Selected topics in Dirichlet problems for linear parabolic stochastic partial differential equations

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by

Vasile Nicolae Stanciulescu
Department of Mathematics,
University of Leicester,
United Kingdom.

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In memory of my mother (1940-2007)

and

to my father

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Vasile Nicolae Stanciulescu

Abstract

This thesis is devoted to the study of Dirichlet problems for some linear parabolic SPDEs. Our aim in it is twofold.

First, we consider SPDEs with deterministic coefficients which are smooth up to some order of regularity. We establish some theoretical results in terms of existence, uniqueness and regularity of the classical solution to the considered problem. Then, we provide the probabilistic representations (the averaging-over-characteristic formulas) of its solution. We, thereafter, construct numerical methods for it. The methods are based on the averaging-over-characteristic formula and the weak-sense numerical integration of ordinary stochastic differential equations in bounded domains. Their orders of convergence in the mean-square sense and in the sense of almost sure convergence are obtained. The Monte Carlo technique is used for practical realization of the methods. Results of some numerical experiments are presented. These results are in agreement with the theoretical findings.

Second, we construct the solution of a class of one dimensional stochastic linear heat equations with drift in the first Wiener chaos, deterministic initial condition and which are driven by a space-time white noise and the white noise. This is done by giving explicitly its Wiener chaos decomposition. We also prove its uniqueness in the weak sense. Then we use the chaos expansion in order to show that the unique weak solution is an analytic functional with finite moments of all orders. The chaos decomposition is also utilized as a very useful tool for obtaining a continuity property of the solution.

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Introduction

In the last decades, the need to describe important phenomena appearing in control theory, chemistry, physics and biology strongly motivated the study of stochastic partial differential equations (SPDEs). As a special class of SPDEs, stochastic evolution equations can model a broad spectrum of problems in science and engineering. In this respect, there are several important examples of applications such as reaction diffusion equation [4], [38], Krylov equation or backward diffusion equation (see [111] and the references therein), Zakai's equation which arises in filtering theory [126, 64, 106]. SPDEs are utilized to simulate a random motion of a string [40] or changes in the structure of population [24]. They are also used to model a free field connected with the relativistic quantum theory [58] or an electrical potential of nerve cells utilized in neurophysiology [57], [88]. Other important examples of SPDEs are the equation of the lift of a diffusion process [13], the Markovian lifting equation which arises in the study of stochastic delay equations [121, 14, 28], the Helmholtz parabolic equation which models the diffraction in a random nonuniform medium [110, 66, 67], the equation of the number of particles related to continuous branching models with geographical structure used in chemistry and population biology [36] or the equation of stochastic quantization [8, 63, 32].

Concerning the existence, uniqueness and regularity of solutions to SPDEs, important contributions are due to Krylov and Rozovskii [73, 69, 68, 75], Krylov [70, 74, 72, 71], Rozovskii [111], Pardoux [105, 107, 106], Kunita [77], Da Prato and Zabzyck

[22], Nualart and Zakai [101], Nualart [103], Flandoli [34], Tubaro [119], Lototskii and Rozovskii [84], Walsh [123], Brzezniak [9] and others.

Taking into account that explicit solutions to SPDEs are rarely available, there has recently been a growing interest in the construction and development of numerical methods for SPDEs. Several important results in this direction were obtained by Milstein and Tretyakov [94, 95, 96], Crisan and Lyons [20], Crisan [18], [19], Gyöngy and Krylov [51], [55], Gyöngy [49], [50], Itô and Rozovskii [62], Kushner [79], Picard [108], Clark [17], Del Moral and Miclo [27], Grecksch and Kloeden [43], Allen, Novosel and Zhang [1], Le Gland [80], Yoo [125], Debussche and Printems [25] and others.

This thesis is concerned with the investigation of Dirichlet problems for some linear parabolic SPDEs. It is divided into two parts.

In the first part of the thesis, we consider the problem assuming that the coefficients are deterministic and sufficiently smooth up to some order of regularity. The starting point is to present some results concerning existence and uniqueness of the classical solution to the Dirichlet problem for the considered SPDEs. Moreover, we give the probabilistic representation of the solution. Next, we construct some numerical methods for the SPDEs and we study their order of convergence. The results of some numerical experiments which support the theoretical results close this part of thesis.

The second part of the thesis deals with the study of some anticipating heat equations with the drift in the first chaos and space-time noise or white noise potentials.

Utilizing the Wiener chaos decomposition, we construct their solutions. We also prove an uniqueness result. Last, we exploit the Wiener chaos expansion to obtain some moment estimates and continuity properties of the solutions to SPDEs.

The area of SPDEs is very broad and fast developing. Here we concentrate on the Dirichlet problems for both nonanticipating and anticipating SPDEs of some specific types. We demonstrate different theoretical approaches to the notion of solution of SPDEs in both parts of the thesis as well as different techniques in addressing existence and uniqueness of the solutions. We propose new numerical methods in the nonanticipating case. Even within our study of a relatively limited scope, we have tried to demonstrate how rich the field of SPDEs is.

Part I

Numerical solution of the Dirichlet problem for linear parabolic SPDEs based on averaging over characteristics

Introduction

Stochastic linear evolution equations are used in various applications which are arising in physics, chemistry or biology. They are exploited for describing, observing and modeling different kind of phenomena of nature and technical problems. One of this important applications, referred often as the filtering problem, consists in estimation of an unobservable signal which could represent the true coordinates of the object by using an observed one which describes the position of a moving object. Taking into account that this problem appears, for example, in radar detection, it is crucial to know the behaviour of the solutions of SPDEs particularly those of parabolic type which are modeling this problem. The famous result related to the nonlinear filtering problem, referred to as Zakai's equation (see [126]), states that under suitable conditions given the observation process the unnormalized density for the signal process satisfies a class of evolution SPDEs. Due to their importance in applications, a lot of attention has been paid to SPDEs in terms of existence, uniqueness and regularity of their solutions. Some important results in this direction are presented in [74, 72, 111, 22, 77, 34, 106, 122, 84, 9, 7] (see also references therein).

Looking at the importance of SPDEs from the practical point of view, it is of main interest to simulate their solutions. Unfortunately the examples of SPDEs for which one can give their solutions explicitly are very rare but this difficulty may be overcome through the help of numerical methods. In the recent decades several approaches for constructing numerical approximations of solutions to linear and semilinear SPDEs have been exploited.

In this part of the thesis our goal is to construct some numerical methods for the Dirichlet problem for a class of linear parabolic SPDEs. It is organized as follows.

In the first chapter we briefly present a survey of some important numerical results for different classes of SPDEs.

The second chapter gives some preliminary materials needed throughout the first part of the thesis.

The third chapter contains three sections. In the first section we introduce the framework used in this part of the thesis and we set the problem which we will study in what follows. In the second one we establish an existence and uniqueness result of the classical solution to our problem. The third section deals with the probabilistic representation of its solution.

In Chapter 4 we propose numerical schemes for our studied object introduced in Section 3.1 and we obtain their order of convergence in the mean-square sense and in the sense of almost convergence.

The last chapter is devoted to some numerical experiments which support the results obtained.

This part of the thesis is based on our joint work with M. Tretyakov (to be submitted).

Chapter 1

A survey of numerical methods for SPDEs

We give a brief survey of some results in the direction of numerical schemes for SPDEs.

One of the important tools arising in the theory of SPDEs is the method of characteristics (see [69, 68, 106, 111, 77, 34]), which has been used to construct and to develop some approximation schemes for the solutions to different classes of SPDEs. It seems that the first study in this direction was done in [108]. The considered approach is based on the direct Monte Carlo simulation of the numerator and denominator in the Kallianpur-Striebel formula. When approached the Cauchy problem for SPDEs, Milstein and Tretyakov in [94, 95] proposed a number of approximation methods and studied their mean-square and almost sure convergence. The numerical schemes approximate solution of the characteristic system of SDEs through the help of mean-square and weak-sense numerical integration. Crisan and Lyons in [20] and Crisan in [18, 19] utilized the method of characteristics to propose a particle method and to evaluate its speed order of convergence in the weak mean-square norm and in particular in mean for the Kushner-Stratonovich and Zakai equations. Gobet, Pages, Pham and Printems in [42] considered the Zakai equation of the nonlinear filtering problem with space-dependent coefficients. They employed a Feynmann-Kac formula for it to obtain an Euler-type scheme, for which they got an error estimate in the L^p-quantization norm using techniques of the Malliavan calculus. It is to be noted that using the averaging-over-characteritics formula together with the help of the innovation process, some numerical methods for the nonlinear filtering problem and their mean-square and almost sure convergence are also presented in [96]. Kurtz and Xiong in [78] also used the probabilistic representations of solution to SPDEs to obtain other approaches when they constructed numerical methods for some classes of SPDEs. The Feynmann-Kac formula for SPDEs was exploited by Brzezniak, Capinski and Flandoli in [10] in the case of a class of elliptic SPDEs in the Stratonovich form with space dependent coefficients in a bounded domain. It was also utilized by Brzezniak and Flandoli in [11] in the case of a class of parabolic SPDEs in the Stratonovich form with space time coefficients in R^d, $d \geq 1$. In both cases, the authors proved the mean-square and almost sure convergence of the approximation of Wong-Zakai type for the corresponding solution to SPDEs.

Other ways have been used by several authors for approximating solutions of various classes of SPDEs. Yoo in [125] and Germani and Piccini in [41] considered the Cauchy problem for parabolic SPDEs. Using the weighted Sobolev L^p-theory, they obtained a sup-norm error estimate for a finite-difference approximation scheme and a mean-square error estimate for a finite-element method, respectively. Davie and Gaines in [23] were concerned by the convergence of some finite-difference approximations to solutions of a class of parabolic SPDEs with the assumptions that the SPDEs have an initial condition and periodic boundary conditions on $[0, 1]$, the noise which is driven the SPDEs is the space-time white noise and the parabolic operator is the Laplacian. In this case, they gave an L^p-evaluation of the error of convergence. Gyöngy and Krylov in [55] considered the Cauchy problem for a couple of two parabolic SPDEs in the Stratonovich form driven by the same multidimensional martingale noise and by different increasing processes. They studied the convergence of an approximate scheme using the splitting-up method and get a sup-norm estimate in the Sobolev norm for its error. Le Gland in [80] and Bensoussan, Glowinski and Rascanu in [6] also presented approaches through a splitting-up method and proved its rate of convergence in some appropriate norms. Wiener chaos decomposition, used by Lototsky, Mikulevicius and Rozovskii in [85], provides another method to obtain approximation schemes for the solution to SPDEs and to compute their error estimates in the mean square norm. Another technique, often called the Wong-Zakai approximation, was utilized by Gyöngy and Shmatkov in [53] and Gyöngy and Michaletzki in [54] to estimate the rate of convergence of the approximations for the solution to the Cauchy problem for SPDEs in the Sobolev norm. Using a finite-element method in space and an implicit Euler scheme in time, Printems and Debussche obtained in [25] a weak method which approximates the solution of a class of SPDEs particularly the stochastic heat equation. In [42] Printems used an implicit Euler scheme to semi-discretize in time a nonlinear evolution SPDE. He defined the order of convergence in probability and proved that this order of convergence is less than 1/4.

Concerning the Dirichlet problem for SPDEs, several numerical methods and convergence for them have been obtained in different frameworks. Gyöngy and Nualart in [52] and Gyöngy in [49, 50] studied one dimensional SPDEs which are driven by the space-time white noise, have nonlinear coefficients and the parabolic operator is the Laplacian. Utilizing an implicit approximation scheme based on the finite-difference technique, they estimated the rate of convergence of the approximations in the L^p norm and also proved their almost sure uniform convergence to the solution. The same type of object as that of the previous authors but in higher dimensions was considered by Chow and Jian in [15]. They constructed an approximation scheme via discretization in time and another one by an eigenfunction expansion and showed that they converge almost surely to the solution of equation. Yan in [124] and Shardlow in [112] treated the stochastic equation in some more general settings concerning the parabolic operator and the noise which drives the stochastic equation. Yan dealt with a stochastic differential equation with additive noise which takes values in a Hilbert space H and assumed that the parabolic operator, denoted by A, is linear, positive definite, selfadjoint, not necessarily bounded but with compact inverse and with its domain $D(A)$ densely defined. Under further assumptions concerning the coefficients of the stochastic differential equation, he constructed with the help of Galerkin's method an approximate scheme for its solution and estimated in the mean square the rate of its convergence with respect to H-norm and $\dot{H}^{-1}$-norm (i.e., $||A^{-1/2}\cdot||$-associated norm of the space H). As an immediate consequence for the Dirichlet problem when the operator is the Laplacian the mean-square error of the convergence scheme is evaluated in the L^2-norm and fractional Sobolev space $H^{-1/2}$-norm with respect to the spatial variables. In Shardlow's paper [112] the SPDE is driven by the space-time noise, the operator is the Laplacian and the coefficients of the SPDE have some nonlinearity. The numerical method for the solution was constructed with the help of finite difference technique and spectral decomposition for the approximated noise. The mean-square estimation of the order of convergence is computed in L^2-norm with respect to the spatial variables. Millet and Morien in [90] considered the Laplacian for the parabolic operator but the noise which forces the SPDEs is a more general one. They approached a class of SPDEs perturbed by a space correlated valued centered Gaussian noise. The error of convergence of the proposed explicit and implicit time-space discretization schemes was estimated uniformly in $L^{2p}-$norm with $p \geq 1$.

We are interested in investigating approximations of the solution to the Dirichlet problem for SPDEs when the noise is a standard scalar Wiener process. Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{T_0 \le t \le T})$ be a filtered probability space and

$$(w(t), \mathcal{F}_t) = ((w_1(t), \dots, w_q(t))^\top, \mathcal{F}_t)$$

be a q-dimensional standard Wiener process. Throughout the first part of the thesis we will treat the following class of first boundary problem for SPDEs:

$$\begin{aligned} -dv &= [\mathcal{L}v + f(t,x)]\,dt + \sum_{r=1}^{q} [\mathcal{M}_r v + \gamma_r(t,x)] * dw_r(t), && (1.1) \\ (t,x) &\in [T_0, T) \times G, \\ v(T,x) &= \psi(x)\ ,\ x \in G\ ,\ v(t,x) = \varphi(t,x),\ (t,x) \in [T_0, T] \times \partial G, && (1.2) \end{aligned}$$

where G is a bounded domain in R^d, $d \ge 1$, "$*dw_r$" means backward Itô integral. The operators $\mathcal{L}$ and $\mathcal{M}_r$, for $r = 1, \dots, q$, have the following forms

$$\begin{aligned} \mathcal{L}v(t,x) &:= \frac{1}{2} \sum_{i,j=1}^{d} a^{ij}(t,x) \frac{\partial^2}{\partial x^i \partial x^j} v(t,x) + \sum_{i=1}^{d} b^i(t,x) \frac{\partial}{\partial x^i} v(t,x) + c(t,x) v(t,x), \\ \mathcal{M}_r v(t,x) &:= \beta_r(t,x) v(t,x) + \gamma_r(t,x),\ r = 1, \dots, q. \end{aligned}$$

The coefficients which appear in the above SPDE and the boundary ∂G are the subjects of some compatibility conditions which will be given later.

When $\gamma_r = 0$, $r = 1, \dots, q$, one can see that the SPDE from above is nothing that backward Zakai's equation in a bounded domain. It is known that, for a given backward SPDE, one can write down the corresponding forward SPDE, and vice versa. Also the theory of backward Itô equations is identical with the one developed for the forward ones (see [111, 77]). Therefore, by duality arguments, the methods for the backward SPDEs (1.1)-(1.2) used in the thesis can be used for solving the forward ones as well.

Let us go back to the framework of Dirichlet problem for SPDEs with scalar Wiener noise. To the best of our knowledge, numerical methods in this direction have been obtained by Grecksch and Kloeden in [43], Chow, Jiang and Menaldi in [16] and Hausenblas in [56]. In the brief review below we preserve the notation, and we present the results as they formulated in the original works.

We start with the paper by Grecksch and Kloeden [43]. They considered a bounded domain D in R^d with sufficiently smooth boundary ∂D and the following parabolic SPDE

$$dU_t = \{AU_t + f(U_t)\}dt + g(U_t)dW_t, \tag{1.3}$$

where $\{W_t,\, t \geq 0\}$ is a standard scalar Wiener process, with the Dirichlet boundary condition

$$U|_{\partial D} = 0 \tag{1.4}$$

and initial condition $U_0 \in H_0^{1,2}(D)$. Here $H_0^{1,2}(D)$ is the space of functions $u : D \to R^1$ which vanish on ∂D such that u and its first-order generalized derivatives Du belong to $L_2(D)$ with norm $||\cdot||$ and A is a linear operator which is densely defined in $L_2(D)$ by $\{v \in H_0^{1,2}(D) : Av \in L_2(D)\}$ such that $-A$ is strongly monotone, that is there is a positive constant α such that

$$(-Au, u)_{H_0^{1,2}(D)} \geq \alpha |u|^2_{H_0^{1,2}(D)},\ u \in H_0^{1,2}(D),$$

and

$$(-Au, u)_{L^2(D)} \geq \alpha ||u||^2_{L^2(D)},\ u \in L_2(D),$$

where $(\cdot,\cdot)_{L^2(D)}$ is the inner product and $||\cdot||_{L^2(D)}$ the associated norm of the space $L_2(D)$ and $(\cdot,\cdot)_{H_0^{1,2}(D)}$ is the inner product and $|\cdot|_{H_0^{1,2}(D)}$ the associated norm of the space $H_0^{1,2}(D)$, respectively. In addition f and g, which map either $L_2(D)$ or $H_0^{1,2}(D)$ into itself, are formed from real valued functions of a real variable with uniformly bounded derivatives of an appropriate order, into which the numerical values $U = U(t, x, \omega)$ are inserted.

The eigenvalues λ_j and corresponding eigenfunctions $\varphi_j \in H_0^{1,2}(D)$ of the operator $-A$, solving

$$-A\varphi_j = \lambda_j \varphi_j,\ j = 1,\ 2, \ldots,$$

form an orthonormal basis in $L_2(D)$ with $\lambda_j \to \infty$ as $j \to \infty$.

Consider the N dimensional subspace χ_N of $H_0^{1,2}(D)$ spanned by $\{\varphi_1, \ldots, \varphi_N\}$ and let P_N denote the projection of $L_2(D)$ or $H_0^{1,2}(D)$ onto χ_N . Write U^N synonymously for $(U^{N,1}, \ldots, U^{N,N}) \in R^N$ and $\sum_{j=1}^{N} U^{N,j} \in \chi_N$ according to the context and define $A_N = P_N A|_{\chi_N}$, $f_N = P_N f|_{\chi_N}$ and $g_N = P_N g|_{\chi_N}$, where f and g are now interpreted as mappings of $L_2(D)$ or $H_0^{1,2}(D)$ into itself. The N dimensional Itô-Galerkin SDE corresponding to the SPDE (1.3) with its boundary condition (1.4) is then

$$dU_t^N = \{A_N U_t^N + f_N(U_t^N)\}dt + g_N(U_t^N)dW_t. \tag{1.5}$$

Let

$$\mathcal{M} = \{\alpha = (j_1, \ldots, j_l) \in \{0, 1, 2, \ldots, m\}^l : l \in \mathbf{N}\} \cup \{\nu\}$$

be the set of all multi-indices. The lenght $l(\alpha)$ of a multi-indidex $\alpha = (j_1, \ldots, j_l)$ is defined as $l(\alpha) = l$ whereby ν is the multi-index of length 0. In addition let $n(\alpha)$ denote the number of components of a multi-index α which are equal to 0. For $\alpha = (j_1, \ldots, j_l) \in \mathcal{M}$ with $l(\alpha) \geq 1$, we define $-\alpha = (j_2, \ldots, j_l)$ and $\alpha- = (j_2, \ldots, j_{l-1})$ by deleting the first and the last component of α, respectively. A subset $\mathcal{A} \subset \mathcal{M}$ is called a hierarchical set if $\mathcal{A} \neq \emptyset$ and if

$$\sup_{\alpha \in \mathcal{A}} l(\alpha) < \infty$$

and

$$-\alpha \in \mathcal{A} \text{ for each } \alpha \in \mathcal{A} \backslash \{\nu\}.$$

The corresponding remainder set $\mathcal{R}(\mathcal{A})$ for the hierachical set $\mathcal{A}$ is defined as

$$\mathcal{R}(\mathcal{A}) = \{\alpha \in \mathcal{M} \backslash \mathcal{A} : -\alpha \in \mathcal{A}\}$$

and consists of all next following multi-indices with respect to the given hierarchical set $\mathcal{A}$.

Consider the hierarchical set

$$A_\gamma = \{\alpha \in \mathcal{M} : l(\alpha) + n(\alpha) \leq 2\gamma \text{ or } l(\alpha) = n(\alpha) = \gamma + \frac{1}{2}\}$$

where γ takes possible values 0.5 (Euler scheme), 1.0 (Milstein scheme), etc.

According to Kloeden and Platen [65], an order γ strong Taylor scheme with constant time-step Δ for the SDE (1.5) has the form

$$Y_{k+1}^N = Y_k^N + \sum_{\alpha \in A_\gamma \setminus \{v\}} F_\alpha^N(Y_k^N) I_{\alpha,k,\Delta}, \; Y_0^N = U_0^N, \tag{1.6}$$

with coefficient functions F_α^N and multiple stochastic integrals

$$I_{\alpha,k,\Delta} = \int_{t_k}^{t_{k+1}} \int_{t_k}^{s_l} \dots \int_{t_k}^{s_1} dW_{s_1}^{\alpha_1} \dots dW_{s_l}^{\alpha_l}$$

for all $\alpha \in A_\gamma$. The multi-indices $\alpha = (j_1, \dots, j_l) \in A_\gamma \setminus \{v\}$ have components $j_i = 0$ or 1 corresponding to integration with respect to "dt" or "dW_t", respectively, while the jth component of F_α^N is defined by

$$F_\alpha^{N,j}(Y^N) = L_N^{j_1} ... L_N^{j_l} F_j(Y^N), \tag{1.7}$$

where the operators

$$L_N^0 = \sum_{j=1}^N (-\lambda_j U^{N,j} + f^{N,j}(U^N)) \frac{\partial}{\partial U^{N,j}} + \frac{1}{2} \sum_{i,j=1}^N g^{N,i}(U^N) g^{N,j}(U^N) \frac{\partial^2}{\partial U^{N,i} \partial U^{N,j}}$$

and

$$L_N^1 = \sum_{j=1}^N g^{N,j}(U^N) \frac{\partial}{\partial U^{N,j}}$$

are applied successively to $F_j(Y^N) = Y^{N,j}$ and the result evaluated at Y^N.

After the authors have combined the Galerkin approximation scheme with the numerical scheme (1.7) the mean estimation of the error of the global space-time scheme was presented as the following main result.

Theorem 1 *The global space-time discretization error of the order γ strong Taylor scheme (1.7) with constant time-step Δ applied to the N-dimensional Itô-Galerkin approximation (1.5) of the SPDE (1.3) has the form*

$$E|U_{k\Delta} - Y_k^N| \leq K(\lambda_{N+1}^{-1/2} + \lambda_N^{[\gamma+1/2]+1}\Delta^\gamma),$$

where $[x]$ is the integer part of the real number x and the constant K depends on $E||U_0||^2$, bounds on the f, g coefficients of the SPDE and the length of the time interval $0 \leq k\Delta \leq T$ under consideration.

Let us now look at the work by Chow, Jiang and Menaldi [16]. They considered a bounded domain $D \subset R^d$ and the following Zakai equation which satisfies the homogeneous boundary condition $u_t|_{\partial D} = 0$

$$du_t(x) = L^* u_t(x)dt + u_t(x)[g(x), dw_t], \tag{1.8}$$

$$u_0(x) = v(x), \tag{1.9}$$

where $f = (f_1, \ldots, f_d) : R^d \to R^d$, $g = (g_1, \ldots, g_r) : R^d \to R^r$, $\sigma : R^d \to R^d \times R^d$, $a = \sigma\sigma^* = (a_{ij})_{i,j=1}^d$, w_t, $t \geq 0$ is a standard Brownian motion in R^r and

$$L^* u = \frac{1}{2}\sum_{i,j=1}^d \frac{\partial^2}{\partial x_i \partial x_j}(a_{ij}u) - \sum_{i=1}^d \frac{\partial}{\partial x_i}(f_i u).$$

Let $H = L^2(D)$ and $V = H_0^1(D)$, where $H_0^1(D)$ is the set of functions in Sobolev space $H^1(D)$ vanishing on ∂D. Denote by $(\cdot,\cdot)$ the inner product in H. Let $< \cdot, \cdot >$ the duality pairing between V and its dual space $V' = H^{-1}(D)$. Also $|\cdot|$ and $||\cdot||$ will denote the norms in H and V, respectively.

Let us split the operator L in the following way

$$L^* = A + B,$$

where

$$\begin{aligned}
Au &= \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial u}{\partial x_j}) - \gamma u, \\
Bu &= \sum_{i=1}^{d} \frac{\partial}{\partial x_i}(b_i u) + \gamma u, \\
b_i &= \frac{1}{2}\sum_{j=1}^{d} \frac{\partial}{\partial x_j} a_{ij} - f_i,
\end{aligned}$$

for a fixed strictly positive constant γ. After denoting by G the linear operator from $K = R^r$ into H which is defined by

$$(G(u)y)(x) = [g(x), y]u(x), \;\; x \in D,$$

we can rewrite the SPDE (1.8)-(1.9) for a fixed time $l > 0$ when $t \in (0, l]$ as a standard Itô equation in V':

$$du_t = (Au_t + Bu_t)dt + G(u_t)dw_t, \tag{1.10}$$

$$u_0 = v. \tag{1.11}$$

Denote by $C_B^k(D)$ the set of continuous functions on D which have bounded derivatives of order k. Let us assume the following conditions:

(A.1) $a_{ij} \in C_B^3(D)$, $f_i \in C_B^2(D)$ and $g_k \in C_B^1(D)$ for i, $j = 1, 2, \ldots, d$ and $k = 1, 2, \ldots, r$.

(A.2) $\exists\ \delta > 0$ such that $\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq \delta|\xi|^2$, $\forall\ x \in D$ and $\xi \in R^d$;

(A.3) $v \in C^1_B(D)$.

The operator A generates an analytic semigroup T_t and, therefore, equation (1.10)-(1.11) has the following evolutional integral form:

$$u_t = T_t v + \int_0^t T_{t-s} B u_s ds + \int_0^t T_{t-s} G(u_s) dw_s. \tag{1.12}$$

For each positive integer n let us consider the linear subspace $V_n \subset H$ spanned by the set of eigenfunctions $\{e_1, ..., e_n\}$ of the operator A. Let $P_n : H \to V_n$ be the corresponding orthogonal projection. Using the Galerkin method, we construct a sequence $\{u^n_t\}$ of approximate solution to the equation (1.10)-(1.11) defined by

$$u^n_t = v_n + \int_0^t (A + B_n) u^n_s) ds + \int_0^t G_n(u^n_s) dw_s, \tag{1.13}$$

where $u^n_t \in V_n$, $B_n = P_n B$, $G_n(u) = P_n G(u)$.

Equation (1.13) can be rewritten in the following form

$$u^n_t = T_t v_n + \int_0^t T_{t-s} B_n u^n_s) ds + \int_0^t T_{t-s} G_n(u^n_s) dw_s. \tag{1.14}$$

Partition the time interval $[0, l]$ into m equidistant intervals and define

$$\pi_m s = \frac{i}{m} l = t_i \text{ for } s \in [\frac{i}{m} l, \frac{i+1}{m} l),\ i = 0, 1, \ldots, m-1,$$

and

$$\pi_m u_s = u_{\pi_m s},\ s \in [0, l].$$

Let $\{u^m_t\}$ denote the Euler approximation which corresponds to the equation (1.14):

$$u^m_t = T_t v_n + \int_0^t T_{t-\pi_m s} B_m(u^m_s) ds + \int_0^t T_{t-\pi_m s} G_m(u^m_s) dw_s,$$

where $B_m(u_s) = B\pi_m u_s$ and $G_m(u_s) = G(\pi_m u_s)$. For $t \in (\frac{k}{m}l, \frac{k+1}{m}l]$ we have

$$u_t^m = u_{t_k}^m + T_{t-t_k} B u_{t_k}^m (t - t_k) + T_{t-t_k} G(u_{t_k}^m)(w_t - w_{t_k}), \text{ for } k = 0, 1, \ldots, m-1.$$

Denote by $C_l(H)$ the Banach space of continuous H-valued functions on $[0, l]$ with the sup-norm.

Let $u_t^{n,m}$ be the combined Euler-Galerkin approximation obtained from the above ones:

$$u_t^{n,m} = T_t v_n + \int_0^t T_{t-\pi_m s} B_{n,m}(u_s^{n,m}) ds + \int_0^t T_{t-\pi_m s} G_{n,m}(u_s^{n,m}) dw_s,$$

where

$$v_n = P_n v, \ B_{n,m}(u_t) = P_n B(P_n u_{\pi_m t}) \text{ and } G_{n,m}(u_t) = P_n G(P_n u_{\pi_m t}).$$

The order of convergence of the $u_t^{n,m}$ approximation scheme to the mild solution u_t of the equation (1.12) is evaluated pathwise in the sup-norm of $C_l(H)$ and the main result from [16] is presented as the theorem below.

Theorem 2 *Let the conditions A(1)-A(3) be satisfied. Then the sequence of finite-dimensional approximations $\{u_t^{n,m}\}$ converges almost surely to the solution u_t in $C_l(H)$ so that*

$$\sup_{0 \le t \le l} |u_t - u_t^{n,m}| = O(n^{-\alpha}) + O(m^{-\beta}) \ a.s.,$$

for any $\alpha \in (0, \frac{1}{d})$ and $\beta \in (0, \frac{1}{2})$.

Last but not the least let us now look at the work done by Hausenblas in [56]. In her paper she considered a more general framework which in particular has direct implications for a special class of the first boundary problem for SPDEs.

Assume that we have an X-valued Wiener process $W(t)$ with nuclear covariance operator Q where X is a separable Hilbert space. Also let A be the infinitesimal generator of an analytic semigroup and $D((-A)^\gamma)$ be the domain of $(-A)^\gamma$ equipped with the norm $||\cdot||_\gamma := ||(-A)^\gamma \cdot||$ for $\gamma > 0$. Consider the following evolution equation

$$du(t) = (Au(t) + f(t, u(t)))dt + \sigma(u(t))dW(t), \tag{1.15}$$

$$u(0) = u_0 \in D((-A)^\gamma). \tag{1.16}$$

Defining the space of Hilbert-Schmidt operators $L_2^0 := L_2(U_0, X)$ from $U_0 = Q^{1/2}(X)$ into X equipped with the norm $||\Psi||_{L^2}^0 := ||\Psi Q^{1/2}||_{L^2}^2$ =trace$[\Psi Q \Psi^*]$ and the subspace of all Hilbert-Schmidt operators $L_2^{0,\gamma} := L_2(U_0, D((-A)^\gamma))$ equipped with the norm $||\Psi||_{L_2^{0,\gamma}}^0 := ||(-A)^\gamma \Psi||_{L_2^0}^2$. For $\gamma > 0$, $0 \le \theta \le 1/2$, $0 \le \rho \le \min(1/2, \gamma)$ and $\rho + \theta_\sigma \le \theta$ let the following set of assumptions hold.

(H1) Suppose that $f(t, \cdot) : X \to X$ is well defined such that we have

(i) it is Lipschitz continuous, i. e.

$$||f(t,x) - f(t,y)||_\delta \le C||x-y||_\delta \text{ for } \delta \in [\rho, \gamma], x, y \in D((-A)^\gamma);$$

(ii) it satisfies the linear growth condition i.e.,

$$||f(t,x)||_\delta^2 \le K(1 + ||x||_\delta^2) \text{ for } \delta \in [\rho, \gamma], x, y \in D((-A)^\gamma);$$

(iii) it is Hölder continuous in t variable with exponent $\min(1/2, \gamma - \rho - \theta_\sigma)$

in the sense of

$$||f(t,x) - f(s,x)||_\delta \leq C|t-s|^{min(1/2,\gamma-\rho-\theta_\sigma)}||x||_{\min(1+\rho+\theta+\theta_\sigma,\gamma+\theta)},$$

$$\text{for any } x \in D((-A)^{\gamma+\theta});$$

(H2) Assume that σ is an operator such that $(-A)^{-\theta}\sigma : X \to L_2^0$ is bounded and satisfies the following hypotheses:

(i) $(-A)^{\theta_\sigma}\sigma : D((-A)^\delta) \to L_2^{0,\delta}$ is Lipschitz continuous in space, i.e.,

$$||(-A)^{\theta_\sigma}[\sigma(x) - \sigma(y)]||_{L_2^{0,\delta}} \leq C||x-y||_\delta \text{ for } \delta \in (\rho,\gamma],\ x,y \in D((-A)^\gamma;$$

(ii) a linear growth condition, i.e.,

$$||(-A)^{\theta_\sigma}\sigma(x)||_{L_2^{0,\delta}} \leq C||x||_\delta \text{ for } \delta \in (\rho,\gamma],\ x \in D((-A)^\gamma,$$

(iii) $(-A)^{-\theta}\sigma$ is globally Lipschitz, i.e., it satisfies

$$||(-A)^{-\theta}[\sigma(x) - \sigma(y)]||_{L_2^{0,\delta}} \leq \xi||x-y||_\delta \text{ for } \delta \in [\rho,\gamma],\ x,y \in D((-A)^\gamma).$$

(H3) Further we assume that the semigroup $T_A(t)$ associated to the operator A satisfies

$$\int_0^t ||T_A(s)(x-y)||^2_{\theta+\delta} ds \leq (\zeta + \phi(t))||x-y||^2_\delta, \text{ for } x,y \in D((-A)^\gamma,$$

for all $\delta \in [0,\gamma]$, for a constant $\zeta \in [0,\xi^{-1})$(for ξ see (H2)(iii)) and some function $\phi(t)$, such that $\phi(t) \to 0$ as $t \to 0$.

Every function $f \in X$ has the following representation

$$f = \sum_{i=1}^\infty f_i\phi_i,$$

where the sequence of functions $\{\phi_i\}_{i=1}^{\infty}$ represents a complete set of basis functions in X.

Denote by X_n the d_n-dimensional subspace spanned by the set of functions $\{\phi_i\}_{i=1}^{d_n}$. If we substitute now the approximation of f which is $\sum_{i=1}^{d_n} f_i\phi_i$ into the operator equation and if we take the inner product in the resulted equality, with a finite family of weighting functions or testing functions $\{\chi_i\}_{i=1}^{d_n}$ we obtain the following relation

$$\sum_{i=1}^{d_n} f_i < A\phi_i, \chi_j >=< g, \chi_j >, \ 1 \leq j \leq d_n$$

which is equivalent to the following matrix relation $[a_{ij}][f_i] = [g_j]$, where $a_{ij} =< A\phi_i, \chi_j >$, $1 \leq i, j \leq d_n$ and $g_j =< g, \chi_j >$, $1 \leq j \leq d_n$. Consider the approximation operator $A_n := (a_{ij})_{i,j=1}^{d_n}$. Then $A_n = P_n A E_n$ where the projection operator P_n is defined by $(P_n f)_i =< f, \chi_i >$, $i = 1, \ldots, d_n$ and the interpolation or embedding operator E_n is defined by

$$E_n c = \sum_{i=1}^{d_n} c_i \phi_i, \ c \in R^{d_n}.$$

Denote by $k(n)$ the condition number of A_n, i.e.,

$$k(n) := ||A_n||||A_n^{-1}||. \tag{1.17}$$

If we consider an approximation by eigenfunction, that is $A(I - E_n P_n) \supset I - E_n P_n$ or $A^{-1} E_n P_n \subset E_n P_n$, we take

$$k(n) := 1 \tag{1.18}$$

Also the basis function and the testing function should be chosen so that $P_n E_n x =$

x for all $x \in X_n$. The approximation satisfies the following relations

$$du_n(t) + A_n u_n(t)dt = P_n f(t, E_n u_n(t))dt + \sigma_n(E_n u_n(t))dP_n W(t), \quad (1.19)$$

$$u_n(0) = P_n u_0, \quad (1.20)$$

where σ_n is a bounded operator on X_n approximating σ such that ($\sigma_n Q_n \sigma_n^T$) is exact on $E_n P_n X$, that is,

$$< trace(\sigma Q \sigma^T)\phi_j, \xi_i >=< trace(\sigma_n Q_n \sigma_n^T)\phi_j, \xi_i >,$$

for $i,\ j = 1, \ldots, d_n$, where $Q_n = P_n Q E_n$. The space discretization satisfies the following assumptions:

(A) (a) $X,\ X_1,\ X_2, \ldots$ are all real or all complex valued Banach spaces. All the norms will be denoted by $||\cdot||$.

(b) P_n is a bounded linear operator, satisfying $||P_n x|| \leq p||x||$ for all $n \geq 1, x \in X$ and for some $p \geq 0$.

(c) E_n is a bounded linear operator, satisfying $||E_n x|| \leq q||x||$ for all $n \geq 1, x \in X$ and for some $q \geq 0$.

(d) $P_n E_n x = x$ for all $n \geq 1$ and $x \in X$.

(B) A_n is a bounded operator and there exists some $M < \infty$, and for $\omega \in R$ such that

$$||e^{A_n t}|| \leq M e^{\omega t} \text{ for } t \geq 0,\ n \geq 1.$$

Let τ_n be the time-step size corresponding to the space X_n. Consider the explicit Euler scheme, that is, if v_n^k denotes the approximation of $u_n(k\tau_n)$ then v_n^{k+1} is given by

$$
\begin{aligned}
v_n^{k+1} &= (1+\tau_n A_n)v_n^k + \tau_n P_n f(k\tau_n, E_n v_n^k) + \sqrt{\tau_n}\sigma_n(v_n^k)[\xi_n^k], \\
v_n^0 &= P_n u_0
\end{aligned}
$$

where ξ_n^k, $k = 1, \dots$ are d_n-dimensional standard Gaussian random variables distributed according to $N(0, Q_n)$; the implicit Euler scheme, that is,

$$
\begin{aligned}
v_n^{k+1} &= (1-\tau_n A_n)^{-1}v_n^k + \tau_n P_n f(k\tau_n, v_n^k) + \sqrt{\tau_n}\sigma_n(v_n^k)[\xi_n^k], \\
v_n^0 &= P_n u_0,
\end{aligned}
$$

and the Crank-Nicholson scheme, that is,

$$
\begin{aligned}
v_n^{k+1} &= (1-\frac{\tau_n}{2}A_n)^{-1}(1+\frac{\tau_n}{2}A_n)v_n^k + \tau_n P_n F(k\tau_n, v_n^k) + \sqrt{\tau_n}P_n\sigma(v_n^k)[\xi_n^k], \\
v_n^0 &= P_n u_0.
\end{aligned}
$$

Let us denote by $F_\tau(A) := (1+\tau A)$ for the Euler scheme, $F_\tau(A) := (1+\tau A)^{-1}$ for the implicit Euler scheme and $F_\tau(A) := (1-\frac{\tau}{2}A)^{-1}(1+\frac{\tau}{2}A)$ for the Crank-Nicholson scheme.

We are now in the position to state the main result from [56] which gives the estimation of the order of mean-square convergence of the approximation of the mild solution u_t of the problem (1.15)-(1.16) computed in the $||\cdot||_\gamma$-associated norm:

Theorem 3 *Let X be a separable Hilbert space, and A be the infinitesimal generator of an analytic semigroup of negative type and $\gamma > 0$ arbitrary. Further, assume that*

$W(t)$ is a Wiener process in X with nuclear covariance operator Q. Let u be a solution to

$$
\begin{aligned}
du(t) &= (Au(t) + f(t, u(t)))dt + \sigma(u(t))dW(t), \\
u(0) &= u_0 \in D((-A)^\gamma),
\end{aligned}
$$

where σ, f and A satisfy the assumptions (H1),(H2) and (H3) with parameters γ, ρ, θ and θ_σ. Let X_n, $n \geq 1$, be a finite-dimensional approximations of X discussed above and $k(n)$ defined by (1.17), resp. (1.18). Define $F_{\tau_n}(A)$ as above and suppose that the stability conditions

$$
\begin{aligned}
||F_{\tau_n}(A_n)^k|| &\leq Me^{\breve{\omega}\tau_n k} \text{ for some } k \geq 1 \text{ and } \breve{\omega} \in R, \\
||F_{\tau_n}(A_n)^k||^2 + \tau_n||(-A_n)^{\theta_\sigma}||^2 &\leq 1 \text{ for } \theta_\sigma > 0
\end{aligned}
$$

are satisfied. Assume additionally to (H3) that there exists a constant $\varsigma_n \in [0, \xi^{-1}]$ and some function $\phi_n(t)$, such that $\phi_n(t) \to 0$ as $t \to 0$

$$
\int_0^t ||T_{A_n}(s)(-A_n)^\theta P_n x||^2 ds \leq (\varsigma_n + \phi_n(t))||P_n x||, \ \forall x \in D(A),
$$

uniformly in $n \in N$. Moreover assume that there exists a function $\eta_\delta : N \to [0, 1]$, $\delta \in (0, \gamma]$, satisfying the following properties

$$
\begin{aligned}
(i)\ ||[I - E_n P_n]u|| &\leq \eta_\delta(n)||u||_\delta, \\
(ii)||A[I - E_n P_n]u|| &\leq ||A_n||\eta_\delta(n)||u||_\delta, \\
(iii)||[\sigma_n(P_n u) - P_n\sigma(u)||_{L_2^0} &\leq ||A_n||^\theta \eta_\delta(n)||u||_\delta.
\end{aligned}
$$

Fix $\epsilon > 0$ arbitrary. Then a priori error of the implicit Euler scheme can be estimated at $t = k\tau_n$ by

$$
\begin{aligned}
E[||v_n^k - u(k\tau_n)||_\rho^2] \;\; &\leq \;\; C_1\tau_n^{\min(1,2(\gamma-\rho-\theta_\sigma))} + k(n)^2(\frac{C_2}{\epsilon^2}\eta_{\gamma-\epsilon}^2(n) + \\
&+ (C_3(k\tau_n)^{-2p} + C_4)\eta_\gamma^2(n)).
\end{aligned}
$$

In case of the explicit Euler scheme we get

$$
\begin{aligned}
E[||v_n^k - u(k\tau_n)||_\rho^2] \;\; &\leq \;\; C_1(\tau_n^{\min(1,2(\gamma-\rho-\theta_\sigma))} + \tau_n^2||(-A)^{\max(0,1+\rho+\theta_\sigma+\gamma)}||^2) + \\
&+ k(n)^2(\frac{C_2}{\epsilon^2}\eta_{\gamma-\epsilon}^2(n) + (C_3(k\tau_n)^{-2p} + C_4)\eta_\gamma^2(n)),
\end{aligned}
$$

subject to the stability condition $\tau_n||A_n|| \leq 2$. The constants C_1, C_2, C_3 and C_4 are given by

$$
\begin{aligned}
C_1 \;\; &\backsim \;\; ||u_0||^2_{\min(1+\rho+\theta_\sigma,\gamma)} + E[\sup_{0<s\leq t}||u(s)||^2_{\min(1+\rho+\theta_\sigma,\gamma)}] + \\
&+ E[\tau_n\sum_{i=0}^{k}||v_n^i||^2_{\min(1+\rho+\theta_\sigma,\gamma)}] + E[\sup_{0<i\leq k}||v_n^i||^2_{\min(1+\rho+\theta_\sigma,\gamma)}] + \\
&+ E[\int_0^t||u(s)||^2_{\min(1+\rho+\theta+\theta_\sigma,\gamma+\theta)}ds],
\end{aligned}
$$

and

$$C_4 \backsim \zeta + \zeta_n + \phi(t) + \phi_n(t),$$

where

$$\breve{C} = \min((\zeta + \phi(t))(\zeta_n + \phi_n(t)), 1/(1 - 2\rho + 2\epsilon).$$

If it is assumed that $\sigma : X \to X$ is unbounded and

$$||(I + \frac{\tau_n}{2}A_n)^k|| \leq M\exp(k\tau_n)$$

then under the conditions of Theorem 3 the Crank-Nicholson scheme gives the same order of convergence as the implicit Euler scheme.

The Dirichlet problem for SPDEs is touched by Hausenbals in [56] with the help of an example given below as a direct application of Theorem 3.

Example (see [56]) Let $\theta_\sigma = \theta = 1/2$. Let us define the differential operators A and B, $j \in N$.

$$
\begin{aligned}
A(\xi)u(\xi) \quad &: \; = \sum_{i,j=1}^{d} \frac{\partial}{\partial \xi_i}(a_{ij}(\xi)\frac{\partial}{\partial \xi_j}u(\xi)) + \sum_{i=1}^{d} a_i(\xi)\frac{\partial}{\partial \xi_i}u(\xi) + a(\xi)u(\xi), \\
B^j(\xi)u(\xi) \quad &: \; = \sum_{i=1}^{d} b_i^j(\xi)\frac{\partial}{\partial \xi_i}u(\xi) + c^j(\xi)u(\xi),
\end{aligned}
$$

with bounded C^∞ coefficients. Let O be a bounded domain in R^d with smooth boundary. We consider the following parabolic SPDE

$$
\begin{aligned}
du(t) &= Au(t) + \sum_{i=1}^{\infty} B^j u(t) d\beta_t^i, \\
u(0) &= u_0 \in H_0^1(O), \\
u(t,\xi) &= 0, \; t \in [0,T], \; \xi \in \partial O,
\end{aligned}
$$

where $\{\beta_t^j : t \geq 0, \; j \in N\}$ is a sequence of independent one-dimensional standard Wiener processes. Set $\gamma = 1/2$. Take $X = L^2(O)$, $H_0^2(O) = D(A) = \{u \in H^2(O); u = 0 \text{ on } \partial O\}$. It follows $D((-A)^\gamma) = H_0^1(O) = \{u \in H^1(O); u = 0 \text{ on } \partial O\}$. If the joint parabolicity condition is satisfied, i.e.

$$
\mu < Ax, x > + \frac{1}{2}\sum_{i=1}^{\infty} |B^j x|^2 \leq c|x|^2 \text{ for all } x \in D(A),
$$

for some constant $c \geq 0$ and $\mu \in (0,1)$, condition (H3) is satisfied. Assuming addition that B_p^j is a tangent to ∂O. We know B^j maps $H_0^2(O)$ into $H_0^1(O)$ (see [34], Chapter 5.3.2), and we can apply Theorem 3.

In comparison with all the methods from [43, 16, 56] which have been constructed with the help of similar techniques used in numerical analysis of PDEs our method is inspired by the probabilistic representations of solution to SPDEs. It gives a sharper result in terms of space-time pathwise approximation and a better result concerning its order of convergence. That means (as we will see in Chapter 4) for each time-space point (t, x), almost every trajectory $w(\cdot)$ and any $\varepsilon > 0$ there exists a constant $C > 0$ such that

$$|\bar{v}(t,x) - v(t,x)| \leq Ch^{1-\varepsilon},$$

where $\bar{v}$ is the approximation scheme of the solution v of problem (1.1)-(1.2) and h is the time step of the discretization scheme. In addition, the results are supported by numerical examples.

Chapter 2

Preliminaries

In this chapter we recall some preliminary material which we need throughout the first part of this thesis. It is organized as follows.

The first section is devoted to a short review of the theory of backward stochastic integration (see [111, 77] and the references therein).

In Section 2.2, we discuss some properties related to the continuity and differentiability of continuous semimartingales with spatial parameter in more general settings (see [77] and the references therein).

In the last section, we review some results on the existence and uniqueness of the solution to a class of second order parabolic equations (see [87] and the references therein).

2.1 Backward Itô stochastic calculus

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $\mathcal{F}_t$, $T_0 \leq t \leq T$, be a filtration satisfying the usual hypotheses, and $(w(t), \mathcal{F}_t) = ((w_1(t), \ldots, w_q(t))^\top, \mathcal{F}_t)$ be a q-dimensional standard Wiener process.

Definition 4 *The function $f(.,.) : [T_0, T] \times \Omega \to R$ is said to be backward predictable on the compact interval $[T_0, T]$ (or $\overleftarrow{P}_{[T_0,T]}-$measurable) relative to the family $\{\mathcal{F}_T^t\}$, if the function $f(T - .,.)$ is predictable with respect to the family $\{\mathcal{F}_T^{T-t}\}$ for $t \in [T_0, T]$.*

Denote $w_T(t) := w(T) - w(t)$. It is assumed that $w_T(t)$ is a standard Wiener martingale relative to the family $\{\mathcal{F}_T^{T-t}\}$, where $t \in [0, T]$.

The $\sigma-$algebra generated by the "future" of the Wiener process $w(t)$ (i.e. by increments $w(T) - w(s)$ for $s \leq t$) and completed with respect to the measure P usually stands for $\mathcal{F}_T^t$. Obviously, in this case $\mathcal{F}_T^{T-t}$ coincides with the $\sigma-$algebra generated by values of the process $w_T(s)$ for $s \leq t$, which is completed with respect to the measure P. It is clear that the process $w_T(s)$ is a Wiener martingale with respect to the family of P-completed $\sigma-$algebras generated by itself.

Let f be a backward predictable function on $[T_0, T]$ with respect to the family $\mathcal{F}_T^t$. Suppose that for every s, $t \in [T_0, T]$ such that $s \leq t$ and $T - t \leq T_0$

$$\int_{[T_s, T_t]} |f(r)|^2 dr < \infty \quad (P - \text{a.s.})$$

where $T_r := T - r$.

Definition 5 *The backward stochastic integral with respect to a one-dimensional Wiener process* $w^i(t)$, $i = 1, 2, \ldots, d_1$ *is defined as the Itô integral with respect to* $w^i_T(t)$*:*

$$\int_{[s,t]} f(r) * w^i(r) := \int_{[T_s,T_t]} f(T_r) dw^i_T(r).$$

Clearly, this definition does not depend on the choice of T.

Suppose that $\sigma^{il}(\cdot,\cdot)$, $b^i(\cdot,\cdot)$, $i = 1, 2, \ldots, d$, $l = 1, 2, \ldots, q$, are backward predictable (relative to the family $\{\mathcal{F}^t_T\}$), real-valued functions on $[0,T]$. Suppose also that $\sigma^{il}(\cdot,\omega) \in L^2((0,T))$ and $b^i(\cdot,\omega) \in L^1((0,T))$ for $l = 1, 2, \ldots, q$ and $i = 1, 2, \ldots, d$ (P-a.s.).

Definition 6 *Let* τ *be a stopping time with respect to the family* $\{\mathcal{F}^t_T\}$ *such that* $\tau \leq T$ *taking values in* R^d*. We shall say that* $\xi(t)$ *possesses the backward stochastic differential*

$$-d\xi^i(t) = b^i(t)dt + \sigma^{il}(t) * dw^l(t), \; t \in (\tau, T], \; i = 1, \; 2, \ldots, d, \tag{2.1}$$

if for every $i = 1, 2, \ldots, d$ *it satisfies the equality*

$$P(\sup_{t\in[\tau,T]} |\xi^i(t) - \xi^i(T) - \int_{[t,T]} b^i(s)ds - \int_{[t,T]} \sigma^{il}(s)dw^l(s)| = 0) = 1.$$

It is to be noted that in the following theorem the convention of summation upon repeated indices is used.

Theorem 7 *(see [111]) (Backward Itô's formula) Let* H *be a function from* $C^{1,2}(R_+, R^d)$ *and* $\xi(t)$ *be a stochastic process possessing the backward stochastic differential (2.1)* *then $H(t,\xi(t))$ also possesses a backward stochastic differential of the form*

$$
\begin{aligned}
& -dH(t,\xi(t)) \\
= \ & [-\frac{\partial}{\partial t}H(t,x)|_{x=\xi(t)} + \frac{1}{2}\sigma^{il}\sigma^{jl}\frac{\partial^2}{\partial x_i \partial x_j}H(t,\xi(t)) + b^i\frac{\partial}{\partial x_i}H(t,\xi(t))]dt \\
& +\sigma^{il}\frac{\partial}{\partial x_i}H(t,\xi(t)) * dw^l(t),
\end{aligned}
$$

where $t \in [\tau, T]$.

We can look at a backward Itô's equation in a completely analogous way as it is done for a "forward" one. Taking into account the fact that the theory of backward Itô's equations is identical to the one developed for the forward ones, from now on, when considering backward Itô's equations we shall use the corresponding results for the forward ones without special references.

2.2 Stochastic calculus with respect to spatial parameters

In this section we discuss some properties related to the continuity and differentiability of continuous semimartingales with spatial parameter in more general settings (see Kunita [77] and the references therein). To this end, we first introduce various function spaces which are used to formulate the properties.

Let D be a domain in R^d and let R^l be another Euclideian space. Let m be a non-negative integer. Denote by $C^m(D : R^l)$ or C^m the set of all maps $f : D \to R^l$ which are m-times continuously differentiable. When $m = 0$, it is denoted by $C(D : R^l)$. In this section, we consider a multi-index of non-negative integers $\alpha = (\alpha_1, \ldots, \alpha_d)$ and $|\alpha| = \sum_{i=1}^{d} \alpha_i$.Let K be a subset of D. Equipped with the seminorms

$$\begin{aligned} & ||f||_{m:K} \\ = \ & \sup_{x \in K} \frac{|f(x)|}{1+|x|} + \sum_{1 \le |\alpha| \le m} \sup_{x \in K} \left| \frac{\partial^{|\alpha|}}{(\partial x^1)^{\alpha_1} \ldots (\partial x^d)^{\alpha_d}} f(x) \right|, \end{aligned}$$

$C^m(D : R^l)$ is a Frechet space when K is a compact in D. If $K = D$, we write $|| \cdot ||_{m:K}$ as $|| \cdot ||_m$. Denote by $C_b^m(D : R^l)$ or C_b^m the set $\{f \in C^m : ||f||_m < \infty\}$ which is a Banach space with the norm $|| \cdot ||_m$.

For any $0 < \delta \le 1$, let us denote by $C^{m,\delta}(D : R^l)$ or simply by $C^{m,\delta}$ the set of all f of C^m such that $\frac{\partial^{|\alpha|}}{(\partial x^1)^{\alpha_1} \ldots (\partial x^d)^{\alpha_d}} f(x)$, $|\alpha| = m$ are δ-Hölder continuous. Endowed with the seminorms

$$\begin{aligned} & ||f||_{m+\delta:K} \\ = \ & ||f||_{m:K} + \sum_{|\alpha|=m} \sup_{x,y \in K,\ x \ne y} \frac{\left| \frac{\partial^{|\alpha|}}{(\partial x^1)^{\alpha_1} \ldots (\partial x^d)^{\alpha_d}} f(x) - \frac{\partial^{|\alpha|}}{(\partial y^1)^{\alpha_1} \ldots (\partial y^d)^{\alpha_d}} f(y) \right|}{|x-y|^{\alpha}}, \end{aligned}$$

$C^{m,\delta}$ is a Frechet space. If $K = D$, we write $|| \cdot ||_{m+\delta:K}$ as $|| \cdot ||_{m+\delta}$. Denote by $C_b^{m,\delta}(D : R^l)$ or $C_b^{m,\delta}$ the set $\{f \in C^m : ||f||_{m+\delta} < \infty\}$.

A continuous function $f(x,t)$, $x \in D$, $t \in [0,T]$ is said to belong to the class $C^{m,\delta}$ if for every t, $f(t) := f(\cdot, t)$ belongs to $C^{m,\delta}$ and $||f(t)||_{m+\delta:K}$ is integrable on $[0,T]$ with respect to t for any compact subset K. If the set K is replaced by D, f is said to belong to class $C_{ub}^{m,\delta}$.

We denote by $\widetilde{C}^m$ the set of all R^l-valued functions $g(x,y)$, $x, y \in D$ which are m-

times continuously differentiable with respect to each variable x and y. For $g \in \widetilde{C}^m$, we define a seminorm

$$\begin{aligned} & ||g||^{\sim}_{m:K} \\ = & \sup_{x,y\in K} \frac{|g(x,y)|}{(1+|x|)(1+|y|)} \\ & + \sum_{1\le|\alpha|\le m} \sup_{x,y\in K} \Big|\frac{\partial^{|\alpha|}}{(\partial x^1)^{\alpha_1}...(\partial x^d)^{\alpha_d}} \frac{\partial^{|\alpha|}}{(\partial y^1)^{\alpha_1}...(\partial y^d)^{\alpha_d}} g(x,y)\Big|, \end{aligned}$$

and for $0 < \delta \le 1$,

$$\begin{aligned} & ||g||^{\sim}_{m+\delta:K} \\ = & ||g||^{\sim}_{m:K} \\ & + \sum_{|\alpha|=m} \Big|\Big|\frac{\partial^{|\alpha|}}{(\partial x^1)^{\alpha_1}...(\partial x^d)^{\alpha_d}} \frac{\partial^{|\alpha|}}{(\partial y^1)^{\alpha_1}...(\partial y^d)^{\alpha_d}} g\Big|\Big|^{\sim}_{\delta:K}, \end{aligned}$$

where

$$||g||^{\sim}_{\delta:K} = \sup_{x,y,x',y'\in K,\ x\neq y,\ x'\neq y'} \frac{|g(x,y)-g(x',y)-g(x,y')+g(x',y')|}{|x-x'|^{\delta}|y-y'|^{\delta}}.$$

The function g is said to belong to the space $\widetilde{C}^{m,\delta}$ if $||g||^{\sim}_{m+\delta:K} < \infty$ for any compact set K in D. We denote $||\cdot||^{\sim}_{m:D}$ and $||\cdot||^{\sim}_{m+\delta:D}$ by $||\cdot||^{\sim}_{m}$ and $||\cdot||^{\sim}_{m+\delta}$, respectively. We set $\widetilde{C}^m_b = \{g : ||g||^{\sim}_m < \infty\}$ and $\widetilde{C}^{m,\delta}_b = \{g : ||g||^{\sim}_{m+\delta} < \infty\}$.

A continuous function $g(x,y,t)$, $x,y \in D$, $t \in [0,T]$ is said to belong to the class $\widetilde{C}^{m,\delta}$ if for every t, $g(t) := g(\cdot,\cdot,t)$ belongs to the space $\widetilde{C}^{m,\delta}$ and $||g(t)||^{\sim}_{m+\delta:K}$ is integrable on $[0,T]$ with respect to t for any compact subset K. The classes $\widetilde{C}^{m,\delta}_b$ and $\widetilde{C}^{m,\delta}_{ub}$ are defined similarly to $C^{m,\delta}_b$ and $C^{m,\delta}_{ub}$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{F}_t,\ 0 \le t$, be a filtration satisfying the usual hypotheses, D be a domain in R^d and $\xi(x,t)$ be a family of real valued processes on it with parameter $x \in D$. We regard it as a random field with double parameters x and t. If $F(x,t,\omega)$ is a continuous function of x for almost all ω for any t, we regard $F(\cdot,t)$ as a stochastic process with values in $C = C(D : R^1)$ or a C-valued process. If $F(x,t,\omega)$ is m-times continuously differentiable with respect to x a.s. for any t, it can be regarded as a stochastic process with values in C^m or a C^m-valued process. If $F(x,t)$ is a continuous process with values in C^m, then it is called a continuous C^m- process. A $C^{m,\delta}$-valued process and a continuous $C^{m,\delta}$-valued process are defined similarly.

Let $\varsigma(x,y,t)$ be a stochastic process with parameters $x, y \in D$. If it is m-times continuously differentiable with respect to each x and y a.s. for any t, then it is called a stochastic process with values in $\widetilde{C}^m$ or a $\widetilde{C}^m$-valued process. The $\widetilde{C}^{m,\delta}$-valued process and continuous $\widetilde{C}^{m,\delta}$-valued process are defined similarly.

The following theorem gives the Hölder continuity of a family of continuous localmartingal with spatial parameter in connection with the Hölder continuity of the joint quadratic variation.

Theorem 8 *(see [77]) Let $M(x,t)$, $x \in D$, be a family of continuous localmartingales such that $M(x,0) = 0$. Assume that the joint quadratic variation $< M(x,t), M(y,t) >$ has a modification of a continuous $\widetilde{C}^{0,\delta}$-process. Then $M(x,t)$ has a modification of a continuous $\widetilde{C}^{0,\epsilon}$-process for any $\varepsilon < \delta$.*

The next theorem provides (in terms of differentiability) regularity of a continuous localmartingale with spatial parameter in connection with regularity of the joint quadratic variation.

Theorem 9 *(see [77]) Let $M(x,t)$, $x \in D$, be a family of continuous localmartingales such that $M(x,0) = 0$. Assume that its joint quadratic variation has a continuous modification $A(x,y,t)$ of a continuous $\widetilde{C}^{m,\delta}$- process for some $m \geq 1$ and $\delta \in (0,1]$. Then $M(x,t)$ has a modification of a continuous $\widetilde{C}^{m,\varepsilon}$-process for any $\varepsilon < \delta$. Furthermore, for each α with $|\alpha| \leq m$,*

$$\frac{\partial^{|\alpha|}}{(\partial x^1)^{\alpha_1} ... (\partial x^d)^{\alpha_d}} M(x,t), x \in D,$$

is a family of continuous localmartingales with the joint quadratic variation

$$\frac{\partial^{|\alpha|}}{(\partial x^1)^{\alpha_1} ... (\partial x^d)^{\alpha_d}} \frac{\partial^{|\alpha|}}{(\partial y^1)^{\alpha_1} ... (\partial y^d)^{\alpha_d}} A(x,y,t).$$

The following result is a straightforward application of the Theorems 8 and 9 when the continuous localmartingale has a representation in terms of Itô stochastic integral.

Proposition 10 *(see Exercise 3.1.5 from [77]) Let M_t be a continuous localmartingale and $f(x,t)$, $x \in D$ be a predictable process with values in $C^{m,\delta}$ such that*

$$||f(s)||_{m+\delta:K} \in L^2(< M >)$$

holds for any compact set K. Then for any $\varepsilon < \delta$, $M(x,t) = \int\limits_0^t f(x,s) dM_s$ has a *modification of a $C^{m,\epsilon}$-localmartingale which satisfies*

$$\frac{\partial^{|\alpha|}}{(\partial x^1)^{\alpha_1}...(\partial x^d)^{\alpha_d}} \int_0^t f(x,s)dM_s$$
$$= \int_0^t \frac{\partial^{|\alpha|}}{(\partial x^1)^{\alpha_1}...(\partial x^d)^{\alpha_d}} f(x,s)dM_s, \ \textit{for } |\alpha| \leq m.$$

2.3 Existence and uniqueness of the classical solution to the Dirichlet problem for linear parabolic PDEs

This section deals with the existence and uniqueness of the solution to a class of second order parabolic equations (see [87] and the references therein). Again, we start with introduction of some functional spaces to formulate the main result, Theorem 11, at the end of the section.

Let X be a real or complex Banach space with norm $||\cdot||$ and let $I \subset R$ be a (possibly unbounded) interval. We consider the functional spaces $B(I;X)$, $C(I;X)$, $C^m(I;X)$ $(m \in N)$, $C^\infty(I;X)$, consisting respectively of the bounded, continuous, m times continuously differentiable, infinitely many times differentiable, functions $f : I \to X$. $B(I;X)$ is endowed with the sup norm

$$||f||_{B(I;X)} = \sup_{t\in I}||f(t)|| \ .$$

Denote by $C_0^\infty(I;X)$ the subset of $C^\infty(I;X)$ consisting of the functions with their support contained in the interior of I. Let

$$\begin{aligned} C_b(I;X) &= B(I;X)\cap C(I;X),\ ||f||_{C(I;X)} = ||f||_{B(I;X)}, \\ C_b^m(I;X) &= \{f\in C^m(I;X) : f^{(k)}\in C_b(I;X),\ k=0,\dots,m\}, \\ ||f||_{C_b^m(I;X)} &= \sum_{k=0}^m ||f^{(k)}||_{B(I;X)}. \end{aligned}$$

We write $||f||_{B(X)}$ or simply $||f||_\infty$ instead of $||f||_{B(I;X)}$ for any bounded function f; moreover, if $X = R$ or C, we write $B(I)$, $C(I)$, etc., instead of $B(I;X)$, $C(I;X)$, etc.

If U is any open set in R^d, $C(\overline{U})$ (respectively, $UC(\overline{U})$) denotes the Banach space of all the continuous (respectively, uniformly continuous) and bounded functions in $\overline{U}$, endowed with the sup norm, and $C^m(\overline{U})$ (respectively, $UC^m(\overline{U})$) denotes the set of all m times continuously differentiable functions in U, with derivatives up to the order m bounded and continuously (respectively, uniformly continuously) extendable up to the boundary. For the multi-index of non-negative integers $\alpha = (\alpha_1,\dots,\alpha_d)$, we endow this space with the norm

$$||f||_{C^m(\overline{U})} = \sum_{|\alpha|\le m} ||\frac{\partial^{|\alpha|}}{(\partial x^1)^{\alpha_1}\dots(\partial x^d)^{\alpha_d}} f(x)||_\infty,$$

where $|\alpha| = \sum \alpha_i$.

Let us consider the following Banach spaces of Hölder continuous functions

$$\begin{aligned}
C^\alpha(I;X) &= \{f \in C_b(I;X) : [f]_{C^\alpha(I;X)} = \sup_{t,s\in I,\ s<t} \frac{||f(t)-f(s)||}{(t-s)^\alpha} < \infty\},\\
||f||_{C^\alpha(I;X)} &= ||f||_\infty + [f]_{C^\alpha(I;X)};\\
C^{k+\alpha}(I;X) &= \{f \in C_b^k(I;X) : f^{(k)} \in C^\alpha(I;X)\},\\
||f||_{C^{k+\alpha}(I;X)} &= ||f||_{C_b^k(I;X)} + [f^{(k)}]_{C^\alpha(I;X)}.
\end{aligned}$$

Next we define some spaces of functions which are Hölder continuous or continuously differentiable either with respect to time or with respect to the space variables. For a fixed $\alpha > 0$ and $a < b$ we set

$$\begin{aligned}
C^{\alpha,0}([a,b]\times\overline{U}) &= \{f \in C([a,b]\times\overline{U}) : f(\cdot,x) \in C^\alpha([a,b])\ \forall\ x \in \overline{U},\\
||f||_{C^{\alpha,0}} &= \sup_{x\in\overline{U}} ||f(\cdot,x)||_{C^\alpha([a,b])} < \infty\},\\
C^{0,\alpha}([a,b]\times\overline{U}) &= \{f \in C([a,b]\times\overline{U}) : f(t,\cdot) \in C^\alpha(\overline{U})\ \forall\ t \in [a,b],\\
||f||_{C^{0,\alpha}} &= \sup_{a\le t\le b} ||f(t,\cdot)||_{C^\alpha(\overline{U})} < \infty\}.
\end{aligned}$$

We introduce the classical space of functions where one looks for solutions.

$$\begin{aligned}
& C^{1,2}([a,b]\times\overline{U})\\
={}& \{f \in C([a,b]\times\overline{U}) : \exists\ \frac{\partial}{\partial t}f,\ \frac{\partial^2}{\partial x_i \partial x_j} f \in C([a,b]\times\overline{U}),\ i,j = 1,\ldots,n\},\\
& ||f||_{C^{1,2}([a,b]\times\overline{U})}\\
={}& ||f||_\infty + \sum_{i=1}^n ||\frac{\partial}{\partial x_i} f||_\infty + ||\frac{\partial}{\partial t} f||_\infty + \sum_{i,j=1}^n ||\frac{\partial^2}{\partial x_i \partial x_j} f||_\infty.
\end{aligned}$$

In some cases one needs more regular solutions. Therefore we define next some function spaces in this respect.

For a fixed $0 < \alpha < 1$ we set

$$
\begin{aligned}
& C^{1,2+\alpha}([a,b] \times \overline{U}) \\
= \;& \{f \in C^{1,2}([a,b] \times \overline{U}) : \frac{\partial}{\partial t} f,\ \frac{\partial^2}{\partial x_i \partial x_j} f \in C^{0,\alpha}([a,b] \times \overline{U}),\ \forall\ i,j\}, \\
& ||f||_{C^{1,2+\alpha}([a,b]\times\overline{U})} \\
= \;& ||f||_\infty + \sum_{i=1}^n ||\frac{\partial}{\partial x_i} f||_\infty + ||\frac{\partial}{\partial t} f||_{C^{0,\alpha}} + \sum_{i,j=1}^n ||\frac{\partial^2}{\partial x_i \partial x_j} f||_{C^{0,\alpha}}.
\end{aligned}
$$

Let us introduce the "parabolic" Hölder spaces in which usually one gets the classical solutions.

For any $0 < \alpha < 2$ these spaces are defined by

$$
\begin{aligned}
C^{\alpha/2,\alpha}([a,b] \times \overline{U}) &= C^{\alpha/2,0}([a,b] \times \overline{U}) \cap C^{0,\alpha}([a,b] \times \overline{U}), \\
||f||_{C^{\alpha/2,\alpha}([a,b]\times\overline{G})} &= ||f||_{C^{\alpha/2,0}} + ||f||_{C^{0,\alpha}}
\end{aligned}
$$

and

$$
\begin{aligned}
& C^{1+\alpha/2,2+\alpha}([a,b] \times \overline{U}) \\
= \;& \{f \in C^{1,2}([a,b] \times \overline{U}) : \frac{\partial}{\partial t} f,\ \frac{\partial^2}{\partial x_i \partial x_j} f \in C^{\alpha/2,\alpha}([a,b] \times \overline{U}),\ \forall\ i,j\}, \\
& ||f||_{C^{1+\alpha/2,2+\alpha}([a,b]\times\overline{U})} \\
= \;& ||f||_\infty + \sum_{i=1}^n ||\frac{\partial}{\partial x_i} f||_\infty + ||\frac{\partial}{\partial t} f||_{C^{\alpha/2,\alpha}} + \sum_{i,j=1}^n ||\frac{\partial^2}{\partial x_i \partial x_j} f||_{C^{\alpha/2,\alpha}([a,b]\times\overline{U})}.
\end{aligned}
$$

If U is any open set in R^n, $m \in N$,

$$B(0,1) = \{x \in R^n : |x| < 1\}$$

we say that the boundary ∂U is uniformly C^m if there exists $r > 0$ and a (at most countable) collection of open balls

$$U_j = \{x \in R^n : |x - x_j| < r\}, \ j \in \boldsymbol{N},$$

covering ∂U and such that there exists an integer k with the property that $\bigcap\limits_{j \in J} U_j = \emptyset$ for all $J \subset \boldsymbol{N}$ with more than k elements. Moreover, we assume that there is $\epsilon > 0$ such that the balls centered at x_j with radius $r/2$ still cover an $\epsilon-$neighborhood of ∂U, and there exist coordinate transformations φ_j such that

$$\varphi_j : \overline{U_j} \to B(0,1) \subset R^n$$

is a C^m diffeomorfism, mapping $\overline{U_j} \cap U$ onto the upper half ball

$$B_+(R^n) = \{y \in B(0,1) : y_n > 0\},$$

and mapping $U_j \cap U$ onto the basis

$$\Sigma_n = \{y \in B(0,1) : y_n = 0\}.$$

All the coordinate transformations φ_j and their inverses are supposed to have uniformly bounded derivatives up to order m,

$$\sup_{j \in \boldsymbol{N}} \sum_{1 \le |\alpha| \le m} (|| \frac{\partial^{|\alpha|}}{(\partial x^1)^{\alpha_1} ... (\partial x^d)^{\alpha_d}} \varphi_j(x) ||_\infty + || \frac{\partial^{|\alpha|}}{(\partial x^1)^{\alpha_1} ... (\partial x^d)^{\alpha_d}} \varphi_j^{-1}(x) ||_\infty) \le M,$$

where $M > 0$ is fixed. The definition of uniformly $C^{m+\alpha}$ boundary, with $0 < \alpha < 1$ is analogous to the definition of uniformly C^m boundary.

We are now ready to state the main result which closes this chapter.

Theorem 11 *(see Theorem 5.1.16 in [87]) Let ∂U be uniformly $C^{2+\alpha}$, with $0 < \alpha < 1$ and let $a_{ij}, b_i, c, f \in C^{\alpha/2,\alpha}([0,T] \times \overline{U})$, $g \in C^{1+\alpha/2,2+\alpha}([0,T] \times \partial U)$, $u_0 \in C^{2+\alpha}(\overline{U})$ be such that*

$$g(0,x) = u_0(x), \ \frac{\partial}{\partial t} g(0,x) = \mathcal{A}(0,x)u_0(x) + f(0,x), \ x \in \partial U \tag{2.2}$$

where for $0 \le t \le T$, $x \in \overline{U}$,

$$\begin{aligned} & \mathcal{A}(t,x)\varphi(t,x) \\ = & \sum_{i,j=1}^{n} a_{ij}(t,x) \frac{\partial^2}{\partial x_i \partial x_j} \varphi(t,x) + \sum_{i=1}^{n} b_i(t,x) \frac{\partial}{\partial x_i} \varphi(t,x) + c(t,x)\varphi. \end{aligned}$$

Let moreover the following ellipticity condition be satisfied

$$\sum_{i,j=1}^{n} a_{ij}(t,x)\xi_i \xi_j \ge \nu |\xi|^2,$$

$0 \le t \le T$, $x \in \overline{U}$, $\xi \in R^n$, for some $\nu > 0$. Then the problem

$$\begin{aligned} \frac{\partial}{\partial t} u(t,x) &= \mathcal{A}(t,x)u(t,x) + f(t,x), \ 0 < t \le T, \ x \in U, \\ u(0,x) &= u_0(x), \ x \in \overline{U}, \ u(t,x) = g(t,x), \ 0 < t \le T, \ x \in \partial U, \end{aligned}$$

has a unique solution u belonging to $C^{1+\alpha/2,2+\alpha}([0,T] \times \overline{U})$ and

$$||u||_{C^{1+\alpha/2,2+\alpha}} \le C(||u_0||_{C^{2+\alpha}(\overline{U})} + ||f||_{C^{\alpha/2,\alpha}} + ||g||_{C^{1+\alpha/2,2+\alpha}}).$$

Chapter 3

Auxiliary knowledge

3.1 Setting of the problem

Let G be a bounded domain in $\mathbf{R}^d$, $Q = [T_0, T) \times G$ be a cylinder in $\mathbf{R}^{d+1}$, and $\Gamma = \bar{Q} \backslash Q$ be the part of the cylinder's boundary consisting of the upper base and lateral surface. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $\mathcal{F}_t,\ T_0 \le t \le T$, be a filtration satisfying the usual hypotheses, and $(w(t), \mathcal{F}_t) = ((w_1(t), \ldots, w_r(t))^\top, \mathcal{F}_t)$ be a r-dimensional standard Wiener process. We consider the Dirichlet problem for the backward stochastic partial differential equation (SPDE):

$$-dv = [\mathcal{L}v + f(t,x)]\, dt + \left[\beta^\top(t,x)v(t,x) + \gamma^\top(t,x)\right] * dw(t), \quad (t,x) \in Q, \tag{3.1}$$

$$v \mid_\Gamma = \varphi(t,x), \tag{3.2}$$

where

$$\mathcal{L}v(t,x) := \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(t,x)\frac{\partial^2}{\partial x^i \partial x^j}v(t,x) + b^\top(t,x)\nabla v(t,x) + c(t,x)v(t,x), \tag{3.3}$$

$a(t,x) = \{a^{ij}(t,x)\}$ is a $d \times d$-matrix; $b(t,x)$ is a d-dimensional column-vector composed of the coefficients $b^i(t,x)$; $c(t,x)$ and $f(t,x)$ are scalar functions; $\beta(t,x)$ and $\gamma(t,x)$ are r-dimensional column-vectors composed of the coefficients $\beta^i(t,x)$ and $\gamma^i(t,x)$, respectively. The notation "$*dw$" in (3.1) means backward Itô integral [69, 106, 111] (see also Section 2.1).

We already remarked in the first chapter that by changing the time $s := T-(t-T_0)$ one can re-write the backward SPDE problem (3.1)-(3.2) in the one with forward time (see [111]), and vice versa. Also the theory of backward Itô equations is identical with the one developed for the forward ones (see [111], [77] and the references therein). Therefore by duality arguments the methods for the backward SPDEs (3.1)-(3.2) used in here can be used for solving the forward ones as well.

We impose the following conditions on the problem (3.1)-(3.2).

Assumption 2.1. (smoothness) We assume that the coefficients in (3.1)-(3.2) are sufficiently smooth in $\bar{Q}$ and the domain G has the sufficiently smooth boundary ∂G.

Assumption 2.2. (ellipticity) We assume that $a = \{a^{ij}\}$ is symmetric and positive definite in $\bar{Q}$.

Assumption 2.3. (on the solution) We assume that the problem (3.1)-(3.2) has the classical solution $v(t,x)$, which has derivatives in x up to a sufficiently high order for all $(t,x) \in \bar{Q}$; and for a fixed $p \geq 1$ we have

$$\max_{(t,x)\in\overline{Q},\ i_1+...+i_d=k,\ k=\overline{0,\ 4}} E|\frac{\partial^k v(t,x)}{\partial^{i_1}x_1...\partial^{i_d}x_d}|^{2pl} \leq C,$$

where l is a sufficiently large positive integer and the constant C is finite.

Also, we require the following compatibility conditions to hold.

Compatibility conditions. We assume that $\gamma(t,x)$, and its partial derivatives up to a sufficiently high order equal zero for $x \in \partial G$, $t \in [T_0, T]$; we also require that either $\beta(t,x)$ and its partial derivatives up to a sufficiently high order equal zero for $x \in \partial G$, $t \in [T_0, T]$ and

$$-\frac{\partial \varphi}{\partial t}(T,x) = \mathcal{L}\varphi(T,x) + f(T,x),\ \ x \in \partial G \tag{3.4}$$

or

$$f(T,x) = 0,\ \ x \in \partial G \text{ and } \varphi(t,x) = 0 \text{ for } x \in \partial G,\ \ t \in [T_0,T]. \tag{3.5}$$

We note that Assumptions 2.1-2.3 together with the above compatibility conditions from above are sufficient for all the statements of this part of thesis.

3.2 Existence and uniqueness of the classical solution to the backward Dirichlet problem

Introduce the processes for $T_0 \leq t \leq T$:

$$-d\eta = \beta^\top(t,x) * dw(t),\ \ \ \eta(T;x) = 0, \tag{3.6}$$

and

$$-d\varsigma = \gamma^{\top}(t,x)e^{-\eta(t;x)} * dw(t), \quad \varsigma(T;x) = 0. \tag{3.7}$$

Thanks to Assumption 2.1, the processes $\eta(t;x)$ and $\varsigma(t;x)$ are smooth in x (see Section 2.2 or [77, p. 78]). It is evident that $\eta(t;x)$ can be written in the form of a usual (forward) Itô integral:

$$\eta(t;x) = \int_t^T \beta^{\top}(s,x) * dw(s) = \int_t^T \beta^{\top}(s,x)dw(s). \tag{3.8}$$

Consider the Dirichlet problem for the linear PDE with random coefficients:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(t,x)\frac{\partial^2 u}{\partial x^i \partial x^j} + B^{\top}(t,x)\nabla u + C(t,x)u + F(t,x) = 0, \quad (t,x) \in Q, \tag{3.9}$$

$$u\mid_{\Gamma} = \Phi(t,x), \tag{3.10}$$

where

$$B(t,x) = b(t,x) + a(t,x)\nabla\eta(t;x), \tag{3.11}$$

$$\begin{aligned} C(t,x) &= c(t,x) - \frac{1}{2}|\beta(t,x)|^2 + b^{\top}(t,x)\nabla\eta(t;x) + \frac{1}{2}\nabla^{\top}\eta(t;x)a(t,x)\nabla\eta(t;x) \\ &+ \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(t,x)\frac{\partial^2}{\partial x^i \partial x^j}\eta(t;x), \end{aligned} \tag{3.12}$$

$$\begin{aligned} F(t,x) &= e^{-\eta(t;x)}f(t,x) - \frac{1}{2}|\beta(t,x)|^2\varsigma(t;x) - e^{-\eta(t;x)}\gamma^{\top}(t,x)\beta(t,x) + \mathcal{L}\varsigma(t;x) \\ &+ \nabla^{\top}\varsigma(t;x)a(t,x)\nabla\eta(t;x) + \frac{1}{2}\varsigma(t;x)\nabla^{\top}\eta(t;x)a(t,x)\nabla\eta(t;x) \\ &+ \frac{1}{2}\varsigma(t;x)\sum_{i,j=1}^{d} a^{ij}(t,x)\frac{\partial^2}{\partial x^i \partial x^j}\eta(t;x), \end{aligned} \tag{3.13}$$

and

$$\Phi(t,x) = \varphi(t,x)e^{-\eta(t;x)}. \tag{3.14}$$

We have the following result.

Theorem 12 *Under the assumptions and compatibility conditions from Section 3.1, the problem (3.9)-(3.10) admits a unique strong classical solution which is backward adapted.*

Proof. Since the coefficients of (3.9)-(3.10) contain the Wiener process, they are a.s. Hölder continuous with exponent $1/2-\epsilon$, $0<\epsilon<1/2$, in the time variable t. Assumption 2.1 together with Theorem 9 or Proposition 10 (see also [77]) imply that they are also sufficiently smooth in x. Thus they fulfill the conditions of Theorem 11 from Section 2.3 (see also Theorem 5.1.16 from [87]). We note that due to the **Compatibility conditions** from Section 3.1 we have that $\beta(t,x)$ and its partial derivatives up to a sufficiently high order equal zero for $x\in\partial G$, $t\in[T_0,T]$. Consequently, $\eta(t;x)$ and its partial derivatives up to a sufficiently high order equal zero for $x\in\partial G$, $t\in[T_0,T]$ Therefore, again due to the **Compatibility conditions** from Section 3.1 (see (3.4) or relation (3.5)) Φ checks the compatibility conditions (2.2) of Theorem 11 from Section 2.3 (see also Theorem 5.1.16 from [87]). The backward adaptivity of the solution to the problem (3.9)-(3.10) follows from the fact that the main tool in the proof of Theorem 11 from Section 2.3 (see also Theorem 5.1.16 from [87]).is the contraction principle (see also Remark 2.3 in [7]). The proof is now complete. ■

Using Theorem 9 or Proposition 10 (see also [77]), Theorem 12, the backward Itô formula (see Section 2.1 or [111]) and the transform

$$v(t,x) = e^{\eta(t;x)}u(t,x) + e^{\eta(t;x)}\varsigma(t;x) \tag{3.15}$$

we have the following equivalence result in terms of strong classical solution. The details concerning the transformed PDE with random coefficients are given in Appendix A.

Theorem 13 *Under the assumptions and compatibility conditions from Section 3.1, the problem (3.1)-(3.2) has a unique classical solution in the strong sense.*

3.3 Probabilistic representations of the solution to SPDEs

The probabilistic representations (or in other words, the averaging-over-characteristic formulas) for the solution of the problem (3.1)-(3.2) presented in this section are analogous to the ones in [35] and also to the representations for solutions of the Cauchy problem for linear SPDEs from [69, 111, 77, 106] (see also [94, 95]).

Let a $d \times d$-matrix $\sigma(t,x)$ be obtained from the equation

$$\sigma(t,x)\sigma^{\top}(t,x) = a(t,x).$$

The solution of the problem (3.1)-(3.2) has the following probabilistic representation:

$$v(t,x) = E^{w}\left[\varphi(\tau, X_{t,x}(\tau))Y_{t,x,1}(\tau) + Z_{t,x,1,0}(\tau)\right], \quad T_0 \le t \le T, \tag{3.16}$$

where $X_{t,x}(s)$, $Y_{t,x,y}(s)$, $Z_{t,x,y,z}(s)$, $s \geq t$, $(t,x) \in Q$, is the solution of the SDEs

$$dX = b(s,X)ds + \sigma(s,X)dW(s), \quad X(t) = x, \tag{3.17}$$

$$dY = c(s,X)Yds + \beta^{\top}(s,X)Ydw(s), \quad Y(t) = y, \tag{3.18}$$

$$dZ = f(s,X)Yds + \gamma^{\top}(s,X)Ydw(s), \quad Z(t) = z, \tag{3.19}$$

$W(s) = (W_1(s), \ldots, W_d(s))^{\top}$ is a d-dimensional standard Wiener process independent of $w(s)$ and $\tau = \tau_{t,x}$ is the first exit time of the trajectory $(s, X_{t,x}(s))$ to the boundary Γ.

The expectation E^w in (3.16) is taken over the realizations of $W(s)$, $t \leq s \leq T$, for a fixed $w(s)$, $t \leq s \leq T$; in other words, $E^w(\cdot)$ means the conditional expectation $E\left(\cdot | w(s) - w(t), t \leq s \leq T\right)$. We note that the exit time $\tau_{t,x}$ does not depend on $w(\cdot)$.

To verify that (3.16)-(3.19) is a probabilistic representation for the solution $v(t,x)$ of (3.1)-(3.2), one can proceed as follows. We write down the probabilistic representation for the solution $u(t,x)$ of deterministic PDEs with random coefficients (3.9)-(3.10) [31, 37]:

$$u(t,x) = E^w\left[\Phi(\tau, X_{t,x}(\tau))\mathbb{Y}_{t,x,1}(\tau) + \mathbb{Z}_{t,x,1,0}(\tau)\right], \quad T_0 \leq t \leq T, \tag{3.20}$$

where $X_{t,x}(s)$, $Y_{t,x,y}(s)$, $Z_{t,x,y,z}(s)$, $s \geq t$, $(t,x) \in Q$, is the solution of the system

$$dX = b(s,X)ds + \sigma(s,X)dW(s), \quad X(t) = x, \tag{3.21}$$

$$d\mathbb{Y} = C(s,X)\mathbb{Y}ds + \nabla^{\top}\eta(s;X)\sigma(s,X)\mathbb{Y}dW(s), \quad \mathbb{Y}(t) = y, \tag{3.22}$$

$$d\mathbb{Z} = F(s,X)\mathbb{Y}ds, \quad \mathbb{Z}(t) = z. \tag{3.23}$$

and the coefficients η, C, F and Φ were defined in Section 3.2 by relations (3.6), (3.12), (3.13) and (3.14), respectively. We note that we have the probabilistic representation (3.20)-(3.23) so that its $X(t)$ component is driven by the same SDE as in the SPDE representation (3.16)-(3.19). Consequently, $\tau = \tau_{t,x}$ is the same first exit time in both representations. Using the relation (3.15), we transform the probabilistic representation (3.20)-(3.23) into the one for $v(t,x)$ and arrive at (3.16)-(3.19). The corresponding details are given in Appendix B.

Chapter 4

Numerical methods

To construct numerical methods for the Dirichlet problem for the SPDE (3.1)-(3.2), we use ideas of the simplest random walks for the deterministic Dirichlet problem from [92] (see also [93, Chapter 6]) together with the approach to solving SPDEs via averaging over characteristics developed in [95] (see also [94, 96]).

We propose three algorithms: two of them (Algorithm 1A and Algorithm 1B) are of order one and Algorithm 2 is of order 1/2.

Difficulties arising in realization of the probabilistic representation for solving deterministic Dirichlet problems were discussed in [91, 92, 93]. They are inherited in the case of the Dirichlet problem for SPDEs. For instance, the difference $\tau - t$ in (3.16) can take arbitrary small values and, consequently, it is impossible to integrate numerically the system (3.17) with a fixed time step. In particular, we cannot use mean-square Euler approximations for simulating (3.17). Thanks to the fact that in

(3.16) we average realizations of the solution $X(t)$ of (3.17), we can exploit simple weak approximations $\bar{X}$ of X imposing on them some restrictions related to nonexit of $\bar{X}$ from the domain $\bar{Q}$ as in the case of deterministic PDEs [91, 92, 93]. Namely, we will require that Markov chains approximating in the weak sense the solution of (3.17) remain in the domain $\bar{Q}$ with probability one.

In all the algorithms considered here we apply the weak explicit Euler approximation with the simplest simulation of noise to the sub-system (3.17):

$$X_{t,x}(s+h) \approx \bar{X} = x + hb(s,x) + h^{1/2}\sigma(s,x)\,\xi\,, \tag{4.1}$$

where $h > 0$ is a step of integration (a sufficiently small number), $\xi = (\xi^1, \ldots, \xi^d)^\intercal$, ξ^i, $i = 1, \ldots, d$, are mutually independent random variables taking the values ± 1 with probability $1/2$. As we will see, this approximation is used "inside" the space domain G.

Let us now introduce the boundary zone $S_{t,h} \subset \bar{G}$ for the time layer t while $\bar{G} \backslash S_{t,h}$ will become the corresponding "inside" part of G. Clearly, the random vector $\bar{X}$ in (4.1) takes 2^d different values. Introduce the set of points close to the boundary (a boundary zone) $S_{t,h} \subset \bar{G}$ on the time layer t : we say that $x \in S_{t,h}$ if at least one of the 2^d values of the vector $\bar{X}$ is outside $\bar{G}$. It is not difficult to see that due to compactness of $\bar{Q}$ there is a constant $\lambda > 0$ such that if the distance from $x \in G$ to the boundary ∂G is equal to or greater than $\lambda\sqrt{h}$ then x is outside the boundary zone and, therefore, for such x all the realizations of the random variable $\bar{X}$ belong to $\bar{G}$.

Since restrictions connected with nonexit from the domain $\bar{G}$ should be imposed on an approximation of the system (3.17), the formula (4.1) can be used only for the points $x \in \bar{G} \backslash S_{t,h}$ on the layer t, and a special construction is required for points from the boundary zone. In Algorithm 1A and Algorithm 1B we use a construction based on linear interpolation while in Algorithm 2 we just stop the Markov chain as soon as it reaches the boundary zone $S_{t,h}$. In the deterministic case these constructions were exploited in [92, 93].

Below we also use the following notation. Let $x \in S_{t,h}$. Denote by $x^{\pi} \in \partial G$ the projection of the point x on the boundary of the domain G in the normal direction (the projection is unique because h is sufficiently small and ∂G is smooth) and by $n(x^{\pi})$ the unit vector of internal normal to ∂G at x^{π}.

4.1 Methods of order one

To define our approximation of $X(t)$ in the boundary zone, we introduce the random vector $X_{x,h}^{\pi}$ taking two values x^{π} and $x+h^{1/2}\lambda n(x^{\pi})$ with probabilities $p = p_{x,h}$ and $q = q_{x,h} = 1 - p_{x,h}$, respectively, where

$$p_{x,h} = \frac{h^{1/2}\lambda}{|x + h^{1/2}\lambda n(x^{\pi}) - x^{\pi}|} \, .$$

We have that $v(t,x)$ is a twice continuously differentiable function with the domain of definition $\bar{G}$, then an approximation of $v(t,x)$ by the expectation $Ev(t, X_{x,h}^{\pi})$ corresponds to linear interpolation and

$$v(t,x) = Ev(t, X^{\pi}_{x,h}) + O(h) = pv(t, x^{\pi}) + qv(t, x + h^{1/2}\lambda n(x^{\pi})) + O(h), \qquad (4.2)$$

where $O(h)$ is asymptotically equivalent with

$$h \max_{(t,x)\in\overline{Q},\ i,j=1,\ldots,d} |\frac{\partial^2 v(t,x)}{\partial x_i \partial x_j}|.$$

We emphasize that the second value $x + h^{1/2}\lambda n(x^{\pi})$ does not belong to the boundary zone. We also note that p is always greater than $1/2$ (since the distance from x to ∂G is less than $h^{1/2}\lambda$) and that if $x \in \partial G$ then $p = 1$ (since in this case $x^{\pi} = x$).

Consider the partly weak, partly mean-square explicit Euler approximation (see also [95]) to the rest of the probabilistic representation, i.e. to (3.18)-(3.19)

$$\begin{aligned} Y_{t,x,y}(s+h) \ &\approx \ \overline{Y} = y + hc(s,x)y + \beta^{\top}(s,x)y\Delta w(s) \qquad (4.3) \\ &+\frac{1}{2}y\sum_{i=1}^{r}[\beta^i(s,x)]^2([\Delta_i w(s)]^2 - h) \\ &+y\sum_{i=1}^{r}\sum_{j=i+1}^{r}\beta^{i}(s,x)\beta^{j}(s,x)\Delta w_i(s)\Delta w_j(s), \\ Z_{t,x,y,z}(s+h) \ &\approx \ \overline{Z} = z + hf(s,x)y + \gamma^{\top}(s,x)y\Delta w(s) \qquad (4.4) \\ &+y\sum_{i,j=1}^{r}\beta^i(s,x)\gamma^j(s,x)I_{ij}(s), \end{aligned}$$

where $\Delta w(s) = w(s+h) - w(s)$ and $I_{ij}(s)$ is the Itô integral

$$I_{ij}(s) = \int_{s}^{s+h}(w_i(s') - w_i(s))dw_j(s').$$

Remark 14 *We note that unless* $\beta^{i}\gamma^{j} = \beta^{j}\gamma^{i}$ *we face the difficulty in simulating* $\bar{Z}$ *from* (4.4) *efficiently due to the presence of the integral* $I_{ij}(s)$ (*see various approaches* *to its approximation in, e.g. [93] and also [21]). As is well known, to realize* (4.4) *in the commutative noise situation* $\beta^{i}\gamma^{j} = \beta^{j}\gamma^{i}$, *it suffices to simulate the Wiener increments* $\Delta w_i(s)$ *only, avoiding simulation of* $I_{ij}(s)$. *We also note that in the important case of* $\gamma \equiv 0$ *the term with* $I_{ij}(s)$ *in* (4.4) *vanishes.*

Now we are ready to propose an algorithm for solving the SPDE problem (3.1)-(3.2). Let a point $(t_0, x_0) \in Q$. We would like to find the value $v(t_0, x_0)$. Introduce a discretization of the interval $[t_0, T]$, for definiteness the equidistant one:

$$t_0 < t_1 < \cdots < t_N = T, \quad h := (T - t_0)/N.$$

Let ξ_k, $k = 0, 1, \ldots$, be i.i.d. random variables with the same law as defined for the ξ above. We formulate the algorithm as follows (see its deterministic counterpart in [93, p. 355]).

Algorithm 1A

STEP 0. $X_0' = x_0,\ Y_0 = 1,\ Z_0 = 0,\ k = 0.$

STEP 1. If $X_k' \notin S_{t_k,h}$ then $X_k = X_k'$ and go to STEP 3. If $X_k' \in S_{t_k,h}$ then either $X_k = X_k'^{\pi}$ with probability $p_{X_k',h}$ or $X_k = X_k' + h^{1/2}\lambda n(X_k'^{\pi})$ with probability $q_{X_k',h}$.

STEP 2. If $X_k = X_k'^{\pi}$ then STOP and $\varkappa = k$, $X_\varkappa = X_k'^{\pi}$, $Y_\varkappa = Y_k$, $Z_\varkappa = Z_k$.

STEP 3. Simulate ξ_k and find X_{k+1}', Y_{k+1}, Z_{k+1} according to (4.1)-(4.3)-(4.4) for $s = t_k,\ x = X_k,\ y = Y_k,\ z = Z_k,\ \xi = \xi_k$, $\Delta w(s) = \Delta w(t_k)$.

STEP 4. If $k + 1 = N$, STOP and $\varkappa = N$, $X_\varkappa = X_N'$, $Y_\varkappa = Y_N$, $Z_\varkappa = Z_N$, otherwise $k := k + 1$ and return to STEP 1.

Having obtained the end points of the chain $(t_\varkappa, X_\varkappa, Y_\varkappa, Z_\varkappa)$ with $(t_\varkappa, X_\varkappa) \in \Gamma$, we get the approximation of the solution to the SPDE problem (3.1)-(3.2):

$$v(t_0, x_0) \approx \bar{v}(t_0, x_0) = E^w \left[\varphi(t_\varkappa, X_\varkappa) Y_\varkappa + Z_\varkappa\right], \tag{4.5}$$

where the approximate equality corresponds to the numerical integration error (see Theorem 20 below). To realize the approximation (4.5) in practice, one case use the Monte Carlo technique:

$$\bar{v}(t_0, x_0) \approx \hat{v}(t_0, x_0) = \frac{1}{M} \sum_{m=1}^{M} \left[\varphi(t_\varkappa^{(m)}, X_\varkappa^{(m)}) Y_\varkappa^{(m)} + Z_\varkappa^{(m)}\right], \tag{4.6}$$

where $(t_\varkappa^{(m)}, X_\varkappa^{(m)}, Y_\varkappa^{(m)}, Z_\varkappa^{(m)})$ are independent realizations of $(t_\varkappa, X_\varkappa, Y_\varkappa, Z_\varkappa)$, each obtained by Algorithm 1A. The approximate equality in (4.6) corresponds to the Monte Carlo (or in other words, statistical) error.

Remark 15 *It may happen so that it is more rational to choose both h and λ depending on the chain's state: h_k and λ_k. Then, in Theorem 20 (see below) one should put $h = \max_{0 \le k < N} h_k$. In practice, one can take $\lambda_k = |\sigma(t_k, X_k)|$, possibly with small corrections.*

As it was noted, e.g. in [94, Remark 3.4], it can be computationally more efficient to approximate (3.18) as

$$Y_{t,x,y}(s+h) \approx \bar{Y} = y \exp\left(c(s,x)h - \beta^\top(s,x)\beta(s,x)h/2 + \beta^\top(s,x)\Delta w(s)\right), \tag{4.7}$$

rather than by (4.3). We remark that in comparison with (4.3) the approximation (4.7) preserves the property of positivity of Y.

By *Algorithm 1B* we denote the algorithm which coincides with Algorithm 1A but uses (4.7) to approximate the component $Y(t)$ instead of (4.3). The approximation of the solution to the SPDE problem (3.1)-(3.2) in the case of Algorithm 1B has the same form (4.5) as for Algorithm 1A but with the new $Y_\varkappa$ and $Z_\varkappa$ (note that simulation of Z_k depends on simulation of Y_k).

4.2 Proof of convergence of Algorithms of order one

We start with some lemmas which are used in the proof of the convergence theorem for Algorithms 1A and 1B.

Denote by ν_{t_0,x_0} the number of those t_k at which X_k' gets into the set $S_{t_k,h}$. The following lemma is proved in [93].

Lemma 16 *(see [93]) The following inequalities hold:*

$$P\{\nu_{t_0,x_0} > n\} \le \frac{1}{2^n}\,. \tag{4.8}$$

$$P\{\nu_{t_0,x_0} = n\} \le \frac{1}{2^{n-1}}\,, \tag{4.9}$$

where K does not depend on t_0, x_0, h.

Lemma 16 implies that for $1 \le p < \infty$

$$E\nu_{t_0,x_0}^p \le K\,, \tag{4.10}$$

where K does not depend on t_0, x_0, h.

We extend the definition of the constructed chain for all k by the rule: if $k > \varkappa$, then $(t_k, X_k, Y_k, Z_k) = (t_\varkappa, X_\varkappa, Y_\varkappa, Z_\varkappa)$.

Lemma 17 *For any $1 \leq p < \infty$ the following inequality holds:*

$$E(\max_{k=0,\dots,\ N-1} Y_k)^{2p} \leq K, \tag{4.11}$$

where K does not depend on t_0, x_0, h and Y_k is defined by relation (4.3).

Proof. Consider

$$\overline{Y_k} = Y_k / \exp(\overline{c}hk),\ \ k = 0, \dots,\ N-1$$

where

$$\overline{c} = \min_{(t,x)\in\overline{Q}} c(t,x).$$

Let us prove that $(\overline{Y_k}, \sigma_k)$ is a positive submartingale where for any $k \geq 1$, σ_k is the $\sigma-$algebra generated by the random variables $X_0, \dots,\ X_{k-1},\ w(t_0), \dots,\ w(t_k)$.

We have:

$$\overline{Y_{k+1}} \geq \overline{Y_k} \exp(-\beta^\top(t_k, X_k)\beta(t_k, X_k)(t_{k+1} - t_k)/2 \tag{4.12}$$

$$+\beta^\top(t_k, X_k)\Delta w(t_k)), 1 \leq k \leq N-1, Y_0 = 1.$$

Taking the conditional expectation in relation (4.12) and utilizing the fact that $\overline{Y_k}$ is measurable with respect to σ_k we obtain

$$E[\overline{Y_{k+1}}|\sigma_k] \geq \overline{Y_k}E[\exp(-\beta^\top(t_k, X_k)\beta(t_k, X_k)(t_{k+1} - t_k)/2$$

$$+\beta^\top(t_k, X_k)\Delta w(t_k))|\sigma_k], 1 \leq k \leq N-1, Y_0 = 1.$$

Therfore in order to show that $(\overline{Y_k}, \sigma_k)$ is a positive submartingale it is enough to prove that

$$E[\exp(-\beta^{\top}(t_k, X_k)\beta(t_k, X_k)(t_{k+1} - t_k)/2 + \beta^{\top}(t_k, X_k)\Delta w(t_k))|\sigma_k] = 1. \qquad (4.13)$$

Let $\sigma(R^d)$ be the Borel $\sigma-$ algebra on R^d and $\sigma(R^r)$ be the Borel $\sigma-$ algebra on R^r, respectively. Using the monotone class theorem in order to prove (4.13) it will be enough to show that for any for any $A_0, \ldots, A_{k-1} \in \sigma(R^d)$ and $B_0, \ldots, B_k \in \sigma(R^r)$ we have:

$$\begin{aligned}
& \int_{\Omega} 1_{A_0}(X_0) \times ... \times 1_{A_{k-1}}(X_{k-1}) 1_{B_0}(w_{t_0}) \times ... \times 1_{B_k}(w_{t_k}) \qquad (4.14) \\
& \times \exp(-\beta^{\top}(t_k, X_k)\beta(t_k, X_k)(t_{k+1} - t_k)/2 + \beta^{\top}(t_k, X_k)\Delta w(t_k))dP \\
= & \int_{\Omega} 1_{A_0}(X_0) \times ... \times 1_{A_{k-1}}(X_{k-1}) 1_{B_0}(w_{t_0}) \times ... \times 1_{B_k}(w_{t_k})dP
\end{aligned}$$

Using the independence of the increments of brownian motion and the fact that X_k is independent of w we have:

$$\begin{aligned}
& \int_{\Omega} 1_{A_0}(X_0) \times ... \times 1_{A_{k-1}}(X_{k-1}) 1_{B_0}(w_{t_0}) \times ... \times 1_{B_k}(w_{t_k}) \\
& \times \exp\{-\beta^{\top}(t_k, X_k)\beta(t_k, X_k)(t_{k+1} - t_k)/2 + \beta^{\top}(t_k, X_k)\Delta w(t_k)\}dP \\
= & \int_{\Omega} 1_{A_0}(X_0(\varpi')) \times ... \times 1_{A_{k-1}}(X_{k-1}(\varpi')) 1_{B_0}(w_{t_0}(\varpi')) \times ... \times 1_{B_k}(w_{t_k}(\varpi')) \\
& \times \exp\{-\beta^{\top}(t_k, X_k(\varpi'))\beta(t_k, X_k(\varpi'))(t_{k+1} - t_k)/2)\} \times \\
& \times \int_{\Omega} \exp\{\beta^{\top}(t_k, X_k(\varpi'))\Delta w(t_k)(\varpi)\}dP(\varpi)dP(\varpi')
\end{aligned}$$

$$= \int_{\Omega} 1_{A_0}(X_0(\varpi')) \times ... \times 1_{A_{k-1}}(X_{k-1}(\varpi'))1_{B_0}(w_{t_0}(\varpi')) \times ... \times 1_{B_k}(w_{t_k}(\varpi'))$$
$$\times \exp\{-\beta^{\top}(t_k, X_k(\varpi'))\beta(t_k, X_k(\varpi'))(t_{k+1} - t_k)/2)\} \exp\{\beta^{\top}(t_k, X_k(\varpi'))$$
$$\times \beta(t_k, X_k(\varpi'))(t_{k+1} - t_k)/2)\} dP(\varpi')$$
$$= \int_{\Omega} 1_{A_0}(X_0(\varpi')) \times ... \times 1_{A_{k-1}}(X_{k-1}(\varpi'))1_{B_0}(w_{t_0}(\varpi')) \times ... \times 1_{B_k}(w_{t_k}(\varpi'))dP(\varpi')$$

which is exactly relation (4.14). Thus we proved that $(\overline{Y_k}, \sigma_k)$ is a positive submartingale. Now the inequality (4.11) is an easy consequence of Doob's maximal inequality for positive submartingales and the smoothness of the coefficients in (3.1)-(3.2) on $\bar{Q}$.

■

A similar result holds when Y_k is defined by the relation (4.7).

Lemma 18 *For any $1 \leq p < \infty$ the following inequality holds:*

$$E(\max_{k=0,...,\ N-1} |Y_k|)^{2p} \leq K, \tag{4.15}$$

where K does not depend on t_0, x_0, h and Y_k is defined by relation (4.7).

Proof. Consider

$$\overline{Y_k} = Y_k / \prod_{s=1}^{k-1} (1 + hc(t_s, X_s)),\ k = 2, ...,\ N-1,\ \overline{Y_0} = 1.$$

Let us prove that $(\overline{Y_k}, \sigma_k)$ is a martingale where for any $k \geq 1$, σ_k is the $\sigma-$algebra generated by the random variables X_0, ..., X_k, $w(t_0)$, ..., $w(t_k)$.

Using the independence of the increments of brownian motion and the fact that

X_k is independent of w we have

$$
\begin{aligned}
& E[\beta^{\top}(t_k, X_k)\Delta w(t_k) + \frac{1}{2}\sum_{i=1}^{r}[\beta^i(t_k, X_k)]^2([\Delta_i w(t_k)]^2 - h) \\
& + \sum_{i=1}^{r}\sum_{j=i+1}^{r}\beta^i(t_k, X_k)\beta^j(t_k, X_k)\Delta w_i(t_k)\Delta w_j(t_k)|\sigma_k] \\
= \; & 0.
\end{aligned}
$$

Therefore, utilizing the fact that $\overline{Y_k}$ is measurable with respect to σ_k we obtain

$$E[\overline{Y_{k+1}}|\sigma_k] = \overline{Y_k}.$$

Thus $(\overline{Y_k}, \sigma_k)$ is a martingale.

For h small enough, similar to the proof from above, the result is an easy consequence of Doob's maximal inequality for positive submartingales, Jensen, Hölder and Burkhölder-Davis-Gundy inequalities and the smoothness of the coefficients in (3.1)-(3.2) on $\bar{Q}$. ■

Before proceeding to the proof of the convergence theorem (Theorem 20) let us first provide some intuitive guidance to its proof. "Inside" the domain, i.e., when $X_k \notin S_{t_k,h}$, the Markov chain (t_k, X_k, Y_k, Z_k) is analogous to the one used in the case of the Cauchy problem for the linear SPDE (3.1), for which the first mean-square order of convergence was proved in [94] (see also [95, 96]). Then it is reasonable to expect that the approxiamtion "inside" the domain contributes $O(h)$ to the global error of Algorithm 1B. Near the boundary, i.e., when $X_k \in S_{t_k,h}$, the local error is of order $O(h)$ as it follows from the interpolation relation (4.2) while the number of steps ν_{t_0,x_0} on which we should count contributions of this local error is such that its any moment is uniformly (with respect to the time step) bounded (see 4.10), i.e., roughly speaking, the local error $O(h)$ is counted finite number of times to the global error. Consequently, we can expect that the total error "inside" the domain and the total error near the boundary should sum up to $O(h)$.

We now proceed to the formal proof. Following the idea of the proofs of convergence theorems in the corresponding deterministic case [92, 93], we introduce

$$\begin{aligned} d_k &= (v(t_k, X_k) - v(t_k, X_k'))Y_k, \\ d_k' &= v(t_{k+1}, X_{k+1}')Y_{k+1} - v(t_k, X_k)Y_k + Z_{k+1} - Z_k, \\ k &= 0, \ldots,\ N-1. \end{aligned}$$

We recall that X_k' belongs to the layer $t = t_k$; the variable d_k can be nonzero in the case of $X_k' \in S_{t_k,h}$ only, i.e., d_k has the meaning of the local error in the boundary zone. If $k < \varkappa$, then $X_k \notin S_{t_k,h}$ and all the 2^d realizations of the random variable X_{k+1}' belong to $\overline{G}$; if $\varkappa \leq k$, then $t_k = t_{k+1} = t_\varkappa$, $X_{k+1}' = X_k = X_\varkappa$, $Y_{k+1} = Y_k = Y_\varkappa$, $Z_{k+1} = Z_k = Z_\varkappa$ and consequently, $d_k' = 0$. The random variable d_k' has the meaning of the local error "inside" the domain.

The following lemma is used to prove the convergence theorem, Theorem 20 below.

Lemma 19 *For any $1 \leq p < \infty$ and any of the weak Euler approximation schemes*

(4.1)-(4.3)-(4.4) or (4.1)-(4.7)-(4.4) the following relations hold

$$\sum_{k=0}^{N-1} E^w[d_k] = O_1(h), \tag{4.16}$$

$$\sum_{k=0}^{N-1} E^w[d'_k] = \sum_{k=0}^{N-1}\sum_{r=1}^{q}(E^w[Y_kF_r^1(t_{k+1},X_k)]hI_{rk} + E^w[Y_kF_r^2(t_{k+1},X_k)]I_{0rk}) \tag{4.17}$$
$$+\sum_{k=0}^{N-1}\sum_{r=1}^{q}\sum_{n=1}^{q}\sum_{l=1}^{q} E^w[Y_kF_{rnl}(t_{k+1},X_k)]I_{rnlk} + O_2(h),$$

where

$$I_{rk} = w^r(t_{k+1}) - w^r(t_k),$$
$$I_{0rk} = \int_{t_k}^{t_{k+1}} (w^r(s) - w^r(t_k))ds,$$
$$I_{rnlk} = \int_{t_k}^{t_{k+1}}\int_{s}^{t_{k+1}}\int_{u}^{t_{k+1}} 1 * dw^l(z) * dw^n(u) * dw^r(s),$$

$F_r^1(t_{k+1},x)$, $F_r^2(t_{k+1},x)$ and $F_{rnl}(t_{k+1},x)$ are combinations of the coefficients in (3.1)-(3.2) and their partial derivatives and $v(t_{k+1},x)$ and its partial derivatives in spatial variables up to the second order and $O_i(h)$, $i = 1,2$ are random variables such that

$$E|O_i(h)|^{2p} \le Kh^{2p},\ \ i = 1,2$$

with K being independent of t_0, x_0, h.

Proof. From the interpolation relation (4.2) and Jensen and Hölder inequalities we obtain

$$E(\sum_{k=0}^{N-1} E^w[d_k])^{2p} \tag{4.18}$$
$$\le\ Kh^{2p}E|\max_{(t,x)\in\overline{Q},\ i,j=1,\dots,\ d}|\frac{\partial^2 v(t,x)}{\partial x_i \partial x_j}|\max_{k=0,\dots,\ N-1}|Y_k|$$
$$\times\sum_{k=0}^{N-1}\mathbf{1}_{S_{t_k,h}}(X'_k)|^{2p}$$

$$\leq \quad Kh^{2p}(E((\max_{(t,x)\in\overline{Q},\ i,j=1,\ldots,\ d}|\frac{\partial^2 v(t,x)}{\partial x_i \partial x_j}|)^{2pr_1}))^{1/r_1}$$
$$\times(E(\max_{k=0,\ldots,\ N-1}|Y_k|)^{2pr_2})^{1/r_2}(E(\nu_{t_0,x_0})^{2pr_3})^{1/r_3},$$

where $r_1, r_2, r_3 > 0$, $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = 1$ and K does not depend on t_0, x_0, h. Now the relation (4.16) follows from the above estimate (4.18), Lemma 16, Lemma 17 or Lemma 18 and Assumption 2.3.

Introduce

$$d_k^1 = v(t_{k+1}, X_k')Y_{k+1} - v(t_{k+1}, X_k)Y_k,$$
$$d_k^2 = (v(t_{k+1}, X_k) - v(t_k, X_k))Y_k + Z_{k+1} - Z_k,$$
$$k = 0, \ldots,\ N-1.$$

We have

$$d_k' = d_k^1 + d_k^2,\ k = 0, \ldots,\ N-1.$$

Let us now continue the proof for the case of Algorithm 1B. The case of Algorithm 1A can be treated in an analogous way.

Applying Taylor's formula for d_k^1 and expanding around (X_k, Y_k) up to the fourth derivatives and utilizing the convention of summation upon repeated indices, we obtain

$$d_k^1 \tag{4.19}$$
$$= \sum_{i=1}^{d} \frac{\partial v}{\partial x_i}(t_{k+1}, X_k)\{hb^i(t_k, X_k) + h^{1/2}\sigma^{ip}(t_k, X_k)\xi_k^p)Y_k + v(t_{k+1}, X_k)Y_k$$

$$\times(h(c(t_k,X_k)-\frac{1}{2}\sum_{r=1}^{q}\beta_r^2(t_k,X_k))+\sum_{r=1}^{q}\beta_r(t_k,X_k)(w_r(t_{k+1})-w_r(t_k))\}$$
$$+\frac{1}{2}\sum_{i=1}^{d}\frac{\partial^2 v}{\partial x_i^2}(t_{k+1},X_k)(hb^i(t_k,X_k)+h^{1/2}\sigma^{ip}(t_k,X_k)\xi_k^p)^2 Y_k$$
$$+\frac{1}{2}\sum_{i=1}^{d}\frac{\partial^2 v}{\partial x_i^2}(t_{k+1},X_k)(hb^i(t_k,X_k)+h^{1/2}\sigma^{ip}(t_k,X_k)\xi_k^p)^2 Y_k$$
$$+\sum_{\substack{i,j=1\\ i\neq j}}^{d}\frac{\partial^2 v}{\partial x_i\partial x_j}(t_{k+1},X_k)(hb^i(t_k,X_k)+h^{1/2}\sigma^{ip}(t_k,X_k)\xi_k^p)(hb^j(t_k,X_k)$$
$$+h^{1/2}\sigma^{jp}(t_k,X_k)\xi_k^p)Y_k+\sum_{i=1}^{d}\frac{\partial v}{\partial x_i}(t_{k+1},X_k)(hb^i(t_k,X_k)+h^{1/2}\sigma^{ip}(t_k,X_k)\xi_k^p)$$
$$\times Y_k\{h(c(t_k,X_k)-\frac{1}{2}\sum_{r=1}^{q}\beta_r^2(t_k,X_k))+\sum_{r=1}^{q}\beta_r(t_k,X_k)(w_r(t_{k+1})-w_r(t_k))\}$$
$$+\frac{1}{2}v(t_{k+1},X_k)Y_k\{h(c(t_k,X_k)-\frac{1}{2}\sum_{r=1}^{q}\beta_r^2(t_k,X_k))+\sum_{r=1}^{q}\beta_r(t_k,X_k)$$
$$\times(w_r(t_{k+1})-w_r(t_k))\}^2+\frac{1}{3!}\sum_{i=1}^{d}\frac{\partial^3 v}{\partial x_i^3}(t_{k+1},X_k)(hb^i(t_k,X_k)+h^{1/2}$$
$$\times\sigma^{ip}(t_k,X_k)\xi_k^p)^3 Y_k+\frac{1}{3!}v(t_{k+1},X_k)Y_k\{h(c(t_k,X_k)-\frac{1}{2}\sum_{r=1}^{q}\beta_r^2(t_k,X_k))$$
$$+\sum_{r=1}^{q}\beta_r(t_k,X_k)(w_r(t_{k+1})-w_r(t_k))\}^3+\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{d}\frac{\partial^3 v}{\partial x_i^2\partial x_j}(t_{k+1},X_k)(hb^i(t_k,X_k)$$
$$+h^{1/2}\sigma^{ip}(t_k,X_k)\xi_k^p)^2(hb^j(t_k,X_k)+h^{1/2}\sigma^{jp}(t_k,X_k)\xi_k^p)Y_k$$
$$+\frac{1}{2}\sum_{i=1}^{d}\frac{\partial^2 v}{\partial x_i^2}(t_{k+1},X_k)(hb^i(t_k,X_k)+h^{1/2}\sigma^{ip}(t_k,X_k)\xi_k^p)^2 Y_k\{h(c(t_k,X_k)$$
$$-\frac{1}{2}\sum_{r=1}^{q}\beta_r^2(t_k,X_k))+\sum_{r=1}^{q}\beta_r(t_k,X_k)(w_r(t_{k+1})-w_r(t_k))\}$$
$$+\frac{1}{2}\sum_{i=1}^{d}\frac{\partial v}{\partial x_i}(t_{k+1},X_k)(hb^i(t_k,X_k)+h^{1/2}\sigma^{ip}(t_k,X_k)\xi_k^p)Y_k\{h(c(t_k,X_k)$$
$$-\frac{1}{2}\sum_{r=1}^{q}\beta_r^2(t_k,X_k))+\sum_{r=1}^{q}\beta_r(t_k,X_k)(w_r(t_{k+1})-w_r(t_k))\}^2$$
$$+\sum_{\substack{i,j,l=1\\ i\neq j,i\neq l,j\neq l}}^{d}\frac{\partial^3 v}{\partial x_i\partial x_j\partial x_l}(t_{k+1},X_k)(hb^i(t_k,X_k)+h^{1/2}\sigma^{ip}(t_k,X_k)\xi_k^p)(hb^j(t_k,X_k)$$
$$+h^{1/2}\sigma^{jp}(t_k,X_k)\xi_k^p)(hb^l(t_k,X_k)+h^{1/2}\sigma^{lp}(t_k,X_k)\xi_k^p)Y_k$$

$$
\begin{aligned}
&+\sum_{\substack{i,j=1\\ i\neq j}}^{d}\frac{\partial^2 v}{\partial x_i\partial x_j}(t_{k+1},X_k)(hb^i(t_k,X_k)+h^{1/2}\sigma^{ip}(t_k,X_k)\xi_k^p)(hb^j(t_k,X_k)+h^{1/2}\\
&\times\sigma^{ip}(t_k,X_k)\xi_k^p)(hb^j(t_k,X_k)+h^{1/2}\sigma^{jp}(t_k,X_k)\xi_k^p)Y_k\{h(c(t_k,X_k)\\
&-\frac{1}{2}\sum_{r=1}^{q}\beta_r^2(t_k,X_k))+\sum_{r=1}^{q}\beta_r(t_k,X_k)(w_r(t_{k+1})-w_r(t_k))\}\\
&+\frac{1}{4!}\sum_{i=1}^{d}\frac{\partial^4 v}{\partial x_i^4}(t_{k+1},X_k^h)(hb^i(t_k,X_k)+h^{1/2}\sigma^{ip}(t_k,X_k)\xi_k^p)^4\exp(U_k^h)\\
&+\frac{1}{4!}\sum_{i=1}^{d}v(t_{k+1},X_k^h)\exp(U_k^h)\{h(c(t_k,X_k)-\frac{1}{2}\sum_{r=1}^{q}\beta_r^2(t_k,X_k))\\
&+\sum_{r=1}^{q}\beta_r(t_k,X_k)(w_r(t_{k+1})-w_r(t_k))\}^4+\frac{1}{3!}\sum_{\substack{i,j=1\\ i\neq j}}^{d}\frac{\partial^4 v}{\partial x_i^3\partial x_j}(t_{k+1},X_k^h)\\
&\times(hb^i(t_k,X_k)+h^{1/2}\sigma^{ip}(t_k,X_k)\xi_k^p)^3(hb^j(t_k,X_k)+h^{1/2}\sigma^{jp}(t_k,X_k)\xi_k^p)\exp(U_k^h)\\
&+\frac{1}{3!}\sum_{i=1}^{d}\frac{\partial^3 v}{\partial x_i^3}(t_{k+1},X_k^h)(hb^i(t_k,X_k)+h^{1/2}\sigma^{ip}(t_k,X_k)\xi_k^p)^3\exp(U_k^h)\\
&\times\{h(c(t_k,X_k)-\frac{1}{2}\sum_{r=1}^{q}\beta_r^2(t_k,X_k))+\sum_{r=1}^{q}\beta_r(t_k,X_k)(w_r(t_{k+1})-w_r(t_k))\}\\
&+\frac{1}{3!}\sum_{i=1}^{d}\frac{\partial v}{\partial x_i}(t_{k+1},X_k^h)(hb^i(t_k,X_k)+h^{1/2}\sigma^{ip}(t_k,X_k)\xi_k^p)\exp(U_k^h)\\
&\times\{h(c(t_k,X_k)-\frac{1}{2}\sum_{r=1}^{q}\beta_r^2(t_k,X_k))+\sum_{r=1}^{q}\beta_r(t_k,X_k)(w_r(t_{k+1})-w_r(t_k))\}^3\\
&+\frac{1}{2!2!}\sum_{i=1}^{d}\frac{\partial^2 v}{\partial x_i^2}(t_{k+1},X_k^h)(hb^i(t_k,X_k)+h^{1/2}\sigma^{ip}(t_k,X_k)\xi_k^p)^2\exp(U_k^h)\\
&\times\{h(c(t_k,X_k)-\frac{1}{2}\sum_{r=1}^{q}\beta_r^2(t_k,X_k))+\sum_{r=1}^{q}\beta_r(t_k,X_k)(w_r(t_{k+1})-w_r(t_k))\}^2\\
&+\frac{1}{2!}\sum_{\substack{i,j=1\\ i\neq j}}^{d}\frac{\partial^2 v}{\partial x_i\partial x_j}(t_{k+1},X_k^h)(hb^i(t_k,X_k)+h^{1/2}\sigma^{ip}(t_k,X_k)\xi_k^p)(hb^j(t_k,X_k)\\
&+h^{1/2}\sigma^{jp}(t_k,X_k)\xi_k^p)\exp(U_k^h)\{h(c(t_k,X_k)-\frac{1}{2}\sum_{r=1}^{q}\beta_r^2(t_k,X_k))\\
&+\sum_{r=1}^{q}\beta_r(t_k,X_k)(w_r(t_{k+1})-w_r(t_k))\}^2+\sum_{\substack{i,j,l=1\\ i\neq j,i\neq l,j\neq l}}^{d}\frac{\partial^3 v}{\partial x_i\partial x_j\partial x_l}(t_{k+1},X_k^h)
\end{aligned}
$$

$$\times(hb^i(t_k,X_k)+h^{1/2}\sigma^{ip}(t_k,X_k)\xi_k^p)(hb^j(t_k,X_k)+h^{1/2}\sigma^{jp}(t_k,X_k)\xi_k^p)$$
$$\times(hb^l(t_k,X_k)+h^{1/2}\sigma^{lp}(t_k,X_k)\xi_k^p)\exp(U_k^h)\{h(c(t_k,X_k)$$
$$-\frac{1}{2}\sum_{r=1}^{q}\beta_r^2(t_k,X_k))+\sum_{r=1}^{q}\beta_r(t_k,X_k)(w_r(t_{k+1})-w_r(t_k))\}$$
$$+\sum_{\substack{i,j,l,s=1\\ i\neq j,i\neq l,j\neq l,i\neq s,j\neq s,l\neq s}}^{d}\frac{\partial^4 v}{\partial x_i\partial x_j\partial x_l}(t_{k+1},X_k^h)(hb^i(t_k,X_k)+h^{1/2}\sigma^{ip}(t_k,X_k)\xi_k^p)$$
$$\times(hb^j(t_k,X_k)+h^{1/2}\sigma^{jp}(t_k,X_k)\xi_k^p)(hb^l(t_k,X_k)+h^{1/2}\sigma^{lp}(t_k,X_k)\xi_k^p)$$
$$\times(hb^s(t_k,X_k)+h^{1/2}\sigma^{sp}(t_k,X_k)\xi_k^p)\exp(U_k^h)+\frac{1}{2!2!}\sum_{\substack{i,j=1\\ i\neq j}}^{d}\frac{\partial^4 v}{\partial x_i^2\partial x_j^2}(t_{k+1},X_k^h)$$
$$\times(hb^i(t_k,X_k)+h^{1/2}\sigma^{ip}(t_k,X_k)\xi_k^p)^2(hb^j(t_k,X_k)+h^{1/2}\sigma^{jp}(t_k,X_k)\xi_k^p)^2$$
$$\times\exp(U_k^h),$$

where X_k^h, U_k^h are random variables between X_k and X_k', U_k and U_{k+1}, respectively. Taking the conditional expectation in the above relation (4.19) and using Assumption 2.3**,** Lemma 17, the smoothness of the coefficients of (3.1)-(3.2) in $\bar{Q}$, the Jensen and Hölder inequalities,the fact that (X_k,Y_k), $v(t_{k+1},x)$ are independent and ξ_k, w are independent, respectively, we get

$$E^w[d_k^1] \tag{4.20}$$
$$= E^w[Y_k\{hb^T(t_k,X_k)\nabla v(t_{k+1},X_k)+v(t_{k+1},X_k)[hc(t_k,X_k)$$
$$-\frac{h}{2}\sum_{r=1}^{q}\beta_r^2(t_k,X_k)+\sum_{r=1}^{q}\beta_r(t_k,X_k)(w_r(t_{k+1})-w_r(t_k))]$$
$$+\frac{h}{2}\sum_{i,j=1}^{d}\frac{\partial^2 v}{\partial x_i\partial x_j}(t_{k+1},X_k)a^{ij}(t_k,X_k)+hb^T(t_k,X_k)\nabla v(t_{k+1},X_k)$$
$$\times\sum_{r=1}^{q}\beta_r(t_k,X_k)(w_r(t_{k+1})-w_r(t_k))+\frac{1}{2}v(t_{k+1},X_k)[2h(c(t_k,X_k)$$

$$-\frac{1}{2}\sum_{r=1}^{q}\beta_r^2(t_k,X_k))\sum_{r=1}^{q}\beta_r(t_k,X_k)(w_r(t_{k+1})-w_r(t_k))$$
$$-(\sum_{r=1}^{q}\beta_r(t_k,X_k)(w_r(t_{k+1})-w_r(t_k)))^2]+\frac{1}{3!}v(t_{k+1},X_k)$$
$$+(\sum_{r=1}^{q}\beta_r(t_k,X_k)(w_r(t_{k+1})-w_r(t_k)))^3+\frac{h}{2}\sum_{i,j=1}^{d}\frac{\partial^2 v}{\partial x_i\partial x_j}(t_{k+1},X_k)$$
$$\times a^{ij}(t_k,X_k)\sum_{r=1}^{q}\beta_r(t_k,X_k)(w_r(t_{k+1})-w_r(t_k))\}]+O(h^2),$$

where $O(h^2)$ is a random variable such that $E|O(h^2)|^{2p}\le Kh^{4p}$.

Using relation (3.1) and the backward Itô formula (see Section 2.1 or [111]), we obtain

$$(v(t_{k+1},X_k)-v(t_k,X_k))Y_k \tag{4.21}$$
$$= Y_k\{-\int_{t_k}^{t_{k+1}}[(\mathcal{L}v)(s,X_k)+f(s,X_k)]ds-\sum_{r=1}^{q}\int_{t_k}^{t_{k+1}}(\beta_r(s,X_k)v(s,X_k)$$
$$+\gamma_r(s,X_k))*dw^r(s)\}$$
$$= Y_k\{-\int_{t_k}^{t_{k+1}}[\frac{1}{2}\sum_{i,j=1}^{d}a^{ij}(s,X_k)\frac{\partial^2 v}{\partial x_i\partial x_j}(t_{k+1},X_k)+b^T(s,X_k)\nabla v(t_{k+1},X_k)$$

$$+c(s,X_k)v(t_{k+1},X_k)+f(s,X_k)]ds-\int_{t_k}^{t_{k+1}}[\frac{1}{2}\sum_{i,j=1}^{d}a^{ij}(s,X_k)\int_{s}^{t_{k+1}}\frac{\partial^2(\mathcal{L}v)}{\partial x_i\partial x_j}(u,X_k)du$$
$$+b^T(s,X_k)\int_{s}^{t_{k+1}}\nabla(\mathcal{L}v)(u,X_k)du+c(s,X_k)\int_{s}^{t_{k+1}}(\mathcal{L}v+f)(u,X_k)du]ds$$
$$-\int_{t_k}^{t_{k+1}}[\frac{1}{2}\sum_{i,j=1}^{d}a^{ij}(s,X_k)\sum_{r=1}^{q}\int_{s}^{t_{k+1}}\frac{\partial^2(\beta_r v+\gamma_r)}{\partial x_i\partial x_j}(u,X_k)*dw^r(u)$$
$$+b^T(s,X_k)\sum_{r=1}^{q}\int_{s}^{t_{k+1}}\nabla(\beta_r v+\gamma_r)(u,X_k)*dw^r(u)+c(s,X_k)$$
$$\times\sum_{r=1}^{q}\int_{s}^{t_{k+1}}(\beta_r v+\gamma_r)(u,X_k)*dw^r(u)]ds-\sum_{r=1}^{q}\int_{t_k}^{t_{k+1}}(\beta_r v+\gamma_r)(s,X_k)*dw^r(s)\}.$$

By the backward Itô formula (see Section 2.1 or [111]), we get for the following term from (4.21):

$$
\begin{aligned}
& \int_s^{t_{k+1}} \frac{\partial^2(\beta_r v+\gamma_r)}{\partial x_i \partial x_j}(u, X_k) * dw^r(u) \qquad (4.22)\\
= & \int_s^{t_{k+1}} \frac{\partial^2(\beta_r v+\gamma_r)}{\partial x_i \partial x_j}(t_{k+1}, X_k) * dw^r(u) \\
& + \int_s^{t_{k+1}} \int_u^{t_{k+1}} [-\frac{\partial}{\partial u_1} \frac{\partial^2 \beta_r}{\partial x_i \partial x_j}(u_1, X_k)) v(u_1, X_k) - (\frac{\partial}{\partial u_1} \frac{\partial \beta_r}{\partial x_j}(u_1, X_k)) \frac{\partial v}{\partial x_i}(u_1, X_k) \\
& -(\frac{\partial}{\partial u_1} \frac{\partial \beta_r}{\partial x_i}(u_1, X_k)) \frac{\partial v}{\partial x_j}(u_1, X_k) - \frac{\partial \beta_r}{\partial u_1}(u_1, X_k) \frac{\partial^2 v}{\partial x_i \partial x_j}(u_1, X_k) \\
& -\frac{\partial}{\partial u_1} \frac{\partial^2 \gamma_r}{\partial x_i \partial x_j}(u_1, X_k) + \frac{\partial^2 \beta_r}{\partial x_i \partial x_j}(u_1, X_k)(\mathcal{L}v+f)(u_1, X_k) + \frac{\partial \beta_r}{\partial x_j}(u_1, X_k) \\
& \times \frac{\partial(\mathcal{L}v+f)}{\partial x_i}(u_1, X_k) + \frac{\partial \beta_r}{\partial x_i}(u_1, X_k) \frac{\partial(\mathcal{L}v+f)}{\partial x_j}(u_1, X_k) + \beta_r(u_1, X_k) \\
& \times \frac{\partial^2(\mathcal{L}v+f)}{\partial x_i \partial x_j}(u_1, X_k))] du_1 * dw^r(u) + \int_s^{t_{k+1}} [\sum_{l=1}^{q} \int_u^{t_{k+1}} (\frac{\partial^2 \beta_r}{\partial x_i \partial x_j}(u_1, X_k) \\
& \times (\beta_l v+\gamma_l)(u_1, X_k) + \frac{\partial \beta_r}{\partial x_j}(u_1, X_k) \frac{\partial(\beta_l v+\gamma_l)}{\partial x_i}(u_1, X_k) + \frac{\partial \beta_r}{\partial x_i}(u_1, X_k) \\
& \times \frac{\partial(\beta_l v+\gamma_l)}{\partial x_j}(u_1, X_k) + \beta_r(u_1, X_k) \frac{\partial^2(\beta_l v+\gamma_l)}{\partial x_i \partial x_j}(u_1, X_k)] * dw^l(u_1) * dw^r(u).
\end{aligned}
$$

Utilizing Assumption 2.3**,** Lemma 17, Lemma 18, the smoothness of the coefficients on (3.1)-(3.2) in $\bar{Q}$, the fact that (X_k, Y_k), $v(t_{k+1}, x)$ are independent and Jensen, Hölder and Burkhölder-Davis-Gundy inequalities, we obtain for the first integral in the right-hand side of (4.21)

$$
\begin{aligned}
& E^w[Y_k\{\int_{t_k}^{t_{k+1}} [\frac{1}{2} \sum_{i,j=1}^{d} a^{ij}(s, X_k)(\sum_{r=1}^{q} \int_s^{t_{k+1}} \frac{\partial^2(\beta_r v+\gamma_r)}{\partial x_i \partial x_j}(u, X_k) * dw^r(u))] ds\} \qquad (4.23)\\
= \quad & E^w[Y_k \int_{t_k}^{t_{k+1}} (\frac{1}{2} \sum_{i,j=1}^{d} a^{ij}(s, X_k)(\sum_{r=1}^{q} \int_s^{t_{k+1}} \frac{\partial^2(\beta_r v+\gamma_r)}{\partial x_i \partial x_j}(t_{k+1}, X_k) * dw^r(u))) ds] \\
& + O(h^2),
\end{aligned}
$$

where $O(h^2)$ is a random variable such that $E|O(h^2)|^{2p} \le Kh^{4p}$.

Substituting (4.23) and (4.22) in (4.21) together with the use of iterated backward Itô formula and arguments similar to those which we used in deriving (4.23) we get

$$
\begin{aligned}
& E^w[d_k^2] \qquad (4.24) \\
= \; & E^w[Y_k\{-\frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial^2 v}{\partial x_i \partial x_j}(t_{k+1},X_k)\int_{t_k}^{t_{k+1}} a^{ij}(s,X_k)ds - \nabla v(t_{k+1},X_k) \\
& \times \int_{t_k}^{t_{k+1}} b^T(s,X_k)ds - v(t_{k+1},X_k)\int_{t_k}^{t_{k+1}} c(s,X_k)ds - \int_{t_k}^{t_{k+1}} f(s,X_k)ds \\
& -\frac{1}{2}\sum_{i,j=1}^{d}\int_{t_k}^{t_{k+1}} a^{ij}(s,X_k)\sum_{r=1}^{q}\int_{s}^{t_{k+1}}\frac{\partial^2(\beta_r v+\gamma_r)}{\partial x_i \partial x_j}(t_{k+1},X_k) * dw^r(u)ds \\
& -\int_{t_k}^{t_{k+1}} b^T(s,X_k)\sum_{r=1}^{q}\int_{s}^{t_{k+1}}\nabla(\beta_r v+\gamma_r)(t_{k+1},X_k) * dw^r(u)ds \\
& -\int_{t_k}^{t_{k+1}} c(s,X_k)\sum_{r=1}^{q}\int_{s}^{t_{k+1}}(\beta_r v+\gamma_r)(t_{k+1},X_k) * dw^r(u)ds \\
& -\sum_{r=1}^{q}\int_{t_k}^{t_{k+1}}(\beta_r v+\gamma_r)(t_{k+1},X_k) * dw^r(s) + \sum_{r=1}^{q}\int_{t_k}^{t_{k+1}}\int_{s}^{t_{k+1}}(\frac{\partial \beta_r}{\partial u}v \\
& +\frac{\partial \gamma_r}{\partial u})(t_{k+1},X_k)du * dw^r(s) - \sum_{r=1}^{q}\int_{t_k}^{t_{k+1}}\int_{s}^{t_{k+1}}\frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(t_{k+1},X_k) \\
& \times(\beta_r\frac{\partial^2 v}{\partial x_i \partial x_j})(t_{k+1},X_k)du * dw^r(s) - \sum_{r=1}^{q}\int_{t_k}^{t_{k+1}}\int_{s}^{t_{k+1}}(b^T\beta_r\nabla v)(t_{k+1},X_k) \\
& du * dw^r(s) - \sum_{r=1}^{q}\int_{t_k}^{t_{k+1}}\int_{s}^{t_{k+1}}(\beta_r(cv+f))(t_{k+1},X_k)du * dw^r(s) \\
& -\sum_{r=1}^{q}\int_{t_k}^{t_{k+1}}\sum_{l=1}^{q}\int_{s}^{t_{k+1}}(\beta_r(\beta_l v+\gamma_l))(t_{k+1},X_k) * dw^l(u) * dw^r(s) \\
& -\sum_{r=1}^{q}\int_{t_k}^{t_{k+1}}\sum_{l=1}^{q}\int_{s}^{t_{k+1}}\sum_{n=1}^{q}\int_{u}^{t_{k+1}}(\beta_r\beta_l(\beta_n v+\gamma_n))(t_{k+1},X_k) * dw^n(z) \\
& * dw^l(u) * dw^r(s) + f(t_k,X_k)h + \sum_{r=1}^{q}\gamma_r(t_k,X_k)(w^r(t_{k+1})-w^r(t_k)) \\
& +\sum_{i,j=1}^{q}(\beta_i\gamma_j)(t_k,X_k)\int_{t_k}^{t_{k+1}}(w^i(s)-w^i(t_k))dw^j(s)\}] + O(h^2),
\end{aligned}
$$

where $O(h^2)$ is a random variable such that $E|O(h^2)|^{2p} \le Kh^{4p}$.

Combining (4.20) and (4.24), we arrive at

$$
\begin{aligned}
& E^w[d'_k] \qquad\qquad (4.25) \\
= \;& E^w[Y_k\{\sum_{r=1}^{q}[\frac{\partial \gamma_r}{\partial t}(t_k, X_k) - c(t_k, X_k)(\beta_r v + \gamma_r)(t_{k+1}, X_k) - b^T(t_k, X_k) \\
& \times \nabla(\beta_r v + \gamma_r)(t_{k+1}, X_k) - \frac{1}{2}\sum_{i,j=1}^{d} a^{i,j}(t_k, X_k)\frac{\partial^2(\beta_r v + \gamma_r)}{\partial x_i \partial x_j}(t_{k+1}, X_k) \\
& +\nabla v(t_{k+1}, X_k)(b^T\beta_r)(t_k, X_k) + v(t_{k+1}, X_k)(c\beta_r)(t_k, X_k) - \frac{1}{2}v(t_{k+1}, X_k) \\
& \times \beta_r(t_k, X_k)\sum_{r=1}^{q}\beta_r(t_k, X_k) + \frac{1}{2}\sum_{i,j=1}^{d} a^{i,j}(t_k, X_k)\frac{\partial^2 v}{\partial x_i \partial x_j}(t_{k+1}, X_k)\beta_r(t_k, X_k) \\
& -v(t_{k+1}, X_k)\frac{\partial \beta_r}{\partial t}(t_k, X_k) + \frac{1}{2}v(t_{k+1}, X_k)\beta_r(t_{k+1}, X_k)\sum_{r=1}^{q}\beta_r^2(t_k, X_k)] \\
& \times h(w^r(t_{k+1}) - w^r(t_k)) + \sum_{r=1}^{q}[\frac{1}{2}\sum_{i,j=1}^{d} a^{i,j}(t_k, X_k)\frac{\partial^2(\beta_r v + \gamma_r)}{\partial x_i \partial x_j}(t_{k+1}, X_k) \\
& +b^T(t_k, X_k)\nabla(\beta_r v + \gamma_r)(t_{k+1}, X_k) + c(t_k, X_k)(\beta_r v + \gamma_r)(t_{k+1}, X_k) + \\
& +(\frac{\partial \beta_r}{\partial t}v + \frac{\partial \gamma_r}{\partial t})(t_{k+1}, X_k) - \frac{1}{2}\sum_{i,j=1}^{d}(a^{ij}\beta_r\frac{\partial^2 v}{\partial x_i \partial x_j})(t_{k+1}, X_k) \\
& -(b^T\beta_r\nabla v)(t_{k+1}, X_k) - (\beta_r(cv + f))(t_{k+1}, X_k)]\int_{t_k}^{t_{k+1}}(w^r(s) - w^r(t_k))ds \\
- & \sum_{r=1}^{q}\sum_{l=1}^{q}\sum_{n=1}^{q}(\beta_r\beta_l\gamma_n)(t_{k+1}, X_k)\int_{t_k}^{t_{k+1}}\int_{s}^{t_{k+1}}\int_{u}^{t_{k+1}} * dw^n(z) * dw^l(u) * dw^r(s)\}] + O(h^2),
\end{aligned}
$$

where $O(h^2)$ is a random variable such that $E|O(h^2)|^{2p} \leq Kh^{4p}$.

Now Jensen's inequality, the smoothness of the coefficients on (3.1)-(3.2) in $\bar{Q}$, and relation (4.25) imply relation (4.17) with the corresponding F_r^1, F_r^2 and F_{rnl}. ■

We are now ready to state and prove the main result concerning the mean-square order of convergence for the numerical schemes (4.1)-(4.3)-(4.4) and (4.1)-(4.7)-(4.4).

Theorem 20 *Any of the methods (4.1)-(4.3)-(4.4) or (4.1)-(4.7)-(4.4) satisfies the* *inequality for* $1 \le p < \infty$

$$(E|E^w[\varphi(t_\varkappa, X_\varkappa)Y_\varkappa + Z_\varkappa] - v(t_0, x_0)|^{2p})^{1/2p} \le Kh, \tag{4.26}$$

where K *does not depend on* t_0, x_0 *and the discretization step* h.

Proof. Using the standard technique (see [92, 93]) of re-writing the global error as a sum of local errors, we obtain

$$E^w[\varphi(t_\varkappa, X_\varkappa)Y_\varkappa] - v(t_0, x_0)$$
$$= E^w[v(t_0, X_0)Y_0 + Z_0] - v(t_0, x_0) + \sum_{k=0}^{N-1} E^w[v(t_{k+1}, X_{k+1})Y_{k+1} + Z_{k+1}$$
$$- v(t_k, X_k)Y_k - Z_k]$$
$$= \sum_{k=0}^{N-1} E^w[d_k] + \sum_{k=0}^{N-1} E^w[d_k'].$$

Now using Jensen's inequality and Lemma 19 we obtain

$$E^w[\varphi(t_\varkappa, X_\varkappa)Y_\varkappa + Z_\varkappa] - v(t_0, x_0)$$
$$= \sum_{k=0}^{N-1}\sum_{r=1}^{q}(E^w[Y_k F_r^1(t_{k+1}, X_k)]hI_{rk} + E^w[Y_k F_r^2(t_{k+1}, X_k)]I_{0rk}) +$$
$$+ \sum_{k=0}^{N-1}\sum_{r=1}^{q}\sum_{n=1}^{q}\sum_{l=1}^{q} E^w[Y_k F_{rnl}(t_{k+1}, X_k)]I_{rnlk} + O(h),$$

where $O(h)$ is a random variable such that

$$E|O(h)|^{2p} \le Kh^{2p},$$

with K being independent of t_0, x_0, h.

To complete the proof we use in an analogous way the same ideas which were utilized in STEPs 2-5 from the proof of Theorem 3.3 presented in [94].

Let

$$\psi_{kr}^1 = E^w[Y_k F_r^1(t_{k+1}, X_k)],$$

$$\psi_{kr}^2 = E^w[Y_k F_r^2(t_{k+1}, X_k)],$$

$$\psi_{kml} = E^w[Y_k F_{rnl}(t_{k+1}, X_k)].$$

Introduce the family of $\sigma-$subalgebras $\mathcal{F}_t^w$, $T_0 \leq t \leq T$, induced by the Wiener process w. Then we have

$$\begin{aligned} & E^w[\varphi(t_\varkappa, X_\varkappa)Y_\varkappa + Z_\varkappa] - v(t_0, x_0) \\ = & \sum_{k=0}^{N-1}\sum_{r=1}^{q}\psi_{kr}^1 h I_{rk} + \sum_{k=0}^{N-1}\sum_{r=1}^{q}\psi_{kr}^2 I_{0rk} + \\ & + \sum_{k=0}^{N-1}\sum_{r=1}^{q}\sum_{n=1}^{q}\sum_{l=1}^{q}\psi_{kml} I_{rnlk} + O(h), \end{aligned}$$

where ψ_{kr}^1, ψ_{kr}^2 and ψ_{kml} are $\mathcal{F}_T^w-$measurable random variables which have finite moments of sufficiently high order bounded by a constant independent of h and $O(h)$ is a random variable such that

$$E|O(h)|^{2p} \leq K h^{2p},$$

with K being independent of t_0, x_0, h.

Consider the martingales

$$\psi_{kr}^1(t) = E(\psi_{kr}^1|\mathcal{F}_t^w),\ \psi_{kr}^2(t) = E(\psi_{kr}^2|\mathcal{F}_t^w),$$

$$\psi_{kml}(t) = E(\psi_{kr}^2|\mathcal{F}_t^w),\ 0 \leq t \leq T,\ k = 0, ...,\ N-1.$$

From the martingale representation theorem it follows that they can be written in the form

$$\psi^1_{kr}(t) = E\psi^1_{kr} + \int\limits_0^t Z^T_{kr}(s)dw(s), \ a.s.$$

$$\psi^2_{kr}(t) = E\psi^1_{kr} + \int\limits_0^t U^T_{kr}(s)dw(s), \ a.s.$$

$$\psi_{kml}(t) = E\psi_{kml} + \int\limits_0^t V^T_{krnl}(s)dw(s), \ a.s.,$$

where $Z_{kr}(s)$, $U_{kr}(s)$ and $V_{krl}(s)$ are square-integrable $\mathcal{F}^w_s-$adapted processes. As a result, we have

$$\begin{aligned}
&\sum_{k=0}^{N-1}\sum_{r=1}^{q}\psi^1_{kr}hI_{rk} + \sum_{k=0}^{N-1}\sum_{r=1}^{q}\psi^2_{kr}I_{0rk} + \sum_{k=0}^{N-1}\sum_{r=1}^{q}\sum_{n=1}^{q}\sum_{l=1}^{q}\psi_{kml}I_{rnlk} \\
= \ & S^1 + S^2 + S^3,
\end{aligned}$$

where

$$\begin{aligned}
S^1 &= \sum_{r=1}^{q}\sum_{k=0}^{N-1}\{\psi^2_{kr}(t_k)I_{0rk} + I_{0rk}\int\limits_{t_k}^{t_{k+1}} U^T_{kr}(s)dw(s) + I_{0rk}\int\limits_{t_{k+1}}^{T} U^T_{kr}(s)dw(s)\}, \\
S^2 &= \sum_{r=1}^{q}\sum_{k=0}^{N-1}\{\psi^1_{kr}(t_k)hI_{rk} + hI_{rk}\int\limits_{t_k}^{t_{k+1}} Z^T_{kr}(s)dw(s) + hI_{rk}\int\limits_{t_{k+1}}^{T} Z^T_{kr}(s)dw(s)\}, \\
S^3 &= \sum_{r=1}^{q}\sum_{k=0}^{N-1}\{\sum_{n=1}^{q}\sum_{l=1}^{q}\psi_{kml}(t_k)I_{rnlk} + I_{rnlk}\int\limits_{t_k}^{t_{k+1}} V^T_{kr}(s)dw(s) + I_{rnlk}\int\limits_{t_{k+1}}^{T} V^T_{kr}(s)dw(s)\}.
\end{aligned}$$

Let us estimate the term S^1. The terms S^2 and S^3 can be treated in an analogous way. In the next sequel we will proceed as the authors did in [94].

Introduce

$$S_{r,1} = 0,\ S_{r,k} = \sum_{i=0}^{k-1} \psi_{ir}^{1}(t_i) I_{0ri},$$
$$P_{r,1} = 0,\ P_{r,k} = \sum_{i=0}^{k-1} I_{0ri} \int_{t_k}^{t_{k+1}} Z_{ir}^{T}(s)dw(s),$$
$$Q_{r,1} = 0,\ Q_{r,k} = \sum_{i=0}^{k-1} \sum_{l=0}^{i-1} I_{0rl} J_{ril},$$

where

$$J_{ril} = \int_{ti}^{t_{i+1}} Z_{lr}^{T}(s)dw(s)$$

and $k = 2, ..., N$, $r = 1, ..., q$. Then we have

$$S^1 = \sum_{r=1}^{q} (S_{r,N} + P_{r,N} + Q_{r,N}). \tag{4.27}$$

Utilizing the properties of the Itô integral, the boundedness of moments of ψ_{kr}^2 and the discrete version of Gronwall lemma (see, e.g. [93, Lemma 1.1.6, p. 7]), they proved that

$$E|S_{r,N}|^{2p} \leq Kh^{2p}. \tag{4.28}$$

Since we also have the following estimate obtained in [94, p. 80]

$$E|I_{0ri} \int_{t_k}^{t_{k+1}} Z_{ir}^{T}(s)dw(s)|^{2p} \leq Kh^{2p},$$

we obtain

$$E|P_{r,N}|^{2p} \leq Kh^{2p}. \tag{4.29}$$

In order to get an estimation for the third term in (4.27) the authors firstly showed the following estimate (see [94, Lemma 3.3, p. 80]).

We have, for $p \geq 1$,

$$E|\sum_{l=0}^{k-1} I_{0rl}J_{ril}|^{2p} \leq Kh^{3p}, \tag{4.30}$$

where K is independent of h.

Again using the properties of the Itô integral, the boundedness of moments of ψ^2_{kr}, the discrete version of Gronwall lemma (see, e.g. [93, Lemma 1.1.6, p. 7]) and the estimate (4.30), they proved that

$$E|Q_{r,N}|^{2p} \leq Kh^{2p}. \tag{4.31}$$

Thus, it results from (4.28), (4.29) and (4.31) that

$$E|S^1|^{2p} \leq Kh^{2p}. \tag{4.32}$$

Utilizing arguments similar to those used in deriving (4.32), we also obtain

$$E|S^2|^{2p} \leq Kh^{2p}, E|S^1|^{2p} \leq Kh^{2p},$$

and, therefore, $E|E^w[\varphi(t_\varkappa, X_\varkappa)Y_\varkappa + Z_\varkappa] - v(t_0, x_0)|^{2p} \leq kh^{2p}$. Theorem 20 is proved.

■

Theorem 20 together with the Markov inequality and Borel-Cantelli lemma (see also [94, 95]) implies the a.s. convergence as stated in the next theorem.

Theorem 21 *For almost every trajectory $w(\cdot)$, $h = (T - t_0)/N$ any $\varepsilon > 0$ there exists $C(\omega) > 0$ such that*

$$|\bar{v}(t,x) - v(t,x)| \leq C(\omega)h^{1-\varepsilon}, \tag{4.33}$$

where v is the solution to the problem (3.1)-(3.2), $\bar{v}$ is the approximation of v given by (4.5) with $(t_\varkappa,\ X_\varkappa,\ Y_\varkappa,\ Z_\varkappa)$ either due to Algorithm 1A or Algorithm 1B; and C does not depend on the discretization step h; i.e., both Algorithm 1A and 1B converge with order $1-\varepsilon$ a.s. .

Proof. Now denote $R := |\bar{v}(t,x) - v(t,x)|$. The Markov inequality together with (4.26) implies

$$P(R > h^\gamma) \leq \frac{ER^{2p}}{h^{2p\gamma}} \leq Kh^{2p(1-\gamma)}.$$

Then for any $\gamma = 1-\varepsilon$ there is a sufficiently large $p \geq 1$ such that (recall that $h = (T-t_0)/N$)

$$\sum_{N=1}^{\infty} P\left(R > \frac{(T-t_0)^\gamma}{N^\gamma}\right) \leq K(T-t_0)^{2p(1-\gamma)} \sum_{N=1}^{\infty} \frac{1}{N^{2p(1-\gamma)}} < \infty.$$

Hence, due to the Borel-Cantelli lemma, the random variable $\varsigma := \sup\limits_{h>0} h^{-\gamma}R$ is a.s. finite which implies (4.33). ■

4.3 Method of order 1/2

The next algorithm (Algorithm 2) is obtained by a simplification of Algorithm 1A. In Algorithm 2 as soon as X_k gets into the boundary domain $S_{t_k,h}$, the random walk terminates, i.e., $\varkappa = k$, and $\bar{X}_\varkappa = X_k^\pi$, $Y_\varkappa = Y_k$, $Z_\varkappa = Z_k$ is taken as the final state of the Markov chain. Since such algorithm obviously cannot be of order higher than 1/2 (it is already of order 1/2 in the case of deterministic PDEs [92, 93]), we also simplify the approximations (4.3)-(4.4). Namely, instead of (4.3)-(4.4) we use the approximations

$$Y_{t,x,y}(s+h) \approx \bar{Y} = y + hc(s,x)\,y + \beta^{\top}(s,x)y\Delta w(s), \tag{4.34}$$

$$Z_{t,x,y,z}(t+h) \approx \bar{Z} = z + hf(s,x)\,y + \gamma^{\top}(s,x)y\Delta w(s). \tag{4.35}$$

Let us write this algorithm formally.

Algorithm 2

STEP 0. $X_0 = x_0,\ Y_0 = 1,\ Z_0 = 0,\ k = 0.$

STEP 1. If $X_k \notin S_{t_k,h}$ then go to STEP 2. If $X_k \in S_{t_k,h}$ then STOP and $\varkappa = k$, $\bar{X}_\varkappa = X_k^\pi$, $Y_\varkappa = Y_k$, $Z_\varkappa = Z_k$.

STEP 2. Simulate ξ_k and find $X_{k+1},\ Y_{k+1},\ Z_{k+1}$ according to (4.1), (4.34)-(4.35) for $s = t_k,\ x = X_k,\ y = Y_k,\ z = Z_k,\ \xi = \xi_k$, $\Delta w(s) = \Delta w(t_k)$.

STEP 3. If $k+1 = N$, STOP and $\varkappa = N$, $\bar{X}_\varkappa = X_N$, $Y_\varkappa = Y_N$, $Z_\varkappa = Z_N$, otherwise $k := k+1$ and return to STEP 1.

We form the approximation of the solution to the SPDE problem (3.1)-(3.2) as (cf. (4.5)):

$$v(t_0,x_0) \approx \tilde{v}(t_0,x_0) = E^w\left[\varphi(t_\varkappa, X_\varkappa)Y_\varkappa + Z_\varkappa\right] \tag{4.36}$$

with $(t_\varkappa, X_\varkappa, Y_\varkappa, Z_\varkappa)$ obtained by Algorithm 2.

We extend the definition of the constructed chain for all k by the rule: if $k > \varkappa$, then $(t_k, X_k, Y_k, Z_k) = (t_\varkappa, X_\varkappa, Y_\varkappa, Z_\varkappa)$.

As in the case of methods of order one, the following lemma is used to prove the convergence theorem, Theorem 22 below.

Lemma 22 *For any $1 \leq p < \infty$ and the weak Euler approximation scheme which is given by Algorithm 2 the following relation hold*

$$\begin{aligned} &\sum_{k=0}^{N-1} E^w[v(t_{k+1}, X_{k+1})Y_{k+1} + Z_{k+1} - v(t_k, X_k)Y_k - Z_k] \\ = &\sum_{k=0}^{N-1} \sum_{r=1}^{q} \sum_{l=1}^{q} E^w[Y_k F_r(t_{k+1}, X_k)] I_{rlk} + O(h^{1/2}), \end{aligned} \tag{4.37}$$

where

$$I_{rlk} = \sum_{r=1}^{q} \sum_{l=1}^{q} \int_{t_k}^{t_{k+1}} \int_{s}^{t_{k+1}} * \, dw^l(u) * dw^r(s),$$

$F_r(t_{k+1}, x)$ are combinations of the coefficients of the SPDE (3.1)-(3.2) and the SPDE solution $v(t_{k+1}, x)$, and $O(h^{1/2})$ is a random variable such that

$$E|O(h^{1/2})|^{2p} \leq Kh^p$$

with K being independent of t_0, x_0, h.

Proof. Taking into account the fact that the last non-zero term in the sum

$$\sum_{k=0}^{N-1} E^w[v(t_{k+1}, X_{k+1})Y_{k+1} + Z_{k+1} - v(t_k, X_k)Y_k - Z_k]$$

is a random variable $O_1(h^{1/2})$ such that

$$E|O_1(h^{1/2})|^{2p} \leq Kh^p,$$

it only remains to evaluate the remaining terms in the above sum.

As in Lemma 19, let

$$\begin{aligned} d_k^1 &= v(t_{k+1}, X_{k+1})Y_{k+1} - v(t_{k+1}, X_k)Y_k, \\ d_k^2 &= (v(t_{k+1}, X_k) - v(t_k, X_k))Y_k + Z_{k+1} - Z_k. \end{aligned}$$

We have

$$\begin{aligned} & E^w[v(t_{k+1}, X_{k+1})Y_{k+1} + Z_{k+1} - v(t_k, X_k)Y_k - Z_k] \\ = \;& E^w[d_k^1 + d_k^2]. \end{aligned}$$

Applying Taylor formula for d_k^1 and expanding around (X_k, Y_k) up to the fourth derivatives and using Assumption 2.3**,** Lemma 17, the smoothness of the coefficients of (3.1)-(3.2) in $\bar{Q}$, the fact that (X_k, Y_k), $v(t_{k+1}, x)$ are independent and the Jensen and Hölder inequalities, we get

$$\begin{aligned} & E^w[d_k^1] \qquad (4.38) \\ = \;& E^w[Y_k\{hb^T(t_k, X_k)\nabla v(t_{k+1}, X_k) + v(t_{k+1}, X_k)[hc(t_k, X_k) \\ & + \sum_{r=1}^q \beta_r(t_k, X_k)(w_r(t_{k+1}) - w_r(t_k))] + \frac{h}{2}\sum_{i,j=1}^d \frac{\partial^2 v}{\partial x_i \partial x_j}(t_{k+1}, X_k)a^{ij}(t_k, X_k) \\ & + hb^T(t_k, X_k)\nabla v(t_{k+1}, X_k)\sum_{r=1}^q \beta_r(t_k, X_k)(w_r(t_{k+1}) - w_r(t_k)) \\ & + \frac{h}{2}\sum_{i,j=1}^d \frac{\partial^2 v}{\partial x_i \partial x_j}(t_{k+1}, X_k)a^{ij}(t_k, X_k)\sum_{r=1}^q \beta_r(t_k, X_k)(w_r(t_{k+1}) - w_r(t_k))\}] \\ & + O(h^{3/2}), \end{aligned}$$

where $O(h^{3/2})$ is a random variable such that $E|O(h^{3/2})|^{2p} \leq Kh^{3p}$.

Using relation (4.38) together with the use of iterated backward Itô formula and arguments similar to those which we used in deriving (4.23) we get

$$\begin{aligned} & E^w[d_k^1 + d_k^2] \qquad (4.39) \\ = \; & E^w[-Y_k(\sum_{r=1}^{q} \int_{t_k}^{t_{k+1}} \sum_{l=1}^{q} \int_{s}^{t_{k+1}} (\beta_r(\beta_l v + \gamma_l))(t_{k+1}, X_k) * dw^l(u) * dw^r(s))] \\ & + O(h^{3/2}), \end{aligned}$$

where $O(h^{3/2})$ is a random variable such that

$$E|O(h^{3/2})|^{2p} \leq Kh^{3p}.$$

Now Jensen's inequality, the smoothness of the coefficients on (3.1)-(3.2) in $\bar{Q}$, and relation (4.39) imply relation (4.37) with the corresponding F_r. ■

Utilizing Lemma 22, the following convergence theorem can be proved analogously to Theorem 20 and Theorem 21.

Theorem 23 *Algorithm 2 satisfies the inequality for* $1 \leq p < \infty$:

$$\left(E\left|\tilde{v}(t,x) - v(t,x)\right|^{2p}\right)^{1/2p} \leq Kh^{1/2}, \qquad (4.40)$$

where $K > 0$ *does not depend on the discretization step* h*, i.e., in particular, Algorithm 2 is of mean-square order* $1/2$.

For almost every trajectory $w(\cdot)$ *and any* $\varepsilon > 0$ *there exists* $C(\omega) > 0$ *such that*

$$|\tilde{v}(t,x) - v(t,x)| \leq C(\omega)h^{1/2-\varepsilon}, \qquad (4.41)$$

where v *is the solution to the problem (3.1)-(3.2),* $\tilde{v}$ *is the approximation of* v *which is given by (4.36) and* C *does not depend on the discretization step* h*, i.e., Algorithm 2 converges with order* $1/2 - \varepsilon$ *a.s. .*

Chapter 5

Numerical experiments

For our numerical tests, we take the model which solution can be simulated exactly. We consider the following problem

$$-dv = \frac{\sigma^2}{2}\frac{\partial^2 v}{\partial x^2}dt + \beta v * dw(t), \quad (t,x) \in [T_0, T) \times (-1,1), \tag{5.1}$$

$$v(t, \pm 1) = 0, \ (t,x) \in [T_0, T), \tag{5.2}$$

$$v(T,x) = \varphi(x), \quad x \in (-1,1), \tag{5.3}$$

where $w(t)$ is a standard scalar Wiener process, and σ and β are constants. The solution of this problem is given by (see (3.16)-(3.19)):

$$v(t,x) = E^w\left[\mathbf{1}_{\{\tau \geq T\}}\varphi(X_{t,x}(T))Y_{t,x,1}(T)\right], \tag{5.4}$$

$$dX = \sigma dW(s), \tag{5.5}$$

$$dY = \beta Y dw(s), \tag{5.6}$$

where $W(s)$ is a standard Wiener process independent of $w(s)$ and $\tau = \tau_{t,x}$ is the first exit time of the trajectory $X_{t,x}(s)$, $s \geq t$, from the interval $(-1, 1)$.

For the experiments, we choose

$$\varphi(x) = A \cdot (x^2 - 1)^3 \tag{5.7}$$

with some constant A. We note that the coefficients of (5.1)-(5.3), (5.7) satisfy the assumptions made in Section 3.1.

The problem (5.1)-(5.3) with (5.7) has the explicit solution, which can be written in the form

$$\begin{aligned} v(t,x) = {} & \frac{18\,432}{\pi^7} A \exp\left(-\frac{\beta^2}{2}(T-t) + \beta(w(T) - w(t))\right) \\ & \times \sum_{k=0}^{\infty} \frac{(-1)^k (\pi^2 (2k+1)^2 - 10)}{(2k+1)^7} \cos\frac{\pi(2k+1)x}{2} \exp\left(-\frac{\sigma^2\pi^2(2k+1)^2(T-t)}{8}\right). \end{aligned} \tag{5.8}$$

We use the three algorithms from Section 4.1 and Section 4.3 to solve (5.1)-(5.3), (5.7). In the tests we fix a trajectory $w(t)$, $0 \leq t \leq T$, which is obtained with a small time step equal to 0.0001. In Table 5.1 we present the errors for Algorithms 1A and 1B from our tests. In the table the "$\pm$" reflects the Monte Carlo error only, it gives the confidence interval for the corresponding value with probability 0.95 while the values before "$\pm$" gives the difference $\hat{v}(0,0) - v(0,0)$ (see $\hat{v}$ in (4.6)). In Table 5.1 one can observe convergence of both algorithms with order one that is in good agreement with our theoretical results. We note that in this example the trajectory of $Y(t)$ is simulated exactly by Algorithm 1B which results in Algorithm 1B being more accurate than Algorithm 1A.

Table 5.1: *Errors of Algorithms 1A and 1B.* Evaluation of $v(0,0)$ from (5.1)-(5.3), (5.7) with various time steps h. Here $\sigma = 0.5$, $\beta = 0.5$, $A = 10$, and $T = 5$. The expectations are computed by the Monte Carlo technique simulating M independent realizations. The "$\pm$"reflects the Monte Carlo error only. All simulations are done along the same sample path $w(t)$. The corresponding reference value is -2.074421, which is found due to (5.8).

h	M	Algorithm 1A	Algorithm 1B
0.1	10^7	$4.823 \cdot 10^{-2} \pm 0.247 \cdot 10^{-2}$	$3.604 \cdot 10^{-2} \pm 0.249 \cdot 10^{-2}$
0.05	10^8	$2.591 \cdot 10^{-2} \pm 0.079 \cdot 10^{-2}$	$2.219 \cdot 10^{-2} \pm 0.079 \cdot 10^{-2}$
0.025	10^8	$1.616 \cdot 10^{-2} \pm 0.079 \cdot 10^{-2}$	$0.815 \cdot 10^{-2} \pm 0.079 \cdot 10^{-2}$
0.01	$2.5 \cdot 10^8$	$0.545 \cdot 10^{-2} \pm 0.050 \cdot 10^{-2}$	$0.338 \cdot 10^{-2} \pm 0.050 \cdot 10^{-2}$
0.005	$2.5 \cdot 10^8$	$0.210 \cdot 10^{-2} \pm 0.050 \cdot 10^{-2}$	$0.135 \cdot 10^{-2} \pm 0.050 \cdot 10^{-2}$

Algorithm 2 produced less accurate results than Algorithms 1 and it was possible to observe its convergence only after choosing very small time steps. The results are presented in Table 5.2 which demonstrate convergence with order 1/2 as it was expected due to our theoretical results.

Table 5.2: *Errors of Algorithm 2.* Evaluation of $v(0,0)$ from (5.1)-(5.3), (5.7) with various time steps h. The parameters here are the same as in Table 5.1.

h	M	Algorithm 2
0.0005	10^7	$0.518 \cdot 10^{-1} \pm 0.025 \cdot 10^{-1}$
0.0002	10^7	$0.404 \cdot 10^{-1} \pm 0.025 \cdot 10^{-1}$
0.0001	10^7	$0.306 \cdot 10^{-1} \pm 0.025 \cdot 10^{-1}$

Part II

Existence and uniqueness of solutions to anticipating heat SPDEs

Introduction

In this part of the thesis we study a certain class of anticipating linear heat equations driven by the space-time white noise and the white noise. We are interested in the case when the drift lies in the first Wiener chaos. Such problems can also be approached when the drift lies in other Wiener chaos with order different than one. If we can manage to obtain the explicit Wiener chaos decomposition of the solution in these different cases, then we can get an extended result concerning the existence and uniqueness of solutions to more general classes of anticipating heat SPDEs when the drift is a random variable with finite moment of second order.

This second part of thesis is organized as follows.

In the first chapter (Chapter 6) we briefly present a variety of theoretical studies about existence, uniqueness and regularity of solutions to different classes of parabolic SPDEs. We are mainly interested in the framework of stochastic heat equation driven by white noise, space-time white noise and fractional noise. We are also interested in some classes of anticipating parabolic SPDEs driven by the standard Wiener process. At the end of Chapter 6 we introduce our studied problem and the results in this direction.

The second chapter (Chapter 7) has two sections. It contains some preliminary material that are needed in Chapter 8. In Section 7.1 we introduce main features of the stochastic calculus with variations or Malliavin calculus. Section 7.2 presents an important result about the equivalence of the notion of weak solution and the notion of mild solution for some classes of SPDEs in the nonanticipating case. This result will be adapted in Chapter 8 to the anticipating case.

Finally, the third chapter (Chapter 8) is devoted to the existence and uniqueness of the solutions to some classes of anticipating linear heat equations with Dirichlet boundary conditions. This chapter consists of two sections. In the first one the driving noise is the space time white noise and in the second one the potential is the space noise. We give the explicit chaos representation of the weak solutions to the considered SPDEs and we prove that they are analitical functionals with finite moments of all orders. Moreover, some continuity properties of them are obtained. The results are also presented in [115] and [114].

Chapter 6

A survey of some classes of SPDEs

We overview below some important results for SPDEs mostly close to our object which will be studied in Chapter 8.

Parabolic SPDEs with martingale measure drift whose solutions are in the distribution spaces were extensively treated by Walsh in [122]. Among various frameworks, he in particular considered the stochastic heat equation

$$\begin{aligned} \frac{\partial v}{\partial t}(t,x) &= \frac{\partial^2 u}{\partial x^2}(t,x) + g(t,v(t,x)) + f(v(t,x),t)\frac{\partial^2 W}{\partial t \partial x}(t,x),\ t>0, \quad (6.1) \\ x &\in (0,L), \\ \frac{\partial v}{\partial x}(t,0) &= \frac{\partial v}{\partial x}(t,L) = 0,\ t>0, \quad (6.2) \\ v(0,x) &= v_0(x),\ x\in(0,L), \quad (6.3) \end{aligned}$$

where $\frac{\partial^2 W}{\partial t \partial x}$ is the space-time white noise. It turns out that if g and f satisfy some Lipschitz conditions then problem (6.1)-(6.3) has a unique L^2-valued weak solution.

Gyöngy and Pardoux in [48] proved the existence and uniqueness of the strong solution to a class of SPDE, namely

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + f(t,x,u(t,x)) + \frac{\partial^2 W}{\partial t \partial x}(t,x), \ t \geq 0, \ x \in (0,1) \tag{6.4}$$

with the initial condition $u(0,x) = u_0(x)$, $u_0 \in C_0([0,1])$ and Neumann boundary conditions

$$\frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,1) = 0, \ t \geq 0,$$

where $\frac{\partial^2 W}{\partial t \partial x}$ denotes space-time white noise and f satisfies some measurability and growth conditions. They showed that their result is also true for the Dirichlet problem.

The result for the Dirichlet case was extended by Bally, Gyöngy and Pardoux in [5]. The authors included a diffusion coefficient g at the noise of (6.4) a diffusion coefficient g which has a locally Lipschitz derivative and satisfies a linear growth condition. Under the assumptions from above, the following first boundary problem

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \frac{\partial^2 u}{\partial x^2}(t,x) + f(t,x,u(t,x)) + g(t,x,u(t,x))\frac{\partial^2 W}{\partial t \partial x}(t,x), \ t \geq 0, && (6.5) \\ x &\in (0,1), && \\ u(0,x) &= u_0(x), \ x \in (0,1), && (6.6) \\ u(t,0) &= u(t,1) = 0, \ t \geq 0, && (6.7) \end{aligned}$$

has a unique strong solution. This result was improved by Gyöngy in [45] when he assumed that it is enough for g to be only locally Lipschitz.

For $t \in [0,T]$, the above problem was also treated by Alòs, León and Nualart in [2], when the Laplacian was replaced by a second-order differential operator A_t whose coefficients are adapted random fields. The diffusion coefficients f and g are assumed to be predictable processes satisfying global Lipschitz conditions. The authors obtained an existence and uniqueness result for the stochastic evolution equation

$$\begin{aligned} u(t,x) &= \int_0^1 \Gamma_{t,0}(x,y)u_0(y)dy + \int_0^t\int_0^1 \Gamma_{t,s}(x,y)f(s,y,u(s,y))dyds \\ &\quad + \int_0^t\int_0^1 \Gamma_{t,s}(x,y)g(s,y,u(s,y))d\frac{\partial^2 W}{\partial s \partial y}(s,y), \end{aligned} \tag{6.8}$$

where $\Gamma_{t,s}(x,y)$ denotes the Green kernel associated to A_t and the stochastic integral is interpreted in the Skorokhod sense. Moreover, they proved that the solution of the equation (6.8) is also a weak solution of the problem (6.5)-(6.7): for every (sufficiently regular) test function φ and for all $t \in [0,T]$ we have

$$\begin{aligned} & \int_0^1 u(t,x)\varphi(x)dx \\ = & \int_0^1 u_0(x)\varphi(x)dx + \int_0^1\int_0^t A_s^*\varphi(x)u(s,x)dsdx \\ & + \int_0^1\int_0^t f(s,x,u(s,x))\varphi(x)dsdx + \int_0^1\int_0^t g(s,x,u(s,x))\varphi(x)d\frac{\partial^2 W}{\partial t \partial x}(t,x), \end{aligned}$$

where A_t^* is the adjoint of A_t.

Alòs, Nualart and Viens in [3] considered the existence and uniqueness of the solution to the following class of stochastic evolution equations driven by a zero mean Gaussian random field

$$u(t,x) = \int_R p(0,t,y,x)u_0(y)dy + \int_R\int_0^t p(s,t,y,x)F(s,y,u(s,y))dW_{s,y}, \tag{6.9}$$

where $F : [0,T] \times R^2 \times \Omega \to R$ is an adapted random field satisfying a Lipschitz type condition, $p(s,t,y,x)$ is a stochastic semigroup measurable with respect to $\sigma\{W_{r,x} -$

$W_{s,x}$, $x \in R$, $r \in [s,t]\}$, and the stochastic integral is anticipating. Using an iteration procedure, the authors proved the existence of an adapted solution to equation (6.9).

The Picard iteration procedure was utilized by Nualart and Viens (see [102]) in order to show the existence of a continuous solution with values in $L^2(R^d)$ for the stochastic evolution equation

$$u(t,x) = \int_{R^d} p(0,t,y,x)u_0(y)dy + \int_{R^d}\int_0^t p(s,t,y,x)F(s,y,u(s,y))W(ds,y)dy, \qquad (6.10)$$

where $p(s,t,y,x)$ is a stochastic semigroup, $F : [0,T] \times R^d \times R \times \Omega \to R$ is a progressively measurable random function satisfying some Lipschitz type conditions, $W(t,y) = \int_0^t \int_S a(\lambda, y)M(ds, d\lambda)$, $(S, \mathcal{S}, \mu)$ is a finite measure space, λ is the Lebesgue measure on a time interval $[0,T]$, $M = \{M(A),\ A \in B([0,T] \times \mathcal{S})\}$ is a centered Gaussian family of random variables with covariation function given by

$$E(M(A)M(B)) = \lambda \times \mu(A \cap B),$$

and a is a deterministic measurable function. The stochastic integral which appears in (6.10) is the Skorokhod one. The authors of [102] also proved the uniqueness of the solution to equation (6.10).

In [99] Nualart and Pardoux considered the white-noise driven SPDE with Dirichlet boundary conditions:

$$
\begin{aligned}
\frac{\partial u}{\partial t}(t,x) &= \frac{\partial^2 u}{\partial x^2}(t,x) + f(t,x,u(t,x)) + \frac{\partial^2 W}{\partial t \partial x}(t,x) + \eta(t,x),\ t \geq 0, \\
x &\in (0,1), \\
u(0,x) &= u_0(x), x \in (0,1), \\
u(t,0) &= u(t,1) = 0, t \geq 0,
\end{aligned}
$$

where W is a Brownian sheet, f satisfies a Lipschitz-type condition and η is an adapted random measure on $R_+ \times (0,1)$ satisfying $\eta(\{t\} \times (0,1)) = 0$ and $\int_0^t \int_0^1 x(1-x)\eta(ds,dx) < \infty$, a.s.. They obtained the existence and uniqueness of a pair (u,η) in the weak formulation and $\int_{R_+ \times (0,1)} u d\eta = 0$, a.s.. Donati-Martin and Pardoux (see [29]) generalized later this result to the case where the diffusion coefficient from the white-noise drift is locally Lipschitz.

Nualart and Rozovskii in [98] considered the stochastic boundary problem

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \mathcal{L}u + u \cdot \dot{W}, && (6.11) \\
u(x,0) &= u_0(x), && (6.12)
\end{aligned}
$$

where $\mathcal{L}$ is a uniformly elliptic second-order differential operator and $\dot{W}$ is the white noise in R^d, $d > 1$. Taking into account the fact that this equation does not have any solution in the usual sense, the authors looked for the solution in a space of generalized random variables. As a result, they defined $L^2_Q(\Omega)$ as the completion of $L^2(\Omega)$ under the norm $||\cdot||^2_Q := \sum_\alpha (\rho^\alpha)^2 E[\cdot \xi_\alpha]$, where $\{\xi_\alpha\}_\alpha$ is a basis of Wick polynomials associated with $\dot{W}$, $\rho = (\rho_1, \rho_2, \ldots)$ denotes the eigenvalues of $Q^{1/2}$ and Q is a positive self-adjoint operator on $L^2(R^d)$ which satisfies some assumptions. In this framework, the authors proved assertion which states that problem (6.11)-(6.12) has a unique solution in $L^2_Q(\Omega)$. They also presented an explicit formula for its Hermite expansion.

Uemura in [120] investigated the stochastic heat equation on $R_+ \times R$, namely

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\Delta u + \dot{\omega}(x)u, \qquad (6.13) \\ u(0,x) &= u_0(x), \qquad (6.14) \end{aligned}$$

where $\dot{\omega}(x)$ is the white noise with respect to x and u_0 is square integrable. Utilizing properties of the Skorokhod integral and the heat kernel integral representation, he constructed a unique L^p-valued, $p \geq 1$, weak solution to the problem (6.13)-(6.14). If it is assumed that u_0 is continuously differentiable with bounded derivative then the solution u has a locally Hölder continuous modification of order α for $0 \leq \alpha < \frac{1}{2}$.

A theoretical study about the existence of the solution to the stochastic heat equation with fractional white noise potential was done by Hu in [59]. In this work, the author treated the SPDE

$$\frac{\partial u_t(x)}{\partial t} = \frac{1}{2}\Delta u_t(x) + w^H(x) \cdot u_t(x), \; x \in R^d, \; t > 0, \qquad (6.15)$$

where w^H is a time independent fractional white noise with Hurst parameter $H = (h_1, \ldots, h_d)$. As Uemura, using the Wiener chaos decomposition and properties of the Skorokhod integral, he proved that if the initial condition u_0 is deterministic and if

$h_1 + \ldots + h_d > d - 1$, $\frac{1}{2} < h_i < 1$ for $i = 1, 2, \ldots, d$ then equation (6.15) admits an L^2-valued mild solution. If the initial condition u_0 is deterministic, $h_0 + \ldots + h_d > d - 2/(2h_0 - 1)$, $\frac{1}{2} < h_i < 1$ for $i = 0, 2, \ldots, d$, and the driving noise in (6.15) is time dependent, then the existence result holds.

Consider the following class of quasilinear SPDE on $[0, T] \times [0, 1]$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + b(u) + \frac{\partial^2 B}{\partial t \partial x} \tag{6.16}$$

with Neuman boundary conditions and $u(0, x) = u_0(x)$, $x \in [0, 1]$. Here the driving noise B is centered Gaussian field which is a Brownian motion in the space variable and a fractional Brownian motion in time, with Hurst parameter $H \in (\frac{1}{4}, 1)$; u_0 is continuous, and b is a function defined on $[0, T] \times [0, 1] \times R$. Adapting the method from [48], Nualart and Ouknine in [100] showed that if $H > \frac{1}{2}$ and the function b satisfies some Hölder conditions or if $H \in (\frac{1}{4}, 1)$ and the function b satisfies some mild integration conditions then equation (6.16) can be solved.

Nualart and Hu in [60] considered a class of stochastic heat equations driven by fractional noise in terms of Wick product:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u \diamondsuit \frac{\partial^2 W^H}{\partial t \partial x}, \tag{6.17}$$

where the initial condition u_0 is a bounded continuous function, the symbol $\diamondsuit$ denotes the Wick product and W^H is a Gaussian noise which is a white noise in the space variable and a fractional Brownian motion with Hurst parameter $H \in (0, 1)$ in the time variable. Denote by p_t the Gaussian kernel associated with $\frac{1}{2}\Delta$. The authors looked for a solution to equation (6.17) in the integral form

$$u_{t,x} = p_t u_0 + \int_0^t \int_{R^d} p_{t-s}(x-y) u_{s,y} \delta W^H_{s,y}, \tag{6.18}$$

where the stochastic integral is the Skorokhod one. It turns out that if $d = 1$ and $H > \frac{1}{2}$ then equation (6.18) has a unique mild solution. They also proved that if $d = 2$ and $H > \frac{1}{2}$ then equation (6.18) has a mild solution in a small time interval. When the authors replaced the Wick product in equation (6.17) by the ordinary one, then the equation $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u \cdot \frac{\partial^2 W^H}{\partial t \partial x}$ was treated in the weak form

$$\int_R u_{t,x}\varphi(x)dx = \int_R u_0\varphi(x)dx + \int_0^t \int_R u_{s,x}\varphi''(x)dxds + \int_0^t \int_R u_{s,x}\varphi(x) \circ dW^H_{s,x},$$

where φ is an arbitrary C^∞-function with compact support on R and the last integral is interpreted in the Stratonovich sense. If $H > \frac{3}{4}$ then an existence result of weak sense solutions was also obtained.

Nualart and Zakai in [101] introduced a class of generalized Wiener functionals, related to those of Watanabe and Hida, in order to study the SPDE on a domain A

$$\frac{\partial Y}{\partial t} = \mathcal{L}Y + Y \cdot \eta + \psi, \tag{6.19}$$

$$Y(x,0) = u(x), \; x \in A, \tag{6.20}$$

$$Y(x,t) = 0, \; x \in \partial A, \; t \in [0,T], \tag{6.21}$$

where A is R^d or a connected bounded open set with smooth, Hölder continuous boundary; $\mathcal{L}$ is a uniformly elliptic second-order operator with bounded and continuously twice differentiable coefficients on $\overline{A}$, the closure of A; the initial condition u

and ψ are bounded and continuous functions; and η is the white noise. The authors introduced a class of test functions Ψ, they considered equation (6.19) in the weak sense, i. e., for all $\varphi \in \Psi$:

$$\int\limits_{A\times[0,T]} Y(\frac{\partial}{\partial t} + \mathcal{L}^*)\phi dxdt = \int\limits_{A} u(x)\varphi(x,0)dx + \delta(\phi Y) + \int\limits_{A\times[0,T]} \varphi\psi dxdt, \tag{6.22}$$

where δ is the Skorokhod integral and $\mathcal{L}^*$ is the adjoint of $\mathcal{L}$. They showed that the above problem admits a unique weak solution which belongs to an appropriate space of generalized random functionals.

Kunita in [76] dealt with the Cauchy problem for a certain stochastic parabolic partial differential equation arising in the nonlinear filtering theory. The nonhomogeneous noise term of the SPDE and the initial condition are given by the Schwartz distributions and its solution was investigated in the distributional sense.

Mohammed and Zhang in [97] obtained an extensive result concerning the existence of solutions to a class of semilinear Stratonovich SPDEs with anticipating initial conditions. Let E and H be two real separable Hilbert spaces. The authors firstly approached the Stratonovich evolution equation in H :

$$\begin{aligned} du(t,x) &= (-Au(t,x) + F(u(t,x)) - \frac{1}{2}\sum_{k=1}^{\infty} B_k^2 u(t,x))dt + Bu(t,x) \circ dW(t), \\ t &> 0, \end{aligned} \tag{6.23}$$

with the initial condition $u(0,x) = x$, $x \in H$. Here W is an E-valued Brownian motion with a separable covariance Hilbert space K which has a complete orthonormal basis $\{f_k : k \geq 1\}$ and B_k are given by $B_k(x) := B(x)(f_k)$, $x \in H$, $k \geq 1$.

Under some assumptions on the coefficients A, F and B, they proved the existence and uniqueness of a mild solution to the Cauchy problem (6.23). Moreover, if we denote by $U(t,\omega)(x) := u(t,\omega,x)$, where $u(t,\omega,x)$ is the solution to (6.23), then U is a cocycle. The anticipating equation which the authors treated in the same paper is

$$\begin{aligned} du(t,Y) &= (-Au(t,Y) + F(u(t,Y)) - \frac{1}{2}\sum_{k=1}^{\infty} B_k^2 u(t,Y))dt + Bu(t,Y) \circ dW(t), \\ t &> 0, \end{aligned} \tag{6.24}$$

with the initial condition $u(0,Y) = Y$, where the random variable Y belongs to an appropriate Sobolev space of Wiener functionals. Utilizing the solution to (6.23) and the cocycle U, they showed that $U(t)(Y)$ is also a mild solution to the anticipating Cauchy problem (6.24).

A systematic treatment for some nonanticipating and anticipating classes of parabolic SPDEs was done by Lototsky and Rozovskii in [84]. The study is based on the Cameron-Martin version of the Wiener chaos decomposition. Firstly, the authors looked at the white noise solutions of stochastic parabolic equations of the form:

$$\frac{\partial u}{\partial t}(t,x) = a(x)\frac{\partial^2 u}{\partial x^2}(t,x) + b(x)\frac{\partial u}{\partial x}(t,x) + \frac{\partial u}{\partial x}(t,x)\lozenge\dot{W}(t,x), \tag{6.25}$$

$$x \in R, \ 0 < t < T,$$

$$u(0,x) = u_0(x), \tag{6.26}$$

where the initial condition u_0 and the coefficients a, b are bounded and have continuous bounded derivatives up to second order; the second derivative of a is uniformly Hölder continuous; there exists a positive number ϵ such that $a(x) \geq \epsilon$, $x \in R$; $\dot{W}$ is the white noise on R^2; and the symbol $\Diamond$ denotes the Wick product. They obtained an existence and uniqueness result which states that problem (6.25)-(6.26) has a white noise solution which lies in the Hida space of distributions and it is unique in an appropriate class of weakly measurable functions. Let $(V,\, H,\, V')$ be a normal triple of Hilbert spaces. The same paper was concerned by the abstract evolution equation

$$du = u_0 + \int_0^t (\mathcal{A} + f)dt + \sum_{k \geq 1} \int_0^t (\mathcal{M}_k u + g_k) dw_k(t),\ t \leq T, \tag{6.27}$$

where the processes f and g are deterministic, $\int_0^T ||f(t)||^2_{V'} dt < \infty$ and $\sum_{k\geq 1} \int_0^T ||g(t)||^2_H dt < \infty$; $w_k,\ k \geq 1$ is a collection of standard Wiener processes; the initial condition u_0 is non-random; $\mathcal{A}(t) : V \to V'$, $\mathcal{M}_k(t) : V \to H$ are linear bounded operators for any $t \in [0, T]$ and there exist $C_1, \delta > 0$ and a real number C_2 such that

$$\begin{aligned} &< \mathcal{A}(t)v, v > +\delta ||v||^2_V \leq C_1 ||v||^2_H,\ v \in V,\ t \in [0,T] \\ 2 &< \mathcal{A}(t)v, v > + \sum_{k\geq 1} ||\mathcal{M}_k(t)v||^2_H \leq C_2 ||v||^2_H,\ v \in V,\ t \in [0,T]. \end{aligned}$$

The authors defined the notion of Wiener chaos solution, $w(A, X)$, where A and X are two Banach spaces, they proved that equation (6.27) admits a unique Wiener chaos solution, $w(V, V')$. Utilizing the Wiener chaos decomposition, further regularity results of the solution to equation (6.27) were also obtained. As an application of this abstract result, they considered some appropriate weighted Sobolev spaces $H^1_{2,(r)}$ and the linear parabolic SPDE

$$du = (a_{ij} D_i D_j u + b_i D_i u + cu + f)dt + (\sigma_{ik} D_i u + v_k u + g_k)dw_k. \tag{6.28}$$

Under some assumptions on the initial condition, on the coefficients and on the linear operators appearing in the above equation, there exists unique $w(H^1_{2,(r)}, H^{-1}_{2,(r)})$ Wiener chaos solution to the SPDE (6.28), where $H^{-1}_{2,(r)}$ is the dual space of $H^1_{2,(r)}$.

Deck and Potthoff in [26] dealt with the existence and uniqueness, in the Hida space of distributions, of the solution to the Cauchy problem

$$\frac{\partial \rho}{\partial t}(t,x) - L\rho(t,x) = \dot{X}^i_t(x)\frac{\partial \rho}{\partial x^i}(t,x), \tag{6.29}$$

$$\rho(0,\cdot) = \rho_0, \tag{6.30}$$

where L is a uniformly elliptic second-order differential operator.

It is to be noted that for $\sigma^i_j \in C^1_b([0,T] \times R^d)$, $X^i_t(x) := \int\limits_0^t \sigma^i_j(s,x)dB^j_s$, $1 \leq i,\ j \leq d$, where B_t realizes on the white noise probability space a d-dimensional Brownian motion, the time derivatives $\dot{X}_t$ are well defined elements in the Hida space of distributions, the product $\dot{X}^i_t \frac{\partial \rho}{\partial x^i}$ is interpreted in the Wick sense as $\dot{X}^i_t(x)\varphi := \sigma^i_j(t,x)\dot{B}^j_t \Diamond \varphi$ for any generalized random variable φ which belongs to the Hida space of distributions, and the derivatives in (6.29) are understood in the weak sense. The main result proved by the authors is that if the initial condition, σ^i_j and the coefficients of the operator L lie in some appropriate Hölder spaces then the Cauchy problem (6.29)-(6.30) has a unique solution $\rho \in C^{1,2}([0,T] \times R^d, \mathcal{S}^*)$, where $\mathcal{S}^*$ is the Hida space of distributions. Using the Wiener chaos expansion, the authors showed that under more restrictive conditions on the coefficients of the operator L, on σ^i_j and on the initial condition the solution $\rho(t,x)$ to (6.29)-(6.30) is L^2-valued for all $(t,x) \in [0,T] \times R^d$.

Our aim in this part of thesis is to study the existence and uniqueness of the solution to the following class of anticipating linear heat equation with Dirichlet boundary conditions:

$$\begin{aligned} \frac{\partial u_t(x)}{\partial t} &= \frac{1}{2}\frac{\partial^2 u_t(x)}{\partial x^2} + I_1(A^t)u_t(x) + b(t)u_t(x)\dot{W}, \quad (6.31)\\ t &\in [0,T],\ x \in (0,1), \\ u_t(0) &= u_t(1) = 0,\ t \in [0,T], \quad (6.32)\\ u_0(x) &= \eta,\ x \in (0,1), \quad (6.33) \end{aligned}$$

where $\dot{W}$ is the space-time white noise or the space white noise and I_1 is the first chaos Wiener drift. Assuming some conditions on the coefficients A^t and $b(t)$ and on the initial deterministic condition η, we construct the explicit Wiener chaos representation of the unique solution to the Dirichlet problem (6.31)-(6.33). Utilizing the Wiener chaos decomposition, we also show that the solution to the above SPDE is an analitical functional with finite moments of all orders. As a final result, we prove a continuity property of the considered solution.

Chapter 7

Preliminaries

7.1 Wiener chaos decomposition and anticipating stochastic calculus

In this section we recall the basic definitions and some important results of the stochastic calculus with variations or Malliavin calculus (see [103] and [104]).

Let $(T, \mathcal{B}, \mu)$ be a measurable space where μ is a $\sigma-$finite atomless measure on it and W an isonormal Gaussian process on $L^2(T, \mathcal{B}, \mu)$ defined in a complete probability space $(\Omega, \mathcal{F}, P)$. We assume that the Hilbert space $L^2(T, \mathcal{B}, \mu)$ is separable. Denote by $\mathcal{G}$ the completion of the $\sigma-$algebra generated by W. Fix $n \geq 1$ and set $\mathcal{B}_0 = \{A \in \mathcal{B} : \mu(A) < \infty\}$. Denote by $\mathcal{E}_m$ the set of elementary functions of the form

$$f(t_1, \ldots, t_n) = \sum_{i_1,\ldots,i_n=1}^{m} a_{i_1 \ldots i_n} 1_{A_{i_1} \times \ldots \times A_{i_n}}(t_1, \ldots, t_n), \tag{7.1}$$

where $A_1, \dots, A_m$ are pairwise-disjoint sets belonging to $\mathcal{B}_0$ and the coefficients $a_{i_1 \dots i_n}$ are zero if any two of the indices $i_1, \dots, i_n$ are equal. For a function of the form (7.1), we define the multiple Wiener-Itô integral (see [103])

$$I_n(f) = \sum_{i_1,\dots,i_n=1}^{m} a_{i_1\dots i_n} W(A_{i_1}) \dots W(A_{i_n}).$$

By a density argument the operator I_n can be extended to a linear and continuous operator from $L^2(T^n)$ to $L^2(\Omega, \mathcal{F}, P)$ (see [103]). We have for any $F \in L^2(\Omega, \mathcal{G}, P)$ the following version of the Wiener chaos decomposition:

$$F = \sum_{n=0}^{\infty} I_n(f_n), \tag{7.2}$$

where $f_0 = E(F) = I_0(f_0)$ and $f_n \in L^2(T^n)$ are symmetric and uniquely determined by F. Let $f \in L^2(T^p)$ and $g \in L^2(T^q)$ be two symmetric functions. The following formula for the multiplication of multiple integrals will play an important role in the proof of our main results in Chapter 8:

$$I_p(f)I_q(g) = \sum_{k=0}^{p\wedge q} k! \binom{p}{k}\binom{q}{k} I_{p+q-2k}(f \otimes_k g), \tag{7.3}$$

where for any $1 \leq k \leq p \wedge q$, the contraction of k indices of f and g is denoted by $f \otimes_k g$ and is defined by

$$f \otimes_k g(t_1, \dots, t_{p+q-2k}) = \int_{T^k} f(t_1, \dots, t_{p-k}, s) g(t_{p+1}, \dots, t_{p+q-k}, s) \mu^k(ds).$$

Let $\mathcal{S}$ denote the class of smooth random variables and defined by $\mathcal{S} = \{F : F = f(W(h_1), \dots, W(h_n)),\ f \in C_p^\infty(R^n), h_1, \dots, h_n \in L^2(T, \mathcal{B}, \mu),\ n \geq 1\}$, where $C_p^\infty(R^n)$

is the set of all infinitely continuously differentiable functions $g : R^n \to R$ such that g and all of its partial derivatives have polynomial growth. We note that $\mathcal{S}$ is dense in $L^2(\Omega)$ (see [103]).

We will now introduce the derivative operator (for details see [103] and [104]).

Definition 24 *The derivative of a random variable $F = f(W(h_1), \ldots, W(h_n))$, where $f \in C_p^\infty(R^n), h_1, \ldots, h_n \in L^2(T, \mathcal{B}, \mu)$, $n \geq 1$, is given by*

$$DF = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(W(h_1), \ldots, W(h_n))h_i. \tag{7.4}$$

Thus the derivative of a smooth random variable F is a stochastic process denoted by $\{D_t F : t \in T\}$.

More generally, for any $k \geq 2$ the $k-$th derivative of a smooth random variable F, denoted by $D^k F$, will be the $(L^2(T, \mathcal{B}, \mu))^{\otimes k}-$valued random variable

$$D^k_{t_1,\ldots,t_k} F = D_{t_1} \ldots D_{t_k} F,$$

where $t_1, \ldots, t_k \in T$.

Let $k \geq 1$ and $p \geq 1$. We introduce the seminorm on $\mathcal{S}$ defined by

$$||F||_{k,p} = (E(|F|^p) + \sum_{j=1}^{k} E(||D^k F||^p_{(L^2(T,\mathcal{B},\mu))^{\otimes j}}))^{\frac{1}{p}}.$$

Denote by $\mathbb{D}^{k,p}$ the completion of $\mathcal{S}$ with respect to the norm $|| \cdot ||_{k,p}$.

For every $\lambda \geq 1$, we define

$$||F||^2_\lambda = \sum_{n=0}^{\infty} n! \lambda^{2n} ||f_n||^2_{L^2(T^n)}$$

and

$$\mathbb{H}_\infty = \{F : ||F||_\lambda < \infty, \ \forall \lambda \geq 1\}.$$

The elements of $\mathbb{H}_\infty$ are called analytic functionals and $\mathbb{H}_\infty$ is included in the space $\mathbb{D}_\infty = \bigcap_{p,k \geq 1} \mathbb{D}^{k,p}$ introduced by Watanabe (see [123]).

We introduce the divergence operator or the Skorohod integral which is the adjoint of the operator D and it will be denoted by δ. For details about the properties of the Skorohod integral operator one can use [103, 104] and the references therein.

The explicit chaos decomposition is a convenient way to solve linear SPDEs and it is very useful for obtaining some important properties of the solutions, such as continuity, estimates of the moments, etc. Thus it is of main interest to have a characterization of the Skorohod integral operator in terms of Wiener chaos expansion.

Any element $u \in L^2(T \times \Omega)$ has a Wiener chaos decomposition of the form

$$u_t = \sum_{n=0}^{\infty} I_n(f_n(\cdot, t)), \tag{7.5}$$

where $t \in T$ and for each $n \geq 1$, $f_n \in L^2(T^{n+1})$ is a symmetric function in the first n variables. For each $n \geq 1$, denote by $\widetilde{f_n}$ the symmetrization of f_n in all its variables. The following result expresses the operator δ in terms of Wiener chaos expansion.

Proposition 25 *(see [103]) Let $u \in L^2(T \times \Omega)$ with the expansion (7.5). Then u belongs to Dom δ if and only if the series*

$$\delta(u) = \sum_{n=0}^{\infty} I_{n+1}(\widetilde{f_n}), \tag{7.6}$$

converges in $L^2(\Omega)$.

Let $\mathbb{L}^{1,2}$ denote the class of processes $u \in L^2(T \times \Omega)$ such that $u(t) \in \mathbb{D}^{1,2}$ for almost all $t \in T$ and there exists a measurable version of the two-parameter process $D_s u_t$ verifying $E\int\limits_T\int\limits_T (D_s u_t)^2 \mu(ds)\mu(dt) < \infty$. Denote by $\mathbb{L}^{2,2}$ the class of processes $u \in L^2(T \times \Omega)$ such that $u(t) \in \mathbb{D}^{2,2}$ for almost all $t \in T$ and there exists a measurable version of the three-parameter process $D_r D_s u_t$ verifying $E\int\limits_T\int\limits_T\int\limits_T (D_r D_s u_t)^2 \mu(dr)\mu(ds)\mu(dt) < \infty$.

Let $T = [0,1]$ and $\{W_t : t \in [0,1]\}$ be a $d-$dimensional standard Wiener process defined on the canonical space $(\Omega, \mathcal{F}, P)$. We will use the notation $\delta(u) = \int\limits_0^1 u_t dW_t$. The following chain rule which generalizes the Itô formula was proved in [104] (see also [103]). We note that the convention of summation upon repeated indices is used.

Theorem 26 *Let $\Phi : R^M \times R^N \to R$ be a continuous function, such that the derivatives,*

$$\frac{\partial}{\partial x_i}\Phi, \frac{\partial}{\partial y_j}\Phi, \frac{\partial^2}{\partial y_j \partial x_i}\Phi, \frac{\partial^2}{\partial y_j \partial y_k}\Phi,$$

exist and are continuous for $1 \le i \le M$, $1 \le j \le N$. Let $\{u^{ij} : 1 \le i \le N,\ 1 \le j \le d\}$ be a set of processes, such that for any $1 \le i \le N$, $1 \le j \le d$, $u^{ij} \in \mathbb{L}^{2,2}$ and there exists $p > 4$ with

$$\int\limits_0^1\int\limits_0^1 |E(D_s u_t)|^p ds dt + E\int\limits_0^1\int\limits_0^1\int\limits_0^1 |D_r D_s u_t|^p dr ds dt < \infty,$$

and $\{V^i : 1 \le i \le M\}$ be another set of processes, such that for any $1 \le i \le M$, V_t^i, $t \in [0,1]$ is a continuous process with a.s. finite variation belonging to $\mathbb{L}^{1,2}$,

$$E\int\limits_0^1\int\limits_0^1 (D_s V_t^i)^4 ds dt < \infty,$$

and the mapping $t \to D_s V_t^i$ *is continuous with values in* $L^4(\Omega)$*, uniformly with respect to* s*. For* $t \in [0,1]$*, we denote by* $U_t = \int_0^t u_s dW_s$ *the* $N-$*dimensional process defined by* $U_t^i = \int_0^t u_s^{ij} dW_s^j$*. We then have:*

$$\begin{aligned}
& \Phi(V_t, U_t) \\
= \; & \Phi(0, U_0) + \int_0^t \frac{\partial}{\partial x_i}\Phi(V_s, U_s) dV_s^i \\
& + \int_0^t \frac{\partial}{\partial y_k}\Phi(V_s, U_s) u_s^{kj} dW_s^j + \frac{1}{2}\int_0^t \frac{\partial^2}{\partial y_k \partial y_l}\Phi(V_s, U_s) u_s^{kj} u_s^{lj} ds \\
& + \int_0^t \left(\frac{\partial^2}{\partial y_k \partial x_i}\Phi(V_s, U_s) D_s^j V_s^i + \frac{\partial^2}{\partial y_k \partial y_l}\Phi(V_s, U_s)\int_0^s D_s^j u_r^{lh} dW_r^h\right) u_s^{kj} ds.
\end{aligned}$$

We conclude this section with some L^p-inequalities which are very important for our results from Chapter 8.

In [104] Nualart and Pardoux proved the following lemma which turns out to be useful for obtaining L^p moment estimates.

Lemma 27 *Let* $u_x \in L^2(R \times \Omega)$ *be given by (7.5) satisfying* $u_x \in \mathbb{D}^{1,2}$ *for a.e.* x *and*

$$\sum_{m=1}^{\infty} m(m!) \int_R ||I_m(f_m(\cdot, x))||_{L^2(R^m)} dx < \infty.$$

Then $u_x \in Dom\ \delta$ *and, for* $2 \leq p < \infty$ *there exists a positive constant* C_p *such that*

$$||\delta(u)||_p \leq C_p[(\int_R (Eu(x))^2)^{1/2} + ||(\int_R\int_R (D_y u(x))^2 dx dy)^{1/2}||_p].$$

Here $||\cdot||_p$ *denotes the* L^p-*norm with respect to* P*.*

Using Lemma 27 Uemura proved in [120] the following proposition.

Proposition 28 *Let $2 \le p < \infty$. Suppose $F = \sum\limits_{m=0}^{\infty} I_m(f_m) \in L^2(\Omega, \mathcal{G}, P)$ satisfies*

$$\sum_{m=1}^{\infty}(m-1)!!C_p^{[m/2]}||f_m||_{L^2(R^m)} < \infty.$$

Here C_p is the same constant as in Lemma 27 and $[x]$ denotes the largest integer which does not exceed x. Then $F \in L^p(\Omega, \mathcal{G}, P)$ and

$$||F||_p \le V_p \sum_{m=1}^{\infty}(m-1)!!C_p^{[m/2]}||f_m||_{L^2(R^m)},$$

where

$$V_p = (\frac{1}{\sqrt{2\pi}}\int_R x^p e^{-x^2/2}dx)^{1/p}.$$

7.2 Formulation of an equivalence result for a class of nonanticipating SPDEs

In this section we present an important result obtained by Gyöngy in [47] (see also [46]).

Consider a stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ carrying an $\mathcal{F}_t-$Brownian sheet $\{W(t,x) : t \ge 0,\ x \in R\}$. That is, W is a continuous centered $\mathcal{F}_t-$adapted Gaussian random field with covariance

$$E(W(s,x)W(t,y)) = (s \wedge t)(x \wedge y),$$

such that $W(s,x) - W(r,x) - W(s,y) + W(r,y)$ and $\mathcal{F}_t$ are independent for every $0 \le t \le r \le s,\ x, y \in [0,1]$. Denote by $\mathcal{B}(V)$ the Borel $\sigma-$algebra on V (for a topological space V) and $\mathcal{P}$ for the predictable $\sigma-$algebra on $R_+ \times \Omega$.

Let $p_t(x,y)$ be the Green function of the heat equation on the interval $[0,1]$ with Dirichlet boundary conditions, that is

$$p_t(x,y)=\frac{1}{\sqrt{2\pi t}}\sum_{n=-\infty}^{\infty}[\exp(-\frac{(y-x-2n)^2}{2t})-\exp(-\frac{(y+x-2n)^2}{2t}].$$

Using some properties of the Green function, the Itô formula, stochastic Fubini theorem and the deterministic one, Gyöngy showed the following result (for details see [47] and [46]).

Proposition 29 *Assume that u_0 is an $\mathcal{F}_0-$measurable random variable in $L^2([0,1])$ and f, g and σ are $\mathcal{P}\otimes\mathcal{B}(L^2([0,1]))-$measurable bounded functions mapping $R_+\times\Omega\times L^2([0,1])$ into $L^2([0,1])$, $L^1([0,1])$ and $L^2([0,1])$, respectively. Assume moreover that for every $T\geq 0$ there is a constant K such that*

$$\begin{aligned}||f(t,v)||_{L^2([0,1])}+||\sigma(t,v)||_{L^2([0,1])} &\leq K||v||_{L^2([0,1])},\\ ||f(t,v)||_{L^1([0,1])} &\leq ||v||^2_{L^2([0,1])},\end{aligned}$$

for all $t\in[0,T]$ and $v\in L^2([0,1])$. Then an L^2-valued $\mathcal{F}_t-$adapted locally bounded stochastic process $\{u(t):t\in[0,T]\}$ has a continuous modification, satisfying

$$\begin{aligned}&\int_0^1 u(t,x)\varphi(x)dx \qquad (7.7)\\ =&\int_0^1 u_0(x)\varphi(x)dx+\int_0^t\int_0^1 u(s,x)\frac{\partial^2}{\partial x^2}\varphi(x)dxds\\ &+\int_0^t\int_0^1 f(s,u(s))(x)\varphi(x)dxds-\int_0^t\int_0^1 g(s,u(s))(x)\frac{\partial}{\partial x}\varphi(x)dxds\\ &+\int_0^t\int_0^1\sigma(s,u(s))(x)\varphi(x)W(ds,dx)\ (a.s.)\end{aligned}$$

for every test function $\varphi \in C^2([0,1])$, $\varphi(0) = \varphi(1) = 0$ *and for all* $t \in [0,T]$ *if and only if one of the following conditions is met:*

(a) For every test function $\varphi \in C^\infty([0,1])$, $\varphi(0) = \varphi(1) = 0$ *and for all* $t \in [0,T]$ *equation (7.7) holds.*

(b) For every $t \in [0,T]$ *and* $\psi \in C^{1,\infty}([0,t] \times [0,1])$, $\psi(s,0) = \psi(s,1) = 0$, $s \in [0,t]$

$$\begin{aligned}
& \int_0^1 u(t,x)\psi(t,x)dx \\
= \quad & \int_0^1 u_0(x)\psi(0,x)dx + \int_0^t\int_0^1 u(s,x)[\frac{\partial^2}{\partial x^2}\psi(s,x)dx + \frac{\partial}{\partial t}\psi(s,x)]dxds \\
& + \int_0^t\int_0^1 f(s,u(s))(x)\psi(s,x)dxds - \int_0^t\int_0^1 g(s,u(s))(x)\frac{\partial}{\partial x}\psi(s,x)dxds \\
& + \int_0^t\int_0^1 \sigma(s,u(s))(x)\psi(s,x)(x)W(ds,dx) \ \ a.s.
\end{aligned}$$

(c) For almost every $\omega \in \Omega$ *and for all* $t \in [0,T]$

$$\begin{aligned}
& u(t,x) \\
= \quad & \int_0^t p_t(x,y)u_0(y)dy + \int_0^t\int_0^1 p_{t-s}(x,y)f(s,u(s))(y)dyds \\
& - \int_0^t\int_0^1 p_{t-s}(x,y)g(s,u(s))(y)dyds + \int_0^t\int_0^1 p_{t-s}(x,y)\sigma(s,u(s))(y)dW(s,y)
\end{aligned}$$

for $dx-$*almost every* $x \in [0,1]$.

Chapter 8

The main results

8.1 The space-time white noise case

Let $\{W_{t,x} : t \geq 0,\, x \in [0,1]\}$ be a centered Gaussian random field with covariance $E(W_{s,x}W_{t,y}) = (s \wedge t)(x \wedge y)$, $s, t \geq 0$, $x, y \in [0,1]$ which is defined on a complete probability space $(\Omega, \mathcal{F}, P)$, where $\mathcal{F}$ is the completion of the Borel $\sigma-$algebra generated by W.

For a process $\{u_t(x)\}_{t \geq 0, 0 \leq x \leq 1}$ with chaos decomposition

$$u_t(x) = \sum_{n=0}^{\infty} I_n(f_n^{t,x}),$$

with $f_n \in L^2((R_+ \times [0,1])^{n+1})$, $f_n^{t,x}(\cdot)$ symmetric, and for $\lambda \geq 1$ put

$$||u||_{\lambda,T}^2 = \sum_{n=0}^{\infty} n! \lambda^{2n} \int_0^T \int_0^1 ||f_n^{t,x}||_{L^2(([0,T]\times[0,1])^{n+1})}^2 dx dt.$$

Define $H_\infty(L^2([0,T])$ as the class of all processes u such that $||u||_{\lambda,T} < \infty$ for all $\lambda \geq 1$.

Let $\delta : Dom\ \delta \subset L^2(R_+ \times [0,1] \times \Omega) \to L^2(\Omega, \mathcal{F}, P)$ be the Skorohod integral operator. We will use the notation $\delta(u) = \int\limits_0^\infty \int\limits_0^1 u_t(x) dW_{t,x}$ for any $u \in Dom\ \delta$.

For a function $g : R \to R$ denote

$$P_t g(x) = \int\limits_0^1 p_t(x,y) g(y) dy,$$

where $p_t(x,y)$ is the Green function of the heat equation on the interval $[0,1]$ with Dirichlet boundary conditions (see also Section 7.2).

Consider the linear stochastic heat equation

$$\frac{\partial}{\partial t} u_t(x) = \frac{1}{2} \frac{\partial^2}{\partial x^2} u_t(x) + I_1(a^t) u_t(x) + b(t) u_t(x) \frac{\partial^2 W_{t,x}}{\partial t \partial x}, \tag{8.1}$$

$$u_t(0) = u_t(1) = 0, \tag{8.2}$$

$$u_0 = \eta, \tag{8.3}$$

where $a^{\cdot}(\cdot) : R_+^2 \to R$, $b : R_+ \to R$ and $\eta : [0,1] \to R$ are measurable functions.

We impose the following assumptions on the coefficients of problem (8.1)-(8.3).

Assumption (#) b and η are bounded and

$$\int\limits_0^T \int\limits_0^\infty |a^t(s)| ds dt < \infty, \ \int\limits_0^T \int\limits_0^\infty |a^t(s)|^2 ds dt < \infty \text{ for all } T > 0.$$

Definition 30 *We say that a measurable process* $\{u_t(x)\}_{t \geq 0,\ x \in [0,1]}$ *is a weak solution of (8.1)-(8.3) if for every* $t \in [0,1]$ *and* $\varphi \in C^2([0,1])$ *with* $\varphi(0) = \varphi(1) = 0$ *we have*

$1_{[0,t]}(\cdot)b(\cdot)u.(*) \in Dom\ \delta$ *and*

$$\begin{aligned} &\int_0^1 \varphi(x)u_t(x)dx - \int_0^1 \varphi(x)\eta(x)dx \qquad (8.4)\\ = &\int_0^t\int_0^1 \frac{1}{2}\frac{\partial^2}{\partial x^2}\varphi(x)u_s(x)dxds + \int_0^t\int_0^1 \varphi(x)I_1(a^s)u_s(x)dxds \\ &+\int_0^t\int_0^1 \varphi(x)b(s)u_s(x)dW_{s,x}. \end{aligned}$$

Using the chain rule (see [104] or Theorem 26), stochastic and deterministic Fubini theorems and the same ideas as in Gyöngy [47] and [46] (see also Proposition 29 from Section 7.2) we obtain the following result (see also [115]).

Theorem 31 *Let* $\{u_t(x)\}_{t\geq 0, x\in[0,1]}$ *be a measurable process which satisfies conditions of Theorem 26 from Section 7.1. Under Assumption (#) the following statements are equivalent.*

(1) u is a weak solution of problem (8.1)-(8.3).

(2) For every $t \geq 0$ *and* $\psi(s,x) \in C^{1,2}([0,t]\times[0,1])$ *with* $\psi(s,0) = \psi(s,1) = 0$, $s \in [0,t]$ *we have*

$$\begin{aligned} &\int_0^1 \psi(t,x)u_t(x)dx - \int_0^1 \psi(0,x)\eta(x)dx \qquad (8.5)\\ = &\int_0^t\int_0^1 [\frac{1}{2}\frac{\partial^2}{\partial x^2}\psi(s,x) + \frac{\partial}{\partial s}\psi(s,x)u_s(x)]dxds \\ &+\int_0^t\int_0^1 \psi(s,x)I_1(a^s)u_s(x)dxds + \int_0^t\int_0^1 \psi(s,x)b(s)u_s(x)dW_{s,x}. \end{aligned}$$

(3) For almost every $\omega \in \Omega$ *and for all t we have*

$$\begin{aligned} &u_t(x) \qquad (8.6)\\ = &P_t\eta(x) + \int_0^t\int_0^1 p_{t-s}(x,y)I_1(a^s)u_s(y)dyds + \int_0^t\int_0^1 p_{t-s}(x,y)b(s)u_s(x)dW_{s,y}. \end{aligned}$$

for almost all x.

Proof. Assume first (1). Let $\xi(t)$ denote the right-side of (8.4). Then, utilizing the chain rule (see [104] or Theorem 26) we have for every $v \in C^1([0,T])$

$$v(t)\xi(t) - v(t)\xi(t) = \int_0^t v(s)d\xi(s) + \int_0^t \xi(s)\frac{d}{ds}v(s)ds,$$

which by (8.4) gives (8.5) with $v(s)\varphi(x)$ in place of $\psi(s,x)$. Hence by linearity we obtain (8.5) first with $\sum_{i=1}^n v_i(s)\varphi_i(x), v_i \in C^1([0,T]), \varphi_i \in C^\infty([0,1])$ in place of $\psi(s,x)$ and then for any $\psi \in C^{1,2}([0,T]\times[0,1])$, by approximating ψ by its Fourier expansion

$$\begin{aligned}\psi_n(s,x) &= 2\sum_{i=1}^n v_i(s)\sin(i\pi x),\\ v_i(s) &= \int_0^1 \psi(s,y)\sin(i\pi y)dy.\end{aligned}$$

Condition (2) obviously implies (1).

Assume now that (2) is satisfied. Let $t \in [0,T]$ and define

$$\begin{aligned}\psi(s,y) &= \int_0^1 p_{t-s}(z,y)\varphi(z)dzds,\ (s,y) \in (0,t] \times [0,1],\\ \psi(t,y) &= \varphi(y),\ y \in [0,1],\end{aligned}$$

for a function $\varphi \in C^2([0,1])$. Then $\psi \in C^{1,2}([0,t] \times [0,1])$ and

$$\begin{aligned}\frac{\partial}{\partial y^2}\psi(s,y) + \frac{\partial}{\partial s}\psi(s,y) &= 0,\ (s,y) \in (0,t) \times (0,1),\\ \lim_{s\uparrow t}\psi(t,y) &= \varphi(y) \text{ in } L^2([0,1]).\end{aligned}$$

Thus from (8.5) we have

$$\begin{aligned}&\int_0^1 u_t(x)\psi(t,x)dx-\int_0^1\eta(x)\psi(0,x)dx\\ =\ &\int_0^t\int_0^1\psi(s,x)I_1(a^s)u_s(x)dxds+\int_0^t\int_0^1\psi(s,x)b(s)u_s(x)dW_{s,x}.\end{aligned}$$

Using Fubini's theorem (deterministic and stochastic) we obtain

$$\int_0^1 u_t(x)\varphi(x)dx=\int_0^1\varsigma(t,x)\varphi(x)dx$$

for any $\varphi\in C^\infty([0,1])$, where $\varsigma(t,x)$ denotes the right side of (8.6). Hence we get (8.6).

Assume now (3). Let $\varphi\in C^2([0,1])$, $t\in[0,T]$ and

$$\begin{aligned}&\zeta\\ =\ &\int_0^1\varphi(x)u_t(x)dx-\int_0^1\varphi(x)\eta(x)dx-\int_0^t\int_0^1\frac{1}{2}\frac{\partial^2}{\partial x^2}\varphi(x)u_s(x)dxds\\ &-\int_0^t\int_0^1\varphi(x)I_1(a^s)u_s(x)dxds-\int_0^t\int_0^1\varphi(x)b(s)u_s(x)dW_{s,x}.\end{aligned}$$

We have

$$\zeta=\zeta_1+\zeta_2+\zeta_3,$$

where

$$\begin{aligned}&\zeta_1\\ =\ &\int_0^1\int_0^1 p_t(x,y)\varphi(x)\eta(y)dydx-\int_0^1\varphi(x)\eta(x)dx\\ &-\int_0^t\int_0^1\int_0^1 p_s(x,y)\varphi''(x)\eta(y)dydxds,\end{aligned}$$

$$
\begin{aligned}
&\zeta_2\\
= \;& \int_0^1\int_0^t\int_0^1 p_{t-s}(x,y) I_1(a^s) u_s(y)dydsdx\\
&-\int_0^t\int_0^1\int_0^s p_{s-r}(x,y)I_1(a^r)u_r(y)\varphi''(x)dydrdxds - \int_0^t\int_0^1 I_1(a^s)u_s(y)\varphi(y)dyds,
\end{aligned}
$$

$$
\begin{aligned}
&\zeta_3\\
= \;& \int_0^1\int_0^t\int_0^1 p_{t-s}(x,y)b(s)u_s(y)\varphi(x)dW_{s,y}dx\\
&-\int_0^t\int_0^1\int_0^s p_{s-r}(x,y)b(r)u_r(y)\varphi''(x)dW_{r,y}dxds - \int_0^t\int_0^1 b(s)u_s(y)\varphi(y)dydr.
\end{aligned}
$$

Using Fubini's theorem (for ordinary and stochastic integrals) we obtain

$$
\begin{aligned}
\zeta_1 &= \int_0^1 \eta(y)\{G(t,\varphi,y)dy - \varphi(y) - \int_0^t G(s,\varphi'',y)ds\}dy,\\
\zeta_2 &= \int_0^t\int_0^1 I_1(a^r)u_r(y)\{G(t-r,\varphi,y)dy - \varphi(y) - \int_r^t G(s-r,\varphi'',y)ds\}dydr,\\
\zeta_3 &= \int_0^t\int_0^1 b(r)u_r(x)\{G(t-r,\varphi,y) - \varphi(y) - \int_r^t G(s-r,\varphi'',y)ds\}dW_{r,y},
\end{aligned}
$$

where

$$
G(r,\varphi,y) = \int_0^1 p_r(x,y)\varphi(x)dx,
$$

for every φ. Hence noting that for every $\varphi \in C^2([0,1])$, $\varphi(0) = \varphi(1) = 0$

$$
G(t-r,\varphi,y) - \varphi(y) - \int_r^t G(s-r,\varphi'',y)ds\} = 0,
$$

for all $r < t$, we get $\zeta_1 = \zeta_2 = \zeta_3 = 0$, which proves (3). The proof of the theorem is complete. ∎

We introduce the following notation. Given $t_1, \dots, t_n$ and a symmetric function of $n-j$ variables with $n < j$, we denote by $f_{n-j}(\widehat{t_{i_1}}, \dots, \widehat{t_{i_j}})$ the f_{n-j} evaluated at t's other than $t_{i_1}, \ldots, t_{i_j}$. Next we set

$$
\begin{aligned}
\Delta_{j,n} &= \{(i_1, \ldots, i_j) : i_k \neq i_l \text{ if } k \neq l,\ i_k = 1, \ldots, n\},\ j \leq n, \\
h_t(u) &= \int_0^t a^r(u)dr,\ Y_t = \exp\{\frac{1}{2}||h_t(\cdot)||_{L^2(R_+)}\}, \\
C_t &= \exp\{\int_0^t \langle a^s(\cdot), 1_{[0,s]}(\cdot)b(\cdot)\rangle_{L^2(R_+)}\}ds,
\end{aligned}
$$

where $\langle \cdot, \cdot \rangle_{L^2(R_+)}$ denotes the scalar product over $L^2(R_+)$.

We have the following straightforward relations

$$
\begin{aligned}
\frac{d}{dt} h_t^{\otimes n}(t, \ldots, t_n) &= n\ sym(a^t \otimes h_t^{\otimes n-1})(t, \ldots, t_n), && (8.7) \\
\frac{dY_t}{dt} &= \langle a^t(\cdot), h_t(\cdot)\rangle_{L^2(R_+)} Y_t, && (8.8) \\
\frac{dC_t}{dt} &= \langle a^t(\cdot), 1_{[0,t]}(\cdot)b(\cdot)\rangle_{L^2(R_+)} C_t. && (8.9)
\end{aligned}
$$

We are now ready to state the main result.

Theorem 32 *Under assumption (#) the problem (8.1)-(8.3) has a unique solution in $H_\infty(L^2([0,T])$ for all $T > 0$ given by the chaos expansion*

$$
\begin{aligned}
u_t(x) &= \sum_{n=0}^{\infty} I_n(f_n^{t,x}), && (8.10) \\
f_0^{t,x} &= C_t Y_t P_t \eta(x) && (8.11)
\end{aligned}
$$

and, for $n \geq 1$,

$$
\begin{aligned}
& f_n^{t,x}(t_1,\ldots,t_n,x_1,\ldots,x_n) \qquad (8.12)\\
= \;& C_tY_t\{\frac{Y_tP_t\eta(x)h_t^{\otimes n}(t_1,\ldots,t_n)}{n!}\\
& + sym\;[\sum_{j=1}^{n} 1_{t_1<\ldots<t_j<t}p_{t-t_j}(x,x_j)b(t_j)p_{t_j-t_{j-1}}(x_j,x_{j-1})b(t_{j-1})\ldots\\
& \times p_{t_2-t_1}(x_2,x_1)b(t_1)P_{t_1}\eta(x_1)h_t^{\otimes(n-j)}(t_{j+1},\ldots,t_n)]\}\\
= \;& \frac{1}{n!}C_tY_t[\sum_{j=1}^{n}\sum_{(i_1,\ldots,i_j)\in\Delta_{j,n}} 1_{t_{i_1}<\ldots<t_{i_j}<t}p_{t-t_{i_j}}(x,x_{i_j})b(t_{i_j})p_{t_{i_j}-t_{i_{j-1}}}(x_{i_j},x_{i_{j-1}})\\
& \times b(t_{i_{j-1}})\ldots P_{t_{i_1}}\eta(x_{i_1})h_t^{\otimes(n-j)}(\widehat{t_{i_1}},\ldots,\widehat{t_{i_j}}) + P_t\eta(x)h_t^{\otimes n}(t_1,\ldots,t_n)].
\end{aligned}
$$

Moreover, for all $t \geq 0$, $x \in [0,1]$ *and* $n \geq 1$,

$$
\begin{aligned}
& ||f_n^{t,x}||^2_{L^2((R_+\times[0,1])^n)} \qquad (8.13)\\
\leq \;& ||\eta||^2_\infty C_t^2Y_t^2\frac{n+1}{n!}\sum_{j=0}^{n}\frac{||h_t||^{2(n-j)}_{L^2(R_+)}(\frac{t||b||^4_\infty}{4})^{j/2}}{(n-j)!\Gamma(\frac{j}{2}+1)}.
\end{aligned}
$$

Proof. *Existence.* Utilizing Theorem 31, the expression of the Skorohod integral (see [103], [104] or Proposition.25), the product formula for multiple Wiener integrals (7.3) (see [103], [61]) and by identifying the kernels of multiple Wiener integrals in (8.6) we obtain that $u \in H_\infty(L^2([0,T]), T > 0$ is a weak solution to Dirichlet problem (8.1)-(8.3) if and only if the sequence of kernels $\{f_n^{t,x}\}_{n\geq 0}$ is a solution of the infinite system of deterministic integrals

$$
\begin{aligned}
& f_0^{t,x} \qquad (8.14)\\
= \;& P_t\eta(x) + \int_0^t\int_0^1 p_{t-s}(x,y)\langle a^s\otimes 1, f_1^{s,y}\rangle_{L^2(R_+\times[0,1])}
\end{aligned}
$$

and, for $n \geq 1$,

$$
\begin{aligned}
& f_n^{t,x}(s_1,\ldots,s_n,x_1,\ldots,x_n) && (8.15)\\
= \; & \frac{1}{n}\sum_{i=1}^{n}\int_0^t\int_0^1 p_{t-s}(x,y)a^s(s_i)f_{n-1}^{s,y}(\widehat{s_i},\widehat{x_i})dyds \\
& +(n+1)\int_0^t\int_0^1 p_{t-s}(x,y)\langle a^s\otimes 1, f_{n+1}^{s,y}(s_1,\ldots,s_n,x_1,\ldots,x_n)\rangle_{L^2(R_+\times[0,1])}dyds \\
& +\frac{1}{n}\sum_{i=1}^{n}1_{s_i<t}p_{t-s_i}(x,x_i)b(s_i)f_{n-1}^{s_i,x_i}(\widehat{s_i},\widehat{x_i}).
\end{aligned}
$$

Let us prove that $f_n^{t,x}$ given by (8.11) and (8.12) is a solution of (8.14) and (8.15).

We have

$$
\begin{aligned}
f_1^{s,x}(s_1,x_1) &= A_1(s,x,s_1,x_1)+A_2(s,x,s_1),\\
A_1(s,x,s_1,x_1) &= C_sY_sp_{s-s_1}(x,x_1)b(s_1)P_{s_1}\eta(x_1)1_{s_1<s},\\
A_2(s,x,s_1) &= C_sY_sP_{s_1}\eta(x)h_s(s_1).
\end{aligned}
$$

Then

$$
\begin{aligned}
& \int_0^t\int_0^1 p_{t-s}(x,y)\langle a^s\otimes 1, A_1\rangle_{L^2(R_+\times[0,1])}dyds && (8.16)\\
= \; & \int_0^t\int_0^1 p_{t-s}(x,y)C_sY_s[\int_0^s\int_0^1 a^s(s_1)p_{s-s_1}(y,y_1)b(s_1)P_{s_1}\eta(y_1)ds_1dy_1]dyds \\
= \; & \int_0^t\int_0^1 C_sY_s[\int_0^1\int_0^s p_{t-s}(x,y)p_{s-s_1}(y,y_1)dy]a^s(s_1)b(s_1)P_{s_1}\eta(y_1)dy_1ds_1ds \\
= \; & \int_0^t C_sY_s\int_0^s[\int_0^1 p_{t-s_1}(y,y_1)P_{s_1}\eta(y_1)dy_1]a^s(s_1)b(s_1)ds_1ds
\end{aligned}
$$

$$
\begin{aligned}
&= P_t\eta(x)\int_0^t C_sY_s\int_0^s a^s(s_1)b(s_1)ds_1ds\\
&= P_t\eta(x)\int_0^t Y_sC'_sds.
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
& \int_0^t\int_0^1 p_{t-s}(x,y)\langle a^s\otimes 1, A_2(s,y)\rangle_{L^2(R_+\times[0,1])}dyds \qquad (8.17)\\
= & \int_0^t\int_0^1 p_{t-s}(x,y)C_sY_s[\int_0^\infty\int_0^1 a^s(s_1)P_{s_1}\eta(y)h_s(s_1)dy_1ds_1dyds\\
= & \int_0^t C_sY_s\int_0^\infty[\int_0^1 p_{t-s}(x,y)P_{s_1}\eta(y)dy]a^s(s_1)h_s(s_1)ds_1ds\\
= & P_t\eta(x)\int_0^t Y_s'C_sds.
\end{aligned}
$$

From (8.16) and (8.17) we obtain (8.14). Using the expression of f_2 given by (8.15) we have

$$
\begin{aligned}
& 2\int_0^t\int_0^1 p_{t-s}(x,y)\langle a^s\otimes 1, f_2^{s,y}(s_1,x_1)\rangle_{L^2(R_+\times[0,1])}dyds \qquad (8.18)\\
= & \int_0^t C_sY_s\int_0^\infty[\int_0^1 p_{t-s}(x,y)P_{s_1}\eta(y)dy]a^s(s_1)h_s(s_1)ds_1ds\\
= & 2\int_0^t\int_0^1 p_{t-s}(x,y)[\int_0^\infty\int_0^1 a^s(\theta)f_2^{s,y}(\theta,s_1,z,x_1)dzd\theta]dyds\\
= & \sum_{i=1}^5 J_i,
\end{aligned}
$$

where

$$
\begin{aligned}
J_1 = & \int_0^t\int_0^1 p_{t-s}(x,y)\int_0^\infty\int_0^1 a^s(\theta)C_sY_sp_{s-s_1}(y,x_1)b(s_1)p_{s_1-\theta}(x_1,z)\\
& \times b(\theta)P_\theta\eta(z)1_{0<\theta<s_1<s}dzd\theta dyds,\\
J_2 = & \int_0^t\int_0^1 p_{t-s}(x,y)\int_0^\infty\int_0^1 a^s(\theta)C_sY_sp_{s-\theta}(y,z)b(\theta)p_{\theta-s_1}(z,x_1)\\
& \times b(s_1)P_{s_1}\eta(x_1)1_{0<s_1<\theta<s}dzd\theta dyds,
\end{aligned}
$$

$$
\begin{aligned}
J_3 &= \int_0^t\int_0^1 p_{t-s}(x,y)\int_0^\infty\int_0^1 a^s(\theta)C_sY_sp_{s-\theta}(y,z)b(\theta)1_{0<\theta<s_1}P_\theta\eta(z)h_s(s_1)dzd\theta dyds,\\
J_4 &= \int_0^t\int_0^1 p_{t-s}(x,y)\int_0^\infty\int_0^1 a^s(\theta)C_sY_sp_{s-s_1}(y,x_1)b(s_1)1_{0<<s_1<\theta}P_{s_1}\eta(x_1)h_s(\theta)dzd\theta dyds,\\
J_5 &= P_t\eta(x)\int_0^t\int_0^1 a^s(\theta)C_sY_sh_s(\theta)h_s(s_1)d\theta ds.
\end{aligned}
$$

Next, using the semigroup property of p we obtain

$$
\begin{aligned}
J_1 &= 1_{s_1<t}p_{t-s_1}(x,x_1)b(s_1)P_{s_1}\eta(x_1)\int_{s_1}^t C_sY_s\int_0^{s_1} a^s(\theta)b(\theta)d\theta ds,\\
J_2 &= 1_{s_1<t}p_{t-s_1}(x,x_1)b(s_1)P_{s_1}\eta(x_1)\int_{s_1}^t C_sY_s\int_{s_1}^{s} a^s(\theta)b(\theta)d\theta ds,
\end{aligned}
$$

and then

$$
J_1+J_2 = 1_{s_1<t}p_{t-s_1}(x,x_1)b(s_1)P_{s_1}\eta(x_1)\int_{s_1}^t\int_0^s C_s'Y_sds. \tag{8.19}
$$

Similarly, we have

$$
\begin{aligned}
J_3 &= P_t\eta(x)\int_0^t C_s'Y_sh_s(s_1)ds, &(8.20)\\
J_4 &= 1_{s_1<t}p_{t-s_1}(x,x_1)b(s_1)P_{s_1}\eta(x_1)\int_{s_1}^t C_sY_s'ds, &(8.21)\\
J_5 &= P_t\eta(x)\int_0^t C_s'Y_s'h_s(s_1)ds. &(8.22)
\end{aligned}
$$

Replacing (8.19)-(8.22) and the expression of f_1 given by (8.15) in (8.18), we obtain (8.15) for $n=1$.

For arbitrary n and $j=2,\ldots,n$, we write

$$
f_{n+1}^{s,y}(s_1,\ldots,s_{n+1},x_1,\ldots,x_{n+1}) = A^{s,y}(s_1,\ldots,s_{n+1},x_1,\ldots,x_{n+1}) \tag{8.23}
$$

$$
+\sum_{j=1}^{n}\sum_{(i_1,\ldots,i_j)\in\Delta_{j,n}}[A_1^{s,y,i_1,\ldots,i_j}(s_1,\ldots,s_{n+1},x_1,\ldots,x_{n+1})
$$

$$+\sum_{k=2}^{j} A_{2,k}^{s,y,i_1,\dots,i_j}(s_1,\dots,s_{n+1},x_1,\dots,x_{n+1}) + A_3^{s,y,i_1,\dots,i_j}(s_1,\dots,s_{n+1},x_1,\dots,x_{n+1})$$

$$+B_j^{s,y,i_1,\dots,i_j}(s_1,\dots,s_{n+1},x_1,\dots,x_{n+1})] + C^{s,y}(s_1,\dots,s_{n+1}),$$

$$\begin{aligned} & (n+1)!A^{s,y}(s_1,\dots,s_{n+1},x_1,\dots,x_{n+1}) \\ = \; & C_s Y_s 1_{s_{n+1}<s} p_{s-s_{n+1}}(y,x_{n+1}) b(s_{n+1}) P_{s_{n+1}} \eta(x_{n+1}) h_s^{\otimes n}(s_1,\dots,s_n), \end{aligned}$$

$$\begin{aligned} & (n+1)!A_1^{s,y,i_1,\dots,i_j}(s_1,\dots,s_{n+1},x_1,\dots,x_{n+1}) \\ = \; & C_s Y_s 1_{s_{n+1}<s_{i_1}<\dots<s_{i_j}<s} p_{s-s_{i_j}}(y,x_{i_j}) b(s_{i_j}) \dots p_{s_{i_2}-s_{i_1}}(x_{i_2},x_{i_1}) b(s_{i_1}) \\ & \times p_{s_{i_1}-s_{n+1}}(x_{i_1},x_{n+1}) b(s_{n+1}) P_{s_{n+1}} \eta(x_{n+1}) h_s^{\otimes(n-j)}(\widehat{s_{i_1}},\dots,\widehat{s_{i_j}}), \end{aligned}$$

$$\begin{aligned} & (n+1)!A_{2,k}^{s,y,i_1,\dots,i_j}(s_1,\dots,s_{n+1},x_1,\dots,x_{n+1}) \\ = \; & C_s Y_s 1_{s_{i_1}<\dots<s_{i_{k-1}}<s_{n+1}<si_k<\dots<s_{i_j}<s} p_{s-s_{i_j}}(y,x_{i_j}) b(s_{i_j}) \dots p_{s_{i_k}-s_{n+1}}(x_{i_k},y) b(s_{n+1}) \\ & \times p_{s_{n+1}-s_{i_{k-1}}}(x_{n+1},x_{i_{k-1}}) b(s_{i_{k-1}}) \dots p_{s_{i_2}-s_{i_1}}(x_{i_2},x_{i_1}) b(s_{i_1}) P_{s_{i_1}} \eta(x_{i_1}) \\ & \times h_s^{\otimes(n-j)}(\widehat{s_{i_1}},\dots,\widehat{s_{i_j}}), \end{aligned}$$

$$\begin{aligned} & (n+1)!A_3^{s,y,i_1,\dots,i_j}(s_1,\dots,s_{n+1},x_1,\dots,x_{n+1}) \\ = \; & C_s Y_s 1_{s_{i_1}<\dots<s_{i_j}<s_{n+1}<s} p_{s-s_{n+1}}(y,x_{n+1_j}) b(s_{n+1}) p_{s_{n+1}-s_{i_j}}(x_{n+1},x_{i_j}) b(s_{i_j}) \\ & \times \dots p_{s_{i_2}-s_{i_1}}(x_{i_2},x_{i_1}) b(s_{i_1}) P_{s_{i_1}} \eta(x_{i_1}) h_s^{\otimes(n-j)}(\widehat{s_{i_1}},\dots,\widehat{s_{i_j}}), \end{aligned}$$

$$\begin{aligned} & B_j^{s,y,i_1,\dots,i_j}(s_1,\dots,s_{n+1},x_1,\dots,x_{n+1}) \\ = \; & C_s Y_s 1_{s_{i_1}<\dots<s_{i_j}<s} p_{s-s_{i_j}}(y,x_{i_j}) b(s_{i_j}) \dots P_{s_{i_1}} \eta(x_{i_1}) h_s(s_{n+1}) h_s^{\otimes(n-j)}(\widehat{s_{i_1}},\dots,\widehat{s_{i_j}}), \end{aligned}$$

$$(n+1)!C^{s,y}(s_1,\dots,s_{n+1}) = C_sY_sP_s\eta(y)h_s^{\otimes(n+1)}(s_1,\dots,s_{n+1}),$$

$$(n+1)!A_j^{s,y}(s_1,\dots,s_{n+1},x_j) = C_sY_s1_{s_1<s}p_{s-s_j}(y,x_j)b(s_j)P_{s_j}\eta(x_j)h_s(s_j)h_s^{\otimes(n-1)}(\widehat{s_j}).$$

By the semigroup property of p and by Fubini theorem, we obtain

$$\begin{aligned} J &= (n+1)\int_0^t\int_0^1 p_{t-s}(x,y)\langle a^s\otimes 1, A^{s,y}(s_1,\dots,s_n)\rangle_{L^2(R_+\times[0,1])}dyds \\ &= \frac{1}{n!}P_s\eta(x)\int_0^t C_s'Y_sh_s^{\otimes n}(s_1,\dots,s_n)ds, \end{aligned} \tag{8.24}$$

$$\begin{aligned} J_j &= (n+1)\int_0^t\int_0^1 p_{t-s}(x,y)\langle a^s\otimes 1, A_j^{s,y}(s_1,\dots,s_{n+1})\rangle_{L^2(R_+\times[0,1])}dyds \\ &= \frac{1}{n!}P_{s_j}\eta(x)\int_0^t C_s'Y_sh_s^{\otimes n}(\widehat{s_j})ds, \end{aligned} \tag{8.25}$$

$$\begin{aligned} J_1^{i_1,\dots,i_j} &= (n+1)\int_0^t\int_0^1 p_{t-s}(x,y) \\ &\quad\times\langle a^s\otimes 1, A_1^{s,y,i_1,\dots,i_j}(s_1,\dots,s_n,x_1,\dots,x_n)\rangle_{L^2(R_+\times[0,1])}dyds \\ &= \frac{1}{n!}1_{s_{i_1}<\dots<s_{i_j}<t}p_{t-s_{i_j}}(x,x_{i_j})b(s_{i_j})\dots p_{s_{i_2}-s_{i_1}}(x_{i_2},x_{i_1})b(s_{i_1})P_{s_{i_1}}\eta(x_{i_1}) \\ &\quad\times\int_{s_{i_j}}^t C_sY_s[\int_0^{s_{i_1}} a^s(s_{n+1})b(s_{n+1})ds_{n+1}]h_s^{\otimes(n-j)}(\widehat{s_{i_1}},\dots,\widehat{s_{i_j}})ds, \end{aligned} \tag{8.26}$$

$$\begin{aligned} J_{2,k}^{i_1,\dots,i_j} &= (n+1)\int_0^t\int_0^1 p_{t-s}(x,y) \\ &\quad\times\langle a^s\otimes 1, A_{2,k}^{s,y,i_1,\dots,i_j}(s_1,\dots,s_n,x_1,\dots,x_n)\rangle_{L^2(R_+\times[0,1])}dyds \\ &= \frac{1}{n!}1_{s_{i_1}<\dots<s_{i_j}<t}p_{t-s_{i_j}}(x,x_{i_j})b(s_{i_j})\dots p_{s_{i_2}-s_{i_1}}(x_{i_2},x_{i_1})b(s_{i_1})P_{s_{i_1}}\eta(x_{i_1}) \\ &\quad\times\int_{s_{i_j}}^t C_sY_s[\int_{s_{i_{k-1}}}^{s_{i_k}} a^s(s_{n+1})b(s_{n+1})ds_{n+1}]h_s^{\otimes(n-j)}(\widehat{s_{i_1}},\dots,\widehat{s_{i_j}})ds, \end{aligned} \tag{8.27}$$

$$
\begin{aligned}
J_3^{i_1,\ldots,i_j} &= (n+1)\int_0^t\int_0^1 p_{t-s}(x,y) \\
&\quad \times\langle a^s\otimes 1, A_3^{s,y,i_1,\ldots,i_j}(s_1,\ldots,s_n,x_1,\ldots,x_n)\rangle_{L^2(R_+\times[0,1])}dyds \\
&= \frac{1}{n!}1_{s_{i_1}<\ldots<s_{i_j}<t}p_{t-s_{i_j}}(x,x_{i_j})b(s_{i_j})\ldots p_{s_{i_2}-s_{i_1}}(x_{i_2},x_{i_1})b(s_{i_1})P_{s_{i_1}}\eta(x_{i_1}) \\
&\quad \times\int_{s_{i_j}}^t C_sY_s[\int_{s_{i_j}}^s a^s(s_{n+1})b(s_{n+1})ds_{n+1}]h_s^{\otimes(n-j)}(\widehat{s_{i_1}},\ldots,\widehat{s_{i_j}})ds.
\end{aligned} \tag{8.28}
$$

From (8.26)-(8.28) we deduce that

$$
\begin{aligned}
& J_1^{i_1,\ldots,i_j}+\sum_{k=2}^{j}J_{2,k}^{i_1,\ldots,i_j}+J_3^{i_1,\ldots,i_j} \\
= \; & \frac{1}{n!}1_{s_{i_1}<\ldots<s_{i_j}<t}p_{t-s_{i_j}}(x,x_{i_j})b(s_{i_j})\ldots p_{s_{i_2}-s_{i_1}}(x_{i_2},x_{i_1})b(s_{i_1})P_{s_{i_1}}\eta(x_{i_1}) \\
& \times\int_{s_{i_j}}^t C_s'Y_sh_s^{\otimes(n-j)}(\widehat{s_{i_1}},\ldots,\widehat{s_{i_j}})ds.
\end{aligned} \tag{8.29}
$$

Also, we have

$$
\begin{aligned}
J_4^{i_1,\ldots,i_j} &= (n+1)\int_0^t\int_0^1 p_{t-s}(x,y) \\
&\quad \times\langle a^s\otimes 1, B_j^{s,y,i_1,\ldots,i_j}(s_1,\ldots,s_{n+1},x_1,\ldots,x_{n+1})\rangle_{L^2(R_+\times[0,1])}dyds \\
&= \frac{1}{n!}1_{s_{i_1}<\ldots<s_{i_j}<t}p_{t-s_{i_j}}(x,x_{i_j})b(s_{i_j})\ldots p_{s_{i_2}-s_{i_1}}(x_{i_2},x_{i_1})b(s_{i_1})P_{s_{i_1}}\eta(x_{i_1}) \\
&\quad \times\int_{s_{i_j}}^t C_sY_s'h_s^{\otimes(n-j)}(\widehat{s_{i_1}},\ldots,\widehat{s_{i_j}})ds.
\end{aligned} \tag{8.30}
$$

Next,

$$
\begin{aligned}
JJ &= (n+1)\int_0^t\int_0^1 p_{t-s}(x,y)\langle a^s\otimes 1, C^{s,y}(s_1,\ldots,s_{n+1})\rangle_{L^2(R_+\times[0,1])}dyds \\
&= \frac{1}{n!}P_s\eta(x)\int_0^t C_sY_s'h_s^{\otimes n}(s_1,\ldots,s_n)ds.
\end{aligned} \tag{8.31}
$$

Then from (8.24)-(8.31) we get

$$
\begin{aligned}
& J + JJ && (8.32)\\
= \ & \frac{1}{n!}P_s\eta(x)\int_0^t (C_sY_s)' h_s^{\otimes n}(s_1,\dots,s_n)ds \\
= \ & \frac{1}{n!}P_s\eta(x)C_tY_th_t^{\otimes n}(s_1,\dots,s_n) - \frac{1}{n!}\int_0^t n\ sym(a^s\otimes h^{\otimes(n-1)})(s_1,\dots,s_n)ds.
\end{aligned}
$$

Using relations (8.29) and (8.30) we have

$$
\begin{aligned}
& J_1^{i_1,\dots,i_j} + \sum_{k=2}^{j} J_{2,k}^{i_1,\dots,i_j} + J_3^{i_1,\dots,i_j} + J_4^{i_1,\dots,i_j} && (8.33)\\
= \ & \frac{1}{n!}1_{s_{i_1}<\dots<s_{i_j}<t}p_{t-s_{i_j}}(x,x_{i_j})b(s_{i_j})\dots p_{s_{i_2}-s_{i_1}}(x_{i_2},x_{i_1})b(s_{i_1})P_{s_{i_1}}\eta(x_{i_1})\\
& \times\int_{s_{i_j}}^t (C_sY_s)'h_s^{\otimes(n-j)}(\widehat{s_{i_1}},\dots,\widehat{s_{i_j}})ds\\
= \ & \frac{1}{n!}1_{s_{i_1}<\dots<s_{i_j}<t}p_{t-s_{i_j}}(x,x_{i_j})b(s_{i_j})\dots p_{s_{i_2}-s_{i_1}}(x_{i_2},x_{i_1})b(s_{i_1})P_{s_{i_1}}\eta(x_{i_1})\\
& \times C_tY_th_t^{\otimes(n-j)}(\widehat{s_{i_1}},\dots,\widehat{s_{i_j}}) - \frac{1}{n!}1_{s_{i_1}<\dots<s_{i_j}<t}p_{t-s_{i_j}}(x,x_{i_j})\\
& \times b(s_{i_j})\dots p_{s_{i_2}-s_{i_1}}(x_{i_2},x_{i_1})C_{s_{i_j}}Y_{s_{i_j}}h_{s_{i_j}}^{\otimes(n-j)}(\widehat{s_{i_1}},\dots,\widehat{s_{i_j}})\\
& -\frac{1}{n!}1_{s_{i_1}<\dots<s_{i_j}<t}p_{t-s_{i_j}}(x,x_{i_j})b(s_{i_j})\dots p_{s_{i_2}-s_{i_1}}(x_{i_2},x_{i_1})b(s_{i_1})P_{s_{i_1}}\eta(x_{i_1})\\
& \times\int_{s_{i_j}}^t C_sY_s(n-j)\ sym(a^s\otimes h^{\otimes(n-j-1)})(\widehat{s_{i_1}},\dots,\widehat{s_{i_j}})ds.
\end{aligned}
$$

Finally, (8.25), (8.32) and (8.33) prove the validity of (8.15) for any n.

Let us prove that $u\in H_\infty(L^2([0,T])$ for any $T>0$.

For $t\geq 0$, $x\in[0,1]$ and $j<n$ we have

$$
\begin{aligned}
& ||sym[1_{t_1<\dots<t_j<t}p_{t-t_j}(x,x_j)b(t_j)\dots P_{t_j}\eta(x_j)h_t^{\otimes(n-j)}(t_{j+1},\dots,t_n)||^2_{L^2((R_+\times[0,1])^n)}\\
= \ & \frac{||h1_{[0,t]}||^{2(n-j)}_{L^2(R_+)}}{(n-j)!}\int_{t_1<\dots<t_j<t}\int_{[0,1]^j}|p_{t-t_j}(x,x_j)b(t_j)\dots P_{t_j}\eta(x_j)|^2dx_1\dots dx_jdt_1\dots dt_j
\end{aligned}
$$

$$\leq \frac{||\eta||_\infty^2 ||b||_\infty^{2j} ||h1_{[0,t]}||_{L^2(R_+)}^{2(n-j)}}{(n-j)!}$$

$$\times \int_{t_1<\ldots<t_j<t} \int_{[0,1]^j} p_{t-t_j}^2(x,x_j) p_{t_j-t_{j-i}}^2(x_j,x_{j-1}) p_{t_2-t_i}^2(x_2,x_1) dx_1\ldots dx_j dt_1 \ldots dt_j$$

$$= \frac{||\eta||_\infty^2 ||h1_{[0,t]}||_{L^2(R_+)}^{2(n-j)}}{(n-j)!} \frac{(\frac{t||b||_\infty^4}{4})^{j/2}}{\Gamma(\frac{j}{2}+1)},$$

which implies (8.13).

From (8.13) we obtain

$$||u||_{\lambda,T}^2 = \sum_{n=0}^{\infty} n!\lambda^{2n} \int_0^T \int_0^1 ||f_n^{t,x}||_{L^2((R_+\times[0,1])^{n+1})}^2 dxdt$$

$$\leq C_t^2 Y_t^2 ||\eta||_\infty^2 \{ \sum_{n=1}^{\infty} (n+1)\lambda^{2n} \sum_{j=0}^{n} \frac{||h1_{[0,t]}||_{L^2(R_+)}^{2(n-j)} (\frac{t||b||_\infty^4}{4})^{j/2}}{(n-j)!\Gamma(\frac{j}{2}+1)} \}$$

$$= C_t^2 Y_t^2 ||\eta||_\infty^2 \{ \sum_{j=0}^{n} \sum_{n=j}^{\infty} \frac{(n+1)\lambda^{2n} ||h||_{L^2(R_+)}^{2(n-j)} (\frac{t||b||_\infty^4}{4})^{j/2}}{(n-j)!\Gamma(\frac{j}{2}+1)} + \exp(\lambda^2 ||h||_{L^2(R_+)}^2)$$

$$= C_t^2 Y_t^2 ||\eta||_\infty^2 \{ \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} \frac{(j+k+1)\lambda^{2(j+k)} ||h||_{L^2(R_+)}^{2k}}{k!} \frac{(\frac{t||b||_\infty^4}{4})^{j/2}}{\Gamma(\frac{j}{2}+1)} \}$$

$$= C_t^2 Y_t^2 ||\eta||_\infty^2 \{ \sum_{k=0}^{\infty} \frac{\lambda^{2k} ||h||_{L^2(R_+)}^{2k}}{k!} \sum_{j=0}^{\infty} \frac{j(\frac{t\lambda^4||b||_\infty^4}{4})^{j/2}}{\Gamma(\frac{j}{2}+1)} + \lambda^2 ||h||_{L^2(R_+)}^2$$

$$\times \sum_{k=0}^{\infty} \frac{\lambda^{2(k-1)} ||h||_{L^2(R_+)}^{2(k-1)}}{k!} \sum_{j=0}^{\infty} \frac{j(\frac{t\lambda^4||b||_\infty^4}{4})^{j/2}}{\Gamma(\frac{j}{2}+1)} + \sum_{k=0}^{\infty} \frac{\lambda^{2k} ||h||_{L^2(R_+)}^{2k}}{k!} \sum_{j=0}^{\infty} \frac{j(\frac{t\lambda^4||b||_\infty^4}{4})^{j/2}}{\Gamma(\frac{j}{2}+1)} \},$$

and then

$$||u||_{\lambda,T}^2 \tag{8.34}$$

$$\leq C_t^2 Y_t^2 ||\eta||_\infty^2 (2+\lambda^2 ||h||_{L^2(R_+)}^2) \exp(\lambda^2 ||h||_{L^2(R_+)}^2) \sum_{j=1}^{\infty} \frac{(j+1)(\frac{t\lambda^4||b||_\infty^4}{4})^{j/2}}{\Gamma(\frac{j}{2}+1)}.$$

Clearly, (8.34) implies that $u \in H_\infty(L^2([0,T])$.

Uniqueness. Let $u \in H_\infty(L^2([0,T])$ be a weak solution. Define

$$\begin{aligned} v_t &= I_1(h_t) - \int_0^t h_r(r)b(r)dr = \int_0^t I_1(a^r)dr - \int_0^t h_r(r)b(r)dr, \\ y_t(x) &= \exp(-v_t)u_t(x), \\ X_t &= \int_0^t\int_0^1 \frac{1}{2}\frac{\partial^2}{\partial x^2}\varphi(x)u_s(x)dxds \\ &\quad + \int_0^t\int_0^1 \varphi(x)I_1(a^s)u_s(x)dxds + \int_0^t\int_0^1 \varphi(x)b(s)u_s(x)dW_{s,x}. \end{aligned}$$

We have

$$v_t' = I_1(a^t)h_t(t)b(t),\ \ D_{t,x}\exp(-v_t) = -h_t(t)\exp(-v_t).$$

From the definition for $\varphi \in C^2([0,1])$ with $\varphi(0) = \varphi(1) = 0$ we have

$$X_t = \int_0^1 \varphi(x)u_t(x)dx - \int_0^1 \varphi(x)\eta(x)dx.$$

Using the chain rule (see [103], [104] or Theorem 26) we obtain

$$\begin{aligned} &\int_0^1 \varphi(x)y_t(x)dx - \int_0^1 \varphi(x)\eta(x)dx \\ = &\int_0^t\int_0^1 \frac{1}{2}\frac{\partial^2}{\partial x^2}\varphi(x)y_s(x)dxds + \int_0^t\int_0^1 \varphi(x)b(s)y_s(x)dW_{s,x}. \end{aligned}$$

Thus we get that y is a weak solution of the following problem

$$\frac{\partial}{\partial t}y_t(x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}y_t(x) + b(t)y_t(x)\frac{\partial^2 W_{t,x}}{\partial t\partial x}, \tag{8.35}$$

$$y_t(0) = y_t(1) = 0, \tag{8.36}$$

$$y_0 = \eta. \tag{8.37}$$

To complete the proof we use Theorem 4.1 of Nualart and Zakai in [101] which states that problem (6.19)-(6.21) admits a unique weak solution. This implies that problem (8.35)-(8.37) has a unique solution. ■

We close this section with a result about some continuity property of solution to (8.1)-(8.3) and its bounded moment estimates.

Proposition 33 *Let u be the unique solution from Theorem 32. Then the following properties are satisfied.*

(i) For all $t \geq 0,\ x \in [0,1],$ we have

$$\sup_{0\leq t\leq T,\ 0\leq x\leq 1} ||u_t(x)||_\lambda < \infty \text{ for every } T>0,\ \lambda \geq 1. \tag{8.38}$$

(ii) For any $T>0,$ the mapping

$$(t,x) \to \{u_t(x)\}_{(t,x)\in[0,T]\times[0,1]} : [0,T]\times[0,1] \to H_\infty(L^2([0,T])$$

is continuous.

(iii) For every $1 \leq p < \infty,\ T>0,$ we have

$$\sup_{0\leq t\leq T,\ 0\leq x\leq 1} E|u_t(x)|^p < \infty\ . \tag{8.39}$$

Proof. The property (8.38) is a consequence of (8.34) while (8.39) follows from (8.34) and Proposition 28 (see also Proposition 2.2 of Uemura [120]). Consider a sequence (t^k, x^k) converging to (t,x). Then $f_n^{t^k,x^k}(t_1,\ldots,t_n,x_1,\ldots,x_n)$ converges to $f_n^{t,x}(t_1,\ldots,t_n,x_1,\ldots,x_n)$ as $k\to\infty$ for almost all $t_i, x_i,\ 1\leq i\leq n$. Then (8.13) and the dominated convergence theorem yield

$$||f_n^{t^k,x^k}(t_1,\ldots,t_n,x_1,\ldots,x_n) - f_n^{t,x}(t_1,\ldots,t_n,x_1,\ldots,x_n)||_{L^2((R_+\times[0,1])^n)} \overset{k\to\infty}{\to} 0. \tag{8.40}$$

Next, (8.40) and (8.38) imply that

$$||u_{t^k}(x) - u_t(x)||_\lambda^2 \overset{k\to\infty}{\to} 0,$$

and therefore the mapping

$$(t,x) \to u_t(x) : [0,T] \times [0,1] \to H_\infty(L^2([0,T])$$

is continuous. ■

8.2 The white noise case

Let $\{W_x : x \in [0,1]\}$ be a Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$, where $\mathcal{F}$ is the completion of the Borel $\sigma-$algebra generated by W.

In the next sequel we use some of the notations from Chapter 7 and Section 8.1.

Consider the linear stochastic heat equation

$$\frac{\partial}{\partial t} u_t(x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} u_t(x) + a(t) W_1 u_t(x) + b(t) u_t(x) \dot{W}, \tag{8.41}$$

$$u_t(0) = u_t(1) = 0, \tag{8.42}$$

$$u_0 = \eta, \tag{8.43}$$

where $a, b : R_+ \to R$ and $\eta : [0,1] \to R$ are measurable functions.

We impose the following assumptions on the coefficients of problem (8.1)-(8.3).

Assumption (##) The functions b and η are bounded and

$$\int_0^T |a(t)|^2 dt < \infty \text{ for all } T > 0.$$

Definition 34 *We say that a measurable process* $\{u_t(x)\}_{0 \le t \le 1,\ x \in [0,1]}$ *is a weak solution of (8.41)-(8.43) if for every* $t \in [0,1]$ *and* $\varphi \in C^2([0,1])$ *with* $\varphi(0) = \varphi(1) = 0$

we have $u_t(\cdot) \in Dom\ \delta$ *and*

$$\begin{aligned}
&\int_0^1 \varphi(x)u_t(x)dx - \int_0^1 \varphi(x)\eta(x)dx \\
= &\int_0^t\int_0^1 \frac{1}{2}\frac{\partial^2}{\partial x^2}\varphi(x)u_s(x)dxds + W_1\int_0^t\int_0^1 \varphi(x)a(s)u_s(x)dxds \\
&+\int_0^t[\int_0^1 \varphi(x)b(s)u_s(x)dW_x]ds.
\end{aligned} \tag{8.44}$$

We introduce

$$\begin{aligned}
h_t &= \int_0^t a(u)du,\ Y_t = \exp\{\frac{1}{2}h_t^2\}, \\
C_t &= \exp\{\int_0^t a(s)\int_0^s b(s_1)ds_1 ds\}.
\end{aligned}$$

Using similar ideas and techniques, the following results can be proved analogously to the ones from Section 8.1 (see further details in [114]).

The next theorem is about the equivalence of the notion of weak solution and the notion of mild solution for problem (8.41)-(8.43).

Theorem 35 *Let* $\{u_t(x)\}_{0\le t\le 1,\ x\in[0,1]}$ *be a measurable process which satisfies conditions of Theorem 26 from Section 7.1. Under Assumption (##) the following statements are equivalent.*

(1) u *is a weak solution of problem (8.41)-(8.43).*

(2) For every $t \ge 0$ *and* $\psi(s,x) \in C^{1,2}([0,t]\times[0,1])$ *with* $\psi(s,0) = \psi(s,1) = 0,$

$s \in [0,t]$ we have

$$\int_0^1 \psi(t,x)u_t(x)dx - \int_0^1 \psi(0,x)\eta(x)dx \tag{8.45}$$
$$= \int_0^t\int_0^1 [\frac{1}{2}\frac{\partial^2}{\partial x^2}\psi(s,x) + \frac{\partial}{\partial s}\psi(s,x)u_s(x)]dxds$$
$$+W_1\int_0^t\int_0^1 \psi(s,x)a(s)u_s(x)dxds + \int_0^t\int_0^1 \psi(s,x)b(s)u_s(x)dW_x ds.$$

(3) For almost every $\omega \in \Omega$ and for all t we have

$$u_t(x) \tag{8.46}$$
$$= P_t\eta(x) + W_1\int_0^t\int_0^1 p_{t-s}(x,y)a(s)u_s(y)dyds + \int_0^t\int_0^1 p_{t-s}(x,y)b(s)u_s(x)dW_{s,y}.$$

for almost all x.

Now we state the theorem on existence and uniqueness of solution to problem (8.41)-(8.43).

Theorem 36 *Under Assumption (##) the problem (8.41)-(8.43) has a unique solution in $H_\infty(L^2([0,T])$ for all $T > 0$ given by the chaos expansion*

$$u_t(x) = \sum_{n=0}^{\infty} I_n(f_n^{t,x}), \tag{8.47}$$
$$f_0^{t,x} = C_t Y_t P_t \eta(x) \tag{8.48}$$

and, for $n \geq 1$,

$$f_n^{t,x}(x_1,\ldots,x_n) \tag{8.49}$$
$$= \frac{1}{n!}C_tY_t[\sum_{j=1}^{n} h_t^{n-j} \sum_{(i_1,\ldots,i_j)\in\Delta_{j,n}} \int_{0<t_{i_1}<\ldots<t_{i_j}<t} p_{t-t_{i_j}}(x,x_{i_j})b(t_{i_j})p_{t_{i_j}-t_{i_{j-1}}}(x_{i_j},x_{i_{j-1}})$$
$$\times b(t_{i_{j-1}})\ldots P_{t_{i_1}}\eta(x_{i_1})dt_{i_2}\ldots dt_{i_j} + P_t\eta(x)h_t^n].$$

Moreover, for all $t \geq 0$, $x \in [0,1]$ *and* $n \geq 1$,

$$\begin{aligned} & ||f_n^{t,x}||^2_{L^2([0,1]^n)} \\ \leq \;& ||\eta||^2_\infty C_t^2 Y_t^2 \frac{n+1}{n!} \sum_{j=0}^{n} \frac{{h_t}^{2(n-j)}(\Gamma^2(\frac{3}{4})t^{3/2}||b||^2_\infty)^j}{[(n-j)!]^2\Gamma(\frac{3j}{4}+1)}. \end{aligned} \tag{8.50}$$

Finally, we formulate the proposition related to finite moment estimates and continuity property of solution to problem (8.41)-(8.43).

Proposition 37 *Let* u *be the unique solution from Theorem 36. Then the following properties are satisfied.*

(i) For all $t \geq 0$, $x \in [0,1]$, *we have*

$$\sup_{0 \leq t \leq T,\ 0 \leq x \leq 1} ||u_t(x)||_\lambda < \infty \text{ for every } T > 0,\ \lambda \geq 1. \tag{8.51}$$

(ii) For any $T > 0$, *the mapping*

$$(t,x) \to \{u_t(x)\}_{(t,x)\in[0,T]\times[0,1]} : [0,T] \times [0,1] \to H_\infty(L^2([0,T])$$

is continuous.

(iii) For every $1 \leq p < \infty$, $T > 0$, *we have*

$$\sup_{0 \leq t \leq T,\ 0 \leq x \leq 1} E|u_t(x)|^p < \infty\ . \tag{8.52}$$

The details of the above statements can be found in [114]. Since they are based on the same ideas and techniques used in Section 8.1, we skipped them here.

Chapter 9

Conclusions and Outlook

9.1 Conclusions

The objective of the thesis was the treatment of some classes of Dirichlet problems for linear parabolic SPDEs. The problems were considered in the thesis as follows.

Revision of existence and uniqueness results and probabilistic representations for solutions to the Dirichlet problem for linear parabolic SPDEs was done through the help of the transform which gave the correspondence between our SPDEs and the transformed PDEs with random coefficients. Construction of new approximations of solutions to the Dirichlet problem for linear parabolic SPDEs was obtained by approximating the stochastic characteristics; to this end, we exploited ideas of simplest random walks for deterministic Dirichlet problem from [92] together with partly weak, partly mean-square Euler method for the rest of the system of characteristics.

Error analysis (in the mean-square and almost sure senses) for these approximations was done and the implementation of the theoretical results through an illustrative numerical example yielded the expected order of convergence. Construction of the solution of a class of one dimensional anticipating linear heat equations was carried over the approach of Wiener chaos decomposition. Utilizing this chaos expansion, moment estimates and continuity property for these unique weak solutions were also obtained.

It is to be noted that we gave the talk which had the title 'Numerical solution of the Dirichlet problem for linear parabolic SPDEs based on averaging over characteristics' at NUMDIFF-12 (Numerical treatment of differential equations) hosted by Martin Luther University, Halle Wittenberg in September 2009. In there, we communicated all the results from the first part of this thesis which were also submitted as a paper (joint with M. Tretyakov; see also.[113]). The main results from the second part of this thesis are published in [114] and [115].

9.2 Outlook

We note that the assumptions together with compatibility conditions made in the first part of the thesis have ensured the existence of the classical solution with the properties described. They are sufficient for all the statements of this part of thesis. At the same time, they are not necessary and the numerical methods of the thesis can be used under broader conditions. These rather strong assumptions allow us to prove convergence of the proposed methods in a strong norm and with optimal orders. Weakening the conditions (especially, substituting the compatibility assumptions by using weighted spaces [74, 83, 89]) requires further study. Another area of further research could be the exploiting the other probabilistic representation of solution to SPDEs. In general the stochastic integral from the Feynmann-Kac representation in which two independent Brownian motions are involved may be difficult to handle. Among various representation formulas for the solutions to SPDEs there is the result obtained by Truman and Zhao in [118]. The authors gave some simplified representations for a class of stochastic heat equations by using the stochastic Hamilton Jacobi equation. This might lead to a new numerical method based on averaging over the characteristics.

Concerning the studied problem from the second part of this thesis, we were focused on the case when the drift lies in the first Wiener chaos. We demonstrated the utility of Wiener chaos expansion as a tool not only for the construction of the solution but also for some order estimates of its moments and further continuity properties of it. Therefore, further investigation is needed when the drift lies in other Wiener chaos with order different than one. Thus, utilizing again the Wiener chaos decomposition, it may be possible to achieve a general result when the drift is a random variable with finite moment of second order. Moreover, taking into account that the Wiener chaos expansion provides an approximation of the solution in the mean-square sense, some numerical experiments will need to be investigated. A suitable choice of the coefficients of the studied SPDEs it may provide results in this direction. Another possible numerical treatment of the considered anticipating SPDEs is to adapt and to exploit the method of characteristics for them in order to obtain new approximation schemes. It is also desirable to extend this technique to more general classes of anticipating SPDEs. Further investigation in this respect may be possible due to the results obtained by Lototsky and Rozovskii in [84]. In their work, the authors also studied a probabilistic representation of the Wiener chaos solution to a class of abstract stochastic evolution equation driven by a cylindrical Brownian motion. Under some assumptions on the initial condition and coefficients of the considered stochastic evolution equation, they derived a Feynmann-Kac type formula. There is also a need to extend numerical approximations based on the method of characteristics to SPDEs with fractional white noise potential. A Feynmann-Kac formula for the Cauchy problem for the heat equation driven by the fractional Brownian sheet was communicated by Jiang Song from University of Kansas at The Third Annual Graduate Student Conference in Probability hosted by the Department of Statistics and Operations Research at UNC- Chapel Hill and the Department of Mathematics at Duke University in May 2009. To sum up, utilizing the mentioned results, it will be useful to see how the method of characteristics can be exploited to obtain new numerical methods to other anticipating SPDEs.

Appendix A

In this appendix we present some details concerning the transformed PDE (3.9)-(3.10) with random coefficients from Chapter 3, Section 3.2.

For instance, let v be the solution of problem (3.1)-(3.2), $d = 3$, $q = r$, $\xi^1 = v$, $\xi^2 = \eta$ and $\xi^3 = \varsigma$, where v, η and ς were defined in Section 3.2 by the relations (3.1)-(3.2), (3.6) and (3.7), respectively.

Consider

$$H(y_1, y_2, y_3) = y_1 \exp(-y_2) - y_3.$$

By Theorem 7 (the backward Itô formula-see also [111]) from Section 2.1, we obtain that u defined in Section 2.1 by the transform (3.15) verifies the transformed PDE (3.9)-(3.10) with the corresponding random coefficients B, C, F and Φ given by relations (3.11), (3.12), (3.13) and (3.14), respectively. Vice-versa if problem (3.9)-(3.10) admits a classical solution u, it is possible to apply Theorem 7 (the backward Itô formula-see also [111]) from Section 2.1 and thus we get that problem (3.1)-(3.2) has a solution with the required regularity.

Appendix B

In this appendix we present some details concerning the probabilistic representations from Section 3.3.

We would like to prove that the solution of the problem (3.1)-(3.2) has the following probabilistic representation:

$$v(t,x) = E^w\left[\varphi(\tau, X_{t,x}(\tau))Y_{t,x,1}(\tau) + Z_{t,x,1,0}(\tau)\right], \quad T_0 \le t \le T, \tag{9.1}$$

where $X_{t,x}(s)$, $Y_{t,x,y}(s)$, $Z_{t,x,y,z}(s)$, $s \ge t$, $(t,x) \in Q$, is the solution of the SDEs

$$dX = b(s,X)ds + \sigma(s,X)dW(s), \quad X(t) = x, \tag{9.2}$$

$$dY = c(s,X)Yds + \beta^{\top}(s,X)Ydw(s), \quad Y(t) = y, \tag{9.3}$$

$$dZ = f(s,X)Yds + \gamma^{\top}(s,X)Ydw(s), \quad Z(t) = z, \tag{9.4}$$

$W(s) = (W_1(s), \ldots, W_d(s))^{\top}$ is a d-dimensional standard Wiener process independent of $w(s)$, $\tau = \tau_{t,x}$ is the first exit time of the trajectory $(s, X_{t,x}(s))$ to the boundary Γ and $\sigma(t,x)$ is a $d \times d$-matrix obtained from the equation

$$\sigma(t,x)\sigma^{\top}(t,x) = a(t,x).$$

The expectation E^w in (9.1) is taken over the realizations of $W(s)$, $t \le s \le T$, for a fixed $w(s)$, $t \le s \le T$; in other words, $E^w(\cdot)$ means the conditional expectation $E\left(\cdot | w(s) - w(t), t \le s \le T\right)$. We note that the exit time $\tau_{t,x}$ does not depend on $w(\cdot)$.

As it is known [31, 37], the solution $u(t,x)$ of deterministic PDEs with random coefficients (3.9)-(3.10) has the probabilistic representation:

$$u(t,x) = E^{w}\left[\Phi(\tau, X_{t,x}(\tau))\mathbb{Y}_{t,x,1}(\tau) + \mathbb{Z}_{t,x,1,0}(\tau)\right], \quad T_0 \le t \le T, \tag{9.5}$$

where $X_{t,x}(s)$, $Y_{t,x,y}(s)$, $Z_{t,x,y,z}(s)$, $s \ge t$, $(t,x) \in Q$, is the solution of the system

$$dX = b(s,X)ds + \sigma(s,X)dW(s), \quad X(t) = x, \tag{9.6}$$

$$d\mathbb{Y} = C(s,X)\mathbb{Y}ds + \nabla^{\top}\eta(s;X)\sigma(s,X)\mathbb{Y}dW(s), \quad \mathbb{Y}(t) = y, \tag{9.7}$$

$$d\mathbb{Z} = F(s,X)\mathbb{Y}ds, \quad \mathbb{Z}(t) = z. \tag{9.8}$$

and the coefficients Φ, C, η and F were introduced in Section 3.2.

Let $\sigma(R^q)$ be the Borel $\sigma-$ algebra on R^q. Using the monotone class theorem in order to prove (9.1), (9.2)-(9.4) it will be enough to show that for any $t \le s_1 \le \ldots \le s_n \le T$ and for any $A_1, \ldots, A_n \in \sigma(R^q)$ we have:

$$\begin{aligned} & \int_{\Omega} \mathbf{1}_{\{w(s_1)-w(t)\in A_1,\ldots,w(s_n)-w(s_{n-1})\in A_n\}} v(t,x)dP \\ = & \int_{\Omega} \mathbf{1}_{\{w(s_1)-w(t)\in A_1,\ldots,w(s_n)-w(s_{n-1})\in A_n\}} [\varphi(\tau, X_{t,x}(\tau))Y_{t,x,1}(\tau) + Z_{t,x,1,0}(\tau)]dP \end{aligned} \tag{9.9}$$

Using the transform (3.15) defined in Section 3.2 (see also Appendix A), we obtain:

$$\begin{aligned} & \int_{\Omega} \mathbf{1}_{\{w(s_1)-w(t)\in A_1,\ldots,w(s_n)-w(s_{n-1})\in A_n\}} v(t,x)dP \\ = & \int_{\Omega} \mathbf{1}_{\{w(s_1)-w(t)\in A_1,\ldots,w(s_n)-w(s_{n-1})\in A_n\}} [e^{\eta(t;x)}u(t,x) + e^{\eta(t;x)}\varsigma(t;x)]dP, \end{aligned}$$

where $\eta(t;x)$ and $\varsigma(t;x)$ are defined in (3.6) and (3.7), respectively.

Now from the probabilistic representation (9.5), (9.6)-(9.8) and by the independence of W of w we get:

$$\begin{aligned}
&\int\limits_\Omega \mathbf{1}_{\{w(s_1)-w(t)\in A_1,\ldots,w(s_n)-w(s_{n-1})\in A_n\}} v(t,x)dP \qquad (9.10)\\
=\ &\int\limits_\Omega\int\limits_\Omega \mathbf{1}_{\{w(s_1)(\omega)-w(t)(\omega)\in A_1,\ldots,w(s_n)(\omega)-w(s_{n-1})(\omega)\in A_n\}}\{e^{\eta(t;x)(\omega)}[\Phi(\tau(\omega'),X_{t,x}(\tau(\omega')))\\
&\times\mathbb{Y}_{t,x,1}(\tau(\omega'))+\mathbb{Z}_{t,x,1,0}(\tau(\omega'))]+e^{\eta(t;x)(\omega)}\varsigma(t;x)(\omega)\}dP(\omega')dP(\omega).
\end{aligned}$$

As a result, in order to prove relation (9.9) it will be enough to show

$$\begin{aligned}
&\int\limits_\Omega \mathbf{1}_{\{w(s_1)-w(t)\in A_1,\ldots,w(s_n)-w(s_{n-1})\in A_n\}}\varphi(\tau,X_{t,x}(\tau))Y_{t,x,1}(\tau)dP \qquad (9.11)\\
=\ &\int\limits_\Omega \mathbf{1}_{\{w(s_1)(\omega)-w(t)(\omega)\in A_1,\ldots,w(s_n)(\omega)-w(s_{n-1})(\omega)\in A_n\}}e^{\eta(t;x)(\omega)}\Phi(\tau(\omega'),X_{t,x}(\tau(\omega')))\\
&\times\mathbb{Y}_{t,x,1}(\tau(\omega'))dP(\omega')dP(\omega),\\
&\int\limits_\Omega \mathbf{1}_{\{w(s_1)-w(t)\in A_1,\ldots,w(s_n)-w(s_{n-1})\in A_n\}}Z_{t,x,1,0}(\tau)]dP \qquad (9.12)\\
=\ &\int\limits_\Omega\int\limits_\Omega \mathbf{1}_{\{w(s_1)(\omega)-w(t)(\omega)\in A_1,\ldots,w(s_n)(\omega)-w(s_{n-1})(\omega)\in A_n\}}\{e^{\eta(t;x)(\omega)}\mathbb{Z}_{t,x,1,0}(\tau(\omega'))\\
&+e^{\eta(t;x)(\omega)}\varsigma(t;x)(\omega)\}dP(\omega')dP(\omega).
\end{aligned}$$

Let us prove relation (9.11). Relation (9.12) can be proved using similar arguments in an analogous way.

We can solve the scalar equation (9.3) and write

$$\begin{aligned}
Y(s) &= y\exp\{\int\limits_t^s[c(u,X(u))-\frac{1}{2}\sum_{k=1}^r\beta_r^2(u,X(u))]du\\
&\quad+\sum_{k=1}^r\int\limits_t^s\beta_r(u,X(u))dw_r(u)\},\ \ s\geq t.
\end{aligned}$$

We can also solve the scalar equation (9.7) and write

$$
\begin{aligned}
\mathbb{Y}(s) &= y\exp\{\int_t^s [C(u,X(u)) - \frac{1}{2}\sum_{k=1}^{r}\beta_r^2(u,X(u)) + b^T(u,X(u))\nabla\eta(u;X(u)) \\
&\quad + \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(u,X(u))\frac{\partial^2}{\partial x_i \partial x_j}\eta(u;X(u))]du \\
&\quad + \int_t^s \nabla^\top\eta(u;X(u))\sigma(u,X(u))dW(u)\},\ s \geq t.
\end{aligned}
$$

Therefore in order to prove relation (9.11), it will be enough to prove

$$
\begin{aligned}
&\int_\Omega \mathbf{1}_{\{w(s_1)-w(t)\in A_1,\ldots,w(s_n)-w(s_{n-1})\in A_n\}}\varphi(\tau, X_{t,x}(\tau)) \\
&\times\exp\{\int_t^\tau [c(u,X(u)) - \frac{1}{2}\sum_{k=1}^{r}\beta_r^2(u,X(u))]du + \sum_{k=1}^{r}\int_t^\tau \beta_r(u,X(u))dw_r(u)\}]dP \\
= &\int_\Omega\int_\Omega \mathbf{1}_{\{w(s_1)(\omega)-w(t)(\omega)\in A_1,\ldots,w(s_n)(\omega)-w(s_{n-1})(\omega)\in A_n\}} e^{\eta(t;x)(\omega)}\varphi(\tau(\omega'), X_{t,x}(\tau(\omega'))) \\
&\times\exp\{-\eta(\tau(\omega'), X_{t,x}(\tau(\omega'))) + \int_t^{\tau(\omega')}[c(u,X(u)(\omega')) - \frac{1}{2}\sum_{k=1}^{r}\beta_r^2(u,X(u)(\omega'))]du \\
&+\int_t^{\tau(\omega')}[b^T(u,X(u)(\omega'))\nabla\eta(u;X(u)(\omega')) + \frac{1}{2}\sum_{i,j=1}^{d}a^{ij}(u,X(u)(\omega')) \\
&\times\frac{\partial^2}{\partial x_i\partial x_j}\eta(u;X(u)(\omega'))]du + \int_t^{\tau(\omega')}\nabla^\top\eta(u;X(u)(\omega')) \\
&\times\sigma(u,X(u)(\omega'))dW(u)(\omega')\}dP(\omega')dP(\omega).
\end{aligned}
\tag{9.13}
$$

Next let us fix r and let $g(t,x_1,\ldots,x_d,x_{d+1}) = \beta_r(t,x_1,\ldots,x_d)x_{d+1}$.

For

$$d\xi_i = \alpha^i + \sum_{s=1}^{q}\sigma'^{is}dw'_s,\ i = 1,\ldots,d+1,$$

we have the following forward Itô's formula which is also true for time t replaced by a stopping time τ (see [82]):

$$\begin{aligned} dg(t,\xi) &= (\frac{\partial g}{\partial t}(t,\xi) + \frac{1}{2}\sum_{i,j=1}^{d+1}\frac{\partial^2 g}{\partial x_i \partial x_j}(t,\xi)(\sum_{s=1}^{q}\sigma'^{is}\sigma'^{js}) \\ &\quad + \sum_{i=1}^{d+1}\frac{\partial g}{\partial x_i}(t,\xi)\alpha^i)dt + \sum_{s=1}^{q}\sum_{i=1}^{d+1}\frac{\partial g}{\partial x_i}(t,\xi)\sigma'^{is}dw'_s(t). \end{aligned}$$

Let us take $q = d+1$ and

$$\begin{aligned} \alpha^i &= b^i,\ i=\overline{1,d},\ \sigma'^{is} = \sigma^{is}, i = \overline{1,d},\ s=\overline{1,d}, \\ \sigma'^{id+1} &= 0,\ i=\overline{1,d},\ \alpha^{d+1}=0,\ \sigma'^{d+1s}=0,\ s=\overline{1,d}, \\ \sigma'^{d+1d+1} &= 1,\ w'_s = W_s,\ s=\overline{1,d},\ w'_{d+1} = w_r. \end{aligned}$$

We have the following relations

$$\begin{aligned} \frac{\partial g}{\partial t}(t,x_1,\ldots,x_d,x_{d+1}) &= \frac{\partial \beta_r}{\partial t}(t,x_1,\ldots,x_d)x_{d+1}, \\ \frac{\partial g}{\partial x_i}(t,x_1,\ldots,x_d,x_{d+1}) &= \frac{\partial \beta_r}{\partial x_i}(t,x_1,\ldots,x_d)x_{d+1},\ i=\overline{1,d}, \\ \frac{\partial g}{\partial x_{d+1}}(t,x_1,\ldots,x_d,x_{d+1}) &= \beta_r(t,x_1,\ldots,x_d), \\ \frac{\partial^2 g}{\partial x_i \partial x_j}(t,x_1,\ldots,x_d,x_{d+1}) &= \frac{\partial^2 \beta_r}{\partial x_i \partial x_j}(t,x_1,\ldots,x_d)x_{d+1},\ i,j=\overline{1,d}, \\ \frac{\partial^2 g}{\partial x_i \partial x_{d+1}}(t,x_1,\ldots,x_d,x_{d+1}) &= \frac{\partial \beta_r}{\partial x_i}(t,x_1,\ldots,x_d),\ i=\overline{1,d}, \\ \frac{\partial^2 g}{\partial x_{d+1}^2}(t,x_1,\ldots,x_d,x_{d+1}) &= 0. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} &\sum_{r=1}^{q}\beta_r(\tau,X(\tau))w_r(\tau) \qquad (9.14) \\ =\ &\sum_{r=1}^{q}\beta_r(t,X(t))w_r(t) + \sum_{r=1}^{q}\int_t^\tau(\frac{\partial \beta_r}{\partial s}(s,X(s))w_r(s) \\ &+\frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial^2\beta_r}{\partial x_i\partial x_j}(s,X(s))w_r(s)a^{ij}(s,X(s)) + \sum_{i=1}^{d}\frac{\partial \beta_r}{\partial x_i}(s,X(s))w_r(s) \end{aligned}$$

$$\times b^i(s,X(s)) + \sum_{r=1}^{q}\sum_{k=1}^{d}\int_t^{\tau}\sum_{i=1}^{d}\frac{\partial\beta_r}{\partial x_i}(s,X(s))w_r(s)\sigma^{ik}(s,X(s))dW_k(s)$$
$$+\sum_{r=1}^{q}\int_t^{\tau}\beta_r(s,X(s)))dw_r(s).$$

From the Itô formula or integration by parts formula, we have

$$\eta(s;x) = \sum_{r=1}^{q}\{\beta_r(T,x)w_r(T) - \beta_r(s,x))w_r(s) - \int_s^T\frac{\partial\beta_r}{\partial u}(u,x)w_r(u)du\}. \tag{9.15}$$

Again let us fix r. Applying Itô formula to

$$f(t,x) = \int_t^T\frac{\partial\beta_r}{\partial u}(u,x)w_r(u)du$$

and utilizing the independence of w of W, we obtain

$$\begin{aligned}
&\int_s^T\frac{\partial\beta_r}{\partial u}(u,X(s))w_r(u)du \qquad (9.16)\\
= \;&\int_t^T\frac{\partial\beta_r}{\partial u}(u,x)w_r(u)du + \int_t^s\{-\frac{\partial\beta_r}{\partial u}(u,X(u))w_r(u) + b^T(u,X(u))\nabla\int_u^T\frac{\partial\beta_r}{\partial u_1}(u_1,X(u))\\
&\times w_r(u_1)du_1 + \frac{1}{2}\sum_{i,j=1}^{d}a^{ij}(u,X(u))\frac{\partial^2}{\partial x_i\partial x_j}[\int_u^T\frac{\partial\beta_r}{\partial u_1}(u_1,X(u))w_r(u_1)du_1]\}du\\
&+\int_t^s\nabla^\top[\int_u^T\frac{\partial\beta_r}{\partial u_1}(u_1,X(u))w_r(u_1)du_1]\sigma(u,X(u))dW(u).
\end{aligned}$$

From the Itô formula we also have

$$\begin{aligned}
&\sum_{r=1}^{q}\beta_r(T,X(\tau)) \qquad (9.17)\\
= \;&\sum_{r=1}^{q}\beta_r(T,x) + \sum_{r=1}^{q}\int_t^{\tau}\{\frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial^2\beta_r}{\partial x_i\partial x_j}(T,X(s))w_r(s)a^{ij}(s,X(s))\\
&+b^T(T,X(s))\nabla^T\beta_r(T,X(s))\}ds + \sum_{r=1}^{q}\int_t^{\tau}\nabla^T\beta_r(T,X(s))\sigma(s,X(s))dW(s).
\end{aligned}$$

Now using the fact that w is independent of W, the relation (9.13) follows from relations (9.14), (9.15), (9.16) and (9.17).

Thus, we proved the relation (9.11).

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