Construction operations to create new aperiodic tilings: local isomorphism classes and simplified matching rules

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Abstract

Title: 'Construction operations to create new aperiodic tilings: local isomorphism classes and simplified matching rules'

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This thesis studies several constructions to produce aperiodic tilings with particular properties. The first chapter of this thesis gives a constructive method, exchanging edge to edge matching rules for a small atlas of permitted patches, that can decrease the number of prototiles needed to tile a space. We present a single prototile that can only tile $\mathbb{R}^3$ aperiodically, and a pair of square prototiles that can only tile $\mathbb{R}^2$ aperiodically.

The thesis then details a construction that superimposes two unit square tilings to create new aperiodic tilings. We show with this method that tiling spaces can be constructed with any desired number of local isomorphism classes, up to (and including) an infinite value. Hyperbolic variants are also detailed.

The final chapters of the thesis apply the concept of Toeplitz arrays to this construction, allowing it to be iterated. This gives a general method to produce new aperiodic tilings, from a set of unit square tilings. Infinite iterations of the construction are then studied. We show that infinite superimpositions of periodic tilings are describable as substitution tilings, and also that most Robinson tilings can be constructed by infinite superimpositions of given periodic tilings. Possible applications of the thesis are then briefly considered.

Acknowledgements

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Chapter 1

Introduction

This thesis addresses part of two major questions in the field of aperiodic tilings. ‘What is the least number of prototiles that can describe an aperiodic tiling?’ ‘What new aperiodic tilings can be found, and what can we understand about their tiling spaces?’

Of course we are not assuming that the reader understands these questions at this point in the thesis. In order to explain these questions, and put them in their proper context, we will briefly describe the field of aperiodic tilings and its history.

The most familiar tilings are called *periodic* tilings. These consist of a finite number of tile types (or *prototiles*), which are placed together to form a pattern which is invariant under n linearly independent translations, where n is the dimension of the space being tiled. This repeating pattern is called a *fundamental domain* of the tiling.

A question studied early in the modern branch of tiling theory was if you were given a finite number of prototiles (and rules on how to put them together), when could you form a tiling of the plane from them? Mathematician and logician Hao

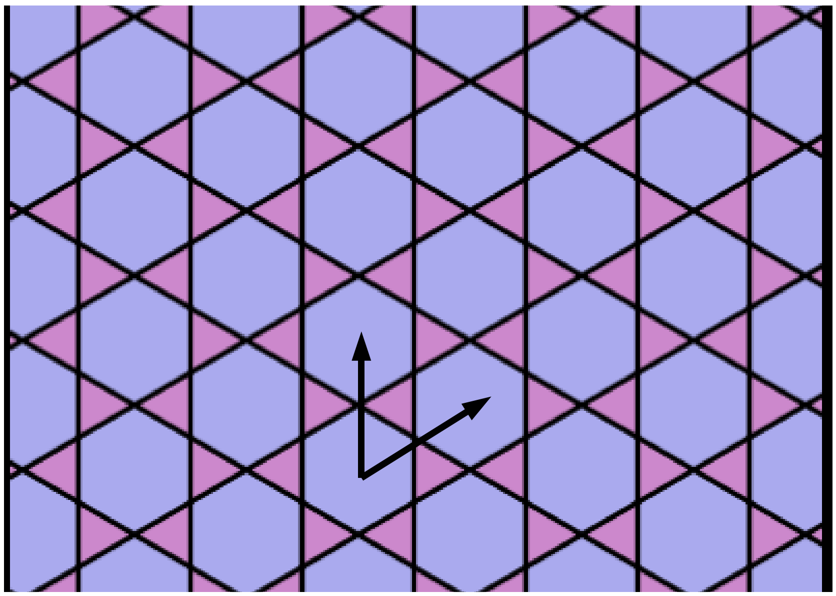


Figure 1.1: Periodic tiling invariant under two translations

Wang addressed this problem in 1961. Define a set of Wang (proto)tiles as a set of unit square tiles with coloured edges. Tiles may be placed next to each other only if they share a full edge (no partial edges) and their adjoining edges match colours.

Wang conjectured [41] an algorithm to detect whether a given set of Wang (proto)tiles could tile the plane. This algorithm took the set of prototiles, and attempted to cover a disk of radius 1 using those prototiles (with no gaps or overlaps allowed). If this was successful, the algorithm then attempted to cover a disk of radius 2, then a disk of radius 3, and so on. If a disk of a certain radius could not be covered, then obviously a tiling of $\mathbb{R}^2$ could not be created. Wang argued that eventually one of these disks of increasing size would be large enough to contain a fundamental domain of the prototiles, and the domain could thus be extended to produce a tiling of the whole plane. This would imply that his algorithm would terminate in a finite time.

His student, Robert Berger, proved in 1966 that Wang's algorithm was incorrect [7], and in fact no such algorithm *could* exist. He did this by constructing a set of *aperiodic* prototiles, which could produce a tiling which could fill the entire plane, but could not do so with a repeating pattern. A tiling associated to such a set of prototiles will be called an *aperiodic tiling*. As such, testing this set of prototiles with Wang's algorithm would cause the algorithm never to finish.

Note that aperiodic prototile sets are different from prototile sets that *can* tile the plane without a repeating pattern, but can also tile it with a repeating pattern. For an example take a right angled isosceles triangle. Two of these triangles can be used to form two different unit squares, with a dividing line running down different diagonals of the square. By flipping a coin as to which direction the diagonal should run, you can create a nonperiodic random tiling. By choosing one of the two possible unit squares and repeating it, you can create a periodic tiling [36].

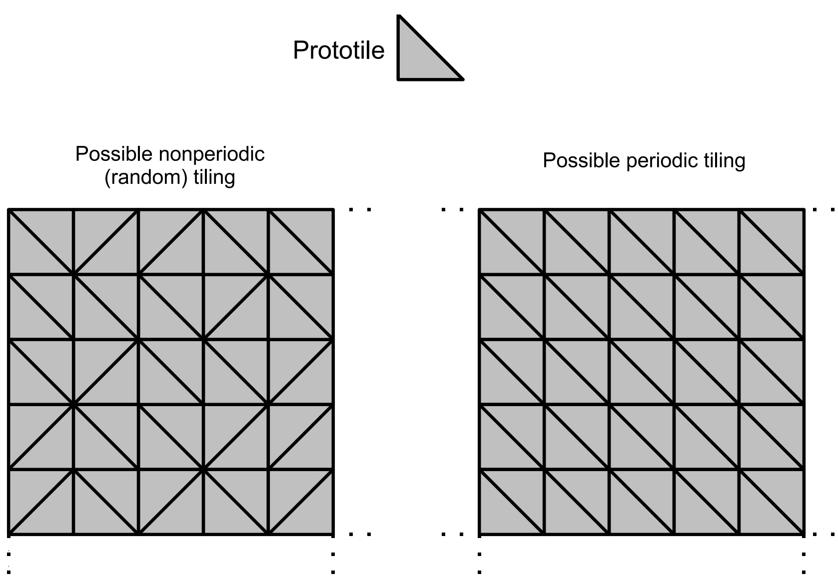


Figure 1.2: Periodic and random tiling

(It was also shown, in Berger's (sadly unpublished) thesis, that Turing machines could be translated into prototiles, with the prototiles tiling the plane if and only if the related Turing machine never halted. This result implies that determining whether a set of tiles could tile the plane is an uncomputable problem. See [8] for more details on Berger's construction.)

The aperiodic set of prototiles discovered by Berger was very complex, with 20426 prototiles. New examples were discovered over time, with Berger's unpublished thesis reducing the number of prototiles needed to 104, Raphael M. Robinson discovering an aperiodic tiling [33] with 6 prototiles in 1971 (which will be heavily used in this thesis), and Roger Penrose reducing the number to 2 prototiles, two years later [31].

This led naturally to considerations about the possible existence of a single prototile that could only tile the plane aperiodically, referred to as a *monotile*. While a simple example has not been forthcoming, by relaxing requirements on the monotile (such as being defined by shape alone, or connectedness) development has occurred. In [39] Socolar studied a more general problem, 'k-isohedral' monotiles, which would have an aperiodic monotile as a limiting case. Relaxing conditions on edge-colouring or non-connected tiles provided partial positive results. Myers has produced many examples of monotiles with high isohedral numbers, such as 10-isohedral polyhexagons and 6-isohedral polyominoes [29]. In 1996 Gummelt [45] considered tiles that are allowed to overlap, and produced a decorated tile which could force aperiodicity. Most recently in early 2010, Socolar and Taylor [40] produced a disconnected tile that could force aperiodicity.

The second chapter of this thesis is a continuation of this research theme, expanding on an idea by Chaim Goodman-Strauss in [24]. The chapter considers what happens if we allow matching rules defined by a set of allowable neighbourhoods around a tile. The chapter produces a single prototile that can force aperiodicity in $\mathbb{R}^3$ under these new 'atlas' matching rules, and perhaps more interestingly gives an algorithm that can decrease the number of prototiles needed to create a tiling by switching to atlas matching rules (under a wide set of conditions). This chapter covers the same material as an article published by the author, [14].

1.1 Types of aperiodic tiling

A wide variety of aperiodic tilings was needed to mature this field. Three major methods have been created to generate aperiodic tilings; local matching rules, projections and substitutions.

The projection method will not be used in this thesis, but for the reader's understanding we will give a brief example (with description) of a projection from 2 dimensions, which forms a 1 dimensional tiling. For further details and proofs see [12] [30].

Choose a lattice Λ of points in $\mathbb{R}^2$, and draw a line L of irrational slope (with respect to the lattice directions) through $\mathbb{R}^2$. Consider $L^{\perp}$, the line orthogonal to L. Take an compact section W of $L^{\perp}$ which is the closure of its interior, and does not contain any point from Λ in its boundary. Call W the *window*. There will be a collection of points from Λ in $L \times W$, and the points will not have a repeating pattern, due to L having irrational slope. Thus projecting points in $L \times W$ to L, and defining the interval between two adjacent points as a tile, will give you an aperiodic pattern.

This method can be used to generate an aperiodic pattern of $\mathbb{R}^2$ by applying a similar method to higher dimensional spaces. In higher dimensions, you cannot define a tile as the interval between two points, thus more care has to be taken.

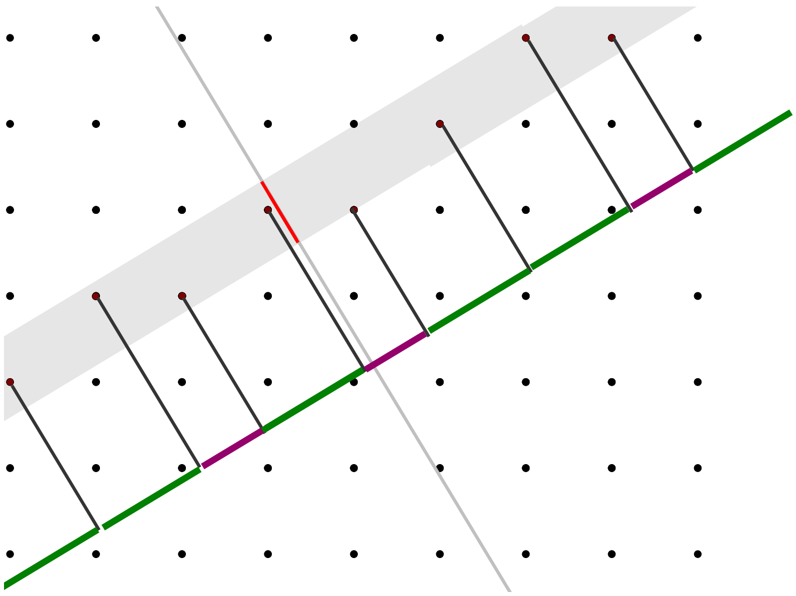


Figure 1.3: Projection of $L \times W$ to L

For example consider an unit 5-cube tessellation T of $\mathbb{R}^5$ where the vertices of the tessellation are the points of the lattice $\mathbb{Z}^5$. Choose a plane P embedded in $\mathbb{R}^5$ and a window W in $P^{\perp}$ (the orthogonal complement to P), such that the window is a projection of an unit 5-cube of $\mathbb{R}^5$ onto $P^{\perp}$ (to ensure points in $P \times W$ are connected by 2-facets in the tessellation). Project the tessellation in $P \times W$ onto P. The 2-facets of $P \times W$ will be projected onto P, and form tiles of an aperiodic tiling of $\mathbb{R}^2$.

The second major family of aperiodic tilings are tilings generated from a substitution rule. Let a *patch* of tiles be a collection of tiles, usually connected and of finite size. Substitutions are created by choosing a set of prototiles $P_1 \dots P_n$ and an *expansion rule* σ sending each prototile to a patch of tiles, as shown in figure 1.4.

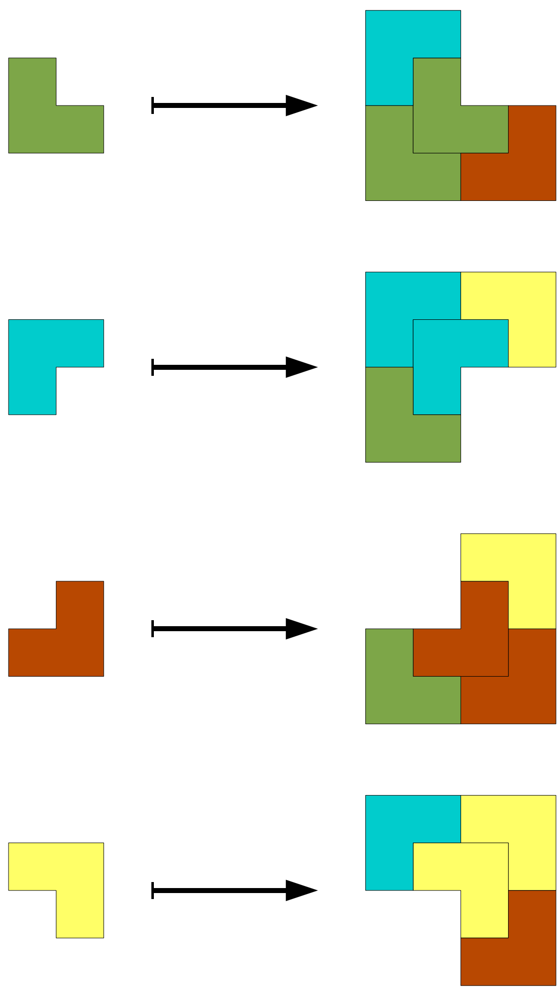


Figure 1.4: Four prototiles with their associated expansion rule.

A tiling T is generated from a set of prototiles and an expansion rule if for every compact patch $P(T)$ of T, there exists a $N \in \mathbb{N}$ such that $P(T) \subset \sigma^N(P_i)$ for some $i \in \{1, \ldots, n\}$. In this thesis we will be concentrating on *primitive* substitutions, where there exists a $N \in \mathbb{N}$ such that $P(T) \subset \sigma^N(P_i)$ for *all* $i \in \{1, \ldots, n\}$.

We will use substitution tilings as examples on which to build our constructions. Please note that in one dimension, you can represent each tile with a letter, and the expansion rule as a map from letters to words. Thus one dimensional tilings can be considered as sequences. In particular we will use the concept of Toeplitz sequences in some of our later proofs in chapters 7 and 8.

The oldest method of constructing aperiodic tilings (and the one most used in this thesis) is that of *local matching rules*.

For local matching rules you construct a rule saying which tiles may be placed next to each other. A legal tiling is one where all tiles obey the rules. For example, the tiles used by Wang and Berger had coloured edges, and two tiles could be placed next to each other only if the matching edges had matching colours and the edges overlapped fully. This is equivalent to placing bumps and notches in the tiles, in the same manner as a child's jigsaw puzzle. Thus the matching rules can be expressed in terms of tile shape only.

Many variations on this have occurred as mentioned earlier. Tiles have been allowed to overlap [45], be disconnected [40], or be defined in terms of allowable local 'atlases' of nearby tiles [24].

Note that there is considerable interest in tilings which can be constructed by more than one of these methods (indeed the original paper [12] describing the projection method described its use to construct a substitution tiling). For a more recent example, see [27].

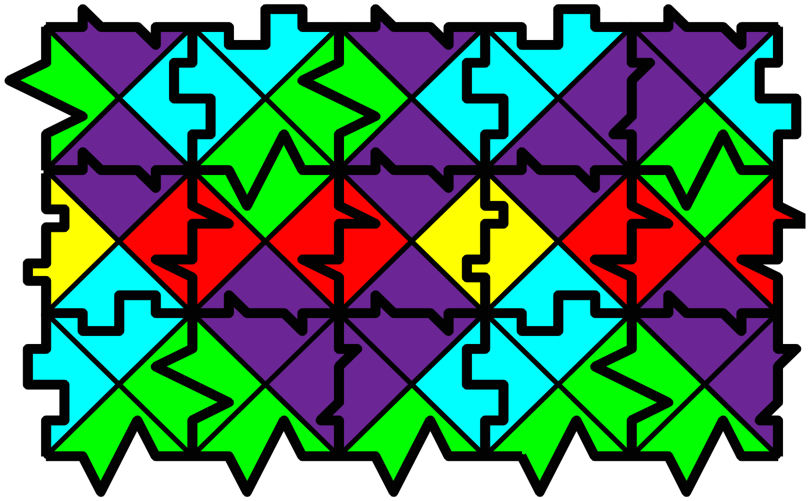


Figure 1.5: Wang tiles expressed as bumps and notches.

In the later chapters of this thesis (particularly the last two chapters) we will describe a variation on the matching rule construction which lets you 'superimpose' two tilings with unit square tiles to create a third new tiling (with variations for $\mathbb{R}^n$ available). This construction method is examined further, and links to substitution tilings are detailed.

1.2 Tiling spaces

One of the major difficulties in this field is finding some way to classify tilings effectively.

Consider figure 1.6 (pictures from [44]). This figure shows us four patches, each representing a substitution tiling of the plane. The tilings represented by the top two patches have the same prototiles and substitution rule, but one of them is rotationally symmetric and the other is not. The tilings that the bottom two patches represent use different prototiles (and thus substitution rules), but there appears to be some correlation between the local patterns in each one (and in fact, the north-east tiling as well).

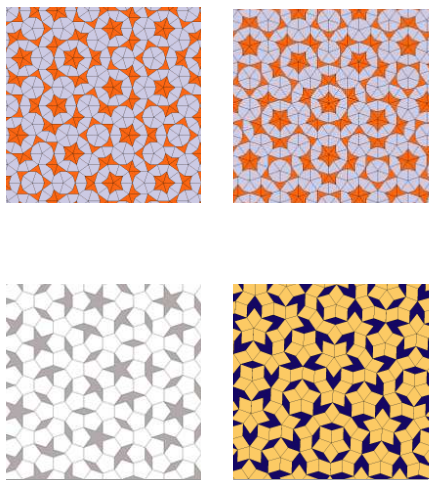


Figure 1.6: Are these four tilings different?

In fact all these tilings are *locally derivable* from each other; for any two tilings T, T', there exists a $R > 0$ such that the properties of T at any point $x \in \mathbb{R}^2$ are determined by the properties of T' in a ball of radius R about x. (Two tilings locally derivable from each other are *mutually locally derivable* or *MLD*.)

As can be gathered from the above four tilings, figuring out whether tilings are MLD is a non-trivial problem (since all possible patches must be considered, for any given tiling).

To sidestep these issues of figuring out when tilings are locally derivable, mathematicians in this field often concentrate on *tiling spaces*. While the precise definition will be given in chapter 3, definition 15, at this point the reader can consider the tiling space of a set of prototiles P to be the set of tilings constructible from prototiles in P, under a given set of matching rules. This is topologised by an associated metric, where two tilings T_1 and T_2 are ϵ-close if they agree on a ball of radius $(\frac{1}{\epsilon})$ around the origin, up to small translation (the full definition can be found in chapter 3, definition 14).

This differs slightly from the more common definition of a tiling space in the literature, which centres around the closure of the set of all tilings which are translates of a given tiling T under a fixed metric. This version of a tiling space is also called the (continuous) hull, and we will refer to it as such in this thesis.

The reason for our alternative definition links in with the main method of studying tiling spaces; topological invariants.

Much progress has been made on finding topological invariants for substitution and projection tilings. Čech cohomology [36], C^*-algebras [21], and dynamical spectrum have all provided useful invariants to distinguish between tilings, with unforeseen applications of the invariants to real-world phenomena. (See the preface of [36] for an overview). However these invariants tend not to be applicable to local matching rule tiling spaces, and calculating them for complex substitution tiling spaces can be strenuous.

For this reason, weaker invariants are a valid field of research. One such invariant is the number of 'Local Isomorphism Classes' of the tiling space. Two tilings are in the same Local Isomorphism (LI) Class if a copy of each finite patch of one tiling occurs in the other tiling and vice versa.

This invariant has been an area of study, for example in [34] and [2], but has suffered from a relative paucity of concrete examples of tilings with more than one

LI class. This is reflected in the common definition of tiling space ('hull' in this thesis) which only detects tilings in one LI class.

Thus in this thesis we will construct a family of local matching rule tilings that have n LI classes, for any $n \in \mathbb{N}$. Furthermore, we will show examples of tilings with an infinite number of LI classes, created as a precursor to our construction that generates a third tiling by superimposing two old ones.

For the final section of our thesis, we will describe possible applications of our work, including links to Winfree's model of a method of creating self-assemblying machines out of Wang tiles [46], and links to computer graphics.

1.3 Summary of chapters

Chapter 1. The introduction to the thesis, and a list of basic definitions used throughout.

Chapter 2. This chapter gives a constructive method that can decrease the number of prototiles needed to tile a space, in a wide variety of situations, in exchange for more complicated matching rules.

Chapter 3. The Robinson tiling space is introduced, and a construction to produce a new tiling space of the plane with 2 LI classes is illustrated. The construction combines Robinson tilings and periodic tilings to produce new aperiodic tilings. The construction can be applied to any tiling of the plane by unit square tiles (assuming tiles meet edge-to-edge).

Chapter 4. The construction is generalized to produce tiling spaces with n LI classes. In doing so we ensure that our matching rules are defined via sets of edges allowed to touch (meeting matching rules).

Chapter 5. Examples of tiling spaces in the hyperbolic plane $\mathbb{H}^2$ with more than 1 LI class are described. The examples are variations on tiling spaces described in the previous two chapters.

Chapter 6. Using the existing notion of substitution tilings, a tiling space in $\mathbb{R}$ with an infinite number of LI classes is described. This is then expanded to a two dimensional example, taking inspiration from the Chair tiling. Following this there is a short section detailing what we can deduce about a tiling space from the number of LI classes.

Chapter 7. We introduce the concept of Toeplitz sequences, and use it to rephrase our construction as one of a family of superposition operations, $\cap^n$, $n \in \mathbb{N}$. This allows us to consider iterations of our operations. We then proceed to show links between properly defined limits of these operations and substitution tilings in the 1 dimensional case.

Chapter 8. We use $\mathbb{Z}^d$ Toeplitz arrays to consider a two dimensional version of the $\cap$ operation. In a link back to earlier chapters, most Robinson tilings are shown to be constructible via infinite $\cap$ operations applied to a pair of periodic tilings. Possible applications of the thesis are then described.

1.4 Basic definitions

We will give a list of basic definitions used in this thesis. These are all standard definitions in the field, found in [36] and other sources. Further definitions will be introduced in relevant chapters.

Definition 1 (Ball)**.** A (closed) ball of radius r about a point x is the set $B_r(x) = \{y|d(x,y) \leq r\}$.

Definition 2 (Wang Tiles)**.** Wang tiles are unit square tiles with coloured edges. A *Wang prototile set* is a finite set of Wang tiles. We consider tilings covering the infinite Euclidean plane using arbitrarily many copies of the tiles in a given Wang prototile set. Tiles are placed with their edges oriented horizontally and vertically. The tiles may not be rotated. The tiling is legal if every pair of contiguous edges has the same colour. [From [1]]

Definition 3 (Prototile set)**.** For our purposes, a prototile is defined as a labelled nonempty compact subset of $\mathbb{R}^d$ which is the closure of its interior. Sometimes a prototile is required to be connected, or to be homeomorphic to a closed ball.

In this thesis we require that prototiles have a cell complex as their boundary, and are connected. Note that Wang tiles are prototiles by this definition, thus Wang prototile sets are a special type of prototile set. Thus there are vertices (or *0-facets*). Similarly on the boundary of the prototile there are edges (or *1-facets*), faces and so on, up to $(d-1)$-facets.

Definition 4 (Tile, patch, tiling)**.** Let G be a group of isometries of $\mathbb{R}^d$, which includes all translations of $\mathbb{R}^d$. The groups we will be using most in this thesis are the group of translations G_{Tr} and the group of all isometries G_I. We will be using these groups to define legal isometries from prototiles to tiles.

Define $g(U)$ for some $U \subseteq \mathbb{R}^d$ and $g \in G$ as the set of points $p \in \mathbb{R}^d$ such that $g^{-1}(p) \in U$. For a given set of prototiles $\mathcal{P}$ and a group of isometries G, define a *tile* t as the image $g(P)$, for some $g \in G$, $P \in \mathcal{P}$. A tile inherits vertices, edges and other facets from its prototile.

A *patch* for $\mathcal{P}$ is a set of tiles with pairwise disjoint interiors and the *support* of a patch is the union of its tiles. A *tiling* T with prototiles $\mathcal{P}$ is a patch with support $\mathbb{R}^d$. We shall refer to the support of a prototile P as *supp*(P).

Definition 5. A tiling T is *edge-to-edge* if for any facets f_1, f_2 in the tiling, $f_1 \cap f_2$ is either f_1, f_2 or $\emptyset$. This corresponds to adjacent tiles only sharing full sides with each other.

All tilings considered in this thesis will have the edge-to-edge property.

Tilings may be required to obey additional sets of rules, termed *matching rules*.

Definition 6 (Matching rules)**.** A tile t has *local matching rules* if you can determine whether the tiles placement at any point $p \in \mathbb{R}^2$ in a tiling is legal by considering the tiles in $B_r(p)$ (where r is not dependent on the point p). A tile t has *meeting matching rules* if you can determine whether its placement in $\mathbb{R}^2$ is legal by only considering points within ϵ of t, for all $\epsilon > 0$.

A legal tiling is a tiling which obeys its matching rules, and where tiles meet edge to edge. We will only consider legal tilings henceforth.

Note that matching rules are often defined on a prototile, by colouring or deforming edges. This corresponds to having meeting matching rules for all tiles which are preimages of that prototile.

There are a number of distinct uses in the field of aperiodic tilings of the term aperiodic. We shall use the following definitions, originating from [28] and in the format of [24];

Definition 7. A tiling $T \subset \mathbb{R}^d$ is *weakly periodic* if there exists an infinite cyclic subgroup H of isometries of $\mathbb{R}^d$ such that $HT = T$ (i.e. for all $h \in H$, $hT = T$). A tiling that is not weakly periodic is said to be *strongly non-periodic*. A set of prototiles $\mathcal{P}$ which can only construct strongly non-periodic tilings, is said to be *strongly aperiodic*.
A tiling T of $\mathbb{R}^d$ is *strongly periodic* if there exists a discrete subgroup H of isometries of $\mathbb{R}^d$ with $\mathbb{R}^d/H$ compact and $HT = T$. A tiling that is not strongly periodic is *weakly non-periodic*. A set of prototiles is *weakly aperiodic* if it can only construct weakly non-periodic tilings.

Note that in two dimensions the two definitions of aperiodicity are equivalent (see Theorem 3.7.1 in [26] for proof). Thus we refer to 'aperiodicity' in two dimensions, and 'strong aperiodicity' in three or higher.

Finally, let us define a *Delone set* [44] and the related concept of a *Voronoi cell* [34] .

Definition 8. A point set S in $\mathbb{R}^d$ is called a *Delone set*, if it is uniformly discrete and relatively dense; i.e., if there are numbers $R > r > 0$, such that each ball of radius r contains at most one point of S, and every ball of radius R contains at least one point of S.
Let $\Lambda \in \mathbb{R}^2$ be any Delone set (or even a finite point set). The Voronoi cell of a point $x \in \Lambda$ is the set of points in $\mathbb{R}^2$ that lie at least as close to x as to any other point of Λ.

Chapter 2

Constructions to reduce complexity of matching rules

This chapter will give a constructive method that can decrease the number of prototiles needed to tile a space. In doing so we will exchange facet matching rules (a type of meeting matching rule) for a matching rule defined by a small atlas of permitted patches (which is thus only a local matching rule). We will also describe when meeting matching rules can be expressed in terms of alterations to the shape of a prototile. A paper based on these concepts has been published, [14].

The constructive method is illustrated with Wang tiles, and we apply our method to present via these rules a single prototile that can only tile $\mathbb{R}^3$ aperiodically, and a pair of square tiles that can only tile $\mathbb{R}^2$ aperiodically. This ties in with work by Gummelt in 1996 [45], and Socolar and Taylor [40] in early 2010, to produce prototiles that can only tile aperiodically under certain conditions (Overlapping tiles and disconnected tiles respectively). This paper extends work by Goodman-Strauss on 'atlas matching rules'. In an aside in [24], Goodman-Strauss describes how by requiring a tiling be covered by a suitable finite atlas of permitted bounded configurations, a domino can serve as a monotile. The aside only produced a weak upper bound on the size of the atlas, but Goodman-Strauss postulated the bound could be reduced considerably.

We have already defined our use of the terms prototile, tiling and (strong) aperiodicity (in the last section). We will now formalise the concepts of matching rules based on colours (including a generalisation) or atlases. We will then describe a method of altering coloured matching rules to atlas matching rules with very small patches. If two or more of the tiles have the same shape, the number of prototiles needed is decreased. This method can be applied to tilings in general, not just aperiodic ones. We will restrict ourselves to connected prototiles in this chapter for clarity, but the general method can be applied to disconnected prototiles as well.

In subsection 2.2 we use our method to construct a pair of square tiles which can only tile $\mathbb{R}^2$ aperiodically, and a single cubic tile that can only tile $\mathbb{R}^3$ in a (strongly) aperiodic manner. Further improvements to the method are described in subsection 2.3.

2.1 The atlas matching rule construction

For clarity we will be limiting the spaces we are tiling in this chapter to $\mathbb{R}^d$ for some $d \in \mathbb{N}$. With minor alterations the method will work in any homogeneous space (for example hyperbolic space $\mathbb{H}^d$).

In this chapter, we require that tiles meet in whole facets and that tiles are connected. Similar ideas can be applied to tiles which are disconnected.

We will be using patches defined by the '1-corona' about a tile t.

Definition 9. The '1-corona' of a tile t is the set of tiles with non-empty intersection with t (see [24]).

As an example, see picture 2.1. This picture shows a tile t, and its 1-corona (all blue tiles, including t itself).

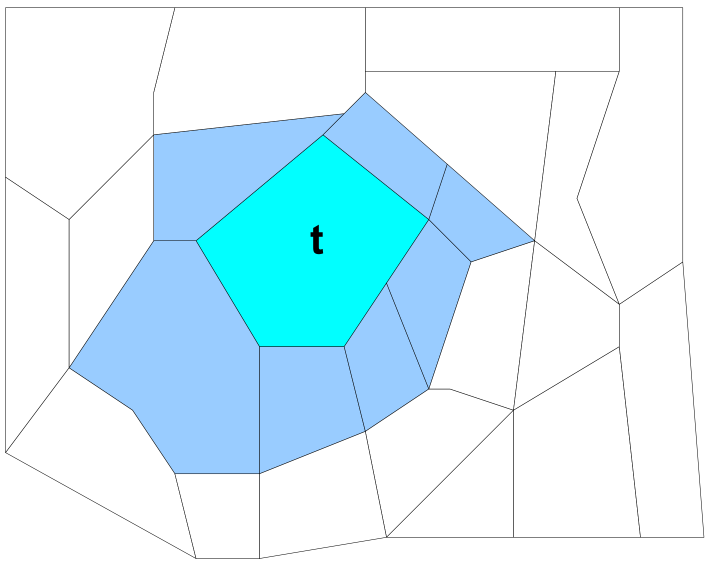


Figure 2.1: A tile t, and its 1-corona

We now want to introduce the notion of ‘colouring’ a prototile’s boundary, and hence all tiles produced from it. In this chapter we only care about the highest dimensional parts of a boundary. Thus for 2 dimensional tiles we only care about edges of a tile, not vertices. For 3 dimensional tiles we only care about faces, not edges or vertices, and so on. Thus for an d dimensional tile, we will assign to each $(d-1)$-dimensional facet of the boundary of the tile an element of a given set, as follows. Throughout this (and future) chapters, we will be using the definitions from the previous section entitled ’basic definitions’.

Let $\mathcal{P}$ be a set of prototiles in $\mathbb{R}^d$. Construct a function $\lambda : \{(d-1)\text{-}facets\ of\ P \in \mathcal{P}\} \to C$ where C is a non-empty set. A facet x of the prototile P is *c-coloured* if $\lambda(x) = c$.

Extend λ to facets of any given tile $t = g(P)$ by $\lambda_t(x) = \lambda g^{-1}(x)$ for each $(n-1)$-facet x of t.

Definition 10. A coloured tiling (T, λ) of $\mathbb{R}^d$ satisfies the *identical facet (matching) rule* if for all tiles t_1, t_2 $\lambda_{t_1}(x) = \lambda_{t_2}(x)$ for each $(d-1)$-facet x that t_1 and t_2 share.

This covers cases where two tiles 'match' if they have the same colour on the interior of their shared boundary (for example Wang tiles, which match when they share edges of a common colour).

We will be using a slightly more general version of this rule in the rest of this chapter, which allows tiles to match under wider conditions, as follows.

Definition 11. A *facet (matching) rule* is a function on pairs of colours $r : C \times C \mapsto \{0, 1\}$ such that $r(x, y) = r(y, x)$. A coloured tiling (T, λ) satisfies the *facet (matching) rule* r if for all tiles t_1, t_2 (where $t_1 \neq t_2$),

$$r(\lambda_{t_1}(x), \lambda_{t_2}(x)) = 1$$

for all facets x of $t_1 \cap t_2$.

The obvious question is when do these facet matching rules coincide with matching rules defined only in terms of shape of tile boundary. As stated at the start of the section, we are only using connected tiles which match facet-to-facet.

Let us consider the matrix corresponding to the facet matching rule r, where $a_{ij} = r(c_i, c_j)$ for some fixed enumeration of the colours which can be associated to the tiles' edges. Note that the matrix must be symmetric, since $r(x, y) = r(y, x)$.

We will assume colours have been made 'distinct' in the sense that if for $c_i, c_j \in C$, $r(c_i, x) = r(c_j, x)$ for all $x \in C$, then $i = j$. This condition is equivalent to not having two different colours which match in precisely the same way (and could thus be identified together). In this matrix this corresponds to ensuring no row is a copy of another row, by removing rows until this is not the case. Similarly we will ignore colours which cannot match any colour, including themselves, because any tile with such a colour on its boundary cannot occur in a tiling satisfying that facet matching rule.

Then a matching rule r can be expressed in terms of shape of tile boundary if and only if there is only one 1 entry in every row and column of its matrix.

We shall illustrate why this is true, starting with the two dimensional case. If you have two tiles with a common edge meeting at vertices u, v then for a given curved edge on one tile, there exists precisely one curved edge which meets it at every point. Similarly in $\mathbb{R}^d$, for any $(d-1)$-facet on the boundary of a tile, there exists precisely one $(d-1)$-facet which can meet it. Thus for any coloured boundary facet, there is only one other colour of facet that can meet it. Thus any row or column of the matrix must have only one non-zero entry.

Let us consider some examples.

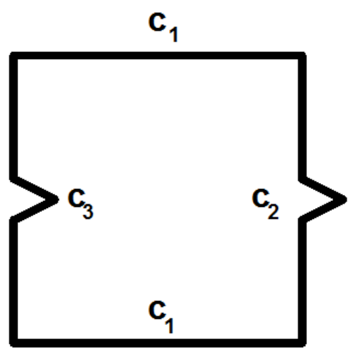
Example 1. *Consider the facet rule matrix in the following figure.*

This matrix has only one entry with the value 1 *in each row, thus it can be expressed in terms of curved edges. One way of doing this would be to set* c_1 *as a straight edge,* c_2 *as a edge with a protrusion out from it, and* c_3 *as the unique edge that can fit to it.*

Example 2. *Let the set of colours on a tiling be the set of cards in a normal* 52 *playing card deck.*

Let $r(c_i, c_j) = 1$ *iff* c_i *and* c_j *are from the same suit, or have the same face value.*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$



Every colour matches to a unique subset of colours, thus we do not need to remove rows or columns from the matrix. Furthermore, each colour matches to 16 *other colours (the* 13 *in the same suit, and* 3 *other cards with the same face value). Thus this matching rule cannot be expressed in terms of curved edges.*

We describe below a way of translating from a facet matching rule to a matching rule of the following type.

Definition 12. A tiling T satisfies an *atlas (matching) rule* $\mathcal{U}$ if there exists a finite atlas of compact patches $U \in \mathcal{U}$ such that for every tile $t \in T$, there exists a patch U_t about t (with t being in the strict interior of U_t) such that $U_t = g(U)$ for some $g \in G_{Tr}$ and $U \in \mathcal{U}$.

A *1-corona atlas rule* is an atlas rule where every patch $U \in \mathcal{U}$ is the 1-corona of some tile t.

Definition 13. A tiling T is a $(\mathcal{P}, G, \lambda, r)$*-tiling* if it has a prototile set $\mathcal{P}$ with allowable isometries G and colouring λ, and satisfies the facet rule r.

A tiling T is a $(\mathcal{X}, G, \mathcal{U})$*-tiling* if it has a prototile set $\mathcal{X}$ with allowable isometries G, and satisfies the atlas matching rule $\mathcal{U}$.

A tiling A is *locally derivable* from a tiling B if there exists a length R such that, if $z_1, z_2 \in \mathbb{R}^d$ and $A - z_1$ agrees with $A - z_2$ on a ball of radius R around the origin, then $B - z_1$ agrees with $B - z_2$ on a ball of radius 1 around the origin. Thus the tile at a point z in B depends only on a finite patch around z in A. If B is locally derivable from A and A is locally derivable from B, then A and B are said to be *mutually locally derivable* (MLD) tilings. (The definition of MLD originates in [6], but we are using the equivalent variation found in [9]).

Theorem 1. A $(\mathcal{P}, G_{Tr}, \lambda, r)$-tiling T is MLD to a $(\mathcal{X}, G_I, \mathcal{U})$-tiling for some 1-corona atlas rule $\mathcal{U}$ and a prototile set $\mathcal{X}$ with $|\mathcal{X}| \leq |\mathcal{P}|$.

Construction 1. Take $\mathcal{P}$ and partition it into a set of equivalence classes $\mathcal{P} = \coprod \mathcal{P}_s$, $s \in \{1, \ldots, m\}$ where $P_i \sim P_j$ iff $supp(P_i) = supp(P_j)$ up to the action of an element of G_{Tr}. For each $\mathcal{P}_s$, let Ψ_s be the group of automorphisms of any of the prototiles $P \in \mathcal{P}_s$. Enumerate the elements of $\mathcal{P}_s$ as $P_1^s, \ldots, P_r^s \in \mathcal{P}_s$.

Ideally we would now construct an injective function from $\mathcal{P}_s$ to ordered pairs of P_1^s and some automorphism of P_1^s. This cannot always be done, since in some cases $|\mathcal{P}_s| > |\Psi_s|$. To cover this possibility, we can attempt to construct an injective function onto $(P_1^s \times \Psi_s) \cup (P_2^s \times \Psi_s)$. If an injective function cannot be constructed under those conditions, we can attempt to construct an injective function onto $(P_1^s \times \Psi_s) \cup (P_2^s \times \Psi_s) \cup (P_{3s}^s \times \Psi_s)$, and so on.

Formally, choose the smallest k you can so as to construct an injective function $e_s : \mathcal{P}_s \to \{(P_i^s, \psi_j) | 1 \leq i \leq k, \psi_j \in \Psi_s\}$. Define $\mathcal{X}_s = \{P_1^s, \ldots, P_k^s\}$.

We now have a construction taking prototiles $P_i^s \in \mathcal{P}_s$ to ordered pairs of a prototile from $\mathcal{X}_s$ and an automorphism of that prototile. Observe that $\mathcal{X}_s$ is a subset of $\mathcal{P}_s$.

Proof of Theorem 1

Define a new prototile set $\mathcal{X} = \mathcal{X}_1 \cup \ldots \cup \mathcal{X}_m$, where $\mathcal{X}_s$ is as just defined. Let the set of allowable functions from the prototiles into $\mathbb{R}^d$ be G_I, instead of G_{Tr}. Take the set of allowable 1-coronas in the $(\mathcal{P}, G_{Tr}, \lambda, r)$-tiling T, and replace every tile originating from a translation of a prototile $P_a^s \in \mathcal{P}_s$ with $\psi_j(P_i^s)$, with ψ_j and P_i^s originating from $e_s(P_a^s) = (P_i^s, \psi_j)$. This will give you a set of 1-corona patches of $\mathcal{X}$. Use this set as the atlas rule $\mathcal{U}$ for $\mathcal{X}$.

T has facet rules, which are intrinsic to the set of allowable first coronas (since the set of allowable first coronas list what boundaries are allowed to meet each other). Since our definition of $\mathcal{X}$ and its atlas correspond to the first coronas of tiles in T, with P_a^s replaced by $g_j(P_i^s)$, any tiling by $\mathcal{X}$ is MLD to a tiling from $\mathcal{P}$. Since $|\mathcal{X}_s| \leq |\mathcal{P}_s|$ then $|\mathcal{X}| \leq |\mathcal{P}|$. $\square$

For reasons of clarity, we will give a concrete example of how to move from a given prototile $\widehat{P}$ in $\mathcal{P}$ to its image $g_j(\widehat{P}_i^s)$. Consider figure 2.2. At the top of the picture we have a set of 13 prototiles, $\mathcal{P}$. The first step is to group the prototiles into sets (the $\mathcal{P}_s$ of the construction). A set must contain prototiles which have the same support up to translation, but cannot have more prototiles than there are automorphisms of that support. For example, with the rectangular tiles, there are four automorphisms which will send a rectangle onto itself. Thus the sets containing rectangles cannot contain more than four rectangles. (Note that the two F shapes must be placed in different $\mathcal{P}_s$ sets since they do not have the same support up to translation. In the final section of this chapter we will describe a construction which

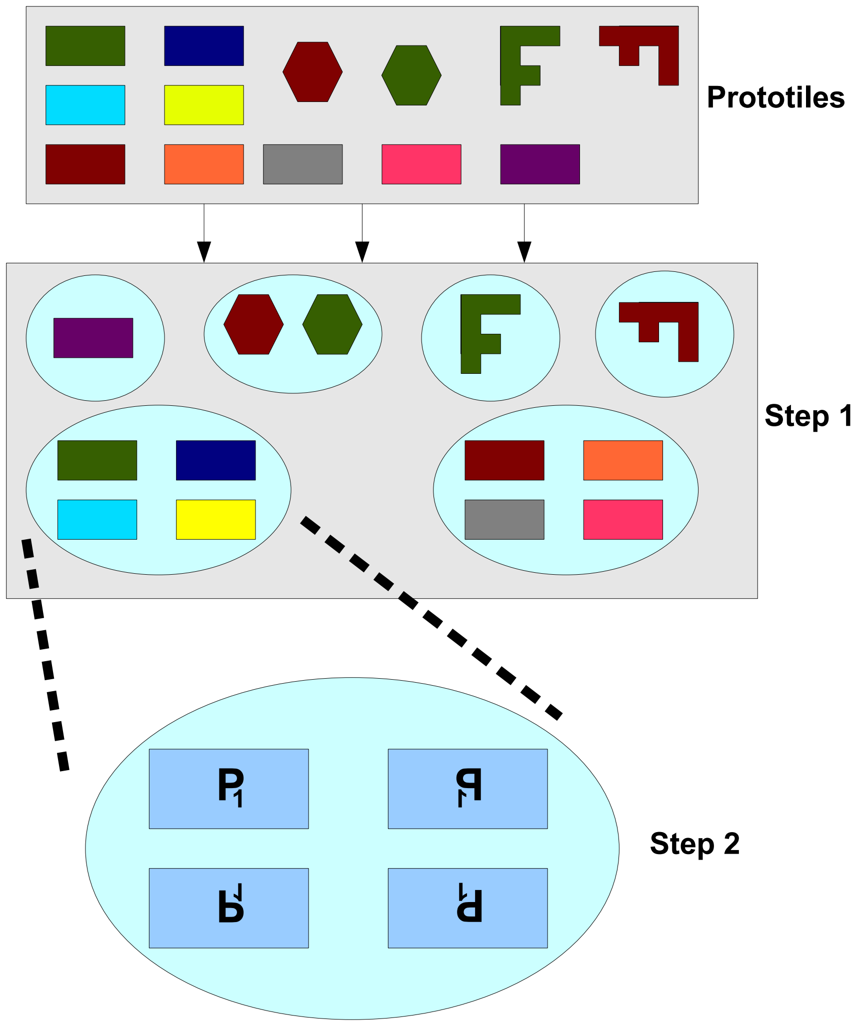


Figure 2.2: Applying the construction to a set of prototiles.

can place them in the same $\mathcal{P}_s$ set.)

The second step is to take one of the new collections of tiles with the same support. In the figure we have taken one of the sets of rectangles. Then replace one of the tiles with a new prototile P_i. For each other prototile, take P_i and apply a different automorphisms of the support of P_i to associate an automorphism of P_i to that prototile. Then substitute the new prototiles you have produced into the allowable 1-coronas of $\mathcal{P}$ to produce a tiling with less prototiles needed.

Corollary. Take a prototile set $\mathcal{P}$ and partition it into a set of equivalence classes $\mathcal{P} = \coprod \mathcal{P}_s,\ s \in \{1, \ldots, m\}$ as in the previous construction. If there exists $\mathcal{P}_s$ such that $|\mathcal{X}_s| < |\mathcal{P}_s|$, there exists a prototile set (with atlas rules) which tiles $\mathbb{R}^d$ with less prototiles than $\mathcal{P}$.

Proof. We know that $|\mathcal{X}_s| < |\mathcal{P}_s|$, thus $|\mathcal{X}| < |\mathcal{P}|$. $\square$

Remark. This method of construction produces a prototile set with cardinality $\sum_{s=1}^{m} \lceil \frac{|\mathcal{P}_s|}{|\Psi_s|} \rceil$.

Remark. $\mathcal{P}$ is (strongly) aperiodic iff $\mathcal{X}$ is (strongly) aperiodic. This is because every tiling in $\mathcal{X}$ is MLD to a tiling in $\mathcal{P}$, and strong aperiodicity is preserved under MLD equivalency.

2.2 Motivating examples and aperiodicity

Example 3. *For a simple illustration of the construction method, let us consider a tiling of the plane by 13 Wang prototiles (unit squares with matching rules defined by matching coloured edges) as given in [11, 18]. Label the Wang prototiles as $\{Q_1, \ldots, Q_{13}\}$. We can apply the above construction to get a function from $\{Q_j\}_{j=1}^{13}$ to $\{(P_i, r) | 1 \leq i \leq 2, r \in D_4\}$, where D_4 is the group of symmetries of the square.*

For example, enumerate the symmetries of the square as $\{r_1, r_2, \ldots, r_8\}$*. Then such a function could send* $\{Q_j | 1 \leq j \leq 8\}$ *to* (P_1, r_j)*, and the remaining tiles* $\{Q_j | 9 \leq j \leq 13\}$ *to* (P_2, r_{j-8})*. The result is shown in diagram 2.3, for a small patch of the tiling.*

As is common with Wang tiles, the colouring of $\{Q_j\}_{j=1}^{13}$ is represented as actual colours superimposed onto the tile. For diagram 2.3 we represent the change of prototile set from $\{Q_j\}_{j=1}^{13}$ to $\{P_1, P_2\}$ by adding a label to P_1 and P_2, which looks like their alphabetical symbols. This label will be visually affected by the automorphisms of P_1 and P_2. This is solely for the diagram, and is intended to assist the reader in distinguishing between the various automorphisms of P_1 and P_2.

The top picture of the diagram shows a tiling with prototiles $Q_1, \ldots, Q_{13}$ with facet matching rules, and translation as an isometry group. The bottom picture shows the resultant tiling, which uses a two element prototile set, with rotations, reflections and translations as a isometry group.

Next we will consider a 3 dimensional example. As mentioned before, aperiodicity in higher dimensions is more complicated and we have to worry about weak and strong aperiodicity. We will solve this problem by extending a known example of a prototile set which only tiles in a strongly aperiodic manner.

Example 4. *Consider Kari's Wang cube prototiles, W [20]. This is a set of 21 unit cube prototiles with facet matching rules, where every tiling of* $\mathbb{R}^3$ *by* W *is strongly aperiodic.*

Choose a unit cube prototile A*.*

Since the set of isometries of the cube (and thus A*) is of cardinality 48, we can choose 21 unique isometries of* A*,* i_k*,* $1 \leq i \leq 21$*. We use the method in Construction* 1 *to replace* $P_k \in W$ *with* $i_k(A)$*.*

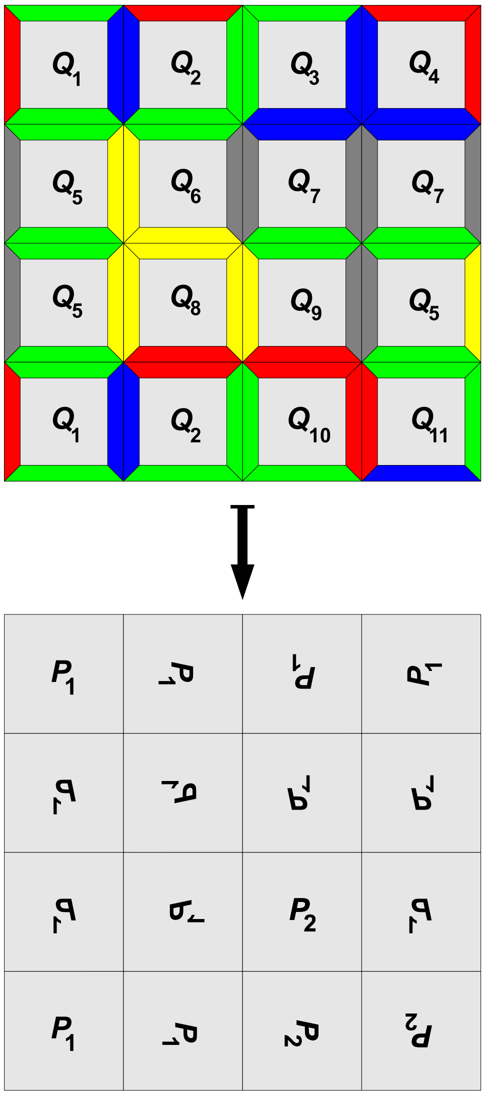


Figure 2.3: Construction applied to a tiling with 13 prototiles.

Thus we have an aperiodic protoset with one prototile which is MLD to Kari's Wang Cubes. Note that we have lost the property of matching rules being determined on faces, and replaced them with a set of legal one corona patches (which cannot be rotated or reflected, of course). We have also had to broaden the set of allowable mappings of the prototiles into the tiled space, from translations to translations and rotation/reflections.

2.3 Further improvements

Remark. The construction can be further improved, by partitioning $\mathcal{P}$ into equivalence classes based on what prototiles have the same support *up to isometry*, not just translation.

Let T be a $(\mathcal{P}, G_{Tr}, \lambda, r)$-tiling as in Construction 1. If there is a prototile $P_i \in \mathcal{P}$ whose support is a non-trivial isometry of another prototile P_j (where $i \neq j$), then the resulting $(\mathcal{X}, G_I, \mathcal{U})$-tiling may have less prototiles than one originating from Construction 1. An example of this is in figure 2.2, where the two F-shape tiles have equivalent support up to rotation.

Construction 2. Partition $\mathcal{P} = \coprod \mathcal{P}^s$, $s = \{1, \ldots, p\}$ where $P_i \sim P_j$ iff $supp(P_i) = supp(P_j)$ up to the action of an element of G_I.

Further partition $\mathcal{P}^s = \coprod \mathcal{P}^s_t$, $t = \{1, \ldots, q\}$ where $P^s_a \sim P^s_b$ iff $supp(P^s_a) = supp(P^s_b)$ up to the action of an element of G_{Tr}.

This two-stage partitioning gives us a collection of equivalence classes ($\coprod_{s,t} \mathcal{P}^s_t$) as per the first construction. Additionally we know that there exist isometries in G_I from elements of $\mathcal{P}^s_i$ to elements of $\mathcal{P}^s_j$. Take the $\mathcal{P}^s_t$ with the largest cardinality and denote it $\mathcal{P}^s_T$. From the definition of $\mathcal{P}^s$ there exists an isometry $\alpha_{P_i P_j}$ such that $\alpha_{P_i P_j}(supp(P_i)) = supp(P_j)$. Furthermore we know that an given isometry can only take elements from one set $\mathcal{P}_i^s$ to $\mathcal{P}_T^s$ (by definition of equivalence class). Thus we can replace any prototile in $\mathcal{P}_t^s$ with a unique isometry of a prototile in $\mathcal{P}_T^s$, since $|\mathcal{P}_t^s| \leq |\mathcal{P}_T^s|$.

By applying the previous construction to $\mathcal{P}_T^s$, we can get a minimal uncoloured prototile set $\mathcal{X}^s$ that can be used to translate prototiles in $\mathcal{P}_T^s$, and hence $\mathcal{P}^s$, to atlas rules.

Example 5. *Take a prototile set $\mathcal{T}$ of equilateral triangles, as shown in figure* 4. *The prototiles have two different orientations, and three (could be up to six) colours.*

We partition $\mathcal{T}$ into $\mathcal{T} = \mathcal{T}_1$, since all prototiles in $\mathcal{T}$ have the same support, up to isometry. We then further partition $\mathcal{T}_1 = \mathcal{T}_1^1 \coprod \mathcal{T}_1^2$, where $\mathcal{T}_1^1$ is the set of prototiles with point upwards, and $\mathcal{T}_1^2$ is the set of prototiles with point downwards. Denote the first prototile of $\mathcal{T}_1^1$ as t_1. Applying the first construction to $\mathcal{T}_1^1$ gives you $f(P_i \in P_1^1) = d_i(t_1)$, for $d_i \in D_3$, and $f(P_i \in P_1^2) = \mathrm{rot}_{\frac{\pi}{3}}(d_i(t_1))$. While this is sufficient to define the tiling, it has the problem that any picture of the tiling needs to include information about the isometries used for each tile. Thus we replace t_1 with a tile x with an uncoloured boundary, but with a coloured interior which is not preserved under any non-identity element of D_3.

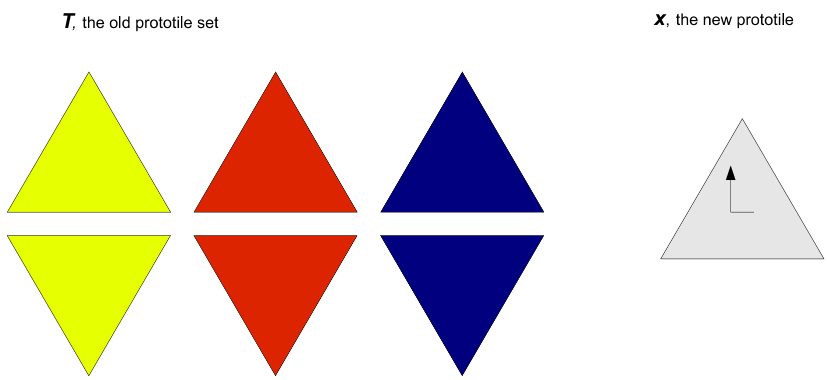


Figure 2.4: New and old prototile set

Chapter 3

Robinson tilings and Local Isomorphism classes

We will now consider *tiling spaces*, and the invariant *Local Isomorphism classes*. This chapter will give necessary definitions, and describe a tiling space (which is a central example used in this thesis), which has two Local Isomorphism classes.

Our first step is to construct a metric for a set of tilings. We will use the most common metric in the field, using the notation of [36].

Definition 14. [36] Consider a set $\mathbb{T}$ of tilings of $\mathbb{R}^d$, and choose any two tilings $T_1, T_2 \in \mathbb{T}$ without loss of generality. Define $R(T_1, T_2)$ as the supremum of all radii r such that there exist vectors x, y with $|x| < \frac{1}{2r}$ and $|y| < \frac{1}{2r}$, and $T_1 - x$ and $T_2 - y$ agree on $B_r(0)$, the ball of radius r about the origin.
Then define the *standard (tiling) metric* on S as $d(T_1, T_2) = min\{1, \frac{1}{R(T_1,T_2)}\}$.

Informally, T_1 and T_2 are ϵ-close if they agree on a ball of radius $(\frac{1}{\epsilon})$ around the origin, up to small translation.

Definition 15. Define a (P, G, λ, r)-*tiling space* as the set of all (P, G, λ, r)-tilings (as defined in definition 13 in the previous chapter), under the standard metric.

This will commonly be referred to as the *tiling space of a set of prototiles*, implicitly assuming standard isometries, matching rules and colours for those prototiles.

Definition 16. The *(continuous) hull* of a tiling T is the closure of the set $O(T) = \{T - x | x \in \mathbb{R}^d\}$ under the standard metric. The symbol Ω_T is often used for this concept.

Note that in many papers in the field, the hull is referred to as the tiling space of a tiling T. This is due to most tilings studied in the field only having one LI class, in which case the two definitions are equivalent.

Note that in practice we will often refer to the prototile set and its tiling space interchangeably.

Definition 17. [36] [2] Let $\mathbb{T}, \mathbb{T}'$ be tiling spaces, with a homeomorphism $f : \mathbb{T} \mapsto \mathbb{T}'$. Then $\mathbb{T}$ and $\mathbb{T}'$ are *MLD* if there exists a radius R such that, whenever two tilings $T_1, T_2 \in \mathbb{T}$ agree on a ball of radius R around x, then $f(T_1)$ and $f(T_2)$ agree on a ball of radius 1 around x.

Note that if two tilings T, T' are MLD as per definition 13, then their hulls Ω_T and $\Omega_{T'}$ will also be MLD (see [36] p.9 for a proof). This result does not hold true for tiling spaces in general.

Definition 18 (Local Isomorphism classes)**.** [23] Two tilings are locally indistinguishable if a copy of each patch of one tiling occurs in the other tiling and vice versa. A *Local Isomorphism Class (LI class)* is an equivalence class of tilings induced by this relationship.

More formally, define a *R-patch* around x in a tiling T as the set of tiles in T with nonempty intersection with the closed ball of radius R around x (Definition from [44]). Then the *Local Isomorphism Class* of T consists of the set of all ϵ-patches in T, for all $\epsilon > 0$. Let us call this $LI(T)$.

Two tilings T_1, T_2 belong to the same *Local Isomorphism Class (LI class)* iff $LI(T_1) = LI(T_2)$.

Remark. Note that all points in the hull of a tiling T will be in the same LI class. Thus this thesis will concentrate on tiling spaces of prototile sets. We will commonly refer to a 'tiling space of a prototile set P' as a 'tiling space' throughout this thesis for brevity.

Definition 19 (Repetitivity)**.** [44] A tiling T is *repetitive* if for every $r > 0$ there is an $R > 0$ such that every patch of radius r is contained in every patch of radius R.

A tiling space is repetitive if all of the tilings within it are repetitive.

We care about Local Isomorphism classes because if a repetitive tiling space has two tilings T_1, T_2 in different LI classes, then there is a minimum distance between translations of those tilings in the metric on the tiling space. Thus if we know the total number of LI classes in a tiling space (and the number of LI classes is finite), we gain information about the number of connected components in the topology. Since we currently only have limited methods ([3] , [5] etc) of understanding tiling spaces, this is a useful step.

Definition 20 (Robinson (*Rob*) tiling)**.** The Robinson prototiles are a variant on Wang prototiles. They consist of the unit prototiles in figure 3.1. (The prototiles originated in [33].)

The group of isometries associated with these prototiles consist of automorphisms of the square, composed with any translation. The colouring associated with these prototiles is the trivial one, with all facets mapping to 0. Thus the matching rule associated with the prototiles will allow any two facets to be placed

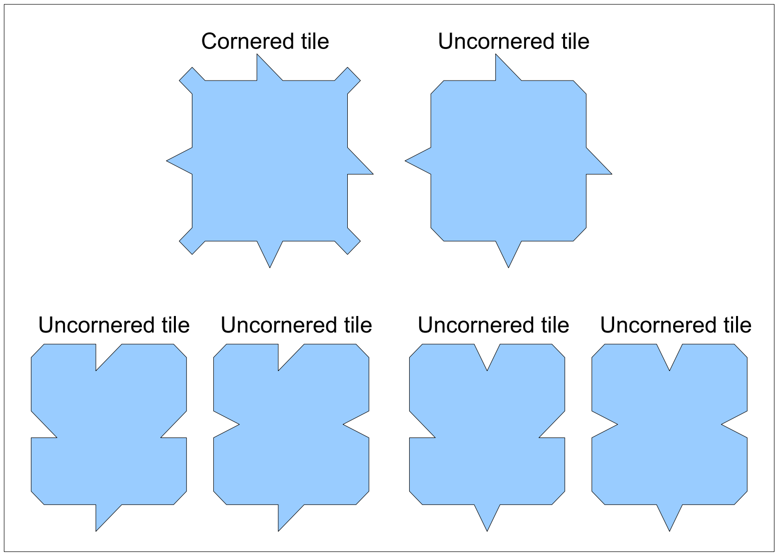


Figure 3.1: Robinson prototile set

next to each other (assuming they can fit edge-to-edge). Thus the tiling can be expressed with matching rules which only reference the shape of tile boundary. The prototiles can be divided into two sets, cornered and un-cornered tiles, depending on what the prototile looks like at a corner (vertex of the original Wang tile).

We know that automorphisms of the square are allowed with these prototiles, which can be slightly non-intuitive. Thus we will be using an alternative (and MLD) depiction of the prototiles, as shown in figure 3.2. (This alternative depiction also originates from [33], with the specific formatting deriving from [26].) Call the second depiction of the Robinson prototiles the 'arrowed' prototiles (R). This depiction adds a non-trivial colouring and matching rule, to simplify the shape of the prototiles. Note that the arrows and corner marks on the prototiles are a representation of the colouring of that prototile.

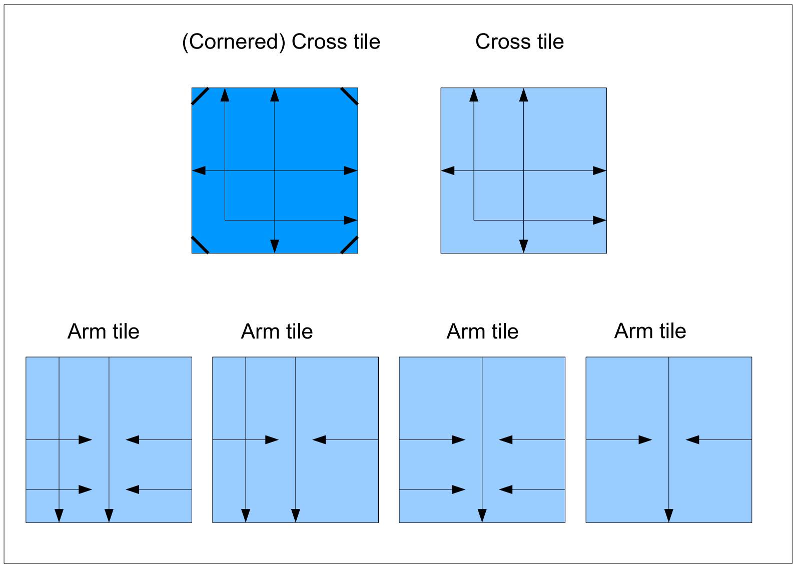


Figure 3.2: Alternate Robinson prototile set R

The group of isometries associated with the arrowed prototiles is also the automorphisms of the square composed with any translation. It is more intuitive in this case, since a legal tile in a tiling will be an automorphism of a prototile.

The colouring and matching rules of the prototiles will be designed so that two edges e_1, e_2 can be placed next to each other if every head of an arrow leading into e_1 meets up with the tail of an arrow leading from e_2, and vice versa.

Additionally every vertex of the tiling must have precisely one tile with a marked corner next to it.

We can divide the prototiles of the arrowed depiction into two sets. Cross prototiles are those in which arrows form an L-shape in the prototile (namely the first two tiles). Arm prototiles consist of the remaining prototiles. Note that there is a one to one isomorphism between the prototiles in one depiction of the Robinson tiling and the other depiction. For this paper, if one prototile has one of the properties cornered, un-cornered, arm or cross; we will consider its twin to have the property as well.

One problem we will have is that the Robinson prototiles can be made into half-plane or quarter-plane tilings of the plane. These partial tilings can be put together to form full-plane tilings, by joining partial tilings together. These partial tilings are joined together by 'faultlines'. In order to define faultlines rigorously, we will first define a coordinate set for our tiles.

Definition 21 (Tile Coordinates)**.** Consider $\underline{0}$ in a unit square tiling T of $\mathbb{R}^2$. If $\underline{0}$ belongs to the interior of a prototile, label that tile with the coordinate $(0, 0)$. Otherwise label as $(0, 0)$ the unique tile t s.t $\exists \epsilon > 0$ such that $(k, k) \in t$ for all $k < \epsilon$. Similarly assign coordinates $(a, b), a, b \in \mathbb{Z}$ to all other tiles.

Definition 22 (Bordering Tiles)**.** Two tiles s (labelled by (s_1, s_2)) and t (labelled by (t_1, t_2)) would then border each other iff $(s_1-t_1, s_2-t_2) \in (1, 0), (-1, 0), (0, 1), (0, -1)$.

Definition 23 (Directly Above, Below, Left, Right)**.** For a given tile s (labelled by (s_1, s_2)), the tile *directly above* it is the tile a bordering s where $(s_1 - a_1, s_2 - a_2) = (0, -1)$. Similarly the tile b *directly below* s has $(s_1 - b_1, s_2 - b_2) = (0, 1)$, the tile r *directly right* of s corresponds to $(s_1 - r_1, s_2 - r_2) = (-1, 0)$, and the tile l *directly left* of s corresponds to $(s_1 - l_1, s_2 - l_2) = (1, 0)$.

This rather cumbersome definition of bordering tiles has been chosen to interact well with future constructions, where we try to superimpose two tilings onto the same copy of $\mathbb{R}^2$, and we do not want our two tilings to interfere with each others' matching rules.

Definition 24 (Faultline)**.** A *Faultline* is a subset of a tiling consisting solely of arm tiles, such that for every tile t with coordinate (t_1, t_2) in the subset either;

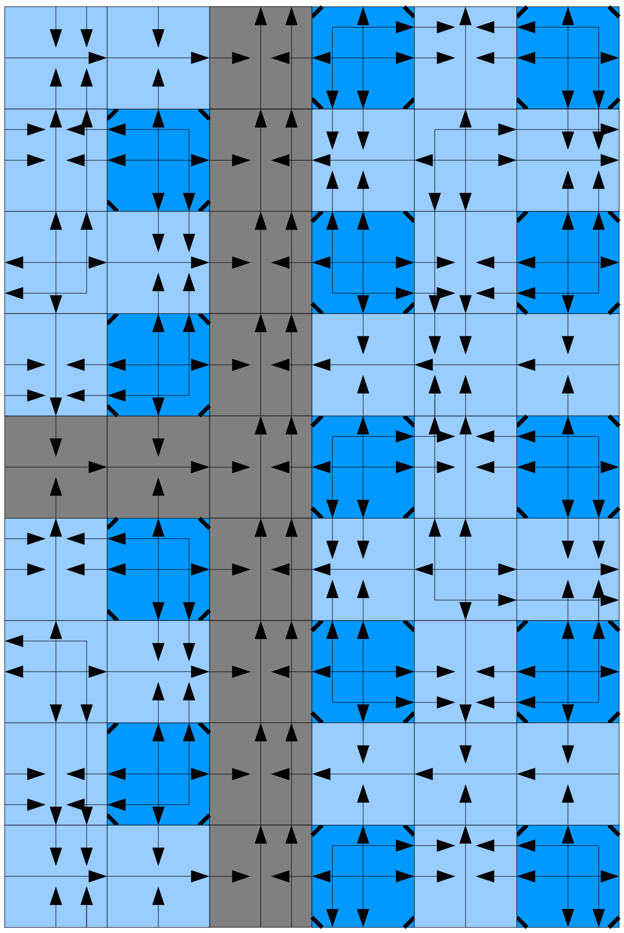


Figure 3.3: Sample faultline

(i) There exists an arbitrarily high number of arm tiles directly above or below t (more precisely, with coordinates $(t_1, t_2 + k)$ or $(t_1, t_2 - k), \forall k \in \mathbb{N}$) which are also in the tiling.

(ii) There exists an arbitrarily high number of arm tiles directly to the right or left of t (more precisely, with coordinates $(t_1 + k, t_2)$ or $(t_1 - k, t_2), \forall k \in \mathbb{N}$) which are also in the tiling.

One valid question is whether our Robinson tilings are repetitive.

Theorem 2. All Robinson tilings without faultlines are repetitive.

Proof. We know that $\forall n \in \mathbb{N}$ there exist $(2^n - 1) \times (2^n - 1)$ square blocks with facing cross tiles in the centres [26], which repeat horizontally and vertically with period 2^{n+1}. If there are no faultlines in a Robinson tiling, any patch in the Robinson tiling must be contained in one of these $(2^n - 1) \times (2^n - 1)$ blocks, for some n. □

If there is only one faultline, then it may still be a repetitive tiling, since rows and columns of arm tiles of arbitrary size appear in any Robinson tiling. A tiling with a faultline is repetitive iff all the hierarchical squares line up. Figure 3.4 is an example of a faultline which is not repetitive.

Interestingly, a tiling space allowing tilings with an infinite vertical faultline will have an infinite number of LI classes. This is because you can shift the half-plane of the tiling on the right side of the vertical faultline by any integer vertically. The hierarchical squares can thus be shifted out of alignment by any amount, thus producing tilings in an infinite number of LI classes. Thus there can be an infinite number of LI classes. Thus any questions about LI classes will need to rule out this kind of situation.

Theorem 3. It is not possible to have a tiling space containing Robinson tilings which does not have Robinson tilings with faultlines.

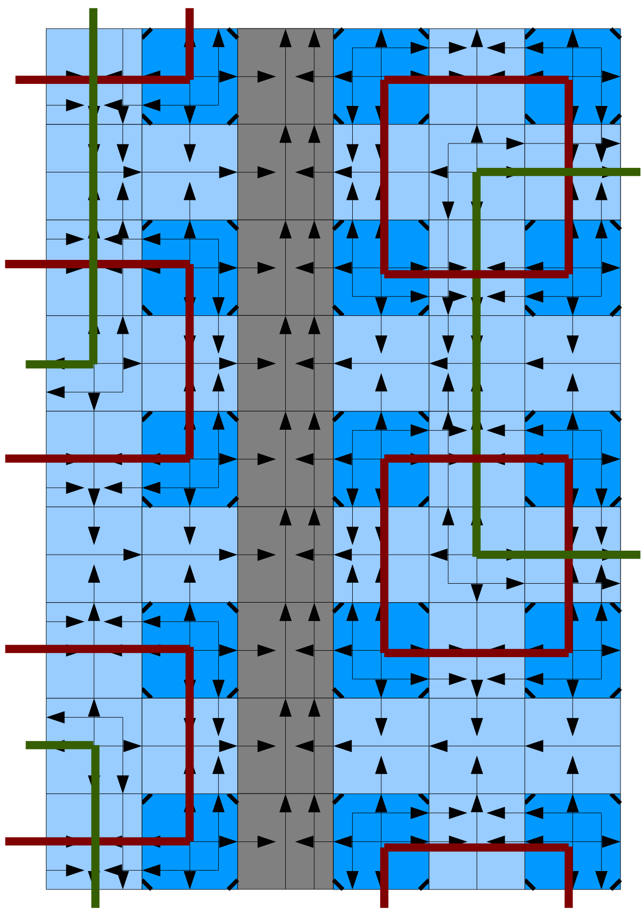


Figure 3.4: Faultline with hierarchical squares which do not line up correctly.

Proof. This proof requires concepts mentioned later in the thesis, namely definition 41 and lemma 11 from the end of chapter 6.

We know that the tiling space T generated by the Robinson prototiles is FLC (in the sense of definition 41), since there are a finite number of tiles which must meet edge to edge. Hence by lemma 11, it is compact, hence complete. Thus if we can find a Cauchy sequence of Robinson tilings which increases the length of a horizontal or vertical row of arm tiles, we can prove a tiling with a faultline is in the tiling space T (and any larger tiling space containing T).
We know that Robinson tilings have a hierarchal structure, where four $(2^{n-1}-1) \times (2^{n-1}-1)$ blocks form a $(2^n-1) \times (2^n-1)$ block, for all $n \in \mathbb{N}$ (See [26] for details, and figure 3.5).

Fix $n \in \mathbb{N}$, and consider one of these $(2^n-1) \times (2^n-1)$ blocks formed from four subblocks (with $n \geq 3$). Such a block consists of a cross tile in the exact centre of the square block, rows of $2^{n-1}-1$ arm tiles radiating out from it in all four directions separating four smaller $(2^{n-1}-1) \times (2^{n-1}-1)$ square blocks filling the rest of the space [26]. The arrows exiting these smaller $(2^{n-1}-1) \times (2^{n-1}-1)$ square blocks will all be single arrows, with the exception of arrows originating in a sub-blocks central cross tile.

Thus the row of 2^{n-1} arm tiles separating two of the sub-blocks will consist of $2^{n-2}-1$ tiles which have a single arrow entering opposite sides of that tile, one tile with a double-arrow entering opposite sides, and then another $2^{n-2}-1$ tiles with a single arrow entering opposite sides of each tile. Thus we can have patches with arbitrarily long strings of arm tiles of the same type. Furthermore, each string of $2^{n-2}-1$ arm tiles will have a 'sub-sub' block of size $(2^{n-2}-1) \times (2^{n-2}-1)$ on each side of it, with the central cross tile facing away from the row of arm tiles. See the yellow patch in figure 3.5.

Thus (due to the hierarchical nature of the Robinson tiling) we have a sequence of patches of increasing radius, which contain previous patches as inclusions. This lets us construct a Cauchy sequence (due to the metric on tilings being defined by tilings being close if they agree on balls of large radius) in the tiling space which increases the length of a row of arm tiles. Thus a tiling with a faultline is in the tiling space T (and any larger tiling space containing T).

□

In later sections we will be trying to construct tiling spaces with a given number of LI classes. Tilings with faultlines introduce an infinite number of LI classes when the tiling is not repetitive. Thus we will limit ourselves to only considering repetitive tilings.

Lemma 1. Any repetitive tiling will have cornered tiles with coordinates of the same parity.

Proof. We will use proof by contradiction. Take a patch P of radius $r > 0$ which does not have this property. If our tiling is repetitive, then $\exists R > r$ such that in any patch of radius R, the patch P exists. Pick a patch of radius R not containing part of a faultline. (We know such a patch must exist because a 3×3 square must exist within the tiling. Such a square can be extended to a 7×7 square, which can be extended again to a 15×15 square. Inductively, $(2^n - 1) \times (2^n - 1)$ squares exist for any $n \in \mathbb{Z}$, which do not contain faultlines). [26]

Any two cornered tiles in this patch will have coordinates of the same parity, from the construction of the Robinson tiling. Thus we have a contradiction. Thus any repetitive tiling will have cornered tiles with coordinates of the same parity. □

Remark. We can divide any repetitive tiling in the tiling space into two subsets, the set T_0 consisting of tiles with coordinates (a, b) such that $(a - b) = 0 \mod 2$,

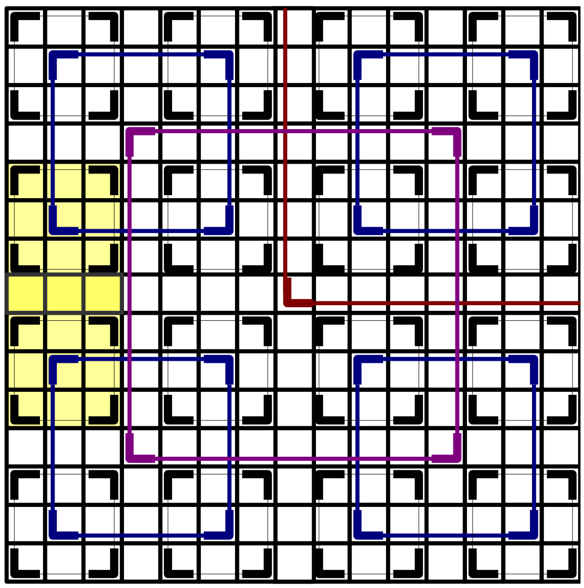


Figure 3.5: Schematic indicating a patch containing a row of $2^{n-2}-1$ arm tiles.

and T_1 consisting of tiles with coordinates (a, b) such that $(a - b) = 1 \mod 2$. If there is a cross tile in T_0 (T_1) then every tile in T_1 (T_0) is an arm tile. The set which contains cross tiles (WLOG T_0) can be further divided up.

Let

$$T_{even} = \{\textit{Tiles with labels } (a, b) | a, b \in 2\mathbb{N}\}$$

$$T_{odd} = \{\textit{Tiles with labels } (a, b) | a, b \in 2\mathbb{N} + 1\}$$

Then one of the two sets T_{even}, T_{odd} will consist solely of cornered (and hence cross) tiles, and the other a mix of uncornered cross and arm tiles.

Informally, cross tiles are restricted to a large subset of the black squares of an chessboard tiling of the plane, with arm tiles forming all of the white squares. Of the black squares, they are again divided into two sets, cornered and non-cornered, depending on whether their coordinates are both even or both odd (respectively) [17]. For an example patch of a Robinson tiling, see figure 3.6.

3.1 Octagon tilings

Consider our altered Robinson prototiles. The property of a prototile being a cross or arm tile does not depend on any decoration of the prototile within ϵ of a vertex of the prototile. Thus we can deform the square prototiles into symmetric octagon prototiles with horizontal and vertical sides of length $(1 - 2\epsilon)$, and diagonal short edges of length $(\sqrt{2}\epsilon)$. The matching rules of the square prototiles's edges are preserved on the horizontal and vertical edges of the new octagon prototiles. Denote these prototiles as R'.

Of course these prototiles do not fully tile the plane. Let us denote the area of the plane they do tile by Oct. (Note that the precise subset $Oct \subseteq \mathbb{R}^2$ may vary for

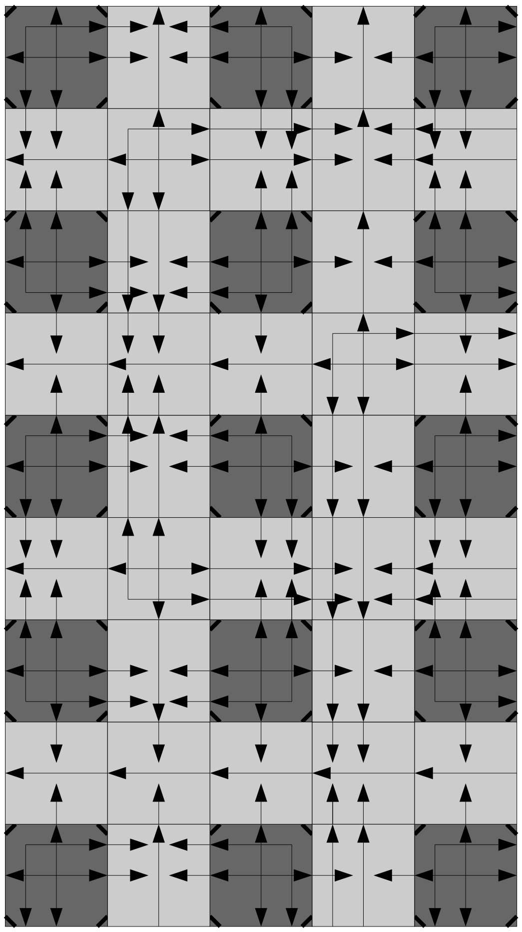


Figure 3.6: Robinson tiling

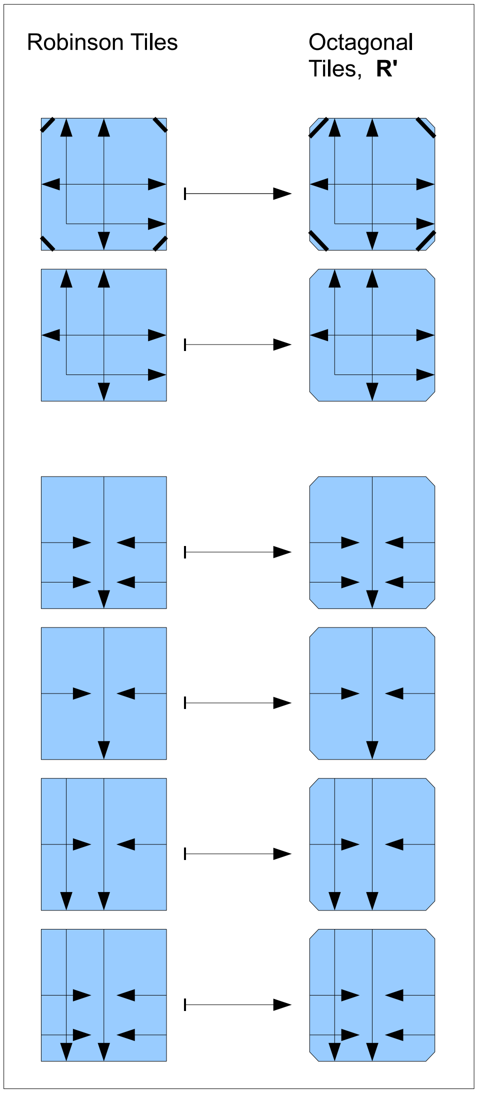


Figure 3.7: Function from R to R'

different tilings in the tiling space of R').

In order to expand this partial tiling to the whole plane, we will 'imprint' a second square tiling into the gaps left between the octagons, as follows;

Definition 25 (P^{shrunk})**.** Take any set of unit square prototiles P which tile the plane. Rotate the prototiles by $-\frac{\pi}{4}$ (ie, 45 degrees anti-clockwise). Then scale each of the prototiles by a factor of ϵ, as used when defining the octagons. The new tiles constructed are defined as P^{shrunk}.

We will now alter the matching rules of P^{shrunk} so that they tile $\mathbb{R}^2 \backslash Oct$.

First apply coordinates to $\mathbb{R}^2 \backslash Oct$. Choose a square tile t of $\mathbb{R}^2 \backslash Oct$, and label it $(0, 0)$. The square tile one unit in the y-axis above $(0, 0)$ will be labelled as $(0, 1)$ and the tile one unit to the right along the x-axis will be labelled as $(1, 0)$. These will form a basis to label all the tiles.

The matching rule between square tiles alters as shown in figure 3.8.

Explicitly, the northwest edge of a tile labelled (a, b) must be the same colour as the south-eastern edge of the tile $(a, b+1)$, and the northeast edge of a tile labelled (a, b) must match the south-western edge of the tile $(a+1, b)$.

This new tiling of $\mathbb{R}^2 \backslash Oct$ is mutally locally derivable to a tiling of the plane by the square prototiles P. Denote the prototiles with these new non-meeting matching rules as P'.

If we combine the prototiles P' with our octagon Robinson tiles R', we can tile the whole plane (R' tiling Oct, P' tiling $\mathbb{R}^2 \backslash Oct$). Note that there are no matching rules linking tiles in R' and P', thus any side from R' can abut a side from P'. We have to introduce an additional matching rule that no tile from P' can share an edge with another tile from P'. This will stop us tiling $\mathbb{R}^2$ using only tiles from P'.

The following definition gives us a more concrete construction.

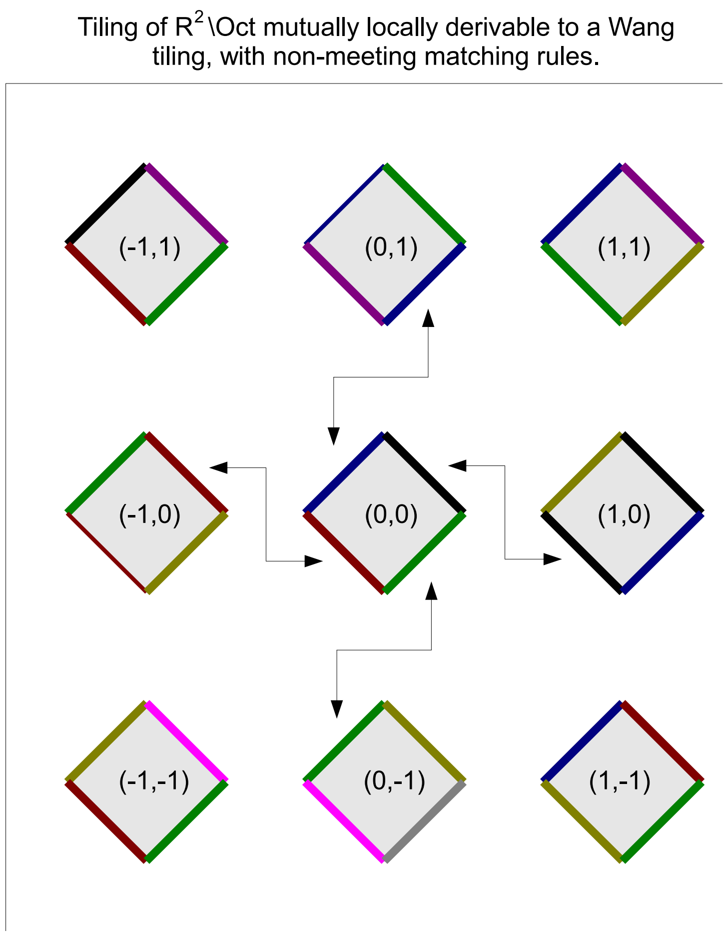


Figure 3.8: Tiling of $\mathbb{R}^2\backslash Oct$ MLD to a Wang tiling.

Definition 26 (Octagon tiling)**.** Let A,B be sets of unit square prototiles. Let $\epsilon \in (0, 0.5)$. Define a function $f_{Oct} : A \to A_{Oct}$ which sends an unit square prototile $a \in A$ to an unique octagon prototile $a_{Oct} \in A_{Oct}$ with horizontal and vertical sides of length $(1 - 2\epsilon)$, and diagonal short edges of length $(\sqrt{2}\epsilon)$. The matching rules on the edges of a now apply to the long edges of a_{Oct}.

Define a function $f_\epsilon : B \to B_\epsilon$ which rotates an unit square prototile $b \in B$ by $\frac{\pi}{4}$ anticlockwise about the centre of b, then scales the rotated b with a factor of ϵ. Call this new prototile b_ϵ. The matching rules on the edges of b now apply to the corresponding edges of b_ϵ.

Define $A \bigcup_{Oct} B$ as the tiling space with prototile set $A_{Oct} \bigcup B_\epsilon$, and function $f : \mathbb{R}^2 \to A_{Oct} \bigcup B_\epsilon$. The first condition is that no two tiles deriving from a prototile in B_ϵ can share a border. This can be forced by colouring the facets f of the prototiles in B_ϵ, and choosing a facet matching rule r such that for any two facets f_i, f_j (with associated colours c_i, c_j), the equation $r(c_i, c_j) = 0$ must hold. (See definition 11 in chapter 2 for terminology).

The second matching rule is that for any tiling if you replace all tiles $t_i \in f^{-1}(A_{Oct})$ (resp. $f^{-1}(B_\epsilon)$) with the corresponding square prototile, and then remove all tiles in B_ϵ (resp. A_{Oct}), you will get an allowable tiling of A (resp. B).

The described layout varies from most tilings present in the literature in that the 'matching rules' for P' are not meeting matching rules. If we widen our definition of a tile to allow non-connected prototiles (like in Socolar and Taylors's recent paper [40]), we can produce a similar set of prototiles which do have meeting matching rules.

In order to get this set of prototiles, start with our octagon Robinson prototiles R' and any set of square prototiles (which tile the plane) S. Deform a square prototile into a prototile consisting of four right-angled triangles of length ϵ as pictured in the following figure;

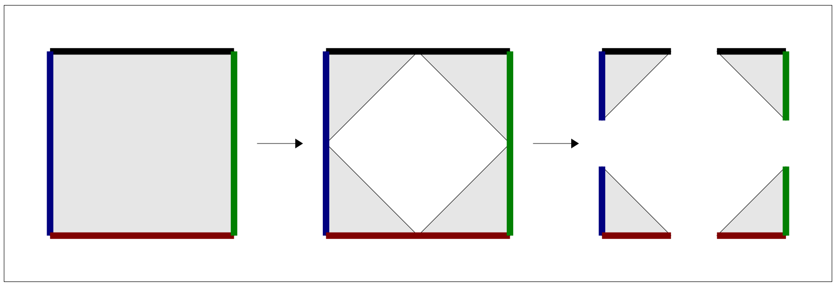


Figure 3.9: Deformation of a square tile into 4 triangles of length ϵ

Note that the new disconnected prototiles retain the same matching rules as S. Denote them as S^{Dis}. Clearly the set $R' \bigcup S^{Dis}$ can tile the plane aperiodically (since R' can tile Oct aperiodically, S^{Dis} can tile $\mathbb{R}^2 \backslash Oct$, and the matching rules of R' and S^{Dis} do not interfere).

Another way of having only meeting matching rules is to turn the octagon tile into a non-connected tile by deleting a rectangle from the NW corner to the SE corner, and from the SW corner to the NE corner. You then can use the space removed to implement matching rules between the 'imprinted' tiles.

3.2 Central Example

We have illustrated a way of superimposing two sets of square prototiles tiling the plane (at least one aperiodically) to get a new set of prototiles which tile the plane aperiodically. This section will describe a set of prototiles which tiles the plane aperiodically, and has members in two Local Isomorphism Classes.

First though, we will have to prove our altered Robinson tiling R only has 1 LI class.

Theorem 4. *R has only 1 LI class.*

Proof. Note that any tiling of $\mathbb{R}^2$ by R is determined by the direction of the non-cornered cross tiles within it. Our altered Robinson tilings can be considered to be constructed from 3×3 squares, four of which form a 7×7 square (with a choice of four non-cornered cross tiles in the centre of the square). These 7×7 squares themselves make up part of a 15×15 square, and so on.

Any patch in a tiling can be contained within a particular $(2^n - 1) \times (2^n - 1)$ square (denote it S^n) with a non-cornered cross tile at the centre of the square. The only variation possible in these $(2^n - 1) \times (2^n - 1)$ squares is the direction of the non-cornered cross tile.

Our S^n square is itself part of a larger S^{n+1} square (in a similar way that a 3×3 square is part of a 7×7 square. This S^{n+1} square is formed from four $(2^n - 1) \times (2^n - 1)$ sub-squares, which are identical to S^n with the sole exception of the central non-cornered cross tile. All four possible variations of central tile occur, in one of the four subsquares. Thus any patch can be found in a $(2^{n+1} - 1) \times (2^{n+1} - 1)$ square. Thus there is only 1 LI class for our altered Robinson tiling. For more details, see [26]. $\square$

Next, we move on to our example of an aperiodic tiling space with 2 LI classes.

Take our Robinson prototiles, altered as above to ensure cross tiles are only on even coordinates. Denote as R as before.

Definition 27 (A new set of unit square prototiles, $Chess$.)**.** Start with two unit square prototiles. Let one have black edges only, the other have only white edges. Denote tiles with only black edges as black tiles, denote tiles with only white edges as white tiles. Let the matching rule between two prototiles be that any for any line on which two tiles meet, one edge must be black, the other white. Clearly, these matching rules and prototiles lead to tilings similar to that of a chessboard, in that if you label a black tile $(0, 0)$, all tiles satisfying (a, b) s.t $a+b = 2n, n \in \mathbb{N}$ are black, and all other tiles are white.

We then deform the Robinson prototiles R to tile Oct as explained in the last section, forming a new set R'. We then deform $Chess$ to tile $\mathbb{R}^2 \backslash Oct$ via the method used in the last section. This produces a set of prototiles $Chess'$. Take the union of the prototile sets R' and $Chess'$, with the additional matching rule that no tile from $Chess'$ can share an edge with another tile from $Chess'$. Call this new prototile set $R' \bigcup_{Oct} Chess'$. Note that $R' \bigcup_{Oct} Chess'$ tiles the plane, since R' tiles Oct and $Chess'$ tiles $\mathbb{R}^2 \backslash Oct$.

Since R' tiles the plane aperiodically, and the matching rules between R' and $Chess'$ do not stop this aperiodicity, $R' \bigcup_{Oct} Chess'$ tiles the plane aperiodically.

Theorem 5. $R' \bigcup_{Oct} Chess'$ produces tilings in two Local Isomorphism Classes.

Proof. Consider any tiling T of the plane created from $R' \bigcup_{Oct} Chess'$. Choose a cross tile of T. Label it $(0, 0)$, and label all other octagon tiles as mentioned previously in definition 21. We know that all cross tiles must have labels (a, b) s.t

$a+b=2n, n \in \mathbb{N}$, and all tiles with labels (a,b) s.t $a+b=2n+1, n \in \mathbb{N}$ are arm tiles. We also know at least one cross tile exists.

Consider the cross tile labelled with $(0,0)$. This tile will have a square tile of width ϵ abutting its southwestern edge. Denote this square tile as $Ch(0)$, and label it $\widehat{(0,0)}$. The hat is to reduce confusion with tiles from R'. Label all other square tiles in T, following the method in the last section. A square tile has a label (a,b) such that $a+b=2n, n \in \mathbb{N}$, iff it has the same colour as $Ch(0)$.

There are no matching rules linking prototiles in R' and prototiles in $Chess'$. Thus $Ch(0)$ can be either black or white. If it is black, then we know that all square tiles labelled (a,b) such that $a+b=2n, n \in \mathbb{N}$ are also black (and no others). Thus every (octogonal) cross tile has a black tile abutting it to the southwest. Call these tilings 'SW-black' tilings.

There can also exist tilings of the plane created from $R' \bigcup_{Oct} Chess'$ where cross tiles have a white tile abutting them to the southwest (simply shift the partial tiling covering $\mathbb{R}^2 \backslash Oct$ by $(0,1)$). Call these tilings 'SW-white' tilings. Any ball B_ϵ s.t $\epsilon > \frac{\sqrt{2}}{2}$ in a SW-white tiling cannot occur in a SW-black tiling, since such a ball will cover an octagon tile and part of the square tile to its southwest.

Thus we have tilings in two distinct LI classes, produced by the prototiles $R' \bigcup Chess'$. Since R' only has one LI class (as does $Chess'$), we only have 2 LI classes. $\square$

Lemma 2. $R' \bigcup_{Oct} Chess'$ contains only repetitive tilings.

Proof. Any tiling which is MLD to a repetitive tiling is itself repetitive. Consider the $R' \bigcup Chess'$ prototile set. This set produces two LI classes, one with black tiles to the south-west of cross tiles, and one with white tiles to the south-west. Each one of these LI classes is MLD to our altered Robinson tiling space R, by definition. As described earlier, we are only considering repetitive tilings in our altered Robinson tiling space. Thus every tiling which can be produced by $R' \bigcup_{Oct} Chess'$ is repetitive. $\square$

Chapter 4

Generalization to n LI classes

The construction of new tilings from two square tilings is not limited to Robinson tilings and chessboard tilings. By imprinting an aperiodic set of prototiles into another set of prototiles (for the octagons), you can produce a family of aperiodic tilings. The number of LI classes this family belongs to depends on the correlation between the two tilings. This appears to be a non-trivial subject to investigate. The Robinson tilings seem to be a good starting point, having some easily understandable long range order.

There is another unanswered question. We have sets of prototiles which have aperiodic tilings of the plane in more than one LI class. However they need to have either non-connected prototiles or non-meeting matching rules. Can we find a set of prototiles which have aperiodic tilings of the plane in more than one LI class, yet do not have non-connected prototiles or non-meeting matching rules?

For one set of prototile sets satisfying this condition, first consider our set of octagon Robinson prototiles R', and the area of $\mathbb{R}^2$ they tile, Oct. The problem with non-meeting matching rules arises from $\mathbb{R}^2 \backslash Oct$ being disconnected. To solve this problem, we alter the prototiles R' from octagons into diamonds, as in figure 4.

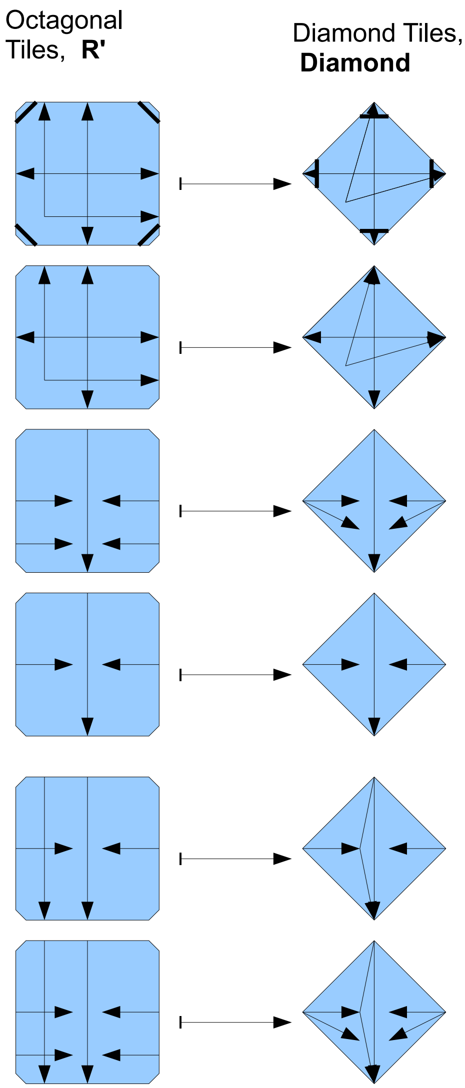


Figure 4.1: Function from R' to *Diamond*

This corresponds to the special case of the octagon construction where $\epsilon = \frac{1}{2}$ on the prototiles (A, B), with the condition that tiles from A cannot border at an edge other tiles from A (and similar for tiles from B). Call this operation $\bigcup$.

The matching rules on the octagons can be transferred onto diamond tiles (Call this set of prototiles $R_{Diamond}$). By replacing a prototile of R' in any tiling of Oct with its corresponding diamond prototile, you can get a mutually locally equivalent tiling which fills $Diamond = \{(a, b)|\exists(x, y) \in \mathbb{Z}^2\}$ s.t $|a - x| + |b - y| \leq \frac{1}{2}\}$. In laymen's terms, it tiles a regular lattice of diamonds. Unlike $\overline{\mathbb{R}^2 \backslash Oct}$, $\overline{\mathbb{R}^2 \backslash Diamond}$ is not disconnected.

To expand this prototile set to one that can aperiodically tile the plane in more than one way, we need a method to transform unit square prototiles (in particular $Chess$) into prototiles which can tile $\mathbb{R}^2 \backslash Diamond$. However $\mathbb{R}^2 \backslash Diamond$ and $Diamond$ are equivalent up to translation. Thus we can transform any unit square prototile into a diamond prototile by the same method we transformed Robinson square prototiles into diamond Robinson prototiles. These diamond prototiles will then tile $\mathbb{R}^2 \backslash Diamond$.

If we transform the set of unit square prototiles $Chess$ into a set of diamond prototiles $Chess_{Diamond}$, and combine them with $R_{Diamond}$, then we can tile the plane aperiodically (with tilings in more than one LI class), since R' and $Chess'$ can tile the plane aperiodically (with tilings in more than one LI class).

Remark. Note that for any two unit square tilings A and B, their superposition $A \bigcup B$ is also a square tiling, with squares of size $\frac{1}{\sqrt{2}}$. Thus you can define a function $f_{super} : Sq \times Sq \longrightarrow Sq$ on the set of all unit square tilings. $f_{super}(A, B) = \gamma(A \bigcup B)$, where $\gamma(x, y) = (\sqrt{2}x, \sqrt{2}y)$. You can extend this to a function on the set of all unit square tiling spaces.

This construction method can also be extended to form an aperiodic tiling with

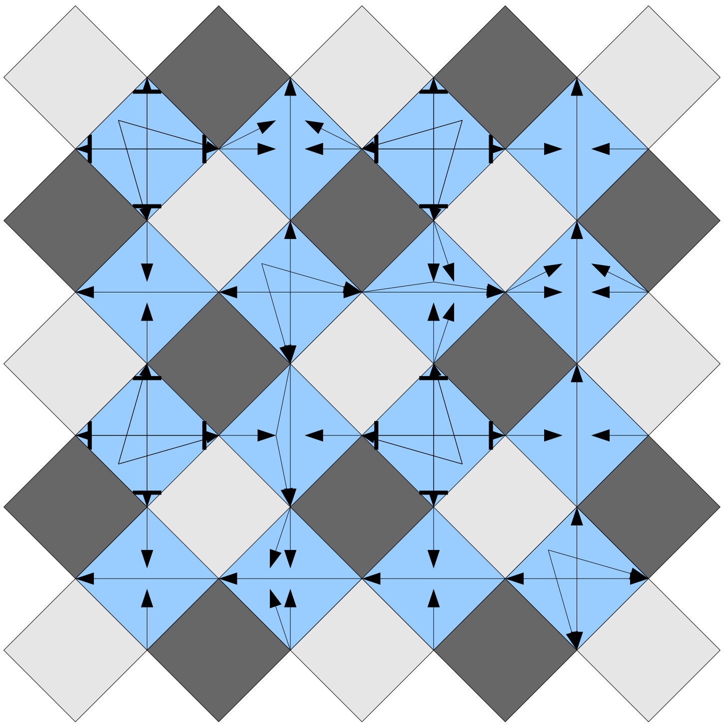


Figure 4.2: Example tiling from a Robinson Tiling Space with 2 LI classes

n LI classes, for $n \in \mathbb{N}$. Starting with our set of square Robinson prototiles R', form a new set of prototiles nR' as follows. Keep the old cornered tiles the same, but swap any old non-cornered cross tiles for an array of $(n-1) \times (n-1)$ unit tiles which together make up an old non-cornered cross tile. Introduce extra matching rules on the unit tiles in the array to force them to match up only to each other. In place of our old horizontal arm tiles, introduce an array of $(n-1) \times 1$ new unit tiles, which together make up the old tile. In place of vertical arm tiles, introduce an array of $1 \times (n-1)$ new unit tiles which together make up the old vertical arm tile. Figures 4.3 and 4.4 will make the tiles and matching rules clearer.

Instead of $Chess$, use the following set of prototiles, $NChess$. Start with n identical unit square prototiles. Let a square prototile be labelled by a element of $\mathbb{Z}_n$, one element to a unit square prototile. To form tilings from these prototiles, force any adjacent tiles to match edge to edge, and for any tile $t_{(i,j)}$ with label $l(t_{(i,j)})$, have the matching rules $l(t_{(i,j)}) - l(t_{(i,j-1)}) = 1 \bmod n$ and $l(t_{(i,j)}) - l(t_{(i-1,j)}) = 1 \bmod n$.

In short, if a chessboard can be considered as a square tiling labelled in 2 colours, the above is a generalisation to n colours. See figure 4.5 for an example of the fundamental domain of one of these tilings, $5Chess$.

To get an aperiodic tiling with n LI classes, simply take nR' and $NChess$ and create a new prototile set $nR' \bigcup NChess$ following the previous method. Tilings produced from this prototile set will be aperiodic and will have n LI classes (since each cornered cross tile in a given tiling will have the same prototile in $NChess$ to its lower left). There are n different possible prototiles which could be placed to the lower left, thus there are at least n LI classes.

By a similar argument to the 2 LI case, these tilings are also repetitive.

Remark. It has been brought to my attention by Edmund Harriss, (joint author

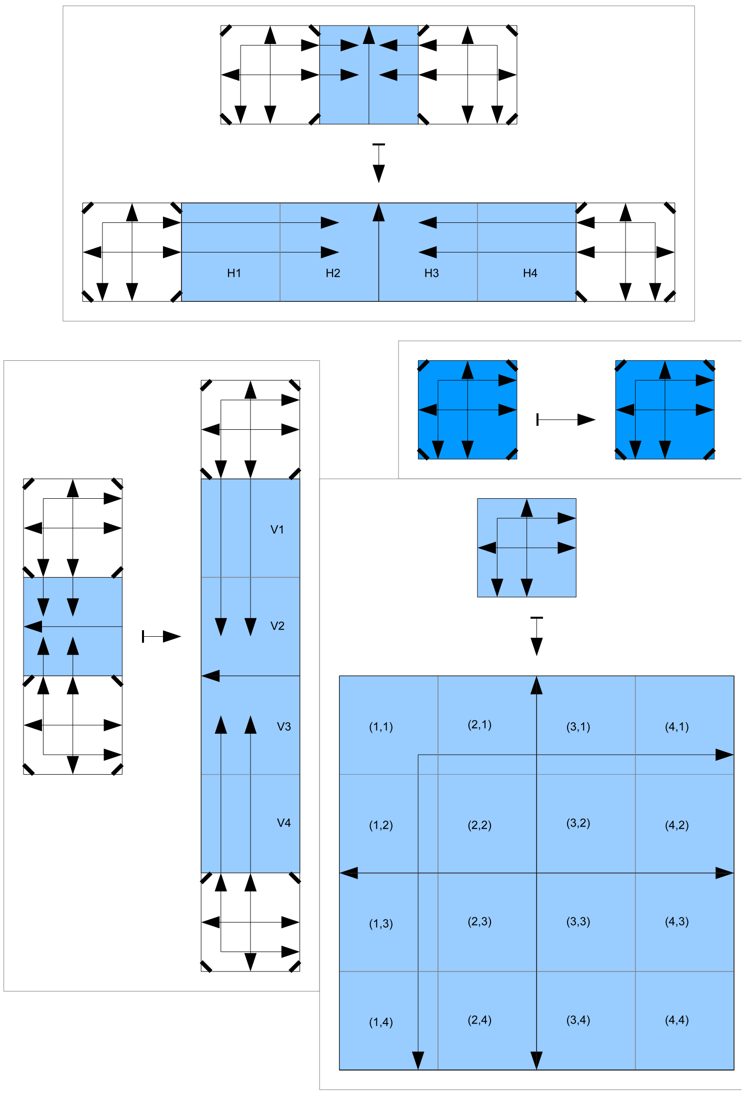


Figure 4.3: Function between prototiles of R' and nR'

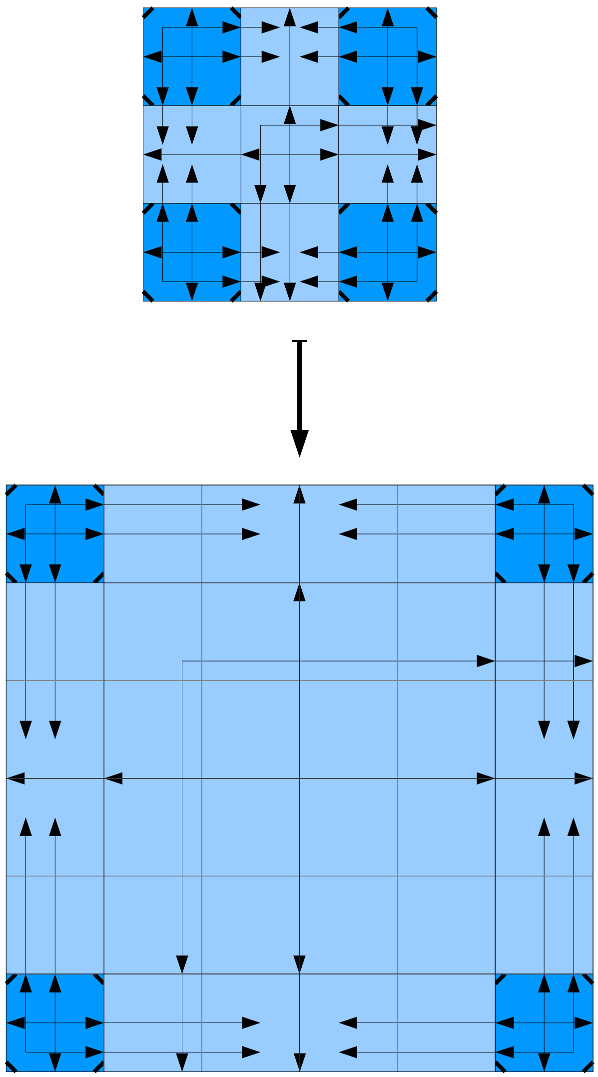


Figure 4.4: Sample patch in R', and its equivalent in nR'

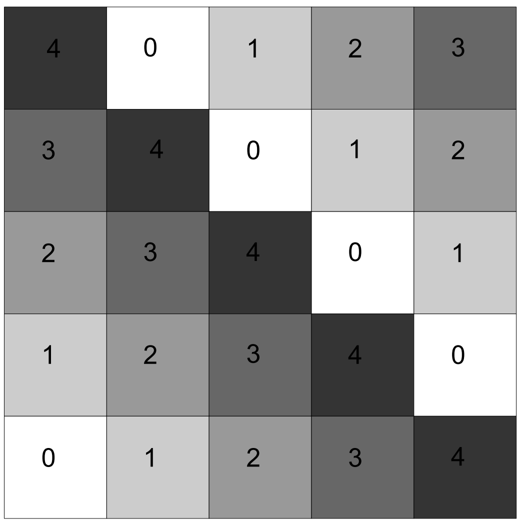


Figure 4.5: Fundamental Domain of $5Chess$

of [44]) that these n LI class constructions can be altered into tilings which do not need facet matching rules by changing the prototiles and introducing new prototiles. Thus we could define the tiling space just by the prototiles, as in figure 3.1. The core idea is that instead of having two tiles in *Diamond* meeting at a point having to obey certain matching rules, you introduce a new tile at the meeting point for each possible matching rule. Figure 4.6 explains this more clearly;

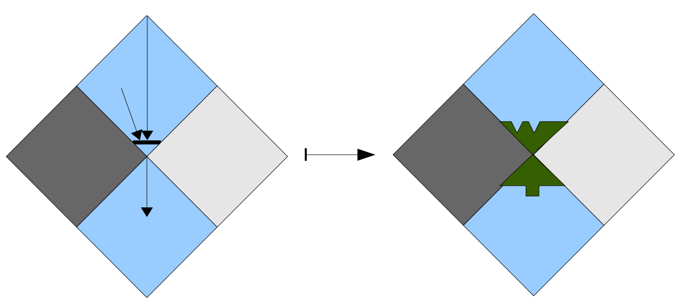


Figure 4.6: Introducing new prototiles to avoid matching rules

This method is not possible for matching rules which are not meeting matching rules, for obvious reasons. This rules out all but the simplest 1 dimension tilings with this property.

Chapter 5

Hyperbolic tilings with more than 1 LI class

In this chapter we will construct (aperiodic) hyperbolic tilings with more than 1 Local Isomorphism class. We will be using J.Kari's method of transferring tilings from $\mathbb{R}^2$ (found in [18] and [19]) as the inspiration for this chapter.

Consider figure 5.1. This figure shows a tiling of the upper half plane model of the hyperbolic plane by a pentagonal prototile. Note that if you consider the figure as a tiling of $\{(x, y) \in \mathbb{R}^2 | y > 0\}$ it consists of horizontal rows tiling the upper half plane.

Remark. This tiling of the hyperbolic plane in figure 5.1 has the property that there is no symmetry that would take a tile onto another tile on the same horizontal row (since tiles in the same row have a precise pattern of tiles 'above' them). However there are symmetries which change the level of tiles. Thus this tiling is weakly aperiodic (as per definition 7).

If we introduce more matching rules to this diagram (via colouring sides in

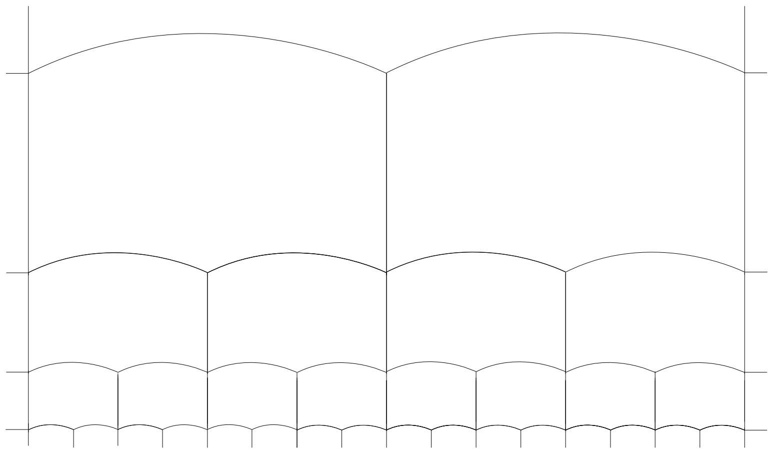


Figure 5.1: D_2, a one prototile hyperbolic tiling with no horizontal isomorphism

a similar manner to Wang tiles, or a different method), then any new tiling we produce will also have no horizontal symmetries. Thus if we can produce a prototile set based on these hyperbolic pentagons, with additional matching rules stopping vertical symmetries, then we will have an aperiodic tiling.

Definition 28 (Mother and daughter tiles)**.** Consider a hyperbolic tiling with 'horizontal rows' partitioning the upper half plane. A tile t in one of these rows will have one row above it and below it (where 'above' and 'below' are defined from the point of view of the tiling being a tiling of $\{(x, y) \in \mathbb{R}^2 | y > 0\}$). For any particular tile t in one of these rows, denote the bordering tile directly above t as t's *mother* and any tiles directly below t as t's *children*.

Definition 29 (2 Daughter tilings)**.** A D_2 tiling is a tiling of H^2, which when illustrated in the upper half plane model has 2 Daughter tiles and 1 Mother tile for

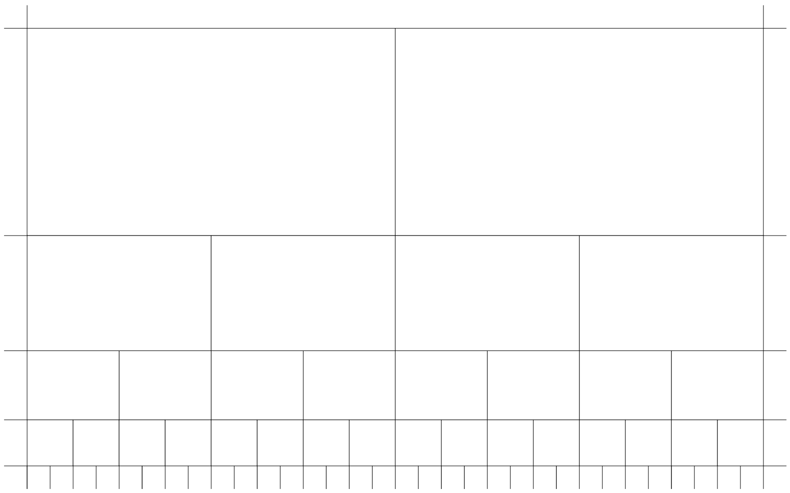


Figure 5.2: MLD hyperbolic tiling with alternate prototiles

each tile in the tiling, and pentagonal tiles.

For clarity in our pictures, we will use the hyperbolic tiling in figure 5.2 instead of our pentagonal one in our pictures. The two tilings are MLD. The only relevant difference is that examples based on this new tiling give clearer pictures.

Note that there are several variations on the basic pentagonal tiling which have the property of barring horizontal symmetries. Instead of having a tiling based on pentagons, base it on a tiling by hexagons. A hexagon in this tiling will have one mother tile, and three daughter tiles. See figure 5.3 for an example. Similarly, you can have tilings with $(n+3)$-gon tiles, with one mother and N daughter tiles.

We will concentrate on the tiling of the hyperbolic plane by pentagons, for simplicity.

Lemma 3. Each prototile set mentioned so far is repetitive.

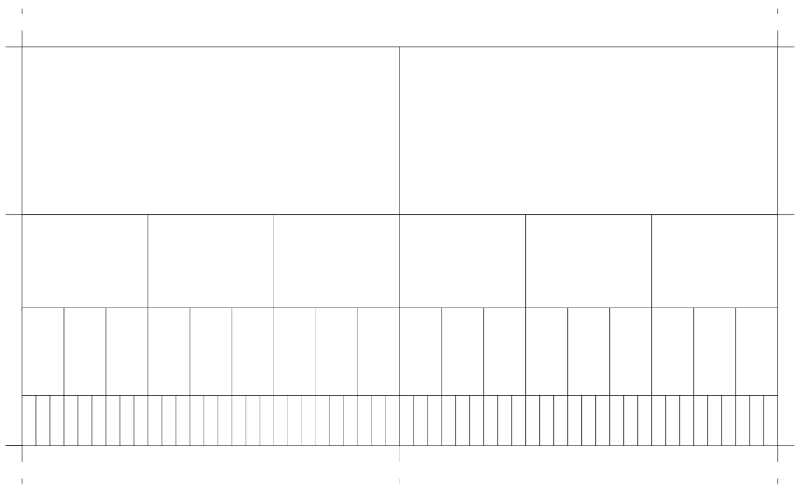


Figure 5.3: Hyperbolic tiling with 3 ‘daughter’ tiles to every ‘mother’ tile

Proof. Any patch p can be contained in a larger patch consisting of k rows of tiles, with each row being directly above or below another row in the patch. More precisely, if we consider the half-plane model of the hyperbolic plane, each patch p can be contained in a Euclidean square $s(p)$ of height n and width m, consisting of one mother tile (of width m when viewed in the half-plane model) and all of its daughter tiles (up to a certain generation $g(p)$ depending on the patch p). Any patch consisting of one mother tile and all of its daughters for $g(p)$ generations will contain the patch p. Since the prototile in any tiling is relatively dense, this implies the prototile set is repetitive. $\square$

Theorem 6. For each prototile set mentioned so far, the related tiling space has tilings in only 1 LI class.

Proof. Fix any previously mentioned hyperbolic prototile set S. Every patch p is contained in a patch $s(p)$ consisting of one mother tile and its daughters up to $g(p)$ generations. This patch $s(p)$ occurs in all possible tilings by the prototile set S. $\square$

Theorem 7. There are prototile sets which produce weakly aperiodic hyperbolic tilings in 2 LI classes.

Proof. In a similar manner to the Euclidean case, we will expand each vertex v of our 2 Daughter tiling (from figure 5.1) into a set homeomorphic to a disk, d_v. More precisely, d_v is a triangle or square, with vertices at the midpoints of each edge of D_2 which enters v. Note that each disk d_v touches another disk d_k iff v and k are on the ends of a edge in D_2. Denote the support of these tiles as V_2.

Note that any tiling based on the tiles of D_2 can be altered to tile $\overline{\mathbb{H}^2 \backslash V_2}$ by changing matching rules from edges of tiles in $\mathbb{H}^2$ to vertices of tiles in $\overline{\mathbb{H}^2 \backslash V_2}$.

Note this will require a change of the pentagonal prototile of D_2 to tile $\overline{\mathbb{H}^2 \backslash V_2}$. We will introduce new tiles to tile V_2. The new tiles introduced will have two distinct shapes, one 'diamond' shape and one 'triangle' shape, as shown in figure 5.4.

The next step is to introduce prototiles to fill the gaps. For your prototile set, pick one black diamond, one white diamond, one black triangle prototile and one white triangle prototile. The matching rules will be that a tile must match to the same colour if they are on the same level, or the different colour if on a different level. This means that the only allowed tilings have black tiles on one row, white tiles on the row above, then black, white and so on. There are a total of two possible square-triangle tilings, as shown in figure 5.5. Note that grey tiles in figure 5.5 are not in V_2.

In order to get multiple LI classes, we need to be able to tell these two tilings apart. In order to do this, we introduce additional matching rules and prototiles for the pentagonal tiles.

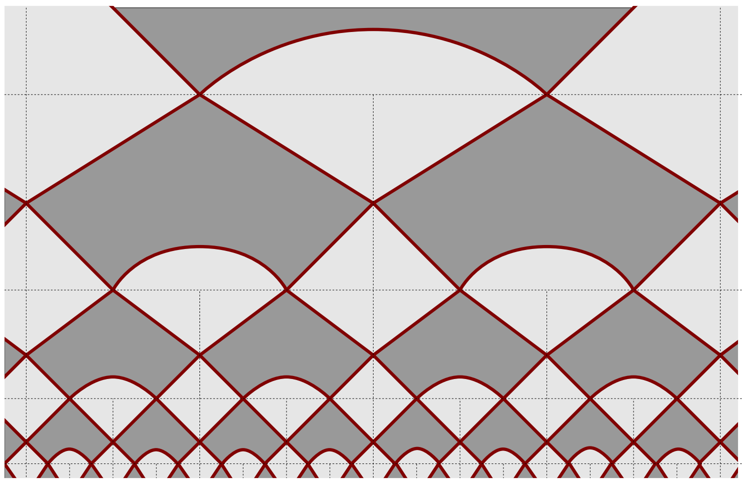


Figure 5.4: Vertices expanded to form 'diamond' and 'triangle' tiles.

Introduce two new pentagonal tiles, a 'marker' pentagonal tile, and a 'transporting' pentagonal tile. We now have three types of pentagonal tiles; marker, transporting and the original 'blank' tiles. The marker tile can only be placed with blank tiles below and above it, 'transporting' tiles to the left and right of it, and it must be the leftmost daughter of its mother tile. A 'transporting' tile must have blank tiles below and above it, 'marked' tiles to the left and right of it, and it must be the rightmost daughter of its mother tile. A 'blank' tile cannot have blank tiles above or below it.

When we combine the two tiles, we get the following two allowable tilings shown in figure 5.7.

These tilings belong to different LI classes, since in the first, all 'marker' tiles have black diamond tiles below it, and in the second one, all 'marker' tiles have white diamond tiles below it. □

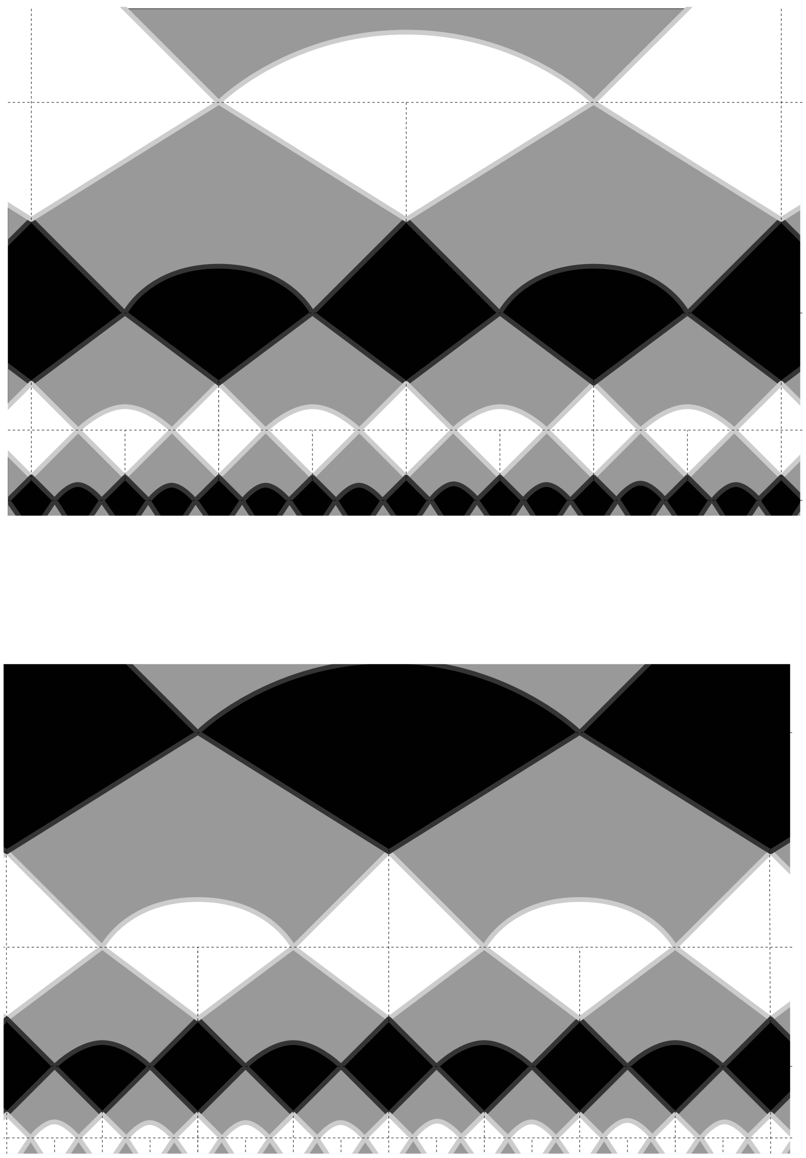


Figure 5.5: Two weakly aperiodic hyperbolic tilings of V_2, in different LI classes.

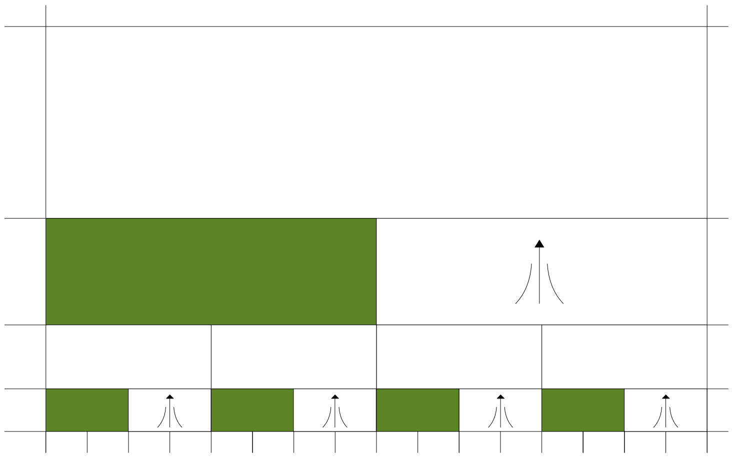


Figure 5.6: Marker (green) and transporting (arrowed) tiles

Note the 'transporting' tiles constructed in the previous proof will be required later for our construction of a strongly aperiodic tiling of $\mathbb{H}^2$. Informally they will 'transport' information between the 'horizontal' layers of the tiling.

Lemma 4. There are prototile sets which produce weakly aperiodic hyperbolic tilings in N LI classes, for $N \in \mathbb{N}$.

Proof. For variations with 3 LI classes or more, the necessary alteration is to change the diamond and triangle prototiles. Instead of having 2 colours of prototile, label the prototiles from the group $\mathbb{Z}_n$. A prototile labelled i must match to a tile labelled $i+1$ above it, tiles labelled i to both horizontal sides of it, and a tile labelled $i-1$ directly below it, where addition is the standard operation inside $\mathbb{Z}_n$. Thus we have n LI classes, distinguished by each marker tile having a diamond tile with a different label from $\{1, \ldots, n\}$ below it. $\square$

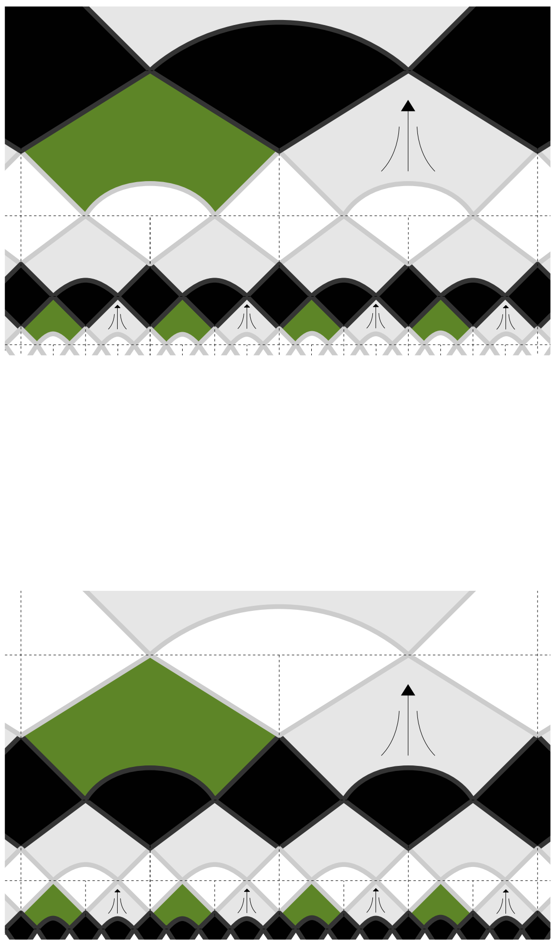


Figure 5.7: Multiple LI classes in the hyperbolic case

While we have multiple LI classes, we still do not have strong aperiodicity. We have several choices in how to do this.

Lemma 5. There are prototile sets which produce aperiodic hyperbolic tilings in N LI classes, $\forall N \in \mathbb{N}$.

Proof. The simplest route is to label each non-marker tile with a number, and then embed a non-periodic 1D tiling into the 2D tiling vertically. In other words take a tile, define s_0 to be its label, then define s_1 to be its mother's label and s_{-1} to be (one of) its daughters. (Clearly all tiles on the same row must have the same label). Repeat this until you have an infinite sequence. That sequence must be equivalent to an aperiodic tiling from a certain tiling space (such as Fib). Since Fib does not have meeting matching rules, any tiling produced by this method will not have meeting matching rules. □

Remark. We believe there are prototile sets which produce aperiodic hyperbolic tilings in N LI classes, $\forall N \in \mathbb{N}$, which have meeting matching rules.

Sketch proof. To insure strong aperiodicity in a less trivial way, we could also use a variation of J.Kari's aperiodic hyperbolic tiling. Use the rows of non-marked, non-transporting tiles to represent Kari's representations of irrational numbers. Use the 'transporting' tile (with various choices as to its decoration) to transmit data between the rows, to keep the meeting matching rules. (Note that the transporting tile will need to pick up data from all tiles in the row below that touch it, including those that only touch it at a vertex. Thus a matching rule only defined on edges will not be sufficient for this proof.) Then apply Kari's [18], [19] matching rules to force aperiodicity. □

Chapter 6

Tilings with an infinite number of LI classes

We have yet to consider if a repetitive aperiodic prototile set with an infinite number of LI classes is possible. We will use the existing notion of substitution tilings in this chapter, and will restrict ourselves to a 1-dimensional case at first (based on the Fibonacci tiling). We then show the existence of a 2-dimensional tiling with an infinite number of LI classes (based on the Chair tiling), and describe what we can deduce from multiple LI classes. The format of the definitions are derived from [36].

Definition 30 (One dimensional substitutions)**.** Pick a finite set (or alphabet) $A = \{a_1, \ldots, a_n\}$. Elements of the alphabet are called *letters*. Finite sequences of letters are called *words*. Denote the set of finite words from A as A^*. Define a function σ' sending each letter to a word. For example, $\sigma'(a) = ab$, $\sigma'(b) = ba$. This function extends to a *substitution* $\sigma : A^* \mapsto A^*$ which replaces every letter l_i in a word with its associated word $\sigma(l_i)$. Using the previously described function we would have $\sigma(aba) = \sigma'(a)\sigma'(b)\sigma'(a) = abbaab$. The function σ' and the substitution σ are often referred to interchangeably in the literature.

A *primitive* substitution σ (with alphabet A) is a substitution where there exists some $k \in \mathbb{N}$ such that for every letter $a_i \in A$, $\sigma^k(a_i)$ contains every letter of σ's alphabet at least once.

A bi-infinite word is σ-*admissible* if each finite sub-word can be found in $\sigma^k(a_1)$ for some $k \geq 0$.

For each letter $a \in A$, associate a prototile to that letter with the same label, forming a prototile set P_A. Given a substitution σ, a (σ-) substitution tiling is a tiling by the prototile set P_A such that the corresponding sequence of letters is a σ-*admissible* word. The tiling space of the substitution, Ω_σ is the set of all σ-substitution tilings.

In higher dimensions, we cannot use the concepts of letters and words to define a substitution, since we cannot put a simple order on points in higher dimensional spaces. We must consider the geometry of the tiles.

Definition 31 (Higher dimensional substitutions)**.** [36] A substitution σ is an operation on a set of prototiles which replaces each tile by a cluster of tiles, with associated rules fixing the positions of the resultant tiles in relation to each other. These clusters are called *supertiles*.

The tiling space of the substitution Ω_σ is the set of tilings T such that every patch of T is found in a supertile of some order (ie, $\sigma^n(P)$, where P is a prototile and $n \in \mathbb{N}$).

Let us now move on to our central examples.

Definition 32 (Labeling unit interval tilings of $\mathbb{R}$)**.** For all tilings T of $\mathbb{R}$ by unit interval prototiles, find the tile containing the origin $\underline{0}$ (if the origin is on the boundary of two tiles, choose the one containing points in $\mathbb{R}^+$). Label this tile as t_0. Denote the tile k units to the right of t_0 as t_k, the tile k units to the left as t_{-k}.

Definition 33 (Fibonacci tiling space Fib)**.** A Fibonacci tiling T is generated from two labelled unit intervals for prototiles, 0 and 1, and a substitution rule defined as $0 \mapsto 01$; $1 \mapsto 0$. The tiling space of all such sequences is denoted as Fib. The elements of Fib thus consist of a series of labelled unit tiles $\{t_s\}_{s=-\infty}^{\infty}$ spanning $\mathbb{R}$.

We will then apply our idea of embedding a second tiling into a Fibonacci tiling, as follows.

Definition 34 ($A \bigcup B$)**.** A tiling T belongs to $A \bigcup B$ if the even unit tiles $\{t_{2k}\}_{k=-\infty}^{\infty}$ correspond to a valid tiling $T_{Even} \in A$, and the odd unit tiles $\{t_{2k+1}\}_{k=-\infty}^{\infty}$ also correspond to a valid tiling $T_{Odd} \in B$. The tilings may be the same, however later on we will need to distinguish tiles from different tilings (ie, even and odd tiles). Thus we will colour even unit tiles black, and odd unit tiles red. $A \bigcup B$ will be given the standard tiling space topology.

Every patch of a tiling T in $A \bigcup B$ can be found in the union of two supertiles of some order (one supertile covering black (even) tiles, one supertile covering red (odd) tiles). Thus $A \bigcup B$ is a minor generalization of a tiling space of a substitution.

Definition 35 ($T \bigcup T^{+(2n+1)}$)**.** Consider a tiling $T \bigcup T^{+(2n+1)}$ in $Fib \bigcup Fib$ which consists of a fixed Fibonacci tiling $T \in Fib$ on the even tiles, and the same tiling T shifted $2n+1$ units to the left, on the odd tiles. The union of these two tilings will cover $\mathbb{R}$, with the only overlaps being at points. Thus it is a valid element of $Fib \bigcup Fib$.

Now consider two tilings, $T \bigcup T^{+(2n+1)}$ and $T \bigcup T^{+(2m+1)}$, where WLOG $n > m$ and n and m are coprime.

We aim to show that these two tilings are in different LI classes. Our first step is to consider the patches shown in figures 6.2 and 6.3. We intend to show that the patch in figure 6.2 can occur in $T \bigcup T^{+(2n+1)}$, and that the patch in figure 6.3 can occur in $T \bigcup T^{+(2m+1)}$.

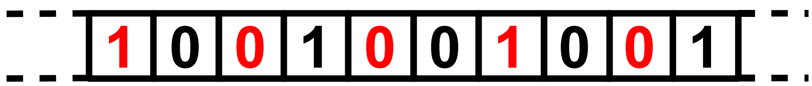


Figure 6.1: T^{+3}

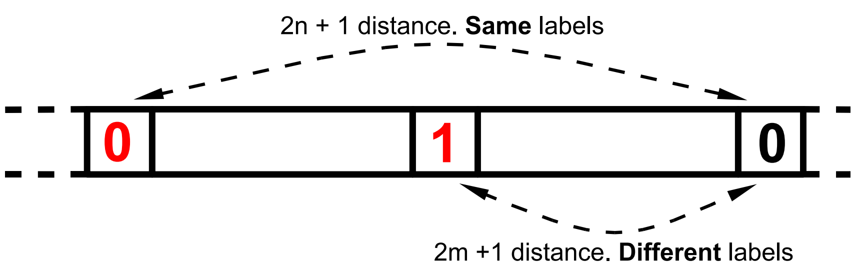


Figure 6.2: Patch in $T \bigcup T^{+(2n+1)}$ which is not in $T \bigcup T^{+(2m+1)}$

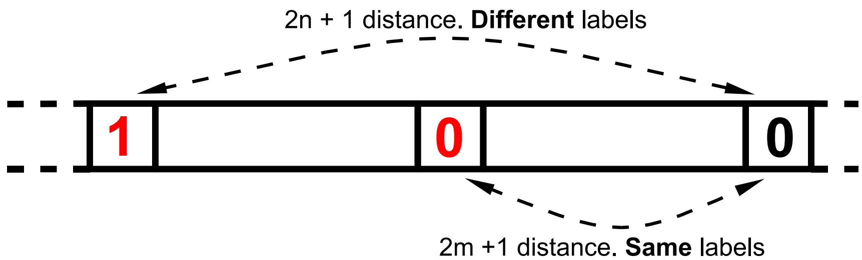


Figure 6.3: Patch in $T \bigcup T^{+(2m+1)}$ which is not in $T \bigcup T^{+(2n+1)}$

Lemma 6. The patch in figure 6.2 belongs to $T \bigcup T^{+(2n+1)}$ iff there is a patch in T with a 1 tile $n - m$ units to the right of a 0 tile. Similarly, the patch in figure 6.3 belongs to $T \bigcup T^{+(2m+1)}$ iff there is a patch in T with a 1 tile $n - m$ units to the left of a 0 tile.

Proof. Note that for any even (black) 0 tile t in $T \bigcup T^{+(2n+1)}$, the tile $2n + 1$ units to the left of t will be a 0 tile (albeit an odd, hence red one) by definition of $T \bigcup T^{+(2n+1)}$.

However we need to show that the tile $2m + 1$ units to the left of t can be a 1 tile to conclude that the patch in figure 6.2 belongs to $T \bigcup T^{+(2n+1)}$. Denote this tile as s.

If s is a 1 tile, then the tile $2n + 1$ units to the right of s must also be a 1 tile, by definition of $T \bigcup T^{+(2n+1)}$. Thus the tile $(2m + 1) - (2n + 1)$ units to the left of t

must be a 1 tile to conclude that the figure 6.2 patch belongs to $T \bigcup T^{+(2n+1)}$. n is greater than m, so we can rewrite as $2(n-m)$ units to the right of t. This property corresponds to there being a patch in T with a 1 tile $n-m$ units to the right of a 0 tile.

The condition on the figure 6.3 patch follows similarly (by swapping the roles of n and m in the proof).

□

We will now prove that there is a patch in T with a 1 tile $n-m$ units to the (left) right of a 0 tile, for all possible values of $n-m \neq 0$. Note that n and m can't be equal, due to an earlier constraint. Thus it is enough to prove for all non-zero integers.

Theorem 8. For all $r \in \mathbb{N}\backslash\{0\}$, for all $T \in Fib$ there exists a 0 tile $t_0 \in T$ such that the tile r units to the left is of type 1. Similarly, there is a 0 tile $t_1 \in T$ such that the tile r units to the right is of type 1.

Proof. Assume that for all even 0 tiles, there can only be 0 tiles r units to the left, not 1 tiles. Thus there are 0's $r.k$ to the left, for all k's. Consider a segment of a Fibonacci tiling T. The ratio of 0's to 1's is bounded (this comes from the substitution rule, more precisely the eigenvalues of the matrix for the substitution rule). Our segment (say of length s) has a certain number of 0's. By our assumption, the segment of length s which is r units to the left must have equal or more 0 tiles in it. This is an ascending sequence, bounded above by r. Thus it reaches a maximum, at a point p. Thus, to the left of the point p, we have a periodic sequence with period r.

Take T and construct a sequence of points in the tiling space

$$\{T, \lambda_1(T), \lambda_2(T), \ldots \lambda_n(T), \ldots\},$$

where λ_k is the translation of a tiling by k to the right. The limit of this sequence is a periodic tiling, and is in Fib. However Fib is an aperiodic tiling space and thus contains no periodic tilings. Contradiction. Thus there must be 1 tiles r units to the left of some even 0 tiles.

A similar argument implies that there must be 1 tiles r units to the right of some even 0 tiles. $\square$

Of course, not all 1D aperiodic tilings have this property.

Remark. Note that we cannot only have 1 tiles r to the left of even 0 tiles in any $T \in Fib$. If this was true, then we would have an equal (or higher) density of 1 tiles then 0 tiles in the tiling. Arguments centred around the substitution matrix tell us that there are $(1 + \sqrt{5})/2$ times more 0's than 1's. Thus there are both 0 tiles and 1 tiles r units to the left of 0 tiles.

Theorem 9. *$T \bigcup T^{+(2n+1)}$ and $T \bigcup T^{+(2m+1)}$ belong to different LI classes, where n and m are coprime.*

Proof. By Lemma 6 and Theorem 8 we know that the patch shown in diagram 6.2 exists somewhere in $T \bigcup T^{+(2n+1)}$ (for certain values of 0,1 in the unlabelled tiles).

However the above patch cannot exist in $T \bigcup T^{+(2m+1)}$, since in $T \bigcup T^{+(2m+1)}$, a black tile must be identical to the red tile $2m + 1$ units to the left of it.

Similarly the patch in figure 6.3 (with two identical tiles $2m + 1$ apart, and two different tiles $2n + 1$ units apart) exists in $T \bigcup T^{+(2m+1)}$ but not in $T \bigcup T^{+(2n+1)}$. Thus $T \bigcup T^{+(2n+1)}$ and $T \bigcup T^{+(2n+1)}$ belong to different LI classes. $\square$

Remark. We need the even and odd unit tiles to be distinguishable for this proof to work though. Otherwise we could effectively flip the colours in the above patch. We would then be left with a patch which could be in either tiling, since there are no constraints on what tiles must be $(2k+1)$ units to the left of a red tile.

Theorem 10. $Fib \bigcup Fib$ has at least a countable number of LI classes.

Proof. Take the set of prime numbers, $P = \{p_1, p_2, \dots\}$. Any two prime numbers p_m, p_n are coprime. Thus the tilings $T \bigcup T^{+(2p_m+1)}$ and $T \bigcup T^{+(2p_n+1)}$ belong to different LI classes, $\forall p_m, p_n \in P$. Thus we have a countably infinite string of examples which all belong to different LI classes. $\square$

While this example is valid, the matching rules are not meeting matching rules. For an example with meeting matching rules, we need to move to two dimensions.

At this point we will consider the 'square chair' tiling, which is MLD to the standard chair tiling.

Definition 36 (Square Chair Tiling)**.** The square chair tiling space $SqCh$ consists of the four prototiles (unit tiles with an arrow pointing to one of the vertices) and the substitution rule shown in figure 6.4. Legal configurations are also shown – note that the only allowable arrangements at the vertices are that either all arrows point out of the vertex, one arrow goes out and three go in, or no arrows enter or leave the vertex.

We will refer to the square chair tiling space as the chair tiling space in this article. More precise definitions are given in [42] and [43].

Sadly I do not know of an expression of the chair tiling with local matching rules and square prototiles (for non-square prototiles, see [25]), and must instead use the substitution map in figure 6.4.

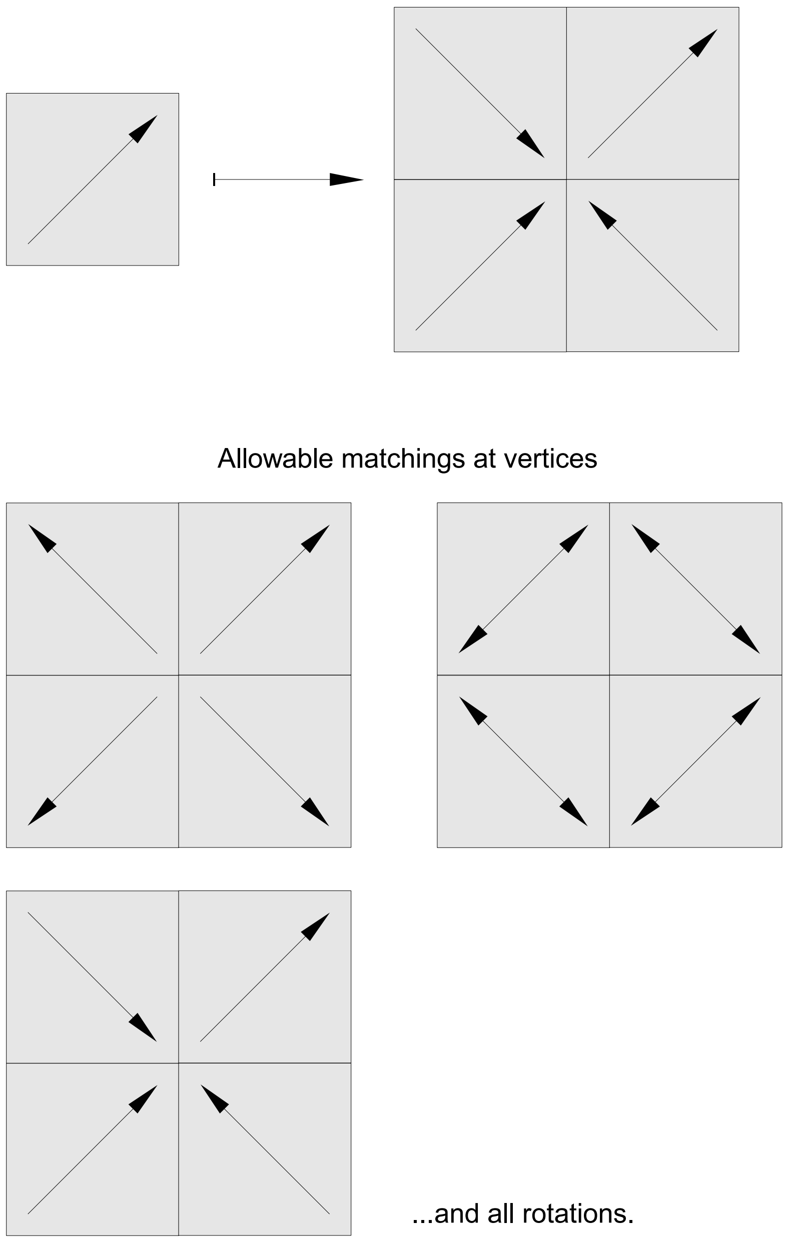


Figure 6.4: Chair matching rules

Definition 37 (CH)**.** For any given tiling of $\mathbb{R}^2$ using the chair tiles and their matching rules, we can get a MLD tiling of *Diamond*, in a similar way that we can transform a Robinson tiling from $\mathbb{R}^2$ to *Diamond*. Denote the set of allowable chair tilings of *Diamond* as CH.

Definition 38 ($CH \bigcup CH$)**.** $CH \bigcup CH$ is the set of tilings of $\mathbb{R}^2$ where any tiling $T \in CH \bigcup CH$ satisfies the following restraints;

T restricted to *Diamond* is some tiling T_i from CH.

T restricted to $\mathbb{R}^2 \backslash Diamond$ is the translation of some tiling T_j from CH.

Furthermore $CH \bigcup CH$ is a tiling space (with the prototile set as CH, with the above additional matching rules).

Definition 39 ($Ch \bigcup Ch^{s,t}$)**.** Fix some chair tiling $Ch \in CH$, which of course tiles *Diamond*. Then translate this chair tiling Ch by $(2(0.5)s + 0.5, 2(0.5t) + 0.5)$. This will map Ch onto $\mathbb{R}^2 \backslash Diamond$. Note that we need to differentiate between the prototiles for the chair tiling on *Diamond* and $\mathbb{R}^2 \backslash Diamond$, since otherwise our proof of an infinite number of LI classes will not work, as in the 1 dimensional case. Thus we must use two sets of prototiles, one (coloured black) for the chair tiling on *Diamond* and another one (coloured red) for the chair tiling on $\mathbb{R}^2 \backslash Diamond$.

Call this particular tiling $Ch \bigcup Ch^{s,t}$.

Lemma 7. $CH \bigcup CH$ consists of only aperiodic tilings.

Proof. There exists a forgetful function f from $CH \bigcup CH$ to CH, by removing all tiles in $\mathbb{R}^2 \backslash Diamond$, and expanding the tiles in *Diamond* to unit squares. Assume $CH \bigcup CH$ contains a periodic tiling. Then there exists a isometry g and periodic tiling $T \in CH \bigcup CH$ such that $g(T) = T$. Furthermore, $g(f(T)) = f(T)$, since f is forgetful. However, CH contains only aperiodic tilings. Thus by contradiction, $CH \bigcup CH$ must contain only aperiodic tilings. □

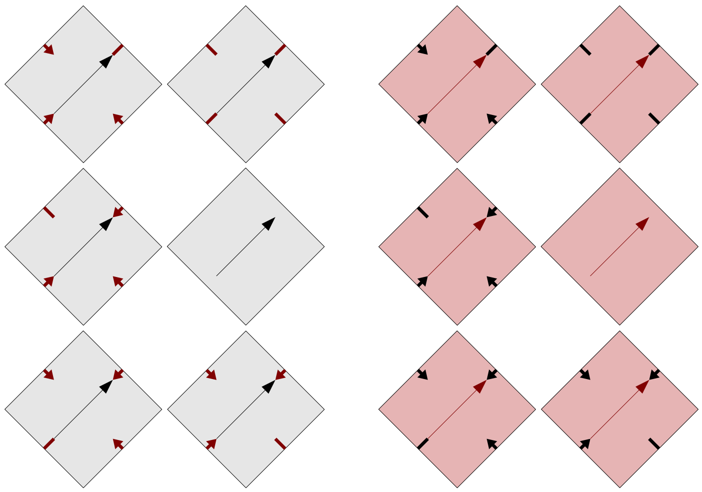


Figure 6.5: Prototiles for $(CH \bigcup CH)^{local}$

Definition 40 ($(CH \bigcup CH)^{local}$)**.** Consider figure 6.5.

Construct $(CH \bigcup CH)^{local}$ from $CH \bigcup CH$ by changing every prototile in a tiling of $CH \bigcup CH$, to the prototile in figure 6.5 of the same colour, with the main arrow pointing in the same direction, with the smaller arrows being chosen to represent the bordering tiles central arrow.

A sample patch of $(CH \bigcup CH)^{local}$ is shown in figure 6.6. Note that $(CH \bigcup CH)^{local}$ has meeting matching rules (with the non-trivial matching rules dependent on edges instead of vertices).

Lemma 8. There exists a homeomorphism f from $(CH \bigcup CH)^{local}$ to $CH \bigcup CH$ such that for all $T \in (CH \bigcup CH)^{local}$, T and $f(T)$ are MLD.

Proof. There is a continuous map f from $(CH \bigcup CH)^{local}$ to $CH \bigcup CH$ by simply forgetting all small arrows, and an continuous inverse f^{-1} from $CH \bigcup CH$ to

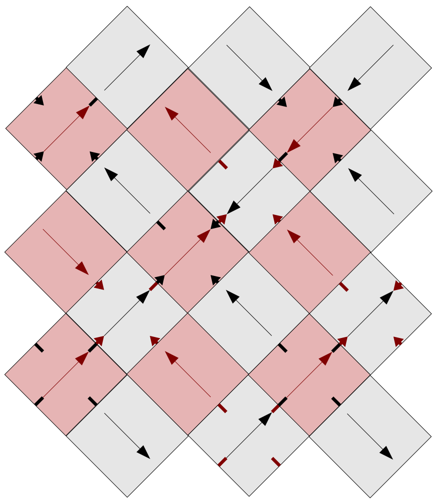


Figure 6.6: Part of a tiling in $(CH \bigcup CH)^{local}$

$(CH \bigcup CH)^{local}$ defined by adding small arrows in the unique way which satisfies matching rules. Thus f is a homeomorphism between $(CH \bigcup CH)^{local}$ to $CH \bigcup CH$.

Consider a tile t in a tiling $T \in (CH \bigcup CH)^{local}$. The markings lost under the homeomorphism f are uniquely determined by the 1-corona of the tile t' in the same position in $(CH \bigcup CH)$. Thus t can be uniquely determined by considering a finite ball in $f(T)$. Thus T is locally derivable from $f(T)$. Similarly, a tile t' in $f(T)$ is uniquely determined by a finite ball in T. Thus T and $f(T)$ are MLD.

$\square$

Remark. A definition of MLD for hulls has been constructed, which implies that if two tilings T, T' are MLD, their hulls Ω_T, $\Omega_{T'}$ are MLD. Lemma 8 would thus imply that every hull in $(CH \bigcup CH)^{local}$ is MLD to a corresponding hull in $CH \bigcup CH$. See page 9 of [36] for details.

Lemma 9. Let $u, v \in \mathbb{Z}$. Then for any black main arrowed tile t in $Ch \bigcup Ch^{(u,0)}$, the tile shifted $(u+0.5, 0.5)$ units from it will have a red arrow pointing in the same direction, by definition. The type of tile $(v + 0.5, 0.5)$ units from that black main arrowed tile does not necessarily have an arrow pointing in the same direction.

Proof. To show this, assume that it does have the same direction. This would imply that in CH, a tile (denote it t') $(v, 0)$ units away from a tile which looks like t would have an arrow in the same direction, regardless of where t' is in the tiling.

There exists a tiling in CH which has an 'infinite chain' of tiles which look like t, as shown in figure 6.7. (The tiling in question is the default iterated expansion of a prototile with an arrow pointing to the upper-right.)

Thus if t' shares the same prototile as t, then we are allowed tilings which look like figure 6.8.

This is because an infinite chain of tiles with all arrows pointing in the same direction acts as a 'faultline' in the tiling. The arrangement of tiles on one side of the line will have no effect on the tiling on the other side of the faultline, since every tile with an edge bordering the infinite chain of tiles must be labelled with an arrow pointing away from the faultline. Thus we are free to have a fixed pattern of tiles between any two infinite chains. In other words, we can have a tiling with a horizontal period.

This tiling has a horizontal period, which contradicts CH (and hence the general chair tiling) being aperiodic. Since the chair tiling is aperiodic, our assumption is false, and the type of tile $(v + 0.5, 0.5)$ units from a black main arrowed tile does not need to have the same direction.

□

We now aim to prove that $CH \bigcup CH$ has elements in a countably infinite number

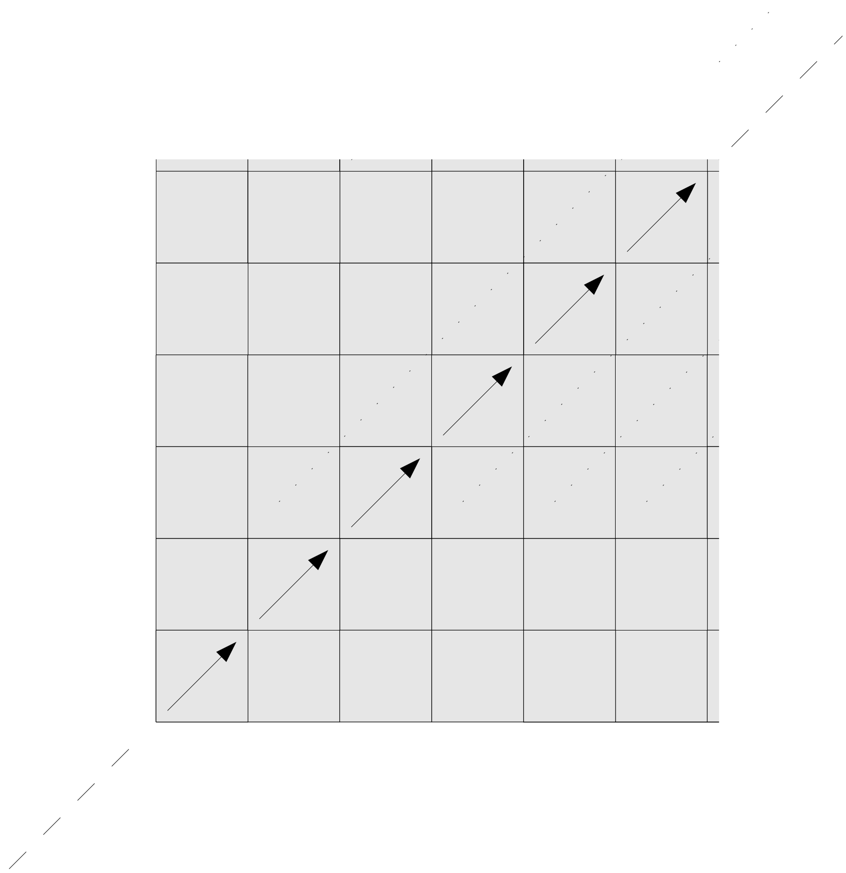


Figure 6.7: Infinite chain

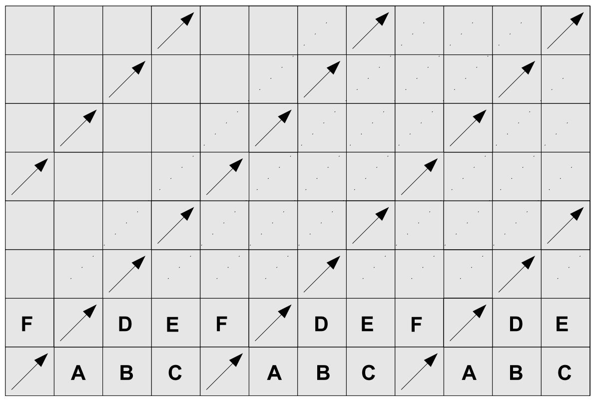


Figure 6.8: Periodic tiling

of LI classes (This of course does not rule out the existence of a higher cardinality of LI classes).

Theorem 11. Consider the tilings $Ch \bigcup Ch^{(s,0)}$ and $Ch \bigcup Ch^{(t,0)}$, where $s > t$, and $s \neq kt$, $\forall k \in N$. $Ch \bigcup Ch^{(s,0)}$ and $Ch \bigcup Ch^{(t,0)}$ are in different LI classes.

Proof. This proof will be a variant on the proof that $T \bigcup T^{+(2n+1)}$ and $T \bigcup T^{+(2m+1)}$ in the one-dimensional case are in different LI classes.

Take the following patch in $Ch \bigcup Ch^{(s,0)}$;

From lemma 9 we know that there exists a patch in $Ch \bigcup Ch^{(s,0)}$ with the following conditions; The patch has a black main arrowed tile at some point, at a point $(t + 0.5, 0.5)$ units away there will be a red arrowed tile facing in the opposite direction, and at a point $(s + 0.5, 0.5)$ units away there will be a red arrowed tile pointing in the same direction.

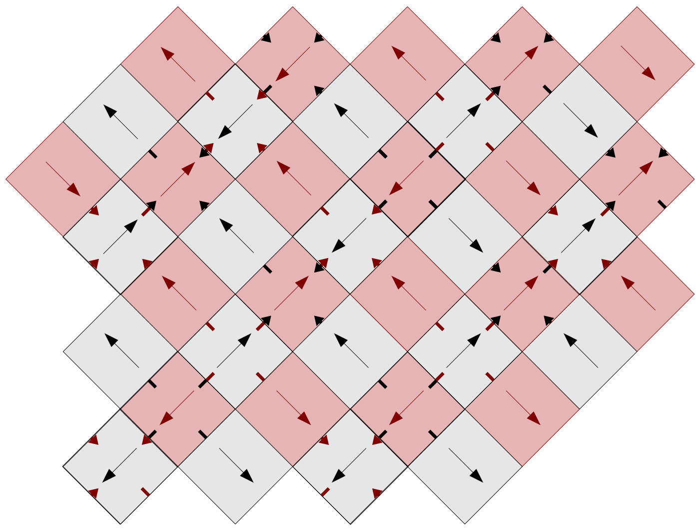


Figure 6.9: Patch in $Ch \bigcup Ch^{(1,0)}$

Similarly in $Ch \bigcup Ch^{(t,0)}$ we can find a patch with a black main arrowed tile at some point, at a point $(s+0.5, 0.5)$ units away there will be a red arrowed tile facing in the opposite direction, and at a point $(t+0.5, 0.5)$ units away there will be a red arrowed tile pointing in the same direction.

Thus $Ch \bigcup Ch^{(t,0)}$ and $Ch \bigcup Ch^{(s,0)}$ are in different LI classes. □

Lemma 10. $CH \bigcup CH$ has elements in a countably infinite number of LI classes.

Proof. From the previous theorem we know that $Ch \bigcup Ch^{(s,0)}$ and $Ch \bigcup Ch^{(t,0)}$ are in different LI classes, if s and t are coprime. By taking the prime numbers $\{p_n\}_{n=1}^{\infty}$, we can construct a countably infinite string of tilings, $\{Ch \bigcup Ch^{(p_i,0)} | i \in \mathbb{N}\}$ which all belong to different LI classes.

Since $(CH \bigcup CH)^{local}$ and $CH \bigcup CH$ are MLD, then $(CH \bigcup CH)^{local}$ also has a countably infinite number of LI classes. □

Remark. Consider Kari's set of 13 Wang prototiles (and the related tiling space, which we shall denote as K). We believe that the proof of an infinite number of LI classes can be adapted to Kari's set of prototiles. As a brief justification, Kari's tilings are constructed from rows of Wang tiles representing real numbers (more precisely balanced representations). In some tilings in the tiling space, these numbers are irrational (and thus have no horizontal period). Then the $Fib \bigcup Fib$ proof should be applicable to this row.

6.1 What we can deduce from multiple LI classes

First we shall show when a tiling space Ω is compact (This is a well known result within the field).

Definition 41. A tiling has *Finite Local Complexity* (FLC) if it contains only finitely many types of patches with diameter less than some given $R > 0$, up to translation.

A tiling has *finite translation classes* if there are a finite number of tile types, up to translation. All tilings used in this thesis have finite translation classes. An example of a tiling which does not is the pinwheel tiling [32]. As a reminder, a tiling is edge-to-edge if any two facets in the tiling cannot partially overlap. A repetitive tiling with finite translation classes, which matches edge-to-edge is automatically of FLC type. All of the tilings used in this thesis match edge-to-edge and have finite translation classes.

We may also say that a tiling 'is FLC'. This is merely a grammatical variation.

Lemma 11. If the tilings in a tiling space Ω are FLC, then Ω is compact.

Proof. Consider the discrete hull $\Omega^0 \subset \Omega$. (The discrete hull of a tiling space is the set of all tilings with the centre of mass of a tile over the origin).

Let $\{T_i, i \in \mathbb{I}\}$ be any given infinite set of elements of Ω^0 (in other words, tilings). Look at the patches of radius 1 about the origin of these elements. Since Ω^0 is FLC, there are a finite number of possible patches of radius 1. Thus there will be (at least) one patch of radius 1 about the origin such that an infinite subset of $\{T_i, i \in \mathbb{N}\}$ contains that patch (about the origin). Renumber this infinite subset as $\{S_i^1, i \in \mathbb{N}\}$.

Now look at patches of radius 2. By a similar argument, there are an infinite subset of $\{S_i^1, i \in \mathbb{N}\}$ with one type of patch around the origin. Renumber this new infinite subset as $\{S_i^2, i \in \mathbb{N}\}$, and iterate the process. Taking a sequence $\{S_1^j\}_{j\in\mathbb{N}}$ will give you a convergent subsequence under the standard metric. Thus for any sequence $\{T_i, i \in \mathbb{N}\}$ in Ω^0 we can find a convergent subsequence. Thus Ω^0 is compact.

But

$$\Omega \twoheadleftarrow \Omega^0 \times D_R$$

where D_R is a disk of radius equal to the maximum possible distance between the centres of mass of two adjacent tiles in any tiling. D_R and Ω^0 are compact. Ω is the continuous image of the product of two compact spaces. Thus it is compact. $\square$

Let us consider the number of connected components of a repetitive tiling space with n LI classes.

Theorem 12. If a repetitive tiling space Ω has exactly n LI classes, that tiling space has n connected components. Furthermore, each LI class is a connected component.

Proof. Choose a labeling of the LI classes as $\{LI_k\}_{k\in\{1,2,\ldots,n\}}$, ensuring that there are no duplicate labels. To prove a LI class LI_k is a connected component, we need to show that L_k is connected, and that there exists an open set U_k around L_k separating LI_k from the rest of Ω.

We will first show that for any $k \in \{1, 2, \ldots, n\}$, LI_k is connected. Our set LI_k is connected if there do not exist non-empty open sets U, V such that $LI_k = U \cup V$, $\overline{U} \cap V = \emptyset$ and $U \cap \overline{V} = \emptyset$. Assume such a pair of open sets exist. Thus both U and V are contained within LI_k. Consider U, and choose some tiling $x \in U$. By definition of open, there exists an ϵ-ball around x contained in U. Thus every tiling which agrees with x on a radius of $\frac{1}{\epsilon}$ about the origin must be in U (due to the metric on the tiling space Ω). Denote the patch of tiles contained in $B_{\frac{1}{\epsilon}}(\underline{0})$ in x as the patch P. Since we are dealing with tilings in an LI class, a tiling $v \in V$ has that patch P in it, centred about some point $\underline{v}$ in the tiling. Thus there exists a path in the tiling space sending v to within ϵ of x, corresponding to the path sending the point $\underline{v}$ to the origin (when considered as translations in the tiling space). This is a contradiction with $\overline{U} \cap V = \emptyset$. Thus LI_k is connected, and similarly all LI classes in this tiling space are connected.

We will now show that the closure of a LI class is that LI class. Without loss of generality, choose a LI class LI_k and any convergent sequence $\{S_i\}_{i=1}^{\infty}$ of tilings inside the LI class. If the limit of $\{S_i\}_{i=1}^{\infty}$ is in LI_k, then the closure of LI_k is LI_k. Our tiling space has a metric and is compact, thus it is complete. Thus the limit of the sequence is in the tiling space.

Take a given patch P from S_1. We know that P will be found somewhere in all tilings in $\{S_i\}_{i=1}^{\infty}$, since the tilings are in the same LI class. We need to show it will also appear in the limit. Consider S_1. Our tiling space is repetitive, thus all tilings are repetitive. Thus P will be within some distance r of the origin, for some $r \in \mathbb{R}$ dependent on P.

Consider S_2. P must be within distance r of the origin of S_2. Otherwise there would be some patch $B_r(\underline{0})$ in S_2 which does not contain P. Since S_1 and S_2 are in the same LI class, this patch would also occur in S_1, which would be a contradiction with S_1 being repetitive (with value r).

Thus P occurs within r of the origin of S_2. By similar argument, P occurs within r of the origin of S_i, for all $i \in \mathbb{N}$. Thus P also occurs (within distance r) of the origin of the limit of $\{S_i\}_{i=1}^{\infty}$. Since we chose an arbitrary patch P, the limit of $\{S_i\}_{i=1}^{\infty}$ is in the same LI class. Thus the closure of a LI class is that LI class.

Thus there is a finite distance between any two LI classes.

We will now show that for any LI class LI_k, we can choose an open set around LI_k, separating LI_k from the rest of Ω. This implies that LI_k is a connected component.

Set $\epsilon = \frac{1}{2} \min \{d(x,y) | x \in LI_k, y \in LI_s, i \neq j; i,j \in \{1,2,\ldots,n\}\}$. Note that we know that $\epsilon > 0$ because the tiling space is repetitive, hence FLC, hence compact. Let $U_k = \{a \in \Omega | d(a,a_k) < \epsilon \textit{ for some } a_k \in LI_k\} = \bigcup_{t \in LI_k} B_\epsilon(t)$. Therefore U_k is open.

But $U_k \cap U_s = \emptyset$ (if $k \neq s$) by choice of ϵ. Thus U_k separates LI_k from the rest of Ω. Thus LI_k is a connected component.

Thus Ω has n connected components.

$\square$

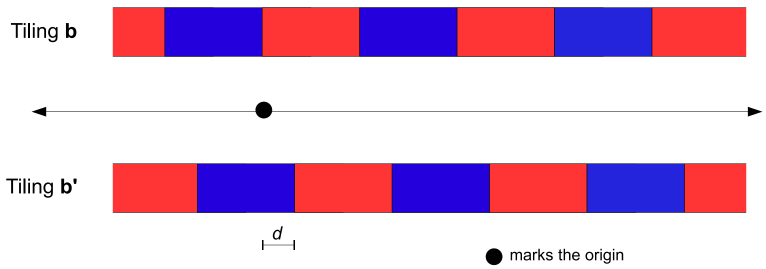
Sadly, we cannot find such a simple relationship between connected components and LI classes when there are an infinite number of LI classes. Consider the following lemma;

Lemma 12. Consider the set S of repetitive tiling spaces which have an infinite number of LI classes. There is no (finite) upper bound on the number of connected components contained in a tiling space from S.

Proof. Take $CH \bigcup CH$ which is a FLC unit square tiling with an infinite number of LI classes. Assume $CH \bigcup CH$ has a finite number of connected components. Consider $(CH \bigcup CH) \bigcup Chess$. This new tiling space doubles the number of connected components. Thus there is no upper limit on the number of connected components possible. $\square$

As a minor aside, we can also consider the general structure of a tiling space of $A \bigcup B = \{a \cup b | a \in A, b \in B\}$. $A \bigcup B$ has strong similarities to a fibre bundle, with B as the fibre, A as the base space and the projection map from $A \bigcup B$ to A being merely the inverse of the $\cup$ operation sending a to $a \cup b$. The complication occurs with translation. As the reader may recall we are using a metric on tilings where two tilings are close if they agree on a large patch about the origin, or are a small translation away from each other. However the function sending a to $a \cup b$ only measures the order that tiles in b are from the origin (so that we can place them in order into a). Consider two tilings b and b', which are an arbitrarily small translation d apart, but have different tiles over the origin. These tilings lead to tilings $a \cup b$ and $a \cup b'$. These two tilings are not translations of each other, and are thus a significant distance apart when considered as points of the tiling space. See figure 6.10. Translations on B are thus discontinuous, when considered with respect to $A \cup B$.

Thus the topology of $A \bigcup B$ is non-trivial.



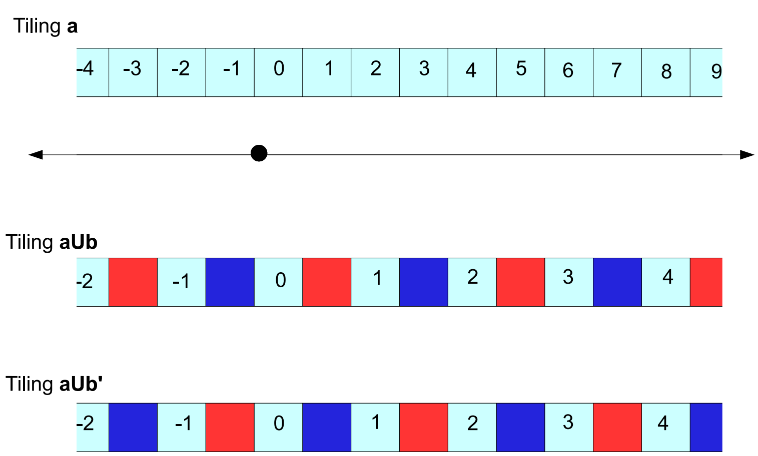


Figure 6.10: b, b', $a \cup b$ and $a \cup b'$

Chapter 7

The $\cap$ operation: An equivalent construction applicable in $\mathbb{R}^d$

The $\cup$ operations used in the last chapter are limited. The 1 dimensional construction uses separate terminology from the 2 dimensional one, both are specific to a particular way of interlacing tilings, and the constructions are unwieldy to iterate.

Thus we will introduce a variation of the $\cup$ operation, which we will denote as the $\cap$ operation. This $\cap$ operation uses the concept of an *array* associated to a tiling. Arrays do not interact with the shape of a tile (unlike the $\cup$ operation), thus we can consider limits of $\cap$ operations with relative ease. The array concept also allows us to consider alternative methods of interlacing tilings ($\cap^n$ operations) with a minor extension of the terminology, producing new tilings we could not derive cleanly from the $\cup$ operation. Furthermore, when we apply a $\cap^n$ operation to an aperiodic tiling, the resultant tiling will be aperiodic.

We will use arrays to describe both the 1 and 2 dimensional cases, using similar terminology for both dimensions. In this chapter we will introduce the (1 dimensional) $\cap$ operation, leaving the 2 dimensional case to the next chapter. Let us consider the following definition of $\mathbb{Z}^d$-arrays. This formulation of $\mathbb{Z}^d$-arrays was first studied in [13], and extended in [10].

Definition 42 ($\mathbb{Z}^d$-array)**.** A $\mathbb{Z}^d$-*array* (with alphabet Σ) is a function

$Z : \mathbb{Z}^d \mapsto \Sigma$

The precise formulation of $\mathbb{Z}^d$-arrays we are using is new to this thesis (since [10] uses many concepts from dynamical systems), but is equivalent to the version found in [10].

Definition 43. Choose a set of basis vectors $\{v_1, \ldots, v_d\}$ of $\mathbb{R}^d$. Define the *underlying lattice* $Latt$ as $\{a_1v_1 + a_2v_2 + \ldots + a_dv_d | a_1, a_2, \ldots \in \mathbb{Z}\}$. Define $p : \mathbb{Z}^d \mapsto Latt$ as $p(a_1, a_2, \ldots, a_d) = a_1v_1 + a_2v_2 + \ldots + a_dv_d$. This is clearly a bijective function.

Then the *array on a lattice* $(Z, Latt)$ is the function;

$Z' : Latt \mapsto \Sigma$ such that $Z' \circ p = Z$.

In other words, Z is the pullback of Z'. Note that if you have an array Z and an underlying lattice $Latt$, you can calculate Z'. Similarly, if you have Z' you can calculate Z. Thus we will often refer to 'arrays on a lattice' as 'arrays', when the lattice is fixed.

$$\begin{tikzcd}
\mathbb{Z}^d \arrow[d, "p"'] \arrow[dr, "Z"] & \\
Latt \arrow[r, "Z'"'] & \Sigma
\end{tikzcd}$$

These definitions correspond to assigning an element from an alphabet Σ to every point on a lattice embedded in $\mathbb{R}^d$. The underlying lattice $Latt$ encodes which lattices we choose and the $\mathbb{Z}^d$-array encodes what elements from the alphabet get assigned to the particular lattice points. From now on we will use the convention that when no underlying lattice is mentioned, we are using the integer points in $\mathbb{R}^d$ for our underlying lattice.

Note that the underlying lattices form Delone sets, i.e. any lattice is relatively dense and uniformly discrete. We can consider the Voronoi diagram of a lattice. As a reminder, the Voronoi cell of a point p in a disconnected set $A \subset \mathbb{R}^d$ is the set of all points $x \in \mathbb{R}^d$ such that $d(x,p) = d(x,A)$.

Since we are using a lattice, the Voronoi cells of a lattice will be all the same shape (a unit interval for lattices in $\mathbb{R}$, a parallelogram for lattices in $\mathbb{R}^2$, parallelepiped for lattices in $\mathbb{R}^3$, and so on).

Thus we can move between a tiling by unit intervals/paralleolograms/parallelepipeds (etc) and an array on a lattice, as follows.

Remark. Tilings by unit intervals/paralleolograms/parallelepipeds (etc) can be converted into arrays on lattices.

To do this, take such a tiling T of $\mathbb{R}^d$ where the origin is the centrepoint of some tile in the tiling. Mark the centrepoint p_t of each tile t. These new points will form a lattice, namely $\mathbb{Z}^d \subset \mathbb{R}^d$. Choose an alphabet consisting of all possible prototiles in T. Construct an array mapping a point in $\mathbb{Z}^d$ to the prototile that point corresponds to. The array will thus be a map from $\mathbb{Z}^d$ to an alphabet of all possible tile types.

Arrays can be turned into tilings by taking the Voronoi diagram of the underlying lattice of the array, and labelling each Voronoi cell with the corresponding element from the array. While there exist multiple tilings that correspond to the same underlying lattice, if we restrict ourselves to unit paralleolograms in $\mathbb{R}^2$ (parallelepipeds in $\mathbb{R}^3$, intervals in $\mathbb{R}$, etc), there will be one unique tiling, and the shape of the tiles will be the same as the Voronoi cells of the lattice. Thus these two functions are invertible when restricted to unit tilings with centrepoints of tiles over the origin of $\mathbb{R}^d$.

Note that this method can only convert tilings with the origin as a centrepoint of some tile. This method can be extended to all tilings by allowing underlying lattices which are offset from the origin by some vector. This chapter will not require such tilings, so for the sake of clarity we will use our current definition.

We will limit ourselves to studying the one and two dimensional cases in this thesis. While the two dimensional case seems more general, there are operations applicable to the 1 dimensional case which are not applicable to the 2 dimensional (and higher) cases.

7.1 1 dimension

The major advantage of studying the 1D case is that we can express the array as a bi-infinite *sequence*, enabling us to use results applicable to sequences. Thus we will define the $\cap$ operation for 1D arrays, and *Toeplitz sequences*, a class of examples originating in the field of dynamical systems [16] [22].

Definition 44 ($A \cap B$, the Superposition operation)**.** Let A and B be $\mathbb{Z}$-arrays, namely $A : \mathbb{Z} \mapsto \Sigma$, $B : \mathbb{Z} \mapsto \Sigma$.

Then define $A \cap B$ as follows;

$$(A \cap B)(v) = \begin{cases} B(n) & \text{if } v = 2n, \\ A(n) & \text{if } v = 2n - 1. \end{cases}$$

Definition 45. Define $f_X(A)$ as the function sending an array A to $A \cap X$, where X is another array.

Theorem 13. If A or B is an aperiodic tiling, then $A \cap B$ is an aperiodic tiling.

Proof. Let us use proof by contrapositive. Assume $A \cap B$ is periodic. Then there exists a period p such that $(A \cap B)(v) = (A \cap B)(v + kp)$, for all $k \in \mathbb{Z}$. If p is a period of $A \cap B$, then $2p$ will also be a period of $A \cap B$. The elements of B are introduced to $A \cap B$ at the even positions. Thus B will be periodic, of period p, since $A \cap B$ is periodic with period $2p$. A will also be periodic, of period $2p - \frac{2p}{2}$, which simplifies to p. Thus if $A \cap B$ is periodic, A and B must be periodic. $\square$

7.2 Motivation

We can consider $A \cap B$ in one dimension as the overlaying, or merger of two separate tilings. Imagine the tilings A and B as two infinite translucent physical strips with the strip representing each tiling. The operation $A \cap B$ corresponds with placing A down over the origin normally, then shifting it half a unit to the left. You then place the strip for B down over the origin normally, effectively interlacing points from A and B. Looking through A and B's strips, the tiling $A \cap B$ can be read off. Note that the distance between consecutive points needs to be scaled back up to 1 (by expanding about the origin by a factor of 2). See picture 7.1 for a schematic motivation of how to create $A \cap B$ from two tilings A and B.

We will now give the definition of a Toeplitz sequence. We will also define a 'null' array, a concept used in the field of Toeplitz sequences. Note that there are many equivalent definitions of a Toeplitz sequence. We will use one more easily applicable to this thesis, from the reference [16] (with a very similar format to [15]).

Definition 46. Let A be a finite set of at least two elements. Let A^* be the set of finite sequences, or words over A. If $w \in A^*$, let $|w|$ denote its length. Let $\Omega = A^{\mathbb{Z}}$. If $S \in \Omega$, $n \in \mathbb{Z}$ and $p \geq 1$, then let

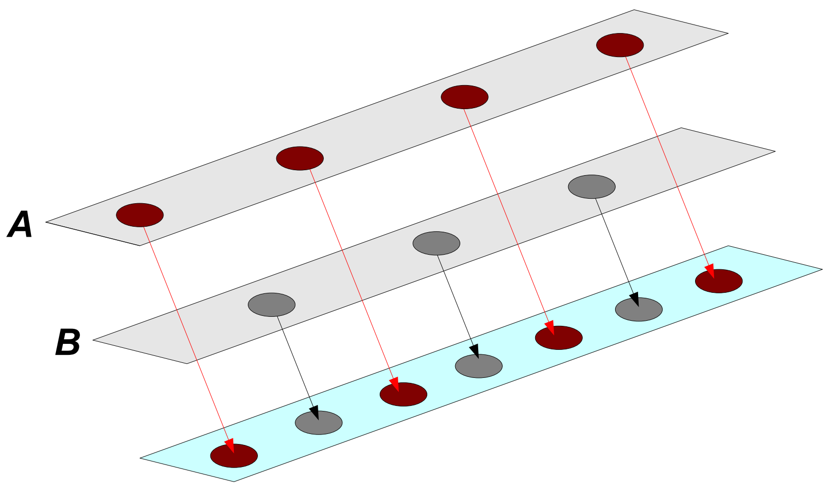


Figure 7.1: A schematic motivation for $A \cap B$.

$$S_n S_{n+1} \ldots S_{n+p-1}$$

denote the word of length p appearing in S starting at position n. Thus S_n is the nth letter in the sequence.

Definition 47. An element $S \in \Omega$ is called a *periodic* sequence with period $p \in \mathbb{N}$ if $S_t = S_{t+p}$ for all $t \in \mathbb{Z}$.

Definition 48. An element $S \in \Omega$ is called a *Toeplitz sequence* if it is not a periodic sequence, and satisfies the following condition;

$$\forall n \in \mathbb{Z}, \exists p \geq 2 \text{ such that } \forall k \in \mathbb{Z}, S_{n+kp} = S_n$$

Definition 49. A *p-periodic part* of a bi-infinite sequence $S \in A^{\mathbb{Z}}$ is;

$Per_p(S) = \{n \in \mathbb{Z} : \forall k \in \mathbb{Z}, S_{n+pk} = S_n\}$

Remark. Consider a sequence T, which is not periodic. If every point $T_k \in T$ is in a p_t-periodic part for some p_t, T is a Toeplitz sequence. This is because if a point T_n belongs to a p_t-periodic part, then by definition of periodic part, $T_{n+kp} = T_n$.

Definition 50. An *almost Toeplitz sequence* is a sequence where all but a finite number of points are in a periodic part.

In this chapter we will be converting certain non-periodic tilings into (almost) Toeplitz sequences. The points not in a periodic part will be a finite word directly to the right of the origin. We will refer to this word as the *seed*, for reasons which will be apparent later.

Definition 51. A *null* array $*$ is the unique array with alphabet $\{*\}$.

Regarding the $\cap$ operation, let us define it to be an operation that is conventionally evaluated, as follows;

Definition 52. $A \cap B \cap C := ((A \cap B) \cap C)$ Furthermore; $A_1 \cap A_2 \cap \ldots \cap A_{n+1} := (A_1 \cap A_2 \cap \ldots \cap A_n) \cap A_{n+1})$ for all $n \in \mathbb{N}$.

Definition 53. $(A \cap B)^2 = A \cap B \cap A \cap B$

$(f_A \circ f_B)^2(X) = f_A \circ f_B \circ f_A \circ f_B(X) = X \cap B \cap A \cap B \cap A$

Definition 54. If $B_1, B_2, B_3, \ldots$ is a sequence of arrays $\mathbb{Z} \mapsto \Sigma$, say they have the limit $B : \mathbb{Z} \mapsto \Sigma$ if $\forall n \in \mathbb{N}\ \exists m$ such that $B(x) = B_i(x)$ for all $-n \leq x \leq n$ and $i \geq m$.

For studying infinite $\cap$ operations, we need to be careful when looking at the limit.

Consider two arrays, $\underline{0}$ and $\underline{1}$, where $\underline{0}(v) = 0$ and $\underline{1}(v) = 1$ for all $v \in \mathbb{Z}$. Now consider the series $\{X_i\}_{i=1}^{\infty} = \underline{0}, \underline{0} \cap \underline{1}, \underline{0} \cap \underline{1} \cap \underline{0}, \ldots$ as shown in figure 7.2.

Position in lattice $\rightarrow$	-2	-1	0	1	2	3	4	5	6	7
$\underline{0}$	0	0	0	0	0	0	0	0	0	0
$\underline{0} \cap \underline{1}$	1	0	1	0	1	0	1	0	1	0
$\underline{0} \cap \underline{1} \cap \underline{0}$	0	1	0	0	0	1	0	0	0	1
$\underline{0} \cap \underline{1} \cap \underline{0} \cap \underline{1}$	1	0	1	0	1	0	1	1	1	0
$f_{\underline{0}}(*)$	0	$*$	0	$*$	0	$*$	0	$*$	0	$*$
$f_{\underline{1}}(*)$	1	$*$	1	$*$	1	$*$	1	$*$	1	$*$

Figure 7.2: The first few points of $\underline{0}, \underline{0} \cap \underline{1}$ and other arrays.

By examining definition 44, the reader can observe that points in the even positions of an array $A \cap B$ are only dependent on the array B. Thus the same result applies for a function $f_B(A)$, since it is merely different terminology.

We will use this fact to show that no limit of the series $\{X_i\}_{i=1}^{\infty}$ exists.

The sequence $\{X_i\}_{i=1}^{\infty} = \underline{0}, \underline{0} \cap \underline{1}, \underline{0} \cap \underline{1} \cap \underline{0}, \ldots$ can be rewritten as

$$\underline{0}, f_{\underline{1}}(\underline{0}), f_{\underline{0}}(f_{\underline{1}}(\underline{0})), \ldots, f_{\underline{0}}(X_{2i+1}), f_{\underline{1}}(X_{2(i+1)}), f_{\underline{0}}(X_{2(i+1)+1})) \ldots$$

More precisely, $X_{2i} = f_{\underline{1}}(X_{2i-1})$, $X_{2i+1} = f_{\underline{0}}(X_{2i})$ and $X_1 = \underline{0}$.

The value of even positions in $X_{2i} = f_{\underline{1}}(X_{2i-1})$ is 1. The value of even positions in X_{2i+1} is 0. Thus the value of points in even positions will be different in X_{2i} and X_{2i+1}, for any $i \in \mathbb{N}$. Thus there does not exist a limit of this sequence.

Limits of $\cap$ operations do not (usually) exist, but we can instead use the common idea of convergent subsequences, in an attempt to get interesting results. For example with the limit of $\underline{0} \cap \underline{1} \cap \underline{0} \cap \underline{1} \cap \ldots$ we could consider the limit of

$$\underline{0} \cap \underline{1}, (\underline{0} \cap \underline{1})^2, (\underline{0} \cap \underline{1})^3, \ldots$$

or

$$\underline{0}, \underline{0} \cap \underline{1} \cap \underline{0}, (\underline{0} \cap \underline{1})^2 \cap \underline{0}, \ldots$$

(These subsequences do have properly defined limits, which will be proven in a later section). Considering different subsequences may give us different values for a limit. Note that the work in this section implies that only superpositions of the form $A_1 \cap A_2 \cap \ldots \cap A_n \cap A \cap A \cap A \ldots$ could have limits when considered in the naive way.

7.3 $A \cap^n B$

We will now define an operation without obvious equivalent in higher dimensions, $\cap^n$. We will then describe the limit of a sequence of $\cap^n$ superposition operations.

Definition 55. Let A and B be one dimensional sequences (possibly derived from tilings as described in the previous section). Define $A \cap^n B : \mathbb{Z} \mapsto \Sigma$ as follows;

$$(A \cap^n B)(v) = \begin{cases} B(\frac{v}{n+1}) & \text{if } v = k(n+1) \text{ for } k \in \mathbb{Z}, \\ A(v - \lfloor \frac{v}{n+1} \rfloor) & \text{otherwise.} \end{cases}$$

Theorem 14. If A or B is an aperiodic tiling, then $A \cap^n B$ is an aperiodic tiling.

Proof. Assume $A \cap^n B$ is periodic. Then there exists a period p such that $(A \cap^n B)(v) = (A \cap^n B)(v + kp)$, for all $k \in \mathbb{Z}$. If p is a period of $A \cap B$, then $(n+1)p$ will also be a period of $A \cap^n B$. The elements of B are introduced to $A \cap B$ at every $(n+1)$th position. Thus B will be periodic, of period p, since $A \cap^n B$ is periodic with period $(n+1)p$. A will also be periodic, of period $(n+1)p - \frac{(n+1)p}{(n+1)}$, which simplifies to np. Thus if $A \cap^n B$ is aperiodic, at least one of A or B must be aperiodic. □

Intuitively you place $B_0 \in B$ at the origin, bumping A_0 left one unit, and this bumping all points left of the origin one unit left. You then add B_1 to the right of A_n, bumping tiles A_N for $N > n$ right by 1. Similarly you add B_{-1} to the right of A_{-n}, bumping tiles away from the origin to make space. You then place B_2 to the right of A_{2n} and B_{-2} to the right of A_{-2n}, and iterate for B_3 and B_{-3}, and so on.

Definition 56. Define $f_X^n(A)$ as the function sending an array A to an array $A \cap^n X$.

Remark. Note that the $\cap^1$ operation is equivalent to the $\cap$ operation defined previously.

Lattice Positions		-5	-4	-3	-2	-1	0	1	2	3	4	5	6
Array	A	A_{-5}	A_{-4}	A_{-3}	A_{-2}	A_{-1}	A_0	A_1	A_2	A_3	A_4	A_5	A_6
Array	B	B_{-5}	B_{-4}	B_{-3}	B_{-2}	B_{-1}	B_0	B_1	B_2	B_3	B_4	B_5	B_6
Array	$A \cap B$	A_{-2}	B_{-2}	A_{-1}	B_{-1}	A_0	B_0	A_1	B_1	A_2	B_2	A_3	B_3
Array	$A \cap^2 B$	A_{-3}	A_{-2}	B_{-1}	A_{-1}	A_0	B_0	A_1	A_2	B_1	A_3	A_4	B_2
Array	$A \cap^3 B$	A_{-3}	B_{-1}	A_{-2}	A_{-1}	A_0	B_0	A_1	A_2	A_3	B_1	A_4	A_5
Array	$A \cap^4 B$	B_{-1}	A_{-3}	A_{-2}	A_{-1}	A_0	B_0	A_1	A_2	A_3	B_2	A_4	A_5
Array	$A \cap^5 B$	A_{-4}	A_{-3}	A_{-2}	A_{-1}	A_0	B_0	A_1	A_2	A_3	A_4	A_5	B_1
	$\vdots$			$\vdots$				$\vdots$				$\vdots$	

Figure 7.3: The arrays A, B, and $A \cap^i B$ for $1 \leq i \leq 5$

7.4 $(\cap A \cap B)^\infty$

We will now consider a subset of infinite compositions of the $\cap$ and $\cap^n$ operations. Note that the $\cap$ operation is equivalent to the operation $\cap^1$. Thus, in the 1 dimensional case, $\cap^n$ can be considered to be a more general operation. We will start with the basic definitions, before giving illuminating examples.

Definition 57 ($(\cap^s A \cap^t B)^m$)**.** Consider two sequences A and B. Let $s, t, m \in \mathbb{N}$. Then define $(\cap^s A \cap^t B)^m$ as follows;

$$(\cap^s A \cap^t B)^m := \underbrace{A \cap^t B \cap^s A \cap^t B \cap^s \ldots\ldots \cap^s A \cap^t B}_{2m}$$

Similarly (for sequences A_i, and integers $a_i \in \mathbb{N}$).

$$(\cap^{a_1} A_1 \cap^{a_2} A_2 \ldots \cap^{a_n} A_n)^m := \underbrace{A_1 \cap^{a_2} A_2 \ldots \cap^{a_n} A_n \cap^{a_1} A_1 \cap^{a_2} A_2 \ldots\ldots \cap^{a_{n-1}} A_{n-1} \cap^{a_n} A_n}_{mn}$$

Definition 58 ($(\cap^s A \cap^t B)^\infty$)**.** Let A and B be sequences. Let $s, t \in \mathbb{N}$. Then define;

$$(\cap^s A \cap^t B)^\infty := \lim_{m\to\infty} (\cap^s A \cap^t B)^m$$

if the limit exists.

Similarly (for sequences A_i, and integers $a_i \in \mathbb{N}$),

$$(\cap^{a_1} A_1 \cap^{a_2} A_2 \ldots \cap^{a_n} A_n)^\infty := \lim_{m\to\infty} (\cap^{a_1} A_1 \cap^{a_2} A_2 \ldots \cap^{a_n} A_n)^m$$

if the limit exists.

We will now work towards calculating the limit of repetitively applying a series of $\cap^n N$ operators, with different periodic arrays N. We will eventually prove that you can create aperiodic substitution sequences by this method. To improve clarity, we will run through a simple example (with associated proofs) first, to give the reader some intuition into this area.

7.5 $\underline{0} \cap \underline{1}$ and its limit $(\underline{0} \cap \underline{1})^\infty$

Recall that $(\underline{0} \cap \underline{1})^m$ is defined as $\underbrace{\underline{0} \cap \underline{1} \cap \underline{0} \cap \underline{1} \ldots \cap \underline{0} \cap \underline{1}}_{2m}$. Similarly, $(\underline{0} \cap \underline{1})^\infty$ is defined as the limit of $(\underline{0} \cap \underline{1})^m$ as $m \to \infty$.

We will show that $(\underline{0} \cap \underline{1})^\infty$ exists, and is equivalent to an aperiodic primitive substitution tiling.

Consider the sequence of arrays $*, f_{\underline{1}}(*), f_{\underline{0}} \circ f_{\underline{1}}(*), \ \ldots, (f_{\underline{0}} \circ f_{\underline{1}})^m(*), f_{\underline{1}} \circ (f_{\underline{0}} \circ f_{\underline{1}})^m(*), \ldots$. Part of this sequence is shown in figure 7.4. Recall that $*$ is the null array.

Arrays \ Array positions	-3	-2	-1	0 (origin)	1	2	3	4	5	6	7	...
$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	...
$f_{\underline{1}}(*)$	$*$	1	$*$	1	$*$	1	$*$	1	$*$	1	$*$	...
$f_{\underline{0}} \circ f_{\underline{1}}(*)$	$*$	0	1	0	$*$	0	1	0	$*$	0	1	...
$(f_{\underline{1}} \circ f_{\underline{0}}) \circ f_{\underline{1}}(*)$	1	1	0	1	$*$	1	0	1	1	1	0	...
$(f_{\underline{0}} \circ f_{\underline{1}})^2(*)$	0	0	1	0	$*$	0	1	0	0	0	1	...
$(f_{\underline{1}} \circ f_{\underline{0}})^2 \circ f_{\underline{1}}(*)$	1	1	0	1	$*$	1	0	1	1	1	0	...
	$\vdots$			$\vdots$			$\vdots$					

Figure 7.4: Sequence of arrays.

As we repetitively apply the $f_{\underline{0}} \circ f_{\underline{1}}$ operation to our array it appears that larger and larger patches of the array are independent of what the starting array was. We will now make this rigorous.

Definition 59. Consider an array $A_1 \cap^{n_2} A_2 \cap^{n_3} A_3 \ldots \cap^{n_m} A_m$. A point p in the underlying lattice of $A_1 \cap^{n_2} A_2 \ldots \cap^{n_m} A_m$ is called an *undefined* point if $* \cap^{n_2} A_2 \cap^{n_3} A_3 \ldots \cap^{n_m} A_m$ maps that point to $*$.

Thus from definition 44, any array of the form $X \cap \underline{1}$ (ie $f_{\underline{1}}(X)$), where X is any array, has undefined points at odd positions.

Lemma 13. Let $m \in \mathbb{N}_0$ and $k \in \mathbb{N}$. If a point is defined in $f_{\underline{1}} \circ (f_{\underline{0}} \circ f_{\underline{1}})^m(*)$, it will be defined, and take the same value, in $f_{\underline{1}} \circ (f_{\underline{0}} \circ f_{\underline{1}})^{m+k}(*)$.

Proof. Consider $f_{\underline{0}}(X)$, for any array X. By the definition of the $\cap$ operation, we know that $f_{\underline{0}}(X)$ will have defined points at positions $2s$ ($s \in \mathbb{Z}$), with value 0. (Similarly, $f_{\underline{1}}(X)$ will have defined points with value 1 in the same positions).

Consider $f_{\underline{1}} \circ f_{\underline{0}}(X)$ which is equivalent to $f_{\underline{1}}$ applied to $f_{\underline{0}}(X)$. By the definition of the $\cap$ operation, a point at position $4s-1$ in $f_{\underline{1}} \circ f_{\underline{0}}(X)$ will share the same value as a point at position $2s$ in $f_{\underline{0}}(X)$. Thus $f_{\underline{1}} \circ f_{\underline{0}}(X)$ will have defined points at positions $4s-1$ with value 0, and defined points at positions $2s$ with value 1.

Let us generalise to an array $f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_n}(X)$. This array will have defined points at positions $2^n s - 2^{n-1} + 1$, with value i_n. We will use induction on n to prove this.

Regarding the initial case, $f_{i_1}(X)$ has defined points at positions $2s$ as required (via definition 44). For the induction step, consider $f_{i_2} \circ f_{i_3} \circ \ldots \circ f_{i_{n+1}}(X)$. This will have defined points at positions $2^n s - 2^{n-1} + 1$, with value i_{n+1}, from our assumption. Call this array X_n. From definition 44, $f_{i_1}(X_n)$ will have a point with value $X_n(v)$ at position $2v-1$. Thus the set of points with value i_{n+1} in $f_{i_1}(X_n)$ are $\{x | x \in 2(2^n s - 2^{n-1} + 1) - 1, s \in \mathbb{Z}\}$. This can be rewritten as $S_{n+1} = \{x | x \in 2^{n+1} s - 2^n + 1, s \in \mathbb{Z}\}$, as required. Thus our induction holds.

Thus $f_{\underline{1}} \circ (f_{\underline{0}} \circ f_{\underline{1}})^{m+k}(*)$ will define every point $f_{\underline{1}} \circ (f_{\underline{0}} \circ f_{\underline{1}})^{m}(*)$ defines. This is because $f_{\underline{1}} \circ (f_{\underline{0}} \circ f_{\underline{1}})^{m}(*)$ will have defined points (with associated values) in $\bigcup_{i=1}^{i=2m+1} S_i$ and $f_{\underline{1}} \circ (f_{\underline{0}} \circ f_{\underline{1}})^{m+k}(*)$ will have defined points in $\bigcup_{i=1}^{i=2(m+k)+1} S_i$. Furthermore, the defined points will have the same value, since the sets $S_n = \{2^n s - 2^{n-1} + 1 | s \in \mathbb{Z}\}$ are disjoint for $n \geq 1$, thus the value of a point is solely determined by which set S_i it is contained within.

For an explicit proof that the sets S_n are disjoint, consider S_a and S_b, for $a, b \in \mathbb{N}$, $a \neq b$. Assume WLOG that $a < b$. Then $b = a + c$, for $c > 0$. The sets S_a and S_b will have non-zero intersection if there exists an integer solution to the equation $2^a s - 2^{a-1} + 1 = 2^b t - 2^{b-1} + 1$. This can be rewritten as $2^a s - 2^{a-1} + 1 =$

$2^{a+c}t - 2^{a+c-1} + 1$, and simplified to $2s - 1 = 2^{c+1}t - 2^c$. The left hand side of this equation can only take odd values, and the right hand side can only take even values (since $c > 0$). Thus S_a and S_b are disjoint.

$\square$

Theorem 15. $\lim_{m\to\infty}(\underline{0} \cap \underline{1})^m$ exists.

Proof. Note that another equivalent way of writing $\lim_{m\to\infty}(\underline{0}\cap\underline{1})^m$ is as $\lim_{m\to\infty}[f_{\underline{1}} \circ (f_{\underline{0}} \circ f_{\underline{1}})^{m-1}(\underline{0})]$.

Consider the lattice point at position '1' in any array. For any array X, the f_X operation does not change what value is assigned to this point. An array $(\underline{0} \cap \underline{1})^m$ is constructed by taking a $\underline{0}$ array, and applying multiple $f_{\underline{1}}$ and $f_{\underline{0}}$ operations to it in turn. Thus the lattice point at position '1' in an array $(\underline{0} \cap \underline{1})^m$ will be a point of type 0, for all m. Thus the lattice point at position '1' is well-defined in the limit.

For our next step, we will show that in the limit, the only undefined point is at position '1'. Call this point the *seed*. Consider $f_{\underline{1}}(*)$. The points at positions 2 and 0 are defined. On applying $f_{\underline{0}}$ to this array, the point at position 2 is shifted to position 3, and the point at position 0 is shifted to position -1. Thus in $f_{\underline{0}} \circ f_{\underline{1}}(*)$, the points two units away from the seed point are defined from $f_{\underline{1}}$, and the points one unit away from the seed are now defined by $f_{\underline{0}}$. Every time a new f_i function is applied to the array, the patch of defined points grows by (at least) one. Thus in the limit, every point (barring the seed point) is defined for some $f_{\underline{1}} \circ (f_{\underline{0}} \circ f_{\underline{1}})^m(\underline{0})$. From the previous lemma, we know that every defined point will keep the same value as $m \to \infty$. Thus these lattice points are well-defined in the limit. Thus $\lim_{m\to\infty}(\underline{0} \cap \underline{1})^m$ exists.

$\square$

Theorem 16. $(\underline{0} \cap \underline{1})^\infty$ is an almost Toeplitz sequence, with the only undefined point being at position '1'.

Proof. We will use induction to show that every defined point in $(\underline{0} \cap \underline{1})^\infty$ is in some periodic part. Construct the sequence $\{X_i\}$ where $X_1 = f_{\underline{1}}(\underline{0})$, $X_2 = f_{\underline{0}}(X_1)$, $X_{2i} = f_{\underline{0}}(X_{2i-1})$ and $X_{2i+1} = f_{\underline{1}}(X_{2i})$ for $i \geq 2$.

Consider the defined points of $X_1 = f_{\underline{1}}(\underline{0})$. These points occur at even positions, and thus form a 2-periodic part, as defined in definition 49. Thus every defined point in X_1 is in some periodic part. (In fact, every point is in a periodic part).

Assume every defined point in X_n is in some periodic part. Consider $X_{n+1} = f_{\underline{i}}(X_n)$ (where $f_{\underline{i}}$ is either $f_{\underline{0}}$ or $f_{\underline{1}}$, based on parity of n). (Denote the points of X_n as x'_t, and the points of X_{n+1} as x_t, for $t \in \mathbb{Z}$).

Any defined point in $f_{\underline{i}}(X_n)$ will either correspond to a defined point in X_n, or will be an even point. If it is an even point, it will be assigned a value of 1 (if n is even) or 0 (if n is odd). Since all even points are assigned the same value, the even point of $f_{\underline{i}}(X_n)$ form a 2-periodic part.

Any defined point $x_{2t-1} \in f_{\underline{i}}(X_n)$ will correspond to a point x'_t in X_n. By our assumption, the point x'_t in X_n belongs to some p-periodic part, for some p. This periodic part consists of a series of points x'_{t+kp} with the same label (for $k \in \mathbb{Z}$). Therefore the p-periodic part will be mapped to a $2p$-periodic part in $f_{\underline{i}}(X_n)$, consisting of a series of points $x_{2(t+kp)-1}$ with the same label. Thus any defined point in X_{n+1} belongs to some periodic part. By induction, any defined point in X_i, for any i, must be in a periodic part. We know from the last theorem that if a point is defined in X_i, it will have the same value in all sequences X_{i+2k}. Therefore once a point is defined (and thus belongs to a periodic part), it will always belong to a periodic part. We can therefore conclude that every defined point in the limit $(\underline{0} \cap \underline{1})^\infty$ is in a periodic part.

$\square$

Remark. Any finite composition of periodic arrays under the $\cap$ operation will be periodic. In particular, the sequence $(\underline{0} \cap \underline{1})^m$ is periodic, with period 2^{2m-1}.

As a sketch proof, let $\{X_k\}_{k=0}^{\infty}$ be any arrays with only one letter in each alphabet, possibly different for each array. (A similar proof applies for periodic arrays, but requires more complex notation for little added clarity). Consider some sequence $F_1 = f_{X_1}(X_0)$, $F_i = f_{X_i}(F_{i-1})$. Consider a point in X_0. It will be in a 1-periodic part. Under the f_{X_1} operation, this 1-periodic part will be mapped to a 2-periodic part. Any point in F_1 will either be undefined (and hence in the 2-periodic part inherited from X_0), or it will be defined. If it is defined, it will be in another 2-periodic part, this one consisting of points from X_1 in even positions in the array. Thus F_1 has period 2.

Applying the f_{X_2} operation to F_1 will send the 2-periodic part representing undefined points in F_1 to a 4-periodic part in F_2. The 2-periodic part representing points first defined by f_{X_1} will be sent to a 4-periodic part, and a new 2-periodic part will be created for the points from X_2 which have been placed in even positions in the array F_2. Thus F_2 will have period 4 (the *lcm* of all the periodic parts).

Similarly F_3 will consist of two 8-periodic parts, a 4-periodic part and a new 2-periodic part containing values from X_3. In general, F_i will contain two 2^i-periodic parts, and one 2^j-periodic part for each $0 < j < i$. Thus F_i will have period 2^i. $(\underline{0} \cap \underline{1})^m$ can be expressed as F_{2m-1} for $X_{2k} = \underline{0}$, $X_{2k+1} = \underline{1}$. Thus the result follows.

Theorem 17. The sequence $(\underline{0} \cap \underline{1})^\infty$ is a fixed point of the substitution,

$$\begin{aligned} \sigma : & 0 \mapsto 0101 \\ & 1 \mapsto 1101 \end{aligned}$$

More precisely $\sigma((\underline{0} \cap \underline{1})^\infty) = (\underline{0} \cap \underline{1})^\infty$.

Furthermore $\lim_{m\to\infty}((\underline{0}\cap\underline{1})^m\cap\underline{0})$ is also a fixed point of a substitution, namely;

$$\begin{aligned}\widehat{\sigma} :&0 \mapsto 0010\\ &1 \mapsto 1010\end{aligned}$$

More precisely $\widehat{\sigma}(\lim_{m\to\infty}((\underline{0} \cap \underline{1})^m \cap \underline{0})) = \lim_{m\to\infty}((\underline{0} \cap \underline{1})^m \cap \underline{0})$.

Proof. First consider $\lim((\underline{0} \cap \underline{1})^m \cap \underline{0})$. Note that this limit exists, since it is equivalent to $\lim f_{\underline{0}}((\underline{0} \cap \underline{1})^m)$, and $\lim((\underline{0} \cap \underline{1})^m)$ is well defined.

We know that $\lim_{m\to\infty}((\underline{0}\cap\underline{1})^m\cap\underline{0})$ must be invariant under $f_{\underline{0}} \circ f_{\underline{1}}$, since $f_{\underline{0}} \circ f_{\underline{1}}$ is equivalent to increasing the value of m by one. From theorem 16, we know that the point at position '1' of $\lim((\underline{0} \cap \underline{1})^m \cap \underline{0})$ must be of value 0, since neither $f_{\underline{0}}$ or $f_{\underline{1}}$ can alter the value of the point at position '1 ', and it is of value '0' in the array $\underline{0}$ (the first sequence we are building the limit from). Denote the point at position '1' the *seed* point.

Examining the proof of theorem 16, note that points defined by the first application of $f_{\underline{0}}$ must be in a 2-periodic part, and points defined by the following application of $f_{\underline{1}}$ will be in a (2×2)-periodic part. Thus we can deduce that the highest periodic part in $f_{\underline{0}} \circ f_{\underline{1}}(*)$ is of period 4, thus $f_{\underline{0}} \circ f_{\underline{1}}(*)$ is periodic, with period 4. Furthermore the repeating period is the word $*$ 0 1 0.

Now consider a generic sequence $t = \ldots t_0 t_1 t_2 \ldots$, and what points are defined from it under the application of $f_{\underline{0}} \circ f_{\underline{1}}$. The seed point, t_1 gets sent to the word t_1 0 1 0. Via the periodicity of $f_{\underline{0}} \circ f_{\underline{1}}(*)$ we know that the point t_2 is sent to the word t_2 0 1 0, which is joined on to t_1's word by concatenation. Similarly, t_3's word is concatenated to the end of t_2's word, and so on for all t_i. This is of course the function σ from the definition of the substitution. Since the seed point is fixed for all $m \geq 1$, the points generated from it are fixed for all $m \geq 2$. Similarly, all points generated from those points are fixed for $m \geq 3$, and so on. Thus all points defined via the application of $f_{\underline{0}} \circ f_{\underline{1}}$ are fixed in the limit.

Since every point in the underlying lattice is defined by $f_{\underline{0}} \circ f_{\underline{1}}(t)$, we can conclude that $\lim((\underline{0} \cap \underline{1})^m \cap \underline{0})$ can be generated via a substitution. Since $f_{\underline{0}} \circ f_{\underline{1}}(\lim((\underline{0} \cap \underline{1})^m \cap \underline{0})) = \lim((\underline{0} \cap \underline{1})^m \cap \underline{0})$, it is a fixed point of that substitution.

For $(\underline{0} \cap \underline{1})^\infty$ use a similar proof, using the fact that $(\underline{0} \cap \underline{1})^\infty$ is invariant under $f_{\underline{1}} \circ f_{\underline{0}}$. $\square$

Remark. We now have a way of describing $(\underline{0} \cap \underline{1})^\infty$ as a fixed point of the substitution $0 \mapsto 0101$, $1 \mapsto 1101$. Unlike tiling spaces based on matching rules, there are several methods available to calculate the cohomology of substitution tiling spaces, which are applicable to this example (or more precisely, the tiling space of tilings generated by the associated substitution). We will not give explicit details of a full calculation here, since the concept is well described in papers such as [4].

Informally, we form an cellular complex L consisting of all prototiles and all allowable transitions between prototiles (in the general case, 'collared' prototiles). The tiling space can be represented as an inverse limit of the substitution map on this complex. Thus the cohomology can be calculated from the direct limit of the substitution map applied to the cohomology of the cellular complexes. In this specific case, the subcomplex S formed by the allowable transitions between prototiles is contractible, allowing for simple calculations via the exact sequence of cohomology groups of S and L. The cohomology of this substitution's tiling space calculates as being equivalent to the cohomology of the Thue-Morse tiling space ($H^0(\sigma) = \mathbb{Z}$, $H^1(\sigma) = \mathbb{Z} \oplus \mathbb{Z}[\frac{1}{2}]$, $H^2(\sigma) = 0$).

The cohomology of other substitutions described later on in this chapter can also be calculated via similar methods.

We will now generalize these proofs to arrays of the form $(\cap^{a_1} A_1 \cap^{a_2} A_2 \cap^{a_3} \ldots \cap^{a_n} A_n)^\infty$, where A_i are periodic arrays.

We will also show these types of arrays are (almost) Toeplitz arrays.

7.6 $(\cap^{a_1} A_1 \cap^{a_2} A_2 \cap^{a_3} \ldots \cap^{a_n} A_n)^\infty$

We can extend the proofs of lemma 13 and theorem 15 to cover the more general case, $(\cap^{a_1} A_1 \cap^{a_2} A_2 \cap^{a_3} \ldots \cap^{a_n} A_n)^\infty$. To avoid excessive duplication, we will sketch how to alter the proof to the more general case.

Theorem 18. $(\cap^{a_1} A_1 \cap^{a_2} A_2 \cap^{a_3} \ldots \cap^{a_n} A_n)^\infty$ is well defined.

Sketch proof. Take $\sigma = \min\{a_1, \ldots, a_N\}$. Any point in an array X with position strictly between 0 and σ is invariant under any operation $\cap^{a_i} A_i$. Thus the first $(\sigma - 1)$ points of A_1 form a 'seed' which is invariant, and thus the values of the seed points are well defined in the limit.

Ignoring the seed, the first undefined point in $\cap^{a_1} A_1 \cap^{a_2} A_2 \cap^{a_3} \ldots \cap^{a_n} A_n$ cannot be closer to the origin that position $(\sigma + 1)$, since the first $(\sigma - 1)$ points are in the seed, and a new value will be inserted at position σ during every iteration of $\cap^{a_1} A_1 \cap^{a_2} A_2 \cap^{a_3} \ldots \cap^{a_n} A_n$, by the definition of σ.

Similarly, the first undefined point of $(\cap^{a_1} A_1 \cap^{a_2} A_2 \cap^{a_3} \ldots \cap^{a_n} A_n)^2$ cannot be closer to the origin than $\sigma + 2$, and in general the first undefined point of $(\cap^{a_1} A_1 \cap^{a_2} A_2 \cap^{a_3} \ldots \cap^{a_n} A_n)^m$ cannot be closer than $(\sigma + m)$.

$(\cap^{a_1} A_1 \cap^{a_2} A_2 \cap^{a_3} \ldots \cap^{a_n} A_n)^m$ can be rewritten as;

$$(f_{A_n}^{a_n} \circ f_{A_{n-1}}^{a_{n-1}} \circ \ldots \circ f_{A_1}^{a_1})^k((\cap^{a_1} A_1 \ldots \cap^{a_n} A_n)^{m-k}$$

for $m > k$.

Thus any defined point in $(\cap^{a_1} A_1 \cap^{a_2} A_2 \cap^{a_3} \ldots \cap^{a_n} A_n)^k$ will have the same value for any $(\cap^{a_1} A_1 \cap^{a_2} A_2 \cap^{a_3} \ldots \cap^{a_n} A_n)^m$ for all $m > k$. Thus $(\cap^{a_1} A_1 \cap^{a_2} A_2 \cap^{a_3} \ldots \cap^{a_n} A_n)^\infty$ is well defined.

$\square$

Theorem 19. If A_i are periodic arrays, then $(\cap^{a_1} A_1 \cap^{a_2} A_2 \cap^{a_3} \ldots \cap^{a_n} A_n)^\infty$ is an (almost) Toeplitz array.

Proof. Note that if A_i is a periodic array, every point in A_i is in a periodic part. The maximum period of these periodic parts is equal to the period of the array A_i, denoted $p(A_i)$.

We wish to show that every defined point in $(\cap^{a_1} A_1 \cap^{a_2} A_2 \cap^{a_3} \ldots \cap^{a_n} A_n)^m$ is in a periodic part (ignoring the seed). Construct a sequence $C = \{c(i)\}_{i=1}^\infty$ of arrays where the sth term in the sequence consists of the array A_1 with s $f_{A_i}^{a_i}$ operations applied to it in ascending cyclic sequence.

The first few terms of this sequence are;

$c(1) = A_1 \cap^{a_2} A_2$.

$c(2) = A_1 \cap^{a_2} A_2 \cap^{a_3} A_3$

$c(3) = A_1 \cap^{a_2} A_2 \cap^{a_3} A_3 \cap^{a_4} A_4$

...

$c(n-1) = A_1 \cap^{a_2} \ldots \cap^{a_n} A_n$

$c(n) = A_1 \cap^{a_2} \ldots \cap^{a_n} A_n \cap^{a_1} A_1$

$c(n+1) = A_1 \cap^{a_2} \ldots \cap^{a_n} A_n \cap^{a_1} A_1 \cap^{a_2} A_2$

...

....

Explicitly $c(s)$ is;

$$f_{A_j}^{a_j} \circ \ldots \circ f_{A_1}^{a_1} \circ f_{A_n}^{a_n} \circ f_{A_{n-1}}^{a_{n-1}} \circ \ldots \circ f_{A_1}^{a_1} \circ \ldots \ldots \circ f_{A_2}^{a_2}(A_1)$$

...where j is such that $s = (n-1) + (k-1)n + j$, and $j \leq n$.

The arrays $(\cap^{a_1} A_1 \cap^{a_2} A_2 \cap^{a_3} \ldots \cap^{a_n} A_n)^m$ form a subsequence of this sequence. Thus if we can show that every defined point in any array (from C) is in a periodic part, ignoring the seed, then we are done. We will use induction, showing that this statement holds true for $c(1)$, and that if the statement is true for $c(s)$ it is true for $c(s+1)$.

For the initial step of our induction, consider $f_{A_2}^{a_2}(*)$, alternatively known as $* \cap^{a_2} A_2$. The points defined by $f_{A_2}^{a_2}$ belong to a finite number of periodic parts, with maximum period $(a_2+1).p(A_2)$. Thus every defined point in $f_{A_2}^{a_2}(A_1)$ belongs to a periodic part.

For our induction step, assume every defined point in the array $c(s)$ is in a periodic part. Without loss of generality, take any one of these periodic parts, p, of period x. Consider $f_{A_{j+1}}^{a_{j+1}}$. Consider a section of $c(s)$ of length $\text{lcm}(x, a_{j+1})$. The operation $f_{A_{j+1}}^{a_{j+1}}$ adds a point every a_{j+1} units. Thus under the $f_{A_{j+1}}^{a_{j+1}}$ operation any section *sec* of length $\text{lcm}(x, a_{j+1})$ is expanded to a section *sèc* of length

$$l = \text{lcm}(x, a_{j+1}) + \frac{\text{lcm}(x, a_{j+1})}{a_{j+1}}.$$

Furthermore, after the section of length $\text{lcm}(x, a_{j+1})$ has been expanded, the following section in $c(s)$ of length $\text{lcm}(x, a_{j+1})$ will also be expanded to a section of length l. Via the properties of the lcm function, the positions of points from p will be the same in *sèc*, the following section of length l, and all following sections after that. Thus a x-periodic part in $c(s)$ will be turned into a l-periodic part in

$f^{a_{n+1}}_{A_{n+1}}(c(s))$.

Thus by induction, every defined point (not in the seed) in $(\cap^{a_1} A_1 \cap^{a_2} A_2 \cap^{a_3} \ldots \cap^{a_n} A_n)^\infty$ is in a periodic part. By theorem 7.2, $(\cap^{a_1} A_1 \cap^{a_2} A_2 \cap^{a_3} \ldots \cap^{a_n} A_n)^\infty$ is (almost) Toeplitz. $\square$

We will now introduce a natural extension of the well-known concept of a substitution. The major change will be that the defining function of the substitution is based on *words*, not letters.

Definition 60. Let A be an alphabet, A^* be the set of finite words over A, and $W \subset A^*$. Let A^∞ be the set of bi-infinite words, and $A^\infty w$ be the set of bi-infinite words with unique decomposition into a sequence of elements from W. Let $A^* w$ be the set of finite words with unique decomposition into a sequence of elements from W.

The function $\sigma : W \mapsto A^*$ is a valid substitution on words if it satisfies the following two properties; Firstly it must induce a well defined function $\sigma^\infty : A^* w \mapsto A^* w$. Secondly for all $w \in W$, there exists N such that $|w| < |\sigma^n(w)|$.

A σ-substitution tiling is an element $\Lambda \in A^\infty w$ when $\Lambda \subset Im(\sigma^\infty)^N$ for all $N \in \mathbb{N}$.

For example a function σ defined as $\sigma(01) = 0110$, $\sigma(10) = 1001$ would induce a function σ^∞ sending $011010\ldots$ to $011010011001\ldots$.

Theorem 20. $(\cap^{a_1} A_1 \cap^{a_2} A_2 \cap^{a_3} \ldots \cap^{a_n} A_n)^\infty$ is a fixed point of a substitution on words.

Proof. For brevity, we will define the function $f^{a_n}_{A_n} \circ f^{a_{n-1}}_{A_{n-1}} \circ \ldots \circ f^{a_1}_{A_1}$ as F. Consider $F(*)$. By the proof of theorem 19, points defined by $f^{a_i}_{A_i}$ will be in a p_i-periodic part, where p_i is some finite number. Thus, since we have only applied a finite number of $f_{A_i}^{a_i}$ operations, there are a finite number of periodic parts. Thus $F(*)$ is periodic (with period less than or equal to $p_1 \cdot p_2 \cdot \ldots \cdot p_n$).

Since $F(*)$ is periodic, there must be a repeating sequence S of minimum length (starting at the origin). For example, let K_i be an array with alphabet $\{k_i\}$. Then $f_{K_2} \circ f_{K_1}^2(*)$ would have a repeating sequence $*\ k_2 * k_2\ k_1\ k_2$.

By theorem 18, there exists a finite number of 'seed points' near the origin that are unchanged by $f_{A_i}^{a_i}$, for any $i \in \mathbb{N}$. Call the word formed by these points $w = w_1 w_2 \ldots w_k$. Consider the number n of undefined points in S. If $|w| \geq n$, then we can repetitively apply our function F to the first n points of w to define the entire tiling. This is equivalent to a substitution rule sending $w_1 \ldots w_n$ to the first $|S|$ terms of $F(\grave{w})$ where $\grave{w} = w_1 w_2 \ldots w_n * * * \ldots$. For our earlier example of $f_{K_2} \circ f_{K_1}^2(*)$, this would give a substitution rule $w_1 w_2 \mapsto w_1\ k_2\ w_2\ k_2\ k_1\ k_2$.

If $|w| < n$ then repetitively apply F to $\grave{w}$ until the first n points are defined. Then apply F to these n points as before.

$\square$

Let us consider some examples, in order to improve our intuition regarding this process.

7.7 Examples

Define $\underline{X}$ as the array where all points in the array have type X, where X can be any fixed symbol.

Example 6 $((\cap\underline{0} \cap \underline{1} \cap \underline{2})^\infty)$**.**

$$\begin{aligned}(f_{\underline{2}} \circ f_{\underline{1}} \circ f_{\underline{0}})(*) &= f_{\underline{2}} \circ f_{\underline{1}}(*0 * 0 * 0 \dots) \\ &= f_{\underline{2}}(*101 * 101 \dots) \\ &= *2120212 * 2120212 \dots\end{aligned}$$

The repeating sequence is thus $*2120212$*. Thus the substitution is* $x_i \mapsto x_i 2120212$*.*

The first array in $(\cap\underline{0}\cap\underline{1}\cap\underline{2})^\infty$ *is the* $\underline{0}$ *array, and only the point at the origin in* $\underline{0}$ *is unaffected by* $f_{\underline{0}}$*,* $f_{\underline{1}}$ *and* $f_{\underline{2}}$*. Thus the seed of* $(\cap\underline{0}\cap\underline{1}\cap\underline{2})^\infty$ *is the word '*0*'. '*0*' has one point in it, which is equal to the number of unknown points in* $*2120212$*. Thus* 0 *is our initial word, and* $x_i \mapsto x_i 2120212$ *is our substitution.*

Example 7 $((\cap^2\underline{A}\cap\underline{B}\cap^4\underline{C})^\infty)$**.** *Define a sequence as follows:* $\hat{*} = *_1 *_2 \dots *_n *_{n+1} \dots$ *Now consider* $f^4_{\underline{C}} \circ f_{\underline{B}} \circ f^2_{\underline{A}}(\hat{*})$*.*

$$\begin{aligned}f^4_{\underline{C}} \circ f_{\underline{B}} \circ f^2_{\underline{A}}(\hat{*}) &= f^4_{\underline{C}} \circ f_{\underline{B}}(*_1 *_2 A *_1 *_2 A \dots) \\ &= f^4_{\underline{C}}(*_1 B *_2 BAB *_1 B *_2 BAB \dots) \\ &= *_1 B *_2 BCAB *_1 BC *_2 BABC \dots\end{aligned}$$

Therefore our substitution is:

$$x_{2n-1}x_{2n} \mapsto x_{2n-1}Bx_{2n}BCABx_{2n-1}BCx_{2n}BABC$$

This substitution rule has two unknown points (x_{2n-1} *and* x_{2n}*), but the seed of* $(\cap^2\underline{A}\cap\underline{B}\cap^4\underline{C})^\infty$ *has only one point in it (namely '*A*'). Thus apply* $(f^4_{\underline{C}} \circ f_{\underline{B}} \circ f^2_{\underline{C}}$ *to the seed until the first* 2 *points are defined, as follows.*

$$
\begin{aligned}
f_{\underline{C}}^4 \circ f_{\underline{B}} \circ f_{\underline{C}}^2(A * * * * \ldots) &= f_C^4 \circ f_{\underline{B}}(A * A * * A * * A \ldots) \\
&= f_{\underline{C}}^4(AB * BAB * B * \ldots) \\
&= AB * BCAB * B \ldots
\end{aligned}
$$

Thus the seed of our substitution is AB.

Chapter 8

The $\cap$ operation in two dimensions, and its applications.

We have described the $\cap$ operation in one dimension. We will now extend the $\cap$ operation to higher dimensions. We will describe how to do this, and illustrate how this can be used to recreate tilings from the Robinson tiling space.

Recall the definition of an $\mathbb{Z}^d$-array from definitions 42 and 43 in the last chapter. We now introduce the notion of an $\mathbb{Z}^d$ *Toeplitz array.* This concept was originally introduced in [10] expanding on work done by Downarowicz in [13]. We have changed the basic notation in this thesis, to fit with the particular notation for Toeplitz sequences that we are using.

Definition 61 ($\mathbb{Z}^d$ Toeplitz array)**.** Let Σ be a finite set of at least two elements. Let $Z \subseteq \mathbb{Z}^d$ be a subgroup isomorphic to $\mathbb{Z}^d$. For a given array $X : \mathbb{Z}^d \mapsto \Sigma$ with underlying lattice $\mathbb{Z}^d$ define;

$Per(x, Z, \sigma) = \{w \in \mathbb{Z}^d : x_{w+z} = \sigma \text{ for all } z \in Z\}, \sigma \in \Sigma$

$Per(x, Z) = \bigcup_{\sigma \in \Sigma} Per(x, Z, \sigma)$

(When $Per(x, Z) \neq 0$ we say Z is a 'group of periods' of x.)
We say that x is a $\mathbb{Z}^d$-Toeplitz array (or simply a Toeplitz array) is it satisfies two conditions. The first condition is that there must not be a translation vector t such that $x_{v+t} = x_v$. Secondly, if for all $v \in \mathbb{Z}^d$ there exists a subgroup $Z \subseteq \mathbb{Z}^d$ isomorphic to $\mathbb{Z}^d$ such that $v \in Per(x, Z)$.

This is a generalization of the Toeplitz sequence concept. A Toeplitz array can be converted to a tiling of $\mathbb{R}^d$ (by unit hypercubes) by the same method a generic array can be converted to a tiling.

As the reader may recall (ie figure 7.1), in the 1 dimensional case we used overlaying infinite strips as a physical motivation behind the $\cap$ operation. Any generalization to two dimensions should be roughly equivalent to taking two pictures of tilings on overhead transparencies, overlaying them, and reading off a new tiling from the result.

Consider the following definition.

Definition 62 ($A \cap B$ for $\mathbb{Z}^2$-arrays)**.** Let $A : \mathbb{Z}^d \mapsto \Sigma$ and $B : \mathbb{Z}^d \mapsto \Sigma$ be $\mathbb{Z}^2$ arrays with the standard lattice with points at integer positions.

$A \cap B$ is an array with alphabet Σ. The underlying lattice of the array differs from the standard lattice, and not in an intuitive way.

Let $f_1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $f_2 = (\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

Note that $|f_1| = |f_2| = 1$. Let $Latt_{A \cap B} = \{a_1 f_1 + a_2 f_2 | a_1, a_2 \in \mathbb{Z}^d\}$ be the underlying lattice for $A \cap B$.

Then define;

$$(A \cap B)(m, n) = \begin{cases} A((\frac{m-n}{2}, \frac{m+n}{2})) & \text{if } m+n \text{ is even,} \\ B((\frac{m-n-1}{2}, \frac{m+n-1}{2})) & \text{if } m+n \text{ is odd.} \end{cases}$$

Definition 63. Let X and A be $\mathbb{Z}^2$ arrays. Then define $g_A(X)$ to be the function sending X to $X \cap A$.

These definitions are of the same format as the definition of $A \cap B$ in the one dimensional case. It can be rephrased as follows

A point $xe_1 + ye_2$ in A is sent to the point $xf_1 - yf_2$ in $A \cap B$. A point $xe_1 + ye_2$ in B is sent to the point $(x+1)f_1 + yf_2$ in $A \cap B$. This covers the whole of the array $A \cap B$.

Definition 64. Denote an array with underlying lattice generated by the vectors e_1, e_2 as a *(unit) square array*. Denote an array with underlying lattice generated by the vectors f_1, f_2 as a *diamond array*.

For an intuitive construction of the $(A \cap B)$ operation, see the following remark;

Remark. Take two unit square tilings, A and B such that the origin of $\mathbb{R}^2$ is the centre of a tile in both A and B. Switch to the lattice view of tilings (ie, consider the array with underlying lattice formed by the centres of the unit square tiles). Shift B by the vector $v = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$. Overlap the two lattices. The vector v has been chosen to maximize the distance between the set of tile centres of A, and the set of tile centres from B, i.e. maximizing $d(A, B)$. From the tile viewpoint, this corresponds to choosing the vector v to place the centre of tiles from B at the vertices of tiles from A.

Scale up so that $d(B, A) = 1$. The resultant array is no longer generated by the standard basis of $\mathbb{R}^2$. We will thus change the generating vectors, to a minimum vector between points in A and B, and a vector orthogonal to that minimum vector.

This is reflected in the non-intuitive function '$\frac{n-m-1}{2}$', and the basis vectors e_1 and e_2 changing to f_1 and f_2. See figure 8.1 for a pictorial representation of A, B and $A \cap B$.

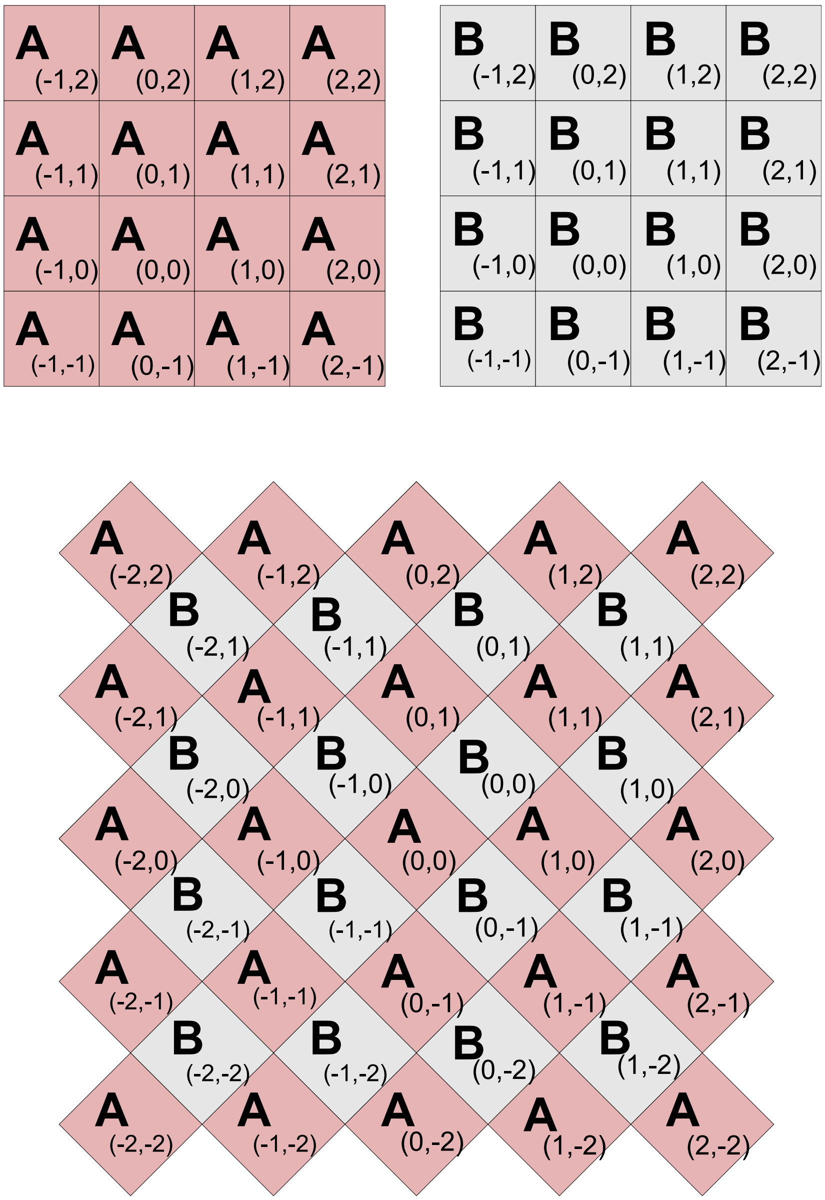


Figure 8.1: Patches of the tilings A, B and $A \cap B$

As the reader will be aware, $A \cap B$ has a different underlying lattice to A or B. The corresponding tiling is not a tiling by unit squares, but by unit diamonds. (This feature has been met before in this thesis, in figure 4.2 in chapter 4.)

If we wish to consider multiple $\cap$ operations, it is much more natural to consider sequential *pairs* of $\cap$ operations.

8.1 $\cap A \cap B$

We already have a function g_A (from definition 63) which applies an unit square array A to some unit square array X to produce a diamond array $X \cap A$.

In order to consider the limit of this g_A function (and hence the $\cap$ function), we must find out how to apply it to a diamond tiling. Let A and B be unit square arrays. Let X be some unit square array.

Remark. Like in the 1D case, we will use a null array to help clarify how the $(\cap A \cap B)$ operation builds up the periodic parts of $(\cap A \cap B)^{\infty}$ through each application of $(\cap A \cap B)$, and to show how much of the array is uniquely defined by the repeated application of $(\cap A \cap B)$.

In the previous section, we showed that $X \cap A$ is a diamond array. Thus $X \cap A$ has different basis vectors to a unit square array. Let us see what effect naively applying the $\cap$ operation to a diamond array will have.

From the previous chapter, the g_B operation applied to any array X, will double the distance of every point from the origin, (and then add points from B).

Our points in $(X \cap A)$ are of the form $\frac{1}{\sqrt{2}}(a-b, a+b)$ for $a, b \in \mathbb{Z}$.

Thus $g_B(X \cap A) = 2(a-b, a+b)$ for $a, b \in \mathbb{Z}$, which, when represented as a tiling, is the subset shown in figure 8.2 (points from X and A are labelled as such).

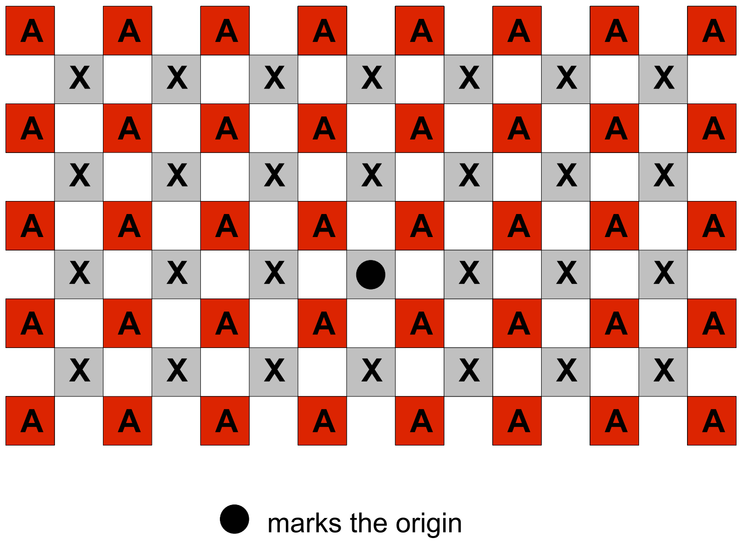


Figure 8.2: The image of $X \cap A$ under the $\cap$ operation

This is precisely half of an unit square array. Thus we need to ensure that the points from array B fill the gaps. If B was a diamond array, this would simply be a case of scaling B by $\sqrt{2}$, and shifting it by a small vector. Of course, B is not a diamond array (though you could easily define a related $\cap$ operation where B is a diamond array.

Instead we will alter how the $\cap$ operation will effect points from the B array, as follows;

Definition 65 ($A \cap B$, if A is a diamond array)**.** Let A be a diamond array, with basis vectors f_1, f_2. Let B be a unit square array, with standard euclidean basis vectors e_1, e_2.

Then

$$A \cap B(m,n) = \begin{cases} A((\frac{m+n}{2}, \frac{-m+n}{2})) & \text{if } m+n \text{ is even,} \\ B((\frac{m+n-1}{2}, \frac{-m+n+1}{2})) & \text{if } m+n \text{ is odd.} \end{cases}$$

The underlying lattice for $A \cap B$ will be $\{n_1e_1 + n_2e_2 | n_1, n_2 \in \mathbb{Z}\}$.

This definition can be rephrased as follows.

A point $xf_1 + yf_2$ in A is sent to the point $(x-y)e_1 + (x+y)e_2$ in $A \cap B$. A point $xe_1 + ye_2$ in B is sent to the point $(x + -y + 1)e_1 + (x+y)e_2$ in $A \cap B$. This covers the whole of the array $A \cap B$.

Effectively, this function takes the underlying lattice of the square array B, rotates it anti-clockwise by $\frac{\pi}{4}$, then moves the point of the lattice which was over the origin, to the point $(0,1) \in \mathbb{Z}^2$. The underlying lattice of A is then used to fill in the gaps in $\mathbb{Z}^2$, as in figure 8.3.

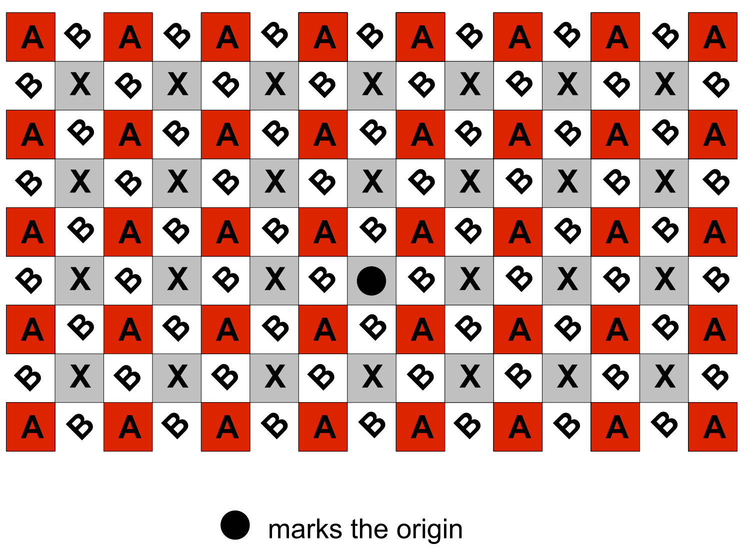


Figure 8.3: $(X \cap A \cap B)$ (schematic)

Obviously the rotation effect is not ideal, but its effect can be ignored by choosing a B tiling which has only one prototile (as we will do later).

8.2 Motivation

As motivation for the $\cap$ operation applied to diamond tilings, let us extrapolate from our physical motivation with square tilings. With square tilings, we aimed to 'overlay' the tilings, placing the second square tiling B such that every centre of a tile in B was as far away from the centre of tiles in A as possible. We would then scale up this new tiling to ensure that the minimum distance between centres of tiles was 1.

The problem we will get by applying this motivation to a diamond tiling $(X \cap A)$ and a square tiling B is that we have to rotate B to ensure the points of B are as far away from points of A as possible (ie, maximising $d(B, A)$).

We will have several choices of how much we rotate B by, namely $(\frac{\pi}{4} + k\frac{\pi}{2}$, $k \in \{0, 3\})$, just as we have several choices of how far we translate B by (any (a, b) such that $a + b$ is odd). We will choose $\frac{-\pi}{4}$ and $(0, 1)$ in this thesis.

We have defined $X \cap A$, when X is a diamond array and A a square array. We will now define $\cap A \cap B$.

Definition 66. Let X be a unit square tiling. Let A and B be unit square tilings. Then $X \cap A \cap B$ is evaluated from left to right, as in the 1 dimensional case. Similarly longer strings of $\cap$ operations are evaluated from left to right, for example; $X(\cap A \cap B)^n = (X(\cap A \cap B)^{n-1}) \cap A) \cap B$

Also, if the limit of $X(\cap A \cap B)^n$ as $n \to \infty$ exists, denote it as $X(\cap A \cap B)^\infty$.

We can also consider the g_B operation as a function from the domain of unit square tiling into the domain of diamond square tilings. Similar functions can be defined from diamond square tilings into unit square tilings (distinguishable by the domain specified).

Definition 67. Define the function $g_B \circ g_A(X)$ as $f_B(X \cap A)$.

Remark. Note that $\cap A \cap B$ doubles the distance of any point $(x, y) \in X$ from the origin. This is because $\cap A$ sends the (x, y) to $\sqrt{2}(x, y)$, then $\cap B$ sends $\sqrt{2}(x, y)$ to $2(x, y)$. Thus in the limit of $(\cap A \cap B)^n$. there will be only one tile from X remaining, at the origin. Thus in the current form, $X \cap (A \cap B)^\infty$ is not quite Toeplitz, since the origin is not defined solely by $\cap A \cap B$. In later examples, we will set $X = B$ because of this problem, to sidestep it.

Theorem 21. If A and B are unit square arrays, $(\cap A \cap B)^\infty$ is well defined.

Proof. Firstly, note that $X(\cap A \cap B)^n$ is a unit square tiling, for all n. Thus all we need to show is that each tile t has an unique label, for all $X(\cap A \cap B)^n$ where $n > N_t$, where $N_t \in \mathbb{N}$.

Divide $X \cap A \cap B$ into three sets; the origin, points (x, y) where $x + y$ is even, and tiles $(x + y)$ where $x + y$ is odd. The origin of $X \cap A \cap B$ is unchanged under the operation $\cap A \cap B$ (being the origin of X).

The odd tiles of $X \cap A \cap B$ only depend on the tile types of B. Since any tiling $X(\cap A \cap B)^n$ can be rewritten as $(X \cap A \cap B \ldots \cap B) \cap A \cap B$, these tiles are fixed for all $n \geq 1$.

Let us consider the tiles where $x + y$ is even. The tiles in $X \cap A \cap B$ where x *and* y are odd are uniquely defined by A. Thus, since $X(\cap A \cap B)^n$ can be rewritten as $(X \cap A \cap B \ldots \cap B) \cap A \cap B$, this holds true for $X(\cap A \cap B)^n$ as well. For the tiles where x and y are *both even*, recall that the $g_B \circ g_A$ function doubles the distance of every point from the origin. We know any point (x, y) where $x + y$ is odd, is fixed for $n \geq 1$. Thus any point $(a, b) = 2^k(x, y)$ where $x + y$ is odd is fixed for $n \geq 1 + k$. Call this the doubling property.

Take a point (x, y), where x and y are even. By the prime factorization theorem, x and y can be decomposed uniquely into factors. Thus (x, y) can be rewritten as $(2^q s, 2^r t)$, where s and t are odd numbers (possibly 1), and q and r are integers. WLOG assume $q > r$. Via the doubling property, we know that after r iterations of $g_B \circ g_A$, $(2^q s, 2^r t)$ will have the same label as $(2^{q-r}s, t)$, which is in the fixed set dependent on B (since $2^{q-r}s$ is even, t is odd, thus $2^{q-r}s + t$ is odd). If $q = r$, then we have $(2^q s, 2^r t)$ having the same label as (s, t), where s and t are both odd numbers and thus fixed by A.

Thus every point in $X \cap A \cap B$ is fixed in the limit. $\square$

For clarification, figure 8.4 illustrates a patch of $X(\cap A \cap B)^\infty$, with colours added to indicate which tiles are defined by which iteration of g_A and g_B.

8.3 Illustrative example: Robinson tilings

Let B be a tiling by blank unit tiles. Let A be the periodic tiling depicted in figure 8.5, with 4 prototiles with corner decorations. Note that the tile over the origin has a marking with a corner in the south west of the tile.

Use B as our seed tiling. If we construct $B \cap A \cap B$, we produce the tiling depicted in figure 8.6.

The reader may notice that this pattern bears a certain similarity to the Robinson tiling from earlier chapters. More precisely, the tiles from A with corner decorations can be identified with cross tiles of the Robinson tiling. However only cross tiles which are corners of 3×3 tiles have a counterpart in $B \cap A \cap B$.

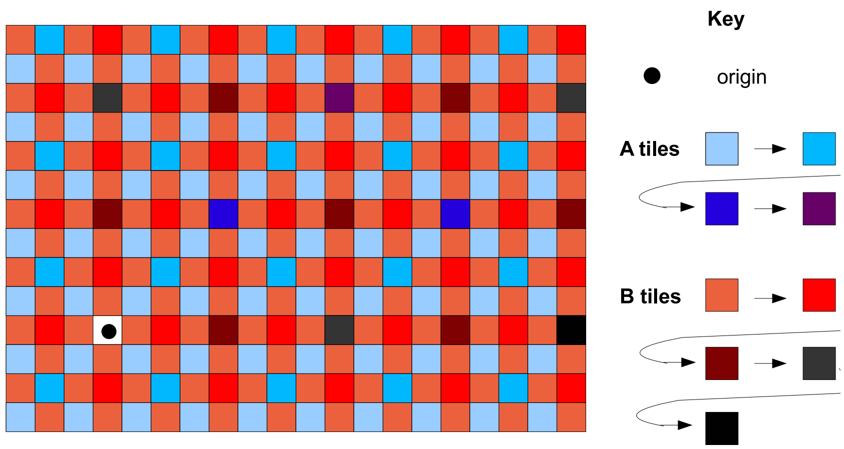


Figure 8.4: Patch of $X(\cap A \cap B)^{\infty}$

Let us consider $B(\cap A \cap B)^2$

As the reader can see, applying $g_B \circ g_A$ again to $B \cap A \cap B$ adds another layer of corner tiles, corresponding to the cross tiles of 7×7 tiles. This is due to the doubling property of the $g_B \circ g_A$ operation sending a 3×3 square onto a 7×7 square. Similarly, $B(\cap A \cap B)^3$ will provide corner tiles corresponding to the cross tiles for the 15×15 squares, and in general $B(\cap A \cap B)^n$ will provide the cross tiles for all $2^n \times 2^n$ squares (as well as cross tiles from smaller squares).

Since $B(\cap A \cap B)^{\infty}$ is well-defined, in the limit you get a tiling with corner tiles corresponding to every cross tile in a Robinson tiling. Since the cross tiles encode the positions of all $2^m \times 2^m$ squares (for $m \in \mathbb{Z}$) in the Robinson tiling, this is enough to encode the specific Robinson tiling. Thus $B(\cap A \cap B)^{\infty}$ is MLD to a Robinson tiling.

To be precise, it is MLD to a Robinson tiling with four faultlines meeting at the

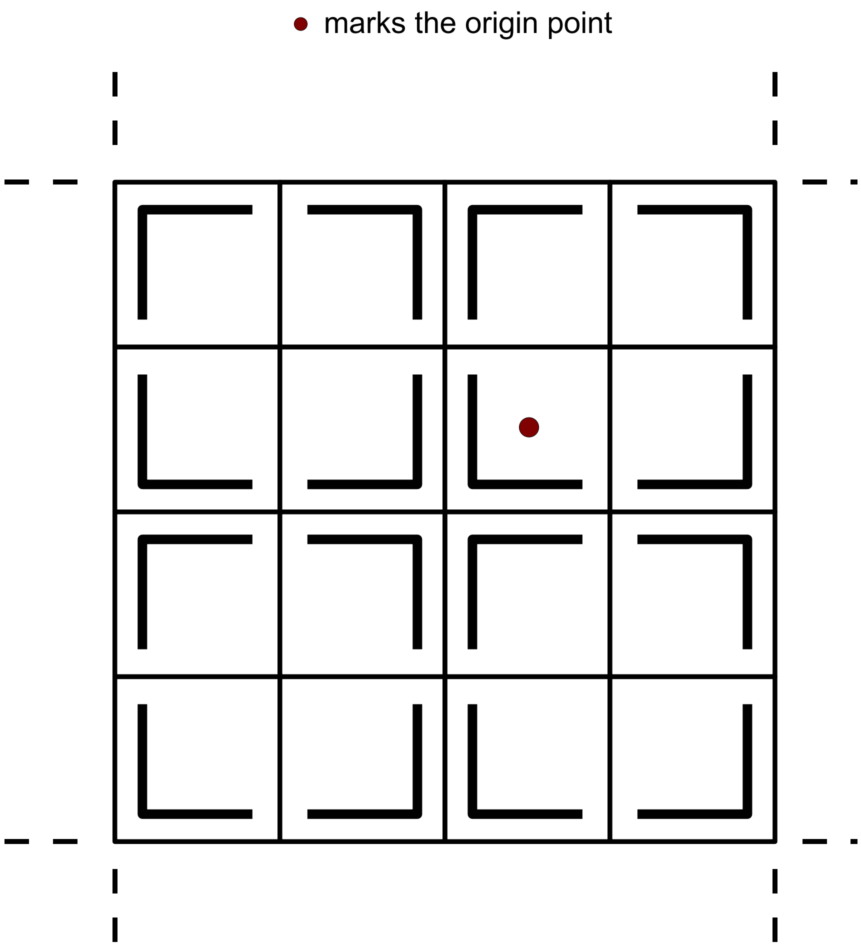


Figure 8.5: A

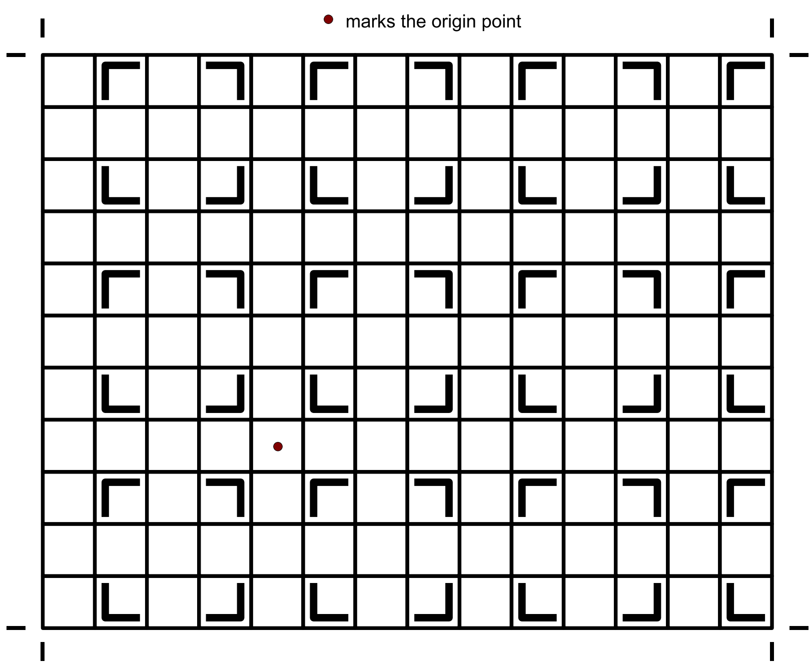


Figure 8.6: $B(\cap A \cap B)$

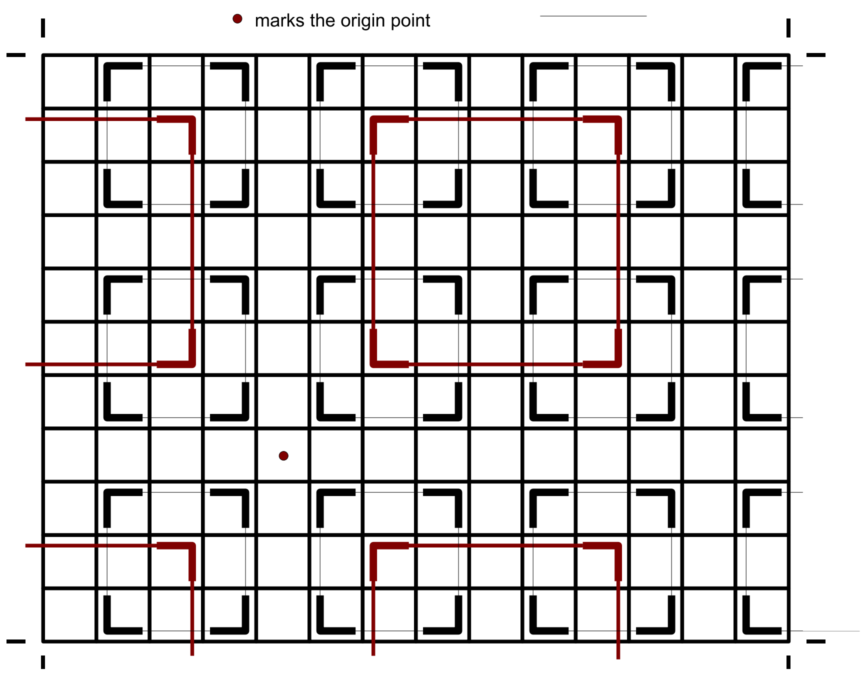


Figure 8.7: $B(\cap A \cap B)^2$

origin.

8.4 Extension to other Robinson tilings

By extending our definition of $g_B \circ g_A$, we can produce alternative Robinson tilings when we move to the limit. Note in our previous example, the second application of $g_B \circ g_A$ to B creates the second (red) layer of squares via the doubling property. But what would happen if the second application of $g_B \circ g_A$ to B used a different point as its origin? If the second application of $g_B \circ g_A$ used the blue point in figure 8.8 as its origin, then we could still have a tiling consistent with a Robinson tiling, but the position of the 7×7 squares will have changed. This is the key concept we will use to cover more Robinson tilings.

First we need to define what we mean by using a different point as an origin.

Definition 68. Define $(g_B \circ g_A)_{(x,y)} = t(x,y) \circ (g_B) \circ (g_A) \circ t(-x,-y)$, where $t(x,y)$ is the translation of the plane by the vector (x,y).

$(g_B \circ g_A)_{(x,y)}$ has a doubling property like $g_B \circ g_A$ does, but it is centred around the point (x,y).

Let us use this new function to construct more general Robinson tilings. For a generic Robinson tiling, consider the 3×3 square closest to the origin. This square will form a corner of a 7×7 square, which will itself form a corner of a 15×15 square, and so on. By choosing a vector $\underline{x} = (x,y)$ such that it is at a particular corner of the 3×3 square, we can choose what 7×7 square our 3×3 square we will mapped to under $(g_B \circ g_A)_{\underline{x}}$. Thus by choosing a series of $\underline{x}_i$ vectors, we can get any infinite sequence of expanding $2^n - 1 \times 2^n - 1$ squares. See diagram 8.9.

Explicitly, let us consider the following definition of a tiling, dependent on a sequence of vectors $\underline{x}_i$.

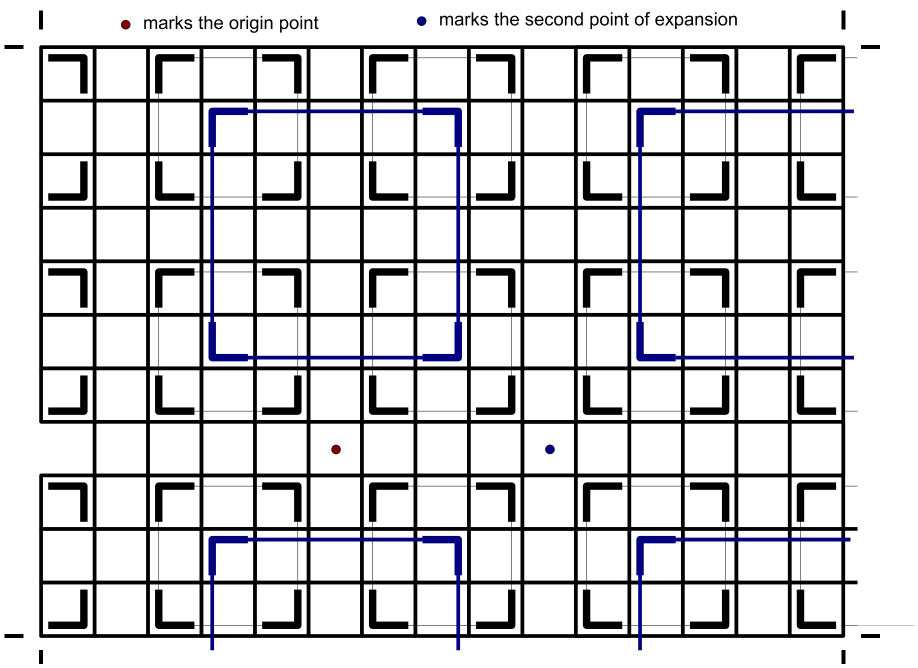


Figure 8.8: $B(\cap A \cap B)^2$ with shifted point of expansion.

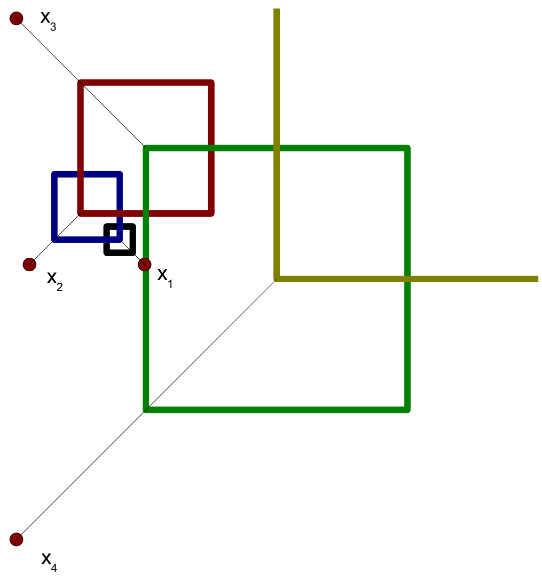


Figure 8.9: A series of points $\underline{x}_i$, creating new layers of a Robinson tiling.

Definition 69. $Rob_{\mathbb{X}=\{\underline{x}_i\}_{i=1}^{\infty}} = (g_B \circ g_A)_{\underline{x}_n} \circ (g_B \circ g_A)_{\underline{x}_{n-1}} \circ \ldots \circ (g_B \circ g_A)_{\underline{x}_2} \circ (g_B \circ g_A)_{\underline{x}_1}(B)$ where $\underline{x}_i = (0 + 4m_i, 0 + 4n_i)$ for $m_i, n_i \in \mathbb{Z}$

We have chosen the above condition on the values of x_i to force the x_i to be at a corner of a 3×3 square. By the doubling property, this ensures that the defining structure of a Robinson tiling (interlocking squares) is preserved, since $\cap A \cap B_{\underline{x}_i}$ will map every $2^k \times 2^k$ square onto a $2^{k+1} \times 2^{k+1}$ square (for $k \leq i$), and create a new layer of 3×3 squares in the correct position. Furthermore, since each patch of radius 2^k about the origin is fixed after k steps, the limit exists (and is MLD to a Robinson tiling).

We know that every tiling $Rob_{\mathbb{X}}$ is MLD to some Robinson tiling, but we have not proven that every Robinson tiling is MLD to a tiling $Rob_{\mathbb{X}}$, for some $\mathbb{X}$. To investigate this further, let us consider hierarchies of squares in the tiling.

As suggested in [26], choosing a 3×3 square closest to the origin, and considering the hierarchical sequence of squares containing it will define an infinite subset of the plane, since the size of the $2^n - 1 \times 2^n - 1$ squares increases without bound. Of course an infinite subset of the plane is different from the plane. We have three possibilities, namely quadrant, half plane, or full plane.

For a choice of vectors $\mathbb{X} = \{\underline{x}_j\}_{j\in\mathbb{N}}$, let us define four subsequences (possibly of finite length); $\mathbb{X}_{NW}$, $\mathbb{X}_{NE}$, $\mathbb{X}_{SW}$ and $\mathbb{X}_{SE}$. $\mathbb{X}_{NW}$ consists of all points $\underline{x}_j$ to the northwest of the largest currently defined hierarchical square, $\mathbb{X}_{NE}$ consists of all points to the northeast, and so on. As an example, in figure 8.9, $\underline{x}_1$ would belong to $\mathbb{X}_{SE}$, $\underline{x}_2$ and $\underline{x}_4$ would belong to $\mathbb{X}_{SW}$, and $\underline{x}_3$ would belong to $\mathbb{X}_{NW}$.

The infinite area of the plane covered by the hierarchical sequence of squares is determined by the four subsequences. In particular, if only one of the subsequences is of infinite length, then the hierarchical sequence of squares will cover one quadrant. Two subsequences of infinite length next to each other (ie $\mathbb{X}_{SW}$ and $\mathbb{X}_{SE}$) will cause the hierarchical sequence of squares to cover a half-plane, and any other result will cause the hierarchical sequence of squares to cover the whole plane.

A hierarchical sequence of squares covering the whole plane corresponds to a Robinson tiling without any faultlines. Thus this allows us to make any Robinson tiling without faultlines (up to a small translation to account for when the origin is not at a corner of a 3×3 square) by choosing our series of $\underline{x}_i$ vectors appropriately.

Let us consider a case where we do not have the whole plane defined by the hierarchical sequence of squares. This will let us determine that subset of the plane (ie, a half plane), but we do not have free choice in determining the rest of the plane.

More precisely, this method will create Robinson tilings with single width faultlines along the x-axis and/or y-axis, where the tiling on opposite sides of the faultlines are reflections of each other, since they are both created by the same sequence of $\underline{x}_i$'s. (The obvious example is if all $\underline{x}_i = (0,0)$, in which case we have shown previously that a Robinson tiling with four faultlines meeting at the origin is created.)

However this construction cannot create tilings MLD to Robinson tilings with more general faultlines, such as figure 3.3 or figure 3.4, since not all tilings with faultlines are reflections about the faultlines.

8.5 Possible applications and avenues of inquiry

There is a current area of active research in trying to build self-assembling tiling systems, which could model a computer or other piece of machinery. The majority of this work centres around the Winfree model [46], which starts with unit square tiles with decorated edges (each edge also having an integer assigned for its 'bond strength'), and a 'seed' of a finite number of connected tiles. Unlike a standard tiling, a tile can be attached to this seed even if some of its edges do not match. However it must have edges matching with a total bond strength equal to or above the 'temperature' being studied (usually 2).

This is meant to be a basic mathematical model of a solution containing molecular structures, bonding to each other via chemical bonds. We can use the concept of 3D Winfree tiles mentioned in [47] to produce a basic method of creating disconnected (possible infinite) seeds, along with a new way of mass producing Winfree structures, which may have advantages over the standard method.

Construction 3. Take a repetitive tiling by unit cubes of a unit thick slice of $\mathbb{R}^3$. For our example we will use a Robinson tiling $\times[0,1]$, because of its strong hierarchical structure. Call this tiling T. Introduce a set of $1\times 3\times 3$ tiles, denoted as W_T, where the bottom of the tiles has a strong bond strength to a 3×3 patch of the Robinson tiling, as shown in figure 8.10;

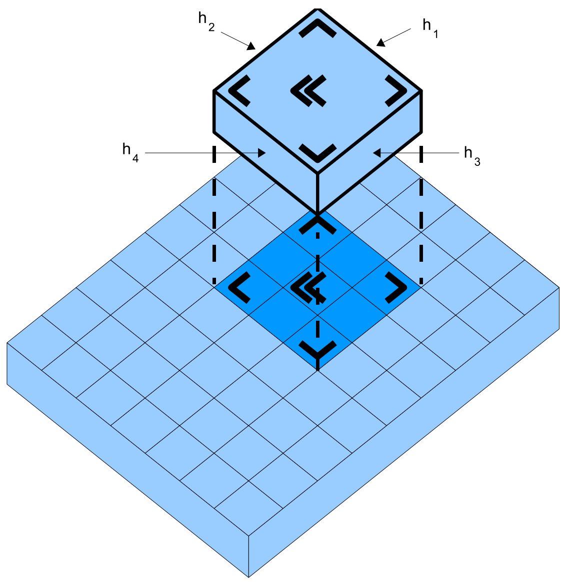


Figure 8.10: A tile from W_T attaching itself to T

Since the Robinson tiling is repetitive, these $3 \times 3 \times 1$ tiles can attach at many spatially distant locations. Thus by varying the decorations and bond strength of the $3 \times 3 \times 1$ tiles on their 1×3 faces (labelled as $h_1, \ldots h_4$ on the diagram), and adding new cubes which do not have a bond strength to the Robinson tiling, you can start the Winfree method with a disconnected infinite seed.

Furthermore, if the Winfree structure is stable at a higher temperature than the strength of the $W_T \to T$ bond, increasing the temperature to above the $W_T \to T$ bond strength will cause the Winfree structure to break off from the tiling T, to be used elsewhere. The tiling T could then be used to create another Winfree structure.

As a separate point, consider the Winfree model as a basic model of a solution of molecular structures, bonding to each other via chemical bonds. Then from my non-specialist viewpoint, the $\cap$ function bears certain similarities to biological DNA manipulation, in particular insertion of genes into a DNA strand via RNA or viruses. Specialists in this field may thus be able to make use of the $\cap$ operation. As motivation, we have shown that the $\cap$ function can produce a Robinson tiling, which could be used via the above method to mass produce Winfree structures.

Another possible use for the $\cap$ operation is in the field of computer graphics. Methods to create computer graphics from Wang tilings are well studied and form one of the most common methods of texture generation (for example [38]). Briefly, a simple texture is placed onto each Wang prototile, and a tiling is formed from the Wang prototiles. The eye cannot see any periodicity in the tiling, thus a convincing graphic can be produced from simple textures. This method breaks down when the graphic is approached, since the original simple textures can be seen more clearly, and pixelation may occur. Methods to stop this problem are currently being investigated (for example [37]).

The two dimensional $\cap$ function lends itself to this problem, since if you remove the rescaling factor of the $\cap$ function, the size of the tiles shrinks with every iteration. Thus by applying the $\cap$ function multiple times as the graphic is approached, new tiles can be introduced, and thus pixelation should be avoided.

On a more mathematical note, the last two chapters have described operations from limits of $\cap$ operations to Toeplitz sequences. Investigating whether a reverse operation can be constructed from a subset of Toeplitz sequences to compositions of $\cap$ operations would be an interesting avenue of inquiry.

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