Supplement to data-adaptive estimation of time-varying spectral densities

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Abstract

This supplement contains technical details on the distributional properties of the adaptive estimator as defined in the main paper "Data-adaptive estimation of time-varying spectral densities". Using the setting of empirical spectral processes (e.g. Dahlhaus and Polonik 2009, Dahlhaus 2009), we conclude that asymptotic properties of the nonadaptive estimator carry over to the adaptive estimator under local homogeneity.

Keywords: Local stationary processes, data-adaptive kernel estimation

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1 Asymptotic properties of $\tilde{f}^{(k)}(u, \lambda)$

The objective of this supplement is to provide some intuition on the distributional properties of the estimator

$$\tilde{f}^{(k)}(u_r, \lambda_i) = \tfrac{1}{\tilde{N}^{(k)}(r,i)} \sum_{(s,j)\in\in B^{(k)}(r,i)} \tilde{W}^{(k)}_{r,i}(s,j)\, J_T\big(u_s, \lambda_j\big) \tag{1}$$

with the adapted kernel weights

$$\tilde{W}^{(k)}_{r,i}(s,j) = K_{\mathrm{f}}\big(\tfrac{\lambda_i - \lambda_j}{b^{(k)}_{\mathrm{f}}}\big)\, K_{\mathrm{t}}\big(\tfrac{u_s - u_r}{b^{(k)}_{\mathrm{t}}}\big)\, P^{(k)}_{r,i}(s,j), \tag{2}$$

where $\tilde{N}^{(k)}(r,i) = \sum_{s,j} \tilde{W}^{(k)}_{r,i}(s,j)$ and $B^{(k)}(r,i) = \{(s,j) : |u_s - u_r| < b^{(k)}_{\mathrm{t},T}, |\lambda_j - \lambda_i| < b^{(k)}_{\mathrm{f},T}\}$ for $k = 0, \ldots, k_{\max}$ and where

$$\tilde{P}^{(k)}_{r,i}(s,j) = K_P\Big(\big(\hat{f}^{(k-1)}(u_r, \lambda_i), \hat{f}^{(k-1)}(u_s, \lambda_j)\big)\Big).$$

We start with some necessary background on empirical spectral processes (e.g. Dahlhaus and Polonik 2009, Dahlhaus 2009). Generally, the empirical spectral process for arbitrary index functions ϕ is defined by

$$E_T(\phi) = \sqrt{T} \int_{-\pi}^{\pi} \big(F_T(\phi) - F(\phi)\big),$$

where

$$F(\phi) = \int_0^1 \int_{-\pi}^{\pi} \phi(u, \lambda)\, f(u, \lambda)\, du\, d\lambda$$

is the generalized spectral measure and

$$F_T(\phi) = \tfrac{1}{T} \sum_{t=1}^{T} \int_{-\pi}^{\pi} \phi\big(\tfrac{t}{T}, \lambda\big)\, J_T(\tfrac{t}{T}, \lambda)\, d\lambda$$

denotes the corresponding empirical spectral measure. For particular classes of index functions independent of T, a functional central limit theorem has been proved (Dahlhaus and Polonik 2009, Theorem 2.11). Additionally, for index functions depending on T a central limit theorem has been derived (Dahlhaus 2009, Theorem 3.2).

Many localized statistics for non-stationary time series can be written in terms of the empirical spectral measure. In particular, we obtain the continuous version of the non-adaptive time-varying spectral estimator

$$\hat{f}_T(u,\lambda) = \frac{1}{C}\sum_{s,j} K_{\mathrm{f}}\Big(\frac{\lambda-\lambda_j}{b_{\mathrm{f},T}}\Big)\, K_{\mathrm{t}}\Big(\frac{u-s/T}{b_{\mathrm{t},T}}\Big)\, J_T\big(\tfrac{s}{T},\lambda_j\big), \tag{3}$$

where $\lambda_j = \frac{\pi j}{T}$ for $j = 1-T,\dots,T$ denote the Fourier frequencies and $C = \sum_{s,j} K_{\mathrm{f}}\big((\lambda-\lambda_j)/b_{\mathrm{f},T}\big)\, K_{\mathrm{t}}\big((u-s/T)/b_{\mathrm{t},T}\big)$, by considering index functions

$$\phi^{(T)}_{u,\lambda}(v,\mu) = \tfrac{1}{b_{\mathrm{t},T}\, b_{\mathrm{f},T}}\, K_{\mathrm{t}}\big(\tfrac{u-v}{b_{\mathrm{t},T}}\big)\, K_{\mathrm{f}}\big(\tfrac{\lambda-\mu}{b_{\mathrm{f},T}}\big). \tag{4}$$

Since the index functions depend on T, asymptotic normality of the estimator $F_T(\phi^{(T)}_{u,\lambda})$ and its discretized version (3) follows from Theorem 3.2 and Example 4.1 of Dahlhaus (2009) under the following additional conditions.

Assumption 1.1.

(i) The time-varying spectral density $f(u,\lambda)$ is twice differentiable in u and λ with uniformly bounded derivatives.

(ii) The bandwidths satisfy $b_{\mathrm{t},T}, b_{\mathrm{f},T} \to 0$ and $b_{\mathrm{t},T} b_{\mathrm{f},T} T \gg \log(T)^2$ as $T\to\infty$,

(iii) The kernels K_{t} and K_{f} are of bounded variation with compact support. Moreover, $\int x\, K_{\mathrm{t}}(x)\, dx = 0$ and $\int K_{\mathrm{t}}(x)\, dx = 1$ and analogously for K_{f}.

In particular, we find that

$$b_{\mathrm{t},T}\, b_{\mathrm{f},T}\, \mathrm{var}\Big(E_T(\phi^{(T)}_{u,\lambda})\Big) \to 2\pi\, f^2(u,\lambda)\, \kappa_{\mathrm{t}}\, \kappa_{\mathrm{f}}.$$

Furthermore, estimators at different points in the time-frequency plane are asymptotically independent.

For the adaptive estimator in (1) similar asymptotic results cannot be derived easily since the final smoothing kernel is iteratively defined and depends on the spectral estimators in previous steps through penalization and the memory step. In the following, we therefore provide at least heuristic arguments that under homogeneity of the spectral density penalization has a negligible effect and hence the estimator remains consistent and asymptotically normal.

More precisely assume that $f(u,\lambda) = f$ for all u and λ and define for fixed $u \in [0,1]$ and $\lambda \in [-\pi,\pi]$ the functions

$$\psi_{\alpha,\beta} = \phi^{(T)}_{u+\alpha b_{\mathrm{t},T},\lambda+\beta b_{\mathrm{f},T}} \tag{5}$$

where $\phi^{(T)}_{u,\lambda}$ is defined as above. Then the family of index functions $\mathscr{F}_0 = \{\psi_{\alpha,\beta} | \alpha,\beta \in [-1,1]\}$ satisfies the conditions of Theorem 2.11 of Dahlhaus and Polonik (2009). Hence the penalty statistic $\Delta(\hat{f}^{(k)}(u,\lambda), \hat{f}^{(k)}(u+\alpha b_{\mathrm{t},T}, \mu + \beta b_{\mathrm{f},T}))$ asymptotically has that same distribution as

$$\frac{b_{\mathrm{t},T}\, b_{\mathrm{f},T}}{4\pi\, \kappa_{\mathrm{f}}\, \kappa_{\mathrm{t}}\, f^2} \big(E(\psi_{0,0}) - E(\psi_{\alpha,\beta})\big)^2, \tag{6}$$

where $E(\psi)$ is a Gaussian process with mean zero and covariances

$$\begin{aligned} b_{\mathrm{t},T}\, b_{\mathrm{f},T} \operatorname{cov}\big(E(\psi_{\alpha,\beta}), E(\psi_{\gamma,\delta})\big) = 2\pi\, f^2 \int_{-1/2}^{1/2} \int_{-\pi}^{\pi} & K_{\mathrm{t}}(\alpha-u)\, K_{\mathrm{t}}(\gamma-u)\, K_{\mathrm{f}}(\beta-\lambda) \\ & \times \big[K_{\mathrm{f}}(\delta-\lambda) + K_{\mathrm{f}}(\delta+\lambda)\big]\, du\, d\lambda + O(b_{\mathrm{f},T}). \end{aligned}$$

The expression shows that under the assumption of homogeneity of the time-varying spectrum over the local neighborhood about the point (u,λ) the distribution of the penalty statistic does not depend on the bandwidth or the sample size but through a term of order $O(b_{\mathrm{f},T})$. Moreover, the strong positive correlation of the Gaussian process $E(\psi)$ leads to at most weak penalization towards the borders of the local neighborhood yielding a total smoothing kernel that differs only slightly from the non-adaptive smoothing kernel. Finally, since $E(\psi_{0,0})$ and $E(\psi_{\alpha,\beta})$ are positively correlated, the variance of their difference can be bounded by $2\operatorname{var}\big(E(\psi_{0,0})\big)$ uniformly for all $\alpha,\beta \in [-1,1]$, which justifies the use of the χ^2_1-distribution for determining the cut-off point of the penalty kernel.

We note that the same covariance structure can be derived from Theorem 3.2 of Dahlhaus (2009) by considering the index functions $\phi^{(T)}_{u+\alpha b_{\mathrm{t},T},\lambda+\beta b_{\mathrm{f},T}}$ directly, that is, taking their dependence on T into account in the asymptotics. However, the result is weaker insofar it does not yield convergence over the whole local neighborhood defined by $\alpha, \beta \in [-1, 1]$ simultaneously. Although the above arguments based on fixed index functions indicate that this result could be strengthened, a derivation of a functional central limit theorem in this setting is beyond the scope of this paper.

Summarizing we find that under the assumption of homogeneity penalization does only modify the shape of the smoothing kernel even if applied iteratively multiple times but will keep the rates approximately the same. In contrast, in case of a non-constant spectral density, the penalty statistic depends quadratically on the difference in levels which leads to more severe penalization as bandwidths in time and frequency direction increase. Accordingly, the resulting smoothing kernel will have in general a smaller support corresponding to a smaller bandwidth than the one actually imposed. Nevertheless, in the setting of locally stationary processes this effect will disappear asymptotically since the level of local homogeneity increases as long as the bandwidths used in the iteration satisfy the conditions in Assumption 1.1. In other words, the adaptive estimator remains consistent with rate $\sqrt{T b_{\mathrm{t},T}^{(k_{\max})} b_{\mathrm{f},T}^{(k_{\max})}}$ since its adaptiveness only shows in finite samples. This is even true when the dynamics of the process exhibit structural breaks and thus should be described by a piecewise locally stationary process. In that case, penalization will be strong in the local neighborhood of a break leading to asymmetric smoothing kernels that seem to be cut off. Again, since the local neighborhoods (in the rescaled time-frequency plane) are shrinking for increasing sample size, the effect will disappear but for the points along the breaks where the time-varying spectral density is not well-defined. Examples of such processes with structural breaks are discussed in Section 4 of the main paper, where we illustrate the final sample behavior of the adaptive estimator by simulations.

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