2 throughout its length this means that the patch will repeat throughout U when S_1 and S_2 have gradients of this form.

Note that the chosen patch will repeat in a relatively dense pattern throughout U because the patches that you get in T_2 defined by having a lattice point within the required distance of the relevant boundary of S_2 are relatively dense in S_2. Or in other words, because T_2 is repetitive.

Therefore when S_1 and S_2 have gradients that are rationally related to each other and irrational relative to the lattice Λ (and positive relative to the lattice L) then the tiling produced (U) will be repetitive.

$\square$

Note that the general version of this result is proved in theorem 5.11.

3.2 Irrational 2:1 X-Projections

We now look at some of the basic properties of irrational 2:1 X-projections.

Lemma 3.7. *In the irrational case the lengths, $|t_1|$ and $|t_2|$, of the two tiles t_1, t_2 in the tilings T_i are irrational multiples of each other.*

Proof. The two lengths are equal to $\sin\theta$ and $\cos\theta$, so depend only on the gradient of the pattern space E. If E has irrational gradient then $\frac{\sin\theta}{\cos\theta} = \tan\theta$ is irrational. $\square$

Lemma 3.8. *There are only finitely many tiles of maximal length in a tiling generated by an irrational 2:1 X-projection scheme.*

Proof. Maximal tiles in the tiling can only arise when the projections onto E of maximal steps in S_1 and S_2 coincide. If a lattice point contained in a strip was projected to the same point in E as a lattice point not contained in that strip then this would imply that there were two lattice points on a line parallel to $E^{\perp}$, which implies that $E^{\perp}$ and hence E have rational gradient.

Hence, the only part of the tiling that could contain tiles of maximal length is the part that corresponds to the projection of the lattice points in $S_1 \bigcap S_2$, and there are only finitely many lattice points in this intersection. Therefore there can be only finitely many tiles of maximal length in the tiling. $\square$

Theorem 3.9. *A tiling generated by an irrational 2:1 X-projection has infinitely many prototiles (i.e., the tiling contains infinitely many different lengths of tile).*

Proof. We have an irrational 2:1 X-projection with strips S_1 and S_2 at different irrational gradients and orthogonal projection onto a pattern space E that is also at an irrational gradient relative to the lattice.

As before, we refer to the set of points in E that are projections of points in S_i as P_i and the corresponding tilings (taking these points as the endpoints of tiles) as T_i. Each of these tilings has two prototiles t_1 and t_2 with $|t_2| = q\ |t_1|$ for some $q > 1$, q irrational. We will call the X-projection tiling U.

By the previous lemma, U has only finitely many tiles of maximal length (i.e. of length $|t_2|$). However there are infinitely many tiles of this length in both T_1 and T_2. Thus there must be infinitely many maximal length tiles in T_1 that are "broken up" into shorter tiles in U by having points from P_2 in between their endpoints. As T_1 and T_2 have only finitely many common points there must be infinitely many tiles in U that have one endpoint from each P_i.

Figure 3.2 shows the situation in a part of tiling U that does not come from the projections of points in $S_1 \bigcap S_2$.

The points at the top of the line are points in P_1 and the points at the bottom are in P_2. The top points are $|\,t_2\,|$ apart, and since these points do

Figure 3.4: Tiles with endpoints projected from different strips.

not coincide with any points from P_2 there must be at least one point from P_2 between them (the points of each P_i are at most $|t_2|$ apart). So for each t_2 tile in T_1 (outside of the tiles corresponding to $S_1 \bigcap S_2$) we must get two tiles in U that have one endpoint in each T_i. These tiles are marked v_1 and v_2 in the above diagram, and at least one of these tiles must have a length that is not equal to $|t_1|$ because they can only have tiles of length $|t_1|$ between them (or no tiles) so if they both had length $|t_1|$ this would imply that $|t_2|$ was an integer multiple of $|t_1|$. However, this is an irrational 2:1 X-projection, so t_1 and t_2 have lengths that are irrational multiples of each other.

Thus there must be infinitely many tiles in U that have one endpoint in each P_i and are not of the same length as either of the prototiles in the tilings T_i.

Let v be such a tile in U. Then v has endpoints a and b, with $a \in T_1$ and $b \in T_2$. Say a prototile with the same length as v reappears in U with endpoints $a' \in T_1$ and $b' \in T_2$ in the same order as before (there are of course two possible orders for the endpoints), then a' and b' must be translates of a and b by the same vector, and they are points of the tilings, so:

$$|a' - a| = c_1 \ |t_1| + c_2 \ |t_2|$$

$$|b' - b| = d_1 \ |t_1| + d_2 \ |t_2|$$

with $c_1, c_2, d_1, d_2 \in \mathbb{N}$.

But the lengths of the prototiles are irrational multiples of each other, so this can only happen when $c_1 = d_1$ and $c_2 = d_2$, or in other words when the tilings

T_1 and T_2 contain the same number of each prototile between these points. However, $\frac{c_1}{c_2}$ and $\frac{d_1}{d_2}$ have different limits, so there can only be finitely many occurrences of tiles of length $|v|$ with endpoints in the same order as for v (and similarly, only finitely many when the order of the endpoints is switched).

So the tiling U contains infinitely many tiles that are not of length $|t_1|$ or $|t_2|$, but a tile of a given length can only appear finitely many times. Therefore there must be infinitely many prototiles in an irrational 2:1 X-projection.

□

3.3 Projections with Strips at Rational Gradients

In this section we look at the tilings that are produced by projecting the lattice points within two non-parallel strips at rational gradients onto a pattern space at irrational gradient.

We will call the strips S_1 and S_2 and the pattern space E. For each S_i there is a tiling T_i which is obtained by projecting the lattice points within S_i orthogonally onto E. The complete tiling (that is, the tiling of $\mathbb{R}$ given by projecting the lattice points from both strips) is denoted by U. Each of the tilings T_i has two prototiles which we will call t_1 and t_2, and these tiles have lengths that are irrationally related due to E having irrational gradient, i.e., $|t_2| = \lambda\,|t_1|$ (with λ irrational).

Lemma 3.10. *T_1 and T_2 are periodic with irrationally related periods.*

Proof. The strips are at rational gradients, say gradient of S_i is equal to $\frac{a_i}{b_i}$. Then given any lattice point $(x_i, y_i) \in S_i$ all lattice points of the form $(x_i + nb_i, y_i + na_i)$, for $n \in \mathbb{Z}$, will be in S_i and will be the same distance from both boundaries of the strip as the original point (x_i, y_i). So the patterns of lattice points within the strips repeat after a fixed number of steps and therefore both T_1 and T_2 must be periodic and consist of repeated patches with a_i tiles of type t_1 and b_i tiles of type t_2.

Thus the period of T_1 is,

$$a_1|t_1| + b_1|t_2| = (a_1 + \lambda b_1)|t_1| = p$$

and the period of T_2 is,

$$a_2|t_1| + b_2|t_2| = (a_2 + \lambda b_2)|t_1| = q\,.$$

Then, $p = xq$, for $x \in \mathbb{Q} \Rightarrow a_1 + \lambda b_1 = x(a_2 + \lambda b_2)$

$$\begin{aligned}
&\Rightarrow a_1 - xa_2 + \lambda(b_1 - xb_2) = 0 \\
&\Rightarrow b_1 - xb_2 = 0 \\
&\Rightarrow a_1 - xa_2 = 0 \\
&\Rightarrow b_1 = xb_2 \text{ and } a_1 = xa_2 \\
&\Rightarrow \tfrac{a_1}{b_1} = \tfrac{xa_2}{xb_2} = \tfrac{a_2}{b_2} \\
&\Rightarrow \text{gradients of } S_1 \text{ and } S_2 \text{ are equal.}
\end{aligned}$$

So the periods must be irrationally related when S_1 and S_2 have different gradients. □

Definition 3.9. For two finite non-empty sets of points $X, Y \subset \mathbb{R}$ the *Hausdorff distance* $d_H(X,Y)$ is equal to $\inf\{r \in \mathbb{R} \,:\, X \subset Y_r \text{ and } Y \subset X_r\}$, where $A_r = \{x \in \mathbb{R} \,:\, |x-a| \leq r \text{ for some } a \in A\}$.

Definition 3.10. Two patches $P_1, P_2 \in U$ (thought of as finite sets of points) are said to be ε-*close* if there exist translates Q_1 and Q_2 of P_1 and P_2 satisfying $d_H(Q_1, Q_2) \leq \varepsilon$.

Proposition 3.11. *For any patch P in U and any $\varepsilon > 0$ the set of patches in U that are ε-close to P is relatively dense.*

Proof. Assume that the patch P in U is defined by lattice points from both strips. All patches will either be of this type or will be subpatches of such patches, so it suffices to prove the proposition in this case.

Choose two lattice points $x_1 \in S_1$ and $x_2 \in S_2$ that are projected into P with their projections being distance δ apart where δ is less than or equal to the shorter of the periods of T_1 and T_2. Then whenever there are lattice points $y_1 \in S_1$ and $y_2 \in S_2$ occupying the same positions within the strips (i.e. the same distance from both edges) as x_1 and x_2 whose projections are within ε of being distance δ apart then the patch around the projections of these points will be ε-close to P.

T_1 and T_2 have periods p and q respectively, and these periods are irrationally related, i.e., $q = rp$ for some irrational r. So the projections of lattice points that occupy the same position within S_1 as x_1 can be found at distances np from x_1 for all $n \in \mathbb{Z}$, and similarly the projections of lattice points in the same position within S_2 as x_2 will be at distances $nq = npr$ from x_2 for all $n \in \mathbb{Z}$.

The fractional part of r is an irrational number r' with $0 < r' < 1$, so the set of fractional parts of nr for all $n \in \mathbb{Z}$ must be a dense subset of $[0, 1]$ and therefore for all $\varepsilon > 0$ the set of intervals of radius ε around these points must cover $[0, 1]$ and so by compactness of $[0, 1]$ there must exist a finite subcover by intervals of this form.

Thus for all $\varepsilon > 0$ whenever two lattice points occupying the same positions within the strips as x_1 and x_2 have projections that are within ε of being distance δ apart there must be only a finite number of periods before two more lattice points, in the same positions relative to the strips, are projected to points that are within ε of being distance δ apart, i.e., there must be patches that are ε-close to P throughout U, never more than a certain distance apart. $\square$

Tilings of this type therefore have a property that is similar to repetitivity, and could perhaps be referred to as being ε-repetitive.

Corollary 3.12. *Tilings of this type have infinitely many prototiles, including prototiles of arbitrarily small length.*

Proof. The strips S_1 and S_2 have at least one lattice point in their intersection and so for any $\varepsilon > 0$ there will be lattice points whose projections are within distance ε of each other, and therefore tiles of length less than ε. $\square$

4 Rational 2:1 X-Projections: Positioning and Translates

In this chapter we will look at rational 2:1 X-projections and the translates of these that can be obtained by altering the positions of the two strips relative to the lattice. The sets of points corresponding to translates that we obtain in this way will be important in the next chapter when we come to look at the tiling spaces associated to these tilings.

We start by recapping the definition of a rational 2:1 X-projection.

Definition 4.1. A *strip* S at gradient q in $\mathbb{R}^2$ is defined to be $K \times F_q$ where F_q is a line at gradient q in $\mathbb{R}_2$, and K is a compact, closed and connected subset of $F_q^{\perp}$ (i.e. a closed interval in $F_q^{\perp}$).

Strip S is said to be of *canonical width* if K has length equal to the projection of a unit square onto $F_q^{\perp}$. Equivalently, the point (α, β) is on the lower boundary of S (for $\alpha, \beta \in \mathbb{R}$) if and only if the point $(\alpha - 1, \beta + 1)$ is on the upper boundary of S.

Definition 4.2. A rational 2:1 X-projection scheme consists of:

- An integer lattice L sitting in $\mathbb{R}^2$.
- Two strips S_1 and S_2 of canonical width at gradients g_1 and g_2 respectively, with g_1 and g_2 satisfying,

$$g_1, g_2 \notin \mathbb{Q}$$

$$g_1, g_2 > 0$$

$$g_1 \neq g_2 .$$

In addition we have that the strips are positioned so that $\partial S_i \bigcap L = \emptyset$ for $i = 1, 2$.

- A line E, known as the pattern space, at gradient $q \in \mathbb{Q}$, with $q > 0$, $q \neq 1$. We will usually write $q = \frac{c}{d}$, with c and d assumed to be coprime.
- Projection $\pi : \mathbb{R}^2 \to E$.

Each strip S_i has an associated pattern of points P_i in E, with

$$P_i = \pi\left(L \bigcap S_i\right) .$$

We thus get a pattern of points $P = P_1 \bigcup P_2$. From this pattern we get a rational 2:1 X-projection tiling by taking the points to be the endpoints of the tiles.

Example 4.1. *An example of a rational 2:1 X-projection setup is shown in figure 4.1.*

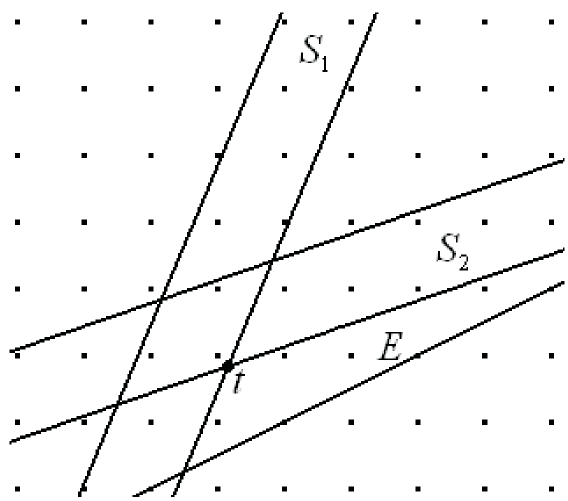

Figure 4.1: A rational 2:1 X-projection scheme.

Definition 4.3. Strips S_1 and S_2 are non-parallel and will thus overlap. We refer to the point at which the lower boundaries of the strips meet as the *intersection point* of S_1 and S_2. We will call this point t, as in the example diagram.

Definition 4.4. The lattice L in which we are interested is an integer lattice (as explained above). We define the *fundamental domain* of L to be the unit square with vertices $(0, 0), (1, 0), (0, 1)$ and $(1, 1)$ and all other fundamental domains to be translates of this by integers both horizontally and vertically.

If we define the origin on E to be at position $\pi(t)$ then translating t by a vector of the form (n, m) with n and m integers (so moving the two strips by integer steps both horizontally and vertically) will result in the same tiling, since every lattice point (i, j) in the original strips will be replaced by a corresponding lattice point $(i, j) + (n, m)$ in the translated strips with the same projection relative to the new origin on E.

Thus, when looking at ways in which the two strips can be positioned we need only consider the positions of their intersection point within a fundamental domain of L.

4.1 Intersection Point Positions

In the definition of a rational 2:1 X-projection we required that the strips be positioned so that there are no lattice points on their boundaries, thus not every point within the fundamental domain of L is a point at which the intersection of the lower boundaries can be placed.

Definition 4.5. We say that a point in the fundamental domain of L is a *forbidden point* if a line drawn through this point parallel to either S_1 or S_2 intersects any point of the lattice L.

If u is a forbidden point within the fundamental domain of L that leads to a lattice point being placed on the boundary of S_i then all points along the line parallel to S_i that pass through u will also be forbidden points.

So the forbidden points for t within a fundamental domain of L are all the points on two sets of infinitely many lines that pass through the fundamental domain, one set parallel to S_1 and the other parallel to S_2. This is all the lines parallel to either strip that pass through both the fundamental domain and a point of lattice L.

For example, the set of forbidden points within the fundamental domain of L may look a bit like those shown in figure 5.3 (though the lines will actually be dense).

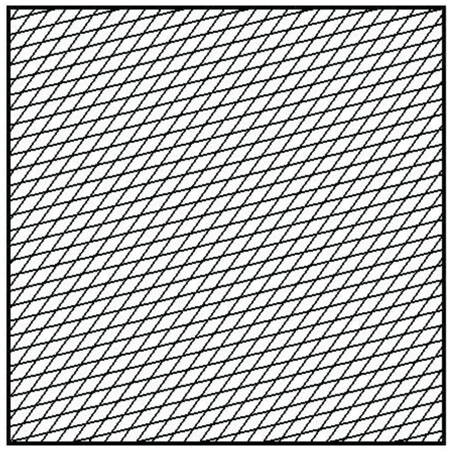

Figure 4.2: The forbidden points in the fundamental domain of L.

Claim 1. *This can equivalently be thought of as just two lines passing through the origin with gradients equal to those of the two strips that each wind round the fundamental domain.*

Proof. We will think of the fundamental domain that we are looking at as having the origin at the bottom-left position, and refer to this fundamental domain as F.

If we take a line parallel to S_1 running through F that also passes through some lattice point (m, n) and call this line $I_{(m,n)}$, then $I_{(m,n)} \bigcap F$ gives a line of forbidden points in F.

A line parallel to S_1 passing through O has the same intersection with F as the line $I_{(m,n)}$ has with the fundamental domain that has (m, n) at its bottom-left corner. So once this line loops round F a certain number of times in the relevant direction we will get the line $I_{(m,n)} \bigcap F$. Thus such a line must cover the intersection with F of any line defined in the same way as $I_{(m,n)}$, for any lattice point (m, n).

Similarly, all intersections of F with lines parallel to S_2 that pass through lattice points must be represented by a single line parallel to S_2 passing through the origin. □

The remaining points, which form a totally disconnected subset of the fundamental domain, give all "allowed" intersection points for S_1 and S_2. So every possible positioning of the two strips is defined by one of these points.

4.2 Translate Points

As we have seen, there are infinitely many points in the fundamental domain of L at which the intersection point t can be positioned, but could some of these alternative positions correspond to translates of the tiling?

In this section we investigate the points in the fundamental domain of L at which we can reposition t to get translates of the original tiling, first explaining how this works and then looking at the various cases that arise.

Lemma 4.1. *We have a rational 2:1 X-projection setup with strips S_1 and S_2 having intersection point t as before with $\pi(t) = O$, the origin on E. We will call the tiling produced by this setup U.*

Now, assume that we also have points t_1 and t_2 on the lower boundaries of S_1 and S_2 respectively satisfying,

$$\pi(t_1) = \pi(t_2)$$

$$t_1 = t_2 + (m, n)$$

for $m, n \in \mathbb{Z}$. Then the tiling U' produced by translating the strips so that they intersect at t_1 and defining $\pi(t_1) = O$ (the origin on E) is a translate of the *tiling U.*

Proof. We first introduce/recap some notation for the proof.

- P_1 and P_2 are the patterns of points corresponding to the projections of lattice points from strips S_1 and S_2 respectively (with $\pi(t) = O$ on E).

- S_1' and S_2' are the translates of strips S_1 and S_2 so that the point t_1 is on their lower boundaries. Note that $S_1' = S_1$.

- P_1' and P_2' are the patterns of points corresponding to the projections of lattice points from strips S_1' and S_2' respectively (with $\pi(t_1) = O$ on E).

Using the above notation we see that,

$$P_1' = \{k + (\alpha, \beta) : k \in P_1, (\alpha, \beta) = \pi(t) - \pi(t_1)\}$$

since $S_1' = S_1$ so they contain the same lattice points and only the position of the origin in the tiling is changed.

We also have that,

$$S_2' = S_2 + (m, n)$$

so,

$$(x, y) \in S_2 \Rightarrow (x + m, y + n) \in S_2' .$$

So every lattice point in S_2 has a corresponding lattice point in S_2', and since we know that,

$$\pi(t_1) = \pi(t_2 + (m, n)) = \pi(t_2)$$

we must have that,

$$\pi((x, y) + (m, n)) = \pi(x, y) .$$

Therefore, as above, we get that,

$$P_2' = \{k + (\alpha, \beta) : k \in P_2, (\alpha, \beta) = \pi(t) - \pi(t_1)\} .$$

Since the tiling U' comes from a combination of the sets of points P_1' and P_2' it must be a translate of U.

$\square$

Figure 4.3 gives an idea of what is happening. Here t_1 and t_2 occupy corresponding positions in two fundamental domains of L, and if the intersection point is taken to be at t_1 or t_2 then the projections of lattice points will be the same but the origin of the tiling will be at O' rather than O, giving a translate.

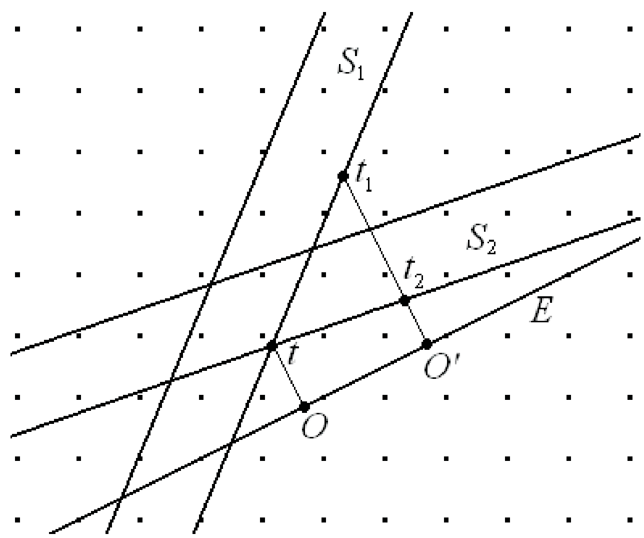

Figure 4.3: A translate point for the tiling.

So for a given X-projection setup with intersection point t there may be alternative positions for the intersection point within the fundamental domain of L that will give translates of the original tiling.

We now briefly recap the definitions of the sublattice Λ and associated system of coordinates given in the previous chapter.

Definition 3.7. Let E be a line with rational gradient $\frac{c}{d}$, with c and d coprime (i.e. the fraction is written in its lowest terms), passing through a point of the lattice L which we will refer to as O.

We define the lattice Λ to be the sublattice of L containing the point O and generated by the vectors (d, c) and $(-c, d)$.

Example 4.2. *When E has gradient $\frac{1}{2}$ the sublattice Λ is as shown in figure 4.4.*

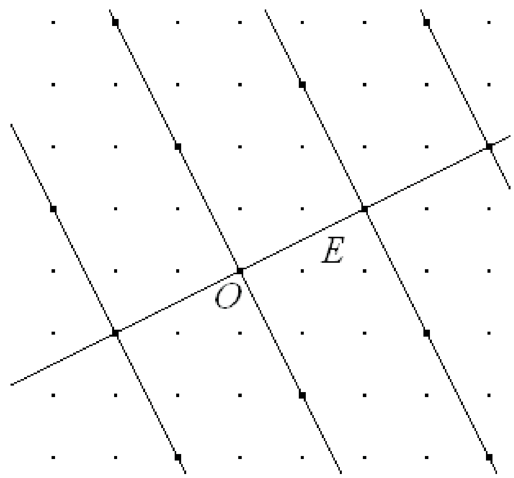

Figure 4.4: The sublattice Λ for E at gradient $\frac{1}{2}$.

Definition 3.8. Translations in L coordinates are denoted by (a, b).

Translations in Λ coordinates (with Λ defined as above) are denoted by $(a, b)_\sim$, where the relationship between the two types of translation is as follows.

$$(1, 0)_\sim = (d, c)$$

$$(0, 1)_\sim = (-c, d)\,.$$

So the translate $(a, b)_\sim = (ad - bc, ac + bd)$.

As before with lattice L we will define the *fundamental domain* of Λ to be the unit square with vertices $(0,0)_\sim, (1,0)_\sim, (0,1)_\sim$ and $(1,1)_\sim$. Whenever we talk about two or more fundamental domains of Λ they will be translates of the fundamental domain by vectors of the form $(m,n)_\sim$ for $m, n \in \mathbb{Z}$.

Definition 4.6. We will be considering the patterns of points given by projecting lattice points from Λ that are within the strips S_i onto the pattern space E. We will denote the resulting sets by P_i^Λ. That is,

$$P_i^\Lambda = \left\{ \pi(r) : r \in S_i \bigcap \Lambda \right\} .$$

Lemma 4.2. *If two points occupy corresponding positions in two different fundamental domains of Λ then they also occupy corresponding positions in two different fundamental domains of L.*

Proof. Λ is a square lattice, and for gradient of E equal to $\frac{c}{d}$ as above we have that,

$$(1,0)_\sim = (d,c)$$

$$(0,1)_\sim = (-c,d)\,.$$

So integer translations in Λ coordinates give integer translations in L coordinates and therefore if two points occupy corresponding positions in two fundamental domains of Λ then they will occupy corresponding positions in two fundamental domains of L. □

Lemma 4.3. *Two points have the same projection onto E if and only if they are translates of each other by a vector of the form $(0,k)_\sim$, for some $k \in \mathbb{R}$.*

Proof. The projection is perpendicular to E, so two points will have the same projection onto E if and only if they are on a line perpendicular to E, and Λ is defined so that vertical lines in the Λ coordinates are perpendicular to E. □

Definition 4.7. We define an *intersection-point tiling* to be a tiling that can be produced by a rational 2:1 X-projection setup satisfying,

$$\pi(t) = O$$

where t is the point of intersection of the lower boundaries of the strips and O is the origin in the tiling (i.e. on E).

Similarly, a *non-intersection-point tiling* is a translate of an intersection-point tiling that is not itself an intersection-point tiling, i.e., a tiling generated by a rational 2:1 X-projection that is not the same as any of the tilings that can be produced by repositioning t so that $\pi(t) = O$.

Note 4.1. We have not yet proved that the set of non-intersection-point tilings is non-empty, i.e., that not all translates of a tiling are intersection point tilings.

Proposition 4.4. *A rational 2:1 X-projection has at least countably many translates that are intersection-point tilings.*

Proof. Note that we will be using Λ coordinates throughout this proof.

We will denote by L_1 and L_2 the lines that form the lower boundaries of S_1 and S_2 respectively. These lines have irrational gradients relative to both L and Λ. If the intersection point t is at coordinates $(\alpha, \beta)_\sim$ then (also in Λ coordinates) the lines can be written as,

$$L_1 = \{(x, px)_\sim + (\alpha, \beta)_\sim : x \in \mathbb{R}\}$$

$$L_2 = \{(x, qx)_\sim + (\alpha, \beta)_\sim : x \in \mathbb{R}\}$$

with p and q irrational.

Now define points t_1 and t_2 on lines L_1 and L_2 respectively to be,

$$t_1 = (x_0 + \alpha, px_0 + \beta)_\sim$$

$$t_2 = (x_0 + \alpha, qx_0 + \beta)_\sim .$$

Then for $z \in \mathbb{Z}$,

$$t_1 = t_2 + (0, z)_\sim \Rightarrow \pi(t_1) = \pi(t_2)$$

and also t_1 and t_2 occupy the same position within the fundamental domain of Λ and therefore within the fundamental domain of L.

The points t_1 and t_2 differ in such a way for all x satisfying,

$$px = qx + z .$$

We also have that,

$$px = qx + z \Leftrightarrow x = \frac{z}{p-q} .$$

So there are countably many values of x at which L_1 and L_2 occupy corresponding positions within fundamental domains of Λ. Thus countably many translates of the tiling must be intersection-point tilings, and these translates are the set of integer multiples of the translate by distance $\frac{1}{p-q}$ (that is, this distance in Λ coordinates). $\square$

Proposition 4.5. *The points described above are the only points on the lower boundaries of S_1 and S_2 that occupy corresponding positions within fundamental domains of L and project to the same points on E.*

Proof. The points described above are all pairs t_1 and t_2 on the lower boundaries of S_1 and S_2 respectively satisfying,

$$t_1 = t_2 + (0, z)_\sim$$

for $z \in \mathbb{Z}$.

Since t_1 and t_2 differ by a vector of the form $(0, y)_\sim$ we must also have that,

$$\pi(t_1) = \pi(t_2)\,.$$

Thus these must be the only pairs of points occupying corresponding positions within fundamental domains of Λ and having the same projections onto E. By lemma 4.2 they must also occupy corresponding points within fundamental domains of L.

All that remains to be shown is, for t_1 and t_2 in the same fundamental domain of Λ satisfying,

$$\pi(t_1) = \pi(t_2)$$

$$t_1 \neq t_2$$

the points t_1 and t_2 cannot occupy corresponding positions within two fundamental domains of L.

Assume that they do occupy corresponding positions within two fundamental domains of L, then (in L coordinates),

$$t_1 = t_2 + (a, b)$$

for some $a, b \in \mathbb{Z}$ with (a, b) parallel to $(-c, d)$ as in the definition of Λ (since projection is along this line). However, the step (a, b) must be less than an integer multiple of the step $(-c, d)$ since t_1 and t_2 are in the same fundamental domain of Λ. But c and d are coprime, so such a step does not exist.

Therefore t_1 and t_2 cannot occupy corresponding positions within two fundamental domains of L. $\square$

Note 4.2. The above argument does not completely rule out the possibility that there could be other intersection points than those listed above that give translates of the tiling. We also need to show that tilings that are produced by projection onto E with the origin not below an intersection point are distinct from intersection point tilings.

Lemma 4.6. *For pattern space E at gradient $\frac{c}{d}$, with c, d coprime, the shortest (non-zero) distance between the projections of lattice points in L (along E) is $\frac{1}{\sqrt{c^2+d^2}}$.*

Proof. The pattern space E is at gradient $\frac{c}{d}$, and thus at angle $\theta = \arctan(\frac{c}{d})$ to the horizontal. So we have the situation illustrated by figure 4.5.

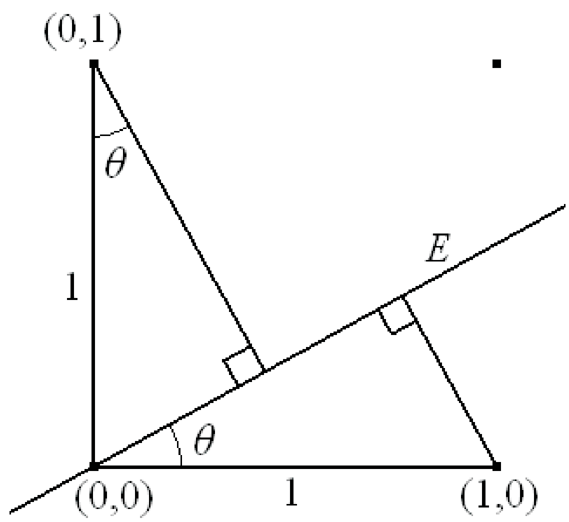

Figure 4.5: The lengths of the projections of horizontal and vertical steps onto E.

Thus,

$$\pi(1,0) = \cos\left(\arctan\left(\frac{c}{d}\right)\right) = \frac{1}{\sqrt{1+(\frac{c}{d})^2}}$$

$$\pi(0,1) = \sin\left(\arctan\left(\frac{c}{d}\right)\right) = \frac{\frac{c}{d}}{\sqrt{1+(\frac{c}{d})^2}} = \frac{1}{\sqrt{1+(\frac{d}{c})^2}}.$$

We will refer to the line segment between $(0,0)$ and $\pi(1,0)$ as t_1 (this segment is of length $|t_1|$), similarly the line between $(0,0)$ and $\pi(0,1)$ will be called t_2 (and have length $|t_2|$), so $|t_2|= \frac{c}{d} |t_1|$, giving two different cases, specifically $c > d$ and $d > c$ (the case $c = d$ is ignored).

Case 1: $c > d$ gives $|t_2|>|t_1|$ and the shortest possible distance between projections is $\frac{1}{d} |t_1|$ (see proposition 3.3). Then we have that,

$$\frac{1}{d} |t_1|= \frac{1}{d\sqrt{1+(\frac{c}{d})^2}} = \frac{1}{\sqrt{d^2(1+(\frac{c}{d})^2}} = \frac{1}{\sqrt{c^2+d^2}} .$$

Case 2: $d > c$ gives $|t_1|>|t_2|$ ($|t_1|= \frac{d}{c} |t_2|$) and the shortest possible distance between projections is $\frac{1}{c} |t_2|$. Then we have that,

$$\frac{1}{c} |t_2|= \frac{1}{c\sqrt{1+(\frac{d}{c})^2}} = \frac{1}{\sqrt{c^2(1+(\frac{d}{c})^2}} = \frac{1}{\sqrt{c^2+d^2}} .$$

Thus the shortest possible (non-zero) distance between the projections of points in L is $\frac{1}{\sqrt{c^2+d^2}}$.

$\square$

Proposition 4.7. *If U is a tiling produced by a rational 2:1 X-projection setup with,*

$$\pi(p_1) = \pi(p_2) = O$$

for points $p_1 \neq p_2 + (0,z)_\sim$ ($z \in \mathbb{Z}$) on the lower boundaries of strips S_1 and S_2 respectively (where O is the origin on E) and U' is an intersection-point tiling with intersection point t', then $U = U'$ implies that t' must be on the line parallel to $E^\perp$ passing through p_1 and p_2 within the fundamental domain of L.

Proof. Let E be at gradient $\frac{c}{d}$, with $c, d \in \mathbb{Z}$ coprime.

We have that,

$$\pi(p_1) = \pi(p_2)$$

so p_1 and p_2 are on a line parallel to $E^\perp$ (which has rational gradient).

Both p_1 and p_2 are within fundamental domains of L (possibly the same fundamental domain, but not necessarily), and they sit on the same line parallel to $E^\perp$, which crosses fundamental domains of L in only a finite number of ways (since it has rational gradient). Therefore p_1 and p_2 must be on one of a finite number of lines within the fundamental domain of L.

For example, we could have the situation show in figure 4.6.

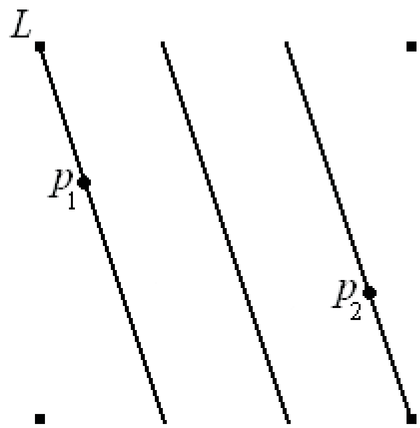

Figure 4.6: Points p_1 and p_2 on a line parallel to $E^\perp$.

If the line through p_1 and p_2 contains a lattice point then all points of the tilings T_1 and T_2 must be in $\left\{\frac{z}{\sqrt{c^2+d^2}} : z \in \mathbb{Z}\right\}$, since $\frac{1}{\sqrt{c^2+d^2}}$ is the minimum distance between projections of lattice points onto E. If t' is not on a line parallel to $E^\perp$ passing through a lattice point, then all lattice points in U' will be in $\left\{\frac{z}{\sqrt{c^2+d^2}} + \varepsilon : z \in \mathbb{Z}\right\}$ for some $\varepsilon < \frac{1}{\sqrt{c^2+d^2}}$, i.e., the tilings cannot coincide.

The case where the line through p_1 and p_2 does not pass through a lattice point is similar.

Thus the tilings cannot coincide when t' is not on the line through p_1 and p_2.

□

So we have that any non-intersection point tiling must either be distinct from all intersection point tilings or the same as an intersection point tiling with intersection point on the line perpendicular to E through the origin.

Note that we can consider p_1 and p_2 to be within the same fundamental domain of Λ, since we can translate strip S_i by a vector of the form $(0,n)_\sim$, with $n \in \mathbb{Z}$, without changing the tiling T_i given by the projection of lattice points from the strip.

Thus we have a situation similar to that illustrated by figure 4.7, where any intersection point tiling that coincides with the original tiling must have intersection point on the line through p_1 and p_2.

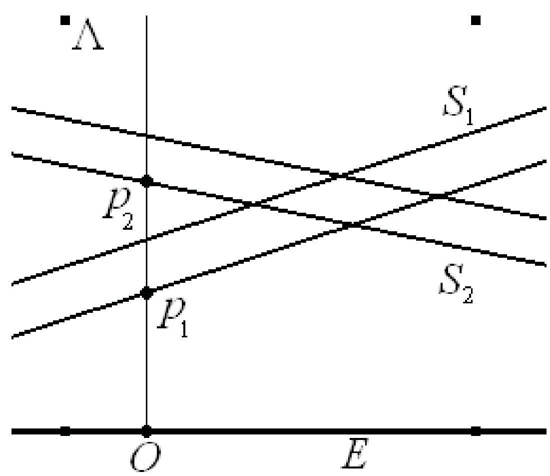

Figure 4.7: The line parallel to $E^\perp$ through p_1 and p_2 in the fundamental domain of Λ.

If we look at the tiling given by placing the intersection point at p_1 then we will get a tiling that is produced by the projection of the points contained in the strip S_1 and the points contained in a new strip S_2', which is the translate of strip S_2 so that it passes through the point p_1. This new strip is a translate of the strip S_2 by some vector $(0, y)_\sim$, with $|y| < 1$, so the two strips may overlap but can neither coincide nor differ by an integer amount in Λ. Thus the projections of the lattice points contained within the two strips cannot be the same. In fact, we will get a thin strip $S_2 \bigcap S_2'$ where the strips S_2 and S_2' intersect (or differ by one vertically in Λ coordinates) and also strips $S_2 \backslash S_2' \subset S_2$ and $S_2' \backslash S_2 \subset S_2'$, satisfying,

$$(S_2 \backslash S_2') \bigcap \left(S_2' \bigcup (S_2' + (0,1)_\sim) \bigcup (S_2' + (0,-1)_\sim) \right) = \emptyset$$
$$(S_2' \backslash S_2) \bigcap \left(S_2 \bigcup (S_2 + (0,1)_\sim) \bigcup (S_2 + (0,-1)_\sim) \right) = \emptyset$$

where $S_i + (0,n)_\sim$ is the strip obtained by translating strip S_i by the vector $(0,n)_\sim$. Note also that $S_2 \bigcap S_2'$ may be empty.

For example, we may have the situation shown in figure 4.8.

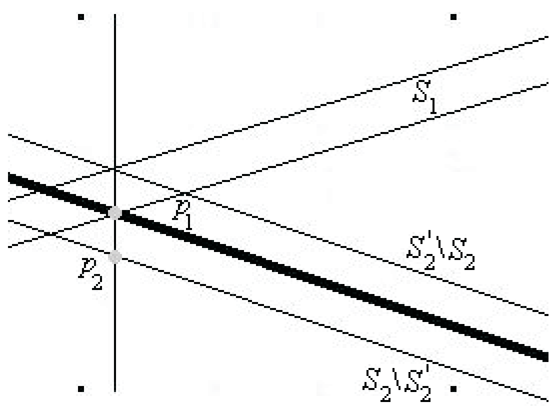

Figure 4.8: The intersection of S_2 and S_2'.

We have that,

$$\left\{ \pi(x) : x \in S_2' \backslash S_2 \bigcap L \right\} \bigcap \left\{ \pi(y) : y \in S_2 \backslash S_2' \bigcap L \right\} = \emptyset\,.$$

This is because these strips do not contain any lattice points that are either the same or differ by a vector of the form $(0,n)_\sim$ (for $n \in \mathbb{Z}$).

We will denote by T_i the tiling associated to strip S_i and by T_i' the tiling associated to the strip S_i'. So for the tiling with intersection point p_1 (that is, the combination of T_1 and T_2') to be the same as the original tiling (the combination of T_1 and T_2) we must have,

$$\left\{\pi(x) : x \in S_2 \backslash S_2' \bigcap L\right\} \subset \left\{\pi(y) : y \in S_1 \bigcap L\right\}$$

and,

$$\left\{\pi(x) : x \in S_2' \backslash S_2 \bigcap L\right\} \subset \left\{\pi(y) : y \in S_1 \bigcap L\right\} .$$

In other words, no points are lost from the original tiling (so all lattice points that are in S_2 but not in S_2' must have equivalents in S_1) and no extra points are added in (so all lattice points that are in S_2' but not in S_2 must have equivalents in S_1).

Thus the strip S_1 must contain lattice points with the same projections as all the lattice points in the thin strips $S_2 \backslash S_2'$ and $S_2' \backslash S_2$. The case where the intersection point is at p_2 is similar.

In the more general case, where the intersection point is not at either p_i, we get strips S_1' and S_2' that do not coincide with S_1 or S_2. Then for each S_i we get a strip $S_i \bigcap S_i'$, which may be empty, and strips $S_i \backslash S_i'$ and $S_i' \backslash S_i$. An argument similar to that above shows that for this altered setup to produce the same tiling we must have that,

$$\left\{\pi(y) : y \in S_1' \backslash S_1 \bigcap L\right\} \subset \left\{\pi(x) : x \in S_2 \bigcap L\right\}$$
$$\left\{\pi(y) : y \in S_1 \backslash S_1' \bigcap L\right\} \subset \left\{\pi(x) : x \in S_2' \bigcap L\right\}$$

and also,

$$\left\{\pi(x) : x \in S_2' \backslash S_2 \bigcap L\right\} \subset \left\{\pi(y) : y \in S_1 \bigcap L\right\}$$

$$\left\{\pi(x) : x \in S_2 \backslash S_2' \bigcap L\right\} \subset \left\{\pi(y) : y \in S_1' \bigcap L\right\} .$$

In the general case the question is whether a canonical width strip at one gradient can contain lattice points with the same projections as all the lattice points contained in a strip of up to canonical width at a different gradient. Thinking just in terms of the lattice Λ this would mean that any time you got a lattice point in the thin strip there would have to be a corresponding lattice point vertically above or below this contained in the other strip.

We will now show that this could be possible in some cases, but cannot happen in certain other cases.

Definition 4.8. For strips S_1 and S_2 of up to canonical width define the lines J_1 and J_2 to be the lines parallel to S_1 and S_2 respectively and positioned in the centres of the two strips.

Note 4.3. The strips S_1 and S_2 are at different gradients and thus the lines J_1 and J_2 will intersect. We will be thinking of the strips as sitting in the lattice Λ, so we will write the intersection point as,

$$J_1 \bigcap J_2 = (\alpha, \beta)_\sim$$

where α and β can be assumed to be between 0 and 1 (by choice of origin for Λ).

Definition 4.9. For a strip S sitting in lattice Λ, define the *height of S (in Λ)* to be the length of the interval given by $S \bigcap F$ where,

$$F = \{(0, y)_\sim : y \in \mathbb{R}\} .$$

So, the height of strip S in figure 4.9 is the length of the interval highlighted within the strip.

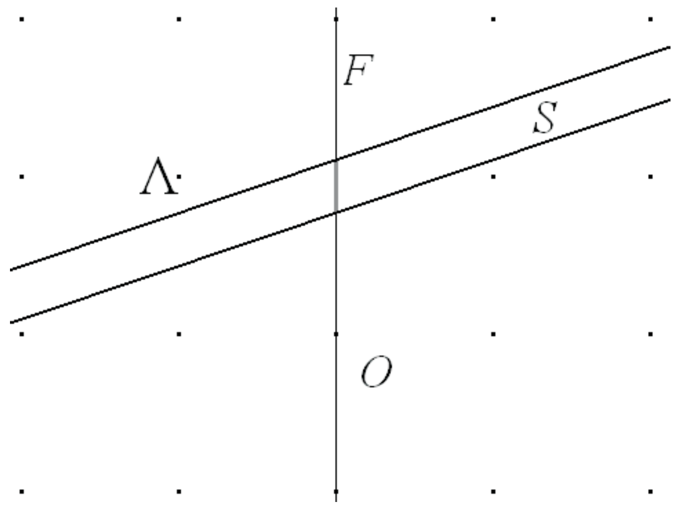

Figure 4.9: The height of strip S in Λ.

Theorem 4.8. *If strips S_1 and S_2 are at irrational gradients g_1 and g_2 respectively (relative to Λ) with,*

$$g_2 = ng_1$$

for $n \in \mathbb{Z}\backslash\{0,1\}$, and are of heights ε and δ in Λ respectively, then when S_2 has height $\delta < \min\{|n|\varepsilon, 1\}$ we have that,

$$P_1^\Lambda \not\subset P_2^\Lambda \,.$$

Proof. As above, we will denote by J_1 and J_2 the centre lines of strips S_1 and S_2 respectively.

These lines intersect at the point $(\alpha, \beta)_\sim$ and have gradients g_1 and g_2 relative to Λ. Thus, line J_1 contains the points,

$$\{(x, g_1(x-\alpha)+\beta)_\sim : x \in \mathbb{R}\}$$

and J_2 contains the points,

$$\{(x, g_2(x-\alpha)+\beta)_\sim : x \in \mathbb{R}\} = \{(x, ng_1(x-\alpha)+\beta)_\sim : x \in \mathbb{R}\}\,.$$

The line J_1 is at irrational gradient in the lattice Λ and will therefore pass arbitrarily close to points of the lattice.

If point $(z, k+\iota)_\sim$ is on line J_1, for $z, k \in \mathbb{Z}$ and $\iota \in \mathbb{R}$ then we have that,

$$\begin{aligned} g_1(z-\alpha)+\beta = k+\iota \Rightarrow g_1(z-\alpha) &= k+\iota-\beta \\ &\Rightarrow ng_1(z-\alpha) = n(k+\iota-\beta) \\ &\Rightarrow ng_1(z-\alpha)+\beta = nk+n\iota-(n-1)\beta \\ &\Rightarrow g_2(z-\alpha)+\beta = nk+n\iota-(n-1)\beta\,. \end{aligned}$$

Therefore J_2 passes through the point $(z, nk+n\iota-(n-1)\beta)_\sim$. Here the value nk is an integer, the $n\iota$ term may be arbitrarily small, and the $(n-1)\beta$ term is some fixed shift, independent of z, k and ι. Note that this final term may be zero, since β can have value zero.

The strip S_1 has height ε in Λ, so a lattice point $(z,k)_\sim \in S_1$ can approach (vertical) distance $\frac{\varepsilon}{2}$ from centre line J_1.

Whilst there will not be any lattice points on the boundaries of strip S_1, by choice of positioning of the strip, there will be lattice points approaching the situation described by points $(z_1, k_1)_\sim$ and $(z_2, k_2)_\sim$ below,

$$g_1(z_1-\alpha)+\beta = k_1 + \frac{\varepsilon}{2}$$
$$g_1(z_2-\alpha)+\beta = k_2 - \frac{\varepsilon}{2}$$

giving,

$$ng_1(z_1-\alpha)+\beta = nk_1 + \frac{n\varepsilon}{2} - (n-1)\beta$$

$$ng_1(z_2-\alpha)+\beta = nk_2 - \frac{n\varepsilon}{2} - (n-1)\beta\,.$$

Thus at integer x-values J_2, the centre line of S_2, contains points having y-values with fractional parts that vary between $[\frac{n\varepsilon}{2}-(n-1)\beta]$ and $[-\frac{n\varepsilon}{2}-(n-1)\beta]$ (here square brackets are used to denote fractional part) with strip S_2 containing lattice points at these x-values.

This requires S_2 to have height in Λ satisfying,

$$\delta \geq \min\{|n|\varepsilon, 1\}\,.$$

□

Theorem 4.9. *If we have strips S_1 and S_2 at irrational gradients g_1 and g_2 relative to Λ satisfying,*

$$g_2 = \frac{a}{b}g_1$$

for $a \in \mathbb{Z}$, $b \in \mathbb{N}$ coprime and $\frac{a}{b}$ not an integer, then if S_2 has height less than $\frac{b-1}{b}$ in Λ we have that,

$$P_1^\Lambda \not\subset P_2^\Lambda\,.$$

Proof. As before, we denote by J_1 and J_2 the centre lines of the strips S_1 and S_2 respectively, with $(\alpha,\beta)_\sim$ the point at which these lines intersect.

Line J_1 contains the points

$$\{(x, g_1(x-\alpha)+\beta)_\sim : x \in \mathbb{R}\}$$

and J_2 contains the points,

$$\{(x, g_2(x-\alpha)+\beta)_\sim : x \in \mathbb{R}\} = \left\{\left(x, \frac{a}{b}g_1(x-\alpha)+\beta\right)_\sim : x \in \mathbb{R}\right\} .$$

Again, in a similar way to the previous proof, if J_1 passes through point $(z, k+\iota)_\sim$ for $z, k \in \mathbb{Z}$ and $\iota \in \mathbb{R}$ then we have that,

$$\begin{aligned} g_1(z-\alpha)+\beta = k+\iota \Rightarrow g_1(z-\alpha) &= k+\iota-\beta \\ &\Rightarrow \frac{a}{b}g_1(z-\alpha) = \frac{a}{b}(k+\iota-\beta) \\ &\Rightarrow \frac{a}{b}g_1(z-\alpha)+\beta = \frac{a}{b}k+\frac{a}{b}\iota-\left(\frac{a}{b}-1\right)\beta \\ &\Rightarrow \frac{a}{b}g_1(z-\alpha)+\beta = \frac{ak}{b}+\frac{a}{b}\iota-\left(\frac{a-b}{b}\right)\beta \\ &\Rightarrow g_2(z-\alpha)+\beta = \frac{ak}{b}+\frac{a}{b}\iota-\left(\frac{a-b}{b}\right)\beta . \end{aligned}$$

Thus J_2 passes through the point $\left(z, \frac{ak}{b}+\frac{a}{b}\iota-\left(\frac{a-b}{b}\right)\beta\right)$.

Now, consider the subsets of Λ of the form $(z, by+r)_\sim$, for $y, z \in \mathbb{Z}$ and $r \in \mathbb{N}$ varying between 0 and $b-1$, i.e., the subsets $\{(z, by)_\sim : y, z \in \mathbb{Z}\}$, $\{(z, by+1)_\sim : y, z \in \mathbb{Z}\}$, etc.

The line J_1 still has an irrational gradient relative to any subset of Λ of this form, and will thus pass arbitrarily close to points in each of these subsets.

If J_1 passes through the point $(z, bk+r+\iota)_\sim$ then J_2 passes through the point $\left(z, ak+\frac{a}{b}r+\frac{a}{b}\iota-\left(\frac{a-b}{b}\right)\beta\right)$.

In the term $ak+\frac{a}{b}r+\frac{a}{b}\iota-(\frac{a-b}{b})\beta$ the value ak is an integer, and the value $\frac{a}{b}\iota$ may be arbitrarily small. Thus the fractional part of this term can be arbitrarily close to the fractional part of $\frac{a}{b}r$ shifted by $(\frac{a-b}{b})\beta$. By looking at different values of r we get the centre line of S_2 having a fractional part arbitrarily close to the fractional part of $-\frac{a-b}{b}\beta, \frac{1}{b}-\frac{a-b}{b}\beta, \frac{2}{b}-\frac{a-b}{b}\beta$ etc. But we also know that when J_2 takes any of these values the strip S_2 must be wide enough to contain a lattice point (if it contains lattice points with the same projections as every lattice point in S_1).

Thus we must examine the distance from an integer of the most distant of the points in,

$$\left\{\frac{r}{b} - \left(\frac{a-b}{b}\right)\beta : r \in \mathbb{N}, 0 \leq r \leq b-1\right\}.$$

Of course, this will depend on β, but we can still put a lower bound on the value.

Now we look at the points $\left\{\frac{r}{b} : r \in \mathbb{N}, 0 \leq r \leq b-1\right\}$. These points are evenly spaced between 0 and 1, and it is this set of points with some fixed shift that we are interested in. Since the points are evenly spaced, we will consider the shift by $(\frac{a-b}{b})\beta$ to be of length less than $\frac{1}{b}$.

If b is even, then $\frac{1}{2}$ is in the set of points, as are $\frac{1}{2}+\frac{1}{b}$ and $\frac{1}{2}-\frac{1}{b}$. Thus if β is zero we get that the furthest point from an integer is $\frac{1}{2}$ and therefore the strip S_2 must have height at least 1 in Λ. Of course, β may be non-zero, but the shift that minimises the distance of the furthest point is $\frac{1}{2b}$ (or $-\frac{1}{2b}$), resulting in a distance of $\frac{b-1}{2b}$ between the most distant point and an integer. Therefore when b is even the strip S_2 must have height at least $\frac{b-1}{b}$ in Λ.

If b is odd, then the points $\frac{1}{2}-\frac{1}{2b}$ and $\frac{1}{2}+\frac{1}{2b}$ are in $\left\{\frac{r}{b} : r \in \mathbb{N}, 0 \leq r \leq b-1\right\}$, and any shift of size less than $\frac{1}{b}$ will result in one of these points moving further away from an integer. So as above if b is odd, the minimum distance of the most distant point from an integer is $\frac{b-1}{2b}$, and again the strip S_2 must have height at least $\frac{b-1}{b}$ in Λ.

$\square$

As explained above, the question of whether a canonical width strip at one gradient can contain lattice points with the same projections as all the lattice points in a strip at a different gradient is important when deciding whether a tiling with the origin not at the projection of the intersection point is the same as one where the intersection point is above the origin.

From the above results, we get the following proposition.

Proposition 4.10. *If we have a rational X-projection with strips S_1 and S_2 at gradients g_1 and g_2 satisfying,*

$$g_1 = \frac{a}{b} g_2$$

with neither $\frac{a}{b}$ nor $\frac{b}{a}$ an integer, and with S_1 and S_2 having heights in Λ of less than $\frac{1}{2}$ then the intersection-point tilings described in proposition 4.4 are all the intersection-point tilings for this setup.

Proof. We have strips S_1 and S_2 with lower boundaries I_1 and I_2 passing through points p_1 and p_2 with,

$$p_1 \neq p_2 + (0, n)_{\sim}$$

for $n \in \mathbb{Z}$, and

$$\pi(p_1) = \pi(p_2) = O \, .$$

Say we have an identical tiling given by translating strips S_1 and S_2 (to get S_1' and S_2', with corresponding lower-boundary lines I_1' and I_2') so that,

$$I_1' \bigcap I_2' = q$$

with $\pi(q) = O$.

Then as explained in the discussion following proposition 4.7, we must have that,

$$\left\{ \pi(y) : y \in S_1' \backslash S_1 \bigcap \Lambda \right\} \subset \left\{ \pi(x) : x \in S_2 \bigcap \Lambda \right\}$$
$$\left\{ \pi(y) : y \in S_1 \backslash S_1' \bigcap \Lambda \right\} \subset \left\{ \pi(x) : x \in S_2' \bigcap \Lambda \right\}$$

and also,

$$\left\{\pi(x) : x \in S_2' \backslash S_2 \bigcap \Lambda\right\} \subset \left\{\pi(y) : y \in S_1 \bigcap \Lambda\right\}$$
$$\left\{\pi(x) : x \in S_2 \backslash S_2' \bigcap \Lambda\right\} \subset \left\{\pi(y) : y \in S_1' \bigcap \Lambda\right\} .$$

It could be the case that q coincides with p_1 or p_2, but of course not both so at least one of $S_1' \backslash S_1$ and $S_2' \backslash S_2$ must exist.

Say that $S_1' \backslash S_1$ exists, then (since the heights of the strips are both less than $\frac{1}{2}$, which is the lowest value $\frac{b-1}{b}$ can take) by theorem 4.9 we have that,

$$\left\{\pi(y) : y \in S_1' \backslash S_1 \bigcap \Lambda\right\} \not\subset \left\{\pi(x) : x \in S_2 \bigcap \Lambda\right\} .$$

Thus the tiling is indeed a non-intersection-point tiling.

□

We now look at the possible height of a canonical width strip in Λ. This will depend on the gradient of the pattern space E.

Proposition 4.11. *If pattern space E is at gradient $\frac{c}{d}$ with c and d coprime, and $\frac{c}{d}$ not equal to n or $\frac{1}{n}$ for $n \in \mathbb{N}$ then a canonical width strip must have height in Λ of less than $\frac{1}{2}$.*

Proof. If E is at gradient $\frac{c}{d}$ then the lattice Λ is generated by the vectors (d, c) and $(-c, d)$.

If we have a canonical width strip S at gradient p (with $p > 0$) with lower boundary passing through O, then the upper boundary of S will pass through the point $(-1, 1)$. What we are interested in is the point at which this upper boundary intersects the line between O and $(-c, d)$.

For S to have height $\frac{1}{2}$ in Λ we would require the upper boundary of S to pass through the point $\left(-\frac{c}{2}, \frac{d}{2}\right)$. Note that we have,

$$-\frac{c}{2} \leq -1$$

$$\frac{d}{2} \geq 1$$

and also note that we cannot have both values being equal to 1 (since E cannot have gradient 1).

However, the upper boundary of S passes through the point $(-1, 1)$ and has strictly positive (finite) gradient, so cannot also pass through any point $(-x, y)$ for both x and y greater than or equal to 1. Thus S cannot have height as much as $\frac{1}{2}$ in Λ. $\square$

Corollary 4.12. *When we are projecting onto a pattern space E at gradient $\frac{c}{d}$ with $\frac{c}{d}$ not equal to n or $\frac{1}{n}$ for $n \in \mathbb{N}$ and the strips S_1 and S_2 have gradients g_1 and g_2 relative to Λ satisfying,*

$$g_1 = \frac{a}{b} g_2$$

for $\frac{a}{b}$ not equal to m or $\frac{1}{m}$ ($m \in \mathbb{Z}$), then the intersection-point tilings identified in proposition 4.4 are all of the intersection-point tilings.

Proof. Follows from proposition 4.10 and proposition 4.11. $\square$

This also provides us with a fundamental result about these tilings:

Corollary 4.13. *Consider the tilings generated by 2:1 X-projection schemes as above, i.e., with pattern space E at gradient $\frac{c}{d}$ with $\frac{c}{d}$ not equal to n or $\frac{1}{n}$, for $n \in \mathbb{N}$, and with strips S_1 and S_2 having gradients g_1 and g_2 relative to Λ satisfying,*

$$g_1 = \frac{a}{b} g_2$$

for $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ coprime and $\frac{a}{b}$ not equal to m or $\frac{1}{m}$ for $m \in \mathbb{Z}$.

Any tiling generated in this way is aperiodic.

Proof. Take a tiling T generated in the above way.

If T is an intersection-point tiling then by the above discussion it is distinct from all translates of T that are non-intersection-point tilings. A similar argument to that presented in proposition 4.10 proves that it is also distinct from all translates that are intersection-point tilings. So T must be aperiodic.

If T is a non-intersection-point tiling then there exists some translation u so that $T + u$ is an intersection-point tiling. Then, as above, $T + u$ must be aperiodic, and therefore T is aperiodic. □

4.3 Rationally Related Gradients

There are two possibilities for the relationship between the gradients of the two strips relative to the lattice Λ, namely that they are either rationally or irrationally related. That is, if we denote by p the gradient of strip S_1 and by q the gradient of strip S_2 (both in Λ coordinates) then we can have that p is either a rational or an irrational multiple of q. In this section we will look at the first of these cases.

Definition 4.10. When we have X-projection giving tiling U with intersection point tilings V_i, $i \in \mathbb{Z}$, corresponding to translates of U then we will refer to the locations of the intersection points of the setups producing the tilings V_i within fundamental domains of L or Λ as *translate points.*

So a translate point is a point at which you can reposition the intersection of the strips in an X-projection setup to get a translate of the original tiling.

Proposition 4.14. *When the gradients of the two strips are rationally related relative to Λ, the translate points form dense subsets of a finite number of horizontal lines in the fundamental domain of Λ.*

Proof. Recall from the proof of proposition 4.4 that we get a translate point of the tiling when we have points t_1 and t_2 on the lower boundaries of strips S_1 and S_2 satisfying,

$$t_1 = t_2 + (0, z)_\sim$$

for some $z \in \mathbb{Z}$. This happens at $x \in \mathbb{R}$ where,

$$px = qx + z\,.$$

In the case where p and q are rationally related we have that,

$$p = aq, \quad a \in \mathbb{Q} \Rightarrow x = \frac{z}{q(a-1)}$$

but,

$$qx = \frac{z}{a-1} \in \mathbb{Q}.$$

So the points of intersection appear at irrational steps along but at rational "heights", relative to Λ. Writing $\frac{1}{a-1}$ as $\frac{\alpha}{\beta}$ (with $\alpha, \beta \in \mathbb{Z}$ coprime) we get that the "heights" of the intersection points within the fundamental domain of Λ are the fractional parts of $\frac{z\alpha}{\beta}$ for $z \in \mathbb{Z}$. Since α and β are coprime we get β different values for the fractional part of $\frac{z\alpha}{\beta}$, and as we consider x values in turn we will cycle through these β values of the fractional part of $\frac{z\alpha}{\beta}$.

So within the fundamental domain of Λ the points that correspond to translates of the tiling are dense within horizontal lines at heights 0, $\frac{1}{\beta}$, $\frac{2}{\beta}$, etc., that is, if we put the original intersection point at the origin. □

Example 4.3. *In this example the gradients of the two strips are rationally related relative to Λ, meaning that we are in the case described by proposition 4.14.*

We take the pattern space E to be at gradient $\frac{2}{3}$ and the two strips to have gradients relative to Λ of $\frac{1}{\sqrt{2}}$ and $\frac{1}{3\sqrt{2}}$, and we put the original intersection point of the lower boundaries of the strips at the origin, thus:

$$x = \frac{z}{2q} = \frac{3\sqrt{2}z}{2}$$

$$qx = \frac{z}{2}\,.$$

So the translate points will form dense subsets of two horizontal lines in the fundamental domain of Λ, one at height 0 and the other at height $\frac{1}{2}$, as illustrated by figure 4.10.

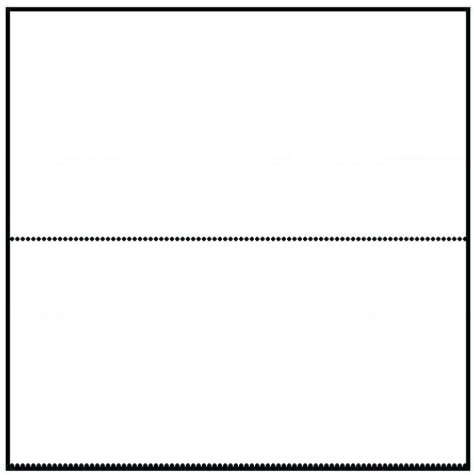

Figure 4.10: The translate points for example 4.3 in the fundamental domain of Λ.

For the original tiling the intersection point t is positioned at the origin. The next point at which the lower boundaries of the two strips occupy the same position within the fundamental domain of Λ comes at distance $\frac{3\sqrt{2}}{2}$ along E (in Λ coordinates that is). At this point the line qx is at height $\frac{1}{2}$ and the line px is at $\frac{3}{2}$, so in the fundamental domain of Λ the point corresponding to this positioning of t is at $([\frac{3\sqrt{2}}{2}], \frac{1}{2})$, where square brackets are used to denote fractional part. Thus, putting the intersection point of the strips here gives you *a translate of the original tiling by* $\frac{3\sqrt{2}}{2}$ *relative to* Λ*, that is a translate by* $\frac{3\sqrt{26}}{2}$ *relative to* L*.*

The next point is at $([3\sqrt{2}], 0)$ *and corresponds to double the translate along* E*, and so on.*

However, these points do not cover all possible translates of the tiling, only all integer multiples of a certain translate, so there are more translates than those corresponding to the points above.

These extra translates may not correspond to a repositioning of t *within the fundamental domain of* Λ*. Assuming that this is the case we will get a picture of the translates that looks a bit like that shown in figure 4.11.*

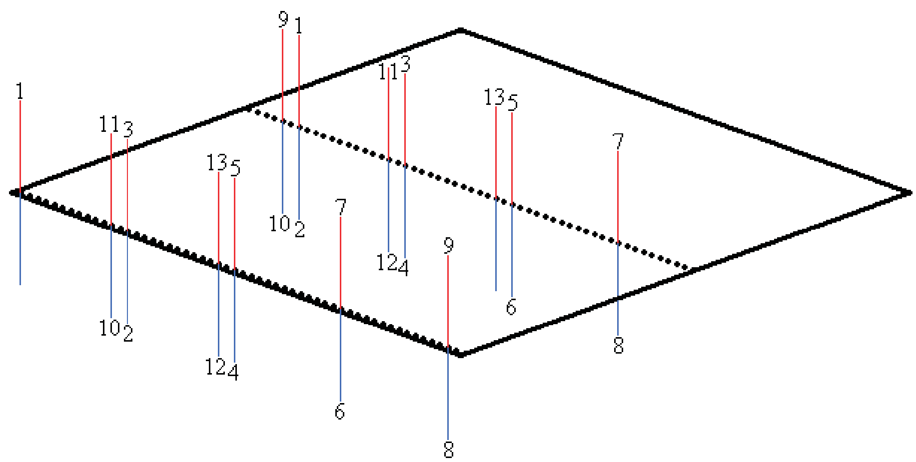

Figure 4.11: The line of translates of the tiling.

The ends of the lines here are identified as denoted by the numbers, though of course the lines will actually be dense, so the set of translates will look like a dense spiral winding round a torus.

4.4 Irrationally Related Gradients

Now we investigate the case where p and q are irrationally related.

I.e., $p = aq$ for some a irrational.

In fact this case breaks down into two separate subcases:

1. $\frac{z}{(a-1)q} \in \mathbb{Q}$.

2. $\frac{z}{(a-1)q} \in \mathbb{R}\backslash\mathbb{Q}$.

These cases correspond to the gradients of L_1 and L_2 being irrationally related but differing by either a rational (case 1) or an irrational (case 2) amount, relative to Λ.

4.4.1 Gradients Differ by a Rational Amount

Proposition 4.15. *When the gradients of the two strips are irrationally related but differ by a rational amount relative to Λ we have translate points that are dense in a finite number of vertical lines in the fundamental domain of Λ.*

Proof. In this subcase we have that $p - q \in \mathbb{Q}$, and so,

$$x = \frac{z}{p-q} \in \mathbb{Q}.$$

Since x always takes a rational value, there will only be finitely many possible fractional parts of x, and therefore only finitely many x-values for the translate points within the fundamental domain of Λ.

However, the values of qx are of course irrational (being irrational multiples of the values of x) and so the steps between consecutive values have irrational fractional parts and you will get dense sets of translate points on finitely many vertical lines. $\square$

Example 4.4. *In this example we look at the points corresponding to translates of a tiling of the type described in proposition 4.15 above, where the gradients of the two strips are irrationally related relative to Λ but differ by a rational amount.*

Once again we will take the pattern space E to be at gradient $\frac{1}{3}$, and if the two strips to have gradients $p = \sqrt{2} + \frac{3}{4}$ and $q = \sqrt{2}$ relative to Λ then:

$$x = \frac{z}{p-q} = \frac{4z}{3}$$
$$qx = \frac{4\sqrt{2}z}{3}\,.$$

Thus the points are contained in three vertical lines within the fundamental domain of Λ, one of which passes through the original intersection point with the other two being translates of this line by $\frac{1}{3}$ and $\frac{2}{3}$ of the side length of the fundamental domain of Λ. So if the intersection point is on the side edge of a fundamental domain of Λ then, in the fundamental domain of Λ, the positions for translate points will be like those shown in figure 4.12.

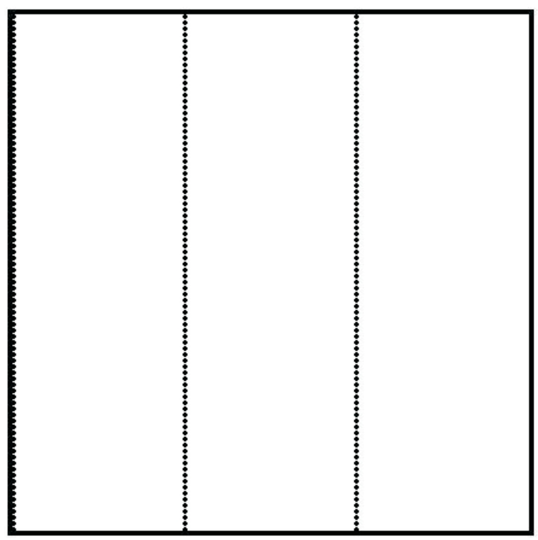

Figure 4.12: The translate points for example 4.4 in the fundamental domain of Λ.

In the fundamental domain of L the vertical lines above will appear as lines perpendicular to E, that is having gradient -3. As with the previous example the three lines traverse the fundamental domain of L several times giving a diagram *that appears to contain more lines of points, as shown in figure 4.13.*

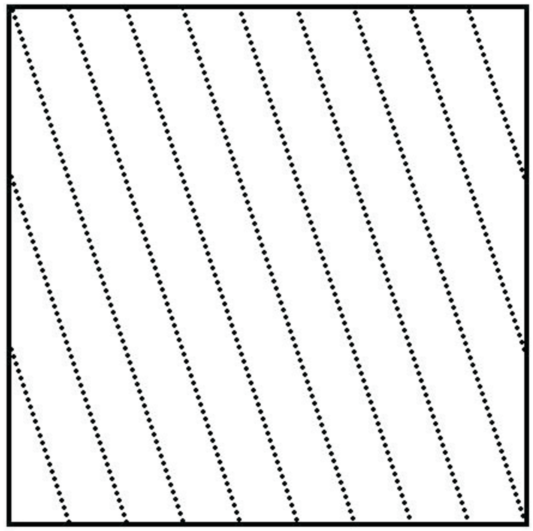

Figure 4.13: The translate points for example 4.4 in the fundamental domain of L.

4.4.2 Gradients Differ by an Irrational Amount

Lemma 4.16. *When the gradients of the two strips are irrationally related and differ by an irrational amount then each translate point is both an irrational step along and an irrational step up from the previous one in the fundamental domain of* Λ.

Proof. The gradients of the strips differ by an irrational amount, so $p - q$ is irrational and therefore,

$$x = \frac{z}{p-q} \notin \mathbb{Q}$$

so the horizontal step from one point to the next is irrational, but also because $p = aq$ with a irrational we get that,

$$qx = \frac{qz}{(a-1)q} = \frac{z}{a-1} \notin \mathbb{Q}$$

and so the vertical step from one point to the next is also irrational. $\square$

Proposition 4.17. *If the fractional parts of x and qx are rationally related then all the translate points will be on a line of rational gradient within the fundamental domain of Λ.*

Proof. The step from one translate point to the next must be along the line that has gradient equal to the fractional part of qx divided by the fractional part of x, and if these values are rationally related then this line must have rational gradient. $\square$

Example 4.5. *This example shows the set of translate points of a tiling generated by a rational 2:1 X-projection scheme of the type described in proposition 4.17, where the gradients of the two strips are irrationally related relative to Λ and differ by an irrational amount, but the fractional parts of the horizontal an vertical steps between consecutive translate points are rationally related.*

Take the gradients of L_1 and L_2 relative to Λ to be $\frac{3+\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}}$ and $\frac{2+\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}}$ respectively.

I.e., p and q take these values.

Then,

$$a = \frac{p}{q} = \frac{3+\frac{1}{\sqrt{2}}}{2+\frac{1}{\sqrt{2}}} = \frac{2+\frac{1}{\sqrt{2}}}{2+\frac{1}{\sqrt{2}}} + \frac{1}{2+\frac{1}{\sqrt{2}}} = 1 + \frac{1}{2+\frac{1}{\sqrt{2}}}.$$

So we have that,

$$a - 1 = \frac{1}{2+\frac{1}{\sqrt{2}}} \Rightarrow (a-1)q = \frac{1}{1+\frac{1}{\sqrt{2}}}.$$

Therefore,

$$x = \frac{1}{(a-1)q} = 1 + \frac{1}{\sqrt{2}}$$

and

$$qx = \frac{1}{a-1} = 2 + \frac{1}{\sqrt{2}}\,.$$

So the fractional parts of $\frac{z}{(a-1)q}$ *and* $\frac{z}{a-1}$ *are equal for all integers* z *and so all intersection points of the lower boundaries that are translates of the original intersection point appear on the line* $y = x$ *within the fundamental domain of* Λ. *Giving a diagram that looks a bit like figure 4.14.*

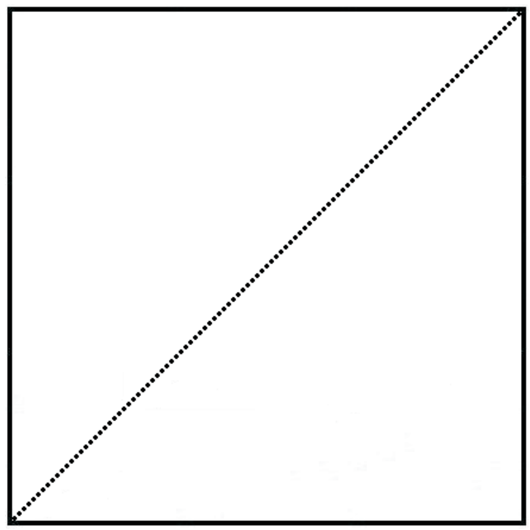

Figure 4.14: The translate points for example 4.5 in the fundamental domain of Λ.

If we once again take E *to be at gradient* $\frac{1}{3}$ *then if the bottom left corner of the fundamental domain of* Λ *is situated at the origin we will have that the top right corner is at* $(2, 4)$ *in the lattice* L, *thus the points lie on the line of gradient* 2 *in the fundamental domain of* L, *as shown in figure 4.15.*

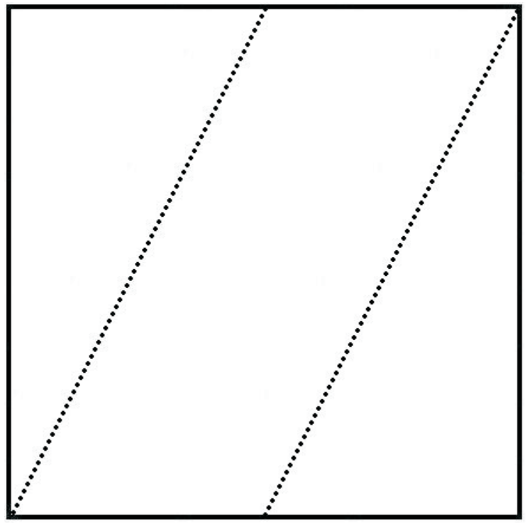

Figure 4.15: The translate points for example 4.5 in the fundamental domain of L.

Example 4.6. *In this example, as in example 4.5, the translate points are not contained in either horizontal or vertical lines in the fundamental domain of* Λ*, however this time the translate points appear in more than one line at rational gradient in the fundamental domain.*

If we have strips with gradients (relative to Λ*) of,*

$$p = 3\sqrt{2} - 2$$

$$q = \sqrt{2}$$

then a *is equal to* $3 - \sqrt{2}$ *and so is irrational. Thus* qx *will be irrational and more precisely we have that,*

$$x = \frac{z}{q(a-1)} = \frac{z}{2(\sqrt{2}-1)}$$

$$qx = \frac{\sqrt{2}z}{\sqrt{2}(2-\sqrt{2})} = \frac{z}{2-\sqrt{2}}\,.$$

The fractional parts of x and qx (at $z = 1$) are $\frac{\sqrt{2}-1}{2}$ and $\frac{1}{\sqrt{2}}$ respectively and so are irrationally related (with the fractional part of qx divided by the fractional part of x being equal to $\frac{2}{2-\sqrt{2}}$).

However, the set of points given by these values is not dense in the fundamental domain of Λ. This is because the fractional part of qx is equal to $\frac{1}{2}$ plus the fractional part of x when $z = 1$, as can be seen below.

$$\frac{\sqrt{2}-1}{2} + \frac{1}{2} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}\,.$$

So if we take our initial point to be at $(0,0)$ (which is on the line $y = x$) then the next point will be on the line $y = x + \frac{1}{2}$ within the fundamental domain of Λ and the next point will be back on the line $y = x$, etc. In this way the points will alternate between being on these two lines within the fundamental domain of Λ, giving the situation shown in figure 4.16.

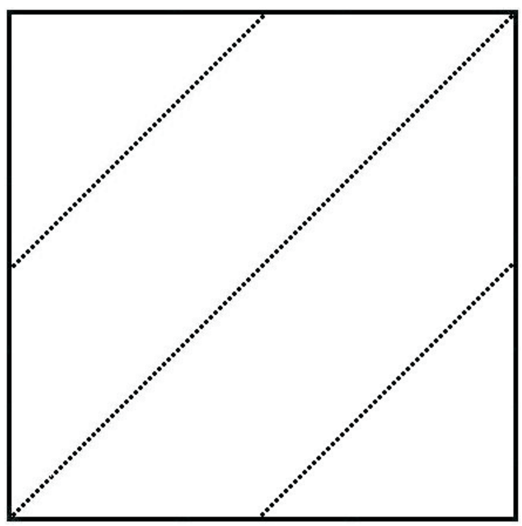

Figure 4.16: Translate points for example 4.6 in the fundamental domain of Λ.

This example comes from another subcase where,

$$[qx] = c[x] + r$$

for $c, r \in \mathbb{Q}$. Here the square brackets are used to denote the *fractional* parts of x and qx.

Of course, if we allow r to be equal to zero then this subcase includes the case described above where $[x]$ and $[qx]$ are rationally related.

Proposition 4.18. *Whenever $[x]$ and $[qx]$ are irrational and $[qx]$ can be expressed in the form $c[x] + r$ for some $c, r \in \mathbb{Q}$ this expression is unique.*

Proof. Assume that $[x]$ and $[qx]$ are irrational and that,

$$[qx] = c_1[x] + r_1$$

$$[qx] = c_2[x] + r_2$$

for $c_1, c_2, r_1, r_2 \in \mathbb{Q}$.

$$\Rightarrow c_1[x] + r_1 = c_2[x] + r_2$$

$$\Rightarrow c_1[x] - c_2[x] = r_2 - r_1$$

$$\Rightarrow (c_1 - c_2)[x] = r_2 - r_1\ .$$

However $[x]$ is irrational, so $c_1 - c_2$ and $r_2 - r_1$ must both be equal to zero. Thus the expression is unique. □

Proposition 4.19. *If we can write $[qx] = c[x] + r$ for some $c, r \in \mathbb{Q}$ (at $z = 1$) then the points are contained in a finite set of lines at gradient c in the fundamental domain of Λ.*

Proof. If we can write $[qx] = c[x] + r$ for some $c, r \in \mathbb{Q}$ (at $z = 1$) then as before if we think of our original point as being at $(0, 0)$, which is on the line $y = cx$, the next point will be on the line $y = cx + [r]$ within the fundamental domain of Λ, and so on. Of course, because r is rational this results in the points being contained in a finite set of lines at gradient c within the fundamental domain of Λ. $\square$

There is now one more case that we will look at.

Proposition 4.20. *If $[x]$ and $[qx]$ are irrationally related but $[qx]$ cannot be written in the form $c[x]+r$ for $c, r \in \mathbb{Q}$ then the translate points are not contained in any finite set of parallel lines with rational gradients in the fundamental domain of Λ.*

Proof. If $[qx]$ cannot be written in this way then writing $[qx] = c[x] + r$ for any rationally valued c means that r must be irrational.

The initial point can always be considered to be positioned on a line parallel to $y = cx$ in the fundamental domain of Λ, but then the next point will be on a line parallel to $y = cx$ but shifted by an irrational amount (the fractional part of r), and the lines of gradient c containing all subsequent points will be shifted by integer multiples of the same irrational amount, thus each line at gradient c can contain at most one of the points.

Since any rational value of c will give an irrational value for r we have that the points cannot be contained in any finite number of parallel lines at rational gradient in the fundamental domain of Λ. $\square$

Corollary 4.21. *A line drawn between any two translate points must have irrational gradient.*

Proof. If we have an initial point that we think of as sitting on a line with rational gradient c then the next point in the sequence is sitting on a line with gradient c that is shifted by an irrational amount, and the same is true for all subsequent points, so none of these points can be on the original line.

Thus any line with rational gradient through any translate point must not pass through any other point in the set, so a line drawn between two points of the set must have irrational gradient. □

Of course, going from one translate point to another just involves taking a number of steps, say k, and going through another k steps will give another point on the same line that is the same translation along again. So a line drawn between two points will have irrational gradient and will contain infinitely many translate points evenly spaced along its length.

Lemma 4.22. *Under the assumptions of proposition 4.20, for any $\varepsilon > 0$ we can find two translate points that are within distance ε of each other.*

Proof. As explained above in this case the values of x and qx are both irrational, so the step from one point in the set to the next consists of an irrational step along and an irrational step up. Thus the set of all x-coordinates and the set of all y-coordinates are dense.

So fix $\varepsilon > 0$ and choose a point in the set, say at position (x_0, y_0). Then there must be infinitely many points with x-coordinates between x_0 and $x_0 + \frac{\varepsilon}{\sqrt{2}}$, and these points cannot all have y-coordinates that are separated by more than $\frac{\varepsilon}{\sqrt{2}}$ so there must be a pair of points within distance ε of each other. □

Proposition 4.23. *When $[x]$ and $[qx]$ are irrationally related but $[qx]$ cannot be written in the form $c[x] + r$ for $c, r \in \mathbb{Q}$ then the set of translate points forms a dense subset of the fundamental domain of Λ.*

Proof. For any $n \in \mathbb{N}$ we can divide the fundamental domain of Λ into squares of side $\frac{1}{n}$. Then we can find two translate points that are within distance $\frac{1}{2n}$ of each other and so we have a line with irrational gradient between them that has translate points evenly spaced along its length with the gap between any two being less than $\frac{1}{2n}$.

As the line is at irrational gradient it will intersect all of the squares of side $\frac{1}{n}$ (countably many times) and because of the distance between points on the line there must be at least one translate point within each of the little squares.

So for any $n \in \mathbb{N}$, when dividing the fundamental domain of Λ into an n by n grid there must always be at least one point in each square, and therefore the set of translate points is dense in the fundamental domain of Λ. $\square$

So all but one of the subcases where the gradients of the two strips are irrationally related relative to Λ result in sets of translation points that are dense on a finite number of lines that have rational gradient (or are vertical). These subcases should therefore give sets of translates that are similar to that seen in the case where the gradients of the two strips are rationally related relative to Λ.

The final subcase gives a dense set of points in the fundamental domain of Λ and as such the set of translates should look quite different from the other cases.

Note that the final case is in fact the general case, with the other cases requiring some rational relationship between the gradients of the strips or the steps between lattice points.

4.5 Summary

Figure 4.17 gives a summary of the results in this section.

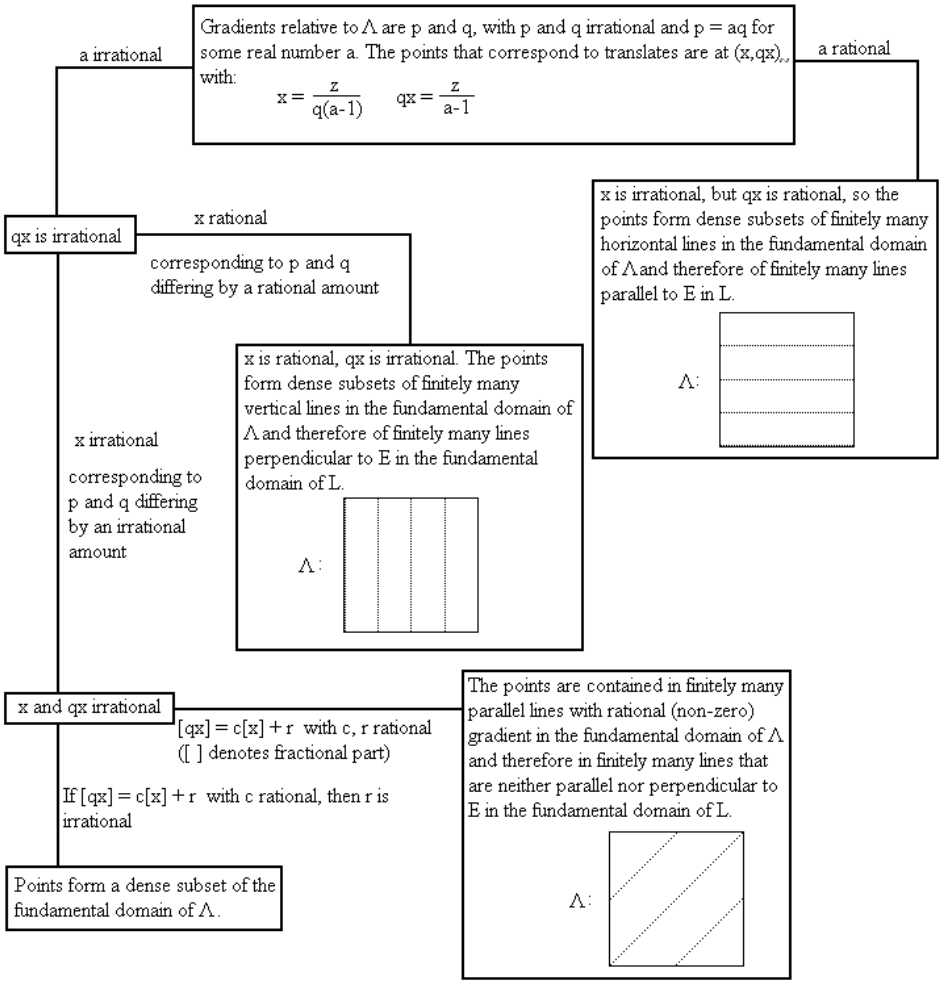

Figure 4.17: The sets of translate points given by rational 2:1 X-projections.

4.6 Explanation of Diagrams

In this section we will present some further explanation of the diagrams, like those above, that are produced by rational 2:1 X-projections. In particular we will be looking at the case where the gradients of the two strips are rationally related to each other, relative to Λ.

The setup is as before, with points from integer lattice L being projected onto pattern space E at gradient $\frac{c}{d}$. We have strips S_1 and S_2 at gradients g_1 and g_2 (relative to Λ).

We are looking at the case where g_1 and g_2 are rationally related, so we say that,

$$g_1 = ag_2$$

for some $a \in \mathbb{Q}$. We will assume that $|a| \geq 1$, and because a is rational we will sometimes write,

$$a = \frac{a_1}{a_2}\,.$$

Note that the above fraction is assumed to be expressed in its lowest terms, so a_1 and a_2 are coprime.

4.6.1 Gradients of Lines

As explained above, in the case where g_1 and g_2 are rationally related the translate points appear in horizontal lines in the fundamental domain of Λ. Due to Λ being rotated relative to L these lines will have gradient equal to that of E in the fundamental domain of L.

It is also worth noting that each line in the fundamental domain of Λ will give a loop in the fundamental domain of L, and will thus look like several lines crossing the fundamental domain.

4.6.2 Number of Lines

The value a (the ratio of the gradients of the two strips) determines the number of lines in the fundamental domain of Λ in which the translate points sit.

The following proposition explains this relationship.

Proposition 4.24. *With the above setup, the translate points are contained in* $|a_1 - a_2|$ *lines in the fundamental domain of* Λ *.*

Proof. As explained above, in the fundamental domain of Λ we have an irrational horizontal step and a rational vertical step between consecutive translate points.

The vertical step is (the fractional part of),

$$\frac{1}{a-1}\,.$$

Thus the translate points can be found in a number of lines equal to the number of distinct fractional parts of multiples of the above term.

If we have $a_1 - a_2 = z$, for some $z \in \mathbb{Z}$, then we get,

$$\frac{1}{a-1} = \frac{1}{\frac{a_1}{a_2} - 1} = \frac{a_2}{a_1 - a_2} = \frac{a_2}{z}\,.$$

We now look at the fractional parts of all multiples of this value to determine the number of lines in the fundamental domain of Λ.

There will be $|z|$ distinct fractional parts provided a_2 and z are coprime. However, they must be coprime, since $a_1 = a_2 + z$ and the values a_1 and a_2 are coprime.

□

A single horizontal line in the fundamental domain of Λ will cross several fundamental domains of L, and since the translate $(1,0)_\sim$ is equal to the translate (d,c), a line across the fundamental domain of Λ will give a loop around the fundamental domain of L.

We will now look at the number of loops containing translate points in the fundamental domain of L. This will often be the same as the number of lines in the fundamental domain of Λ, however the two numbers are not necessarily the same, as the following results show.

Lemma 4.25. *Within the fundamental domain of Λ there are $c^2 + d^2$ points of L (counting the four lattice points at the corners as a single lattice point).*

Proof. Due to the way that it is defined, the fundamental domain of Λ does not contain any lattice points from L with the same projections (with the exceptions of the points at the corners).

To see this, consider any lattice point (r, s) within the fundamental domain of Λ. The closest points of L with the same projections as (r, s) are the points $(r - c, s + d)$ and $(r + c, s - d)$. However, neither of these points lie in the same fundamental domain of Λ as the point (r, s) (unless (r, s) is one of the corner points).

Note that for similar reasons if we have a point on E that can be the projection of a lattice point of L then each fundamental domain of Λ above and below that point will contain a lattice point with that projection.

Now, by lemma 4.6, the shortest distance between the projections of lattice points of L is $\frac{1}{\sqrt{c^2+d^2}}$. Thus, by the above arguments, the fundamental domain of Λ contains precisely one point of L projecting to each of the points $\frac{n}{\sqrt{c^2+d^2}}$ for n varying between 0 and $c^2 + d^2$ (with these values giving the points at the edges of the fundamental domain of Λ).

So we get that the fundamental domain of Λ contains $c^2 + d^2 - 1$ points of L within its interior. ☐

Proposition 4.26. *In the fundamental domain of L the points appear in a number of loops equal to $|a_1 - a_2|$, provided $|a_1 - a_2|$ and $c^2 + d^2$ are coprime.*

Proof. We have $|a_1 - a_2| = n$ evenly-spaced lines in the fundamental domain of Λ. If we have one of the lines being at height zero, then the others must be at heights $\frac{1}{n}$, $\frac{2}{n}$, etc. With the line at height 1 being the same as the line at height 0.

Each of these lines gives a loop in the fundamental domain of L, but the question is whether all of these loops are distinct.

The first line (at height zero) runs through a lattice point, so if any other lines run through lattice points then they will be overlaid in the fundamental domain of L.

The lattice points within the fundamental domain of Λ are evenly spaced at heights $\frac{1}{\sqrt{c^2+d^2}}$, $\frac{2}{\sqrt{c^2+d^2}}$, etc. So they sit on $m = c^2 + d^2$ evenly-spaced lines in the fundamental domain of Λ, again with the line at height 1 ignored since it is the same as the line at height 0.

When m and n are coprime only the lines at height 0 can coincide. Thus the loop in L corresponding to the line at height 0 in Λ is not overlaid by any other loop. We also cannot have two other lines of translate points giving overlaid loops in the fundamental domain of L since the translate points are on evenly-spaced lines and would need to differ in height by some multiple of $\frac{1}{\sqrt{c^2+d^2}}$. However, since m and n are coprime this cannot happen.

The proof is similar when none of the lines of translate points is at height zero in the fundamental domain of Λ. $\square$

Example 4.7. *If we have E at gradient $\frac{1}{2}$, and the translate points sitting on 3 lines in the fundamental domain of Λ then the situation will look like those shown in figure 4.18 (for one of the lines of translate points being at height 0).*

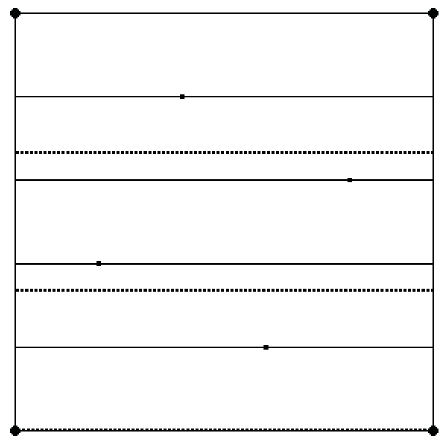

Figure 4.18: The translate points in the fundamental domain of Λ, with points of L also marked.

So here $n = 3$ and $m = 5$, and the only place that two of the lines coincide is at height 0 in the fundamental domain of Λ.

Proposition 4.27. *Let $k \in \mathbb{N}$ be the highest common factor of $|a_1 - a_2|$ and $c^2 + d^2$, then the translate points appear in a number of loops equal to,*

$$\frac{|a_1 - a_2|}{k}$$

in the fundamental domain of L.

Proof. Say that,

$$|a_1 - a_2| = \alpha k$$

$$c^2 + d^2 = \beta k$$

for some $\alpha, \beta \in \mathbb{N}$. Then the translate points appear on αk lines in the fundamental domain of Λ and the lattice points appear on βk lines. As before, these lines have evenly-spaced heights in the fundamental domain of Λ, so a step of $(0, \frac{1}{k})_\sim$ moves you from one line running through a lattice point to another line running through a lattice point.

The translate points sit on αk evenly-spaced lines, so if we have a line at height 0 then there will be a line at height $\frac{1}{k}$ in the fundamental domain of Λ (the line α steps up from the first line) and similarly lines at heights $\frac{2}{k}$, $\frac{3}{k}$, etc. In a similar way there will be lines at heights $\frac{1}{n}$, $\frac{1}{n}+\frac{1}{k}$, $\frac{1}{n}+\frac{2}{k}$ and so on, with all of these line being overlaid in the fundamental domain of L.

Thus the n distinct lines in the fundamental domain of Λ will become at most α distinct loops in the fundamental domain of L.

Indeed, there will be exactly α distinct loops in the fundamental domain of L because α and β are coprime (since k is the highest common factor), in a similar way to the previous proof. $\square$

Figure 4.19 shows the lines that are referred to in the previous results in the case where E is at gradient $\frac{2}{3}$.

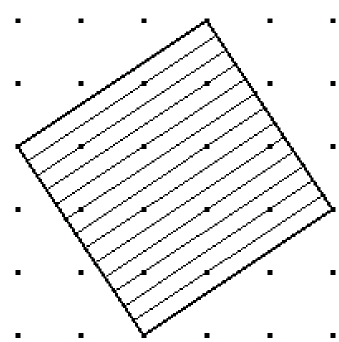

Figure 4.19: The lines through points of L in the fundamental domain of Λ.

Each of these lines passes through a lattice point, and they are overlaid in the fundamental domain of L.

4.6.3 Size of Rotation

We have already established that the diagram corresponding to a tiling setup of this type consists of dense sets of points on a finite number of loops at gradient equal to E in the fundamental domain of L. We now look at the length of the steps along the loops between these lattice points. Since we are referring to the lines as loops we will call this step a rotation, and as explained earlier in the chapter, this will be an irrational rotation.

So, as was mentioned earlier in this chapter, the horizontal step between consecutive translate points is,

$$\frac{1}{g_1 - g_2}$$

in Λ-coordinates. Thus in the case where we have the translate points in a single loop in the fundamental domain of L the rotation will be $\frac{2\pi}{g_1-g_2}$ (or rather, the fractional part of this).

When the translate points are contained in more than one loop then consecutive translate points will be on different loops. However, we will cycle through all the loops in turn, so if there are k loops then the rotation on each one will be (the fractional part of) $\frac{2k\pi}{g_1-g_2}$.

4.6.4 Comparing Diagrams

Having described the diagrams produced by these tilings in some more detail it seems natural to investigate whether two different tiling schemes will give different diagrams. This turns out not to be the case, as the following discussion reveals.

We will be comparing diagrams associated to two tiling schemes, as before these will be rational 2:1 X-projections with the gradients of the two strips rationally related relative to Λ.

So, we have setup 1 with characteristics,

- Pattern space E at gradient $\frac{c}{d}$ (and associated lattice Λ).
- Strips S_1 and S_2 at gradients g_1 and g_2 respectively (relative to Λ).
- Gradients satisfying, $g_1 = ag_2$, with $a = \frac{a_1}{a_2} \in \mathbb{Q}$.

Similarly, setup 2 consists of,

- Pattern space E' at gradient $\frac{c'}{d'}$ (and associated lattice Λ').
- Strips S_1' and S_2' at gradients g_1' and g_2' respectively (relative to Λ').
- Gradients satisfying, $g_1' = a'g_2'$, with $a' = \frac{a_1'}{a_2'} \in \mathbb{Q}$.

For the diagrams produced by these setups to look identical they must have translate points on the same number of lines, at the same gradient and with the same rotation on each line.

By the previous parts, the gradients of the lines are the same if and only if the gradients of E and E' are the same, so we must have that,

$$\frac{c}{d} = \frac{c'}{d'} \,.$$

Recall that c and d (also c' and d') are coprime, so in fact we must have that $c = c'$ and $d = d'$. Note that this also means that the lattices Λ and Λ' will be the same.

We next look at the conditions required for the number of loops in which the translate points sit to also be the same. If the gradients are the same then lattices Λ and Λ' are the same, and the setups will give the same number of lines in the fundamental domain of Λ (Λ') if and only if,

$$|a_1 - a_2| = |a_1' - a_2'| \ \ .$$

This would of course give the same number of loops in the fundamental domain of L. However, as we have seen above, it is possible to have different numbers of lines in Λ giving the same number of loops in L. The following two corollaries to proposition 4.27 describe the circumstances in which this can happen.

Corollary 4.28. *If we have two tiling setups as above (specifically with E and E' at the same gradient) having,*

$$|a_1 - a_2| = n$$

$$c^2 + d^2 = m$$

with n and m coprime then if $|a_1' - a_2'| = n$ or nm we get that the number of loops containing translate points in the fundamental domain of L is the same for both tiling setups.

Proof. This is clear when $|a_1 - a_2| = |a_1' - a_2'|$. When $|a_1' - a_2'| = nm$ the result follows from proposition 4.27. $\square$

Corollary 4.29. *If we have,*

$$c^2 + d^2 = \beta k$$

$$|a_1 - a_2| = \alpha k$$

with k the highest common factor (so α and β coprime), then the second tiling setup will give the same number of loops in the fundamental domain of L provided that $|a_1' - a_2'|$ is equal to α, αk or $\alpha\beta k$.

Proof. Again, this follows from proposition 4.27. $\square$

All that remains now is to look at when the rotations are also the same. As mentioned before, the horizontal step between translate points in Λ coordinates is of length $\frac{1}{g_1 - g_2}$. Thus if both setups give the same number of lines in Λ we will have the same rotation provided,

$$\frac{1}{g_1 - g_2} = \frac{1}{g_1' - g_2'}\,.$$

We can also get rotations that differ by some integer number of full rotations if we have,

$$\frac{1}{g_1 - g_2} = \frac{1}{g_1' - g_2'} + z$$

for some $z \in \mathbb{Z}$.

Indeed, these results also hold when we have projection setups giving unequal numbers of lines in Λ but equal numbers of loops in L. This is because we will still cycle through all the loops in some order, so even though some of the lines get overlaid we still need the same horizontal step between translate points.

Proposition 4.30. *If we have,*

$$g_2' = \frac{a-1}{a'-1} g_2$$

then the horizontal steps of the two tilings are equal.

Proof. The horizontal steps (and therefore the rotations if everything else is equal) are the same if,

$$\frac{1}{g_1 - g_2} = \frac{1}{g_1' - g_2'}\,.$$

But since $g_1 = a g_2$ and $g_1' = a' g_2'$ we have that,

$$\begin{aligned} \frac{1}{g_1 - g_2} = \frac{1}{g_1' - g_2'} &\Leftrightarrow g_1 - g_2 = g_1' - g_2' \\ &\Leftrightarrow g_2(a-1) = g_2'(a'-1) \\ &\Leftrightarrow g_2' = \frac{a-1}{a'-1} g_2\,. \end{aligned}$$

$\square$

From the above results, we can conclude that it is possible to get two different tiling setups producing identical diagrams. This will be true if, for example, we have,

- $E = E'$.
- $|a_1 - a_2| = |a'_1 - a'_2|$.
- $g'_2 = \frac{a-1}{a'-1} g_2$.

Example 4.8. *For a specific example of when this can happen, consider the setup where* $E = E' = \frac{1}{3}$ *and we have in tiling scheme 1 that* $g_1 = \frac{1}{\sqrt{2}}$ *and* $g_2 = \frac{1}{2\sqrt{2}}$. *This gives us a value of 2 for* a, *so* $|a_1 - a_2| = 1$.

If we have $a' = \frac{5}{4}$, *then the number of lines of translate points in* Λ *produced by each setup is the same, since we have that* $|a'_1 - a'_2| = 1$.

For the rotations to also be equal we must have that,

$$g'_2 = \frac{a-1}{a'-1} g_2 = 4g_2 = \sqrt{2}\,.$$

Then of course we get,

$$g'_1 = a' g'_2 = \frac{5}{4}\sqrt{2}\,.$$

So if you have a tiling scheme where the gradients of the strips are $\frac{1}{\sqrt{2}}$ *and* $\frac{1}{2\sqrt{2}}$ *and another where the strips have gradients* $\sqrt{2}$ *and* $\frac{5}{4}\sqrt{2}$ *they will produce identical diagrams if the projection is onto the same pattern spaces (up to the lines of points being translated of course).*

The tilings produced by these two schemes are not the same. Figure 4.20 and figure 4.21 show patches of these two tilings. Note that here we have taken a patch from each tiling and drawn it both horizontally and vertically along the sides of the square, then combined to get a two dimensional diagram. The intention is to make it easier to see differences between the tilings.

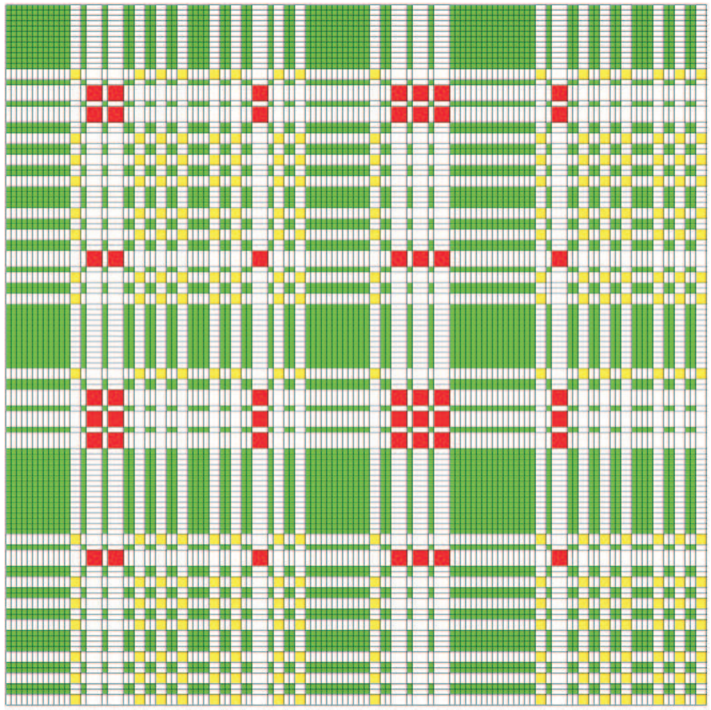

Figure 4.20: A patch of the tiling $T_{\left(\frac{9+10\sqrt{2}}{17},\frac{27+20\sqrt{2}}{71},\frac{1}{3}\right)}$.

The third number in the caption shows the gradient of the pattern space that we are projecting onto, the first two numbers denote the gradients of the two strips in normal coordinates, so these will give gradients $\frac{1}{\sqrt{2}}$ and $\frac{1}{2\sqrt{2}}$ in Λ-coordinates.

The patches already look quite different, but to see that these two tilings cannot be the same we look at the gradients of the strips involved.

Strip S_1 is at gradient $\frac{9+10\sqrt{2}}{17}$, which has a value between 1 and 2. Therefore the tiling T_1 associated to this strip will, due to the gradient of pattern space E being $\frac{1}{3}$, have prototiles of lengths "1" and "3" with the longer prototiles always appearing flanked by shorter prototiles and the shorter prototiles appearing in blocks of length 1 or 2.

Strip S_2 is at gradient $\frac{27+20\sqrt{2}}{71}$, which has a value between $\frac{1}{2}$ and 1. Tiling

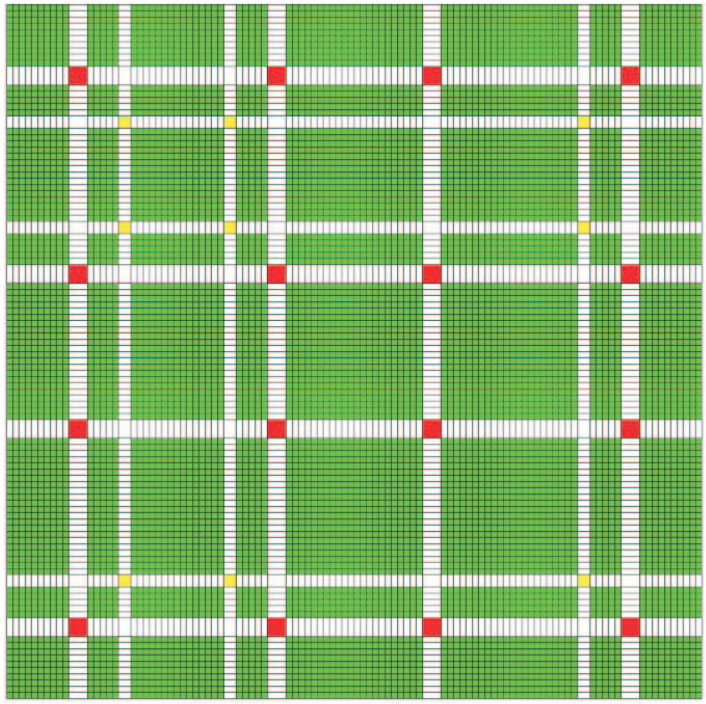

Figure 4.21: A patch of the tiling $T_{\left(\frac{99+100\sqrt{2}}{47},\frac{9+10\sqrt{2}}{7},\frac{1}{3}\right)}$.

T_2 will therefore have the same prototiles, but with the long prototiles appearing in blocks of length 1 or 2 and the shorter ones appearing in singles.

Since the longer tiles in T_1 only appear in singles we will not get a patch of two consecutive long tiles in the combined tiling. However, we can get patches that consist of a long tile followed by a short tile followed by another long tile (and indeed some of these can be seen in figure 4.20).

Looking at the other tiling we have strip S_1' at gradient $\frac{99+100\sqrt{2}}{47}$, which has a value between 5 and 6, and strip S_2' at gradient $\frac{9+10\sqrt{2}}{7}$, which is between 3 and 4. Thus in tiling T_1' we must have at least 5 short tiles between appearances of long tiles, and in T_2' we must have at least 3 short tiles between long tiles. Therefore we cannot have a "long-short-long" patch in the combined tiling.

5 Tiling Spaces

In this chapter we will investigate the spaces associated to rational 2:1 X-projections.

We will do this by first examining the standard 2:1 canonical case, then looking at an intermediate one-strip non-parallel case before finally moving onto rational 2:1 X-projections.

We start by recapping some definitions first given in chapter 2 (see also [14]).

Definition 2.12. Given two tilings U_1 and U_2 of $\mathbb{R}^n$ we define the distance between these two tilings, $d(U_1, U_2)$, to be equal to,

$$\inf \left\{ \{1\} \bigcup \left\{ \varepsilon : U_1 + s_1 = U_2 + s_2 \text{ on } B_{\frac{1}{\varepsilon}} \text{ with } s_1, s_2 \in \mathbb{R}^n, \|s_1\|, \|s_2\| < \frac{\varepsilon}{2} \right\} \right\}$$

where $B_{\frac{1}{\varepsilon}}$ denotes the ball of radius $\frac{1}{\varepsilon}$ centred at the origin.

Note that here $U + s$ is the tiling obtained by translating tiling U by vector s (or equivalently moving the origin by $-s$).

The metric here is defined on the set of all tilings of $\mathbb{R}^n$, though we will be interested in the 1-dimensional analogue of this definition, defined on the set of all 1-dimensional tilings (i.e., tilings of $\mathbb{R}$).

With this metric two tilings will be close if they agree up to a small translation on a large ball about the origin.

We can now look at the translates of a tiling and how far these are from the original tiling in the tiling metric.

Definition 2.13. The *orbit* of a tiling U of $\mathbb{R}^n$ is defined to be,

$$\mathcal{O}(U) = \{U + s : s \in \mathbb{R}^n\}.$$

That is, the set of all translates of the tiling U.

Definition 2.14. A *tiling space* Ω is a set of tilings that is closed under translation and complete in the tiling metric, i.e., if $U \in \Omega$ then $\mathcal{O}(U) \subset \Omega$, and every Cauchy sequence of tilings in Ω has a limit in Ω.

Definition 2.15. The *hull* or *orbit closure* Ω_U of a tiling U is the closure of $\mathcal{O}(U)$.

As above, we will be interested in tilings of $\mathbb{R}$, so the closure will be in the space of all 1-dimensional tilings.

The hull of a tiling U is the set of tilings that locally look like U. A tiling U' is in Ω_U if and only if every patch of U' is found in a translate of U.

We will of course be interested in the tiling spaces of projection tilings, in particular rational 2:1 X-projections, therefore we may also be interested in the space Ω', which is the set of all tilings given by allowed positions of intersection points, and any other translates of these tilings, completed as with Ω_U.

The spaces Ω_U and Ω' may not be the same in all cases. In fact, at this point it is not clear what their relationship is.

5.1 The Canonical 2:1 Case

We will begin by examining the standard 2:1 projection case. There follows a short recap of the canonical 2:1 projection setup (first defined in chapter 2).

Definition 2.8. A *canonical 2:1 projection scheme* is a cut-and-project scheme with lattice $L = \mathbb{Z}^2$ and acceptance domain K being a closed interval, where the width of this interval, and therefore the strip that lattice points are projected from, is taken to be equal to the projection of a unit square onto $E^{\perp}$. In addition, the acceptance domain K is chosen so that the boundaries of the strip do not intersect any points of L.

So in a strip S of *canonical width*, the point (α, β) (for $\alpha, \beta \in \mathbb{R}$) is on the lower boundary of S if and only if the point $(\alpha - 1, \beta + 1)$ is on the upper boundary.

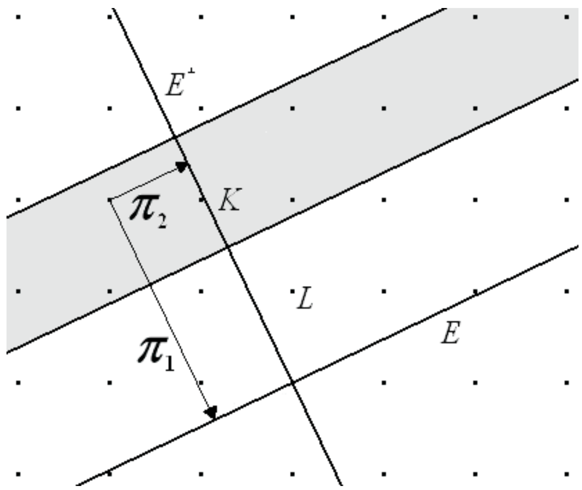

Figure 5.1: A canonical 2:1 projection scheme.

Note that the strip must have an irrational gradient. This is because a projection with rational gradient would not result in $\pi_1|_L$ being injective nor would $\pi_2(L)$ be dense in $E^\perp$.

As in the previous chapter with rational 2:1 X-projections, we will look at the ways in which we can position the strip within the fundamental domain of L and the tilings these different positions will give us (note that as in definition 4.4 when we refer to the fundamental domain of L we mean the unit square with vertices $(0,0),(1,0),(0,1)$ and $(1,1)$).

Definition 5.1. Given a canonical 2:1 projection scheme with strip S producing a tiling T with origin O we say that the point t, within the fundamental domain of L, on the lower boundary of strip S satisfying $\pi(t) = O$ is the point corresponding to tiling T. When we talk about a strip being placed at a point in the fundamental domain of L then this point will be the point on the lower boundary of the strip that projects to the origin in the tiling produced by our projection scheme.

In the definition of a canonical 2:1 projection scheme we require that the strip to be positioned so that its boundaries do not pass through any points of the lattice L. Thus for a given setup there are certain points in the fundamental domain of L at which the strip cannot be placed.

In a similar way, if we start with a tiling, with a corresponding point in the fundamental domain of L, then we can look at the translates of this tiling, which will be produced by effectively sliding the corresponding point along the strip.

We thus get the sets of translates and forbidden points shown in figure 5.2 in the case of a standard 2:1 projection.

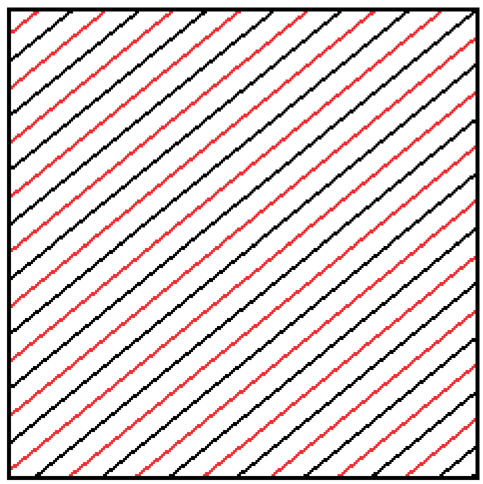

Figure 5.2: The canonical 2:1 projection case.

Here, the black line shows the points at which the strip cannot be positioned, and the red line shows the points that correspond to translates of our tiling. Note that these lines are actually dense.

We will make a distinction between points through which the boundaries of the strip can run and points that are forbidden.

Definition 5.2. For a given 2:1 projection setup we say that a point (x, y) in the fundamental domain of L is *singular* if positioning the strip with (x, y) on the boundary results in the boundary of the strip passing through a point of lattice L.

Equivalently, the point (x, y) in the fundamental domain of L is singular if there exists a point $(m, n) \in L$ such that,

$$(m, n) = (x, y) + r(1, g)$$

for some $r \in \mathbb{R}$, where g is the gradient of the strip.

All other points in the fundamental domain of L (i.e. the points through which the edges of the strip may run without intersecting a lattice point) are called *non-singular*.

All the translate points of a tiling are of course non-singular, but not all non-singular points will correspond to translates of the original tiling.

Since we only have one strip of canonical width in this setup, we only get a single line of singular points in the fundamental domain of L, and similarly the translates of a given tiling appear in a single line. Both of these lines are at the gradient of the strip, which is irrational, and therefore wind round the fundamental domain of L. Thus, as mentioned above, they appear as dense sets of lines in the fundamental domain.

Proposition 5.1. *(see [4]) There is a continuous map, f, from the tiling space of a canonical 2:1 projection to a 2-torus satisfying,*

- *f is one-to-one over non-singular points.*
- *f is two-to-one over singular points.*

From this we know that if we have a convergent sequence of non-singular points within the fundamental domain of L with non-singular limit point x then we get a corresponding sequence of tilings converging to the tiling that corresponds to point x.

However, if we have two sequences of non-singular points in the fundamental domain of L converging to the same singular point, y, but from opposite sides then these sequences will have different limits. This is because a strip with point y on the lower boundary will have some lattice points, (m,n) and $(m-1,n+1)$ on its boundaries, therefore a sequence converging to y from one side will give tilings containing the point $\pi(m,n)$ but not the point $\pi(m-1,n+1)$ after a certain stage, resulting in $\pi(m,n)$ appearing in the limit, whereas a sequence converging to y from the other side will give a tiling at the limit that contains $\pi(m-1,n+1)$ but not $\pi(m,n)$.

Thus the line of singular points in the fundamental domain of L corresponds to a double line in the tiling space.

5.2 One-Strip Non-Parallel Projections

We will now look at the case where we have a single strip at canonical width (still at irrational gradient) but we project onto a line at a positive, finite rational gradient (not equal to 1). This is an intermediate step between standard 2:1 projections and rational 2:1 X-projections.

The tiling produced by changing the gradient of the pattern space (and therefore altering the projection), but leaving the strip unchanged is combinatorially the same as in the standard case, with the alteration to the projection only affecting the lengths of the two prototiles.

If we have a standard 2:1 projection tiling T having prototiles t_1 and t_2 we will get a corresponding one-strip non-parallel projection tiling T' having prototiles t_1' and t_2' with the origin in T' being at the equivalent point in the prototile corresponding to the prototile over the origin in T. For example, if the origin in T is at the midpoint of a t_1 tile then the origin in T' will be at the midpoint of a t_1' tile. Of course, we can also do this in the other direction to get the standard 2:1 projection tiling corresponding to a one-strip non-parallel projection tiling.

Thus we have a bijection between the set of tilings generated by a standard

2:1 projection scheme and the corresponding tilings obtained by projecting onto a line at rational gradient rather than one that is parallel to the strip.

Proposition 5.2. *Two standard projection tilings T_1 and T_2 are close in the tiling metric if and only if their corresponding one-strip non-parallel projection tilings T_1' and T_2' are close in the tiling metric.*

Proof. We have four tilings, T_1, T_2, T_1' and T_2'. The tilings T_1 and T_2 are made up of prototiles t_1 and t_2 and the tilings T_1' and T_2' have prototiles t_1' and t_2'. Altering the line that we are projecting onto changes the lengths of the prototiles, and will have the effect of lengthening one and shortening the other. So let us assume that,

$$|t_1'| = \alpha|t_1|$$

$$|t_2'| = \beta|t_2|$$

with $\alpha > 1$ and $\beta < 1$.

Now, if T_1 and T_2 are within distance ε of each other then after some translate of up to distance ε all the points within $\frac{1}{\varepsilon}$ of the origin of tilings T_1 and T_2 coincide.

Thus all the corresponding points within tilings T_1' and T_2' will coincide after some translate of distance less than $\alpha\varepsilon$, since the distances between points can be scaled by at most α.

However, the radius of the patch containing these points may also vary. The patch in tilings T_1' and T_2' is made up of t_1' and t_2' tiles rather than t_1 and t_2 tiles, and therefore has a minimum possible radius of $\frac{\beta}{\varepsilon}$. So the tilings T_1' and T_2' coincide on a ball of radius $\frac{\beta}{\varepsilon}$ after a translate of up to $\alpha\varepsilon$.

Now, if $\beta \geq \frac{1}{\alpha}$ then after a translate of up to $\alpha\varepsilon$ the tilings T_1' and T_2' must coincide on a ball of radius $\frac{\beta}{\varepsilon} \geq \frac{1}{\alpha\varepsilon}$ about the origin, and hence the two tilings are within distance $\alpha\varepsilon$ in the tiling metric.

If $\beta < \frac{1}{\alpha}$ then after a translate of up to $\alpha\varepsilon < \frac{\varepsilon}{\beta}$ the tilings T_1' and T_2' coincide on a ball of radius $\frac{\beta}{\varepsilon}$ about the origin. Thus the two tilings are within distance $\frac{\varepsilon}{\beta}$ in the tiling metric.

So if we have standard projection tilings T_1 and T_2 that are close in the tiling metric then their corresponding one-strip non-parallel projection tilings T_1' and T_2' must be close in the tiling metric. The proof of the converse is similar. $\square$

From this we can see that if we have a standard 2:1 projection tiling T then there is a homeomorphism between Ω_T and $\Omega_{T'}$, where T' is the tiling corresponding to T but with the projection onto a pattern space at some rational gradient.

Therefore, as with the standard 2:1 projection case, we will have a continuous map from $\Omega_{T'}$ to the 2-torus that is one-to-one on non-singular points and two-to-one on singular points. As with the standard case, the singular points of a one-strip non-parallel projection tiling appear in a single line at irrational gradient winding round the fundamental domain of L. This line is the same as for the corresponding standard 2:1 projection tiling, since forbidden points are determined by the strip and are independent of the projection. Also, as with the standard 2:1 projection, the line of forbidden points will be a double line in the hull of a 2:1 one-strip non-parallel projection tiling.

5.3 Rational 2:1 X-Projections

We now move on to looking at tilings generated by rational 2:1 X-projection schemes. The pattern of points given by such a tiling scheme is a combination of the point patterns given by two projections of the type shown above, with strips at different gradients but the same pattern space.

We will first recap the possible positions of the intersection point of the two strips. That is, the points in the fundamental domain of L at which the point at the intersection of the lower boundaries of the two strips may be positioned. As explained in chapter 4, for a given rational 2:1 X-projection scheme, we will get a corresponding tiling for every allowed choice of intersection point t, namely the tiling with $\pi(t)$ at the origin and endpoints of tiles being the projections of the lattice points from within the two strips.

Note that the "allowed" points are the points in the fundamental domain of L at which the intersection point can be positioned without any points of the lattice L appearing on the boundaries of either strip. We will be using the terms singular and non-singular to describe points in the fundamental domain of L once again in this section.

Definition 5.3. Given a rational 2:1 X-projection scheme with strips S_1 and S_2 at gradients g_1 and g_2 respectively we say that a point (x, y) in the fundamental domain of L is *singular* if positioning the strips so that the intersection point of their lower boundaries is at (x, y) results in the lower boundary of either strip passing through a point of the lattice L.

Equivalently, the point (x, y) in the fundamental domain of L is singular if there exists some lattice point $(m, n) \in L$ satisfying,

$$(m, n) = (x, y) + r(1, g_1)$$

or,

$$(m, n) = (x, y) + r(1, g_2)$$

for some $r \in \mathbb{R}$.

As with the single-strip construction, all other points in the fundamental domain of L (i.e. all points at which the intersection point of the two strips may be positioned without a lattice point appearing on the boundaries of either strip) are called *non-singular*.

A rational 2:1 X-projection scheme is the combination of two one-strip non-parallel projection schemes as described in the previous section. Thus a rational 2:1 X-projection scheme with strips S_1 and S_2 will have a set of singular points that is the union of the sets of singular points associated with the strips S_1 and S_2, since a point in the fundamental domain of L is a singular point of the X-projection scheme if it is a singular point for either S_1 or S_2.

Thus the set of singular points in the fundamental domain of L will look like two lines at gradients g_1 and g_2 (the gradients of the two strips) winding round the fundamental domain. Note that since the strips have irrational gradients this will be a pair of dense lines, as shown in figure 5.3.

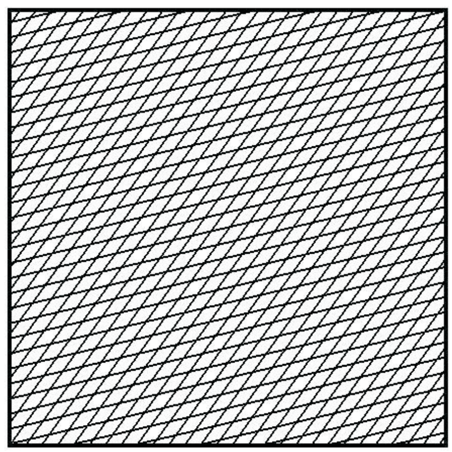

Figure 5.3: The singular points in the fundamental domain of L.

In a similar way, the set of non-singular points in the fundamental domain of L is the intersection of the sets of non-singular points associated to the strips S_1 and S_2. This is a dense set of points in the fundamental domain of L, so any point in the fundamental domain can be expressed as the limit of a convergent sequence of such points.

We will now look at sequences of non-singular points and their corresponding tilings.

Proposition 5.3. *A convergent sequence of non-singular points (for a given X-*

projection scheme) converging to a non-singular point u gives a corresponding convergent sequence of tilings whose limit is the tiling corresponding to the point u.

Proof. This follows from the one-strip non-parallel projection case. The sequence of non-singular points for the X-projection scheme gives two convergent sequences of one-strip non-parallel projection tilings, converging to the two tilings associated to the point u. Therefore the sequence of X-projection tilings will converge to the tiling that has tiles with endpoints given by the union of the endpoints from these two tilings. That is, the X-projection tiling corresponding to the point u. $\square$

Looking at the above proposition, we might expect to get a corresponding result about convergent sequences of non-singular points with limits that are singular. In particular, we might expect to get "double points" at some singular points (where the placement of the intersection point results in a lattice point appearing on the boundary of one strip but not the other) and "quadruple points" at the other singular points (where the placement of the intersection point results in lattice points appearing on the boundaries of both strips). That is, we might expect points at the intersection of the black lines to have four corresponding points in the tiling space (the space Ω'), and points that are only on one black line to have two corresponding points in the tiling space. However, the situation is slightly more complicated than this.

Theorem 5.4. *A singular point in the fundamental domain of L that is not at the intersection of the lines of singular points associated to the strips S_1 and S_2 can correspond to a single point or a double point in the tiling space.*

Proof. Say that we have such a point v on the line of singular points associated to strip S_1, but a non-singular point for strip S_2. Thus the placement of S_1 at this point results in some lattice point (m, n) appearing on the lower boundary of S_1 and the lattice point $(m-1, n+1)$ appearing on the upper boundary (since S_1 has canonical width).

Since v is non-singular for the strip S_2, having the lower boundary of S_2 run through this point does not result in the boundaries of S_2 passing through any points of L. So if we take any convergent sequence of points in the fundamental domain of L with limit v then placing the strip S_2 at these points will give a corresponding sequence of tilings converging to the tiling that corresponds to the point v. We will denote by P_2 the point set corresponding to this tiling (i.e. the set of endpoints of the tiles).

As before, approaching this point from different sides will give different limits for the sequences of tilings generated by the projection of points in translates of the strip S_1. In particular, approaching from one direction will result in the point $\pi(m, n)$ appearing in P_1 (the point set at the limit of the sequence), while the point $\pi(m-1, n+1)$ will not appear in P_1, and approaching from the other direction will result in $\pi(m-1, n+1)$ appearing in the limit (which we will call P_1') whilst $\pi(m, n)$ does not appear in P_1'.

Now, the point set that is the limit of the corresponding sequences of X-projection tilings is the union of the points above, so for tilings approaching from one side we will get $P_1 \bigcup P_2$ and from the other we will have $P_1' \bigcup P_2$. The strip S_2 may contain a lattice point with the same projection as (m, n) or $(m-1, n+1)$, or both, so the point set P_2 may contain the points $\pi(m, n)$ or $\pi(m-1, n+1)$, or both of these. For example, if the pattern space E is at gradient $\frac{1}{2}$ then the points $(0,0)$ and $(-1,2)$ project to the same point on E, and whilst a canonical width strip cannot have both the points $(0,1)$ and $(-1,2)$ in its interior it may have both $(0,0)$ and $(0,1)$, which have the same projections as $(0,1)$ and $(-1,2)$.

If P_2 contains both $\pi(m, n)$ and $\pi(m-1, n+1)$ then $P_1 \bigcup P_2$ and $P_1' \bigcup P_2$ will be the same, and v will only correspond to a single point in the tiling space. Otherwise, the limits will differ, with at least one of $\pi(m, n)$ and $\pi(m-1, n+1)$

failing to appear in one of $P_1 \bigcup P_2$ and $P_1' \bigcup P_2$ but appearing in the other, resulting in v corresponding to a double point in the tiling space. $\square$

We now look at the different cases that arise when we examine sequences of points converging to a point that is singular for both strips in a rational 2:1 X-projection.

Here there will be four different ways in which a sequence of points might approach such a limit point, and placing the strips S_1 and S_2 with such a point on their lower boundaries will result in lattice points (m, n) and $(m - 1, n + 1)$ appearing on the boundaries of S_1, as well as lattice points (m', n') and $(m' - 1, n' + 1)$ appearing on the boundaries of S_2. Figure 5.4 shows the situation that we have, with the lower boundaries of the strips intersecting at the singular point v. Here the different directions from which the point may be approached by sequences of non-singular points are labeled with numbers 1, 2, 3 and 4.

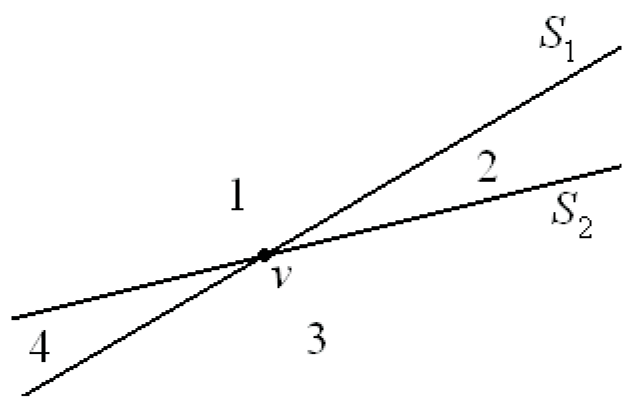

Figure 5.4: The different directions from which a singular point can be approached.

This potentially gives four different cases, depending on which of the points $\pi(m, n)$, $\pi(m - 1, n + 1)$, $\pi(m', n')$ and $\pi(m' - 1, n' + 1)$ can be found in the patterns P_1^i and P_2^i that are the limits of the point patterns associated to strips S_1 and S_2 respectively as we approach the point v from direction i.

As before, approaching v from above strip S_1 will result in $\pi(m-1, n+1)$ appearing in P_1 while $\pi(m,n)$ does not, with a similar result for S_2. The following tables show the points contained in each P_j^i.

	$\pi(m,n)$	$\pi(m-1,n+1)$
P_1^1	no	yes
P_1^2	yes	no
P_1^3	yes	no
P_1^4	no	yes

	$\pi(m,n)$	$\pi(m-1,n+1)$
P_2^1	no	yes
P_2^2	no	yes
P_2^3	yes	no
P_2^4	yes	no

Thus we seem to get four distinct limits and therefore a quadruple point. However, as with the double point case above, the number of distinct limits will depend on whether P_2 contains either the point $\pi(m,n)$ or $\pi(m-1,n+1)$, and whether P_1 contains either $\pi(m',n')$ or $\pi(m'-1,n'+1)$.

Definition 5.4. We will denote by Q^i the point pattern that is given by the union of the patterns P_1^i and P_2^i. This is the pattern of points that form the endpoints of the tiles in the tiling at the limit of a sequence of tilings corresponding to non-singular points converging to v.

Theorem 5.5. *If the points $\pi(m,n)$, $\pi(m-1,n+1)$, $\pi(m',n')$ and $\pi(m'-1,n'+1)$ (as defined above) are all distinct then a point in the fundamental domain of L that is a singular point for both tilings T_i corresponds to a quadruple point in* *the tiling space if exactly one of the following, or any pair except for numbers 1 and 2 or numbers 3 and 4, holds:*

1. $\pi(m, n) \in P_2$

2. $\pi(m-1, n+1) \in P_2$

3. $\pi(m', n') \in P_1$

4. $\pi(m'-1, n'+1) \in P_1$

5. $\pi(m, n), \pi(m-1, n+1) \notin P_2$ *and* $\pi(m', n'), \pi(m'-1, n'+1) \notin P_1$.

In addition, we get a double point if statements 1 and 2, statements 3 and 4, or any three of the first four statements hold, and a single point if all of the first four statements are true.

Proof. The relevant points contained in the combined limit point patterns are shown in the following table.

	$\pi(m, n)$	$\pi(m-1, n+1)$	$\pi(m', n')$	$\pi(m'-1, n'+1)$
Q^1	no	yes	no	yes
Q^2	yes	no	no	yes
Q^3	yes	no	yes	no
Q^4	no	yes	yes	no

If any point from the top row of the table is contained in the other point pattern (for example if $\pi(m, n) \in P_2$) then the corresponding column can be ignored. Ignoring either no columns or any single column results in different limits for each Q^i, as does covering any pair of columns with the exceptions of both columns 1 and 2 or both columns 3 and 4.

Covering columns 1 and 2 or columns 3 and 4, or any three of the four columns results in two different possible limits, and if all the points are contained in every Q^i then they will all be the same. $\square$

It is of course also possible that we will be in a situation where (m, n) and (m', n') project to the same point on E (or a similar result holds for some other pair of the relevant lattice points). The possible implications of this are explored in the following propositions.

Proposition 5.6. *If we have a point in the fundamental domain of L that is a singular point for both constituent tilings T_i and if the lattice points we get on the lower boundaries of the strips S_i satisfy,*

$$\pi(m, n) = \pi(m', n')$$

then this point corresponds to a triple point in the tiling space.

Proof. If the projections of (m, n) and (m', n') are equal then of course the projections of $(m-1, n+1)$ and $(m'-1, n'+1)$ are also equal. Thus for our different cases we combine columns 1 and 3, and columns 2 and 4 from the table in the proof of theorem 5.5.

	$\pi(m, n)$	$\pi(m-1, n+1)$
Q^1	no	yes
Q^2	yes	yes
Q^3	yes	no
Q^4	yes	yes

Note that none of the P_j^i can contain any other lattice points that project to $\pi(m, n)$ or $\pi(m-1, n+1)$, since this would require the strips to have greater than canonical width. Therefore the three possible limits shown above are all the different limits, and the point corresponds to a triple point in the tiling space. $\square$

The final case to be examined is when,

$$\pi(m,n) = \pi(m'-1, n'+1)$$

or,

$$\pi(m',n') = \pi(m-1, n+1)\,.$$

Proposition 5.7. *If we have a point in the fundamental domain of L that is a singular point for both constituent tilings T_i and if this point gives lattice points (m,n) and (m',n') on the lower boundaries of the strips S_i in such a position that,*

$$\pi(m,n) = \pi(m'-1, n'+1)$$

or,

$$\pi(m',n') = \pi(m-1, n+1)$$

then this point can correspond to a quadruple, triple or double point in the tiling space.

Proof. Assume that we are in the case where $\pi(m,n)$ and $\pi(m'-1,n'+1)$ are equal. The other case, where $\pi(m',n')$ and $\pi(m-1,n+1)$ are the same, is similar.

Note that we cannot also have,

$$\pi(m',n') = \pi(m-1, n+1)$$

since we would then get,

$$\pi(m',n') = \pi(m-1, n+1)$$

$$= \pi(m,n) + \pi(-1,1)$$

$$= \pi(m'-1,n'+1) + \pi(-1,1)$$

$$= \pi(m',n') + \pi(-1,1) + \pi(-1,1)$$

$$= \pi(m',n') + 2\pi(-1,1)\,.$$

This means that $(-1,1)$ would have to have the same projection as $(0,0)$, which cannot happen unless the pattern space E has gradient 1, and this case was disallowed in the definition.

However, it is still possible that $\pi(m',n')$ is in P_1 and/or $\pi(m-1,n+1)$ is in P_2. As with the proof of the previous proposition we can combine columns from the table in the proof of theorem 5.5 to get,

	$\pi(m,n)$	$\pi(m-1,n+1)$	$\pi(m',n')$
Q^1	yes	yes	no
Q^2	yes	no	no
Q^3	yes	no	yes
Q^4	no	yes	yes

As before this gives four possible limits, and therefore a quadruple point, unless $\pi(m',n')$ is in P_1 or $\pi(m-1,n+1)$ is in P_2. If exactly one of these conditions holds then we can ignore either column 2 or column 3 of the table and we will have a triple point, if both hold then we ignore both of these columns and we get a double point. $\square$

Now, in theorem 5.4 and theorem 5.5 we found that a point that is a singular point for one strip will give a double point and a point that is a singular point for both strips will give a quadruple point (when the points $\pi(m,n), \pi(m-1,n+1), \pi(m',n')$ and $\pi(m'-1,n'+1)$ are all distinct) unless we are in the situation where S_1 contains lattice points with projections equal to $\pi(m', n')$ and $\pi(m'-1, n'+1)$ and/or S_2 contains lattice points with projections equal to $\pi(m, n)$ and $\pi(m-1, n+1)$.

Note that when S_1 or S_2 (or both) contains only one such point then we still get double/quadruple points as normal. We will therefore now look at when it is possible for S_1 (or S_2) to contain both the required points.

Proposition 5.8. *If we have a one-strip non-parallel 2:1 projection onto pattern space E, at gradient $\frac{a}{b}$, with a and b coprime and $a, b > 1$ then a canonical width strip S with positive gradient cannot contain any pair of lattice points with projections equal to $\pi(m, n)$ and $\pi(m-1, n+1)$ in its interior, for any $(m, n) \in \mathbb{Z}^2$.*

Proof. We will label the points (m, n) and $(m-1, n+1)$ as $(0, 0)$ and $(-1, 1)$ and assume that the point $(-1, 1)$ is contained in S. The proof will be similar with S containing $(0, 0)$, or any other lattice point with the relevant projection onto E.

If S contains the point $(-1, 1)$, then it cannot also contain $(0, 0)$, since S has canonical width. However, since the projection is onto a line at rational gradient there will be other points of the lattice L (that is, $\mathbb{Z}^2$) that project to the same point as $(0, 0)$. In fact, the set of points with the same projection as $(0, 0)$ is $\{z(-b, a) : z \in \mathbb{Z}\}$.

Since S has positive gradient and does not contain the point $(0, 0)$ it will also not contain the point $(b, -a)$. Therefore the point that we will be interested in is $(-b, a)$. Now we look at the two cases, namely the case where a or b is equal to 1, and the case where $a, b > 1$. We examine these two cases below.

1. If a or b is equal to 1 then it is possible for S to contain both $(-1, 1)$ and $(-b, a)$, since $(-b, a)$ is a point of the form $(-1, a)$ or $(-b, 1)$, and the strip S could have an arbitrarily large number of horizontal or vertical steps extending from the point $(-1, 1)$ (depending on the gradient of S).

2. If $a, b > 1$ and S contains the points $(-1, 1)$ and $(-b, a)$ then S must contain the step $(b-1, 1-a)$, which is of the form $(c, -d)$, for both c and d greater than or equal to 1, requiring S to either have negative gradient or greater than canonical width.

Thus when we have a pattern space E that is not at gradient n or $\frac{1}{n}$ (for $n \in \mathbb{N}$) we find that the point pattern corresponding to a canonical width strip cannot contain both $\pi(m, n)$ and $\pi(m-1, n+1)$ for any lattice point (m, n). $\square$

From the above propositions we can conclude that when we have a rational 2:1 X-projection scheme with a pattern space E that is not at gradient k or $\frac{1}{k}$ (for $k \in \mathbb{N}$) then the singular points in the fundamental domain of L correspond to double points in the tiling space Ω' if they are singular for only one of the two strips, and quadruple points if they are singular for both strips.

Figures 5.5 and 5.6 combine all the results about singular points in the fundamental domain of L given above. As before, we're looking at a rational 2:1 X-projection scheme with projection onto a line E at positive rational gradient (not equal to 1), the two strips being called S_1 and S_2, and the pattern of points in E given by the projections of the lattice points within strip S_i being denoted by P_i.

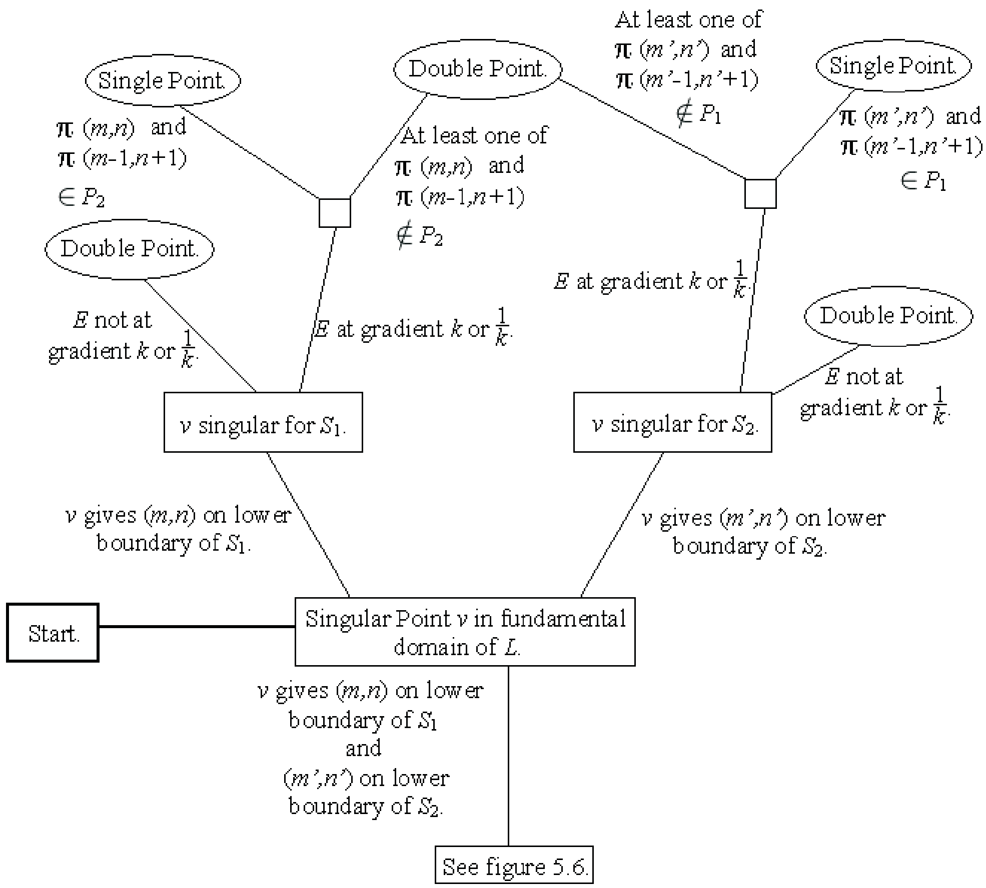

Figure 5.5: The cases for points that are singular for only one of the strips.

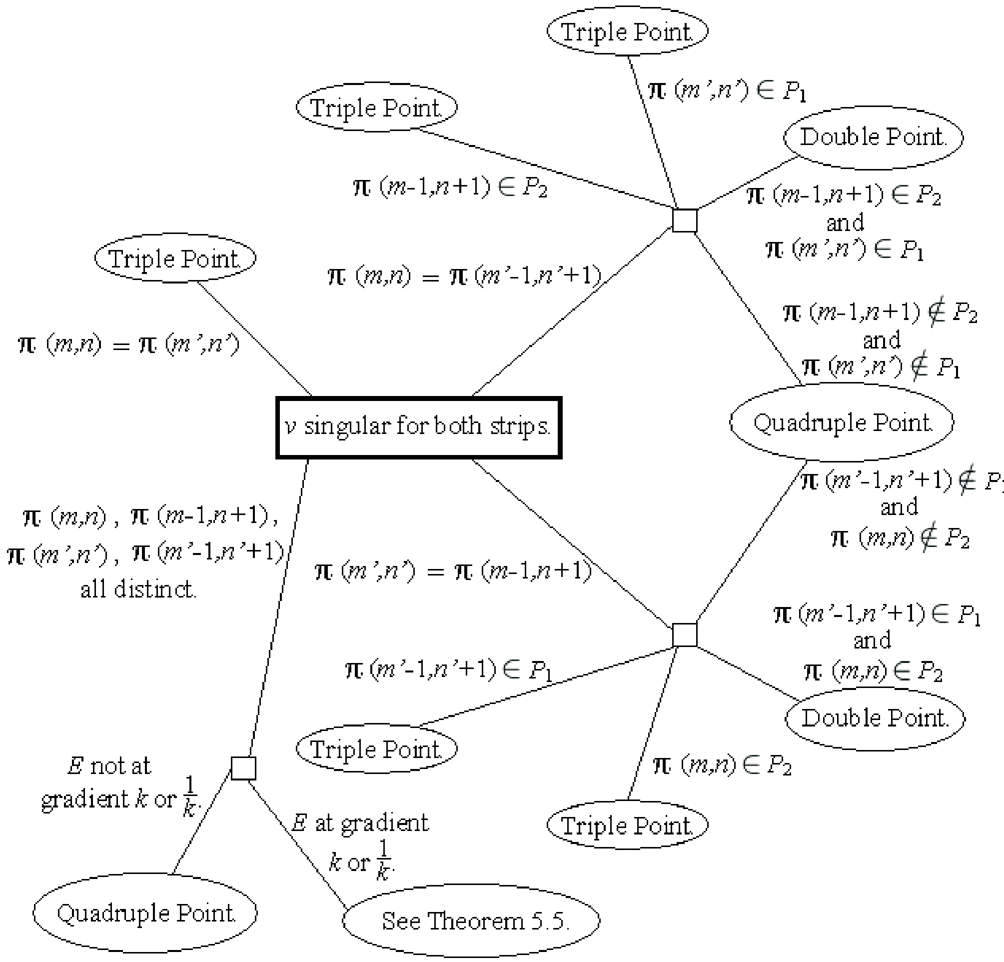

Figure 5.6: The cases for points that are singular for both strips.

5.4 The Space Ω_U

We have so far effectively been looking at the space Ω' for a rational 2:1 X-projection, by looking at all the points in the fundamental domain of L at which the intersection point of the lower boundaries of the two strips can be placed. In this section we will examine the hull of a rational 2:1 X-projection tiling, Ω_U, by looking at the points in the fundamental domain of L that correspond to translates of a given tiling U and limits of convergent sequences of such points.

As in chapter 4, we will be referring to the translate points of our tiling U (see definition 4.10), where a translate point for U is a point in the fundamental domain of L at which the intersection point of the lower boundaries of the two strips can be placed to give a translate of the original tiling.

As we also saw in chapter 4, the set of translate points for a tiling generated by a given X-projection scheme depends on the relationship between the gradients of the two strips relative to the sublattice Λ (and is therefore also dependent on the gradient of the pattern space E). Varying the relationship between the gradients of the two strips relative to Λ gives several different possibilities for the pattern of translate points (see section 4.5), which we summarise below.

1. The gradients of the two strips are rationally related relative to Λ.

   This results in the translate points being dense in a finite number of lines that are parallel to E (and therefore have strictly positive gradient).

2. The gradients of the two strips are irrationally related relative to Λ, but differ by a rational amount.

   Again, this gives sets of translate points that are dense in a finite number of lines, however this time the lines are perpendicular to E (and therefore have strictly negative gradients).

3. The gradients of the two strips are irrationally related relative to Λ and differ by an irrational amount, but we have that,

$$[qx] = c[x] + r$$

for $c, r \in \mathbb{Q}$, where $[.]$ denotes fractional part (see chapter 4).

Once again the translate points are dense in a finite number of parallel lines, though the gradient of these lines in not equal to that of E.

4. The gradients of the two strips are irrationally related relative to Λ, differ by an irrational amount and the quantity $[qx]$ cannot be expressed as $c[x] + r$ with $c, r \in \mathbb{Q}$.

   Here we get that the translate points are dense in the fundamental domain of L.

We will now look at these separate cases, beginning with case 4.

5.4.1 Dense Set of Translate Points

The case where the set of translate points of our rational 2:1 X-projection tiling U is dense in the fundamental domain of L is similar to the previous section.

Every non-singular point u in L will be the limit of some sequence of translate points of U, and therefore the tiling corresponding to u will be a limit of some convergent sequence of tilings that are translates of U. Thus Ω_U will contain every tiling that corresponds to a non-singular point in the fundamental domain of L.

The singular points in L will have corresponding single, double, triple or quadruple points in Ω_U, as described in the previous section, since these are all limits of translate points of U from every direction.

In addition, there should also be more translates of the tiling U that are not represented by points in the fundamental domain of L, since the translate points in the fundamental domain correspond to translates of U by integer multiples of some fixed distance. This should give a line segment of translates of the tiling passing through each translate point (and therefore also each non-singular point) with the end of one segment connected to the start of the segment that runs through the next translate point. At singular points we would expect to see potentially double, triple or quadruple line segments in the same way.

5.4.2 Translate Points Dense on Finite Set of Lines

We now look at the cases where the translate points of our tiling U are dense on a finite set of lines in the fundamental domain of L. This covers tilings of types 1, 2 and 3 above.

As already explained, each of the three types of tiling gives translate points that are dense on some finite set of lines in the fundamental domain of L. Thus any non-singular point, u, on any of these lines will be the limit of some sequence of translates of the tiling U, and so the tiling corresponding to u will be in Ω_U. Non-singular points that do not lie on the lines are of course not at the limits of any sequences of the translates that lie on the lines. However, in most cases we have not proved that the translate points on the line are the only non-singular points in the fundamental domain of L that correspond to translates of the tiling. This is discussed further below.

All three types of tiling give similar sets of translate points, but there are slight differences when we come to look at singular points on the lines on which all these translate points lie.

A singular point on the line of translate points cannot correspond to a triple or quadruple point, since it can only be approached from two directions (i.e. from either direction along the line). However, such a point does not necessarily have to be a double point.

Proposition 5.9. *For a tiling U of type 2, i.e., a tiling where the gradients of S_1 and S_2 are irrationally related relative to Λ but differ by a rational amount, all singular points on the lines of translate points correspond to double points in* *the tiling space provided the pattern space E has gradient not equal to k or $\frac{1}{k}$ (for $k \in \mathbb{N}$).*

Proof. If the projection is onto a pattern space E that is not at gradient k or $\frac{1}{k}$ then all points that are singular for one of the strips, and all points that are singular for both but that result in $\pi(m,n)$, $\pi(m-1,n+1)$, $\pi(m',n')$ and $\pi(m'-1,n'+1)$ being distinct must be double and quadruple points in Ω' respectively (see proposition 5.8).

The remaining two types of singular point are those points where,

$$\pi(m,n) = \pi(m',n')$$

or,

$$\pi(m,n) = \pi(m'-1,n'+1)\,.$$

However, note that for a type 2 tiling the translate points appear in lines that are perpendicular to the pattern space E, and thus have negative gradient. Therefore the singular points can only be approached from directions 1 or 3.

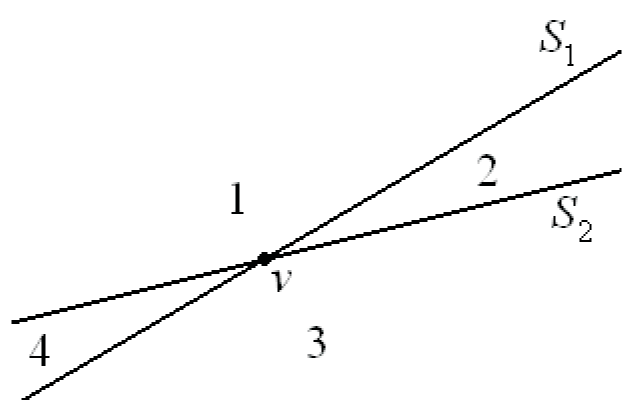

Figure 5.7: The directions from which a singular point can be approached.

Thus the limits are Q^1 and Q^3, and if you look at the tables in the proofs of propositions 5.6 and 5.7 it is clear that these limits are always distinct. ☐

Whilst this result will hold for all tilings of type 2, the tilings of types 1 and 3 may have translate points contained in lines that approach singular points from directions 2 and 4 in the above diagram.

For the type 1 tilings, the translate points are always contained in lines parallel to E, so singular points on these lines will be approached from directions 1 and 3 if g_1 and g_2, the gradients of the two strips relative to Λ, are either both positive or both negative. Otherwise the singular points on the lines will be approached from directions 2 and 4, with the results described in the following proposition.

Proposition 5.10. *If we have pattern space E not at gradient k or $\frac{1}{k}$ (for $k \in \mathbb{N}$) and a tiling of type 1 (U) with strips S_1 and S_2 at gradients g_1 and g_2 respectively, relative to Λ, then:*

- *If g_1 and g_2 are either both positive or both negative all the singular points on the lines of translate points correspond to double points in Ω_U.*

- *If one of g_1 and g_2 is positive and the other negative then any singular points on the lines of translate points that are singular for both strips and satisfy,*

$$\pi(m,n) = \pi(m',n')$$

  correspond to single points in Ω_U.

- *If one of g_1 and g_2 is positive and the other negative then any singular points on the lines of translate points that are singular for both strips and satisfy,*

$$\pi(m,n) = \pi(m'-1,n'+1)$$

correspond to double points in Ω_U *unless at least one of the following holds,*

$$\pi(m,n) \in P_2$$

$$\pi(m'-1,n'+1) \in P_1 \, .$$

With a similar result for the case where,

$$\pi(m',n') = \pi(m-1,n+1) \, .$$

Proof. As in the previous proposition, if the projection is onto a pattern space E that is not at gradient k or $\frac{1}{k}$ then all points that are singular for one of the strips, and all points that are singular for both but that result in $\pi(m,n)$, $\pi(m-1,n+1)$, $\pi(m',n')$ and $\pi(m'-1,n'+1)$ being distinct must be double and quadruple points in Ω' respectively (again, see proposition 5.8).

If g_1 and g_2 are either both positive or both negative then all singular points are approached by translate points from directions 1 and 3, so we are in the same situation as we were with tilings of type 2, thus all singular points correspond to double points in Ω_U.

If one of g_1 and g_2 is positive and the other is negative, then the singular points are approached by translate points from directions 2 and 4, so we are interested in limits Q^2 and Q^4 and the results follow from examining the tables in the proofs of propositions 5.6 and 5.7. $\square$

As was mentioned above, in most cases there may be more translate points in the fundamental domain of L for a tiling U than those already given, however in corollary 4.12 we saw that these translate points are indeed all the translate points in the fundamental domain of L when we have a type 1 tiling with,

- Pattern space E not at gradient k or $\frac{1}{k}$ (for $k \in \mathbb{N}$).

- Strips S_1 and S_2 at gradients g_1 and g_2, relative to Λ, satisfying

$$g_1 = \frac{a}{b} g_2$$

for $a \in \mathbb{Z}$, $b \in \mathbb{N}$ and $\frac{a}{b}$ not equal to c or $\frac{1}{c}$ ($c \in \mathbb{Z}$).

If we insist that $a \in \mathbb{N}$ then such a tiling U will also satisfy the conditions given in the first part of proposition 5.10 and thus all the translate points of such a tiling can be found in a finite number of lines in the fundamental domain of L, and all the singular points on those lines correspond to double points in Ω_U.

In addition, since the translate points are only the translates of U by integer multiples of some fixed distance there will also be lines of translates joining each of the translate points (and indeed all non-singular points on the lines), perhaps best thought of as being a line segment passing through each non-singular point on the line with the ends of the segments identified in the appropriate way. The singular points will therefore give double lines in a similar way.

5.4.3 Repetitivity Revisited

Following the results in this chapter and chapter 4 we can now prove repetitivity in the general case, i.e., that any rational 2:1 X-projection scheme produces repetitive tilings.

Recall that a tiling U is repetitive if any patch P in U appears throughout U, and a copy of P can be found within some fixed distance (dependent on P) of any point in the tiling.

Theorem 5.11. *Tilings generated by rational 2:1 X-projection schemes are repetitive.*

Proof. We have a tiling U generated by a rational 2:1 X-projection scheme. Assume that U has a corresponding point u in the fundamental domain of L.

Note that U may not correspond to any point in the fundamental domain, but if not it will be a small translate of a tiling corresponding to such a point.

Any patch of tiles, P, in the tiling U is contained in some larger patch, Q, about the origin, so if Q appears throughout the tiling then so will P.

There exists some $\varepsilon > 0$ such that any tiling U' within distance ε of U will have the patch Q about the origin (after some small translate). Thus there exists $\delta > 0$ such that all non-singular points within δ of u will give tilings with patch Q near the origin.

Depending on the relationship between the gradients of the two strips, we have points corresponding to translates of U that are either dense in a finite set of lines in the fundamental domain of L (see propositions 4.14, 4.15, 4.17 and 4.19 for the various subcases) or dense in the whole fundamental domain (see proposition 4.23). In all cases there are countably many translates of U corresponding to points within distance δ of u.

Since the step between consecutive translate points is fixed there must be some maximum number of steps that we can have between occurrences of translate points of U within distance δ of u.

Thus the patch Q (and therefore also the patch P) must appear throughout the tiling U and within some fixed distance of any point in the tiling. $\square$

6 Further Work

In this chapter we will look at some examples of rational 2:1 X-projection tilings and the proportions in which the prototiles in these tilings appear. We will also look at the proportions that we might expect in some cases, and how closely the expected values resemble the values observed in our examples.

In the second section we will look at a few examples of tilings generated by irrational 2:1 X-projection schemes.

6.1 Proportions of Prototiles in Rational 2:1 X-Projection Tilings

This section contains some examples of (patches of) tilings generated by rational 2:1 X-projection schemes.

In all of the following examples the two strips have a common point on their lower boundaries (which I have defined to be the origin) and therefore also a common point on their upper boundaries (the point $(-1, 1)$), with the origin being projected onto the pattern space E whilst the point $(-1, 1)$ is not.

In the definition of a 2:1 X-projection the strips S_1 and S_2 were chosen so that they do not have any points of L on their boundaries, so these examples do not give full valid tilings, but since all the strips are at irrational gradients there will only be one point on each boundary and a suitable (very small) translation of the strips will result in a valid scheme with an identical patch to that given by this setup.

Example 6.1. *We start with the pattern space E at gradient 0.5, a setup that gives two possible tile lengths. For simplicity we will refer to the tiles as being of lengths 1 and 2 rather than their actual lengths of $\frac{1}{\sqrt{5}}$ and $\frac{2}{\sqrt{5}}$. Similarly, for the other rational 2:1 X-projection examples that have more possible tile lengths we will say that the the shortest tile has length 1 and all other tiles will be labeled with integers showing their lengths relative to the shortest tile.*

We then take strips S_1 and S_2 at gradients $\frac{1}{\sqrt{2}}$ and $\frac{1}{2\sqrt{2}}$ relative to the sublattice Λ.

Figure 6.1 gives some idea of what a patch of this tiling around the origin looks like. Here each dot represents a tile in the tiling with the scale at the left showing the lengths of the tiles and the scale at the bottom showing where they appear in the tiling. The origin is at the meeting point of the tiles numbered 100 and 101 in this patch, so approximately at the centre of this diagram.

We will refer to the resulting tiling as $T_{\left(\frac{1}{\sqrt{2}},\frac{1}{2\sqrt{2}},0.5\right)_\sim}$ with the first two numbers representing the gradients of the two strips relative to Λ and the third number showing the gradient of the pattern space. Later tilings will be labeled in a similar way.

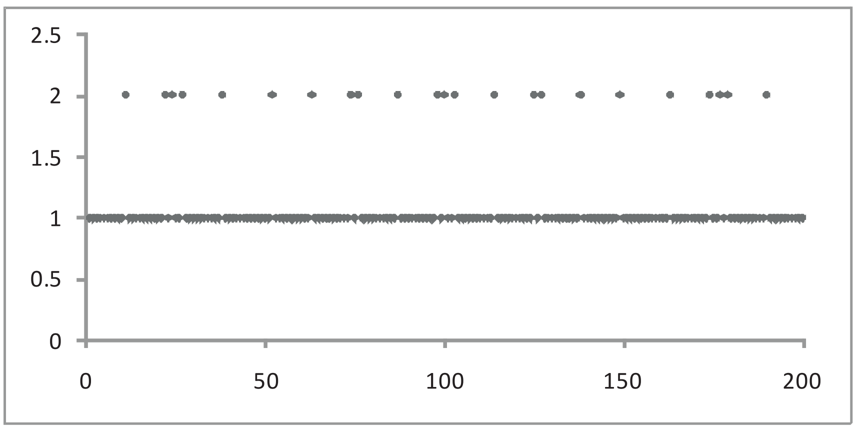

Figure 6.1: Diagram of a patch of tiling $T_{\left(\frac{1}{\sqrt{2}},\frac{1}{2\sqrt{2}},\frac{1}{2}\right)_\sim}$.

There are 23 tiles of length 2 in this patch, so each of the points at height 2 in the diagram represents a single tile.

Changing the gradient of the pattern space to 0.6 and looking at the setup with strips at the same gradients relative to the (now altered) sublattice Λ gives the diagram shown in figure 6.2.

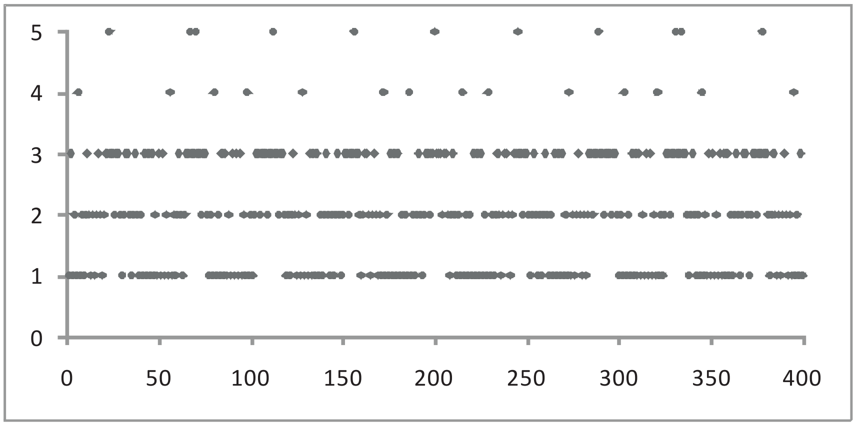

Figure 6.2: Diagram of a patch of tiling $T_{\left(\frac{1}{\sqrt{2}},\frac{1}{2\sqrt{2}},\frac{3}{5}\right)_{\sim}}$.

Once again the origin is approximately at the centre of this diagram (the meeting point of tiles 200 and 201).

Having the pattern space at a gradient of 0.6 $\left(\frac{3}{5}\right)$ *means that there are 5 possible tile lengths, and as can be seen from figure 6.2 all of these tile lengths appear in the resulting tiling.*

For the final example the pattern space E is at gradient 0.3, which gives 10 possible tile lengths, all of which appear in the tiling as can be seen in figure 6.3.

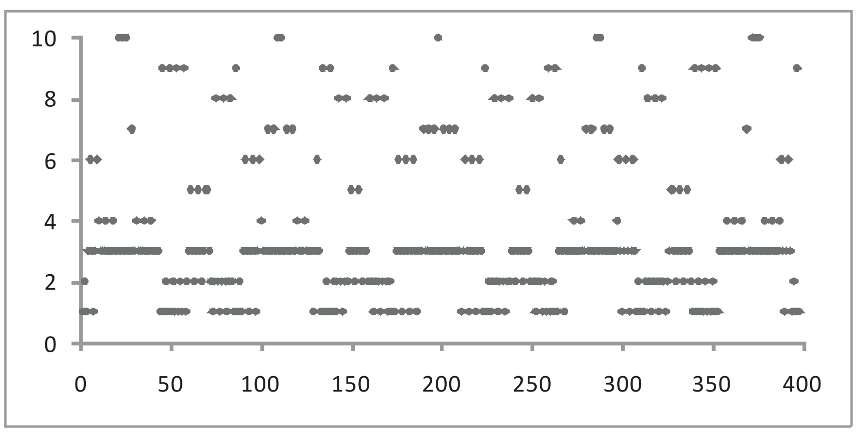

Figure 6.3: Diagram of a patch of tiling $T_{\left(\frac{1}{\sqrt{2}},\frac{1}{2\sqrt{2}},\frac{3}{10}\right)_{\sim}}$.

As before the origin is located approximately at the centre of this diagram (between tiles 200 and 201).

Looking at the diagrams above it is evident that the different prototiles do not exist in equal numbers within these patches. It is perhaps unsurprising that there should be fewer prototiles of maximum length, particularly in the later examples, given that these can only arise from the lining up of longer tiles from the constituent tilings T_1 and T_2. And indeed in each of the above examples the maximum length prototile appears to be the least common.

The most common prototile in the examples given above is the prototile with the same length as the "short" tile from the tilings corresponding to the individual strips (i.e. 1 for E at gradient 0.5 and 3 for E at gradient 0.6 or 0.3).

The following tables give some approximate ratios of numbers of prototiles relative to the number of maximum length tiles in 100000-tile patches of the tilings. The gradients of the strips given are those relative to Λ. Fixing the gradients relative to Λ will still result in them varying relative to L as the gradient of E is altered. The columns labeled "S_1 Approx." and "S_2 Approx." show the approximate gradients of the two strips relative to the lattice L.

E at Gradient 0.5					
S_1 Gradient	S_2 Gradient	S_1 Approx.	S_2 Approx.	1 Tiles	2 Tiles
$\frac{1}{\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	1.867	1.037	9.199	1
$\frac{1}{\sqrt{5}}$	$\frac{5}{12\sqrt{5}}$	1.220	0.757	6.867	1
$\frac{1}{\pi}$	$\frac{1}{3\pi}$	0.973	0.640	5.673	1
$\frac{1}{\sqrt{2}}+\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	4.306	1.867	21.022	1
$\frac{1}{\sqrt{5}}$	$\frac{1}{\sqrt{5}}-\frac{1}{8}$	1.220	0.980	7.593	1
$\frac{1}{\pi}$	$\frac{1}{\pi}-\frac{1}{10}$	0.973	0.806	6.331	1
$\frac{3+\sqrt{5}}{1+\sqrt{5}}$	$\frac{2+\sqrt{5}}{1+\sqrt{5}}$	11.090	5.236	63.433	1
$\frac{9}{10}+\frac{1}{\sqrt{2}}$	$1+\frac{1}{2\sqrt{2}}$	10.726	5.735	96.561	1
$1-\frac{1}{\pi}$	$1-\frac{2}{\pi}$	1.793	1.055	8.922	1
$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{3}}$	1.867	1.515	11.601	1
$\frac{1}{\sqrt{5}}$	$\frac{1}{\sqrt{7}}$	1.220	1.083	7.925	1
$\frac{1}{e}$	$\frac{1}{\pi}$	1.063	0.973	7.106	1

In the above examples, the first three have gradients that are rationally related relative to Λ and the next three are irrationally related but differ by a rational amount.

Recall that in chapter 4 we defined x and qx for strips at gradients p and q (with $p = aq$ for some $a \in \mathbb{R}$) relative to Λ to be,

$$x = \frac{z}{q(a-1)}$$

$$qx = \frac{z}{a-1}\,.$$

The third set of three examples above have gradients giving values of x and qx satisfying,

$$[qx] = c[x] + r$$

for some $c, r \in \mathbb{Q}$, and for $z = 1$ (where the square brackets denote fractional part).

The final three give values of x and qx that are irrationally related and have fractional parts that cannot be expressed in the way given above. Thus there are three examples corresponding to each of the different types of rational 2:1 X-projection given in the summary diagram in chapter 4.

Since the pattern space E is at gradient 0.5 we have that the projection of a $(0,1)$ step in the lattice L gives a tile of length 1 and $(1,0)$ step gives a tile of length 2. So we would expect the X-projection setups with strips at higher gradients to produce tilings with a higher proportion of short tiles, and this appears to be the case.

E at Gradient 0.6						
S_1	S_2	1	2	3	4	5
$\frac{1}{\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	14.404	14.394	15.135	1.994	1
$\frac{1}{\sqrt{5}}$	$\frac{5}{12\sqrt{5}}$	9.162	9.160	10.216	1.998	1
$\frac{1}{\pi}$	$\frac{1}{3\pi}$	7.511	7.517	8.594	2.003	1
$\frac{1}{\sqrt{2}} + \frac{1}{2}$	$\frac{1}{\sqrt{2}}$	48.688	48.688	42.725	1.960	1
$\frac{1}{\sqrt{5}}$	$\frac{1}{\sqrt{5}} - \frac{1}{8}$	10.392	10.393	11.334	1.993	1
$\frac{1}{\pi}$	$\frac{1}{\pi} - \frac{1}{10}$	8.333	8.326	9.320	2.032	1
$\frac{3+\sqrt{5}}{1+\sqrt{5}}$	$\frac{2+\sqrt{5}}{1+\sqrt{5}}$	813.787	812.702	499.170	1.000	1
$\frac{9}{10} + \frac{1}{\sqrt{2}}$	$1 + \frac{1}{2\sqrt{2}}$	783.714	782.694	472.408	1.000	1
$1 - \frac{1}{\pi}$	$1 - \frac{2}{\pi}$	14.066	14.053	14.837	2.000	1
$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{3}}$	19.997	20.166	20.192	2.137	1
$\frac{1}{\sqrt{5}}$	$\frac{1}{\sqrt{7}}$	11.061	11.041	11.910	2.011	1
$\frac{1}{e}$	$\frac{1}{\pi}$	9.611	9.609	10.570	2.008	1

The second table shows the proportions of prototiles for a pattern space at gradient $\frac{3}{5}$. This means that each T_i has a shorter tile of length 3 and a longer tile of length 5.

E at Gradient 0.3											
S_1	S_2	1	2	3	4	5	6	7	8	9	10
$\frac{1}{\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	7.89	7.88	19.01	2.00	2.00	2.00	2.00	1.99	2.00	1
$\frac{1}{\sqrt{5}}$	$\frac{5}{12\sqrt{5}}$	5.69	5.69	13.56	2.02	2.01	2.02	2.02	2.01	2.02	1
$\frac{1}{\pi}$	$\frac{1}{3\pi}$	4.76	4.76	11.07	1.99	1.99	1.99	1.98	2.00	1.99	1
$\frac{1}{\sqrt{2}}+\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	15.32	15.33	34.16	2.00	2.00	2.00	2.00	2.00	2.00	1
$\frac{1}{\sqrt{5}}$	$\frac{1}{\sqrt{5}}-\frac{1}{8}$	6.31	6.31	15.05	2.01	2.00	2.00	2.01	2.00	2.01	1
$\frac{1}{\pi}$	$\frac{1}{\pi}-\frac{1}{10}$	5.22	5.22	12.28	2.00	2.00	2.00	2.00	2.00	1.99	1
$\frac{3+\sqrt{5}}{1+\sqrt{5}}$	$\frac{2+\sqrt{5}}{1+\sqrt{5}}$	34.33	34.32	62.51	1.99	1.99	1.99	1.99	1.99	1.99	1
$\frac{9}{10}+\frac{1}{\sqrt{2}}$	$1+\frac{1}{2\sqrt{2}}$	35.52	35.52	64.08	2.00	2.01	1.99	2.00	2.00	2.01	1
$1-\frac{1}{\pi}$	$1-\frac{2}{\pi}$	7.73	7.72	18.56	1.98	1.99	1.99	1.97	1.98	1.98	1
$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{3}}$	9.41	9.41	22.12	2.01	2.01	2.00	2.00	2.00	2.00	1
$\frac{1}{\sqrt{5}}$	$\frac{1}{\sqrt{7}}$	6.57	6.57	15.69	2.00	2.00	2.00	2.00	2.00	2.00	1
$\frac{1}{e}$	$\frac{1}{\pi}$	5.87	5.89	13.95	2.00	1.99	2.00	1.98	2.00	1.99	1

The third table displays the proportions of prototiles for pattern space at gradient $\frac{3}{10}$. In this case the shorter tile in each T_i has length 3 with the longer tile having length 10.

The tables appear to show that generally the tiles with lengths between those of the short tile and long tile from the tilings T_i appear twice as often as the maximum length tiles. The tilings $T_{\left(\frac{3+\sqrt{5}}{1+\sqrt{5}},\frac{2+\sqrt{5}}{1+\sqrt{5}},0.6\right)_\sim}$ and $T_{\left(\frac{9}{10}+\frac{1}{\sqrt{2}},1+\frac{1}{2\sqrt{2}},0.6\right)_\sim}$ appear to be exceptions to this. In these two cases the number of maximum length tiles in a patch of 100000 tiles is relatively small, so we might see a different result with a larger patch. However, it may be that there is some sort of relationship between the strips that is causing this discrepancy.

Also noticeable from the tables is that the tiles of lengths 1 and 2 (so tiles with lengths less than that of the short tile from the tilings T_i) appear in the same proportions.

The following section is an attempt at an explanation of these observations.

Note that in the later examples the observed proportions of prototiles appear to differ considerably from the proportions we would see from a non-degenerate canonical $N : 1$ projection tiling, as discussed in chapter 2. For example, a non-degenerate 10:1 projection scheme will give tilings with 10 prototiles, but with the longest of these appearing in the highest proportion and with no two prototiles appearing in the same proportions, unlike what we seem to be seeing above.

6.1.1 Possible Explanation

If we think of the X-projection tiling U as being two standard projection tilings overlaid then thinking about the different ways in which each prototile can arise seems to give a reasonable explanation of the above observations.

Firstly, maximal length tiles can only occur in U when t_2 tiles from both T_1 and T_2 line up, as shown in figure 6.4.

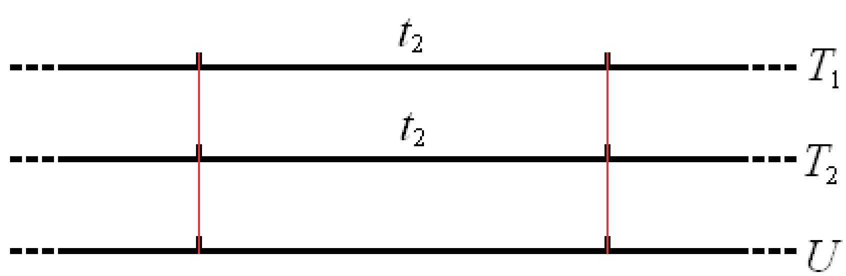

Figure 6.4: Two maximal length tiles that are lined up.

Tiles $u \in U$ satisfying $|t_1|<|u|<|t_2|$ can only arise on the overlap of two t_2 tiles, but, as we can see from figure 6.5, there are two “different” ways in which this overlap can happen.

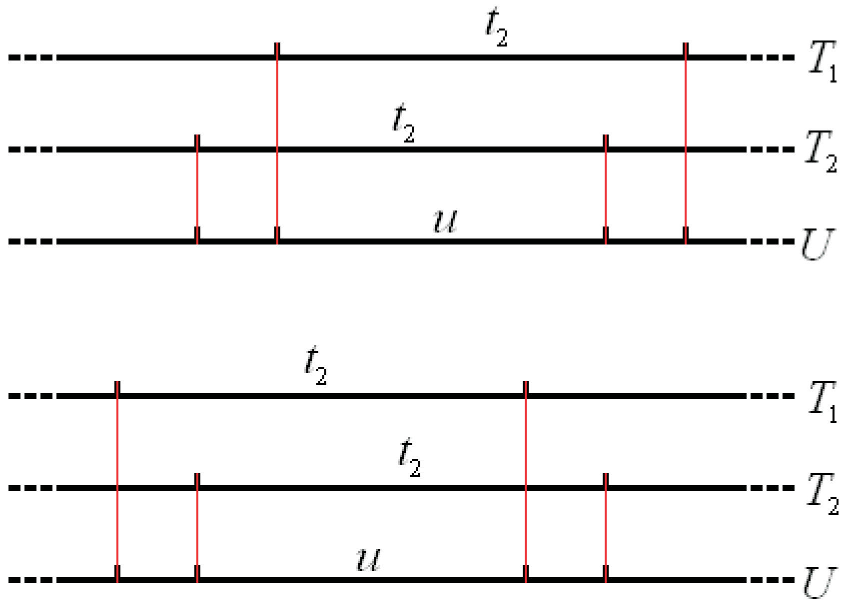

Figure 6.5: Two different ways in which t_2 tiles can overlap to give u tiles.

So it would seem reasonable to expect there to be twice as many of this type of tile in U as there are tiles of length $|t_2|$.

Tiles $v \in U$ satisfying $|v|<|t_1|$ can arise as overlaps of either t_2 tiles with each other, t_1 tiles with each other or on overlaps between t_1 and t_2 tiles. In a similar way to the u tiles, all tiles of type v should be expected to appear in the same proportions due to arising from the same number of possible overlaps, but these proportions should be higher than the u tiles because there are more overlaps than for the u tiles.

Finally, the tiles in U with the same length as t_1 can often arise in yet more ways: as the overlap of two t_2 tiles, when two t_1 tiles line up, or when a t_1 tile from either tiling is projected "inside" a t_2 tile from the other. For example, if the gradient of E is 0.3 then there are 8 different ways that a t_1 tile from T_1 (that has length 3) can be projected into a t_2 tile from T_2 (of length 10).

6.1.2 Expected Proportions

We now look at the simplest case given above where the pattern space E is at gradient 0.5, giving tilings with two prototiles. We will look at the proportions of length 1 and 2 tiles that we might expect to find in a tiling generated by a rational 2:1 X-projection setup where we are projecting onto a pattern space at this gradient.

As before, we have two strips S_1 and S_2 giving corresponding tilings T_1 and T_2. Since the pattern space is at gradient 0.5 each of these tilings also has two prototiles of lengths $\frac{1}{\sqrt{5}}$ and $\frac{2}{\sqrt{5}}$ (which, for simplicity, we will call lengths 1 and 2).

The two tilings T_1 and T_2 are combinatorially just standard 2:1 projection tilings. If we have strip S_i at gradient g_i then by proposition 2.5 we know that a canonical 2:1 projection tiling with g_i less than 1 has,

$$\frac{\text{Proportion of Short Tiles}}{\text{Proportion of Long Tiles}} = g_i$$

and if g_i is greater than 1 then,

$$\frac{\text{Proportion of Short Tiles}}{\text{Proportion of Long Tiles}} = \frac{1}{g_i}.$$

However, if g_i is less than one then horizontal steps between lattice points in S_i correspond to long tiles, and if g_i is greater than 1 then the long tiles arise from the projections of vertical steps. Thus in either case we have g_i vertical steps for each horizontal step.

When we take the projection to be onto a pattern space at gradient 0.5 we get that horizontal steps give long tiles, regardless of the gradient of the strip we are projecting from. Therefore each T_i will have g_i length 1 tiles for each length 2 tile, and thus in tiling T_i we have,

$$\text{proportion of length 2 tiles} = \frac{1}{g_i + 1}$$

$$\text{proportion of length 1 tiles} = \frac{g_i}{g_i + 1} \, .$$

Now, to work out the proportions of 1-tiles and 2-tiles in the combined tiling we first look at the proportions of points that are covered in each of the tilings T_i. The tiles in these tilings have lengths 1 and 2 with both tiles covering the point on their left but the 2-tile also leaving another point not covered.

So the proportion of points that are not covered in each of the tilings T_i is:

$$\frac{\text{proportion of 2-tiles}}{2(\text{proportion of 2-tiles}) + \text{proportion of 1-tiles}} \, .$$

For tiling T_i this is equal to,

$$\frac{\frac{1}{g_i+1}}{\frac{2}{g_i+1} + \frac{g_i}{g_i+1}} = \frac{1}{g_i + 2} \, .$$

Now, uncovered points in the combined tiling only arise from having uncovered points in both tilings T_i lining up. Thus we might expect the proportion of uncovered points in the combined tiling to be,

$$\left(\frac{1}{g_1 + 2}\right)\left(\frac{1}{g_2 + 2}\right) \, .$$

These uncovered points in the combined tiling correspond to the 2-tiles of course, but every 2-tile not only has an uncovered point but also a covered point on its left edge. Thus the expected proportion of all points that are contained in 2-tiles (on the left side or in the middle) is:

$$2\left(\frac{1}{g_1 + 2}\right)\left(\frac{1}{g_2 + 2}\right) \, .$$

From this we get that the expected proportion of points in 1-tiles (that is, on the left side of 1-tiles) is:

$$1 - 2\left(\frac{1}{g_1+2}\right)\left(\frac{1}{g_2+2}\right).$$

Therefore the expected proportions of each tile are given by,

$$\text{proportion of 2-tiles} = \frac{\text{proportion of points at start of 2-tiles}}{\text{proportion of points at start of any tile}}$$

$$= \frac{\left(\frac{1}{g_1+2}\right)\left(\frac{1}{g_2+2}\right)}{1 - \left(\frac{1}{g_1+2}\right)\left(\frac{1}{g_2+2}\right)}$$

$$= \frac{1}{(g_1+2)(g_2+2)-1}$$

$$\text{proportion of 1-tiles} = \frac{(g_1+2)(g_2+2)-2}{(g_1+2)(g_2+2)-1}.$$

So the expected relative proportion of 1-tiles is equal to $(g_1+2)(g_2+2)-2$.

We will now look at how closely the predicted values resemble the observed proportions from the previous examples. The table below shows the predicted proportions of 1-tiles relative to 2-tiles and the values observed in a patch of 100000 tiles of various 2:1 X-projection tilings.

E at Gradient 0.5			
S_1 Gradient	S_2 Gradient	1 Tiles Observed	1 Tiles Predicted
$\frac{1}{\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	9.199	9.744
$\frac{1}{\sqrt{5}}$	$\frac{5}{12\sqrt{5}}$	6.867	6.877
$\frac{1}{\pi}$	$\frac{1}{3\pi}$	5.673	5.849
$\frac{1}{\sqrt{2}}+\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	21.022	22.387
$\frac{1}{\sqrt{5}}$	$\frac{1}{\sqrt{5}}-\frac{1}{8}$	7.593	7.596
$\frac{1}{\pi}$	$\frac{1}{\pi}-\frac{1}{10}$	6.331	6.344
$\frac{3+\sqrt{5}}{1+\sqrt{5}}$	$\frac{2+\sqrt{5}}{1+\sqrt{5}}$	63.433	92.721
$\frac{9}{10}+\frac{1}{\sqrt{2}}$	$1+\frac{1}{2\sqrt{2}}$	96.561	96.431
$1-\frac{1}{\pi}$	$1-\frac{2}{\pi}$	8.922	9.587
$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{3}}$	11.601	11.592
$\frac{1}{\sqrt{5}}$	$\frac{1}{\sqrt{7}}$	7.925	7.926
$\frac{1}{e}$	$\frac{1}{\pi}$	7.106	7.108

As you can see, some of the observed values are very close to the predicted values, whereas some display a large discrepancy.

Several of these tilings have a smaller proportion of 1-tiles than we might expect, showing that the 2-tiles in the two component tilings T_i are lining up more often than we might expect by chance. However, these patches are of course from a specific tiling with strips at the corresponding gradients, i.e., the tiling produced by the given setup, where the intersection point of the lower boundaries of the two strips is placed at the origin. If the intersection point was shifted so as to alter the projections of the points from strip S_1 along by one step relative to the projections of the points from S_2 then all the 2-tiles in the patch would become 1-tiles.

The following example uses tilings produced by the projection of points from strips at rational gradients, and therefore gives periodic tilings, but may illustrate what is going on in the case of rational 2:1 X-projections.

Example 6.2. *Say we have strips S_1 at gradient $\frac{1}{2}$ and S_2 at gradient $\frac{1}{7}$ relative to the integer lattice L, and we project onto a line at gradient $\frac{1}{2}$.*

Both strips are at rational gradients and thus produce periodic tilings. Therefore the combined tiling will be periodic, so we can work out the proportions of the two prototiles by simply drawing the repeating part of the tiling.

If the intersection point of the two strips is positioned so that a length 1 tile in T_2 lines up with a length 1 tile in T_1 then we will have the tiling shown in figure 6.6.

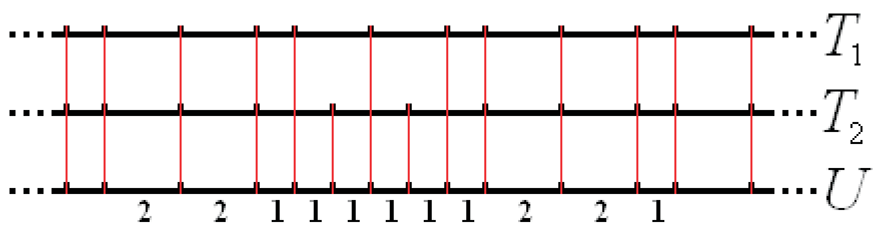

Figure 6.6: Position 1 for T_1 and T_2.

The repeating patch in this tiling is 22111111221 as marked in the diagram, so we can immediately say that the proportion of 1-tiles relative to 2-tiles is $\frac{7}{4}$ (or 1.75).

However, if the position of the intersection point of S_1 and S_2 was altered so that T_1 and T_2 lined up in a different way we could end up with a different tiling, as shown in figure 6.7.

Here the projection of lattice points from strips at the same gradients as before onto the same pattern space has produced a tiling with repeating patch 1111122111111, and therefore with the proportion of 1-tiles relative to 2-tiles being equal to $\frac{11}{2}$ (or 5.5).

There are 3 other ways in which T_1 and T_2 can line up, these ways giv-

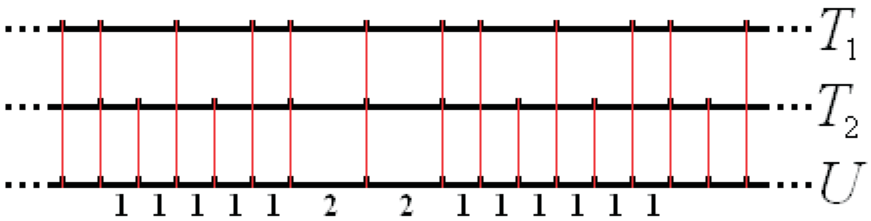

Figure 6.7: Position 2 for T_1 and T_2.

ing combined tilings with repeating patches 121111112211, 111221111112 and 1111111221111 (this last one being the same as the second one).

Over all these patches we get 1-tiles appearing 47 times and 2-tiles appearing 14 times. The expected proportion of 1-tiles relative to 2-tiles for this tiling is,

$$(g_1 + 2)(g_2 + 2) - 2 = \left(\frac{5}{2}\right)\left(\frac{15}{7}\right) - 2 = \frac{47}{14}.$$

This example suggests that our expected value is perhaps correct when we talk about all possible tilings that can be generated by a certain setup, looking at all the different positions at which the intersection point can be placed.

The final three tilings in the table above have sets of translate points that are dense in the unit square (proposition 4.23), so any positioning of the intersection point will be arbitrarily close to a position that gives a translate of the tiling. This may explain why the proportion of 1-tiles observed in the patches of these tilings so closely matches the expected values.

6.2 Irrational 2:1 X-Projection Examples

In this section we will look at a few examples of tilings generated by irrational 2:1 X-Projection schemes. The irrational 2:1 X-projection scheme was first defined in chapter 3. The setup differs from the rational 2:1 X-projection setup in that the projection is onto a pattern space at irrational gradient relative to the lattice.

As was proved in chapter 3, this setup gives tilings with an infinite number of prototiles. We also proved that the prototile with length equal to the shorter of the two prototiles t_i appearing in the constituent tilings T_i is the only prototile that can appear throughout the combined tiling.

Example 6.3. *For the first example we have strip S_1 at gradient $\frac{1}{\sqrt{2}}$, S_2 at gradient $\frac{1}{\sqrt{3}}$ and pattern space E at gradient $\frac{1}{\sqrt{5}}$ (here all gradients are relative to the integer lattice L). A diagram of a large patch of this tiling is shown in figure 6.8.*

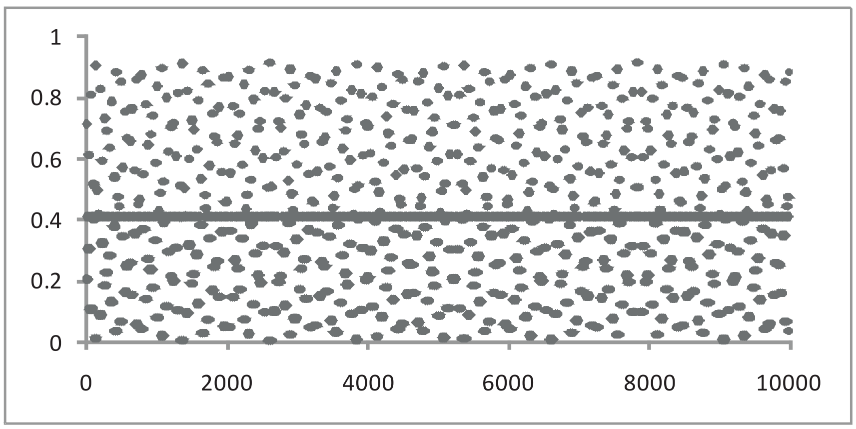

Figure 6.8: Diagram of a patch of tiling $T_{\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{5}}\right)}$.

The patch shown here is considerably larger than the patches of rational 2:1 X-projections seen earlier in the chapter. This is because of the much higher number of prototiles that we get in patches of irrational 2:1 X-projection tilings, meaning that a diagram of a patch of 10000 tiles, like the one shown here, will still display some structure and not merely look like several horizontal lines.

There is what appears to be a horizontal line in this diagram at a height of slightly more than 0.4. This is due to the large number of prototiles of length equal to $\frac{1}{\sqrt{6}}$ in this tiling, which is the length of the shorter of the two prototiles *from the constituent tilings T_i. As explained before, tiles of this length can appear throughout the tiling.*

Example 6.4. *We now look at a second example, where the two strips S_1 and S_2 are at gradients $2\sqrt{5}$ and $\sqrt{5}$ respectively, with the pattern space E at gradient $\frac{1}{\sqrt{5}}$ as before (see figure **??**).*

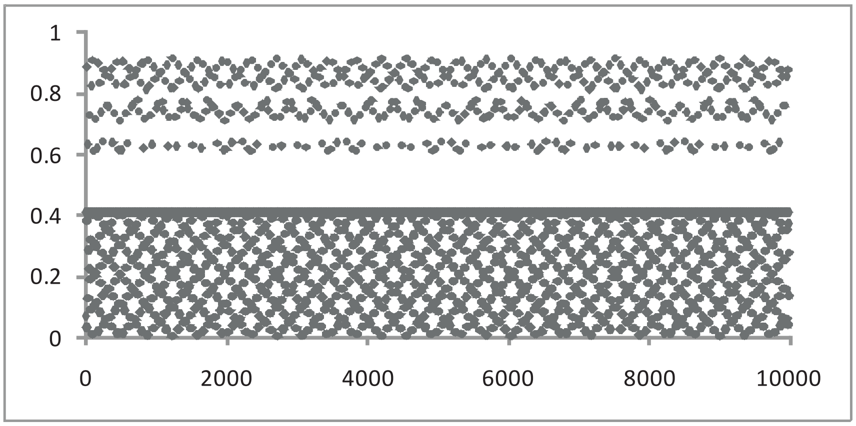

Figure 6.9: Diagram of a patch of tiling $T_{\left(2\sqrt{5},\sqrt{5},\frac{1}{\sqrt{5}}\right)}$.

There are significant gaps in this diagram, indicating that there are no tiles with those particular lengths within the patch. The gaps correspond to tiles with lengths greater than that of the shorter prototile from the tilings T_i. Tiles with these lengths can only arise on the overlap of longer prototiles in each of the constituent tilings, so the gaps suggest that there are ranges of ways in which these tiles cannot overlap in this tiling.

It may be that the situation changes further along the tiling, but these gaps remain on a patch of 300000 tiles, as shown by figure 6.10.

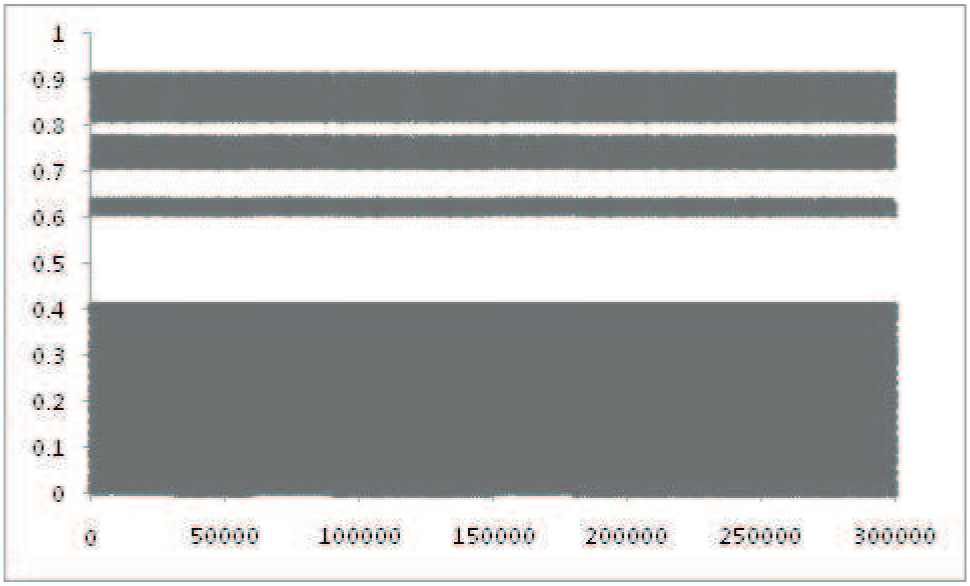

Figure 6.10: Diagram of a patch of tiling $T_{\left(2\sqrt{5},\sqrt{5},\frac{1}{\sqrt{5}}\right)}$.

Figure 6.11 shows a patch of the tiling with S_1 *and* S_2 *at the same gradients as before, but with* E *altered to be at gradient* $\frac{\pi}{7}$*. Again, we have a tile of length slightly greater than 0.4 appearing throughout the patch, though the actual length has altered slightly due to the alteration to the gradient of* E *(this time the length is* $\frac{\pi}{\sqrt{\pi^2+49}}$*).*

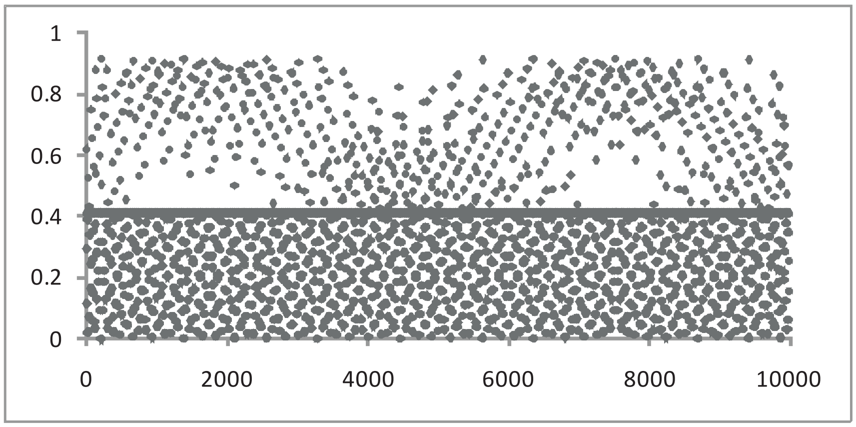

Figure 6.11: Diagram of a patch of tiling $T_{\left(2\sqrt{5},\sqrt{5},\frac{\pi}{7}\right)}$.

The diagram seems to show an "oscillating" pattern to the lengths of tiles above the line. Again, it is possible that the previous tiling is displaying something similar but on a much larger scale.

7 Conclusion

In this document we looked at 2:1 non-parallel projections schemes. Three different types of these were defined in chapter 3, namely rational 2:1 X-projection schemes, irrational 2:1 X-projection schemes and a type of non-parallel projection scheme where the two strips have rational gradients but the projection is onto a line at irrational gradient.

In the same chapter we saw that this final type of non-parallel projection scheme produces tilings with infinitely many prototiles but for any chosen patch of tiles, P, in such a tiling there will be patches that are ε-close to P (definition 3.10) throughout the tiling showing that such tilings have a property that is similar to repetitivity.

Tilings generated by irrational 2:1 X-projection schemes were also seen to have infinitely many prototiles, and several examples of this type of tiling were looked at in chapter 6, including one that appeared to have large "gaps" in the set of prototile lengths observed.

The bulk of this document was concerned with the examination of tilings generated by rational 2:1 X-projection schemes. In contrast to the other two types of non-parallel projection tilings featured these tilings were seen to have only a finite number of prototiles, with an upper bound for this number computed on chapter 3. In the same chapter a certain class of this type of tiling was shown to be repetitive.

In chapter 4 we looked at the points in the fundamental domain of our lattice L that correspond to translates of a tiling generated by a rational 2:1 X-projection scheme. The patterns of these points were found to differ depending on the relationship between the gradients of the two strips relative to the lattice Λ, a sublattice of L that depends on the gradient of the pattern space E. These diagrams were seen to come in four distinct types, with the first three all having translate points appearing as dense subsets of a finite number of lines in the fundamental domain of L and the final type having translate points forming a dense subset of the fundamental domain.

At the end of chapter 4 we looked in greater detail at the diagrams that are produced by rational 2:1 X-projection tilings, particularly in the case where the gradients of the two strips are rationally related relative to Λ.

In chapter 5 we examined the tiling spaces associated to rational 2:1 X-projection tilings. We started by looking at the tiling spaces of canonical 2:1 projection tilings, then proceeded to the intermediate step of one-strip non-parallel 2:1 projection tilings, which we proved to have tiling spaces that are homeomorphic to those of canonical 2:1 projection tilings. Then we moved on to examine the tiling spaces of tilings generated by rational 2:1 X-projection schemes, particularly looking at the multiple points (or lines) that arise in the various different types. Finally we revisited the problem of repetitivity of these tilings, showing that all tilings generated by rational 2:1 X-projections are repetitive.

In chapter 6 we presented some examples of rational 2:1 X-projection tilings and looked at the proportions of prototiles in large patches of these examples. We gave a possible explanation of the observed values and noted that the proportions we predicted the prototiles to appear in most closely matched the observed data in the case where the translate points of the tiling are dense in the fundamental domain of L.

There are also many questions that were not answered (or not fully answered). We saw that irrational 2:1 X-projection tilings have infinitely many prototiles and could not be repetitive, but the question of whether they could be ε-repetitive (as with the non-parallel projections where we had the strips at rational gradients) remains unanswered. We also saw "gaps" in one of the example diagrams for this type of tiling, suggesting that there are ranges of lengths of tiles that do not appear in the tiling, but it is not clear whether these gaps persist throughout. If some examples really do have these "gaps" throughout, then what are the conditions required to give such examples?

In the case of rational 2:1 X-projections more work could be done on describing the tiling spaces, and though we proved aperiodicity in some cases there was no general proof of this. In addition, the question of whether any of these tilings might arise as substitutions was not addressed. Finally, one could investigate the tilings produced by higher dimensional versions of this setup.

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