The cohomology of λ-rings and Ψ-rings

Thesis submitted for the degree of
Doctor of Philosophy
at the University of Leicester

by

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March 2010

Abstract

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In this thesis we develop the cohomology of diagrams of algebras and then apply this to the cases of the λ-rings and the Ψ-rings. A diagram of algebras is a functor from a small category to some category of algebras. For an appropriate category of algebras we get a diagram of groups, a diagram of Lie algebras, a diagram of commutative rings, etc.

We define the cohomology of diagrams of algebras using comonads. The cohomology of diagrams of algebras classifies extensions in the category of functors. Our main result is that there is a spectral sequence connecting the cohomology of the diagram of algebras to the cohomology of the members of the diagram.

Ψ-rings can be thought of as functors from the category with one object associated to the multiplicative monoid of the natural numbers to the category of commutative rings. So we can apply the theory we developed for the diagrams of algebras to the case of Ψ-rings. Our main result tells us that there is a spectral sequence connecting the cohomology of the Ψ-ring to the André-Quillen cohomology of the underlying commutative ring.

The main example of a λ-ring or a Ψ-ring is the K-theory of a topological space. We look at the example of the K-theory of spheres and use its cohomology to give a proof of the classical result of Adams. We show that there are natural transformations connecting the cohomology of the K-theory of spheres to the homotopy groups of spheres. There is a very close connection between the cohomology of the K-theory of the $4n$-dimensional spheres and the homotopy groups of the $(4n-1)$-dimensional spheres.

Acknowledgements

Firstly, I would like to thank my supervisor, Dr Teimuraz Pirashvili, for all of his useful advice and support throughout this project and for introducing me to the subject of λ-rings. His passion for maths has been truly inspirational and encouraging.

I would also like to thank my first supervisor, Prof. Dietrich Notbohm, who is now in Amsterdam. I appreciate the opportunity he gave me to carry out research, for introducing me to the subject of homological algebra, and also for introducing me to Teimuraz.

I would like to thank all of the staff in the Department of Mathematics at the University of Leicester. I would also like to thank the other research students that I met. I am especially grateful to Dr. Mark Parsons for his beneficial advice during the first year.

I am grateful for all the support received from my family; in particular my Mum, Dad, and Brothers. I would like to say a big thank you to Frieda Mann and Dorothy Kinnear for being like Grandmothers to me. A special thanks to John Fergus Burns and John Burn for their invaluable support, and for allowing me to escape Leicester from time to time.

I would like to acknowledge the EPSRC and the University of Leicester for all of the financial support they have provided me.

Contents

Chapter 1

Introduction

λ-rings were first introduced in an algebraic-geometry setting by Grothendieck in 1958, then later used in group theory by Atiyah and Tall. A λ-ring R is a commutative ring with identity, together with operations $\lambda^i : R \to R$, for $i \geq 0$. We require that $\lambda^0(r) = 1$ and $\lambda^1(r) = r$ for all $r \in R$. There are more complicated axioms describing $\lambda^i(r_1+r_2)$, $\lambda^i(r_1 r_2)$ and $\lambda^i(\lambda^j(r))$. The λ-operations behave like exterior powers. The more complicated axioms are difficult to work with, and given a λ-ring R, it is difficult to prove that it is actually a λ-ring.

In 1962 Adams introduced the operations Ψ^i to study vector fields of spheres. These operations give us another type of ring, the Ψ-rings, which are related to the λ-rings by the following formula.

$$\Psi^i(r) - \lambda^1(r)\Psi^{i-1}(r) + ... + (-1)^{i-1}\lambda^{i-1}(r)\Psi^1(r) + (-1)^i i\lambda^i(r) = 0.$$

A Ψ-ring is a commutative ring R, together with ring homomorphisms $\Psi^i : R \to R$, for $i \geq 1$. We only require that $\Psi^1(r) = r$ and $\Psi^i(\Psi^j(r)) = \Psi^{ij}(r)$ for all $r \in R$. The Ψ-rings are much easier to work with, and in several places we will need to pass to Ψ-rings to be able to carry out computations for λ-rings.

Homological algebra is a relatively young discipline, which arose from algebraic topology in the early 20^{th} century. In 1956, Cartan and Eilenberg published their book entitled "Homological Algebra" [5], which was the first book on homological algebra and still remains a standard book of reference today. They found that the cohomology theories for groups, associative algebras and Lie algebras could all be described by derived functors, defined by means of projective and injective resolutions of modules. However the method they used was not enough to define the cohomology of commutative algebras. To overcome this problem, simplicial techniques were developed in homological algebra.

In the 1950's Moore showed that every simplicial group K is a Kan complex whose homotopy groups are the homology of a chain complex called the Moore complex of K. Dold and Kan independently found that there is an equivalence between the category of simplicial abelian groups and the category of non-negative chain complexes of abelian groups given by the Moore complex. Using simplicial methods Dold and Puppe showed that one can define the derived functors of a non-additive functor, since simplicial homotopy doesn't involve addition.

The notion of a monad on a category traces back to R. Godement [9]. Around 1965, Barr and Beck used comonads to define a resolution as a way to compute nonabelian derived functors. In 1967, André and Quillen independently developed what we now call André-Quillen cohomology. The André-Quillen cohomology is defined in general for algebras, using comonads. The λ-rings and Ψ-rings are particular examples which are included in this scheme, so the André-Quillen cohomology is well defined for both λ-rings and Ψ-rings. The main difficulty is that the André-Quillen cohomology is complicated and difficult to compute. Harrison had described a cohomology for commutative algebras in 1962 using a subcomplex of the Hochschild complex. The Harrison cohomology coincides with the André-Quillen cohomology over a field of characteristic zero up to a dimension shift. Our aim is to develop tools which aid computation.

In 2004, Yau [20] defined a cohomology for λ-rings in order to study deformations of the associated Ψ-operations. However, Yau's cohomology for λ-rings is different from the André-Quillen cohomology.

1.1 Outline of the thesis

In Chapter 2, we give a short overview of some of the fundamental concepts of homological algebra. We can trace the roots of these concepts back to Cartan and Eilenberg in the 1950's. We provide the definitions of additive categories, abelian categories and short exact sequences in abelian categories. We outline the construction of the right derived functor Ext^i using projective and injective resolutions. The main references for this part of the chapter are [19] and [17]. We sketch the construction of the cohomology of algebras in general using comonads [19] and we give the example of the André-Quillen cohomology for commutative rings which are the right derived functors of the derivations functor [18]. We provide an overview of the Harrison cohomology of commutative algebras [10] and the Baues-Wirsching cohomology of a small category with coefficients in a natural system [4].

Chapter 2 only provides well known background material which will be required later. It does not contain any original work. The original research can be found in the remaining chapters of the thesis.

In Chapter 3, we turn our attention to Ψ-rings, which are related to λ-rings via the Adam's operations. The first section introduces the basic concept of a Ψ-ring which can be found in [14]. We then develop the Ψ-analogue of modules and the semidirect product. These are then used to develop the Ψ-analogue of derivations and extensions. The results from this chapter are needed in chapter 4 to prove similar results for λ-rings.

In 2005, Donald Yau published a paper entitled, "Cohomology of λ-rings" [20]. In the paper he develops a cohomology of λ-rings in order to study the deformations of the Ψ-ring structure. Yau's cohomology is different from the André-Quillen cohomology. In the last section of Chapter 3 I provide a definition of the deformation of a Ψ-ring which is different to Yau's definition. This alternative definition is related to the André-Quillen cohomology of Ψ-rings.

In Chapter 4, we introduce λ-rings. The first section introduces the basic notions of λ-rings which can be found in [14]. We then develop the λ-analogue of modules and the semidirect product. We then use these to develop the λ-analogue of derivations and extensions. The last section of Chapter 4 provides an overview of Yau's cohomology for λ-rings.

In Chapter 5, we extend the Harrison cochain complex of a commutative algebra to get a bicomplex whose cohomology we define to be the Harrison cohomology of a diagram of a commutative algebra. We then apply this theory to the case of Ψ-rings.

In Chapter 6, we develop a cohomology for diagrams of algebras in general, using comonads. First, we fix a small category I. A diagram of algebras is a functor $I \to \mathfrak{Alg}(T)$, where T is a monad on sets. For appropriate T, we get a diagram of groups, a diagram of Lie algebras, a diagram of commutative rings, etc. The adjoint pair $\mathfrak{Alg}(T) \rightleftarrows \mathfrak{Sets}$ yields a comonad which we denote by $\mathbb{G}$. We can also consider the category I_0, which has the same objects as I, but only the identity morphisms. The inclusion $I_0 \subset I$ yields the functor $\mathfrak{Sets}^I \to \mathfrak{Sets}^{I_0}$ which has a left adjoint given by the left Kan extension. We also have the pair of adjoint functors $\mathfrak{Alg}(T)^I \rightleftarrows \mathfrak{Sets}^I$ which comes from the adjoint pair $\mathfrak{Alg}(T) \rightleftarrows \mathfrak{Sets}$. By putting these pairs together, we get another adjoint pair

$$\mathfrak{Alg}(T)^I \rightleftarrows \mathfrak{Sets}^{I_0} .$$

This adjoint pair yields a comonad which we denote by $\mathbb{G}_I$. We can then take the cohomology associated to the comonad $\mathbb{G}_I$. Now we have both a global cohomology, $H^*_{\mathbb{G}_I}(A, M)$, and a local cohomology, $H^*_{\mathbb{G}}(A(i), M(i))$. Our main result is that there exists a local to global spectral sequence connecting the two:

$$E_2^{pq} = H^p_{BW}(I, \mathcal{H}^q(A, M)) \Rightarrow H^{p+q}_{\mathbb{G}_I}(A, M),$$

where $H^p_{BW}(I, \mathcal{H}^q(A, M))$ denotes the Baues-Wirsching cohomology of the small category I with coefficients in the natural system $\mathcal{H}^q(A, M)$ on I whose value on $(\alpha : i \to j)$ is given by $H^q_{\mathbb{G}}(A(i), \alpha^* M(j))$.

In Chapter 7, we apply our theory from Chapter 6 to the case of Ψ-rings. A Ψ-ring can be considered as a diagram of a commutative ring, so we can apply our results to get a cohomology for Ψ-rings. We also define the cohomology of λ-rings using comonads. We note that there are homomorphisms connecting the cohomology of λ-rings, the cohomology of the associated Ψ-rings and the André-Quillen cohomology of the underlying commutative rings.

The last Chapter looks at applications of the earlier developed theory. Our main application is in algebraic topology. For any topological space X such that $K^1(X) = 0$, there exists a homomorphism natural in X, $\tau : \pi_{2n-1}(X) \to Ext_\Psi(K(X), \widetilde{K}(S^{2n}))$. We show that the cohomology of λ-rings and Ψ-rings can be used to prove the classical result of Adams. We also show that the Ψ-ring cohomology of $K(S^{2n})$ is related to the stable homotopy groups of spheres via the natural transformation τ.

Chapter 2

Homological algebra

2.1 Category theory

2.1.1 Abelian categories

The material in this section can be found in many textbooks, including [16] and [19]. Before we introduce abelian categories, we start by defining the notion of an additive category.

An *additive category* $\mathfrak{A}$ is a category such that the following holds:

1. for every pair of objects X and Y in $\mathfrak{A}$, the hom-set $\mathrm{Hom}_{\mathfrak{A}}(X, Y)$ has the structure of an abelian group such that morphism composition distributes over addition.

2. $\mathfrak{A}$ has a zero object (an object which is both initial and terminal).

3. for every pair of objects X and Y in $\mathfrak{A}$, their product $X \times Y$ exists.

An abelian category is defined in terms of kernels and cokernels, so first we will recall some other basic definitions from category theory.

In a category $\mathfrak{C}$, a morphism $m : X \to Y$ is called a *monomorphism* if for all morphisms $f_1, f_2 : V \to X$ where $m \circ f_1 = m \circ f_2$ we have $f_1 = f_2$. A morphism

$e : Y \to X$ is called an *epimorphism* if for all morphisms $g_1, g_2 : X \to V$ where $g_1 \circ e = g_2 \circ e$ we have $g_1 = g_2$.

In an additive category $\mathfrak{A}$, a *kernel* of a morphism $f : X \to Y$ is defined to be a map $i : X' \to X$ such that $f \circ i = 0$ and for any morphism $g : Z \to X$ such that $f \circ g = 0$ there exists a unique morphism $g' : Z \to X'$ such that $i \circ g' = g$.

$$\begin{tikzcd} & Z \arrow[dl, dotted, "g'"'] \arrow[d, "g"] \arrow[dr, "0"] & \\ X' \arrow[r, "i"'] & X \arrow[r, "f"'] & Y \end{tikzcd}$$

Dually, in an additive category $\mathfrak{A}$, a *cokernel* of a morphism $f : X \to Y$ is defined to be a map $e : Y \to Y'$ such that $e \circ f = 0$ and for any morphism $g : Y \to Z$ such that $g \circ f = 0$ there exists a unique morphism $g' : Y' \to Z$ such that $g' \circ e = g$.

$$\begin{tikzcd} & Z & \\ X \arrow[ur, "0"] \arrow[r, "f"'] & Y \arrow[u, "g"] \arrow[r, "e"'] & Y' \arrow[ul, dotted, "g'"'] \end{tikzcd}$$

An *abelian category* $\mathfrak{A}$ is an additive category such that the following holds:

1. every morphism in $\mathfrak{A}$ has a kernel and cokernel.
2. every monomorphism in $\mathfrak{A}$ is the kernel of its cokernel.
3. every epimorphism in $\mathfrak{A}$ is the cokernel of its kernel.

The basic example of an abelian category is the category of abelian groups, denoted by $\mathfrak{Ab}$. In the category $\mathfrak{Ab}$, the objects are Abelian groups, and the morphisms are abelian group homomorphisms. In general, module categories which appear throughout algebra, are abelian categories.

If $\mathcal{I}$ is a small category and $\mathfrak{A}$ is an abelian category then the category of functors $\mathfrak{A}^{\mathcal{I}}$ as also an abelian category. The category of sets $\mathfrak{Sets}$ and the category of groups $\mathfrak{Grp}$ are not abelian categories. However, if G is a group then the category of left (or right) G-modules, denoted by $G - \mathfrak{mod}$, is an abelian category. If R is a ring then the category of left (or right) R-modules, denoted by $R - \mathfrak{mod}$, is an abelian category. If R is a Ψ-ring then the category of Ψ-modules over R, denoted by $R - \mathfrak{mod}_\Psi$, is an abelian category. If R is a λ-ring then the category of λ-modules over R, denoted by $R - \mathfrak{mod}_\lambda$, is an abelian category.

In an abelian category $\mathfrak{A}$, a *short exact sequence* is a sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

in which α is a monomorphism, β is an epimorphism and $\mathrm{Ker}(\beta) = \mathrm{Im}(\alpha)$.

In an abelian category $\mathfrak{A}$, a sequence

$$\ldots \longrightarrow X^{n-1} \xrightarrow{f^{n-1}} X^n \xrightarrow{f^n} X^{n+1} \longrightarrow \ldots$$

is said to be *exact at* X^n if $\mathrm{Ker}(f^n) = \mathrm{Im}(f^{n-1})$. The sequence is said to be *exact* if it is exact at X^n for all $n \in \mathbb{Z}$.

2.1.2 Modules

Let $\mathfrak{C}$ be a (not necessarily abelian) category with finite limits, and 1 denote a terminal object in $\mathfrak{C}$. An *abelian group object* of $\mathfrak{C}$ is an object A together with arrows $m : A \times A \to A$, $i : A \to A$ and $z : 1 \to A$ such that the following diagrams commute.
(associativity of multiplication)

$$\begin{tikzcd} A \times A \times A \arrow[r, "m \times id_A"] \arrow[d, "id_A \times m"'] & A \times A \arrow[d, "m"] \\ A \times A \arrow[r, "m"'] & A \end{tikzcd}$$

(left and right units)

$$\begin{tikzcd} A \times 1 \arrow[r, "id_A \times z"] \arrow[dr, "\cong"'] & A \times A \arrow[d, "m"] & 1 \times A \arrow[l, "z \times id_A"'] \arrow[dl, "\cong"] \\ & A & \end{tikzcd}$$

(left and right inverses)

$$\begin{tikzcd} A \arrow[r, "(i, id_A)"] \arrow[d] & A \times A \arrow[d, "m"] & A \arrow[l, "(id_A, i)"'] \arrow[d] \\ 1 \arrow[r, "z"'] & A & 1 \arrow[l, "z"] \end{tikzcd}$$

(commutativity)

$$\begin{tikzcd} A \times A \arrow[rr, "(p_2, p_1)"] \arrow[dr, "m"'] & & A \times A \arrow[dl, "m"] \\ & A & \end{tikzcd}$$

These diagrams say that the arrows satisfy the equations of an abelian group.

Let A, i, m, z and A', m', i', z' be abelian group objects of $\mathfrak{C}$, a *morphism of abelian group objects* is an arrow $f : A \to A'$ such that the following diagram commutes.

$$\begin{tikzcd} A \times A \arrow[r, "m"] \arrow[d, "f \times f"'] & A \arrow[d, "f"] \\ A' \times A' \arrow[r, "m'"'] & A' \end{tikzcd}$$

We denote the category of abelian group objects of $\mathfrak{C}$ by $Ab(\mathfrak{C})$.

Let A be any object of the category $\mathfrak{C}$. The *slice category* of objects of $\mathfrak{C}$ over A, denoted by $\mathfrak{C}/A$, has as objects the arrows of $\mathfrak{C}$ with target A. Given two objects $f : B \to A$ and $g : C \to A$ of $\mathfrak{C}/A$, an arrow of $\mathfrak{C}/A$ from f to g is an arrow $h : B \to C$ which makes the following diagram commute.

$$\begin{tikzcd} B \arrow[rr, "h"] \arrow[dr, "f"'] & & C \arrow[dl, "g"] \\ & A & \end{tikzcd}$$

Definition 2.1. Let A be an object in a category $\mathfrak{C}$. An *A-module* is defined to be an abelian group object in the category $\mathfrak{C}/A$,

$$A - \mathfrak{mod} := Ab(\mathfrak{C}/A).$$

The category $A - \mathfrak{mod}$ is usually an abelian category.

Definition 2.2. Let $p : Y \to A$ be an object and $q : Z \to A$ be an abelian group object of $\mathfrak{C}/A$, then we define the abelian group of *p-derivations*, denoted $\mathrm{Der}(Y, Z)$, to be

$$\mathrm{Der}(Y, Z) := \mathrm{Hom}_{\mathfrak{C}/A}(p, q).$$

2.2 Cohomology

The concepts of complexes and (co)homology began in algebraic topology with simplicial and singular (co)homology. The methods of algebraic topology have been applied extensively throughout pure algebra, and have initiated many developments. Complexes are the basic tools of homological algebra and provide us with a way of computing (co)homology. The following definitions can be found in [17] and [5].

A *cochain complex* (C, δ) of objects of an abelian category $\mathfrak{A}$ is a family $\{C^n, \delta^n\}_{n\in\mathbb{Z}}$ of objects $C^n \in obj(\mathfrak{A})$ and morphisms (called the *coboundary maps* or *differential maps*) $\delta^n : C^n \to C^{n+1}$ such that $\delta^{n+1} \circ \delta^n = 0$ for all $n \in \mathbb{Z}$.

$$\cdots \longrightarrow C^{n-2} \xrightarrow{\delta^{n-2}} C^{n-1} \xrightarrow{\delta^{n-1}} C^n \xrightarrow{\delta^n} C^{n+1} \xrightarrow{\delta^{n+1}} C^{n+2} \longrightarrow \cdots$$

The last condition is equivalent to saying that $\mathrm{Im}(\delta^n) \subseteq \mathrm{Ker}(\delta^{n+1})$ for all $n \in \mathbb{Z}$. Hence, one can define the *cohomology* of C denoted by $H^*(C)$,

$$H^*(C) = \{H^n(C)\}_{n\in\mathbb{Z}} \qquad \text{where } H^n(C) = \frac{\mathrm{Ker}(\delta^n)}{\mathrm{Im}(\delta^{n-1})}.$$

$H^n(C)$ is called the n^{th}-cohomology of C. An *n-coboundary* is an element of $\mathrm{Im}(\delta^{n-1})$. An *n-cocycle* is an element of $\mathrm{Ker}(\delta^n)$.

Let (C, δ) and $(C_\diamond, \delta_\diamond)$ be two cochain complexes of an abelian category $\mathfrak{A}$. A *cochain map* $f : (C, \delta) \to (C_\diamond, \delta_\diamond)$ is a family of morphisms $\{f^n : C^n \to C_\diamond^n\}_{n\in\mathbb{Z}}$ such that $\delta_\diamond^n \circ f^n = f^{n+1} \circ \delta^n$ for all $n \in \mathbb{Z}$. The last condition is equivalent to saying the following diagram commutes.

$$\begin{tikzcd}
\cdots \arrow[r] & C^{n-2} \arrow[r, "\delta^{n-2}"] \arrow[d, "f^{n-2}"] & C^{n-1} \arrow[r, "\delta^{n-1}"] \arrow[d, "f^{n-1}"] & C^n \arrow[r, "\delta^n"] \arrow[d, "f^n"] & C^{n+1} \arrow[r, "\delta^{n+1}"] \arrow[d, "f^{n+1}"] & C^{n+2} \arrow[r] \arrow[d, "f^{n+2}"] & \cdots \\
\cdots \arrow[r] & C_\diamond^{n-2} \arrow[r, "\delta_\diamond^{n-2}"'] & C_\diamond^{n-1} \arrow[r, "\delta_\diamond^{n-1}"'] & C_\diamond^n \arrow[r, "\delta_\diamond^n"'] & C_\diamond^{n+1} \arrow[r, "\delta_\diamond^{n+1}"'] & C_\diamond^{n+2} \arrow[r] & \cdots
\end{tikzcd}$$

A cochain map $f : (C, \delta) \to (C_\diamond, \delta_\diamond)$ induces homomorphisms $H^n(f) : H^n(C) \to H^n(C_\diamond)$. This makes each H^n into a functor.

A *cochain bicomplex* of objects of an abelian category $\mathfrak{A}$ is a family $\{C^{p,q}, \delta^{p,q}, \partial^{p,q}\}_{p,q\in\mathbb{Z}}$ of objects $C^{p,q} \in obj(\mathfrak{A})$ and morphisms $\delta^{p,q} : C^{p,q} \to C^{p+1,q}$ and $\partial^{p,q} : C^{p,q} \to C^{p,q+1}$ such that $\delta^{p+1,q} \circ \delta^{p,q} = 0$ and $\partial^{p,q+1} \circ \partial^{p,q} = 0$ and also $\partial^{p+1,q}\delta^{p,q} + \delta^{p,q+1}\partial^{p,q} = 0$ for all $p, q \in \mathbb{Z}$.

It is useful to visualise a cochain bicomplex as a lattice

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots \arrow[r] & C^{p-1,q+1} \arrow[r, "\delta^{p-1,q+1}"'] \arrow[u] & C^{p,q+1} \arrow[r, "\delta^{p,q+1}"'] \arrow[u] & C^{p+1,q+1} \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & C^{p-1,q} \arrow[r, "\delta^{p-1,q}"'] \arrow[u, "\partial^{p-1,q}"'] & C^{p,q} \arrow[r, "\delta^{p,q}"'] \arrow[u, "\partial^{p,q}"'] & C^{p+1,q} \arrow[r] \arrow[u, "\partial^{p+1,q}"'] & \cdots \\
\cdots \arrow[r] & C^{p-1,q-1} \arrow[r, "\delta^{p-1,q-1}"'] \arrow[u, "\partial^{p-1,q-1}"'] & C^{p,q-1} \arrow[r, "\delta^{p,q-1}"'] \arrow[u, "\partial^{p,q-1}"'] & C^{p+1,q-1} \arrow[r] \arrow[u, "\partial^{p+1,q-1}"'] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

where each row $(C^{*,q}, \delta^{*,q})$ and each column $(C^{p,*}, \partial^{p,*})$ is a cochain complex and each square anticommutes.

The *total complexes* $Tot(C) = Tot^{\prod}(C)$ and $Tot^{\bigoplus}(C)$ of a cochain bicomplex C are given by

$$Tot^{\prod}(C)^n = \prod_{p+q=n} C^{p,q} \qquad \text{and} \qquad Tot^{\bigoplus}(C)^n = \bigoplus_{p+q=n} C^{p,q}.$$

The coboundary maps are given by $d = \delta + \partial$. We note that $Tot^{\prod}(C) = Tot^{\bigoplus}(C)$ if C is bounded, especially if C is a first quadrant bicomplex.

Proposition 2.3. *If C is a first quadrant bicomplex then we have the following convergent spectral sequence*

$$E_2^{p,q} = H_h^p H_v^q(C) \Rightarrow H^{p+q}(Tot(C)),$$

where H_h^* denotes the horizontal cohomology, and H_v^* denotes the vertical cohomology.

2.3 Classical derived functors

A standard method of computing classical derived functors between abelian categories is to take a resolution, apply the functor, then take the (co)homology of the resulting complex. The following material can be found in [5], [19] and [17].

2.3.1 Projective and injective objects

An object P of an abelian category $\mathfrak{A}$ is *projective* if for any epimorphism $e : A \twoheadrightarrow B$ and any morphism $f : P \to B$ there exists a morphism $g : P \to A$ such that $f = e \circ g$, in other words, if the following diagram commutes.

$$\begin{tikzcd}
 & P \arrow[dl, dotted, "g"'] \arrow[d, "f"] & \\
A \arrow[r, two heads, "e"'] & B \arrow[r] & 0
\end{tikzcd}$$

An object Q of an abelian category $\mathfrak{A}$ is *injective* if for any monomorphism $m : A \hookrightarrow B$ and any morphism $f : A \to Q$ there exists a morphism $g : B \to Q$ such that $f = g \circ m$, in other words, if the following diagram commutes.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, hook, "m"] \arrow[d, "f"'] & B \arrow[dl, dotted, "g"] \\ & Q & \end{tikzcd}$$

An object P is projective if and only if $\mathrm{Hom}_{\mathfrak{A}}(P, -) : \mathfrak{A} \to \mathfrak{Ab}$ is an exact functor. In other words, if and only if for any exact sequence $0 \to A \to B \to C \to 0$ in $\mathfrak{A}$ it follows that the following sequence of groups

$$0 \longrightarrow \mathrm{Hom}_{\mathfrak{A}}(P, A) \longrightarrow \mathrm{Hom}_{\mathfrak{A}}(P, B) \longrightarrow \mathrm{Hom}_{\mathfrak{A}}(P, C) \longrightarrow 0$$

is also exact.

An object Q is injective if and only if $\mathrm{Hom}_{\mathfrak{A}}(-, Q) : \mathfrak{A} \to \mathfrak{Ab}$ is an exact functor. In other words, if and only if for any exact sequence $0 \to A \to B \to C \to 0$ in $\mathfrak{A}$ it follows that the following sequence of groups

$$0 \longrightarrow \mathrm{Hom}_{\mathfrak{A}}(C, Q) \longrightarrow \mathrm{Hom}_{\mathfrak{A}}(B, Q) \longrightarrow \mathrm{Hom}_{\mathfrak{A}}(A, Q) \longrightarrow 0$$

is also exact.

2.3.2 Projective and injective resolutions

Let A be an object of an abelian category $\mathfrak{A}$. A *projective resolution* of A is a complex P, where $P_i = 0$ for all $i < 0$ and P_j is projective for all $j \geq 0$, together with a morphism $\epsilon : P_0 \to A$ called the *augmentation* such that the augmented complex

$$\ldots \longrightarrow P_2 \xrightarrow{\partial} P_1 \xrightarrow{\partial} P_0 \xrightarrow{\epsilon} A \longrightarrow 0$$

is exact.

Let A be an object of an abelian category $\mathfrak{A}$. An *injective resolution* of A is a complex Q, where $Q_i = 0$ for all $i < 0$ and Q_j is injective for all $j \geq 0$, together with a morphism $\epsilon : A \to Q_0$ called the *augmentation* such that the augmented complex

$$0 \longrightarrow A \xrightarrow{\epsilon} Q_0 \xrightarrow{\delta} Q_1 \xrightarrow{\delta} Q_2 \longrightarrow \dots$$

is exact.

An abelian category $\mathfrak{A}$ is said to have *enough projectives* if for every object A of $\mathfrak{A}$, there exists a projective object P of $\mathfrak{A}$ and an epimorphism $e : P \to A$.

An abelian category $\mathfrak{A}$ is said to have *enough injectives* if for every object A of $\mathfrak{A}$, there exists an injective object Q of $\mathfrak{A}$ and a monomorphism $m : A \to Q$.

2.3.3 Right derived functors

Let $\mathfrak{A}, \mathfrak{B}$ be abelian categories, where $\mathfrak{A}$ has enough injectives. If $F : \mathfrak{A} \to \mathfrak{B}$ is a covariant left exact functor, then we can construct the *right derived functors* of F, denoted by $R^nF : \mathfrak{A} \to \mathfrak{B}$ for $n \geq 0$. If A is an object of $\mathfrak{A}$, and Q is an injective resolution of A, we define

$$R^nF(A) := H^n(F(Q)).$$

Let $\mathfrak{A}, \mathfrak{B}$ be abelian categories, where $\mathfrak{A}$ has enough projectives. If $G : \mathfrak{A} \to \mathfrak{B}$ is a contravariant left exact functor, then we can construct the *right derived functors* of G, denoted by $R^nG : \mathfrak{A} \to \mathfrak{B}$ for $n \geq 0$. If A is an object of $\mathfrak{A}$, and P is a projective resolution of A, we define

$$R^nG(A) := H^n(G(P)).$$

It is known that the functors $R^nF(A)$ and $R^nG(A)$ are independent of the choice of projective/injective resolution chosen, hence it only depends on A. We always get $R^0F(A) \cong F(A)$ and $R^0G(A) \cong G(A)$. Furthermore, if A is injective then $R^nF(A) = 0$ for $n > 0$, and if A is projective then $R^nG(A) = 0$ for $n > 0$.

Given a covariant left exact functor $F : \mathfrak{A} \to \mathfrak{B}$ between the abelian categories $\mathfrak{A}$ and $\mathfrak{B}$ and given a short exact sequence

$$0 \to A_1 \to A_2 \to A_3 \to 0$$

in $\mathfrak{A}$, then there exists the following long exact sequence.

$$0 \longrightarrow R^0F(A_1) \longrightarrow R^0F(A_2) \longrightarrow R^0F(A_3) \longrightarrow R^1F(A_1) \longrightarrow \dots$$

$$\dots \longrightarrow R^nF(A_1) \longrightarrow R^nF(A_2) \longrightarrow R^nF(A_3) \longrightarrow R^{n+1}F(A_1) \longrightarrow \dots$$

2.3.4 Ext

The main example of right derived functors are the functors Ext^n.

Let R be a ring, and let M, N be left R-modules. The functor $F(-) = \mathrm{Hom}_R(M, -) : R - \mathfrak{mod} \to \mathfrak{Ab}$ is a covariant additive left exact functor, so we can define its right derived functors

$$\mathrm{Ext}^n_R(M, -) = R^n \mathrm{Hom}_R(M, -) : R - \mathfrak{mod} \to \mathfrak{Ab}.$$

Given a left R-module M and a short exact sequence of left R-modules $0 \to N' \to N \to N'' \to 0$ we obtain the following long exact sequence.

$$0 \to \mathrm{Hom}_R(M, N') \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N'') \to \mathrm{Ext}^1_R(M, N') \to \dots$$

$$\dots \to \mathrm{Ext}^n_R(M, N') \to \mathrm{Ext}^n_R(M, N) \to \mathrm{Ext}^n_R(M, N'') \to \mathrm{Ext}^{n+1}_R(M, N') \to \dots$$

Similarly $\mathrm{Hom}_R(-, N) : R - \mathfrak{mod} \to \mathfrak{Ab}$ is also a contravariant additive left exact functor, so we can define its right derived functors $\mathrm{Ext}^n_R(-, N) = R^n \mathrm{Hom}_R(-, N) : R - \mathfrak{mod} \to \mathfrak{Ab}$.

Given a short exact sequence of left R-modules $0 \to M' \to M \to M'' \to 0$ and a left R-module N we obtain the following long exact sequence.

$$0 \to \mathrm{Hom}_R(M'', N) \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M', N) \to \mathrm{Ext}^1_R(M'', N) \to \dots$$

$$\dots \to \mathrm{Ext}^n_R(M'', N) \to \mathrm{Ext}^n_R(M, N) \to \mathrm{Ext}^n_R(M', N) \to \mathrm{Ext}^{n+1}_R(M'', N) \to \dots$$

2.4 Comonad cohomology

Cartan and Eilenberg unified the cohomology theories of groups, Lie algebras and associative algebras by describing them as Ext groups in the appropriate abelian categories. Unfortunately, this approach does not work in all categories, for example in the category of commutative algebras. The right approach is the comonad cohomology using simplicial methods. This material can be found in [3] and [19].

2.4.1 Monads and comonads

A *monad* $\mathbb{T} = (T, \eta, \mu)$ in any category $\mathfrak{C}$ consists of an endofunctor $T : \mathfrak{C} \to \mathfrak{C}$ together with two natural transformations: $\eta : Id_{\mathfrak{C}} \to T$, $\mu : T \circ T = T^2 \to T$ such that the following diagrams commute.

$$\begin{tikzcd}
T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"] & T
\end{tikzcd}$$

$$\begin{tikzcd}
Id_{\mathfrak{C}}T \arrow[r, "\eta T"] \arrow[dr, equal] & T^2 \arrow[d, "\mu"] & TId_{\mathfrak{C}} \arrow[l, "T\eta"'] \arrow[dl, equal] \\
& T &
\end{tikzcd}$$

The natural transformation η is called the unit, and the natural transformation μ is called the multiplication. The diagrams are called the associativity, left unit and right unit laws.

A *comonad* $\mathbb{G} = (G, \varepsilon, \delta)$ in any category $\mathfrak{C}$ consists of an endofunctor $G : \mathfrak{C} \to \mathfrak{C}$ together with two natural transformations: $\varepsilon : G \to Id_{\mathfrak{C}}$, $\delta : G \to G^2$ such that the following diagrams commute.

$$\begin{tikzcd}
G \arrow[r, "\delta"] \arrow[d, "\delta"'] & G^2 \arrow[d, "G\delta"] \\
G^2 \arrow[r, "\delta G"] & G^3
\end{tikzcd}$$

$$\begin{tikzcd}
& G \arrow[dl, equal] \arrow[d, "\delta"] \arrow[dr, equal] & \\
Id_{\mathfrak{C}}G & G^2 \arrow[l, "\varepsilon G"] \arrow[r, "G\varepsilon"'] & GId_{\mathfrak{C}}
\end{tikzcd}$$

A pair of functors $L : \mathfrak{C} \to \mathfrak{B}$ and $R : \mathfrak{B} \to \mathfrak{C}$ are *adjoint* if for all objects A in $\mathfrak{C}$ and B in $\mathfrak{B}$ there exists a natural bijection

$$\mathrm{Hom}_{\mathfrak{B}}(L(A), B) \cong \mathrm{Hom}_{\mathfrak{C}}(A, R(B)).$$

Natural means that for all $f : A \to A'$ in $\mathfrak{C}$ and $g : B \to B'$ in $\mathfrak{B}$ the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathfrak{B}}(L(A'), B) \arrow[r, "Lf^*"] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathfrak{B}}(L(A), B) \arrow[r, "g_*"] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathfrak{B}}(L(A), B') \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathfrak{C}}(A', R(B)) \arrow[r, "f^*"] & \mathrm{Hom}_{\mathfrak{C}}(A, R(B)) \arrow[r, "Rg_*"] & \mathrm{Hom}_{\mathfrak{C}}(A, R(B'))
\end{tikzcd}$$

We say that L is the *left adjoint* of R, and R is the *right adjoint* of L.

Let $\mathfrak{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathfrak{B}$ be an adjoint pair of functors with adjunction morphisms $\eta : Id \to RL$ and $\mu : LR \to Id$. Then $\mathbb{T} = (RL, \eta, R\mu L)$ is a monad on $\mathfrak{C}$ and $\mathbb{G} = (LR, \mu, L\eta R)$ is a comonad on $\mathfrak{B}$.

Example 2.4. *Let $U : \mathfrak{Grp} \to \mathfrak{Sets}$ take a group to the set of its elements forgetting the group structure, and take group morphisms to functions between sets. The left adjoint functor to U, is the functor $L : \mathfrak{Sets} \to \mathfrak{Grp}$ taking a set to the free group generated by the set. The functor $T = UL : \mathfrak{Sets} \to \mathfrak{Sets}$ gives rise to a monad and the functor $G = LU : \mathfrak{Grp} \to \mathfrak{Grp}$ gives rise to a comonad.*

Let $\mathbb{G}$ be a comonad on $\mathfrak{C}$. A morphism $f : X \to Y$ in $\mathfrak{C}$ is called a $\mathbb{G}$-*epimorphism* if the map $\mathrm{Hom}_{\mathfrak{C}}(G(Z), X) \to \mathrm{Hom}_{\mathfrak{C}}(G(Z), Y)$ is surjective for all Z. We require the following useful lemma.

Lemma 2.5. *For all objects X in $\mathfrak{C}$ the morphism $GX \xrightarrow{\varepsilon_X} X$ is a $\mathbb{G}$-epimorphism.*

Proof. For any map $h : GZ \to X$, we wish to find a map $f : GZ \to GX$ such that $f\varepsilon_X = h$. We define f via the following commuting diagram.

$$\begin{tikzcd}
G(GZ) \arrow[r, "G(h)"] & G(X) \arrow[r, "\varepsilon_X"] & X \\
GZ \arrow[u, "\delta(Z)"] \arrow[ur, "f"] & &
\end{tikzcd}$$

Now we need to check that $\varepsilon_X \circ f = h$. By the naturality of ε, the following diagram commutes.

$$\begin{tikzcd}
GX \arrow[r, "\varepsilon_X"] & X \\
G(GZ) \arrow[u, "G(h)"'] \arrow[r, "\varepsilon_{GZ}"'] & GZ \arrow[u, "h"'] \\
GZ \arrow[u, "\delta(Z)"'] \arrow[ur, "id_{GZ}"'] \arrow[uu, bend left, "f"] &
\end{tikzcd}$$

So ε_X is a $\mathbb{G}$-epimorphism. $\square$

An object P of $\mathfrak{C}$ is called $\mathbb{G}$-*projective* if for any $\mathbb{G}$-epimorphism $f : X \to Y$, the map $\mathrm{Hom}_{\mathfrak{C}}(P, X) \to \mathrm{Hom}_{\mathfrak{C}}(P, Y)$ is surjective.

Example 2.6. *For any object X in $\mathfrak{C}$ the object $G(X)$ is $\mathbb{G}$-projective.*

Lemma 2.7. *The coproduct of $\mathbb{G}$-projective objects is $\mathbb{G}$-projective.*

Proof. Let $P = \coprod_i P_i$ where P_i is $\mathbb{G}$-projective for all i. For a map $f : X \to Y$, one applies the functors $\mathrm{Hom}_{\mathfrak{C}}(P, -)$ and $\mathrm{Hom}_{\mathfrak{C}}(P_i, -)$ to get the maps $f_* : \mathrm{Hom}_{\mathfrak{C}}(P, X) \to \mathrm{Hom}_{\mathfrak{C}}(P, Y)$ and $f_{i*} : \mathrm{Hom}_{\mathfrak{C}}(P_i, X) \to \mathrm{Hom}_{\mathfrak{C}}(P_i, Y)$. If f is a $\mathbb{G}$-epimorphism then f_{i*} is surjective for all i. Using the well-known lemma $\mathrm{Hom}_{\mathfrak{C}}(\coprod_i P_i, Z) \cong \prod_i \mathrm{Hom}_{\mathfrak{C}}(P_i, Z)$ one gets that if f is a $\mathbb{G}$-epimorphism then $f_* \cong \prod_i f_{i*}$ is surjective. Hence P is $\mathbb{G}$-projective if P_i is $\mathbb{G}$-projective for all i. $\square$

Lemma 2.8. *An object P is $\mathbb{G}$-projective if and only if P is a retract of an object of the form $G(Z)$.*

Proof. A retract of a surjective map is surjective, so it is sufficient to consider the case $P = G(Z)$, which is clear from the definition of a $\mathbb{G}$-epimorphism. $\square$

2.4.2 Simplicial methods

Definition 2.9. A *simplicial object* in a category $\mathfrak{C}$ is a sequence of objects $X_0, X_1, \ldots, X_n, \ldots$ together with two double-indexed families of arrows in $\mathfrak{C}$. The *face operators* are arrows $d_n^i : X_n \to X_{n-1}$ for $0 \leq i \leq n$ and $1 \leq n < \infty$. The *degeneracy operators* are arrows $s_n^i : X_n \to X_{n+1}$ for $0 \leq i \leq n$ and $0 \leq n < \infty$. The face operators and degeneracy operators satisfy the following conditions:

$$\begin{aligned} d_n^i \circ d_{n+1}^j &= d_n^{j-1} \circ d_{n+1}^i \qquad \text{if } 0 \leq i < j \leq n+1 \\ s_n^j \circ s_{n-1}^i &= s_n^i \circ s_{n-1}^{j-1} \qquad \text{if } 0 \leq i < j \leq n \\ d_{n+1}^i \circ s_n^j &= \begin{cases} s_{n-1}^{j-1} \circ d_n^i, & \text{if } 0 \leq i < j \leq n; \\ 1, & \text{if } 0 \leq i = j \leq n \text{ or } 0 \leq i-1 = j < n; \\ s_{n-1}^j \circ d_n^{i-1}, & \text{if } 0 < j < i-1 \leq n. \end{cases} \end{aligned}$$

An *augmented simplicial object* in the category $\mathfrak{C}$ is a simplicial object X_* together with another object X_{-1} and an arrow $\epsilon : X_0 \to X_{-1}$ such that $\epsilon \circ d_1^0 = \epsilon \circ d_1^1$.

An augmented simplicial object $X_* \to X_{-1}$ is called *contractible* if for each $n \geq -1$ there exists a map $s_n : X_n \to X_{n+1}$ such that $d^0 \circ s = 1$ and $d^i \circ s = s \circ d^{i-1}$ for $0 < i \leq n$ and $s^0 \circ s = s \circ s$ and $s^i \circ s = s \circ s^{i-1}$ for $0 < i \leq n+1$.

Let X_* be a simplicial object in an additive category $\mathfrak{B}$. The associated chain complex to X_*, denoted by $C(X_*)$, is the complex

$$\ldots \longrightarrow X_{n+1} \xrightarrow{d} X_n \xrightarrow{d} X_{n-1} \xrightarrow{d} \ldots \xrightarrow{d} X_0 \longrightarrow 0$$

where the boundary maps $d = \sum_{i=0}^{n} (-1)^i d^i : X_n \to X_{n-1}$.

Proposition 2.10. *If $X_* \to X_{-1}$ is a contractible augmented simplicial object in an abelian category $\mathfrak{A}$, then the associated chain complex $C(X_*)$ is contractible.*

2.4.3 Comonad cohomology

Let $\mathbb{G}$ be a comonad on a category $\mathfrak{C}$. For any object A in $\mathfrak{C}$, we get a functorial augmented simplicial object which we denote by $\mathbb{G}(A)_* \to A$. The object of $\mathbb{G}_*(A)$ in degree n is $G^{n+1}(A)$. We define the face and degeneracy operators by

$$\begin{aligned} \varphi_i &= G^i \varepsilon G^{n-i} : G^{n+1}(A) \to G^n(A), \\ \sigma_i &= G^i \delta G^{n-i} : G^{n+1}(A) \to G^{n+2}(A), \end{aligned}$$

for $0 \leq i \leq n$. The augmenting map is given by ε.

$$\dots \rightrightarrows G^n A \rightrightarrows G^{n-1} A \rightrightarrows \dots \underset{\varepsilon G}{\overset{G\varepsilon}{\rightrightarrows}} GA \xrightarrow{\varepsilon} A$$

We call $\mathbb{G}(A)_*$ the $\mathbb{G}$ *comonad resolution of* A.

Let $E : \mathfrak{C} \to \mathfrak{M}$ be a contravariant functor where $\mathfrak{M}$ is an abelian category. The comonad cohomology of an object A with coefficients in E is $H^*_{\mathbb{G}}(A, E)$ where

$$H^n_{\mathbb{G}}(A, E) := H^n(C(E(\mathbb{G}_*(A)))).$$

By definition, $H^*_{\mathbb{G}}(A, E)$ is the cohomology of the associated cochain complex

$$0 \longrightarrow E(G(A)) \longrightarrow E(G^2(A)) \longrightarrow \dots$$

If $M \in A$-$\mathfrak{mod}$, then we define the cohomology of A with coefficients in M to be the comonad cohomology of A with coefficients in $\mathrm{Der}(-, M) : \mathfrak{C} \to \mathfrak{Ab}$.

$$H^n_{\mathbb{G}}(A, M) := H^n_{\mathbb{G}}(A, \mathrm{Der}(-, M)).$$

Lemma 2.11. $H^0_{\mathbb{G}}(A, M) \cong \mathrm{Der}(A, M)$ *for all* A.

Lemma 2.12. *If* A *is* $\mathbb{G}$*-projective then* $H^n_{\mathbb{G}}(A, M) = 0$ *for* $n > 0$.

Proof. From lemma 2.8, it is sufficient to check the case where $A = G(Z)$. There exists a contracting homotopy $s_n : G^{n+2} \to G^{n+3}$ for $n \geq -1$ given by

$$s_n = G^{n+1}\delta.$$

We get that $\epsilon s_{-1} = id$, $\varphi_{n+1} s_n = id$, $\varphi_0 s_0 = s_{-1}\epsilon$, and $\varphi_i s_n = s_{n-1}\varphi_i$ for all $0 \leq i \leq n$. It follows that $H^n_{\mathbb{G}}(G(Z), M) = 0$, for $n > 0$. $\square$

2.4.4 André-Quillen cohomology

In 1967, M. André and D. Quillen [18] independently introduced a (co)homology theory for commutative algebras. This theory now goes by the name of André-Quillen cohomology.

Fix a commutative ring k and consider the category $\mathfrak{Commalg}$ of commutative k-algebras.

The forgetful functor $U : \mathfrak{Commalg} \to \mathfrak{Sets}$ has a left adjoint which takes a set X to the polynomial algebra $k[X]$ on X. This adjoint pair gives us a comonad $\mathbb{G}$ on $\mathfrak{Commalg}$.

Let R be a commutative k-algebras, and $M \in R - \mathfrak{mod}$. We define the André-Quillen cohomology of R with coefficients in M to be comonad cohomology of R with coefficients in $\mathrm{Der}_k(-, M)$,

$$H^n_{AQ}(R/k, M) := H^n_{\mathbb{G}}(R, \mathrm{Der}_k(-, M)).$$

Note that $\mathrm{Der}_k(-, M))$ is a functor from the category of commutative k-algebras $\mathfrak{Commalg}$ to the category of abelian groups $\mathfrak{Ab}$.

An *extension* of R by M is an exact sequence

$$0 \longrightarrow M \xrightarrow{\alpha} X \xrightarrow{\beta} R \longrightarrow 0$$

where X is a commutative k-algebra, the map β is a commutative k-algebra homomorphism, the map α is a k-module homomorphism and

$$x\alpha(m) = \alpha(\beta(x)m)$$

for all $x \in X$ and all $m \in M$. The map α identifies M with an ideal of square-zero in X.

Two extensions X, X' with R and M fixed are *equivalent* if there exists a k-algebra homomorphism $\phi : X \to \overline{X}$ such that the following diagram commutes.

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "\phi"] & R \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M \arrow[r] & \overline{X} \arrow[r] & R \arrow[r] & 0 \end{tikzcd}$$

We denote the set of equivalence classes of extensions of R by M by $Extalg_k(R, M)$.

Proposition 2.13. *1.* $H^0_{AQ}(R/k, M) \cong \mathrm{Der}_k(R, M)$.

2. If R is a free commutative algebra then $H^n_{AQ}(R/k, M) = 0$ for $n > 0$.

3. $H^1_{AQ}(R/k, M) \cong Extalg_k(R, M)$.

2.5 Harrison cohomology of commutative algebras

In 1962, D.K. Harrison introduced a cohomology of commutative algebras. The Harrison complex is a subcomplex of the Hochschild complex in the case of commutative algebras. The Harrison complex consists of the linear functions which vanish on the shuffles. The Harrison cohomology is isomorphic to the comonad cohomology for a commutative algebra over a field of characteristic 0, however, there is a shift of one dimension. The following material can be found in [15].

2.5.1 Hochschild cohomology

Let k be a ring, R be an associative k-algebra and M be an $R-R$-bimodule. All the tensor products in this section are over the ground ring k. The Hochschild cochain complex of R with coefficients in M is given by

$$C^n_{HH}(R, M) = \mathrm{Hom}_{R^e}(R^{\otimes n}, M),$$

for $n \geq 0$ and $R^e = R \otimes R^{op}$. The coboundary maps $\delta^n : C^n_{HH}(R,M) \to C^{n+1}_{HH}(R,M)$ are given by

$$\begin{aligned}\delta^n(f)(r_0,\dots,r_n) =& r_0 f(r_1,\dots,r_n)\\ &+\sum_{i=0}^{n-1}(-1)^{i+1} f(r_0,\dots,r_i r_{i+1},\dots,r_n)\\ &+(-1)^{n+1} f(r_0,\dots,r_{n-1})r_n.\end{aligned}$$

We can then take the cohomology of the resulting complex to get the Hochschild cohomology which we denote by $HH^n(R,M)$. We get that

$$HH^n(R,M) \cong R^n \operatorname{Hom}_{R^e}(R,M) \cong \operatorname{Ext}^n_{R^e}(R,M).$$

2.5.2 Harrison Cohomology

Let S_n be the symmetric group which acts on the set $\{1,\dots,n\}$. A *(p,q)-shuffle* is a permutation σ in S_{p+q} such that:

$$\sigma(1) < \sigma(2) < \ldots < \sigma(p) \text{ and } \sigma(p+1) < \sigma(2) < \ldots < \sigma(p+q).$$

For any k-algebra A and $M \in A - \mathfrak{mod}$, we let S_n act on the left on $C_n^{HH}(A,M) = M \otimes A^{\otimes n}$ by:

$$\sigma \cdot (m, a_1, \dots, a_n) = (m, a_{\sigma^{-1}(1)} \dots, a_{\sigma^{-1}(n)}).$$

Let A' be another k-algebra, $M' \in A' - \mathfrak{mod}$. The *shuffle product*:

$$- \times - = sh_{pq} : C_p^{HH}(A,M) \otimes C_q^{HH}(A',M') \to C_{p+q}^{HH}(A \otimes A', M \otimes M'),$$

is defined by the following formula:

$$\begin{aligned}(m, a_1, \dots, a_p)\times&(m', a'_1, \dots, a'_q)\\ &= \sum_\sigma sgn(\sigma)\sigma \cdot (m \otimes m', a_1 \otimes 1, \dots, a_p \otimes 1, 1 \otimes a'_1, \dots, 1 \otimes a'_q).\end{aligned}$$

Proposition 2.14. *The Hochschild boundary b is a graded derivation for the shuffle product*

$$b(x \times y) = b(x) \times y + (-1)^{|x|} x \times b(y), \qquad x \in C_p^{HH}(A, M), y \in C_q^{HH}(A', M').$$

where the Hochschild boundary $b : C_n^{HH}(A, M) \to C_{n-1}^{HH}(A, M)$ is given by:

$$\begin{aligned} b(m, a_1, \ldots, a_n) =& (ma_1, a_2, \ldots, a_n) + \sum_{i=1}^{n-1} (-1)^i (m, a_1, \ldots, a_i a_{i+1}, \ldots, a_n) \\ &+ (-1)^n (a_n m, a_1, \ldots, a_{n-1}). \end{aligned}$$

Assume that A is commutative and M is symmetric (symmetric means that $am = ma$ for all $a \in A$ and $m \in M$). The product map $\mu : A \otimes A \to A$ is a k-algebra homomorphism, and $\mu' : A \otimes M \to M$ is a homomorphism of bimodules. Composition of the shuffle map with $\mu \otimes \mu'$ allows us to define the *inner shuffle map*

$$- \times - = sh_{pq} : C_p^{HH}(A, A) \otimes C_q^{HH}(A, M) \to C_{p+q}^{HH}(A, M),$$

given by the formula

$$(a_0, a_1, \ldots, a_p) \times (m, a_{p+1}, \ldots, a_{p+q}) = \sum_{\sigma = (p,q)-shuffle} sgn(\sigma) \sigma \cdot (a_0 m, a_1, \ldots, a_{p+q}).$$

We let J denote $\bigoplus_{n>0} C_n^{HH}(A, A)$. Note that $J \subset C_*^{HH}(A, A)$. We define the Harrison chain complex to be the quotient $C_*^{Harr}(A, M) = C_*^{HH}(A, M)/J.C_*^{HH}(A, M)$.

Note that

$$\begin{aligned} C^n_{HH}(A, M) &= \mathrm{Hom}_{A^e}(A^{\otimes n}, M) \cong \mathrm{Hom}_{A \otimes A^e}(A \otimes A^{\otimes n}, M) \\ &= \mathrm{Hom}_{A \otimes A^e}(C_n^{HH}(A, A), M). \end{aligned}$$

We define the Harrison cochain complex by taking

$$C^*_{Harr}(A, M) := \mathrm{Hom}_{A \otimes A^e}(C_*^{Harr}(A, A), M).$$

For example

$$\begin{aligned}
C^0_{Harr}(A,M) =& M,\\
C^1_{Harr}(A,M) =& C^1_{HH}(A,M),\\
C^2_{Harr}(A,M) =& \{f \in C^2_{HH}(A,M) | f(x,y) = f(y,x)\},\\
C^3_{Harr}(A,M) =& \{f \in C^3_{HH}(A,M) | f(x,y,z) - f(y,x,z) + f(y,z,x) = 0.\}
\end{aligned}$$

We define the Harrison cohomology of A with coefficients in M to be the cohomology of the Harrison cochain complex.

$$Harr^n(A,M) := H^n(C^*_{Harr}(A,M)).$$

Lemma 2.15. $Harr^1(A,M) \cong \mathrm{Der}(A,M)$.

An *additively split extension* of A by M is an extension of A by M

$$0 \longrightarrow M \xrightarrow{q} X \xrightarrow{p} A \longrightarrow 0$$

where there exists $s : A \to X$ which is an additive section of p.

Two additively split extensions $(X), (\overline{X})$ with A, M fixed are said to be *equivalent* if there exists a homomorphism of commutative algebras $\phi : X \to \overline{X}$ such that the following diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "\phi"] & A \arrow[r] \arrow[d, equal] & 0\\
0 \arrow[r] & M \arrow[r] & \overline{X} \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

We denote the set of equivalence classes of additively split extensions of A by M by $\mathrm{AExt}(A,M)$.

Lemma 2.16. $Harr^2(A,M) \cong \mathrm{AExt}(A,M)$.

Proof. Given an additively split extension of A by M

$$0 \longrightarrow M \xrightarrow{q} X \xrightarrow{p} A \longrightarrow 0$$

there is an additive homomorphism $s : A \to X$ which is a section of p. The section induces an additive isomorphism $X \approx A \oplus M$ where multiplication in X is given by $(a, m)(a', m') = (aa', ma' + am' + f(a, a'))$ where the bilinear map $f : A \times A \to M$ is given by

$$f(a, a') = s(a)s(a') - s(aa').$$

The map f is a 2-cocycle. Given two additively split extensions which are equivalent, then the two 2-cocycles we get differ by a 2-coboundary.

A 2-cocycle is a map $f : A \times A \to M$. We get an additively split extension of A by M given by taking the exact sequence

$$0 \longrightarrow M \longrightarrow M \oplus A \longrightarrow A \longrightarrow 0$$

where addition in $M \oplus A$ is given by $(m, a) + (m', a') = (m + m', a + a')$ and multiplication is given by

$$(m, a)(m', a') = (a'm + am' + f(a, a'), aa').$$

Given two 2-cocycles which differ by a 2-coboundary, then the two additively split extensions we get are equivalent. $\square$

A *crossed module* consists of a commutative algebra C_0, a C_0-module C_1 and a module homomorphism

$$C_1 \xrightarrow{\rho} C_0 ,$$

which satisfies the property

$$\rho(c)c' = c\rho(c'),$$

for $c, c' \in C_1$. In other words, a crossed module is a chain algebra which is non-trivial only in dimensions 0 and 1. Since $C_2 = 0$ the condition $\rho(c)c' = c\rho(c')$ is equivalent to the Leibnitz relation

$$0 = \rho(cc') = \rho(c)c' - c\rho(c').$$

We can define a product by

$$c * c' := \rho(c)c',$$

for $c, c' \in C_1$. This gives us a commutative algebra structure on C_1 and $\rho : C_1 \to C_0$ is an algebra homomorphism.

Let $\rho : C_1 \to C_0$ be a crossed module. We let $M = \text{Ker}(\rho)$ and $A = \text{Coker}(\rho)$. Then the image $\text{Im}(\rho)$ is an ideal of C_0, $MC_1 = C_1M = 0$ and M has a well-defined A-module structure. We say such a crossed module is a crossed module over A with kernel M.

A *crossed extension* of A by M is an exact sequence

$$0 \longrightarrow M \xrightarrow{\alpha} C_1 \xrightarrow{\rho} C_0 \xrightarrow{\gamma} A \longrightarrow 0$$

where $\rho : C_1 \to C_0$ is a crossed module, γ is an algebra homomorphism, and the module structure on M coincides with the one induced from the crossed module.

A morphism between two crossed extensions consists of commutative algebra homomorphisms $h_0 : C_0 \to C_0$ and $h_1 : C_1 \to C_1'$ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\alpha"] \arrow[d, equal] & C_1 \arrow[r, "\rho"] \arrow[d, "h_1"] & C_0 \arrow[r, "\gamma"] \arrow[d, "h_0"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r, "\alpha'"] & C_1' \arrow[r, "\rho'"] & C_0' \arrow[r, "\gamma'"] & A \arrow[r] & 0
\end{tikzcd}$$

Let $Cross(A, M)$ denote the category of crossed modules over A with kernel M, and let $\pi_0 Cross(A, M)$ denote the connected components of $Cross(A, M)$.

Definition 2.17. An *additively split crossed extension* of A by M is a crossed extension of A by

$$0 \longrightarrow M \xrightarrow{\alpha} C_1 \xrightarrow{\rho} C_0 \xrightarrow{\gamma} A \longrightarrow 0 \tag{2.1}$$

such that all the arrows in the exact sequence 2.1 are additively split.

We denote the connected components of the category of additively split crossed extensions over A with kernel M by $\pi_0 ACross(A, M)$.

Lemma 2.18. *If $\gamma : C_0 \to A$ is k-algebra homomorphism then*

$$Harr^2(\gamma : C_0 \to A, M) \cong \pi_0 ACross(\gamma : C_0 \to A, M),$$

where $Harr^*(\gamma : C_0 \to A, M)$ *and* $\pi_0 ACross(\gamma : C_0 \to A, M)$ *are defined as follows. Consider the following short exact sequence of cochain complexes:*

$$0 \longrightarrow C^*_{Harr}(A, M) \xrightarrow{\gamma^*} C^*_{Harr}(C_0, M) \xrightarrow{\kappa^*} \mathrm{Coker}(\gamma^*) \longrightarrow 0.$$

We define the cochain complex $C^*_{Harr}(\gamma : C_0 \to A, M) := \mathrm{Coker}(\gamma^*)$. *This allows us to define the relative Harrison cohomology*

$$Harr^*(\gamma : C_0 \to A, M) := H^*(C^*_{Harr}(\gamma : C_0 \to A, M)).$$

We let $ACross(\gamma : C_0 \to A, M)$ *denote the category whose objects are the additively split crossed extensions of* A *by* M

$$0 \longrightarrow M \xrightarrow{\alpha} C_1 \xrightarrow{\rho} C_0 \xrightarrow{\gamma} A \longrightarrow 0$$

with $\gamma : C_0 \to A$ *fixed. A morphisms between two of these crossed extensions consists of a morphism of crossed extensions with the map* $h_0 : C_0 \to C_0$ *being the identity.*

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\alpha"] \arrow[d, equal] & C_1 \arrow[r, "\rho"] \arrow[d, "h_1"] & C_0 \arrow[r, "\gamma"] \arrow[d, equal] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r, "\alpha'"] & C_1' \arrow[r, "\rho'"] & C_0 \arrow[r, "\gamma"] & A \arrow[r] & 0
\end{tikzcd}$$

Note that $ACross(\gamma : C_0 \to A, M)$ *is a groupoid.*

Proof. This proof is very similiar to a proof given in [13] for the crossed modules of Lie algebras. Given any additively split crossed module of A by M,

$$0 \longrightarrow M \xrightarrow{\alpha} C_1 \xrightarrow{\rho} C_0 \xrightarrow{\gamma} A \longrightarrow 0,$$

we let $V = \mathrm{Ker}\,\gamma = \mathrm{Im}\,\rho$. There are k-linear sections $s : A \to C_0$ of γ and $\sigma : V \to C_1$ of $\rho : C_1 \to V$. We define the map $g : A \otimes A \to C_1$ by:

$$g(a, b) = \sigma(s(a)s(b) - s(ab)).$$

We also define the map $\omega : C_0 \to C_1$ by:

$$\omega(c) = \sigma(c - s\gamma(c)).$$

By identifying M with $\operatorname{Ker}\delta$, we define the map $f : C_0 \otimes C_0 \to M$ by:

$$f(c, c') = g(\gamma(c), \gamma(c')) + c'\omega(c) + c\omega(c') - \omega(c) * \omega(c') - \omega(cc').$$

Since $g(c, c') = g(c', c)$, it follows that $f(c, c') = f(c', c)$ and so $f \in C^2_{Harr}(C_0, M)$. We define the map $\varpi \in C^3_{Harr}(A, M)$ by:

$$\varpi(x, y, z) = s(x)g(y, z) - g(xy, z) + g(x, yz) - g(y, x)s(z).$$

Note that ϖ vanishes on the shuffles since $g(x, y) = g(y, x)$.

Consider the following commuting diagram.

$$\begin{tikzcd}
0 \arrow[r] & C^2_{Harr}(A, M) \arrow[r, "\gamma^*"] \arrow[d] & C^2_{Harr}(C_0, M) \arrow[r, "\kappa^*"] \arrow[d, "\delta"] & C^2_{Harr}(\gamma : C_0 \to A, M) \arrow[r] \arrow[d, "\delta"] & 0 \\
0 \arrow[r] & C^3_{Harr}(A, M) \arrow[r, "\gamma^*"] & C^3_{Harr}(C_0, M) \arrow[r, "\kappa^*"] & C^3_{Harr}(\gamma : C_0 \to A, M) \arrow[r] & 0
\end{tikzcd}$$

A direct calculation shows that $\delta f = \gamma^*\varpi \in C^3(C_0, M)$. We also have that $\delta\kappa^* f = \kappa^*\delta f = \kappa^*\gamma^*\varpi = 0$, this tells us that $\kappa^* f$ is a cocycle. If we have two equivalent additively split crossed modules then we can choose sections in such a way that the associated cocycles are the same. Therefore we have a well-defined map:

$$ACross(\gamma : C_0 \to A, M) \longrightarrow H^3_{Harr}(\gamma : C_0 \to A, M).$$

Inversely, assume we have a cocycle in $C^2_{Harr}(\gamma : C_0 \to A, M)$ which we lift to a cochain $f \in C^2_{Harr}(C_0, M)$. Let $V = \operatorname{Ker}\gamma$. We define $C_1 = M \times V$ as a module over k with the following action of C_0 on C_1:

$$c(m, v) := (cm + f(c, v), cv).$$

It is easy to check using the properties of f that this action is well defined and together with the map $\rho : C_0 \to C_1$ given by $\rho(m, v) = v$, we have an additively split crossed module of A by M. □

Lemma 2.19. *If k is a field of characteristic 0 then*

$$Harr^3(A, M) \cong \pi_0 ACross(A, M).$$

Proof. From the definition of $C^*_{Harr}(\gamma : C_0 \to A, M)$ we get the long exact sequence:

$$\ldots \longrightarrow Harr^2(A, M) \longrightarrow Harr^2(C_0, M) \longrightarrow$$

$$Harr^2(\gamma : C_0 \to A, M) \longrightarrow Harr^3(A, M) \longrightarrow \ldots \tag{2.2}$$

Given any additively split crossed module in $\pi_0 ACross(A, M)$,

$$0 \longrightarrow M \xrightarrow{\alpha} C_1 \xrightarrow{\rho} C_0 \xrightarrow{\gamma} A \longrightarrow 0$$

we can lift γ to get a map $P_0 \to A$ where P_0 is a polynomial algebra. We can then use a pullback to construct P_1 to get a crossed module where the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\alpha"] & C_1 \arrow[r, "\rho"] & C_0 \arrow[r, "\gamma"] & A \arrow[r] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[u, equal] & P_1 \arrow[r] \arrow[u] & P_0 \arrow[r] \arrow[u] & A \arrow[r] \arrow[u, equal] & 0.
\end{tikzcd}$$

Note that these two crossed modules are in the same connected component of $\pi_0 ACross(A, M)$. By considering the second crossed module in the long exact sequence, we replace C_0 by P_0 to get the new exact sequence:

$$0 \longrightarrow Harr^2(\gamma : P_0 \to A, M) \longrightarrow Harr^3(A, M) \longrightarrow 0 \tag{2.3}$$

since $Harr^2(P_0, M) = 0$ and $Harr^3(P_0, M) = 0$.

The exact sequence 2.3 tells us that every element in $Harr^3(A, M)$ comes from an element in $Harr^2(\gamma : P_0 \to A, M)$ and the previous lemma tells us that this comes from a crossed module in $\pi_0 ACross(A, M)$. Therefore the map $\pi_0 ACross(A, M) \to Harr^3(A, M)$ is surjective.

Assuming we have two crossed modules which go to the same element in $Harr^3(A, M)$,

$$0 \longrightarrow M \xrightarrow{\alpha} C_1 \xrightarrow{\rho} C_0 \xrightarrow{\gamma} A \longrightarrow 0, \tag{2.4}$$

$$0 \longrightarrow M \xrightarrow{\alpha'} C_1' \xrightarrow{\rho'} C_0' \xrightarrow{\gamma'} A \longrightarrow 0. \tag{2.5}$$

There exist morphisms

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\alpha"] & C_1 \arrow[r, "\rho"] & C_0 \arrow[r, "\gamma"] & A \arrow[r] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[u, equal] & P_1 \arrow[r] \arrow[u] & P_0 \arrow[r] \arrow[u] & A \arrow[r] \arrow[u, equal] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[u, equal] \arrow[d, equal] & P_2 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[u, equal] \arrow[d] & A \arrow[r] \arrow[u, equal] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r, "\alpha'"] & C_1' \arrow[r, "\rho'"] & C_0' \arrow[r, "\gamma'"] & A \arrow[r] & 0
\end{tikzcd}$$

where P_0 is a polynomial algebra and P_1, P_2 are constructed via pullbacks. These give us two elements in $Harr^2(\gamma : P_0 \to A, M)$ which go to the same element in $Harr^3(A, M)$. However the exact sequence 2.3 tells us that the two crossed modules 2.4 and 2.5 have to go to the same element in $Harr^2(\gamma : P_0 \to A, M)$. The previous lemma tells us that the two crossed modules 2.4 and 2.5 go to the same element in $ACross(\gamma : C_0 \to A, M)$ which is a groupoid, so there is a map $P_2 \to P_1$ which makes the following diagram commute:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\alpha"] & C_1 \arrow[r, "\rho"] & C_0 \arrow[r, "\gamma"] & A \arrow[r] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[u, equal] & P_1 \arrow[r] \arrow[u] & P_0 \arrow[r] \arrow[u] & A \arrow[r] \arrow[u, equal] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[u, equal] \arrow[d, equal] & P_2 \arrow[r] \arrow[u] \arrow[d] & P_0 \arrow[r] \arrow[u, equal] \arrow[d] & A \arrow[r] \arrow[u, equal] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r, "\alpha'"] & C_1' \arrow[r, "\rho'"] & C_0' \arrow[r, "\gamma'"] & A \arrow[r] & 0
\end{tikzcd}$$

Therefore the two crossed modules 2.4 and 2.5 are in the same connected component of $\pi_0 ACross(A, M)$ and the map $\pi_0 ACross(A, M) \to Harr^3(A, M)$ is injective. □

2.6 Baues-Wirsching cohomology

The following material can be found in [4]. A category $\mathcal{I}$ is said to be *small* if the collection of morphisms is a set. Consider a small category $\mathcal{I}$. The *category of factorizations* in $\mathcal{I}$, denoted by $\mathcal{FI}$, is the category whose objects are the morphisms $f, g, \ldots$ in $\mathcal{I}$, and morphisms $f \to g$ are pairs (α, β) of morphisms in $\mathcal{I}$ such that the following diagram commutes.

$$\begin{tikzcd} B \arrow[r, "\alpha"] & B' \\ A \arrow[u, "f"] & A' \arrow[l, "\beta"] \arrow[u, "g"'] \end{tikzcd}$$

Composition in $\mathcal{FI}$ is given by $(\alpha', \beta')(\alpha, \beta) = (\alpha'\alpha, \beta\beta')$. A *natural system* of abelian groups on $\mathcal{I}$ is a functor

$$D : \mathcal{FI} \to \mathfrak{Ab}.$$

There exists a canonical functor $\mathcal{FI} \to \mathcal{I}^{op} \times \mathcal{I}$ which takes $f : A \to B$ to the pair (A, B). This functor allows us to consider any bifunctor $D : \mathcal{I}^{op} \times \mathcal{I} \to \mathfrak{Ab}$ as a natural system. Similarly, the projection $\mathcal{I}^{op} \times \mathcal{I} \to \mathcal{I}$ gives us the functor $\mathcal{FI} \to \mathcal{I}$ which takes $f : A \to B$ to B. This allows us to consider any functor $D : \mathcal{I} \to \mathfrak{Ab}$ as a natural system.

Following Baues-Wirsching [4], we define the cohomology $H^*_{BW}(\mathcal{I}, D)$ of $\mathcal{I}$ with coefficients in the natural system D as the cohomology of the cochain complex $C^*_{BW}(\mathcal{I}, D)$ given by

$$C^n_{BW}(\mathcal{I}, D) = \prod_{\alpha_1 \ldots \alpha_n : i_n \to \ldots \to i_0} D(\alpha_1 \ldots \alpha_n),$$

where the product is indexed over n-tuples of composable morphisms and the coboundary map

$$d : C^n_{BW}(\mathcal{I}, D) \to C^{n+1}_{BW}(\mathcal{I}, D),$$

is given by

$$\begin{aligned}(df)(\alpha_1 \dots \alpha_{n+1}) =& (\alpha_1)_* f(\alpha_2, \dots, \alpha_{n+1}) \\ &+ \sum_{j=1}^{n} (-1)^j f(\alpha_1, \dots, \alpha_j\alpha_{j+1}, \dots, \alpha_{n+1}) \\ &+ (-1)^{n+1}(\alpha_{n+1})^* f(\alpha_1, \dots, \alpha_n).\end{aligned}$$

Lemma 2.20. *Let $i_0 \in \mathcal{I}$ be an initial object and $F : \mathcal{I} \to \mathfrak{Ab}$ a functor then*

$$H^n_{BW}(\mathcal{I}, F) = \begin{cases} F(i_0) & \text{for } n = 0 \\ 0 & \text{for } n > 0. \end{cases}$$

Chapter 3

Ψ-rings

3.1 Introduction

In this chapter, only the material in this section is already known and everything from section 3.2 onwards is new and original material. Note that in all of the cited material, including [1], [14] and [20], what the authors call a Ψ-ring is what we call a special Ψ-ring. Also note that in our notation $\mathbb{N}$ does not include 0.

λ-rings are complicated, and given a λ-ring it is often difficult to prove it satisfies the λ-ring axioms. We start by introducing another kind of ring, the Ψ-rings, which are closely related to the λ-rings by the Adams operations. The axioms for the Ψ-rings are a lot simpler than those for the λ-rings.

Definition 3.1. A *Ψ-ring* is a commutative ring with identity, R, together with a sequence of ring homomorphisms $\Psi^i : R \to R$, for $i \in \mathbb{N}$, satisfying

1. $\Psi^1(r) = r$,
2. $\Psi^i(\Psi^j(r)) = \Psi^{ij}(r)$,

for all $r \in R$, and $i, j \in \mathbb{N}$.

We say that a Ψ-ring R is *special* if it also satisfies the property

$$\Psi^p(r) \equiv r^p \mod pR,$$

for all primes p and $r \in R$.

Example 3.2. *Any commutative ring with identity, R, can be given a Ψ-ring structure by setting $\Psi^i : R \to R$ to be $\Psi^i(r) = r$ for all $r \in R$ and $i \in \mathbb{N}$.*

Let R_1, R_2 be Ψ-rings. A *map of Ψ-rings* is a ring homomorphism $f : R_1 \to R_2$, such that $\Psi^i(f(r)) = f(\Psi^i(r))$ for all $r \in R_1$ and $i \in \mathbb{N}$. The class of all Ψ-rings and maps of Ψ-rings form the category of Ψ-rings, which we denote by $\Psi{-}\mathfrak{rings}$.

3.2 Ψ-modules

For usual rings, the modules provide us with the coefficients for the cohomology. In this section we define the Ψ-modules for Ψ-rings which provide us with the coefficients for the Ψ-ring cohomology. We then use this to create the Ψ-analogue of some of the results for rings.

Definition 3.3. We say that M is a Ψ-*module* over the Ψ-ring R if M is an R-module together with a sequence of abelian group homomorphisms $\psi^i : M \to M$, for $i \in \mathbb{N}$, satisfying

1. $\psi^1(m) = m$,
2. $\psi^i(rm) = \Psi^i(r)\psi^i(m)$,
3. $\psi^i(\psi^j(m)) = \psi^{ij}(m)$,

for all $m \in M$, $r \in R$, and $i, j \in \mathbb{N}$.

Let M, N be two Ψ-modules over R. A map of Ψ-modules is a module homomorphism $f : M \to N$ such that $\psi^i f(m) = f\psi^i(m)$ for all $m \in M$ and $i \in \mathbb{N}$. We let $R{-}\mathfrak{mod}_\Psi$ denote the category of all Ψ-modules over R.

We say that M is *special* if R is special and

$$\psi^p(m) \equiv 0 \mod pM,$$

for all primes p and $m \in M$.

Note that any Ψ-ring can be considered as a Ψ-module over itself. Also note that if M is special, then $\psi^i(m) \equiv 0 \bmod iM$ for all $i \in \mathbb{N}$ and $m \in M$.

For the rest of this chapter, we let R denote a Ψ-ring and $M \in R\text{−}\mathfrak{mod}_\Psi$. We let $\underline{R}$ denote the underlying commutative ring of R, and we let $\underline{M}$ denote the underlying $\underline{R}$-module of M.

Lemma 3.4. *The set* $R \rtimes M$ *with*

$$\begin{aligned}(r, m) + (s, n) &= (r + s, m + n),\\ (r, m)(s, n) &= (rs, rn + ms),\end{aligned}$$

together with maps $\Psi^i : R \rtimes M \to R \rtimes M$ *for* $i \in \mathbb{N}$ *given by*

$$\Psi^i(r, m) = (\Psi^i(r), \psi^i(m) + \varepsilon^i(r)),$$

for a sequence of maps $\varepsilon^i : R \to M$ *for* $i \in \mathbb{N}$*, is a* Ψ*-ring if and only if*

1. $\varepsilon^1(r) = 0$,
2. $\varepsilon^i(r + s) = \varepsilon^i(r) + \varepsilon^i(s)$,
3. $\varepsilon^i(rs) = \Psi^i(r)\varepsilon^i(s) + \varepsilon^i(r)\Psi^i(s)$,
4. $\varepsilon^{ij}(r) = \psi^i\varepsilon^j(r) + \varepsilon^i\Psi^j(r)$,

for all $r, s \in R$*, and* $i, j \in \mathbb{N}$.

Proof of lemma. It is known that $\underline{R} \rtimes \underline{M}$ is a commutative ring with identity. Hence it is sufficient to check that $\Psi^i : R \rtimes M \to R \rtimes M$ satisfies the Ψ-ring axioms.

1. $\Psi^1(r, m) = (\Psi^1(r), \psi^1(m) + \varepsilon^1(r)) = (r, m + \varepsilon^1(r))$
   Hence $\Psi^1(r, m) = (r, m)$ if and only if $\varepsilon^1(r) = 0$.

2. $\Psi^i((r,m)+(s,n)) = (\Psi^i(r)+\Psi^i(s), \psi^i(m)+\psi^i(n)+\varepsilon^i(r+s))$
   $\Psi^i(r,m)+\Psi^i(s,n) = (\Psi^i(r)+\Psi^i(s), \psi^i(m)+\psi^i(n)+\varepsilon^i(r)+\varepsilon^i(s))$
   Hence $\Psi^i((r,m)+(s,n)) = \Psi^i(r,m)+\Psi^i(s,n)$ if and only if
   $\varepsilon^i(r+s) = \varepsilon^i(r)+\varepsilon^i(s)$.

3. $\Psi^i((r,m)(s,n)) = (\Psi^i(rs), \psi^i(rn+ms)+\varepsilon^i(rs))$
   $\Psi^i(r,m)\Psi^i(s,n) = (\Psi^i(rs), \psi^i(rn+ms)+\Psi^i(r)\varepsilon^i(s)+\varepsilon^i(r)\Psi^i(s))$
   Hence $\Psi^i((r,m)(s,n)) = \Psi^i(r,m)\Psi^i(s,n)$ if and only if
   $\varepsilon^i(rs) = \Psi^i(r)\varepsilon^i(s)+\varepsilon^i(r)\Psi^i(s)$.

4. $\Psi^i\Psi^j(r,m) = (\Psi^i\Psi^j(r), \psi^i\psi^j(m)+\psi^i\varepsilon^j(r)+\varepsilon^i\Psi^j(r))$
   $\Psi^{ij}(r,m) = (\Psi^i\Psi^j(r), \psi^i\psi^j(m)+\varepsilon^{ij}(r))$
   Hence $\Psi^i\Psi^j(r,m) = \Psi^{ij}(r,m)$ if and only if $\varepsilon^{ij}(r) = \psi^i\varepsilon^j(r)) + \varepsilon^i\Psi^j(r)$.

□

The maps $\varepsilon^i : R \to M$ given by $\varepsilon^i(r) = 0$, for all $r \in R$ and $i \in \mathbb{N}$, satisfy properties 3.4.1-3.4.4 meaning that the maps $\Psi^i : R \rtimes M \to R \rtimes M$ given by $\Psi^i(r,m) = (\Psi^i(r), \psi^i(m))$ give us a Ψ-ring structure on $R \rtimes M$. We call this the *semi-direct product* of R and M, denoted by $R \rtimes_\Psi M$.

We note that if R and M are both special, then $R \rtimes_\Psi M$ is also special.

3.3 Ψ-derivations

The André-Quillen cohomology for commutative rings is given by the derived functors of the derivations functor. For a commutative ring S, the derivations of S with values in an S-module N are in one-to-one correspondence with the sections of $S \rtimes N \xrightarrow{\pi} S$. We define the Ψ-derivations and show that they are in one-to-one correspondence with the sections of $R \rtimes_\Psi M \xrightarrow{\pi} R$.

Definition 3.5. A Ψ*-derivation* of R with values in M is an additive homomorphism $d : R \to M$ such that

1. $d(rs) = rd(s) + d(r)s$,

2. $\psi^i(d(r)) = d(\Psi^i(r))$,

for all $r, s \in R$, and $i \in \mathbb{N}$. We let $\mathrm{Der}_\Psi(R, M)$ denote the set of all Ψ-derivations of R with values in M.

Example 3.6. *Let R and M be such that $\Psi^i = Id = \psi^i$ for all $i \in \mathbb{N}$, then*

$$\mathrm{Der}_\Psi(R, M) = \mathrm{Der}(\underline{R}, \underline{M}).$$

Theorem 3.7. *There is a one-to-one correspondence between the elements of $\mathrm{Der}_\Psi(R, M)$ and the sections of $R \rtimes_\Psi M \xrightarrow{\pi} R$.*

Proof of theorem. Assume we have a section of π, then we have the following

$$R \rtimes_\Psi M \underset{\sigma}{\overset{\pi}{\rightleftarrows}} R,$$

where $\pi\sigma = Id_R$. Hence $\sigma(r) = (r, d(r))$ for some $d : R \to M$. The properties

$$\begin{aligned} d(r+s) =& d(r) + d(s), \\ d(rs) =& d(r)s + rd(s), \end{aligned}$$

follow from σ being a ring homomorphism. However σ also preserves the Ψ-ring structure, so we get that $\Psi^i\sigma(r) = \sigma\Psi^i(r)$. We know that

$$\Psi^i\sigma(r) = \Psi^i(r, d(r)) = (\Psi^i(r), \psi^i(d(r))),$$

$$\sigma\Psi^i(r) = (\Psi^i(r), d(\Psi^i(r))).$$

Hence $\Psi^i\sigma(r) = \sigma\Psi^i(r)$ if and only if $\psi^i d(r) = d\Psi^i(r)$. This tells us that if σ is a section of π, then we have a Ψ-derivation d.

Conversely, if we have a Ψ-derivation $d : R \to M$, then $\sigma(r) = (r, d(r))$ is a section of π. $\square$

3.4 Ψ-ring extensions

We have seen in proposition 2.13 that the André-Quillen cohomology $H^1_{AQ}(\underline{R}, \underline{M})$ classifies the extensions of $\underline{\mathrm{R}}$ by $\underline{\mathrm{M}}$. In this section, we develop the Ψ-analogue of extensions.

Definition 3.8. A Ψ-*ring extension* of R by M is an extension of $\underline{R}$ by $\underline{M}$

$$0 \longrightarrow M \xrightarrow{\alpha} X \xrightarrow{\beta} R \longrightarrow 0$$

where X is a Ψ-ring, β is a map of Ψ-rings and $\alpha\psi^n = \Psi^n\alpha$ for all $n \in \mathbb{N}$.

Two Ψ-ring extensions $(X), (\overline{X})$ with R, M fixed are said to be *equivalent* if there exists a map of Ψ-rings $\phi : X \to \overline{X}$ such that the following diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "\phi"] & R \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] & \overline{X} \arrow[r] & R \arrow[r] & 0
\end{tikzcd}$$

We denote the set of equivalence classes of Ψ-ring extensions of R by M by $\mathrm{Ext}_\Psi(R, M)$.

The Harrison cohomology $Harr^2(\underline{R}, \underline{M})$ classifies the additively split extensions of $\underline{R}$ by $\underline{M}$. We can also define the Ψ-analogue of these types of extensions.

Definition 3.9. An *additively split* Ψ-*ring extension* of R by M is a Ψ-ring extension of R by M

$$0 \longrightarrow M \xrightarrow{\alpha} X \xrightarrow{\beta} R \longrightarrow 0$$

where β has a section which is an additive homomorphism.

Multiplication in $X = R \oplus M$ has the form $(r, m)(r', m') = (rr', mr' + rm' + f(r, r'))$, where $f : R \times R \to M$ is some bilinear map. Associativity in X gives us

$$0 = rf(r', r'') - f(rr', r'') + f(r, r'r'') - f(r, r')r''.$$

Commutativity in X gives us

$$f(r, r') = f(r', r).$$

The Ψ-operations $\Psi^i : R \oplus M \to R \oplus M$ for $i \in \mathbb{N}$ are given by $\Psi^i(r, m) = (\Psi^i(r), \psi^i(m) + \varepsilon^i(r))$ for a sequence of operations $\varepsilon^i : R \to M$ which satisfy the following properties

1. $\varepsilon^1(r) = 0$,
2. $\varepsilon^i(r + s) = \varepsilon^i(r) + \varepsilon^i(s)$,
3. $\varepsilon^i(rs) = \Psi^i(s)\varepsilon^i(r) + \Psi^i(r)\varepsilon^i(s) + f(\Psi^i(r), \Psi^i(s)) - \psi^i(f(r, s))$,
4. $\varepsilon^{ij}(r) = \psi^i\varepsilon^j(r) + \varepsilon^i\Psi^j(r)$,

for all $r, s \in R$ and $i, j \in \mathbb{N}$.

Assuming we have two Ψ-ring extensions $(X, \varepsilon, f), (\overline{X}, \overline{\varepsilon}, \overline{f})$ which are equivalent, together with a Ψ-ring map $\phi : X \to \overline{X}$ with $\phi(r, m) = (r, m + g(r))$ for some $g : R \to M$. We have that ϕ being a homomorphism tells us that

$$g(r + r') = g(r) + g(r'),$$

$$f(r, r') - \overline{f}(r, r') = rg(r') - g(rr') + g(r)r'.$$

We also have $\phi(\Psi^i) = \overline{\Psi}^i(\phi)$ for all $i \in \mathbb{N}$, which tells us that

$$\varepsilon^i(r) - \overline{\varepsilon}^i(r) = \psi^i(g(r)) - g(\Psi^i(r)).$$

We denote the set of equivalence classes of the additively split Ψ-ring extensions of R by M by $\mathrm{AExt}_\Psi(R, M)$.

Definition 3.10. An *additively and multiplicatively split* Ψ*-ring extension* of R by M is a Ψ-ring extension of R by M

$$0 \longrightarrow M \xrightarrow{\alpha} X \xrightarrow{\beta} R \longrightarrow 0$$

where β has a section which is an additive and multiplicative homomorphism.

As a commutative ring $X = R \rtimes M$, i.e. $f = 0$ above. The Ψ-operations $\Psi^i : X \to X$ for $i \in \mathbb{N}$ are given by $\Psi^i(r,m) = (\Psi^i(r), \psi^i(m) + \varepsilon^i(r))$ for a sequence of operations $\varepsilon^i : R \to M$ such that

1. $\varepsilon^1(r) = 0$,
2. $\varepsilon^i(r+s) = \varepsilon^i(r) + \varepsilon^i(s)$,
3. $\varepsilon^i(rs) = \Psi^i(s)\varepsilon^i(r) + \Psi^i(r)\varepsilon^i(s)$,
4. $\varepsilon^{ij}(r) = \psi^i\varepsilon^j(r) + \varepsilon^i\Psi^j(r)$,

for all $r, s \in R$ and $i, j \in \mathbb{N}$. Note that conditions 2 and 3 tell us that $\varepsilon^i \in \mathrm{Der}(\underline{R}, \underline{M^i})$ where M^i denotes the Ψ-module over R with M as an abelian group and the action of R given by $(r,m) \mapsto \Psi^i(r)m$, for $r \in R, m \in M$.

Assume we have two additively and multiplicatively split Ψ-ring extensions (X, ε), $(\overline{X}, \overline{\varepsilon})$ which are equivalent, together with a Ψ-ring map $\phi : X \to \overline{X}$ with $\phi(r,m) = (r, m + g(r))$ for some $g : R \to M$. Since ϕ is a ring homomorphism we get that $g \in \mathrm{Der}(\underline{R}, \underline{M})$. Since ϕ is a map of Ψ-rings we get that

$$\varepsilon^i(r) - \overline{\varepsilon}^i(r) = \psi^i(g(r)) - g(\Psi^i(r)),$$

for all $i \in \mathbb{N}$.

We denote the set of equivalence classes of the additively and multiplicatively split Ψ-ring extensions of R by M by $\mathrm{MExt}_\Psi(R, M)$.

Example 3.11. *Let R and M be such that $\Psi^i = Id = \psi^i$ for all $i \in \mathbb{N}$, then*

$$\mathrm{MExt}_\Psi(R, M) \cong \prod_{p\ prime} \mathrm{Der}(\underline{R}, \underline{M}).$$

Lemma 3.12. *There exist exact sequences*

$$0 \longrightarrow \mathrm{MExt}_\Psi(R, M) \xrightarrow{w} \mathrm{Ext}_\Psi(R, M) \xrightarrow{u} H^1_{AQ}(\underline{R}, \underline{M}) \longrightarrow \frac{H^1_{AQ}(\underline{R}, \underline{M})}{Im(u)} \longrightarrow 0$$

$$0 \longrightarrow \mathrm{MExt}_\Psi(R, M) \xrightarrow{w} \mathrm{AExt}_\Psi(R, M) \xrightarrow{u} Harr^2(\underline{R}, \underline{M}) \longrightarrow \frac{Harr^2(\underline{R},\underline{M})}{Im(u)} \longrightarrow 0$$

where w is the inclusion, and u maps the class of a Ψ-ring extension to the class of its underlying extension.

Proof. We only need to check exactness at $\mathrm{Ext}_\Psi(R, M)$ and $\mathrm{AExt}_\Psi(R, M)$. A class in $\mathrm{Ext}_\Psi(R, M)$ or $\mathrm{AExt}_\Psi(R, M)$ belongs to the kernel of u if the underlying class is the trivial class. The additively and multiplicatively split extensions are precisely the Ψ-ring extensions whose underlying extension is trivial. Exactness follows. □

From the definitions, we see that $\mathrm{Ext}_\Psi(R, M) \supseteq \mathrm{AExt}_\Psi(R, M) \supseteq \mathrm{MExt}_\Psi(R, M)$.

If R and M are both special, then we say that a Ψ-ring extension

$$0 \longrightarrow M \xrightarrow{\alpha} X \xrightarrow{\beta} R \longrightarrow 0$$

is *special* if X is also special.

We denote the set of equivalence classes of the special Ψ-ring extensions of R by M by $\mathrm{Ext}_{\Psi_s}(R, M)$. Similarly, we can define $\mathrm{AExt}_{\Psi_s}(R, M)$ and $\mathrm{MExt}_{\Psi_s}(R, M)$.

3.5 Crossed Ψ-extensions

A *crossed Ψ-module* consists of a Ψ-ring C_0, a Ψ-module C_1 over C_0 and a map of Ψ-modules

$$C_1 \xrightarrow{\partial} C_0 ,$$

which satisfies the property

$$\partial(c)c' = c\partial(c'),$$

for $c, c' \in C_1$. In other words, a crossed Ψ-module is a chain algebra which is non-trivial only in dimensions 0 and 1. Since $C_2 = 0$ the condition $\partial(c)c' = c\partial(c')$ is equivalent to the Leibnitz relation

$$0 = \partial(cc') = \partial(c)c' - c\partial(c').$$

We can define a product by

$$c * c' := \partial(c)c',$$

for $c, c' \in C_1$. This gives us a Ψ-ring structure on C_1 and $\partial : C_1 \to C_0$ is a map of Ψ-rings.

Let $\partial : C_1 \to C_0$ be a crossed Ψ-module. We let $M = \mathrm{Ker}(\partial)$ and $R = \mathrm{Coker}(\partial)$ Then the image $\mathrm{Im}(\partial)$ is an ideal of C_0, $MC_1 = C_1M = 0$ and M has a well-defined Ψ-module structure over R.

A *crossed* Ψ*-extension* of R by M is an exact sequence

$$0 \longrightarrow M \xrightarrow{\alpha} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\gamma} R \longrightarrow 0$$

where $\partial : C_1 \to C_0$ is a crossed Ψ-module, γ is a map of Ψ-rings, and the Ψ-module structure on M coincides with the one induced from the crossed Ψ-module. We denote the category of crossed Ψ-extensions of R by M by $Cross_\Psi(R, M)$. We let $\pi_0 Cross_\Psi(R, M)$ denote the connected components of the category $Cross_\Psi(R, M)$.

An *additively split crossed* Ψ*-extension* of R by M is a crossed Ψ-extension

$$0 \longrightarrow M \xrightarrow{\omega} C_1 \xrightarrow{\rho} C_0 \xrightarrow{\pi} R \longrightarrow 0 \tag{3.1}$$

such that all the arrows in the exact sequence 3.1 are additively split. We denote the connected components of the category of additively split crossed Ψ-extensions of R by M by $\pi_0 ACross_\Psi(R, M)$.

An *additively and multiplicatively split crossed* Ψ*-extension* of R by M is a crossed Ψ-extension

$$0 \longrightarrow M \xrightarrow{\omega} C_1 \xrightarrow{\rho} C_0 \xrightarrow{\pi} R \longrightarrow 0$$

such that π is additively and multiplicatively split. We denote the connected components of the category of additively and multiplicatively split crossed Ψ-extensions of R by M by $\pi_0 MCross_\Psi(R, M)$.

3.6 Deformation of Ψ-rings

In this section, we apply Gerstenhaber and Schack's definition of a deformation of a diagram of algebras [7] to the case of Ψ-rings.

Definition 3.13. Let

$$\alpha_t = \alpha_0 + t\alpha_1 + t^2\alpha_2 + \dots$$

be a deformation of $\underline{R}$, i.e. be a formal power series, in which each $\alpha_k : R \times R \to R$ is a bilinear map, α_0 is the multiplication in R and α_t is associative and commutative.

For each $i \in \mathbb{N}$, let

$$\Psi_t^i = \psi_0^i + t\psi_1^i + t^2\psi_2^i + \dots$$

be a formal power series, in which each ψ_k^i is a function

$$\psi_k^i : R \to R,$$

satisfying

1. $\psi_0^i(r) = \Psi^i(r)$,
2. $\psi_k^1(r) = 0$,
3. $\psi_k^i(r+s) = \psi_k^i(r) + \psi_k^i(s)$,
4. $\sum_{h=0}^{k} \psi_h^i \alpha_{k-h}(r,s) = \sum_{h=0}^{k} \sum_{l=0}^{k-h} \alpha_h(\psi_l^i(r), \psi_{k-h-l}^i(s))$,
5. $\psi_k^{ij}(r) = \sum_{l=0}^{k} \psi_l^i \circ \psi_{k-l}^j(r)$,

for all $i, j, k \in \mathbb{N}$ and $r, s \in R$. We call (α_t, Ψ_t^*) a Ψ-*ring deformation* of R.

We call (α_1, ψ_1^*) the *infinitesimal deformation* of (α_t, Ψ_t^*). The infinitesimal Ψ-ring deformation (α_1, ψ_1^*) is identified with the additively split Ψ-ring extensions of R by R by setting $f = \alpha_1$ and $\varepsilon^i = \psi_1^i$ for all $i \in \mathbb{N}$.

Definition 3.14. We define a *formal automorphism* of the Ψ-ring R to be a formal power series

$$\Phi_t = \phi_0 + t\phi_1 + t^2\phi_2 + \dots$$

where each $\phi_k : R \to R$ such that

1. $\phi_0(r) = r$,
2. $\phi_k(r+s) = \phi_k(r) + \phi_k(s)$.

Two Ψ-ring deformations (α_t, Ψ_t^*) and $(\overline{\alpha}_t, \overline{\Psi}_t^*)$ are *equivalent* if there exists a formal automorphism Φ_t such that $\Phi_t\alpha_t(r,s) = \overline{\alpha}_t(\Phi_t r, \Phi_t s)$ and $\Phi_t\Psi_t^* = \overline{\Psi}_t^*\Phi_t$.

If two Ψ-ring deformations (α_t, Ψ_t^*) and $(\overline{\alpha}_t, \overline{\Psi}_t^*)$ are equivalent, then the differences satisfy $\alpha_1(r,s) - \overline{\alpha}_1(r,s) = r\phi_1(s) - \phi_1(rs) + s\phi_1(r)$ and $\psi_1^i - \overline{\psi}_1^i = \overline{\Psi^i}\phi_1 - \phi_1\Psi^i$ for all $i \in \mathbb{N}$. Hence the equivalence classes of the infinitesimal Ψ-ring deformations are identified with the equivalence classes of the additively split Ψ-ring extensions, $\mathrm{AExt}_\Psi(R,R)$.

Yau [20] defined the cohomology of λ-rings in order to study deformations with respect to the Ψ-operations corresponding to the λ-ring. Here, I provide an alternative definition to Yau's definition. A deformation of the Ψ-operations should be a Ψ-ring deformation (α_t, Ψ_t^*) where α_t is the trivial deformation. If we let $\alpha_k = 0$ for all $k \geq 1$ in the definition of a Ψ-ring deformation then we get the following definition.

Definition 3.15. For each $i \in \mathbb{N}$, let

$$\Psi_t^i = \psi_0^i + t\psi_1^i + t^2\psi_2^i + \dots$$

be a formal power series, in which each ψ_k^i is a function

$$\psi_k^i : R \to R,$$

such that

1. $\psi_0^i(r) = \Psi^i(r)$,

2. $\psi_k^1(r) = 0$ for $k \geq 1.$,

3. $\psi_k^i(r+s) = \psi_k^i(r) + \psi_k^i(s)$,

4. $\psi_k^i(rs) = \sum_{l=0}^{k} \psi_l^i(r)\psi_{k-l}^i(s)$,

5. $\psi_k^{ij}(r) = \sum_{l=0}^{k} \psi_l^i \circ \psi_{k-l}^j(r)$,

for all $i, j, k \in \mathbb{N}$ and $r, s \in R$. We call Ψ_t^* a Ψ-*operation deformation* of R.

The infinitesimal Ψ-operation deformation ψ_1^* is identified with the additively and multiplicatively split Ψ-ring extensions of R by R by setting $\varepsilon^i = \psi_1^i$ for all $i \in \mathbb{N}$.

If two Ψ-operation deformations Ψ_t^* and $\overline{\Psi}_t^*$ are equivalent, then the difference satisfies $\psi_1^i - \overline{\psi}_1^i = \overline{\Psi^i}\phi_1 - \phi_1\Psi^i$ for all $i \in \mathbb{N}$. Note that now $\Phi_t(rs) = \Phi_t(r)\Phi_t(s)$ so we get that $\phi_1 \in \mathrm{Der}(\underline{R}, \underline{R})$. Hence the equivalence classes of the infinitesimal Ψ-operation deformations are identified with the equivalence classes of the additively and multiplicatively split Ψ-ring extensions, $\mathrm{MExt}_\Psi(R, R)$.

Chapter 4

λ-rings

4.1 Introduction

In this chapter, only the material in this section and section 4.6 is already known (see [1], [14] and [20]) and everything else is new and original material. Note that in our notation $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

In this chapter, we start by introducing the concept of a pre-λ-ring. After giving the definition, we will look at some examples of pre-λ-rings. Later, we introduce the definition of λ-rings, which are pre-λ-rings which satisfy some additional axioms. Then we will look at which of the pre-λ-ring structures also give us λ-rings.

Definition 4.1. A *pre-λ-ring* is a commutative ring R with identity 1, together with a sequence of operations $\lambda^i : R \to R$, for $i \in \mathbb{N}_0$, satisfying

1. $\lambda^0(r) = 1$,
2. $\lambda^1(r) = r$,
3. $\lambda^i(r + s) = \Sigma_{k=0}^{i} \lambda^k(r)\lambda^{i-k}(s)$,

for all $r, s \in R$ and $i \in \mathbb{N}_0$.

To be able to describe examples of pre-λ-rings or λ-rings it is often useful to consider, for $r \in R$, the formal power series in the variable t

$$\begin{aligned}\lambda_t(r) &= \sum_{i=0}^{\infty} \lambda^i(r)t^i \\ &= \lambda^0(r) + \lambda^1(r)t + \lambda^2(r)t^2 + \ldots\end{aligned}$$

Note that

$$\begin{aligned}\lambda_t(r+s) &= \lambda^0(r+s) + \lambda^1(r+s)t + \lambda^2(r+s)t^2 + \lambda^3(r+s)t^3 \ldots \\ &= 1 + (r+s)t + \Sigma_{k=0}^{2}\lambda^k(r)\lambda^{2-k}(s)t^2 + \Sigma_{k=0}^{3}\lambda^k(r)\lambda^{3-k}(s)t^3 + \ldots \\ &= (1 + rt + \lambda^2(r)t^2 + \ldots)(1 + st + \lambda^2(s)t^2 + \ldots) \\ &= \lambda_t(r)\lambda_t(s).\end{aligned}$$

This gives us an equivalent definition of a pre-λ-ring.

Definition 4.2. A *pre-λ-ring* is a commutative ring R with identity 1, together with a sequence of operations $\lambda^i : R \to R$, for $i \in \mathbb{N}_0$, satisfying

1. $\lambda^0(r) = 1$,
2. $\lambda^1(r) = r$,
3. $\lambda_t(r+s) = \lambda_t(r)\lambda_t(s)$, where $\lambda_t(r) = \sum_{i\geq 0} \lambda^i(r)t^i$,

for all $r, s \in R$ and $i \in \mathbb{N}_0$.

Example 4.3. *We can get a pre-λ-ring structure on $\mathbb{Z}$ by taking*

$$\lambda_t(r) = (1 + t + n_2t^2 + n_3t^3 + \ldots)^r,$$

where $1 + t + n_2t^2 + n_3t^3 + \ldots$ is a power series with integer coefficients.

We can get a pre-λ-ring structure on $\mathbb{R}$ by taking either

1. *$\lambda_t(r) = (1 + t + n_2t^2 + n_3t^3 + \ldots)^r$, where $1 + t + n_2t^2 + n_3t^3 + \ldots$ is a power series with integer coefficients, or*

2. $\lambda_t(r) = e^{tr}$.

The λ-ring axioms involve some universal polynomials. We are now going to introduce the elementary symmetric functions in order to define these universal polynomials.

Definition 4.4. Let $\xi_1, \xi_2, \dots, \xi_q; \eta_1, \eta_2, \dots, \eta_r$ be indeterminates. Define s_i and σ_j to be the elementary symmetric functions of the $\xi_i' s, \eta_j' s$, i.e.

$$(1 + s_1 t + s_2 t^2 + \dots +) = \Pi_i (1 + \xi_i t),$$

$$(1 + \sigma_1 t + \sigma_2 t^2 + \dots +) = \Pi_j (1 + \eta_j t).$$

Let $P_k(s_1, s_2, \dots, s_k; \sigma_1, \sigma_2, \dots, \sigma_k)$ be the coefficient of t^k in $\Pi_{i,j}(1 + \xi_i \eta_j t)$.
Let $P_{k,l}(s_1, s_2, \dots, s_{kl})$ be the coefficient of t^k in $\Pi_{1 \leq i_1 < \dots < i_l \leq q}(1 + \xi_{i_1} \xi_{i_2} \dots \xi_{i_l} t)$.

Example 4.5. *See also appendix B.*

- $P_1(s_1; \sigma_1) = s_1 \sigma_1$,
- $P_2(s_1, s_2; \sigma_1, \sigma_2) = s_1^2 \sigma_2 - 2 s_2 \sigma_2 + s_2 \sigma_1^2$,
- $P_{1,1}(s_1) = s_1$,
- $P_{1,2}(s_1, s_2) = P_{2,1}(s_1, s_2) = s_2$,
- $P_{2,2}(s_1, s_2, s_3, s_4) = s_1 s_3 - s_4$.

Definition 4.6. A *λ-ring* is a commutative ring R with identity 1, together with a sequence of operations $\lambda^i : R \to R$, for $i \in \mathbb{N}_0$, satisfying

1. R is a pre-λ-ring,
2. $\lambda_t(1) = 1 + t$,
3. $\lambda^i(rs) = P_i(\lambda^1(r), \lambda^2(r), \dots, \lambda^i(r), \lambda^1(s), \dots, \lambda^i(s))$,
4. $\lambda^i(\lambda^j(r)) = P_{i,j}(\lambda^1(r), \dots, \lambda^{ij}(r))$,

for all $r, s \in R$ and $i, j \in \mathbb{N}_0$.

Since λ^1 is the identity, it follows that $P_{k,1}(s_1, \ldots, s_k) = P_{1,k}(s_1, \ldots, s_k) = s_k$. In general, $P_{k,j} \neq P_{j,k}$, so the λ-operations do not commute.

Example 4.7. *The simplest example of a λ-ring is $\mathbb{Z}$, together with binomial coefficients $\lambda^i(r) = \binom{r}{i}$. The additional axioms for λ-rings eliminate the more exotic pre-λ-ring structures. From 4.3, the only λ-rings are taking $\lambda_t(r) = (1+t)^r$, which gives us a λ-ring structure on $\mathbb{Z}$ or $\mathbb{R}$.*

Corollary 4.8 (Some properties of λ-rings)**.** *1. The characteristic of R is zero.*

2. $\lambda^i(1) = 0$ for $i \geq 2$.

Proof of corollary. 1. Let j be any integer.

$$\lambda_t(j) = \lambda_t(\underbrace{1 + 1 + \ldots + 1}_{\text{j times}}) = \lambda_t(1)^j = (1+t)^j \neq 0.$$

2. This follows from 4.6.1.

□

A map of λ-rings $R_1 \to R_2$, is a ring homomorphism $f : R_1 \to R_2$, such that $\lambda^i(f(r)) = f(\lambda^i(r))$ for all $r \in R_1$ and $i \in \mathbb{N}_0$. The class of all λ-rings and maps of λ-rings form the category of λ-rings, which we denote by $\lambda\mathfrak{-rings}$.

The λ-operations are often difficult to work with as they are neither additive nor multiplicative. We can get ring maps from the λ-operations, which are the Adams operations $\Psi^i : R \to R$ for $i \in \mathbb{N}$, defined by the Newton formula

$$\Psi^i(r) - \lambda^1(r)\Psi^{i-1}(r) + \ldots + (-1)^{i-1}\lambda^{i-1}(r)\Psi^1(r) + (-1)^i i\lambda^i(r) = 0.$$

Example 4.9. *See also appendix A.*

$$\begin{aligned}
\Psi^1(r) &= \lambda^1(r), \\
\Psi^2(r) &= \Psi^1(r)\lambda^1(r) - 2\lambda^2(r), \\
\Psi^3(r) &= \Psi^2(r)\lambda^1(r) - \Psi^1(r)\lambda^2(r) + 3\lambda^3(r), \\
\Psi^4(r) &= \Psi^3(3)\lambda^1(r) - \Psi^2(r)\lambda^2(r) + \Psi^1(r)\lambda^3(r) - 4\lambda^4(r). \\
&\vdots
\end{aligned}$$

By rearranging and making substitutions we get the following

$$
\begin{aligned}
\Psi^1(r) &= \lambda^1(r) = r, \\
\Psi^2(r) &= r^2 - 2\lambda^2(r), \\
\Psi^3(r) &= r^3 - 3r\lambda^2(r) + 3\lambda^3(r), \\
\Psi^4(r) &= r^4 - 4r^2\lambda^2(r) + 4r\lambda^3(r) - 4\lambda^4(r) + 2(\lambda^2(r))^2, \\
&\vdots
\end{aligned}
$$

It is known that in general

$$
\Psi^i(r) = det \begin{pmatrix}
r & 1 & 0 & 0 & \dots & 0 \\
2\lambda^2(r) & r & 1 & 0 & \dots & 0 \\
3\lambda^3(r) & \lambda^2(r) & r & 1 & 0 & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & 0 \\
\vdots & \vdots & & \lambda^2(r) & r & 1 \\
i\lambda^i(r) & \lambda^{i-1}(r) & \dots & \dots & \lambda^2(r) & r
\end{pmatrix}.
$$

Theorem 4.10. *If R is a λ-ring then the Adams operations give us a special Ψ-ring structure on R, which we denote by R_Ψ.*

We will require the following useful theorem from [14] (p.49).

Theorem 4.11. *Let R be a torsion-free pre-λ-ring. Let $\Psi^i : R \to R$ be the corresponding Adams operations. If R together with the Ψ-operations form a Ψ-ring, then R is a λ-ring.*

The proof of this theorem can also be found in [14].

Example 4.12. *Consider the simplest example of a λ-ring, $\mathbb{Z}$, together with binomial coefficients $\lambda^i(r) = \binom{r}{i}$. The Adams operations give us $\Psi^i(r) = r$ for all $r \in \mathbb{Z}$ and $i \in \mathbb{N}$, which we have already seen gives us a Ψ-ring structure on $\mathbb{Z}$.*

4.2 λ-modules

For usual rings, we have modules which provide us with the coefficients for the cohomology. We now define the λ-modules for λ-rings which provide us with the coefficients for the λ-ring cohomology.

Definition 4.13. M is a λ-*module* over the λ-ring R if M is an R-module together with a sequence of abelian group homomorphisms $\Lambda^i : M \to M$, for $i \in \mathbb{N}$, satisfying

1. $\Lambda^1(m) = m$,
2. $\Lambda^i(rm) = \Psi^i(r)\Lambda^i(m)$,
3. $\Lambda^{ij}(m) = (-1)^{(i+1)(j+1)}\Lambda^i\Lambda^j(m)$,

for all $m \in M, r \in R$ and $i, j \in \mathbb{N}$.

Let M, N be two λ-modules over R. A map of λ-modules is a module homomorphism $f : M \to N$ such that $\Lambda^i f(m) = f\Lambda^i(m)$ for all $m \in M$ and $i \in \mathbb{N}_0$. We let $R{-}\mathfrak{mod}_\lambda$ denote the category of all λ-modules over R.

The main motivation for our definition of a λ-module is as follows. First we let R and X be two λ-rings and $\beta : X \to R$ be a map of λ-rings. Assume $M = \operatorname{Ker}\beta$ is a square-zero ideal. Since $\lambda^i(0) = 0$, for $i > 0$, there are maps $\Lambda^i : M \to M$, for $i > 0$, which make the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\alpha"] \arrow[d, "\Lambda^i"] & X \arrow[r, "\beta"] \arrow[d, "\lambda^i"] & R \arrow[r] \arrow[d, "\lambda^i"] & 0 \\
0 \arrow[r] & M \arrow[r, "\alpha"] & X \arrow[r, "\beta"] & R \arrow[r] & 0.
\end{tikzcd}$$

The properties of the Λ-operations follow from the properties of the λ-operations. For example,

$$\begin{aligned}
\alpha\lambda^i(rm) =& \lambda^i\alpha(rm) \\
=& \lambda^i(x\alpha(m)),
\end{aligned}$$

for some $x \in X$ with $\beta(x) = r$. Therefore,

$$\alpha\lambda^i(rm) = P_i(\lambda^1(x), \ldots, \lambda^i(x), \lambda^1(\alpha(m)), \ldots \lambda^i(\alpha(m))).$$

However $\alpha(m)\alpha(n) = 0$ for all $m, n \in M$ so most of the terms vanish leaving

$$\alpha\lambda^i(rm) = \alpha\Psi^i(r)\Lambda^i(m).$$

For the rest of this chapter, we let R denote a λ-ring and $M \in R - \mathfrak{mod}_\lambda$. We let $\underline{R}$ denote the underlying commutative ring of R, and $\underline{M}$ denote the underlying $\underline{R}$-module of M.

Example 4.14. *In general, R is not a λ-module over itself unless the multiplication in R is trivial. However we can consider the sequence of operations $\Lambda^i : R \to R$ given by $\Lambda^i(r) = (-1)^{(i+1)}\Psi^i(r)$. With these Λ-operations R is a λ-module over R.*

Theorem 4.15. *The Adams operation $\psi^n : M \to M$ given by*

$$\psi^n(m) = (-1)^{(n+1)} n\Lambda^n(m),$$

give us a special Ψ-module structure on M over R_Ψ, which we denote by M_Ψ.

Proof.

1. $\psi^1(m) = \Lambda^1(m) = m$,
2. $\psi^i(m_1 + m_2) = (-1)^{i+1} i\Lambda^i(m_1 + m_2) = (-1)^{i+1} i\Lambda^i(m_1) + (-1)^{i+1} i\Lambda^i(m_2)$ $= \psi^i(m_1) + \psi^i(m_2)$,
3. $\psi^i(rm) = (-1)^{i+1} i\Lambda^i(rm) = (-1)^{i+1} i\Psi^i(r)\Lambda^i(m) = \Psi^i(r)\psi^i(m)$,
4. $\psi^i(\psi^j(m)) = \psi^i((-1)^{(j+1)} j\Lambda^j(m)) = (-1)^{(i+j)} ij\Lambda^i(\Lambda^j(m))$ $= (-1)^{(ij+1)} ij\Lambda^{ij}(m) = \psi^{(ij)}(m)$.

□

We will require the following useful lemma.

Lemma 4.16.

$$\sum_{i=1}^{\nu-1}[(-1)^{i+1}\chi_i(r,m)\Psi^{\nu-i}(r)+(-1)^{\nu+1}i\lambda^i(r)\Lambda^{\nu-i}(m)]=0,$$

for all $r\in R, m\in M$ *and* $\nu\geq 2$, *where* $\chi_i(r,m)=\sum_{j=1}^{i}\Lambda^j(m)\lambda^{i-j}(r)$.

Proof. We are going to use proof by induction on ν. Consider the case when $\nu=2$.

$$\begin{aligned}LHS =&(-1)^2\chi_1(r,m)\Psi^1(r)+(-1)^3\lambda^1(r)\Lambda^1(m)\\ =&mr-rm\\ =&0.\end{aligned}$$

We are also going to consider the case when $\nu=3$.

$$\begin{aligned}LHS =&\chi_1(r,m)\Psi^2(r)+\lambda^1(r)\Lambda^2(m)-\chi_2(r,m)\Psi^1(r)+2\lambda^2(r)\Lambda^1(m)\\ =&m[r^2-2\lambda^2(r)]+r\Lambda^2(m)-[mr+\Lambda^2(m)]r+2\lambda^2(r)m\\ =&0.\end{aligned}$$

Now assume that

$$\sum_{i=1}^{\nu-k-1}[(-1)^{i+1}\chi_i(r,m)\Psi^{\nu-k-i}(r)+(-1)^{\nu-k+1}i\lambda^i(r)\Lambda^{\nu-k-i}(m)]=0,$$

for $1\leq k\leq\nu-2$.

It follows that

$$\begin{aligned}
&\sum_{i=1}^{\nu-1}[(-1)^{i+1}\chi_i(r,m)\Psi^{\nu-i}(r)+(-1)^{\nu+1}i\lambda^i(r)\Lambda^{\nu-i}(m)]\\
=&\sum_{i=1}^{\nu-1}(-1)^{\nu}(\nu-i)\lambda^{\nu-i}(r)\chi_i(r,m)\\
&+\sum_{i=1}^{\nu-2}(-1)^{i+1}\chi_i(r,m)[\sum_{j=1}^{\nu-i-1}(-1)^{j+1}\lambda^j(r)\Psi^{\nu-i-j}(r)]+\sum_{i=1}^{\nu-1}(-1)^{\nu+1}i\lambda^i(r)\Lambda^{\nu-i}(m)\\
=&\sum_{i=1}^{\nu-2}(-1)^{\nu}i\lambda^i(r)[\sum_{j=1}^{\nu-i-1}\Lambda^j(m)\lambda^{\nu-i-j}(r)]+\sum_{i=1}^{\nu-2}\chi_i(r,m)[\sum_{j=1}^{\nu-i-1}(-1)^{j+i}\lambda^j(r)\Psi^{\nu-i-j}(r)]\\
=&\sum_{k=1}^{\nu-2}\lambda^k(r)[\sum_{i=1}^{\nu-k-1}(-1)^{\nu}i\lambda^i(r)\lambda^{\nu-k-i}(r)]+\sum_{k=1}^{\nu-2}\lambda^k(r)[\sum_{i=1}^{\nu-k-1}(-1)^{i+k}\chi_i(r,m)\Psi^{\nu-k-i}(r)]\\
=&\sum_{k=1}^{\nu-2}(-1)^{k+1}\lambda^k(r)[\sum_{i=1}^{\nu-k-1}[(-1)^{i+1}\chi_i(r,m)\Psi^{\nu-k-i}(r)+(-1)^{\nu-k+1}i\lambda^i(r)\Lambda^{\nu-k-i}(m)]\\
=&0.
\end{aligned}$$

as required. □

Lemma 4.17. *The set $R \rtimes M$ with*

$$(r,m)+(s,n)=(r+s,m+n),$$
$$(r,m)(s,n)=(rs,rn+ms),$$

together with maps $\lambda^i : R \rtimes M \to R \rtimes M$ for $i \in \mathbb{N}_0$ given by

$$\lambda^i(r,m)=(\lambda^i(r),f_i(r,m)),$$

for a sequence of maps $f_i : R \rtimes M \to M$, for $i \in \mathbb{N}_0$, is a pre-λ-ring if and only if

1. *$f_0(r,m)=0$,*
2. *$f_1(r,m)=m$,*
3. *$f_i((r,m)+(s,n))=\sum_{j=0}^{i}(f_j(r,m)\lambda^{i-j}(s)+\lambda^j(r)f_{i-j}(s,n))$.*

Proof of lemma. $\underline{R}$ is a commutative ring with identity, and $\underline{M}$ is an $\underline{R}$-module. Then we know that $\underline{R} \rtimes \underline{M}$ is a commutative ring with identity. So we only have to check the properties of $\lambda^i : R \rtimes M \to R \rtimes M$.

1. $\lambda^0(r,m) = (\lambda^0(r), f_0(r,m))$.
   Hence $\lambda^0(r,m) = (1,0)$ if and only if $f_0(r,m) = 0$,

2. $\lambda^1(r,m) = (\lambda^1(r), f_1(r,m))$.
   Hence $\lambda^1(r,m) = (r,m)$ if and only if $f_1(r,m) = m$,

3. $\lambda^i((r,m)+(s,n)) = \lambda^i(r+s, m+n) = (\lambda^i(r+s), f_i(r+s, m+n))$
   $\sum_{j=0}^{i} \lambda^j(r,m)\lambda^{i-j}(s,n) = \sum_{j=0}^{i}(\lambda^j(r), f_j(r,m))(\lambda^{i-j}(s), f_{i-j}(s,n))$
   $= \sum_{j=0}^{i}(\lambda^j(r)\lambda^{i-j}(s), f_j(r,m)\lambda^{i-j}(s) + \lambda^j(r)f_{i-j}(s,n))$
   Hence
   $\lambda^i((r,m)+(s,n)) = \sum_{j=0}^{i} \lambda^j(r,m)\lambda^{i-j}(s,n)$ if and only if
   $f_i((r,m)+(s,n)) = \sum_{j=0}^{i}(f_j(r,m)\lambda^{i-j}(s) + \lambda^j(r)f_{i-j}(s,n))$.

□

Lemma 4.18. *The set $R \rtimes M$ together with maps $\lambda^i : R \rtimes M \to R \rtimes M$, for $i \in \mathbb{N}_0$, given by*

$$\lambda^i(r,m) = \begin{cases} (1,0) & \textit{for } i = 0, \\ (\lambda^i(r), \sum_{j=1}^{i} \Lambda^j(m)\lambda^{i-j}(r)) & \textit{for } i \in \mathbb{N}, \end{cases}$$

gives us a λ-ring.

We call this λ-ring the *semi-direct product* of R and M, denoted by $R \rtimes_\lambda M$.

Proof. We start by showing this is a pre-λ ring by using lemma 4.17 with

$$f_i(r,m) = \begin{cases} 0 & \text{for } i = 0, \\ \sum_{j=1}^{i} \Lambda^j(m)\lambda^{i-j}(r) & \text{for } i \geq 1. \end{cases}$$

Clearly properties 1 and 2 hold, so we only have to check 3. Let $i \geq 2$ then

$$\begin{aligned}
f_i((r,m)+(s,n)) =& f_i(r+s,m+n) = \sum_{j=1}^{i} \Lambda^j(m+n)\lambda^{i-j}(r+s) \\
=& \sum_{j=1}^{i}((\Lambda^j(m)+\Lambda^j(n))\sum_{k=0}^{i-j}\lambda^k(r)\lambda^{i-j-k}(s) \\
=& \sum_{j=1}^{i}\sum_{k=0}^{i-j}(\Lambda^i(m)\lambda^k(r)\lambda^{i-j-k}(s)+\Lambda^i(n)\lambda^k(r)\lambda^{i-j-k}(s)) \\
=& \sum_{j=1}^{i}\sum_{k=1}^{j}\Lambda^k(m)\lambda^{j-k}(r)\lambda^{i-j}(s)+\sum_{j=1}^{i-1}\sum_{k=1}^{i-j}\lambda^j(r)\Lambda^k(n)\lambda^{i-j-k}(s) \\
&+\sum_{k=1}^{i}\Lambda^k(n)\lambda^{i-k}(s)\lambda^0(r) \\
=& \sum_{j=1}^{i-1}\sum_{k=1}^{j}\Lambda^k(m)\lambda^{j-k}(r)\lambda^{i-j}(s)+\sum_{k=1}^{i}\Lambda^k(m)\lambda^{i-k}(r)\lambda^0(s) \\
&+\sum_{j=1}^{i-1}\sum_{k=1}^{i-j}\lambda^j(r)\lambda^k(n)\lambda^{i-j-k}(s)+\sum_{k=1}^{i}\Lambda^k(n)\lambda^{i-k}(s)\lambda^0(r) \\
=& \sum_{j=1}^{i-1}(f_j(r,m)\lambda^{i-j}(s)\lambda^j(r)f_{i-j}(s,n)) \\
&+f_i(r,m)\lambda^0(s)+f_i(s,n)\lambda^0(r)+\lambda^i(s)f_0(r,m)+\lambda^i(r)f_0(s,n) \\
=& \sum_{j=0}^{i}(f_j(r,m)\lambda^{i-j}(s)+\lambda^j(r)f_{i-j}(s,n)).
\end{aligned}$$

So we have proved that $R \rtimes_\lambda M$ is a pre-λ-ring. Checking the last two axioms is reduced to checking the following the following universal polynomial identities hold.

- $P_i(\lambda^1(r,m),\ldots,\lambda^i(r,m),\lambda^1(s,n),\ldots,\lambda^i(s,n))$
  $= (P_i(\lambda^1(r),\ldots,\lambda^i(r),\lambda^1(s),\ldots,\lambda^i(s)),$
  $\sum_{k=1}^{i-1} P_{i-k}(\lambda^1(r),\ldots,\lambda^{i-k}(r),\lambda^1(s),\ldots,\lambda^{i-k}(s))[\Psi^k(s)\Lambda^k(m)+\Psi^k(r)\Lambda^k(n)]),$
- $P_{i,j}(\lambda^1(r,m),\ldots,\lambda^{ij}(r,m)) = (P_{i,j}(\lambda^1(r),\ldots,\lambda^{ij}(r)),$
  $\sum_{k=1}^{i}\sum_{l=1}^{j}(-1)^{(k+1)(l+1)}\Lambda^{kl}(m)\Psi^k(\lambda^{j-l}(r))P_{(i-k),j}(\lambda^1(r),\ldots,\lambda^{(i-k)j}(r))).$

We are going to start by considering the case where R is a free λ-ring and M is free as a λ-module over R.

Our aim is to show that the Adams operations give us the Ψ-ring structure on $R \rtimes M$ with $\Psi^\nu(r,m) = (\Psi^\nu(r), \psi^\nu(m))$ by using induction on ν. Then theorem 4.11 tells us that $R \rtimes_\lambda M$ is a λ-ring and the universal polynomial identities hold.

Consider the case when $\nu = 1$

$$\Psi^1(r,m) = (r,m) = (\Psi^1(r), \psi^1(m)).$$

Consider the case when $\nu = 2$

$$\Psi^2(r,m) = (r^2, 2rm) - 2\lambda^2(r,m) = (r^2 - 2\lambda^2(r), -2\Lambda^2(m)) = (\Psi^2(r), \psi^2(m)).$$

Assume that $\Psi^{\nu-k}(r,m) = (\Psi^{\nu-k}(r), \psi^{\nu-k}(m))$ for $1 \leq k \leq \nu - 1$. It follows that

$$
\begin{aligned}
\Psi^{\nu}(r,m) =& \sum_{j=1}^{\nu-1}(-1)^{\nu-j+1}(\lambda^{\nu-j}(r), \sum_{k=1}^{\nu-j}\lambda^{\nu-j-k}(r)\Lambda^{k}(m))(\Psi^{j}(r),\psi^{j}(m)) \\
&+ (-1)^{\nu+1}\nu(\lambda^{\nu}(r), \sum_{k=1}^{\nu}\lambda^{\nu-k}(r)\Lambda^{k}(m)) \\
=& \sum_{j=1}^{\nu-1}(-1)^{\nu-j+1}(\lambda^{\nu-j}(r)\Psi^{j}(r), \Psi^{j}(r)\sum_{k=1}^{\nu-j}\lambda^{\nu-j-k}(r)\Lambda^{k}(m) \\
&+ \Psi^{j}(m)\lambda^{\nu-j}(r)) + (-1)^{\nu+1}\nu(\lambda^{\nu}(r), \sum_{k=1}^{\nu}\lambda^{\nu-k}(r)\Lambda^{k}(m)) \\
=& (\Psi^{\nu}(r), \sum_{j=1}^{\nu-1}(-1)^{\nu-j+1}[\lambda^{\nu-j}(r)\Psi^{j}(m) + \Psi^{j}(r)\sum_{k=1}^{\nu-j}\lambda^{\nu-j-k}(r)\Lambda^{k}(m)] \\
&+ (-1)^{\nu+1}\nu[\sum_{k=1}^{\nu}\lambda^{\nu-k}(r)\Lambda^{k}(m)]) \\
=& (\Psi^{\nu}(r), \sum_{j=1}^{\nu-1}[(-1)^{j+1}\Psi^{\nu-i}(r)\chi_{i}(r,m) + (-1)^{\nu+1}j\lambda^{j}(r)\Lambda^{\nu-j}(m)] \\
&+ (-1)^{\nu+1}\nu\Lambda^{\nu}(m)) \\
=& (\Psi^{\nu}(r), (-1)^{\nu+1}\nu\Lambda^{\nu}(m)) = (\Psi^{\nu}(r), \psi^{\nu}(m)),
\end{aligned}
$$

as required.

Now consider the case where R is a free λ-ring and M is an arbitrary λ-module over R. Choose P a free λ-module over R with a surjective homomorphism $P \twoheadrightarrow M$, this gives us a surjective homomorphism $R \rtimes_{\lambda} P \twoheadrightarrow R \rtimes_{\lambda} M$. Since the universal polynomial identities hold on $R \rtimes_{\lambda} P$ they also hold on $R \rtimes_{\lambda} M$.

Now we can consider the case when R is an arbitrary λ-ring and M is a λ-module over R. Any λ-ring is the quotient of a free λ-ring, therefore R is the quotient of a free λ-ring F. There exists a surjective homomorphism $F \rtimes_{\lambda} M \twoheadrightarrow R \rtimes_{\lambda} M$. Since the universal polynomial identities hold on $F \rtimes_{\lambda} M$ they also hold on $R \rtimes_{\lambda} M$. Hence $R \rtimes_{\lambda} M$ is a λ-ring. Moreover we proved that $(R \rtimes_{\lambda} M)_{\Psi} = R_{\Psi} \rtimes_{\Psi} M_{\Psi}$. $\square$

4.3 λ-derivations

Definition 4.19. A λ-*derivation* of R with values in M is an additive homomorphism $d : R \to M$ such that

1. $d(rs) = rd(s) + d(r)s$,
2. $d(\lambda^i(r)) = \Lambda^i(d(r)) + \Lambda^{i-1}(d(r))\lambda^1(r) + \ldots + \Lambda^2(d(r))\lambda^{i-2}(r) + \Lambda^1(d(r))\lambda^{i-1}(r)$,

for all $r, s \in R$, and $i \in \mathbb{N}$. We let $\mathrm{Der}_\lambda(R, M)$ denote the set of all λ-derivations of R with values in M.

Example 4.20. *Let $\mathbb{Z}_\lambda[x]$ be the free λ-ring on one generator x, and let $M \in \mathbb{Z}_\lambda[x]-\mathfrak{mod}_\lambda$.*

$$Der_\lambda(\mathbb{Z}_\lambda[x], M) \cong M.$$

$\mathbb{Z}_\lambda[x] = \mathbb{Z}[x_1, x_2, \ldots]$ together with operations determined by $\lambda^i(x_1) = x_i$. For any λ-derivation, $d : \mathbb{Z}_\lambda[x] \to M$, we have that

$$\begin{aligned} d(x_1) &= m, \\ d(x_i) &= \sum_{j=1}^{i} \Lambda^j(m) x_{i-j}, \end{aligned}$$

where $m \in M$ and $x_0 = 1$.

Theorem 4.21. *There is a one-to-one correspondence between the sections of $R \rtimes_\lambda M \xrightarrow{\pi} R$ and the λ-derivations $d : R \to M$.*

Proof of theorem. Assume we have a section of π, then we have the following

$$\begin{tikzcd} R \rtimes_\lambda M \arrow[r, shift left, "\pi"] & R, \arrow[l, shift left, "\sigma"] \end{tikzcd}$$

where $\pi\sigma = Id_R$. Hence $\sigma(r) = (r, d(r))$ for some $d : R \to M$. The properties

$$\begin{aligned} d(r+s) =& d(r) + d(s), \\ d(rs) =& d(r)s + rd(s), \end{aligned}$$

follow from σ being a ring homomorphism. However σ also preserves the λ-ring structure, meaning that $\lambda^i\sigma(r) = \sigma\lambda^i(r)$. We know that

$$\begin{aligned}
\lambda^i\sigma(r) &= \lambda^i(r, d(r)) = \lambda^i((r,0) + (0, d(r))) = \Sigma_{j=0}^{i}\lambda^j(r,0)\lambda^{i-j}(0, d(r)) \\
&= \Sigma_{j=0}^{i-1}(0, \lambda^j(r)\Lambda^{i-j}(d(r))) + (\lambda^i(r), 0) = (\lambda^i(r), \Sigma_{j=0}^{i-1}\lambda^j(r)\Lambda^{i-j}(d(r))) \\
\sigma\lambda^i(r) &= (\lambda^i(r), d(\lambda^i(r))).
\end{aligned}$$

Hence $\lambda^i\sigma(r) = \sigma\lambda^i(r)$ if and only if $d\lambda^i(r) = \Sigma_{j=0}^{i-1}\lambda^j(r)\Lambda^{i-j}(d(r))$. This tells us that if σ is a section of π, then we have a λ-derivation d.

Conversely, if we have a λ-derivation $d : R \to M$, then $\sigma(r) = (r, d(r))$ is a section of π. $\square$

Theorem 4.22. *The λ-derivations of R with values in M are also Ψ-derivations of R_Ψ with values in M_Ψ.*

Proof. Let $d : R \to M$ be a λ-derivation, we are going to use induction on ν to show $\psi^\nu(d(r)) = d(\Psi^\nu(r))$ for all $\nu \geq 1$.

Consider the case when $\nu = 1$

$$\psi^1(d(r)) = d(r) = d(\Psi^1(r)).$$

Consider the case when $\nu = 2$.

$$d(\Psi^2(r)) = d(r^2 - 2\lambda^2(r)) = 2rd(r) - 2[\Lambda^2(d(r)) + d(r)r] = -2\Lambda^2(d(r)) = \psi^2(d(r)).$$

Also consider the case $\nu = 3$.

$$d(\Psi^3(r)) = d(r^3 - 3r\lambda^2(r) + 3\lambda^3(r)) = 3\Lambda^3(d(r)) = \psi^3(d(r)).$$

Assume that $\psi^{\nu-k}(d(r)) = d(\Psi^{\nu-k}(r))$ for $1 \leq k \leq \nu - 1$.

$$\begin{aligned}
d(\Psi^{\nu}(r)) =& \sum_{i=1}^{\nu-1}(-1)^{i+1}d(\lambda^i(r))\Psi^{\nu-i}(r) + \sum_{i=1}^{\nu-1}(-1)^{i+1}\lambda^i(r)d(\Psi^{\nu-i}(r)) \\
&+ (-1)^{\nu+1}\nu d(\lambda^{\nu}(r)) \\
d(\Psi^{\nu}(r)) - \psi^{\nu}(d(r)) =& \sum_{i=1}^{\nu-1}(-1)^{i+1}[\sum_{j=1}^{i}\Lambda^j(d(r))\lambda^{i-j}(r)]\Psi^{\nu-i}(r) \\
&+ \sum_{i=1}^{\nu-1}(-1)^{i+1}\lambda^i(r)[(-1)^{\nu-i+1}(\nu-i)\Lambda^{\nu-i}(d(r))] \\
&+ (-1)^{\nu+1}\nu[\sum_{j=1}^{\nu-1}\Lambda^j(d(r))\lambda^{\nu-j}(r)] \\
=& \sum_{i=1}^{\nu-1}[(-1)^{i+1}\chi_i(r,d(r))\Psi^{\nu-i}(r) + (-1)^{\nu+1}i\lambda^i(r)\Lambda^{\nu-i}(d(r))] \\
=& 0.
\end{aligned}$$

Hence $d(\Psi^{\nu}(r)) = \psi^{\nu}(d(r))$. $\square$

Theorem 4.23. *If M is $\mathbb{Z}$-torsion-free then the Ψ-derivations of R_{Ψ} with values in M_{Ψ} are also λ-derivations of R with values in M*

$$\mathrm{Der}_{\lambda}(R, M) = \mathrm{Der}_{\Psi}(R_{\Psi}, M_{\Psi}).$$

Proof. Let M be $\mathbb{Z}$-torsion-free and $d : R_{\Psi} \to M_{\Psi}$ be a Ψ-derivation. We are going to use induction on ν to show $d(\lambda^{\nu}(r)) = \sum_{i=1}^{\nu}\Lambda^i(d(r))\lambda^{\nu-i}(r)$ for $\nu \in \mathbb{N}$.

Consider the case when $\nu = 1$.

$$\Lambda^1(d(r)) = d(r) = d(\lambda^1(r)).$$

Consider the case when $\nu = 2$.

$$\begin{aligned} d(\Psi^2(r)) &= \psi^2(d(r)) \\ d(r^2 - 2\lambda^2(r)) &= -2\Lambda^2(d(r)) \\ 2[d(\lambda^2(r)) - rd(r) - \Lambda^2(d(r))] &= 0 \\ 2[d(\lambda^2(r)) - \sum_{i=1}^{2} \Lambda^i(d(r))\lambda^{2-i}(r)] &= 0 \\ d(\lambda^2(r)) - \sum_{i=1}^{2} \Lambda^i(d(r))\lambda^{2-i}(r) &= 0. \end{aligned}$$

Assume that $d(\lambda^{\nu-k}(r)) = \sum_{i=1}^{\nu-k} \Lambda^i(d(r))\lambda^{\nu-i-k}(r)$ for $1 \leq k \leq \nu - 1$, we want to show that $\nu d(\lambda^\nu(r)) = \nu \sum_{i=1}^{\nu} \Lambda^i(d(r))\lambda^{\nu-i}(r)$. From $\psi^v(d(r)) = d(\Psi^v(r))$ we get $\nu(\Lambda^\nu(d(r)) - d(\lambda^\nu(r))) = \sum_{i=1}^{\nu-1}(-1)^{i+v}[d(\lambda^i(r))\Psi^{\nu-i}(r) + \lambda^i(r)d(\Psi^{\nu-i}(r))]$. Therefore we have to show that

$$\begin{aligned} &(-1)^\nu \nu \sum_{i=1}^{\nu-1} \Lambda^i(d(r))\lambda^{\nu-i}(r) \\ =&\sum_{i=1}^{\nu-1}(-1)^{i+1}d(\lambda^i(r))\Psi^{\nu-i}(r) + \sum_{i=1}^{\nu-1}(-1)^{i+1}\lambda^i(r)d(\Psi^{\nu-i}(r)) \\ =&\sum_{i=1}^{\nu-1}(-1)^{i+1}[\sum_{j=1}^{i} \Lambda^j(d(r))\lambda^{i-j}(r)] \cdot [\sum_{k=1}^{\nu-i-1} (-1)^{k+1}\lambda^k(r)\Psi^{\nu-i-k}(r) \\ &+ (-1)^{\nu-i-1}(\nu-i)\lambda^{\nu-i}(r)] + (-1)^\nu \sum_{i=1}^{\nu-1} i\Lambda^i(d(r))\lambda^{\nu-i}(r). \end{aligned}$$

Hence it is sufficient to show that
$\sum_{i=1}^{\nu-2}(-1)^{i+1}\chi_i(r,d(r))[\sum_{k=1}^{\nu-i-1}(-1)^{k+1}\lambda^k(r)\Psi^{\nu-i-k}(r)]$
$+ \sum_{i=1}^{\nu-1}(-1)^{\nu+1}(i-\nu)\lambda^{\nu-i}(r)\chi_i(r,d(r)) + (-1)^{\nu+1}\sum_{i=1}^{\nu-1} i\lambda^i(r)\Lambda^{\nu-i}(d(r))] = 0$,
with χ_i as in lemma 4.16. We get that

$$
\begin{aligned}
&\sum_{i=1}^{\nu-2}(-1)^{i+1}\chi_i(r,d(r))[\sum_{k=1}^{\nu-i-1}(-1)^{k+1}\lambda^k(r)\Psi^{\nu-i-k}(r)]\\
&+\sum_{i=1}^{\nu-1}(-1)^{\nu+1}(i-\nu)\lambda^{\nu-i}(r)\chi_i(r,d(r))+(-1)^{\nu+1}\sum_{i=1}^{\nu-1}i\lambda^i(r)\Lambda^{\nu-i}(d(r))]\\
=&\sum_{i=1}^{\nu-2}\chi_i(r,d(r))[(-1)^{i+1}[\sum_{k=1}^{\nu-i-1}(-1)^{k+1}\lambda^k(r)\Psi^{\nu-i-k}(r)\\
&+(-1)^{\nu-i-1}(\nu-i)\lambda^{\nu-i}(r)-\Psi^{\nu-i}(r)]]\\
=&0,
\end{aligned}
$$

as required. $\square$

4.4 λ-ring extensions

We have seen in proposition 2.13 that the André-Quillen cohomology $H^1_{AQ}(\underline{R},\underline{M})$ classifies the extensions of $\underline{\mathrm{R}}$ by $\underline{\mathrm{M}}$. In this section, we develop the λ-analogue of extensions.

Definition 4.24. A *λ-ring extension* of R by M is an extension of $\underline{R}$ by $\underline{M}$

$$0 \longrightarrow M \xrightarrow{\alpha} X \xrightarrow{\beta} R \longrightarrow 0$$

where X is a λ-ring, β is a map of λ-rings and $\alpha\Lambda^n = \lambda^n\alpha$ for all $n \in \mathbb{N}$.

Two λ-ring extensions $(X), (X')$ with R, M fixed are said to be *equivalent* if there exists a map of λ-rings $\phi : X \to X'$ such that the following diagram commutes.

$$
\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "\phi"] & R \arrow[r] \arrow[d, equal] & 0\\
0 \arrow[r] & M \arrow[r] & X' \arrow[r] & R \arrow[r] & 0
\end{tikzcd}
$$

We denote the set of equivalence classes of λ-ring extensions of R by M by $Ext_\lambda(R, M)$.

The Harrison cohomology $Harr^1(\underline{R}, \underline{M})$ classifies the additively split extensions of $\underline{R}$ by $\underline{M}$. We can also define the λ-analogue of these types of extensions.

Definition 4.25. Let R be a λ-ring and $M \in R$-mod$_\lambda$ then an *additively split λ-ring extension* of R by M is a λ-ring extension of R by M

$$0 \longrightarrow M \xrightarrow{\alpha} X \xrightarrow{\beta} R \longrightarrow 0$$

where β has a section that is an additive homomorphism.

Multiplication in $X = R \oplus M$ has the form $(r, m)(r', m') = (rr', mr' + rm' + f(r, r'))$, where $f : R \times R \to M$ is some bilinear map. Associativity in X gives us

$$0 = rf(r', r'') - f(rr', r'') + f(r, r'r'') - f(r, r')r''.$$

Commutativity in X gives us

$$f(r, r') = f(r', r).$$

The λ-operations $\lambda^\nu : R \rtimes M \to R \rtimes M$ for $\nu \in \mathbb{N}_0$ are given by $\lambda^\nu(r, m) = (\lambda^\nu(r), \sum_{i=1}^{\nu} \Lambda^i(m)\lambda^{\nu-i}(r) + \epsilon^\nu(r))$ for a sequence of operations $\epsilon^\nu : R \to M$ which satisfy the following properties

1. $\epsilon^0(r) = \epsilon^1(r) = 0$,
2. $\epsilon^\nu(r+s) = \sum_{i=0}^{\nu}[\epsilon^i(r)\lambda^{\nu-i}(s) + \epsilon^{\nu-i}(s)\lambda^i(r)]$,
3. $\epsilon^\nu(1) = 0$,
4. $P_i(\lambda^1(r, m), \ldots, \lambda^i(s, n))$
   $= (\lambda^i(rs), \sum_{j=1}^{i}(\Psi^j(s)\Lambda^j(m) + \Psi^j(r)\Lambda^j(n) + \Lambda^j(f(r, r')))\lambda^{i-j}(rs) + \epsilon^j(rs))$,
5. $P_{i,j}(\lambda^1(r, m), \ldots, \lambda^{ij}(r, m))$
   $= (\lambda^i(\lambda^j(r)), \sum_{k=1}^{i} \Lambda^k(\sum_{a=1}^{j}(\Lambda^a(m)\lambda^{j-a}(r) + \epsilon^j(r))\lambda^{i-k}(\lambda^j(r))) + \epsilon^i(\lambda^j(r)))$.

Assuming we have two additively split λ-ring extensions (X, ε, f),(X', ε', f') which are equivalent, together with a λ-ring map $\phi : X \to X'$ with $\phi(r, m) = (r, m + g(r))$

for some $g : R \to M$. We have that ϕ being a homomorphism tells us that

$$g(r + r') = g(r) + g(r'),$$

$$f(r, r') - f'(r, r') = rg(r') - g(rr') + g(r)r'.$$

We also have $\phi(\lambda^\nu) = \lambda^\nu(\phi)$ for all $\nu \in \mathbb{N}_0$, which tells us that

$$\varepsilon^\nu(r) - \varepsilon'^\nu(r) = \sum_{i=1}^{\nu} \Lambda^i(g(r))\lambda^{\nu-i}(r) - g(\lambda^\nu(r)).$$

We denote the set of equivalence classes of additively split λ-ring extensions of R by M by $AExt_\lambda(R, M)$.

In order to describe the properties of λ-ring extensions we need to define the partial derivatives of the universal polynomials, see appendix C for examples.

We can use the universal polynomials to define continuous functions

$$P_i : \mathbb{R}^{2i} \to \mathbb{R},$$

$$P_{i,j} : \mathbb{R}^{ij} \to \mathbb{R}.$$

For example $P_2 : \mathbb{R}^4 \to \mathbb{R}$ is given by

$$P_2(x_1, x_2, x_3, x_4) = x_1^2 x_4 - 2x_2x_4 + x_2x_3^2.$$

We can take the partial derivatives of these functions which are again polynomials. We call these new polynomials the *partial derivatives* of the universal polynomials. For example

$$\begin{aligned}\frac{\partial P_2(x_1, x_2, x_3, x_4)}{\partial x_1} &= 2x_1x_4,\\ \frac{\partial P_2(x_1, x_2, x_3, x_4)}{\partial x_2} &= x_3^2 - 2x_4,\end{aligned}$$

For $1 \le j \le i$, we let

$$\frac{\partial P_i(r,s)}{\partial \lambda^j(r)} := \frac{\partial P_i(\lambda^1(r), \ldots, \lambda^i(r), \lambda^1(s), \ldots, \lambda^i(s))}{\partial \lambda^j(r)}.$$

Since the polynomials P_i are symmetric, we can let

$$\frac{\partial P_i(r,s)}{\partial \lambda^j(s)} := \frac{\partial P_i(s,r)}{\partial \lambda^j(s)}.$$

In our examples

$$\begin{aligned}
\frac{\partial P_2(r,s)}{\partial \lambda^1(r)} &= \frac{\partial P_2(\lambda^1(r), \lambda^2(r), \lambda^1(s), \lambda^2(s))}{\partial \lambda^1(r)} &&= && 2r\lambda^2(s),\\
\frac{\partial P_2(r,s)}{\partial \lambda^2(r)} &= \frac{\partial P_2(\lambda^1(r), \lambda^2(r), \lambda^1(s), \lambda^2(s))}{\partial \lambda^2(r)} &&= && s^2 - 2\lambda^2(s).
\end{aligned}$$

Similarly, for $1 \le k \le ij$, we let

$$\frac{\partial P_{i,j}(r)}{\partial \lambda^k(r)} := \frac{\partial P_i(\lambda^1(r), \ldots, \lambda^{ij}(r))}{\partial \lambda^k(r)}.$$

For example,

$$\frac{\partial P_{2,2}(x_1, x_2, x_3, x_4)}{\partial x_1} = x_3.$$

So it follows that

$$\frac{\partial P_{2,2}(r)}{\partial \lambda^1(r)} = \lambda^3(r).$$

These partial derivatives appear because of the multiplication in $R \rtimes M$. Consider the following

$$(r,m)^2 = (r^2, 2rm),$$

$$(r,m)^3 = (r^3, 3r^2m).$$

Definition 4.26. An *additively and multiplicatively split λ-ring extension* of R by M is a λ-ring extension of R by M

$$0 \longrightarrow M \longrightarrow X \longrightarrow R \longrightarrow 0$$

where β has a section that is an additive and multiplicative homomorphism.

As a commutative ring $X = R \rtimes M$, the sequence of operations $\lambda^\nu : R \rtimes M \to R \rtimes M$ for $\nu \in \mathbb{N}_0$ are given by $\lambda^\nu(r,m) = (\lambda^\nu(r), \sum_{i=1}^{\nu} \Lambda^i(m)\lambda^{\nu-i}(r) + \epsilon^\nu(r))$ for a sequence of operations $\epsilon^\nu : R \to M$ such that

1. $\epsilon^0(r) = \epsilon^1(r) = 0$,
2. $\epsilon^\nu(r+s) = \sum_{i=0}^{\nu}[\epsilon^i(r)\lambda^{\nu-i}(s) + \epsilon^{\nu-i}(s)\lambda^i(r)]$,
3. $\epsilon^\nu(1) = 0$,
4. $\epsilon^\nu(rs) = \sum_{i=1}^{\nu}[\epsilon^i(r)\frac{\partial P_\nu(r,s)}{\partial \lambda^i(r)} + \epsilon^i(s)\frac{\partial P_\nu(r,s)}{\partial \lambda^i(s)}]$,
5. $\epsilon^k(\lambda^\nu(r)) = \sum_{i=1}^{\nu k} \epsilon^i(r)\frac{\partial P_{\nu,k}(r)}{\partial \lambda^i(r)} - \sum_{j=1}^{k} \Lambda^j(\epsilon^\nu(r))\lambda^{k-j}(\lambda^\nu(r))$.

Two additively and multiplicatively split λ-ring extensions (X,ϵ),(X',ϵ') with R, M fixed are said to be *equivalent* if there exists a map of λ-rings $\phi : X \to X'$ such that the following diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "\phi"] & R \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] & X' \arrow[r] & R \arrow[r] & 0
\end{tikzcd}$$

Assuming we have two additively and multiplicatively split λ-ring extensions (X,ϵ),(X',ϵ') which are equivalent, together with a λ-ring map $\phi : X \to X'$ with $\phi(r,m) = (r, m + g(r))$ for some $g : R \to M$. We also have ϕ being a homomorphism which tells us that $g \in Der(R,M)$. We also have $\phi(\lambda^\nu) = \lambda^\nu(\phi)$ for all ν, which tells us that

$$\varepsilon^\nu(r) - \varepsilon'^\nu(r) = \sum_{i=1}^{\nu} \Lambda^i(g(r))\lambda^{\nu-i}(r) - g(\lambda^\nu(r)).$$

We denote the set of equivalence classes of additively and multiplicatively split λ-ring extensions of R by M by $MExt_\lambda(R,M)$.

Theorem 4.27. *If $\epsilon^\nu : R \to M$ gives us an additively and multiplicatively split λ-ring extension of R by M, then $\varepsilon^\nu : R \to M$ with*

$$\varepsilon^\nu(r) = \sum_{i=1}^{\nu-1}(-1)^{i+1}[\epsilon^i(r)\Psi^{\nu-i}(r) + \lambda^i(r)\varepsilon^{\nu-i}(r)] + (-1)^{\nu+1}\nu\epsilon^\nu(r),$$

give us an additively and multiplicatively split Ψ-ring extension of R_Ψ by M_Ψ.

Proof. If $\epsilon^\nu : R \to M$ gives an additively and multiplicatively split λ-ring extension of R by M, then $\lambda^\nu : R \rtimes M \to R \rtimes M$ given by $\lambda^\nu(r,m) = (\lambda^\nu(r), \sum_{i=1}^{\nu} \Lambda^i(m)\lambda^{\nu-i}(r) + \epsilon^\nu(r))$ is a λ-ring and hence the Adams operations give the Ψ-ring with operations $\Psi^\nu : R \rtimes M \to R \rtimes M$ given by $\Psi^\nu(r,m) = (\Psi^\nu(r), \psi^\nu(m) + \varepsilon^\nu(r))$ which is an additively and multiplicatively split Ψ-ring extension of R_Ψ by M_Ψ. $\square$

4.5 Crossed λ-extensions

A *crossed λ-module* consists of a λ-ring C_0, a λ-module C_1 over C_0 and a map of λ-modules

$$C_1 \xrightarrow{\partial} C_0 \ ,$$

which satisfies the property

$$\partial(c)c' = c\partial(c'),$$

for $c, c' \in C_1$. In other words, a crossed λ-module is a chain algebra which is non-trivial only in dimensions 0 and 1. Since $C_2 = 0$ the condition $\partial(c)c' = c\partial(c')$ is equivalent to the Leibnitz relation

$$0 = \partial(cc') = \partial(c)c' - c\partial(c').$$

We can define a product by

$$c * c' := \partial(c)c',$$

for $c, c' \in C_1$. This gives us a λ-ring structure on C_1 and $\partial : C_1 \to C_0$ is a map of λ-rings.

Let $\partial : C_1 \to C_0$ be a crossed λ-module. We let $M = \mathrm{Ker}(\partial)$ and $R = \mathrm{Coker}(\partial)$ Then the image $\mathrm{Im}(\partial)$ is an ideal of C_0, $MC_1 = C_1 M = 0$ and M has a well-defined λ-module structure over R.

A *crossed λ-extension* of R by M is an exact sequence

$$0 \longrightarrow M \xrightarrow{\alpha} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\gamma} R \longrightarrow 0$$

where $\partial : C_1 \to C_0$ is a crossed λ-module, γ is a map of λ-rings, and the λ-module structure on M coincides with the one induced from the crossed λ-module. We denote the category of crossed λ-extensions of R by M by $Cross_\lambda(R, M)$. We let $\pi_0 Cross_\lambda(R, M)$ denote the connected components of the category $Cross_\lambda(R, M)$.

An *additively split crossed λ-extension* of R by M is a crossed λ-extension

$$0 \longrightarrow M \xrightarrow{\omega} C_1 \xrightarrow{\rho} C_0 \xrightarrow{\pi} R \longrightarrow 0 \tag{4.1}$$

such that all the arrows in the exact sequence 4.1 are additively split. We denote the connected components of the category of additively split crossed λ-extensions of R by M by $\pi_0 ACross_\lambda(R, M)$.

An *additively and multiplicatively split crossed λ-extension* of R by M is an additively split crossed λ-extension

$$0 \longrightarrow M \xrightarrow{\omega} C_1 \xrightarrow{\rho} C_0 \xrightarrow{\pi} R \longrightarrow 0$$

such that π is additively and multiplicatively split. We denote the connected components of the category of additively and multiplicatively split crossed λ-extensions of R by M by $\pi_0 MCross_\lambda(R, M)$.

4.6 Yau cohomology for λ-rings

In 2005, Donald Yau published a paper entitled, "Cohomology of λ-rings" [20], in which he developed a cohomology theory for λ-rings. In this section we describe Yau's cochain complex and what it computes.

Let R be a λ-ring. We let $End(R)$ denote the algebra of $\mathbb{Z}$-linear endomorphisms of R, where the product is given by composition. We let $\overline{End}(R)$ denote the subalgebra of $End(R)$ which consists of the linear endomorphisms f of R which satisfy the condition,

$$f(r)^p \equiv f(r^p) \qquad \bmod pR,$$

for each prime p and every $r \in R$.

Yau defined $C^0_{Yau}(R)$ be the underlying group of $\overline{End}(R)$. He defined $C^1_{Yau}(R)$ be the set of functions $f : \mathbb{N} \to End(R)$ satisfying the condition $f(p)(R) \subset pR$ for each prime p. Then for $\nu \geq 2$ he set $C^\nu_{Yau}(R)$ to be the set of functions $f : \mathbb{N}^\nu \to End(R)$. For $\nu \in \mathbb{N}_0$, the coboundary map, $\delta^\nu : C^\nu_{Yau} \to C^{\nu+1}_{Yau}$, is given by the following

$$\begin{aligned}\delta^\nu(f)(m_0, \ldots, m_\nu) =& \Psi^{m_0} \circ f(m_1, \ldots, m_\nu) + \sum_{i=1}^{\nu} (-1)^i f(m_0, \ldots, m_{i-1}m_i, \ldots, m_\nu) \\ &+ (-1)^{\nu+1} f(m_0, \ldots, m_{\nu-1}) \circ \Psi^{m_\nu}.\end{aligned}$$

We say that the ν^{th} cohomology of the cochain complex (C_{Yau}, δ) is the ν^{th} Yau cohomology of R, denoted by $H^\nu_{Yau}(R)$.

From the cochain complex it is clear that

$$H^0_{Yau}(R) = \{f \in \overline{End}(R) : f\Psi^\nu = \Psi^\nu f \text{ for all } \nu \in \mathbb{N}\}.$$

We define the group of *Yau derivations* of R, denoted by $YDer_\lambda(R)$, to consist of the functions $f \in C^1_{Yau}(R)$ such that

$$f(ij) = \Psi^j \circ f(i) + f(j) \circ \Psi^i,$$

for all $i, j \in \mathbb{N}$. We define the group of *Yau inner-derivations* of R, denoted by $YIDer_\lambda(R)$, to consist of the functions $f : \mathbb{N} \to End(R)$ which are of the form

$$f(i) = \Psi^i \circ g - g \circ \Psi^i,$$

for some $g \in \overline{End}(R)$.

The first Yau cohomology is given by the quotient,

$$H^1_{Yau}(R) = \frac{YDer_\lambda(R)}{YIDer_\lambda(R)}.$$

Yau tells us that there exists a canonical surjection,

$$H^2_{Yau}(R) \twoheadrightarrow HH^2(\mathbb{Z}[\mathbb{N}], End(R)),$$

and for $\nu \geq 3$, there exists a canonical isomorphism,

$$H^\nu_{Yau}(R) \cong HH^\nu(\mathbb{Z}[\mathbb{N}], End(R)),$$

where $HH^\nu(\mathbb{Z}[\mathbb{N}], End(R))$ denotes the ν^{th} Hochschild cohomology of $\mathbb{Z}[\mathbb{N}]$ with coefficients in $End(R)$.

Yau defined his cohomology in order to study deformations of λ-rings.

We let

$$\Psi^*_t = \psi^*_0 + t\psi^*_1 + t^2\psi^*_2 + \ldots$$

be a formal power series, in which each ψ^*_i is a function

$$\psi^*_i : \mathbb{N} \to End(R),$$

satisfying the following properties. We let ψ^j_i denote $\psi^*_i(j)$.

1. $\psi^j_0(r) = \Psi^j(r)$,
2. $\psi^1_i = 0 \qquad \text{for } i \geq 1$,
3. $\psi^{kl}_i(r) = \sum_{j=0}^{i} \psi^k_j \circ \psi^l_{i-j}(r) \qquad \text{for } k, l \geq 1 \text{ and } i \geq 0$,
4. $\psi^p_i(r) \subset pR \qquad \text{for } i \geq 1 \text{ and } p \text{ prime}$.

Yau calls Ψ^*_t a *deformation* of R.

Note that the Gerstenhaber and Schack's definition we provided in 3.6 is very similar to Yau's definition but gives a different result. We would like to compare the results in the case when $\alpha_i = 0$ for $i \geq 1$. We omitted the condition $\psi_i^p(r) \subset pR$ for p prime, but introduced the condition $\psi_i^j(rs) = \sum_{k=0}^{i} \psi_k^j(r)\psi_{i-k}^j(s)$. This last condition makes things more complicated and may seem strange, but it is necessary to ensure that

$$\Psi_t^*(rs) = \Psi_t^*(r)\Psi_t^*(s).$$

Yau's condition gives us $\psi_1^i \in End(R)$. Gerstenhaber and Schack's condition gives us $\psi_1^i \in \mathrm{Der}(R, R^i)$ where R^i is the R-module with R as an abelian group and the following action of R

$$(r, a) \mapsto \Psi^i(r)a, \qquad \text{for } r \in R, a \in R^i.$$

Chapter 5

Harrison cohomology of diagrams of commutative algebras

5.1 Introduction

For this chapter we let I denote a small category. A category I is said to be *small* if the collection of morphisms is a set. We let i, j, k denote objects in I and we let $\alpha : i \to j$ and $\beta : j \to k$ denote morphisms in I.

Definition 5.1. A *diagram of commutative algebras* is a covariant functor

$$A : I \to \mathfrak{Com.alg},$$

where I is a small category, and $\mathfrak{Com.alg}$ is some category of commutative algebras. We call I the *shape* of the diagram.

If A, B are two covariant functors from I to $\mathfrak{Com.alg}$, then a map of diagrams is a natural transformation $\mu : A \to B$. We denote the category of diagrams of commutative algebras with shape I by $\mathfrak{Com.alg}^I$.

Definition 5.2. An A-module is a functor $M : I \to \mathfrak{Ab}$ such that for all $i \in I$ we have that $M(i) \in A(i)$-$\mathfrak{mod}$ and for all $\alpha \in I$ we have

$$M(\alpha)(a \cdot m) = A(\alpha)(a) \cdot M(\alpha)(m),$$

for all $a \in A(i), m \in M(i)$. We let $A - \mathfrak{mod}^I$ denote the category of all A-modules.

5.2 Natural System

Let $A : I \to \mathfrak{Com.alg}$ be a diagram of a commutative algebra, and M be an A-module. For any $n \geq 0$ there exists a natural system on I as follows

$$D_\alpha := C^n_{Harr}(A(i), \alpha^* M(j)),$$

where $(\alpha : i \to j) \in I$ and $M(j)$ is considered an $A(i)$-module via α. For any $(\beta : j \to j') \in I$, we have $\beta_* : D_\alpha \to D_{\beta\alpha}$ which is induced by $M(\beta) : M(j) \to M(j')$. For any $(\gamma : i' \to i) \in I$, we have $\gamma^* : D_\alpha \to D_{\alpha\gamma}$ which is induced by $A(\gamma) : A(i') \to A(i)$.

5.3 Bicomplex

Let $A : I \to \mathfrak{Com.alg}$ be a diagram of a commutative algebra, and M be an A-module. For each $i \in I$ we can consider the Harrison cochain complex of the commutative algebra $A(i)$ with coefficients in $M(i)$.

$$C^0_{Harr}(A(i), M(i)) \longrightarrow C^1_{Harr}(A(i), M(i)) \longrightarrow C^2_{Harr}(A(i), M(i)) \longrightarrow \dots$$

We can use this to construct the following bicomplex denoted by $C^{*,*}_{Harr}(I, A, M)$:

$$C^{p,q}_{Harr}(I, A, M) = \prod_{\alpha : i_0 \to \dots \to i_p} C^{q+1}_{Harr}(A(i_0)), \alpha^* M(i_p)),$$

for $p, q \geq 0$. The map $C^{p,q}_{Harr}(I, A, M) \to C^{p+1,q}_{Harr}(I, A, M)$ is the map in the Baues-Wirsching cochain complex, and the map $C^{p,q}_{Harr}(I, A, M) \to C^{p,q+1}_{Harr}(I, A, M)$ is the product of the Harrison coboundary maps.

$$\begin{tikzcd}
\vdots & \vdots \\
\prod_i C^3_{Harr}(A(i),M(i)) \arrow[r, "\delta"] \arrow[u] & \prod_{\alpha:i\to j} C^3_{Harr}(A(i),\alpha^*M(j)) \arrow[r] \arrow[u] & \dots \\
\prod_i C^2_{Harr}(A(i),M(i)) \arrow[r, "\delta"] \arrow[u, "\partial"] & \prod_{\alpha:i\to j} C^2_{Harr}(A(i),\alpha^*M(j)) \arrow[r] \arrow[u, "-\partial"'] & \dots \\
\prod_i C^1_{Harr}(A(i),M(i)) \arrow[r, "\delta"] \arrow[u, "\partial"] & \prod_{\alpha:i\to j} C^1_{Harr}(A(i),\alpha^*M(j)) \arrow[r] \arrow[u, "-\partial"'] & \dots
\end{tikzcd}$$

Let $(\alpha_n : i_n \to i_{n+1}) \in I$, and $\alpha = \alpha_p \dots \alpha_0 : i_0 \to i_{p+1}$. Then the coboundary map $\delta : C^{p,q}_{Harr}(I,A,M) \to C^{p+1,q}_{Harr}(I,A,M)$ is given by

$$\begin{aligned}
\delta(f)_{\alpha_{p+1},\dots,\alpha_0}(x_1,\dots,x_q) =& f_{\alpha_{p+1},\dots,\alpha_1}(A(\alpha_0)(x_1),\dots,A(\alpha_0)(x_q)) \\
&+ \sum_{k=0}^{p}(-1)^{k+1} f_{\alpha_{p+1},\dots,\alpha_{k+1}\alpha_k,\dots,\alpha_0}(x_1,\dots,x_q) \\
&+ (-1)^{p+2} M(\alpha_{p+1})(f_{\alpha_p,\dots,\alpha_0}(x_1,\dots,x_q)).
\end{aligned}$$

The coboundary map $\partial : C^{p,q}_{Harr}(I,A,M) \to C^{p,q+1}_{Harr}(I,A,M)$ is given by

$$\begin{aligned}
\partial(f)_{\alpha_p,\dots,\alpha_0}(x_1,\dots,x_{q+1}) =& A(\alpha)(x_1)\cdot f_{\alpha_p,\dots,\alpha_0}(x_2,\dots,x_{q+1}) \\
&+ \sum_{k=1}^{q}(-1)^k f_{\alpha_p,\dots,\alpha_0}(x_1,\dots,x_k x_{k+1},\dots,x_q) \\
&+ (-1)^{q+1} f_{\alpha_p,\dots,\alpha_0}(x_1,\dots,x_q)\cdot A(\alpha)(x_{q+1}).
\end{aligned}$$

Lemma 5.3. *The maps ∂ and δ are coboundary maps.*

$$\partial^2 = 0 = \delta^2.$$

Proof. $\partial(f) = \sum_{k=0}^{q+1}(-1)^k \partial_k(f)$ where

$$(\partial_k(f))(x_1,\ldots,x_{q+1}) = \begin{cases} A(\alpha)(x_1)\cdot f_{\alpha_p,\ldots,\alpha_0}(x_2,\ldots,x_{q+1}) & k=0, \\ f_{\alpha_p,\ldots,\alpha_0}(x_1,\ldots,x_k x_{k+1},\ldots,x_{q+1}) & 0<k<q+1, \\ f_{\alpha_p,\ldots,\alpha_0}(x_1,\ldots,x_q)\cdot A(\alpha)(x_{q+1}) & k=q+1. \end{cases}$$

$\delta(f) = \sum_{k=0}^{p+2}(-1)^k \delta_k(f)$ where

$$(\delta_k(f))(x_1,\ldots,x_q) = \begin{cases} f_{\alpha_{p+1},\ldots,\alpha_1}(A(\alpha_0)(x_1),\ldots,A(\alpha_0)(x_q)) & k=0, \\ f_{\alpha_{p+1},\ldots,\alpha_k\alpha_{k-1},\ldots,\alpha_0}(x_1,\ldots,x_q) & 0<k<p+2, \\ M(\alpha_{p+1})(f_{\alpha_p,\ldots,\alpha_0}(x_1,\ldots,x_q)) & k=p+2. \end{cases}$$

$\partial^2 = 0 = \delta^2$ follows from:

$$\partial_k\partial_l = \partial_l\partial_{k-1} \qquad 0\le l<k\le q+2,$$

$$\delta_k\delta_l = \delta_l\delta_{k-1} \qquad 0\le l<k\le p+2.$$

□

Lemma 5.4. *The coboundary maps ∂ and δ commute.*

$$\delta\partial = \partial\delta.$$

The proof is given on the next page.

Proof. Let $f \in C^{p,q}_{Harr}(I, A, M)$.

$$\begin{aligned}
\delta\partial(f) =& A(\alpha)(x_1) \cdot f_{\alpha_{p+1},\dots,\alpha_1}(A(\alpha_0)(x_2), \dots, A(\alpha_0)(x_{q+1})) \\
& + \sum_{k=1}^{q} (-1)^k f_{\alpha_{p+1},\dots,\alpha_1}(A(\alpha_0)(x_1), \dots, A(\alpha_0)(x_k x_{k+1}), \dots, A(\alpha_0)(x_{q+1})) \\
& + (-1)^{q+1} f_{\alpha_{p+1},\dots,\alpha_1}(A(\alpha_0)(x_1), \dots, A(\alpha_0)(x_q)) \cdot A(\alpha)(x_{q+1}) \\
& + \sum_{l=0}^{p} (-1)^{l+1} [A(\alpha)(x_1) \cdot f_{\alpha_{p+1},\dots,\alpha_{l+1}\alpha_l,\dots,\alpha_0}(x_2, \dots, x_{q+1}) \\
& + \sum_{k=1}^{q} (-1)^k f_{\alpha_{p+1},\dots,\alpha_{l+1}\alpha_l,\dots,\alpha_0}(x_1, \dots, x_k x_{k+1}, \dots, x_{q+1}) \\
& + (-1)^{q+1} f_{\alpha_{p+1},\dots,\alpha_{l+1}\alpha_l,\dots,\alpha_0}(x_1, \dots, x_q) \cdot A(\alpha)(x_{q+1})] \\
& + (-1)^{p+2} M(\alpha_{p+1})[A(\alpha_p \cdots \alpha_0)(x_1) \cdot f_{\alpha_p,\dots,\alpha_0}(x_2, \dots, x_{q+1}) \\
& + \sum_{k=1}^{q} (-1)^k f_{\alpha_p,\dots,\alpha_0}(x_1, \dots, x_k x_{k+1}, \dots, x_{q+1}) \\
& + (-1)^{q+1} f_{\alpha_p,\dots,\alpha_0}(x_1, \dots, x_q) \cdot A(\alpha_p \cdots \alpha_0)(x_{q+1})] \\
=& A(\alpha)(x_1) \cdot [f_{\alpha_{p+1},\dots,\alpha_1}(A(\alpha_0)(x_2), \dots, A(\alpha_0)(x_{q+1})) \\
& + \sum_{l=0}^{p} (-1)^{l+1} f_{\alpha_{p+1},\dots,\alpha_{l+1}\alpha_l,\dots,\alpha_0}(x_2, \dots, x_{q+1}) \\
& + (-1)^{p+2} M(\alpha_{p+1}) f_{\alpha_p,\dots,\alpha_0}(x_2, \dots, x_{q+1})] \\
& + \sum_{k=1}^{q} (-1)^k [f_{\alpha_{p+1},\dots,\alpha_1}(A(\alpha_0)(x_1), \dots, A(\alpha_0)(x_k x_{k+1}), \dots, A(\alpha_0)(x_{q+1})) \\
& + \sum_{l=0}^{p} (-1)^{l+1} f_{\alpha_{p+1},\dots,\alpha_{l+1}\alpha_l,\dots,\alpha_0}(x_1, \dots, x_k x_{k+1}, \dots, x_{q+1}) \\
& + (-1)^{p+2} M(\alpha_{p+1}) f_{\alpha_p,\dots,\alpha_0}(x_1, \dots, x_k x_{k+1}, \dots, x_{q+1})] \\
& + (-1)^{q+1} [f_{\alpha_{p+1},\dots,\alpha_1}(A(\alpha_0)(x_1), \dots, A(\alpha_0)(x_q)) \\
& + \sum_{l=0}^{p} (-1)^{l+1} f_{\alpha_{p+1},\dots,\alpha_{l+1}\alpha_l,\dots,\alpha_0}(x_1, \dots, x_q) \\
& + (-1)^{p+2} M(\alpha_{p+1})(f_{\alpha_p,\dots,\alpha_0}(x_1, \dots, x_q))] \cdot A(\alpha)(x_{q+1}) \\
=& \partial\delta(f).
\end{aligned}$$

$\square$

5.4 Harrison cohomology of diagrams of commutative algebras

Let $A : I \to \mathfrak{Com.alg}$ be a diagram of commutative algebras, and M be an A-module. We define the *Harrison cohomology* of A with coefficients in M, denoted by $Harr^*(I, A, M)$, to be the cohomology of the total complex of $C^{*,*}_{Harr}(I, A, M)$.

The spectral sequence of a bicomplex yields the following spectral sequence.

$$E_2^{p,q} = H^p_{BW}(I, \mathcal{H}^{q+1}_{Harr}(A, M)) \Rightarrow Harr^{p+q}(I, A, M),$$

where $\mathcal{H}^q_{Harr}(A, M)$ is the natural system on I whose value on $(\alpha : i \to j)$ is given by $Harr^q(A(i), \alpha^* M(j))$.

Definition 5.5. A *derivation* $d : A \to M$ is of the form $d = (d_i)_{i \in I}$ where each $d_i : A(i) \to M(i)$ is a derivation of $A(i)$ with values in $M(i)$ such that for all $(\alpha : i \to j) \in I$ we have that $M(\alpha)(d_i) = d_j(A(\alpha))$. We denote the set of all derivations of A with values in M by $\mathfrak{Der}(A, M)$.

Lemma 5.6.

$$Harr^0(I, A, M) \cong \mathfrak{Der}(A, M),$$

$$H^0_{BW}(I, \mathcal{H}^1_{Harr}(A, M)) \cong \mathfrak{Der}(A, M).$$

Definition 5.7. An *additively split extension* of A by M is an exact sequence of functors

$$0 \longrightarrow M \xrightarrow{q} X \xrightarrow{p} A \longrightarrow 0$$

where $X : I \to \mathfrak{Com.alg}$ such that for all $i \in I$ we get an additively split extension of $A(i)$ by $M(i)$.

$$0 \longrightarrow M(i) \xrightarrow{q(i)} X(i) \xrightarrow{p(i)} A(i) \longrightarrow 0$$

This means that there are additive homomorphisms $s(i) : A(i) \to X(i)$ for all $i \in I$ such that $s(i)$ is a section of $p(i)$. The sections induce additive isomorphisms $M(i) \oplus A(i) \approx X(i)$ where addition is given by $(m, a) + (m', a') = (m + m', a + a')$

and multiplication is given by

$$(m, a)(m', a') = (a'm + am' + f_i(a, a'), aa'),$$

where $f_i : A(i) \times A(i) \to M(i)$ is a bilinear map given by

$$f_i(a, a') = s(i)(a)s(i)(a') - s(i)(aa').$$

Associativity in $X(i)$ gives us

$$0 = af_i(a', a'') - f_i(aa', a'') + f_i(a, a'a'') - f_i(a, a')a''.$$

Commutativity in $X(i)$ gives us

$$f_i(a, a') = f_i(a', a).$$

For all $(\alpha : i \to j) \in I$ we identify $M(j)$ with $\mathrm{Ker}(p(j))$ and $M(\alpha)$ with the restriction of $X(\alpha)$ to get a map $\epsilon_\alpha : A(i) \to M(j)$ given by

$$\epsilon_\alpha(a) = X(\alpha)(s(i)(a)) - s(j)(A(\alpha)(a)),$$

which satisfies the following properties:

1. $\epsilon_{id}(a) = 0$,
2. $\epsilon_\alpha(a + a') = \epsilon_\alpha(a) + \epsilon_\alpha(a')$,
3. $\epsilon_\alpha(aa') = A(\alpha)(a)\epsilon_\alpha(a') + A(\alpha)(a')\epsilon_\alpha(a)$
   $+ f_j(A(\alpha)(a), A(\alpha)(a')) - M(\alpha)(f_i(a, a'))$,
4. $\epsilon_{\beta\alpha}(a) = M(\beta)(\epsilon_\alpha(a)) + \epsilon_\beta(A(\alpha)(a))$.

Two additively split extensions $(X), (X')$ with A, M fixed are said to be *equivalent* if there exists a map of diagrams $\phi : X \to X'$ such that the following diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "\phi"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] & X' \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

For all $i \in I$ we get that $\phi_i : X(i) \to X'(i)$ is a homomorphism of commutative algebras. Hence $\phi_i(m, a) = (m + g_i(a), a)$ for some $g_i : A \to M$ such that

$$g_i(a + a') = g_i(a) + g_i(a'),$$

$$f_i(a, a') - f_i'(a, a') = ag_i(a) - g_i(aa') + g_i(a)a'.$$

For all $\alpha \in I$ we get that

$$\epsilon_\alpha(a) - \epsilon_\alpha'(a) = M(\alpha)(g_i(a)) - g_j(A(\alpha)(a)).$$

We denote the set of equivalence classes of additively split extensions of A by M by $\mathfrak{AExt}(A, M)$.

An *additively and multiplicatively split extension* of A by M is an additively split extension of A by M

$$0 \longrightarrow M \xrightarrow{q} X \xrightarrow{p} A \longrightarrow 0$$

such that for each $i \in I$ the arrow $p(i)$ is additively and multiplicatively split.

We denote the set of equivalence classes of additively and multiplicatively split extensions of A by M by $\mathfrak{MExt}(A, M)$.

Lemma 5.8.

$$Harr^1(I, A, M) \cong \mathfrak{AExt}(A, M).$$

Proof. A 1-cocycle is a pair $(f_i : A(i) \times A(i) \to M(i))_{i \in I}$ and $(\epsilon_\alpha : A(i) \to M(j))_{(\alpha: i \to j) \in I}$. We get an additively split extension of A by M given by taking the exact sequence

$$0 \longrightarrow M \longrightarrow M \oplus A \longrightarrow A \longrightarrow 0$$

where addition in $M \oplus A$ is given by $(m, a) + (m', a') = (m + m', a + a')$ and multiplication is given by

$$(m, a)(m', a') = (a'm + am' + f_i(a, a'), aa').$$

For all $(\alpha : i \to j) \in I$ set the map $(M \oplus A)(\alpha) : (M \oplus A)(i) \to (M \oplus A)(j)$ to be

$$(M \oplus A)(\alpha)(m, a) = (M(\alpha)(m) + \epsilon_\alpha(a), A(\alpha)(a)).$$

Given two 1-cocycles which differ by a 1-coboundary, then the two additively split extensions we get are equivalent.

Given an additively split extension of A by M

$$0 \longrightarrow M \xrightarrow{q} X \xrightarrow{p} A \longrightarrow 0$$

there are additive homomorphisms $s(i) : A(i) \to X(i)$ for all $i \in I$ such that $s(i)$ is a section of $p(i)$.

For all $i \in I$ we define the maps $f_i : A(i) \times A(i) \to M(i)$ to be given by

$$f_i(a, a') = s(i)(a)s(i)(a') - s(i)(aa').$$

For all $(\alpha : i \to j) \in I$ we define the maps $\epsilon_\alpha : A(i) \to M(j)$ to be given by

$$\epsilon_\alpha(a) = X(\alpha)(s(i)(a)) - s(j)(A(\alpha)(a)).$$

Then $(f_i : A(i) \times A(i) \to M(i))_{i \in I}$ and $(\epsilon_\alpha : A(i) \to M(j))_{(\alpha : i \to j) \in I}$ give us a 1-cocycle. Given two additively split extensions which are equivalent, then the two 1-cocycles we get differ by a 1-coboundary. $\square$

Corollary 5.9.

$$H^1_{BW}(I, \mathcal{H}^1_{Harr}(A, M)) \cong \mathfrak{MExt}(A, M).$$

Definition 5.10. An *additively split crossed extension* of A by M is an exact sequence of functors

$$0 \longrightarrow M \xrightarrow{\phi} C_1 \xrightarrow{\rho} C_0 \xrightarrow{\pi} A \longrightarrow 0$$

such that for all $i \in I$ we get an additively split crossed extension of $A(i)$ by $M(i)$.

$$0 \longrightarrow M(i) \xrightarrow{\phi(i)} C_1(i) \xrightarrow{\rho(i)} C_0(i) \xrightarrow{\gamma(i)} A(i) \longrightarrow 0 \tag{5.1}$$

This means that all the arrows in the exact sequence 5.1 are additively split. We let $\pi_0\mathfrak{ACross}(A, M)$ denote the connected components of the category of additively split crossed extensions of A by M.

An *additively and multiplicatively split crossed extension* of A by M is an exact sequence of functors

$$0 \longrightarrow M \xrightarrow{\phi} C_1 \xrightarrow{\rho} C_0 \xrightarrow{\gamma} A \longrightarrow 0$$

such that for all $i \in I$ we get an additively and multiplicatively split crossed extension of $A(i)$ by $M(i)$,

$$0 \longrightarrow M(i) \xrightarrow{\phi(i)} C_1(i) \xrightarrow{\rho(i)} C_0(i) \xrightarrow{\gamma(i)} A(i) \longrightarrow 0 \tag{5.2}$$

where $\gamma(i)$ and $\rho(i)$ are additively and multiplicatively split. We let $\pi_0\mathfrak{MCross}(A, M)$ denote the connected components of the category of additively and multiplicatively split crossed extensions of A by M.

Lemma 5.11. *If $\gamma : C_0 \to A$ is a morphism of diagrams of commutative algebras then*

$$Harr^1(I, \gamma : C_0 \to A, M) \cong \pi_0\mathfrak{ACross}(\gamma : C_0 \to A, M),$$

where $Harr^*(I, \gamma : C_0 \to A, M)$ and $\pi_0\mathfrak{ACross}(\gamma : C_0 \to A, M)$ are defined as follows. Consider the following short exact sequence of cochain complexes:

$$0 \longrightarrow C^*_{Harr}(I, A, M) \xrightarrow{\gamma^*} C^*_{Harr}(I, C_0, M) \xrightarrow{\kappa^*} \mathrm{Coker}(\gamma^*) \longrightarrow 0,$$

where $C^*_{Harr}(I, A, M)$ denotes the total complex of the bicomplex $(C^{*,*}_{Harr}(I, A, M)$. We define the cochain complex $C^*_{Harr}(I, \gamma : C_0 \to A, M) := \mathrm{Coker}(\gamma^*)$. This allows us to define the relative Harrison cohomology

$$Harr^*(I, \gamma : C_0 \to A, M) := H^*(C^*_{Harr}(I, \gamma : C_0 \to A, M)).$$

We let $\mathfrak{ACross}(\gamma : C_0 \to A, M)$ denote the category whose objects are the additively split crossed extensions of A by M

$$0 \longrightarrow M \xrightarrow{\phi} C_1 \xrightarrow{\rho} C_0 \xrightarrow{\gamma} A \longrightarrow 0$$

with $\gamma : C_0 \to A$ fixed. A morphism between two of these crossed extensions consists of a morphism of diagrams of commutative algebras $h_1 : C_1 \to C_1$ such that the following diagram commutes.

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "\phi"] \arrow[d, equal] & C_1 \arrow[r, "\rho"] \arrow[d, "h_1"] & C_0 \arrow[r, "\gamma"] \arrow[d, equal] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M \arrow[r, "\phi'"] & C_1' \arrow[r, "\rho'"] & C_0 \arrow[r, "\gamma"] & A \arrow[r] & 0 \end{tikzcd}$$

Note that $\mathfrak{ACross}(\gamma : C_0 \to A, M)$ is a groupoid.

Proof. We use the method used in [13] for the crossed modules of Lie algebras. Given any additively split crossed module of A by M,

$$0 \longrightarrow M \xrightarrow{\phi} C_1 \xrightarrow{\rho} C_0 \xrightarrow{\gamma} A \longrightarrow 0,$$

we let $V = \operatorname{Ker}\gamma = \operatorname{Im}\rho$. For all objects $i \in I$ there are linear sections $s_i : A(i) \to C_0(i)$ of γ and $\sigma_i : V(i) \to C_1(i)$ of $\rho(i) : C_1(i) \to V(i)$. We define the maps $g_i : A(i) \otimes A(i) \to C_1(i)$ by:

$$g_i(a, b) = \sigma_i(s_i(a)s_i(b) - s_i(ab)).$$

We also define the maps $\omega_i : C_0(i) \to C_1(i)$ by:

$$\omega_i(c) = \sigma_i(c - s_i\gamma_i(c)).$$

By identifying M with $\operatorname{Ker}\rho$, we define the maps $f_i : C_0(i) \otimes C_0(i) \to M(i)$ by:

$$f_i(c, c') = g_i(\gamma_i(c), \gamma_i(c')) + c'\omega_i(c) + c\omega_i(c') - \omega_i(c) * \omega_i(c') - \omega_i(cc').$$

Since $g_i(c, c') = g_i(c', c)$, it follows that $f_i(c, c') = f_i(c', c)$ and so $f_i \in C^2_{Harr}(C_0(i), M(i))$.

For all morphisms $(\alpha : i \to j) \in I$ we define the maps $q_\alpha : A(i) \to C_1(j)$ by:

$$q_\alpha(a) = \sigma_j(C_0(\alpha)(s_i(a)) - s_j(A(\alpha)(a))).$$

By identifying M with $\operatorname{Ker}\rho$, we define the maps $e_\alpha : C_0(i) \to M(j)$ by:

$$e_\alpha(c) = \omega_j(C_0(\alpha)(c)) - C_1(\alpha)(\omega_i(c)) - q_\alpha(\gamma_i(c)).$$

Note that $e_\alpha \in C^1_{Harr}(C_0(i), \alpha^* M(j))$.

For all objects $i \in I$ we define the maps $\theta_i \in C^3_{Harr}(A(i), M(i))$ by:

$$\theta_i(x, y, z) = s_i(x)g_i(y, z) - g_i(xy, z) + g_i(x, yz) - g_i(y, x)s_i(z).$$

For all morphisms $(\alpha : i \to j) \in I$ we define the maps $\vartheta_\alpha \in C^2_{Harr}(A(i), \alpha^* M(j))$ by:

$$\begin{aligned}\vartheta_\alpha(x, y) =& g_j(A(\alpha)(x), A(\alpha)(y)) - C_1(\alpha)g_i(x, y)\\ &+ C_0(\alpha)(s_i(x))q_\alpha(y) - q_\alpha(xy) + q_\alpha(x)s_j(A(\alpha)(y)).\end{aligned}$$

For all pairs of composable morphisms $(\beta\alpha : i \to j \to k) \in I$ we define the maps $\eta_{\beta\alpha} \in C^1_{Harr}(A(i), (\beta\alpha)^* M(k))$ by:

$$\eta_{\beta\alpha}(x) = -q_\beta(A(\alpha)(x)) + q_{\beta\alpha}(x) - C_1(\beta)(q_\alpha(x)).$$

We let $f = (f_i)_{(i\in I)}$ and $e = (e_\alpha)_{(\alpha:i\to j\in I)}$. We also let $\theta = (\theta_i)_{(i\in I)}$, $\vartheta = (\vartheta_\alpha)_{(\alpha:i\to j\in I)}$ and $\eta = (\eta_{\beta\alpha})_{(\beta\alpha:i\to j\to k\in I)}$. Consider the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & C^1_{Harr}(I, A, M) \arrow[r, "\gamma^*"] \arrow[d] & C^1_{Harr}(I, C_0, M) \arrow[r, "\kappa^*"] \arrow[d, "\delta"] & C^1_{Harr}(I, \gamma : C_0 \to A, M) \arrow[r] \arrow[d, "\delta"] & 0 \\
0 \arrow[r] & C^2_{Harr}(I, A, M) \arrow[r, "\gamma^*"] & C^2_{Harr}(I, C_0, M) \arrow[r, "\kappa^*"] & C^2_{Harr}(I, \gamma : C_0 \to A, M) \arrow[r] & 0
\end{tikzcd}$$

Note that $(f, e) \in C^1_{Harr}(I, C_0, M)$ and $(\theta, \vartheta, \eta) \in C^2_{Harr}(I, A, M)$. A direct calculation shows that $\delta(f, e) = \gamma^*(\theta, \vartheta, \eta)$. We also have that $\delta\kappa^*(f, e) = \kappa^*\delta(f, e) = \kappa^*\gamma^*(\theta, \vartheta, \eta) = 0$, this tells us that $\kappa^*(f, e)$ is a cocycle. If we have two equivalent additively split crossed modules then we can choose sections in such a way that the associated cocycles are the same. Therefore we have a well-defined map:

$$\mathfrak{ACross}(\gamma : C_0 \to A, M) \to H^2_{Harr}(I, \gamma : C_0 \to A, M).$$

Inversely, assume we have a cocycle in $C^1_{Harr}(I, \gamma : C_0 \to A, M)$ which we lift to a cochain $(f, e) \in C^1_{Harr}(I, C_0, M)$. Let $V = \operatorname{Ker}\gamma$. For all objects $i \in I$ we define $C_1(i) = M(i) \times V(i)$ as a module over k with the following action of $C_0(i)$ on $C_1(i)$:

$$c(m, v) := (cm + f_i(c, v), cv).$$

The maps $C_1(\alpha) : C_1(i) \to C_1(j)$ are given by:

$$C_1(\alpha)(m, v) := (M(\alpha)(m) + e_\alpha(v), C_0(\alpha)(v)).$$

It is easy to check using the properties of f_i and e_α that this action is well defined and together with the maps $\rho_i : C_0(i) \to C_1(i)$ given by $\rho_i(m, v) = v$, we have an additively split crossed module of A by M. □

Lemma 5.12. *If k is a field of characteristic 0 then*

$$Harr^2(I, A, M) \cong \pi_0\mathfrak{ACross}(\mathfrak{A}, \mathfrak{M}).$$

Proof. From the definition of $C^*_{Harr}(I, \gamma : C_0 \to A, M)$ we get the long exact sequence:

$$\ldots \longrightarrow Harr^1(I, A, M) \longrightarrow Harr^1(I, C_0, M) \longrightarrow \tag{5.3}$$

$$Harr^1(I, \gamma : C_0 \to A, M) \longrightarrow Harr^2(I, A, M) \longrightarrow \ldots$$

Given any additively split crossed module in $\pi_0\mathfrak{ACross}(A, M)$,

$$0 \longrightarrow M \xrightarrow{\phi} C_1 \xrightarrow{\rho} C_0 \xrightarrow{\gamma} A \longrightarrow 0$$

we can lift γ to get a map $P_0 \to A$ where P_0 is free as a diagram of commutative algebras. We can then use a pullback to construct P_1 to get a crossed module where the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "\phi"] & C_1 \arrow[r, "\rho"] & C_0 \arrow[r, "\gamma"] & A \arrow[r] & 0 \\ 0 \arrow[r] & M \arrow[r] \arrow[u, equal] & P_1 \arrow[r] \arrow[u] & P_0 \arrow[r] \arrow[u] & A \arrow[r] \arrow[u, equal] & 0 \end{tikzcd}$$

These two crossed modules are in the same connected component of $\pi_0\mathfrak{ACross}(A, M)$. By considering the second crossed module in the long exact sequence, we replace C_0 by P_0 to get the new exact sequence:

$$0 \longrightarrow Harr^1(I, \gamma : P_0 \to A, M) \longrightarrow Harr^2(I, A, M) \longrightarrow 0 \tag{5.4}$$

since $Harr^1(I, P_0, M) = 0$ and $Harr^2(I, P_0, M) = 0$.

The exact sequence 5.4 tells us that every element in $Harr^2(I, A, M)$ comes from an element in $Harr^1(I, \gamma : P_0 \to A, M)$ and the previous lemma tells us that this comes from a crossed module in $\pi_0\mathfrak{ACross}(A, M)$. Therefore the map $\pi_0\mathfrak{ACross}(A, M) \to Harr^2(I, A, M)$ is surjective.

Assume we have two crossed modules which go to the same element in $Harr^2(I, A, M)$,

$$0 \longrightarrow M \xrightarrow{\phi} C_1 \xrightarrow{\rho} C_0 \xrightarrow{\gamma} A \longrightarrow 0, \tag{5.5}$$

$$0 \longrightarrow M \xrightarrow{\phi'} C_1' \xrightarrow{\rho'} C_0' \xrightarrow{\gamma'} A \longrightarrow 0. \tag{5.6}$$

There exist morphisms

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "\phi"] \arrow[d, equal] & C_1 \arrow[r, "\rho"] \arrow[d] & C_0 \arrow[r, "\gamma"] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & A \arrow[r] & 0, \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "\phi'"] \arrow[d, equal] & C_1' \arrow[r, "\rho'"] \arrow[d] & C_0' \arrow[r, "\gamma'"] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M \arrow[r] & P_2 \arrow[r] & P_0 \arrow[r] & A \arrow[r] & 0, \end{tikzcd}$$

where P_0 is free as a diagram of commutative algebras and P_1, P_2 are constructed via pullbacks. These give us two elements in $Harr^1(I, \gamma : P_0 \to A, M)$ which go to the same element in $Harr^2(I, A, M)$. However the exact sequence 5.4 tells us that the two crossed modules 5.5 and 5.6 have to go to the same element in $Harr^1(I, \gamma : P_0 \to A, M)$. The previous lemma tells us that the two crossed modules 5.5 and 5.6 go to the same element in $\mathfrak{ACross}(\gamma : C_0 \to A, M)$ which is a groupoid, so there is a map $P_2 \to P_1$ which makes the following diagram commute:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\phi"] & C_1 \arrow[r, "\rho"] & C_0 \arrow[r, "\gamma"] & A \arrow[r] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[u, equal] & P_1 \arrow[r] \arrow[u] & P_0 \arrow[r] \arrow[u] & A \arrow[r] \arrow[u, equal] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[u, equal] \arrow[d, equal] & P_2 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[u, equal] \arrow[d] & A \arrow[r] \arrow[u, equal] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r, "\phi'"] & C_1' \arrow[r, "\rho'"] & C_0' \arrow[r, "\gamma'"] & A \arrow[r] & 0
\end{tikzcd}$$

Therefore the two crossed modules 5.5 and 5.6 are in the same connected component of $\pi_0\mathfrak{ACross}(A, M)$ and the map $\pi_0\mathfrak{ACross}(A, M) \to Harr^2(I, A, M)$ is injective. □

Corollary 5.13. *If k is a field of characteristic 0 then*

$$H^2_{BW}(I, \mathcal{H}^1_{Harr}(A, M)) \cong \pi_0\mathfrak{MCross}(A, M).$$

Proof. Given an additively and multiplicatively split crossed extension of A by M we get that (with the notation of lemma 5.11) $g_i = 0$ for all $i \in I$. Since $\rho(i)$ is additively and multiplicatively split for all $i \in I$ it follows that $f = 0$, $\theta = 0$ and $\vartheta = 0$. Therefore η is a cocycle in $C^2_{BW}(I, \mathcal{H}^1_{Harr}(A, M))$.

Inversely, the construction given in lemma 5.11 gives us an additively and multiplicatively split extension. □

5.5 Harrison cohomology of Ψ-rings

Let R be a Ψ-ring, and $M \in R - \mathfrak{mod}_\Psi$. Let I denote the category with one object associated to the multiplicative monoid of the natural numbers $\mathbb{N}^{mult}$. For any

$j \geq 1$, there is a natural system on I as follows:

$$D_f := C^j_{Harr}(R, f^*M),$$

where f^*M is an Ψ-module over R with M as an abelian group and the following action of R

$$(r, m) \mapsto \Psi^f(r)m, \text{ for } r \in R, m \in M.$$

For $u \in \mathcal{F}I$ (the category of factorisations in I), we have $u_* : D_f \to D_{uf}$ which is induced by $\Psi^u : f^*M \to (uf)^*M$. For $v \in \mathcal{F}I$, we have $v^* : D_f \to D_{fv}$ which is induced by $\Psi^v : R \to R$.

The bicomplex in section 5.3 becomes

$$\begin{tikzcd}
C^1_{Harr}(R, M) \arrow[r, "b"] \arrow[d, "d"'] & \prod_{i\in\mathbb{N}} C^1_{Harr}(R, i^*M) \arrow[r, "b"] \arrow[d, "-d"'] & \prod_{i,j\in\mathbb{N}} C^1_{Harr}(R, (ij)^*M) \arrow[r, "b"] \arrow[d, "d"'] & \cdots \\
C^2_{Harr}(R, M) \arrow[r, "b"] \arrow[d, "d"'] & \prod_{i\in\mathbb{N}} C^2_{Harr}(R, i^*M) \arrow[r, "b"] \arrow[d, "-d"'] & \prod_{i,j\in\mathbb{N}} C^2_{Harr}(R, (ij)^*M) \arrow[r, "b"] \arrow[d, "d"'] & \cdots \\
C^3_{Harr}(R, M) \arrow[r, "b"] \arrow[d, "d"'] & \prod_{i\in\mathbb{N}} C^3_{Harr}(R, i^*M) \arrow[r, "b"] \arrow[d, "-d"'] & \prod_{i,j\in\mathbb{N}} C^3_{Harr}(R, (ij)^*M) \arrow[r, "b"] \arrow[d, "d"'] & \cdots \\
\vdots & \vdots & \vdots &
\end{tikzcd}$$

with

$$d : \prod_{t=t_1\ldots t_i\in\mathbb{N}} C^j_{Harr}(R, t^*M) \to \prod_{t=t_1\ldots t_i\in\mathbb{N}} C^{j+1}_{Harr}(R, t^*M),$$

with the product being over i-tuples $(t_1, \ldots, t_i)$ and t is the composite, is given by

$$\begin{aligned}
df_{t_1,\ldots,t_i}(x_1, \ldots, x_{j+1}) = &\Psi^{t_1t_2\ldots t_i}(x_1) f_{t_1,\ldots,t_i}(x_2, \ldots, x_{j+1}) \\
&+ \sum_{k=1}^{j} (-1)^k f_{t_1,\ldots,t_i}(x_1, \ldots, x_k x_{k+1}, \ldots, x_{j+1}) \\
&+ (-1)^{j+1} f_{t_1,\ldots,t_i}(x_1, \ldots, x_j) \Psi^{t_1t_2\ldots t_i}(x_{j+1}).
\end{aligned}$$

and

$$b : \prod_{t=t_1\ldots t_i\in\mathbb{N}} C^j_{Harr}(R, t^*M) \to \prod_{t=t_1\ldots t_{i+1}\in\mathbb{N}} C^j_{Harr}(R, t^*M),$$

being given by

$$\begin{aligned} bf_{t_1,\ldots,t_{i+1}}(x_1,\ldots,x_j) =& \Psi^{t_1} f_{t_2,\ldots,t_{i+1}}(x_1,\ldots,x_j) \\ & + \sum_{k=1}^{i} (-1)^k f_{t_1,\ldots,t_k t_{k+1},\ldots,t_{i+1}}(x_1,\ldots,x_j) \\ & + (-1)^{i+1} f_{t_1,\ldots,t_i}(\Psi^{t_{i+1}}(x_1),\ldots,\Psi^{t_{i+1}}(x_j)). \end{aligned}$$

We let $Harr^i_\Psi(R, M)$ denote the i^{th} cohomology of the total complex of the bicomplex described above.

Theorem 5.14. *There exists a spectral sequence*

$$E_2^{p,q} = H^p_{BW}(I, \mathcal{H}^{q+1}_{Harr}(R, M)) \Rightarrow Harr^{p+q}_\Psi(R, M).$$

where $\mathcal{H}^q_{Harr}(R, M)$ is the natural system on I whose value on $(\alpha : i \to j)$ is given by $Harr^q(R, \alpha^*M)$.

Theorem 5.15.

$$Harr^0_\Psi(R, M) = \mathrm{Der}_\Psi(R, M),$$

$$Harr^1_\Psi(R, M) = \mathrm{AExt}_\Psi(R, M),$$

$$Harr^2_\Psi(R, M) = \pi_0 ACross_\Psi(R, M).$$

5.6 Harrison cohomology and λ-rings

Let R be a λ-ring and $M \in R - \mathfrak{mod}_\lambda$.

Conjecture 5.16. *There exists a cochain bicomplex which starts:*

$$\begin{tikzcd}
C^1_{1-Harr}(\underline{R},\underline{M}) \arrow[r, "b^1_1"] \arrow[d, "d_1"'] & C^1_{2-Harr}(\underline{R},\underline{M})) \arrow[r, "b^1_2"] \arrow[d, "-d_2"'] & C^1_{3-Harr}(\underline{R},\underline{M})) \arrow[r] \arrow[d] & \cdots \\
C^2_{1-Harr}(\underline{R},\underline{M}) \arrow[r, "b^2_1"] \arrow[d, "d_1"'] & C^2_{2-Harr}(\underline{R},\underline{M})) \arrow[r] \arrow[d, "-d_2"'] & \ddots & \\
C^3_{1-Harr}(\underline{R},\underline{M}) \arrow[r] \arrow[d, "d_1"'] & \ddots & & \\
\vdots & & &
\end{tikzcd}$$

where the first column is the Harrison cochain complex.

$$C^i_{1-Harr}(\underline{R},\underline{M}) := C^i_{Harr}(\underline{R},\underline{M}).$$

For all $i \geq 1$ and $j \geq 2$ we have that

$$C^i_{j-Harr}(\underline{R},\underline{M}) \subset \prod_{n_1,\dots,n_{j-1}\in\mathbb{N}} Maps(\underline{R}^{\otimes i},\underline{M}).$$

For example, when $j = 2$, we have

$$\begin{aligned} C^1_{2-Harr}(\underline{R},\underline{M})) = \{f \in \prod_{n\in\mathbb{N}} Maps(\underline{R},\underline{M})| \\ f_n(r+s) = \sum_{j=1}^{n}[f_j(r)\lambda^{n-j}(s) + f_j(s)\lambda^{n-j}(r)]\}. \end{aligned}$$

$$
\begin{aligned}
C^2_{2-Harr}(\underline{R},\underline{M})) = \{f \in \prod_{n\in\mathbb{N}} Maps(\underline{R}\otimes\underline{R},\underline{M}) | f_n(r,s) = f_n(s,r),\\
f_n((r,s)+(t,u)) = \sum_{j=1}^{n}[f_j(r,s)\lambda^{n-j}(tu+ru+ts)\\
+ f_j(t,u)\lambda^{n-j}(rs+ru+ts) + f_j(r,u)\lambda^{n-j}(rs+tu+ts)+\\
f_j(t,s)\lambda^{n-j}(rs+ru+tu)]\}.
\end{aligned}
$$

The coboundary maps $d_2 : C^i_{2-Harr}(\underline{R},\underline{M}) \to C^{i+1}_{2-Harr}(\underline{R},\underline{M})$ *are given by*

$$
\begin{aligned}
(d_2(f))_n(r_1,\dots,r_{i+1}) = &\sum_{j=1}^{n}[\frac{\partial P_n(r_1,r_2\dots r_{i+1})}{\partial\lambda^j(r_2\dots r_{i+1})} f_j(r_2,\dots,r_{i+1})]\\
&+\sum_{j=1}^{i} f_n(r_1,\dots,r_jr_{j+1},\dots,r_{i+1})\\
&+\sum_{j=1}^{n}[\frac{\partial P_n(r_1\dots r_i,r_{i+1})}{\partial\lambda^j(r_1\dots r_i)} f_j(r_1,\dots,r_i)].
\end{aligned}
$$

$$
(b^1_1(g))_n(r) = g(\lambda^n(r)) - \sum_{i=1}^{n}\Lambda^i(g(r))\lambda^{n-i}(r).
$$

$$
(b^1_2(f))_{n,m}(r) = f_m(\lambda^n(r)) - \sum_{i=1}^{nm} f_i(r)\frac{\partial P_{n,m}(r)}{\partial\lambda^i(r)} + \sum_{j=1}^{m}\Lambda^j(f_n(r))\lambda^{m-j}(\lambda^n(r)).
$$

We let $Harr^i_\lambda(R,M)$ *denote the* i^{th} *cohomology of the bicomplex above. Then we get the following*

$$
\begin{aligned}
Harr^0_\lambda(R,M) \cong \mathrm{Der}_\lambda(R,M),\\
Harr^1_\lambda(R,M) \cong \mathrm{AExt}_\lambda(R,M).
\end{aligned}
$$

5.7 Gerstenhaber-Schack cohomology

In the paper [7] Gerstenhaber and Schack describe a cohomology for diagrams of associative algebras which we denote by $H^*_{GS}(I, A, M)$. Let $I = \{i, j, k, \ldots\}$ be a partially ordered set. We can view I as the set of objects of a category in which there exists a unique morphism $i \to j$ when $i \leq j$. They define a diagram to be a contravariant functor $A : I^{op} \to \mathfrak{Com.alg}$. They define an A-module to be a contravariant functor $M : I^{op} \to \mathfrak{Ab}$ such that $M(i) \in A(i) - \mathfrak{mod}$ for all $i \in I$ and for each $i \leq j$ the map $M(i \to j)$ is an $A(j)$-module homomorphism where $A(i)$ is viewed as an $A(j)$-module via the morphism $A(i \to j)$. If we consider A as a covariant functor $A : I \to \mathfrak{Com.alg}$ and M as a covariant functor $M : I \to \mathfrak{Ab}$ then we can apply the theory we developed earlier.

The bicomplex described by Gerstenhaber and Schack coincides with our bicomplex $C^{p,q}_{Harr}(I, A, M)$. Therefore $H^n_{GS}(I, A, M) = Harr^n(I, A, M)$ for $n \geq 0$. Therefore we get a new spectral sequence

$$E_2^{p,q} = H^p_{BW}(I, \mathcal{H}^{q+1}_{Harr}(A, M)) \Rightarrow H^{p+q}_{GS}(I, A, M),$$

where $\mathcal{H}^q_{Harr}(A, M)$ is the natural system on I whose value on $(\alpha : i \to j)$ is given by $Harr^q(A(i), \alpha^* M(j))$.

Chapter 6

André-Quillen cohomology of diagrams of algebras

In this chapter, let $\mathfrak{C}$ denote a category with limits, and I denote a small category.

We have already seen that for algebraic objects, we can get cohomology from monads and comonads. In this chapter, we define a cohomology for diagrams of algebras. Our approach can be described as follows. First, we fix a small category I. A diagram of algebras is a functor $I \to \mathfrak{Alg}(T)$, where T is a monad on sets. For appropriate T, one gets a diagram of groups, a diagram of Lie algebras, a diagram of commutative rings, etc. The adjoint pair $\mathfrak{Alg}(T) \rightleftarrows \mathfrak{Sets}$ yields a comonad which we denote by $\mathbb{G}$. We can also consider the category I_0, which has the same objects as I, but only the identity morphisms. The inclusion $I_0 \subset I$ yields the functor $\mathfrak{Sets}^I \to \mathfrak{Sets}^{I_0}$ which has a left adjoint given by the left Kan extension. We also have the pair of adjoint functors $\mathfrak{Alg}(T)^I \rightleftarrows \mathfrak{Sets}^I$ which comes from the adjoint pair $\mathfrak{Alg}(T) \rightleftarrows \mathfrak{Sets}$. By gluing these diagrams together, one gets another adjoint pair

$$\mathfrak{Alg}(T)^I \rightleftarrows \mathfrak{Sets}^{I_0}.$$

This adjoint pair yields a comonad which we denote by $\mathbb{G}_I$. We will prove that $\mathfrak{Alg}(T)^I$ is monadic in $\mathfrak{Sets}^{I_0}$ and the right cohomology theory of diagrams of algebras is one which is associated to the comonad $\mathbb{G}_I$. These cohomology theories are denoted by $H^*_{\mathbb{G}_I}(A, M)$.

6.1 Base change

Let $\mathfrak{C}$ be a category, and X be an object in $\mathfrak{C}$. An X-*module* in $\mathfrak{C}$ is an abelian group object in the category $\mathfrak{C}/X$,

$$X - \mathfrak{mod} := \mathrm{Ab}(\mathfrak{C}/X).$$

Theorem 6.1. *Let $f : X \to Y$ be a morphism in $\mathfrak{C}$, then there exists a base-change functor $f^* : Y - \mathfrak{mod} \to X - \mathfrak{mod}$ via pullbacks.*

Proof. The product in the slice category is given by pullbacks. The functor we are going to use is $f^* : \mathfrak{C}/Y \to \mathfrak{C}/X$ given by pullbacks.

$$\begin{tikzcd}
f^*(M) \arrow[r] \arrow[d] & M \arrow[d, "p"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

If $M \in Y - \mathfrak{mod}$ then $f^*(M)$ has a canonical X-module structure. In set-theoretic notation,

$$f^*(M) = \{(x, m) | x \in X,\ m \in M,\ f(x) = p(m)\},$$

$$f^*(M) \times_X f^*(M) = \{(x, m, m') | x \in X,\ m, m' \in M,\ f(x) = p(m) = p(m')\},$$

$$f^*(M) \times_X f^*(M) \simeq f^*(M \times_Y M).$$

Consider the following commuting diagram.

$$\begin{tikzcd}
f^*(M \times_Y M) \arrow[rr] \arrow[d] \arrow[ddrr, dotted, "\exists!"] & & M \times_Y M \arrow[d] \arrow[ddr, "mult"] & \\
X \arrow[rr, "f"'] \arrow[ddrr, equal] & & Y \arrow[ddr, equal] & \\
 & & f^*(M) \arrow[r] \arrow[d] & M \arrow[d] \\
 & & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

The unique morphism $f^*(mult) : f^*(M \times_Y M) \to f^*(M)$ exists by the universal property of pullbacks. The isomorphism $f^*(M) \times_X f^*(M) \simeq f^*(M \times_Y M)$ and this unique morphism yield multiplication

$$f^*(mult) : f^*(M) \times_X f^*(M) \to f^*(M),$$

which gives an abelian group object structure on $f^*(M)$. □

6.2 Derivations

For $M \in X - \mathfrak{mod}$, one defines a *derivation* from X to M to be a morphism $d : X \to M$ which is a section of the canonical morphism $M \to X$. Let $\mathrm{Der}(X, M)$ denote the set of derivations $d : X \to M$. This is a special case of 2.2 and there is an abelian group structure. We will require the following useful theorem later.

Theorem 6.2. *If $X = \coprod_{\alpha \in I} X_\alpha$ and $M \in X - \mathfrak{mod}$, then*

$$\mathrm{Der}(X, M) \cong \prod_{\alpha \in I} \mathrm{Der}(X_\alpha, M_\alpha),$$

where M_α is the X_α-module produced from M by the base-change functor from the morphism $i_\alpha : X_\alpha \to X$.

Proof. From the definition of the coproduct one has a morphism $i_\alpha : X_\alpha \to X$. Using this one gets $M_\alpha \in X_\alpha$-$\mathfrak{mod}$ via the following pullback diagram.

$$\begin{tikzcd} M & M_\alpha \arrow[l, "j_\alpha"'] \arrow[d, "p_\alpha"] \\ X \arrow[from=1-1, "p"'] & X_\alpha \arrow[l, "i_\alpha"] \end{tikzcd}$$

Let f be a section of p, this means that $pf = id_X$. Consider the following diagram.

$$\begin{tikzcd}
 & & X_\alpha \arrow[dll, "fi_\alpha"'] \arrow[dl, dotted, "f_\alpha"] \arrow[ddl, "id_{X_\alpha}"] \\
M \arrow[d, shift left, "p"] & M_\alpha \arrow[l, "j_\alpha"'] \arrow[d, "p_\alpha"'] & \\
X \arrow[u, shift left, "f"] & X_\alpha \arrow[l, "i_\alpha"] &
\end{tikzcd}$$

The diagram commutes since $pfi_\alpha = id_X i_\alpha = i_\alpha id_{X_\alpha}$. By the universal property of pullbacks $p_\alpha f_\alpha = id_{X_\alpha}$. So if f is a section of p then f_α is a section of p_α.

Conversely, let f_α be a section of p_α, this means that $p_\alpha f_\alpha = id_{X_\alpha}$. By the definition of the coproduct there exists a unique morphism f such that the following diagram commutes.

$$\begin{tikzcd}
M & \\
X \arrow[u, dotted, "f"] & X_\alpha \arrow[l, "i_\alpha"] \arrow[ul, "j_\alpha f_\alpha"']
\end{tikzcd}$$

This means that $fi_\alpha = j_\alpha f_\alpha$. Composing with p on the left gives us that $pfi_\alpha = pj_\alpha f_\alpha = i_\alpha p_\alpha f_\alpha = i_\alpha id_{X_\alpha} = i_\alpha$ Thus the following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, shift left, "id_X"] \arrow[r, shift right, "pf"'] & X \\
X_\alpha \arrow[u, "i_\alpha"] \arrow[ur, "i_\alpha"'] &
\end{tikzcd}$$

The universal property of the coproduct says that $pf = id_X$. Hence f is a section of p. $\square$

We will require the following useful lemma later.

Lemma 6.3. *For all objects $Z \in \mathfrak{C}^I$, for $M \in G_I(Z) - \mathfrak{mod}$, and $\alpha : i \to j$ in the small category I, one has*

$$\mathrm{Der}(G(Z(i)), \alpha^* M(j)) = \prod_{m \in UZ(i)} p_j^{-1} \alpha_* \gamma_i(m),$$

where p_j is the canonical morphism $p_j : M(j) \to GZ(j)$ and γ_i is the inclusion $\gamma_i : UZ(i) \to GZ(i)$.

Proof. The derivations $\mathrm{Der}(G(Z(i)), \alpha^* M(j))$ are the sections of p_α in the following pullback diagram.

$$\begin{tikzcd} & \alpha^* M(j) \arrow[r] \arrow[d, "p_\alpha"'] & M(j) \arrow[d, "p_j"] \\ UZ(i) \arrow[r, hook, "\gamma_i"'] & GZ(i) \arrow[r, "\alpha_*"'] & GZ(j) \end{tikzcd}$$

By definition, $UZ(i)$ is the basis of the free object $GZ(i)$.

$$\begin{aligned} \mathrm{Der}(G(Z(i)), \alpha^* M(j)) &= \{s : UZ(i) \to M(j) | \alpha_* \gamma_i = p_j s, \ s \text{ is a set map.}\} \\ &= \prod_{m \in UZ(i)} p_j^{-1} \alpha_* \gamma_i(m). \end{aligned}$$

□

6.3 Natural system

We require the following useful theorem.

Theorem 6.4. *Let $A \in \mathfrak{C}^I$ and $M \in A$-$\mathfrak{mod}$. If $\alpha : i \to j$ is a morphism in I then $M(j) \in A(j) - \mathfrak{mod}$ and*

$$\mathfrak{Der}(A, M)(\alpha) = \mathrm{Der}(A(i), \alpha^* M(j)),$$

defines a natural system on I.

Proof. Start by fixing A and M, then let $D(\alpha)$ denote $\mathfrak{Der}(A, M)(\alpha)$. Let $\gamma, \alpha, \beta \in I$ such that

$$i' \xrightarrow{\gamma} i \xrightarrow{\alpha} j \xrightarrow{\beta} j'.$$

We are going to show that we have induced maps as follows.

$$D(\alpha\gamma) \xleftarrow{\gamma^*} D(\alpha) \xrightarrow{\beta^*} D(\beta\alpha) .$$

Let $s \in D(\alpha)$, then the following diagram commutes with $ps = id_{A(i)}$, and $\alpha^*M(j)$ is a pullback, $\alpha^*M(j) \in A(i) - \mathfrak{mod}$.

$$\begin{tikzcd}
\alpha^*M(j) \arrow[r] \arrow[d, shift left, "p"] & M(j) \arrow[d] \\
A(i) \arrow[u, shift left, "s"] \arrow[r, "A(\alpha)"] & A(j)
\end{tikzcd}$$

Consider the following commuting diagram.

$$\begin{tikzcd}
M(i) \arrow[d, dotted, "\exists!"'] \arrow[dr, "M(\alpha)"] \arrow[ddd, bend right=60] & & \\
\alpha^*M(j) \arrow[r] \arrow[d, dotted, "\exists!\tau"] \arrow[dd, bend right=30, "p"'] & M(j) \arrow[d, dotted, "\exists!"'] \arrow[dr, "M(\beta)"] \arrow[dd, bend left=30] & \\
\alpha^*\beta^*M(j') \arrow[r] \arrow[d, "p'"] & \beta^*M(j') \arrow[r] \arrow[d] & M(j') \arrow[d] \\
A(i) \arrow[r, "A(\alpha)"'] \arrow[uu, bend left=30, "s"] & A(j) \arrow[r, "A(\beta)"'] & A(j')
\end{tikzcd}$$

Let $s' : A(i) \to \alpha^*\beta^*M(j')$ be the map $s' = \tau s$. Hence

$$p'\tau s = ps = id_{A(i)}.$$

So define $\beta^*(s) = s'$. Hence $s' \in \mathrm{Der}(A(i), \alpha^*\beta^*M(j')) = \mathrm{Der}(A(i), (\beta\alpha)^*M(j'))$.

Consider the following commutative diagram, with s a section of p.

$$\begin{tikzcd}
(\alpha\gamma)^*M(j) \arrow[r] \arrow[d, "p'"] & \alpha^*M(j) \arrow[r] \arrow[d, shift left, "p"] & M(j) \arrow[d] \\
A(i') \arrow[r, "A(\gamma)"] & A(i) \arrow[r, "A(\alpha)"] \arrow[u, shift left, "s"] & A(j)
\end{tikzcd}$$

There exists a unique $s' : A(i') \to (\alpha\gamma)^*M(j)$ which is a section of p' which would make the above diagram still commute. So define $\gamma^*(s) = s'$. Therefore $s' \in \mathrm{Der}(A(i'), (\alpha\gamma)^*M(j))$. $\square$

Corollary 6.5. *For $q \geq 0$ there exists a natural system $\mathcal{H}^q(A, M)$ on I whose value on $(\alpha : i \to j)$ is given by $H^q_{\mathbb{G}}(A(i), \alpha^*M(j))$.*

This corollary allows us to define, for fixed $q \geq 0$, the Baues-Wirsching cohomology $H^*_{BW}(I, \mathcal{H}^q(A, M))$ of I with coefficients in the natural system $\mathcal{H}^q(A, M)$.

Furthermore, we can consider a natural system on the category of chain complexes $\mathfrak{Chaincomplex}$ as follows. To each morphism $\alpha : i \to j \in I$ we assign the chain complex $\mathrm{Der}(\mathbb{G}_*(A(i)), \alpha^*M(j))$. This gives us a functor,

$$D : \mathcal{FI} \to \mathfrak{Chaincomplex},$$

where $\mathcal{FI}$ denotes the *category of factorizations* in $\mathcal{I}$.

This natural system gives rise to a cosimplicial object in $\mathfrak{Chaincomplex}$:

$$\prod_i D(id_i) \rightrightarrows \prod_{\alpha: i \to j} D(\alpha) \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; \cdots$$

which gives rise to a bicomplex described in the next in the next section.

6.4 Bicomplex

Let $\mathbb{G}$ be a comonad in $\mathfrak{C}$, let $A \in \mathfrak{C}^I$ and $M \in A - \mathfrak{mod}$. Then we can construct the following bicomplex denoted by $C^{*,*}(I, A, M)$.

$$C^{p,q}(I, A, M) = \prod_{\alpha: i_0 \to \ldots \to i_p} \mathrm{Der}(G^{q+1}(A(i_0)), \alpha^* M(i_p)).$$

The map $C^{p,q}(I, A, M) \to C^{p+1,q}(I, A, M)$ is the map in the Baues-Wirsching cochain complex, and the map $C^{p,q}(I, A, M) \to C^{p,q+1}(I, A, M)$ is the product of maps in the comonad cochain complex.

$$\begin{tikzcd}
\vdots & \vdots \\
\prod_i \mathrm{Der}(G^3(A(i)), M(i)) \arrow[r, "\delta"] \arrow[u, "\partial"] & \prod_{\alpha: i \to j} \mathrm{Der}(G^3(A(i)), \alpha^* M(j)) \arrow[r, "\delta"] \arrow[u, "-\partial"] & \ldots \\
\prod_i \mathrm{Der}(G^2(A(i)), M(i)) \arrow[r, "\delta"] \arrow[u, "\partial"] & \prod_{\alpha: i \to j} \mathrm{Der}(G^2(A(i)), \alpha^* M(j)) \arrow[r, "\delta"] \arrow[u, "-\partial"] & \ldots \\
\prod_i \mathrm{Der}(G(A(i)), M(i)) \arrow[r, "\delta"] \arrow[u, "\partial"] & \prod_{\alpha: i \to j} \mathrm{Der}(G(A(i)), \alpha^* M(j)) \arrow[r, "\delta"] \arrow[u, "-\partial"] & \ldots
\end{tikzcd}$$

This bicomplex lives in the category of abelian groups. We let $H^*(I, A, M)$ denote the cohomology of the total complex of $C^{*,*}(I, A, M)$.

We will need the following useful lemmas.

Lemma 6.6. *If A is $\mathbb{G}_I$-projective, then $A(i)$ is $\mathbb{G}$-projective for all $i \in I$.*

Proof. Consider $A = G_I(Z) : I \to \mathfrak{C}$ where $G_I(Z)(i) = \coprod_{x \to i} G(Z(x))$. Since $G(Z(x))$ is $\mathbb{G}$-projective, it follows that $\coprod_{x \to i} G(Z(x))$ is $\mathbb{G}$-projective for all $i \in I$. □

Lemma 6.7. $H^0(I, A, M) \cong \mathfrak{Der}(A, M)$*, furthermore, if A is $\mathbb{G}_I$-projective then $H^n(I, A, M) = 0$ for $n > 0$.*

Proof. It is sufficient to consider the case when $A = G_I(Z)$. When $A = G_I(Z)$, it is known that A is $\mathbb{G}_I$-projective. By lemma 6.6 and lemma 2.12, one gets that the vertical columns in our bicomplex are exact except in dimension 0. There is a well known lemma for bicomplexes which tells us the cohomology of the total complex is isomorphic to the cohomology of the following chain complex.

$$\prod_i \mathrm{Der}(A(i), M(i)) \longrightarrow \prod_{\alpha:i\to j} \mathrm{Der}(A(i), \alpha^* M(j)) \longrightarrow \dots$$

It is known that the cohomology of this cochain complex is just $H^*_{BW}(I, \mathfrak{Der}(A, M))$.

To prove the first statement it is enough to show that

$$0 \to \mathfrak{Der}(A, M) \to \prod_i \mathrm{Der}(A(i), M(i)) \to \prod_{\alpha:i\to j} \mathrm{Der}(A(i), \alpha^* M(j))$$

is exact. Let $\psi \in \prod_i \mathrm{Der}(A(i), M(i))$ and $(\alpha : i \to j) \in I$, then $d\psi(\alpha : i \to j) = \alpha_* \psi(i) - \alpha^* \psi(j)$. Therefore $d\psi(\alpha : i \to j) = 0$ if and only if $\alpha_* \psi(i) = \alpha^* \psi(j)$. However $\alpha_* \psi(i) = \alpha^* \psi(j)$ if and only if $M(\alpha)\psi(i) = \psi(j)A(\alpha)$, i.e. the following diagram commutes.

$$\begin{tikzcd} A(i) \arrow[r, "\psi(i)"] \arrow[d, "A(\alpha)"] & M(i) \arrow[d, "M(\alpha)"] \\ A(j) \arrow[r, "\psi(j)"'] & M(j) \end{tikzcd}$$

Hence $\psi \in \mathfrak{Der}(A, M)$. This tells us that the sequence above is exact. Hence $H^0(I, A, M) = \mathfrak{Der}(A, M)$.

To prove the second statement, let us consider

$$\begin{aligned} D(\alpha : i \to j) :&= \mathfrak{Der}(A(i), \alpha^* M(j)) \\ &= \mathrm{Der}(\coprod_{\beta:y\to i} GZ(y), \alpha^* M(j)) \\ &= \prod_{\beta:y\to i} \mathrm{Der}(GZ(y), \beta^* \alpha^* M(j)), \text{ by lemma 6.2.} \end{aligned}$$

Define D_y for a fixed object $y \in I$ to be a natural system on I (using theorem 6.4) given by:

$$D_y(\alpha : i \to j) = \prod_{\beta:y\to i} \mathrm{Der}(GZ(y), \beta^* \alpha^* M(j)).$$

So one has that

$$D(i \to j) = \prod_{y} D_y(i \to j).$$

Hence,

$$H^*_{BW}(I, D) = \prod_{y \in I} H^*_{BW}(I, D_y).$$

Now consider the cochain complex $C^*_{BW}(I, D_y)$.

$$\begin{aligned} C^*_{BW}(I, D_y) = &\ \textstyle\prod_i D_y(i \to i) \longrightarrow \prod_{\alpha : i \to j} D_y(i \to j) \longrightarrow \dots \\ = &\ \textstyle\prod_i \prod_{\beta : y \to i} \mathrm{Der}(GZ(y), \beta^* M(i)) \longrightarrow \\ &\ \textstyle\prod_{\alpha : i \to j} \prod_{\beta : y \to i} \mathrm{Der}(GZ(y), \beta^* \alpha^* M(j)) \longrightarrow \dots \end{aligned}$$

$UZ(y)$ forms a basis of the free object $GZ(y)$, applying lemma 6.3, one can rewrite the cochain complex as

$$C^*_{BW}(I, D_y) = \textstyle\prod_{y \to i} \prod_{m \in UZ(y)} A_{\beta j(m)} \longrightarrow \prod_{\alpha : i \to j} \prod_{\beta : y \to i} \prod_{m \in UZ(y)} A_{\alpha\beta j(m)} \longrightarrow \dots,$$

where $A_{\beta j(m)}$ = preimage of $\beta\gamma(m)$ in the projection $M(j) \to GZ(j)$. This allows us to rewrite the cochain complex as

$$C^*_{BW}(I, D_y) = \prod_{m \in UZ(y)} C^*_{BW}(y/I, F_m)$$

where $F_m : y/I \to Ab$ is a functor defined by $F_m(\beta : y \to i) = A_{\beta j(m)}$.

Since the category y/I contains an initial object $id_y : y \to y$, by lemma 2.20 the cohomology vanishes in positive dimensions. $\square$

Theorem 6.8. $H^*_{\mathbb{G}_I}(A, M) \cong H^*(I, A, M)$.

Proof. Consider the bicomplex $C^*(I, \mathbb{G}_I(A)_*, M)$ shown below.

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
C^2(I, G_I(A), M) \arrow[r] \arrow[u] & C^2(I, G_I^2(A), M) \arrow[r] \arrow[u] & C^2(I, G_I^3(A), M) \arrow[r] \arrow[u] & \cdots \\
C^1(I, G_I(A), M) \arrow[r] \arrow[u] & C^1(I, G_I^2(A), M) \arrow[r] \arrow[u] & C^1(I, G_I^3(A), M) \arrow[r] \arrow[u] & \cdots \\
C^0(I, G_I(A), M) \arrow[r] \arrow[u] & C^0(I, G_I^2(A), M) \arrow[r] \arrow[u] & C^0(I, G_I^3(A), M) \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

We are going to show that

$$H^*(I, A, M) \cong H^*(Tot(C^\bullet(I, \mathbb{G}_I(A)_\bullet, M))) \cong H^n_{\mathbb{G}_I}(A, M).$$

Since $G_I^p(A)$ is $\mathbb{G}_I$-projective, lemma 6.7 tells us that the vertical cohomology

$$H^n(C^*(I, G_I^p(A), M)) \cong \begin{cases} \mathfrak{Der}(G_I^p(A), M), & n = 0, \\ 0, & \text{otherwise,} \end{cases}$$

so each column of the bicomplex $C^*(I, \mathbb{G}_I(A)_*, M)$ is exact except at $C^0(I, G_I^p(A), M)$. Therefore by the spectral sequence argument

$$\begin{aligned} H^n(Tot(C^\bullet(I, \mathbb{G}_I(A)_\bullet, M))) \cong & H^n(\mathfrak{Der}(G_I(A), M) \to \mathfrak{Der}(G_I^2(A), M) \to \cdots) \\ = & H^n_{\mathbb{G}_I}(A, M). \end{aligned}$$

We are now going to compute the horizontal cohomology. From the definition of $C^*(I, A, M)$ we see that each row of the bicomplex $C^*(I, \mathbb{G}_I(A)_*, M)$ is a product of cochain complexes of the form $\mathrm{Der}(G^p\mathbb{G}_I(A)_*(i), \alpha^*M(j))$.

Consider $\mathbb{G}_I(A)_* \to A$ which is an augmented simplicial object. For all objects $i \in I$ we have $\mathbb{G}_I(A)_*(i) \to A(i)$ which is also an augmented simplicial object. Applying the forgetful functor $U : \mathfrak{Alg}(T) \to \mathfrak{Sets}$ we get $U\mathbb{G}_I(A)_*(i) \to UA(i)$ which is contractible in the category $\mathfrak{Sets}$. Then applying the free functor F : $\mathfrak{Sets} \to \mathfrak{Alg}(T)$ we get $G\mathbb{G}_I(A)_*(i) \to GA(i)$ which is contractible in the category

$\mathfrak{Alg}(T)$. Repeated applications of the functors U and F give us $G^p\mathbb{G}_I(A)_*(i) \to G^pA(i)$ which is contractible in the category $\mathfrak{Alg}(T)$. For any arrow $\alpha : i \to j$ in I we can apply the functor $\mathrm{Der}(-, \alpha^*M(j))$ to get a contractible cosimplicial abelian group $\mathrm{Der}(G^pA(i), \alpha^*M(j)) \to \mathrm{Der}(G^p\mathbb{G}_I(A)_*(i), \alpha^*M(j))$. Therefore each row of the bicomplex $C^*(I, \mathbb{G}_I(A)_*, M)$ is exact except at $C^p(I, G_I(A), M)$. Therefore

$$H^n(C^p(I, G_I(A)_*, M)) \cong \begin{cases} C^p(I, A, M), & n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore by the spectral sequence argument

$$H^n(Tot(C^\bullet(I, \mathbb{G}_I(A)_\bullet, M))) \cong H^n(C^*(I, A, M)) = H^n(I, A, M).$$

$\square$

Now one has both a global cohomology, $H^*_{\mathbb{G}_I}(A, M)$, and a local cohomology, $H^*_{\mathbb{G}}(A(i), M(i))$. One can ask how these two are related; the answer is given by the local to global spectral sequence.

Theorem 6.9. *There exists a spectral sequence*

$$E_2^{pq} = H^p_{BW}(I, \mathcal{H}^q(A, M)) \Rightarrow H^{p+q}_{\mathbb{G}_I}(A, M),$$

where $\mathcal{H}^q(A, M)$ is a natural system on I whose value on $(\alpha : i \to j)$ is given by $H^q_{\mathbb{G}}(A(i), \alpha^*M(j))$.

Definition 6.10. An *extension* of A by M is an exact sequence of functors

$$0 \longrightarrow M \xrightarrow{q} X \xrightarrow{p} A \longrightarrow 0$$

where $X : I \to \mathfrak{Com.alg}$ such that for all $i \in I$ we get an extension of $A(i)$ by $M(i)$

$$0 \longrightarrow M(i) \xrightarrow{q(i)} X(i) \xrightarrow{p(i)} A(i) \longrightarrow 0$$

Two extensions $(X), (X')$ with A, M fixed are said to be *equivalent* if there exists a map of diagrams $\phi : X \to X'$ such that the following diagram commutes.

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "\phi"] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M \arrow[r] & X' \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

We denote the set of equivalence classes of extensions of A by M by $\mathfrak{Ext}(A, M)$.

Theorem 6.11. $H^1_{\mathbb{G}_I}(A, M) \cong \mathfrak{Ext}(A, M)$.

Proof. Suppose we have a free resolution P_* of A and an extension representing a class in $\mathfrak{Ext}(A, M)$.

$$0 \longrightarrow M \xrightarrow{i} X \xrightarrow{u} A \longrightarrow 0$$

The map u is a surjection and P_0 is free, so there exists a lift $h : P_0 \to X$ which makes the following diagram commute.

$$\begin{tikzcd} & 0 \arrow[r] & M \arrow[r, "i"] & X \arrow[r, "u"] & A \arrow[r] & 0 \\ \cdots \arrow[r, shift left=2, "\varphi_0^2"] \arrow[r] \arrow[r, shift right=2, "\varphi_2^2"'] & P_1 \arrow[rr, shift left, "\varphi_0^1"] \arrow[rr, shift right, "\varphi_1^1"'] & & P_0 \arrow[u, dotted, "h"] \arrow[r, "\varepsilon"] & A \arrow[u, equal] \arrow[r] & 0 \end{tikzcd}$$

Then we can get a map $d = i^{-1}(h\varphi_0^1 - h\varphi_1^1) : P_1 \to M$.

d is a derivation, and d is also a 1-cocycle in $\mathfrak{Der}(P_*, M)$ and defines a class in $H^1_{\mathbb{G}_I}(A, M)$. This class is independent of the choice of lifting h. This gives a map $\Phi : \mathfrak{Ext}(A, M) \to H^1_{\mathbb{G}_I}(A, M)$.

Conversely, given a derivation $D : P_1 \to M$ we let

$$X = \mathrm{Coker}(\ P_1 \underset{(\varphi_0^1, 0)}{\overset{(\varphi_1^1, 0)}{\rightrightarrows}} P_0 \oplus M\ ).$$

The cokernel is in the category $A - \mathfrak{mod}$, and we let $p : P_0 \oplus M \to X$ be the canonical projection. If D is a 1-cocycle in $\mathfrak{Der}(P_*, M)$ then we obtain an extension in $\mathfrak{Ext}(A, M)$ where $i : M \to X$ is given by $i(m) = p(0 \oplus m)$ and $u : X \to A$ is given by $u(p(y \oplus m)) = \varepsilon(y)$.

$$\begin{tikzcd}
\cdots \arrow[r, shift left=2, "\varphi_0^2"] \arrow[r] \arrow[r, shift right=2, "\varphi_2^2"'] & P_1 \arrow[r, shift left, "\varphi_0^1"] \arrow[r, shift right, "\varphi_1^1"'] \arrow[d, "D"'] & P_0 \arrow[r, "\varepsilon"] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r, "i"] & X \arrow[r, "u"] & A \arrow[r] & 0
\end{tikzcd}$$

This procedure gives us an inverse to Φ. □

Definition 6.12. A *crossed extension* of A by M is an exact sequence of functors

$$0 \longrightarrow M \xrightarrow{\omega} C_1 \xrightarrow{\rho} C_0 \xrightarrow{\pi} A \longrightarrow 0$$

such that for all $i \in I$ we get a crossed extension of $A(i)$ by $M(i)$

$$0 \longrightarrow M(i) \xrightarrow{\omega(i)} C_1(i) \xrightarrow{\rho(i)} C_0(i) \xrightarrow{\pi(i)} A(i) \longrightarrow 0$$

We let $\pi_0\mathfrak{ACross}(A, M)$ denote the connected components of the category of additively split crossed extensions of A by M.

Lemma 6.13.

$$H^2_{\mathbb{G}_I}(A, M) \cong \pi_0\mathfrak{Cross}(A, M).$$

Proof. We are going to show that the crossed extensions are equivalent to the simplicial groups whose Moore complex is of length one. Given a crossed extension we have a crossed module

$$C_1 \xrightarrow{\partial} C_0.$$

Let $X_0 = C_0$ and $X_1 = C_1 \oplus C_0$ where addition is given by $(c_1, c_0) + (d_1, d_0) = (c_1 + d_1, c_0 + d_0)$ and multiplication is given by $(c_1, c_0)(d_1, d_0) = (0, c_0d_0 + \partial(c_1)d_1 + c_0d_1 + d_0c_1)$. For all $\alpha : i \to j$ then we have $X_1(\alpha)(c_0, d_0) = (C_1(\alpha)(c_1), C_0(\alpha)(c_0))$. This gives us that X_1 is a diagram of algebras. We set $d_1 : X_1 \to X_0$ to be $d_1(c_1, c_0) = c_0$ and $d_0 : X_1 \to X_0$ to be $d_0(c_1, c_0) = \partial(c_1) + c_0$. Then d_0 is a natural transformation.

We define the category $\mathfrak{C}$ to be the category whose objects are the elements of X_0 and whose morphisms are the elements of X_1. The source of the morphism $(c_1, c_0) \in \mathfrak{C}$ is given by $d_0(c_1, c_0) = \partial(c_1) + c_0$ and the target of $(c_1, c_0) \in \mathfrak{C}$ is given by $d_1(c_1, c_0) = c_0$. The composable morphisms in $\mathfrak{C}$ are pairs of morphisms $(c_1, c_0), (c_1', c_0')$ such that $c_0' = \partial c_1 + c_0$. The nerve of the category $\mathfrak{C}$ is a simplicial group whose Moore complex is

$$\ldots \longrightarrow 0 \longrightarrow \operatorname{Ker} d_1 \longrightarrow C_0,$$

which is of length one.

Let K_* be a simplicial object whose Moore complex is of length one. Then the Moore complex

$$\operatorname{Ker} d_1 \xrightarrow{d_0} K_0,$$

is a crossed module.

The category of diagrams of algebras is exact and so the results of Glenn [8] tell us that $H^2_{\mathbb{G}_I}(A, M)$ classifies the simplicial groups whose Moore complexes are of length one.

□

6.5 Cohomology of diagrams of groups

In the paper by Cegarra [6], the cohomology of diagrams of groups is described, which we denote by $H^*_{Cg}(G, M)$. A diagram of groups is a functor $G : I \to \mathfrak{Gp}$ where I is a small category and $\mathfrak{Grp}$ is the category of groups. A G-module is a functor $M : I \to \mathfrak{Ab}$ such that for all objects $i \in I$ we have that $M(i) \in A(i) - \mathfrak{mod}$ and for all morphisms $(\alpha : i \to j) \in I$ we have that $M(\alpha)(gm) = G(\alpha)(g) \cdot M(\alpha)(m)$ for all $g \in G(i)$ and $m \in M(i)$.

A derivation of G into M is a natural transformation $d : G \to M$ such that $d(i) : G(i) \to M(i)$ is a derivation of the group $G(i)$ into $M(i)$. We denote the abelian group of all derivations of G into M by $\operatorname{Der}_I(G, M)$. When G is locally constant then $H^{n+1}_{Cg}(G, M) = R^n \operatorname{Der}_I(G, M)$ and the following spectral sequence exists.

$$E_2^{p,q} = H^p_{BW}(I, \mathcal{H}^{q+1}(G, M)) \Rightarrow H^{p+q+1}_{Cg}(G, M),$$

where $\mathcal{H}^q(G, M)$ is a natural system on I whose value on $(\alpha : i \to j)$ is given by $H^q(G(i), \alpha^* M(j))$. So when G is locally constant the cohomology described by Cegarra coincides with the André-Quillen cohomology described above with a dimension shift.

Chapter 7

André-Quillen cohomology of Ψ-rings and λ-rings

7.1 Cohomology of Ψ-rings

Let I denote the category with one object associated to the multiplicative monoid of the nonzero natural numbers. We can consider Ψ-rings as diagrams of commutative rings; Ψ-rings are functors from I to the category of commutative rings

$$R : I \to \mathfrak{Com.rings}.$$

Therefore we can use the theory we developed in the previous chapter.

We are now going to construct the free Ψ-ring on one generator a. Let A be the free commutative ring generated by $\{a_i | i \in \mathbb{N}\}$. Let the operations $\Psi^i : A \to A$ be given by $\Psi^i(a_j) = a_{ij}$, for $i, j \in \mathbb{N}$. Then A is the *free Ψ-ring on one generator.*

Lemma 7.1. *If R and S are Ψ-rings, then $R \otimes S$ with $\Psi^i : R \otimes S \to R \otimes S$ given by $\Psi^i(r, s) = (\Psi^i(r), \Psi^i(s))$ is the coproduct in the category $\Psi - \mathfrak{rings}$.*

Proof. The coproduct of two commutative rings is given by the tensor product, so we only need to check the Ψ-operations. There is a unique Ψ-ring structure on

$R \otimes S$ such that

$$R \to R \otimes S, \qquad r \mapsto r \otimes 1,$$

$$S \to R \otimes S, \qquad s \mapsto 1 \otimes s,$$

are homomorphisms of Ψ-rings given by

$$\begin{aligned}\Psi^i(r \otimes s) &= \Psi^i((r \otimes 1)(1 \otimes s)) \\ &= \Psi^i(r \otimes 1)\Psi^i(1 \otimes s) \\ &= (\Psi^i(r) \otimes 1)(1 \otimes \Psi^i(s)) \\ &= \Psi^i(r) \otimes \Psi^i(s).\end{aligned}$$

□

Corollary 7.2. *Let A be the free commutative ring generated by $\{a_i, b_i, \ldots, x_i | i \in \mathbb{N}\}$. Let the operations $\Psi^i : A \to A$ be given by $\Psi^i(a_j) = a_{ij}$, $\Psi^i(b_j) = b_{ij}$, $\ldots$, $\Psi^i(x_j) = x_{ij}$ for $i, j \in \mathbb{N}$. Then A is the free Ψ-ring generated by $\{a, b, \ldots, x\}$.*

It is well known that there is an adjoint pair of functors

$$\mathfrak{Sets} \underset{U}{\overset{F}{\rightleftarrows}} \mathfrak{Com.rings}\ ,$$

where U is the forgetful functor and F takes a set S to the free commutative ring generated by S. The adjoint pair gives rise to a comonad $\mathbb{G}$ on $\mathfrak{Com.rings}$ which is monadic and the cohomology with respect to this comonad is the André-Quillen cohomology of commutative rings.

The adjoint pair gives rise to another adjoint pair

$$\mathfrak{Sets} \underset{U_I}{\overset{F_I}{\rightleftarrows}} \mathfrak{Com.rings}^I\ ,$$

where U_I is the forgetful functor and F_I takes a set S to the free Ψ-ring generated by S. This adjoint pair yields a comonad $\mathbb{G}_I$ on $\mathfrak{Com.rings}^I = \Psi - \mathfrak{rings}$ which is monadic. Note that for any $R \in \Psi - \mathfrak{rings}$, we get that $G_I(R) = \bigsqcup_{i \in \mathbb{N}} G(R)$. We define the cohomology of a Ψ-ring R with coefficients in $M \in R - \mathfrak{mod}_\Psi$ to be

$$H^*_\Psi(R, M) := H^*_{\mathbb{G}_I}(R, M) = H^*_{\mathbb{G}_I}(R, \mathrm{Der}_\Psi(-, M)).$$

From theorem 6.4 it follows that for any $n \geq 0$, there is a natural system on I as follows

$$D_f := H^n_{AQ}(R, M^f),$$

where M^f is an R-module with M as an abelian group with the following action of R

$$(r, m) \mapsto \Psi^f(r)m, \text{ for } r \in R, m \in M.$$

For any morphism $u \in I$, we have $u_* : D_f \to D_{uf}$ which is induced by $\Psi^u : M^f \to M^{uf}$. For any morphism $v \in I$, we have $v^* : D_f \to D_{fv}$ which is induced by $\Psi^v : R \to R$.

Therefore theorem 6.9 gives us the following theorem.

Theorem 7.3. *There exists a spectral sequence*

$$E_2^{p,q} = H^p_{BW}(I, \mathcal{H}^q(R, M)) \Rightarrow H^{p+q}_\Psi(R, M),$$

where $\mathcal{H}^q(R, M)$ is the natural system on I whose value on a morphism α in I is given by $H^q_{AQ}(R, M^\alpha)$.

Theorem 7.4. *Let R be a Ψ-ring and $M \in R{-}\mathfrak{mod}_\Psi$, then*

1. $H^0_\Psi(R, M) \cong \mathrm{Der}_\Psi(R, M)$,
2. $H^1_\Psi(R, M) \cong \mathrm{Ext}_\Psi(R, M)$,
3. $H^2_\Psi(R, M) \cong \pi_0 Cross_\Psi(R, M)$,
4. *If R is a free Ψ-ring, then $H^n_\Psi(R, M) = 0$ for $n \geq 1$.*

7.2 Cohomology of λ-rings

We are now going to construct the free λ-ring on one generator a. Let A be the free commutative ring generated by $\{a_i | i \in \mathbb{N}\}$. Let the operations $\lambda^i : A \to A$ be given by $\lambda^i(a_j) = P_{i,j}(a_1, \ldots, a_{ij})$ for $i, j \in \mathbb{N}$. Then A is the *free λ-ring on one generator.*

Lemma 7.5. *If R and S are λ-rings, then $R \otimes S$ with $\lambda^i : R \otimes S \to R \otimes S$ given by $\lambda^i(r, s) = P_i((\lambda^1(r), 1), \ldots, (\lambda^i(r), 1), (1, \lambda^1(s)), \ldots, (1, \lambda^i(s)))$ is the coproduct in the category $\lambda - \mathfrak{rings}$.*

It is known that there is an adjoint pair of functors

$$\mathfrak{Sets} \underset{U}{\overset{F}{\rightleftarrows}} \lambda - \mathfrak{rings}\ ,$$

where U is the forgetful functor and F takes a set S to the free λ-ring generated by S. The adjoint pair gives rise to a comonad $\mathbb{G}$ on $\lambda - \mathfrak{rings}$ which is monadic. We define the cohomology of a λ-ring R with coefficients in $M \in R - \mathfrak{mod}_\lambda$ to be

$$H^*_\lambda(R, M) := H^*_{\mathbb{G}}(R, M) = H^*_{\mathbb{G}}(R, \mathrm{Der}_\lambda(-, M)).$$

Theorem 7.6. *Let R be a λ-ring and $M \in R{-}\mathfrak{mod}_\lambda$, then*

1. $H^0_\lambda(R, M) \cong \mathrm{Der}_\lambda(R, M)$,
2. $H^1_\lambda(R, M) \cong \mathrm{Ext}_\lambda(R, M)$,
3. $H^2_\lambda(R, M) \cong \pi_0 Cross_\lambda(R, M)$,
4. *If R is a free λ-ring, then $H^n_\lambda(R, M) = 0$ for $n \geq 1$.*

Proof. Property 1 follows from lemma 2.12, and property 4 follows from lemma 2.11. We are now going to prove property 2.

Suppose we have a free resolution P_* of R as a λ-ring and an extension representing a class in $\mathrm{Ext}_\lambda(R, M)$.

$$0 \longrightarrow M \xrightarrow{i} X \xrightarrow{u} R \longrightarrow 0$$

The map u is a surjection and P_0 is free, so there exists a lift $h : P_0 \to X$ which makes the following diagram commute.

$$\begin{tikzcd}
 & 0 \arrow[r] & M \arrow[r, "i"] & X \arrow[r, "u"] & R \arrow[r] & 0 \\
\dots \arrow[r, shift left=2, "\varphi_0^2"] \arrow[r] \arrow[r, shift right=2, "\varphi_2^2"'] & P_1 \arrow[r, shift left, "\varphi_0^1"] \arrow[r, shift right, "\varphi_1^1"'] & P_0 \arrow[u, dotted, "h"] \arrow[r, "\varepsilon"] & R \arrow[u, equal] \arrow[r] & 0
\end{tikzcd}$$

Then we can get a map $d = i^{-1}(h\varphi_0^1 - h\varphi_1^1) : P_1 \to M$

d is a Ψ-derivation, and d is also a 1-cocycle in $\mathrm{Der}_\Psi(P_*, M)$ and defines a class in $H^1_\Psi(R, M)$. This class is independent of the choice of lifting h. This gives a map $\Phi : \mathrm{Ext}_\Psi(R, M) \to H^1_\Psi(R, M)$.

Conversely, given a λ-derivation $D : P_1 \to M$ we let

$$X = \mathrm{Coker}(\ P_1 \underset{(\varphi_0^1,0)}{\overset{(\varphi_1^1,0)}{\rightrightarrows}} P_0 \oplus M\ ).$$

The cokernel is in the category $R{-}\mathfrak{mod}_\lambda$, and we let $p : P_0 \oplus M \to X$ be the canonical projection. If D is a 1-cocycle in $\mathrm{Der}_\lambda(P_*, M)$ then we obtain an extension in $\mathrm{Ext}_\lambda(R, M)$ where $i : M \to X$ is given by $i(m) = p(0 \oplus m)$ and $u : X \to R$ is given by $u(p(y \oplus m)) = \varepsilon(y)$.

$$\begin{tikzcd}
\dots \arrow[r, shift left=2, "\varphi_0^2"] \arrow[r] \arrow[r, shift right=2, "\varphi_2^2"'] & P_1 \arrow[d, "D"'] \arrow[r, shift left, "\varphi_0^1"] \arrow[r, shift right, "\varphi_1^1"'] & P_0 \arrow[d] \arrow[r, "\varepsilon"] & R \arrow[d, equal] \arrow[r] & 0 \\
0 \arrow[r] & M \arrow[r, "i"] & X \arrow[r, "u"] & R \arrow[r] & 0
\end{tikzcd}$$

This procedure gives us an inverse to Φ.

We are now going to prove property 3 by showing that the crossed λ-extensions are equivalent to the simplicial groups whose Moore complex is of length one. Given a crossed λ-extension we have a crossed λ-module

$$C_1 \xrightarrow{\partial} C_0.$$

Let $X_0 = C_0$ and $X_1 = C_1 \oplus C_0$ where addition is given by $(c_1, c_0) + (d_1, d_0) = (c_1 + d_1, c_0 + d_0)$ and multiplication is given by $(c_1, c_0)(d_1, d_0) = (0, c_0 d_0 + \partial(c_1) d_1 + c_0 d_1 + d_0 c_1)$. We let $\lambda^n(c_0, d_0) = (\sum_{j=1}^{i} \Lambda^j(c_1)\lambda^{i-j}(c_0), \lambda^i(c_0)$. This gives us that X_1 is a λ-ring. We set $d_1 : X_1 \to X_0$ to be $d_1(c_1, c_0) = c_0$ and $d_0 : X_1 \to X_0$ to be $d_0(c_1, c_0) = \partial(c_1) + c_0$. Then d_0 is a λ-ring map.

We define the category $\mathfrak{C}$ to be the category whose objects are the elements of X_0 and whose morphisms are the elements of X_1. The source of the morphism $(c_1, c_0) \in \mathfrak{C}$ is given by $d_0(c_1, c_0) = \partial(c_1) + c_0$ and the target of $(c_1, c_0) \in \mathfrak{C}$ is given by $d_1(c_1, c_0) = c_0$. The composable morphisms in $\mathfrak{C}$ are pairs of morphisms $(c_1, c_0), (c_1', c_0')$ such that $c_0' = \partial c_1 + c_0$. Hence the nerve of the category $\mathfrak{C}$ is a simplicial group whose Moore complex is

$$\ldots \longrightarrow 0 \longrightarrow \operatorname{Ker} d_1 \longrightarrow C_0,$$

which is of length one.

Let K_* be a simplicial object whose Moore complex is of length one. Then the Moore complex yields

$$\operatorname{Ker} d_1 \xrightarrow{d_0} K_0,$$

which is a crossed λ-module.

The category of λ-rings is exact and so the results of Glenn [8] tell us that $H^2_\lambda(R, M)$ classifies the simplicial groups whose Moore complexes are of length one. $\square$

Lemma 7.7. *Let R be a λ-ring and let $M \in R$-mod$_\lambda$. Then there exist homomorphisms, for $n \geq 0$,*

$$\begin{aligned} \varsigma_n &: H^n_\lambda(R, M) \to H^n_\Psi(R_\Psi, M_\Psi), \\ \rho_n &: H^n_\lambda(R, M) \to H^n_{AQ}(\underline{R}, \underline{M}), \\ \varrho_n &: H^n_\Psi(R_\Psi, M_\Psi) \to H^n_{AQ}(\underline{R}, \underline{M}). \end{aligned}$$

Proof. Let P_* be a projective resolution of R in the category of λ-rings. Then applying the Adams operations we get that $(P_*)_\Psi$ is a (not necessarily projective) resolution of R_Ψ in the category of Ψ-rings. We let L_* be a projective resolution of R_Ψ in the category of Ψ-rings. Since L_* is projective, we can use the lifting property to get a map $\alpha : L_* \to (P_*)_\Psi$, such that the following diagram commutes.

$$\begin{tikzcd}
(P_*)_\Psi \arrow[r] & R_\Psi \arrow[d, equal] \\
L_* \arrow[u, dotted, "\alpha", "\exists"'] \arrow[r] & R_\Psi
\end{tikzcd}$$

We then apply the functor $\mathrm{Der}_\Psi(-, M_\Psi)$ to get the commutative diagram.

$$\begin{tikzcd}
\mathrm{Der}_\Psi((P_*)_\Psi, M_\Psi) \arrow[r] \arrow[d, "\alpha^*"] & \mathrm{Der}_\Psi(R_\Psi, M_\Psi) \arrow[d, equal] \\
\mathrm{Der}_\Psi(L_*, M_\Psi) \arrow[r] & \mathrm{Der}_\Psi(R_\Psi, M_\Psi)
\end{tikzcd}$$

The inclusion $i : \mathrm{Der}_\lambda(R, M) \hookrightarrow \mathrm{Der}_\Psi(R_\Psi, M_\Psi)$ gives us maps which make the following diagram commute.

$$\begin{tikzcd}
\mathrm{Der}_\lambda(P_*, M) \arrow[r] \arrow[d, "i"] & \mathrm{Der}_\lambda(R, M) \arrow[d, "i"] \\
\mathrm{Der}_\Psi((P_*)_\Psi, M_\Psi) \arrow[r] \arrow[d, "\alpha^*"] & \mathrm{Der}_\Psi(R_\Psi, M_\Psi) \arrow[d, equal] \\
\mathrm{Der}_\Psi(L_*, M_\Psi) \arrow[r] & \mathrm{Der}_\Psi(R_\Psi, M_\Psi).
\end{tikzcd}$$

This gives us homomorphisms

$$\varsigma_n : H^n_\lambda(R, M) = H^n(\mathrm{Der}_\lambda(P_*, M)) \xrightarrow{(\alpha^* i)^*} H^n(\mathrm{Der}_\Psi(L_*, M_\Psi)) = H^n_\Psi(R_\Psi, M_\Psi).$$

The homomorphisms ρ_n and ϱ_n are induced by the forgetful functors from $\lambda-\mathfrak{rings}$ and $\Psi - \mathfrak{rings}$ respectively to $\mathfrak{Com.rings}$. □

Chapter 8

Applications

8.1 K-theory

The material covered in this section can be found in [2] and [11].

8.1.1 Vector bundles

In this section we will develop the notion of complex vector bundles. A lot of the basic theory for real vector bundles is the same as for complex vector bundles, however we will only be concerned with complex vector bundles in this chapter.

Definition 8.1. A *complex vector bundle* consists of

1. topological spaces X (called the base space) and E (called the total space.)

2. a continuous map $p : E \to X$ (called the projection.)

3. a finite dimensional complex vector space structure on each

$$E_x = p^{-1}(x) \qquad \text{for } x \in X,$$

(we call the $p^{-1}(x)$ the fibres)

such that the following local triviality condition is satisfied. There exists an open cover of X by open sets U_α and for each there exists a homeomorphism $\varphi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^d$ which takes $p^{-1}(b)$ to $\{b\} \times \mathbb{C}^d$ via a vector space isomorphism for each $b \in U_\alpha$.

Example 8.2. *Let $E = X \times \mathbb{C}^d$, and p be the projection onto the first factor. We call this the product or trivial bundle.*

A *homomorphism* from a complex vector bundle $p : E \to X$ to another complex vector bundle $q : F \to X$ is a continuous map $\varphi : E \to F$ such that

1. $q\varphi = p$,
2. $\varphi : E_x \to F_x$ is a linear map of vector spaces for all $x \in X$.

If φ is a bijection and φ^{-1} is continuous, then we say that φ is an *isomorphism* and that E and F are *isomorphic*. We will let $Vect(X)$ denote the set of isomorphism classes of complex vector bundles on X.

Let E be a complex vector bundle over X. We get that $dim(E_x)$ is locally constant on X, furthermore it is a constant function on each of the connected components of X.

For vector bundles E, F we can define the following corresponding bundles

- $E \oplus F$, the direct sum of E and F,
- $E \otimes F$, the tensor product of E and F,
- $\lambda^k(E)$, the k^{th} exterior power of E.

There exist the following natural isomorphisms

- $E \oplus F \cong F \oplus E$,
- $E \otimes F \cong F \otimes E$,
- $E \otimes (F \oplus F') \cong (E \otimes F) \oplus (E \otimes F')$,
- $\lambda^k(E \oplus F) \cong \bigoplus_{i+j=k}(\lambda^i(E) \otimes \lambda^j(F))$.

8.1.2 K-theory

For any space X, we can consider the set $Vect(X)$ which has an abelian semigroup structure where addition is given by the direct sum. There is also a multiplication, given by tensor products, which is distributive over the addition of $Vect(X)$ (this makes $Vect(X)$ into a semiring.)

If A is an abelian semigroup, we can associate an abelian group $K(A)$ to A. Let $F(A)$ be the free abelian group generated by A, and let $E(A)$ be the subgroup of $F(A)$ generated by elements of them form $a + a' - (a \oplus a')$, where $a, a' \in A$ and $\oplus$ is the addition in A. We define the abelian group $K(A) = F(A)/E(A)$. If A is a semiring, then $K(A)$ is a ring.

If X is a space, then we will write $K(X)$ for the ring $K(Vect(X))$. Let $f : X \to Y$ be a continuous map. Then $f^* : Vect(Y) \to Vect(X)$ induces a ring homomorphism $f^* : K(Y) \to K(X)$ which only depends on the homotopy class of f.

We can define operations $\lambda^k : K(X) \to K(X)$ using the exterior powers. These make $K(X)$ into a λ-ring. We can then use these to define the Adams operations $\Psi^k : K(X) \to K(X)$ which makes $K(X)$ into a Ψ-ring.

If X is a compact space with distinguished basepoint, then we define $\widetilde{K}(X)$ to be the kernel of $i^* : K(X) \to K(x_0)$ where $i : x_0 \to X$ is the inclusion of the basepoint. Let $c : X \to x_0$ be the collapsing map, then c^* induces a natural splitting $K(X) \cong \widetilde{K}(X) \oplus K(x_0)$.

Example 8.3. $\widetilde{K}(S^{2n}) \cong \mathbb{Z}[y]/(y)^2$, *where y is the n-fold external product $(H - 1) * \ldots * (H - 1)$ and H is the canonical line bundle of $S^2 = \mathbb{C}\mathrm{P}^1$. Multiplication in $\widetilde{K}(S^{2n})$ is trivial, and the λ-operations $\lambda^k : \widetilde{K}(S^{2n}) \to \widetilde{K}(S^{2n})$ are given by*

$$\lambda^k(x) = (-1)^{k-1}k^{n-1}x.$$

Hence the Ψ-operations $\Psi^k : \widetilde{K}(S^{2n}) \to \widetilde{K}(S^{2n})$ are given by

$$\Psi^k(x) = k^n x.$$

8.2 Natural transformation

Let X, Y be topological spaces such that $\widetilde{K}(Y) = 0$ and $\widetilde{K}(\Sigma X) = 0$. Let $f : Y \to X$ be a continuous map, then we can consider the Puppe exact sequence

$$Y \xrightarrow{f} X \longrightarrow C_f \longrightarrow \Sigma Y \longrightarrow \Sigma X \longrightarrow \Sigma C_f \longrightarrow \dots$$

where C_f is the mapping cone of f, and ΣX is the suspension of X. After applying the functor $\widetilde{K}(-)$ we get the long exact sequence.

$$\dots \longrightarrow \widetilde{K}(\Sigma X) \longrightarrow \widetilde{K}(\Sigma Y) \longrightarrow \widetilde{K}(C_f) \longrightarrow \widetilde{K}(X) \longrightarrow \widetilde{K}(Y)$$

However, since $\widetilde{K}(\Sigma X) = 0$ and $\widetilde{K}(Y) = 0$ we obtain the short exact sequence.

$$0 \longrightarrow \widetilde{K}(\Sigma Y) \longrightarrow K(C_f) \longrightarrow K(X) \longrightarrow 0$$

This gives us the following proposition.

Proposition 8.4. *If X and Y are topological spaces as above then there exist natural transformations $\tau_\lambda : [Y, X] \to Ext_\lambda(K(X), \widetilde{K}(\Sigma Y))$ and $\tau_\Psi : [Y, X] \to Ext_\Psi(K(X), \widetilde{K}(\Sigma Y))$.*

Corollary 8.5. *If X is a topological space such that $\widetilde{K}(\Sigma X) = 0$ then there exist natural transformations $\tau_{\lambda,n} : \pi_{2n-1}(X) \to Ext_\lambda(K(X), \widetilde{K}(S^{2n}))$ and $\tau_{\Psi,n} : \pi_{2n-1}(X) \to Ext_\Psi(K(X), \widetilde{K}(S^{2n}))$.*

8.3 The Hopf invariant of an extension

We are going to give a proof of the classical result of Adams which was first proved by Adams, and subsequently by Adams-Atiyah [1]. We are going to use the same approach as Adams-Atiyah; using Ψ-rings.

Definition 8.6. Consider the commutative ring R which is free as an abelian group with generators x and y, $R \cong \mathbb{Z}x \oplus \mathbb{Z}y$, where x is the unit of the ring and $y^2 = 0$. Let $M \cong \mathbb{Z}z$ be the R-module such that $y \cdot z = 0$. We can consider the square zero extensions of R by M in the category of commutative rings. All the square zero extensions have the following form

$$0 \longrightarrow M \longrightarrow X \oplus \mathbb{Z}\gamma \longrightarrow R \longrightarrow 0 \tag{8.1}$$

where $X \cong \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$ as an abelian group with α being the image of the generator z, the image of the unit γ is the unit x and the image of β being the generator y. Since $M^2 = 0$ we get that $\alpha^2 = 0$. Since $y^2 = 0$, we get that $\alpha\beta = 0$ and $\beta^2 = h\alpha$ for some integer h. We define h to be the *Hopf invariant* of the extension (8.1).

Let $f : S^{4n-1} \to S^{2n}$ be a continuous map. We define the *Hopf invariant* of the map f to be the Hopf invariant of the short exact sequence

$$0 \longrightarrow \widetilde{K}(S^{4n}) \longrightarrow K(C_f) \longrightarrow K(S^{2n}) \longrightarrow 0$$

obtained from applying the natural transformation τ_Ψ to f.

We are going to consider the extensions of $K(S^{2n})$ by $\widetilde{K}(S^{2n'})$ in the category of Ψ-rings. We are going to prove the following theorem.

Theorem 8.7.

$$Ext_\Psi(K(S^{2n}), \widetilde{K}(S^{2n'})) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_{G_{n,n'}} & \textit{if } n \neq n'; \\ \mathbb{Z} \oplus \prod_{p\ prime} \mathbb{Z} & \textit{if } n = n'. \end{cases}$$

where $G_{n,n'}$ denotes the greatest common divisor of all the integers in the set $\{l^n - l^{n'} | l \in \mathbb{Z}, l \geq 2\}$.

Corollary 8.8. *If $n \neq n'$ then,*

$$Ext_\lambda(K(S^{2n}), \widetilde{K}(S^{2n'})) \cong \{(h,\nu) \in \mathbb{Z} \oplus \mathbb{Z}_{G_{n,n'}} | h \equiv \nu \frac{(2^n - 2^{n'})}{G_{n,n'}}\ mod\ 2\}.$$

If $n = n'$, then

$$Ext_\lambda(K(S^{2n}), \widetilde{K}(S^{2n'})) \cong \{(h, \nu_2, \nu_3, \ldots) \in \mathbb{Z} \oplus \prod_{p\ prime} \mathbb{Z} | h \equiv \nu_2\ mod\ 2,$$
$$\nu_p \equiv 0\ mod\ p,\ p > 2\},$$

All the Ψ-ring extensions of $K(S^{2n})$ by $\widetilde{K}(S^{2n'})$ have the form (8.1). The Ψ-operations on $\Psi^k : X \to X$ are given by

$$\psi^k(m,r) = (k^{n'}m + \nu_k r, k^n r),$$

for some $\nu_k \in \mathbb{Z}$.

$$\Psi^k(\Psi^l(m,r)) = (k^{n'}l^{n'}m + k^{n'}\nu_l r + \nu_k l^n r, k^n l^n r),$$
$$\Psi^l(\Psi^k(m,r)) = (l^{n'}k^{n'}m + l^{n'}\nu_k r + \nu_l k^n r, l^n k^n r).$$

Since the Ψ-operations commute, we get that

$$\nu_l r(k^{n'} - k^n) = \nu_k r(l^{n'} - l^n).$$

If $n = n'$ then there is no restriction on the choice of ν_p for p prime. Otherwise we can rearrange the above to get that

$$\nu_l = \nu_k \frac{(l^{n'} - l^n)}{(k^{n'} - k^n)}.$$

By setting $k = 2$ we get that for all $l \geq 2$

$$\nu_l = \nu_2 \frac{(l^{n'} - l^n)}{(2^{n'} - 2^n)}.$$

We can write all the ν_l's as multiples of ν_2 since

$$\nu_l = \nu_2 \frac{(l^{n'} - l^n)}{(2^{n'} - 2^n)} = \nu_2 \frac{(k^{n'} - k^n)}{(2^{n'} - 2^n)} \frac{(l^{n'} - l^n)}{(k^{n'} - k^n)} = \nu_k \frac{(l^{n'} - l^n)}{(k^{n'} - k^n).}$$

Since ν_2 is an integer, we get that $\nu_2 = \frac{z(2^{n'} - 2^n)}{G_{n,n'}}$ for some integer z.

If we replace the generator β by $\beta + N\alpha$, note that $(\beta + N\alpha)^2 = h\alpha$, then we have to replace ν_k by $\nu_k + N(k^{n'} - k^n)$. We get that

$$\nu_k + N(k^{n'} - k^n) = \nu_2 \frac{k^{n'} - k^n}{2^{n'} - 2^n} + N(k^{n'} - k^n) = \frac{(\nu_2 + N(2^{n'} - 2^n))(k^{n'} - k^n)}{(2^{n'} - 2^n)}.$$

So we only have to be concerned with replacing ν_2 by $\nu_2 + N(2^{n'} - 2^n)$, then our usual formula for ν_k holds. Hence we are replacing $\frac{z(2^{n'}-2^n)}{G_{n,n'}}$ by

$$\frac{z(2^{n'} - 2^n)}{G_{n,n'}} + N(2^{n'} - 2^n) = \frac{(z + NG_{n,n'})(2^{n'} - 2^n)}{G_{n,n'}}.$$

This proves theorem 8.7. The isomorphism depends on n and n'. If we now introduce the property that $\Psi^p(x) \equiv x^p \bmod p$, we get that $\nu_2 r \equiv hr^2 \bmod 2$ and $\nu_p r \equiv 0 \bmod p$ for $p \geq 3$. This proves corollary 8.8.

Proposition 8.9. *If there exists an extension in* $Ext_\lambda(K(S^{2n}), \widetilde{K}(S^{2n'}))$ *whose Hopf invariant is odd, then either* $n = n'$ *or* $min(n, n') \leq g^2_{|n-n'|}$*, where* g^p_j *denotes the multiplicity of the prime* p *in the prime factorisation of the greatest common divisor of the set of integers* $\{(k^j - 1)|\ k \in \mathbb{N} - \{1, qp|\forall q \in \mathbb{N}\}\}$.

Proof. The case when $n = n'$ is clear. Assume that $n \neq n'$, then the special Ψ-ring extensions are given by a pair (h, ν) where h is the Hopf invariant. By 8.8, h can only be odd if 2^n divides $G_{n,n'}$. Assume that $n < n'$, since the other case is analogous. The multiplicity of 2 in the prime factorisation of $G_{n,n'}$ is n if $n \leq g^2_{|n-n'|}$ or $g^2_{|n-n'|}$ if $g^2_{|n-n'|} < n$. It follows that if $n \leq g^2_{|n-n'|}$ then 2^n divides $G_{n,n'}$. $\square$

Note that $g^2_{2n-1} = 1$ for all $n \in \mathbb{N}$. Since $(k^{2n} - 1) = (k^n + 1)(k^n - 1)$ it follows that $g^2_{2n} = \begin{cases} 3, & n \text{ odd} \\ g^2_n + 1, & n \text{ even.} \end{cases}$

Theorem 8.10. *If there exists an extension in* $Ext_\lambda(K(S^{2n}), \widetilde{K}(S^{2n'}))$ *whose Hopf invariant is odd, then one of the following is satisfied*

1. $n = n'$,
2. $n = 1$ *or* $n' = 1$,
3. $n' - n$ *is even and either* $n = 2$ *or* $n' = 2$,
4. $n' > n \geq 3$ *and* $n' = n + 2^{n-2}b$ *for some* $b \in \mathbb{N}_0$,

5. $n > n' \geq 3$ *and* $n = n' + 2^{n'-2}b$ *for some* $b \in \mathbb{N}_0$.

Proof. 1. is clear.
2. follows from $g_n^2 \geq 1$ for all $\mathbb{N}$.
3. follows from $g_{2n}^2 \geq 3$ for all $n \in \mathbb{N}$.
4. and 5. follows from $g_{|n-n'|}^2$ being 2 plus the multiplicity of 2 in the prime factorisation of $|n - n'|$. $\square$

Corollary 8.11. *If there exists an extension in* $Ext_\lambda(K(S^{2n}), \widetilde{K}(S^{2(n+k)}))$ *whose Hopf invariant is odd, then one of the following is satisfied*

1. $k = 0$,

2. $n = 1$,

3. k *is even and* $n = 2$,

4. $n \geq 3$ *and* $k = n + 2^{n-2}b$ *for some* $b \in \mathbb{N}_0$.

Lemma 8.12. *If there exists an extension in* $Ext_\lambda(K(S^{2n}), \widetilde{K}(S^{2an}))$ *for* $a \in \mathbb{N}$ *whose Hopf invariant is odd, then one of the following is satisfied*

1. $n = 1, 2$ *or* 4,

2. $n = 3$ *and* a *is even,*

3. $n \geq 5$ *and* $an = 2n + 2^{n-2}b$ *for some* $b \in \mathbb{N}_0$.

Corollary 8.13. *If there exists an extension in* $Ext_\lambda(K(S^{2n}), \widetilde{K}(S^{4n}))$ *whose Hopf invariant is odd, then* $n = 1, 2$ *or* 4.

Corollary 8.14 (Adams)**.** *If* $f : S^{4n-1} \to S^{2n}$ *is a continuous map whose Hopf invariant is odd, then* $n = 1, 2$ *or* 4.

8.4 Stable Ext groups of spheres

Proposition 8.15. *If $n > k+1$ then $G_{n,n+k} = G_{n+1,n+k+1}$.*

Proof. Let $n > k+1$. We know that $G_{n,n+k} = G_{n+1,n+k+1}$ if and only if the multiplicity of any prime p in the prime factorization of $G_{n,n+k}$ is g_k^p. For all primes $p > 2$ we get that $p^n > 2^k - 1$, so the multiplicity of p in the prime factorisation of $G_{n,n+k}$ is g_k^p. We can easily see that $g_k^2 \leq k+1$ for all k. It follows that the multiplicity of 2 in the prime factorisation of $G_{n,n+k}$ is g_k^2. $\square$

Corollary 8.16. *If $n > k+1$ then*

$$\mathrm{Ext}_\lambda(K(S^{2n}), \widetilde{K}(S^{2(n+k)})) \cong \mathrm{Ext}_\lambda(K(S^{2(n+1)}), \widetilde{K}(S^{2(n+k+1)})).$$

The groups $\mathrm{Ext}_\lambda(K(S^{2n}), \widetilde{K}(S^{2(n+k)}))$ are independent of n for $n > k+1$, we call these the *stable Ext groups of spheres* which we denote by Ext^s_{2k}.

Proposition 8.17. *There are natural transformations*

$$\Upsilon_k : \pi^s_{2k-1} \to \mathrm{Ext}^s_{2k},$$

where π^s_{2k-1} denotes the stable homotopy groups of spheres.

For small k these groups look as follows.

k	π^s_{2k-1}	Ext^s_{2k}
1	$\mathbb{Z}_2$	$2\mathbb{Z} \oplus \mathbb{Z}_2$
2	$\mathbb{Z}_{24} \oplus \mathbb{Z}_3$	$2\mathbb{Z} \oplus \mathbb{Z}_{24}$
3	0	$2\mathbb{Z} \oplus \mathbb{Z}_2$
4	$\mathbb{Z}_{240}$	$2\mathbb{Z} \oplus \mathbb{Z}_{240}$
5	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$	$2\mathbb{Z} \oplus \mathbb{Z}_2$
6	$\mathbb{Z}_{504}$	$2\mathbb{Z} \oplus \mathbb{Z}_{504}$
7	$\mathbb{Z}_3$	$2\mathbb{Z} \oplus \mathbb{Z}_2$
8	$\mathbb{Z}_{480} \oplus \mathbb{Z}_2$	$2\mathbb{Z} \oplus \mathbb{Z}_{480}$

Appendix A

Adams Operations

$$\Psi^k(r) = \sum_{i=1}^{k-1}(-1)^{i+1}\lambda^i(r)\Psi^{k-i}(r) + (-1)^{k+1}k\lambda^k(r)$$

$$\begin{aligned}
\Psi^1(r) =& r\\
\Psi^2(r) =& r^2 - 2\lambda^2(r)\\
\Psi^3(r) =& r^3 - 3r\lambda^2(r) + 3\lambda^3(r)\\
\Psi^4(r) =& r^4 - 4r^2\lambda^2(r) + 4r\lambda^3(r) + 2(\lambda^2(r))^2 - 4\lambda^4(r)\\
\Psi^5(r) =& r^5 - 5r^3\lambda^2(r) + 5r^2\lambda^3(r) + 5r(\lambda^2(r))^2 - 5r\lambda^4(r) - 5\lambda^2(r)\lambda^3(r) + 5\lambda^5(r)\\
\Psi^6(r) =& r^6 - 6r^4\lambda^2(r) + 6r^3\lambda^3(r) + 9r^2(\lambda^2)^2 - 6r^2\lambda^4 - 12r\lambda^2(r)\lambda^3(r)\\
&+ 6r\lambda^5(r) - 2(\lambda^2(r))^3 + 3(\lambda^3(r))^2 + 6\lambda^2(r)\lambda^4(r) - 6\lambda^6(r)\\
\Psi^7(r) =& r^7 - 7r^5\lambda^2(r) + 7r^4\lambda^3(r) + 14r^3(\lambda^2(r))^2 - 7r^3\lambda^4(r) - 21r^2\lambda^2(r)\lambda^3(r)\\
&+ 7r^2\lambda^5(r) - 7r(\lambda^2(r))^3 + 7r(\lambda^3(r))^2 + 14r\lambda^2(r)\lambda^4(r) - 7r\lambda^6(r)\\
&+ 7(\lambda^2)^2\lambda^3(r) - 7\lambda^3(r)\lambda^4(r) - 7\lambda^2(r)\lambda^5(r) + 7\lambda^7(r)\\
\Psi^8(r) =& r^8 - 8r^6\lambda^2(r) + 8r^5\lambda^3(r) + 20r^4(\lambda^2(r))^2 - 8r^4\lambda^4(r) - 32r^3\lambda^2(r)\lambda^3(r)\\
&+ 8r^3\lambda^5(r) - 16r^2(\lambda^2(r))^3 + 12r^2(\lambda^3(r))^2 + 24r^2\lambda^2(r)\lambda^4(r) - 8r^2\lambda^6(r)\\
&+ 24r(\lambda^2(r))^2\lambda^3(r) - 16r\lambda^3(r)\lambda^4(r) - 16r\lambda^2(r)\lambda^5(r) + 8r\lambda^7(r) + 2(\lambda^2(r))^4\\
&- 8\lambda^2(r)(\lambda^3(r))^2 + 4(\lambda^4(r))^2 - 8(\lambda^2(r))^2\lambda^4(r) + 8\lambda^3(r)\lambda^5(r)\\
&+ 8\lambda^2(r)\lambda^6(r) - 8\lambda^8(r)
\end{aligned}$$

Appendix B

Universal Polynomials $P_i, P_{i,j}$

For more information on the universal polynomials, refer to the thesis of Hopkinson [12]. He has several results and gives the polynomial P_i upto $i = 10$, as well as giving several formulas for the polynomial $P_{i,j}$.

- $P_1(s_1; \sigma_1) = s_1\sigma_1$
- $P_2(s_1, s_2; \sigma_1, \sigma_2) = s_1^2\sigma_2 - 2s_2\sigma_2 + s_2\sigma_1^2$
- $P_3(s_1, s_2, s_3; \sigma_1, \sigma_2, \sigma_3) = s_1^3\sigma_3 + s_1s_2\sigma_1\sigma_2 - 3s_1s_2\sigma_3 + s_3\sigma_1^3 - 3s_3\sigma_1\sigma_2 + 3s_3\sigma_3$
- $P_4(s_1, s_2, s_3, s_4; \sigma_1, \sigma_2, \sigma_3, \sigma_4) = -2s_1s_3\sigma_2^2 + 2s_4\sigma_2^2 + 4s_4\sigma_1\sigma_3 - 4s_1^2s_2\sigma_4 - 2s_2^2\sigma_1\sigma_3 - 4s_4\sigma_1^2\sigma_2 + 4s_1s_3\sigma_4 + s_1^2s_2\sigma_1\sigma_3 + s_1s_3\sigma_1^2\sigma_2 - s_1s_3\sigma_1\sigma_3 + s_1^4\sigma_4 + s_2^2\sigma_2^2 + 2s_2^2\sigma_4 + s_4\sigma_1^4 - 4s_4\sigma_4$

- $P_{1,1}(s_1) = s_1$
- $P_{1,j}(s_1, \ldots, s_j) = s_j$
- $P_{i,1}(s_1, \ldots, s_i) = s_i$
- $P_{2,j}(s_1, \ldots, s_{2j}) = \sum_{k=1}^{j-1}(-1)^{k+1}s_{j-k}s_{j+k} + (-1)^{j+1}s_{2j}$

Consider the polynomials

- $P_{2,4}(s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8) = s_3s_5 - s_2s_6 + s_1s_7 - s_8$
- $P_{4,2}(s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8) = s_1s_3s_4 - 3s_1s_2s_5 + s_1^3s_5 - s_4^2 + s_3s_5 - s_1^2s_6 + s_1s_7 + 2s_2s_6 - s_8$
- $P_{5,2}(s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}) = s_1^4s_6 + s_2s_4^2 + 3s_1s_2s_7 + 3s_1s_3s_6 - 4s_1^2s_2s_6 - 2s_1s_4s_5 - 2s_2s_3s_5 + s_1^2s_3s_5 + s_{10} - s_3s_7 + 2s_5^2 - s_1^3s_7 - 2s_4s_6 + 2s_2^2s_6 + s_1^2s_8 - s_1s_9 - 2s_2s_8$

So we can see that in general $P_{i,j} \neq P_{j,i}$.

Appendix C

Universal Polynomial Partial Derivatives

k	$\frac{\partial P_1(r,s)}{\partial \lambda^k(r)}$	$\frac{\partial P_2(r,s)}{\partial \lambda^k(r)}$	$\frac{\partial P_3(r,s)}{\partial \lambda^k(r)}$	$\frac{\partial P_4(r,s)}{\partial \lambda^k(r)}$
1	$\Psi^1(s)$	$r(s^2 - \Psi^2(s))$	$\ddots$	$\ddots$
2	0	$\Psi^2(s)$	$r(s^3 - \Psi^3(s))$	$\ddots$
3	0	0	$\Psi^3(s)$	$r(s^4 - \Psi^4(s))$
4	0	0	0	$\Psi^4(s)$

Conjecture C.1. *For all $i \in \mathbb{N}$*

$$\frac{\partial P_i(r,s)}{\partial \lambda^i(r)} = \Psi^i(s), \ \frac{\partial P_{i+1}(r,s)}{\partial \lambda^i(r)} = r(s^{i+1} - \Psi^{i+1}(s))$$

From the other universal polynomial, we get

$$\frac{\partial P_{1,n}(r)}{\partial \lambda^k(r)} = \begin{cases} 1 & k = n \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\partial P_{2,n}(r)}{\partial \lambda^k(r)} = \begin{cases} 0 & k = n, \text{ or } k > 2n \\ (-1)^{k+1}\lambda^{2n-k}(r) & \text{otherwise} \end{cases}$$

$$\frac{\partial P_{i,j}(r)}{\partial \lambda^{ij}(r)} = (-1)^{(i+1)(j+1)}$$

k	$\frac{\partial P_{4,2}(r)}{\partial \lambda^k(r)}$	$\frac{\partial P_{5,2}(r)}{\partial \lambda^k(r)}$
3	$r\lambda^4(r) + \lambda^5(r)$	$\ddots$
4	$r\lambda^3(r) - 2\lambda^4(r)$	$\ddots$
5	$\Psi^3(r) - 2\lambda^3(r)$	$\ddots$
6	$-\Psi^2(r)$	$\Psi^4(r) - r\lambda^3(r) + 2\lambda^4(r)$
7	r	$-\Psi^3(r) + 2\lambda^3(r)$
8	-1	$\Psi^2(r)$
9	0	$-r$
10	0	1

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