Comparison of disease prevalence in two populations under double-sampling scheme with two fallible classifiers

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SUPPLEMENTARY MATERIALS

Web Appendix

A. The derivation of Score test statistic in two models

The Fisher information matrix under $H_0 : \delta = 0$ is given by

$$I(\pi_1, \eta, \theta) = \begin{pmatrix} I^0_{11} & I^0_{12} & I^0_{13} & I^0_{14} \\ I^0_{12} & I^0_{22} & I^0_{23} & I^0_{24} \\ I^0_{13} & I^0_{23} & I^0_{33} & I^0_{34} \\ I^0_{14} & I^0_{24} & I^0_{34} & I^0_{44} \end{pmatrix},$$

and its inverse matrix is given by

$$I^{-1}(\pi_1, \eta, \theta) = \begin{pmatrix} I^{11} & I^{12} & I^{13} & I^{14} \\ I^{12} & I^{22} & I^{23} & I^{24} \\ I^{13} & I^{23} & I^{33} & I^{34} \\ I^{14} & I^{24} & I^{34} & I^{44} \end{pmatrix},$$

where I^{11} is the first main-diagonal element of the inverse of the Fisher information matrix $I(\pi_1, \eta, \theta)$ under $H_0 : \delta = 0$, which is given by

$$I^{11} = [I^0_{11} - \begin{pmatrix} I^0_{12} & I^0_{13} & I^0_{14} \end{pmatrix} \begin{pmatrix} I^0_{22} & I^0_{23} & I^0_{24} \\ I^0_{23} & I^0_{33} & I^0_{34} \\ I^0_{24} & I^0_{34} & I^0_{44} \end{pmatrix}^{-1} \begin{pmatrix} I^0_{12} \\ I^0_{13} \\ I^0_{14} \end{pmatrix}]^{-1}$$

where

$$I_{11}^0 = \frac{n_2\theta(1-\eta)+N_2\eta}{\pi_1} + \frac{n_2(\eta+\theta-\eta\theta)^2}{1-\pi_1(\eta+\theta-\eta\theta)} + \frac{(N_2-n_2)\eta^2}{1-\pi_1\eta}, \quad I_{12}^0 = I_{11}^0,$$
$$I_{13}^0 = \frac{n_2(1-\theta)}{1-\pi_1(\eta+\theta-\eta\theta)} + \frac{N_2-n_2}{1-\pi_1\eta}, \quad I_{14}^0 = \frac{n_2(1-\eta)}{1-\pi_1(\eta+\theta-\eta\theta)},$$
$$I_{22}^0 = \frac{(n_1+n_2)(\eta+\theta-\eta\theta)+(N_1-n_1+N_2-n_2)\eta}{\pi_1} + \frac{(n_1+n_2)(\eta+\theta-\eta\theta)^2}{1-\pi_1(\eta+\theta-\eta\theta)} + \frac{(N_1-n_1+N_2-n_2)\eta^2}{1-\pi_1\eta},$$
$$I_{23}^0 = \frac{(n_1+n_2)(1-\theta)}{1-\pi_1(\eta+\theta-\eta\theta)} + \frac{N_1-n_1+N_2-n_2}{1-\pi_1\eta}, \quad I_{24}^0 = \frac{(1-\eta)(n_1+n_2)}{1-\pi_1(\eta+\theta-\eta\theta)},$$
$$I_{33}^0 = \frac{(N_1+N_2)\pi_1}{\eta} + \frac{(n_1+n_2)\pi_1\theta}{1-\eta} + \frac{(n_1+n_2)(1-\theta)^2\pi_1^2}{1-\pi_1(\eta+\theta-\eta\theta)} + \frac{(N_1-n_1+N_2-n_2)\pi_1^2}{1-\pi_1\eta},$$
$$I_{34}^0 = -\frac{(n_1+n_2)\pi_1(1-\pi_1)}{1-\pi_1(\eta+\theta-\eta\theta)}, I_{44}^0 = \frac{(n_1+n_2)\pi_1}{\theta} + \frac{(n_1+n_2)\pi_1\eta}{1-\theta} + \frac{(n_1+n_2)\pi_1^2(1-\eta)^2}{1-\pi_1(\eta+\theta-\eta\theta)}$$

in Model I, and

$$I_{11}^0 = \frac{n_2+(N_2-n_2)\eta}{\pi_1} + \frac{n_2}{1-\pi_1} + \frac{(N_2-n_2)\eta^2}{1-\pi_1\eta}, I_{12}^0 = I_{11}^0, I_{13}^0 = \frac{N_2-n_2}{1-\pi_1\eta}, I_{14}^0 = 0,$$
$$I_{22}^0 = \frac{(n_1+n_2)+(N_1-n_1+N_2-n_2)\eta}{\pi_1} + \frac{n_1+n_2}{1-\pi_1} + \frac{(N_1-n_1+N_2-n_2)\eta^2}{1-\pi_1\eta},$$
$$I_{23}^0 = \frac{N_1-n_1+N_2-n_2}{1-\pi_1\eta}, I_{24}^0 = 0,$$
$$I_{33}^0 = \frac{(N_1-n_1+N_2-n_2)\pi_1}{\eta} + \frac{(n_1+n_2)\pi_1}{1-\eta} + \frac{(n_1+n_2)\pi_1}{\eta+\theta-1} + \frac{(N_1-n_1+N_2-n_2)\pi_1^2}{1-\pi_1\eta},$$
$$I_{34}^0 = \frac{(n_1+n_2)\pi_1}{\eta+\theta-1}, I_{44}^0 = \frac{(n_1+n_2)\pi_1}{1-\theta} + \frac{(n_1+n_2)\pi_1}{\eta+\theta-1}$$

in Model II, respectively. Therefore, the Score statistic for testing the null hypothesis $H_0 : \delta = 0$ is given by

$$T_{sc} = \frac{\partial l}{\partial \delta}|_{\delta=0,\pi_1=\tilde{\pi}_1,\eta=\tilde{\eta},\theta=\tilde{\theta}} \sqrt{I^{11}|_{\pi_1=\tilde{\pi}_1,\eta=\tilde{\eta},\theta=\tilde{\theta}}},$$

where $\frac{\partial l}{\partial \delta}$ is given in Section (3.1.2) for Model I and Section (3.2.2) in for Model II, respectively.

B. μ_0 and μ_1 in sample size formula based on Score test statistic T_{sc}

According to I^{11} given in the above Appendix 1, μ_0 and μ_1 can be given by

$$\mu_0 = \bar{B} - (\bar{C} + \bar{D} + \bar{E})/\bar{A}, \quad \mu_1 = \check{B} - (\check{C} + \check{D} + \check{E})/\check{A},$$

where $\bar{A} = \bar{I}_{22}\bar{I}_{33}\bar{I}_{44} + 2\bar{I}_{23}\bar{I}_{24}\bar{I}_{34} - \bar{I}_{24}\bar{I}_{24}\bar{I}_{33} - \bar{I}_{23}\bar{I}_{23}\bar{I}_{44} - \bar{I}_{34}\bar{I}_{34}\bar{I}_{22}$, $\bar{B} = \bar{I}_{11}$, $\bar{C} = (\bar{I}_{12}\bar{A}_{11} + \bar{I}_{13}\bar{A}_{12} + \bar{I}_{14}\bar{A}_{13})\bar{I}_{12}$, $\bar{D} = (\bar{I}_{12}\bar{A}_{21} + \bar{I}_{13}\bar{A}_{22} + \bar{I}_{14}\bar{A}_{23})\bar{I}_{13}, \bar{E} = (\bar{I}_{12}\bar{A}_{31} + \bar{I}_{13}\bar{A}_{32} + \bar{I}_{14}\bar{A}_{33})\bar{I}_{14}$, with $\bar{A}_{11} = \bar{I}_{33}\bar{I}_{44} - \bar{I}_{34}\bar{I}_{34}$, $\bar{A}_{12} = \bar{I}_{24}\bar{I}_{34} - \bar{I}_{23}\bar{I}_{44}$, $\bar{A}_{13} = \bar{I}_{23}\bar{I}_{34} - \bar{I}_{24}\bar{I}_{33}$, $\bar{A}_{21} = \bar{I}_{24}\bar{I}_{34} - \bar{I}_{23}\bar{I}_{44}$, $\bar{A}_{22} = \bar{I}_{22}\bar{I}_{44} - \bar{I}_{24}\bar{I}_{24}$, $\bar{A}_{23} = \bar{I}_{23}\bar{I}_{24} - \bar{I}_{22}\bar{I}_{34}$, $\bar{A}_{31} = \bar{I}_{23}\bar{I}_{34} - \bar{I}_{24}\bar{I}_{33}$, $\bar{A}_{32} = \bar{I}_{23}\bar{I}_{24} - \bar{I}_{22}\bar{I}_{34}$, $\bar{A}_{33} = \bar{I}_{22}\bar{I}_{33} - \bar{I}_{23}\bar{I}_{23}$, and

$\breve{A}=\breve{I}_{22}\breve{I}_{33}\breve{I}_{44}+2\breve{I}_{23}\breve{I}_{24}\breve{I}_{34}-\breve{I}_{24}\breve{I}_{24}\breve{I}_{33}-\breve{I}_{23}\breve{I}_{23}\breve{I}_{44}-\breve{I}_{34}\breve{I}_{34}\breve{I}_{22}$, $\breve{B}=\breve{I}_{11}$, $\breve{C}=(\breve{I}_{12}\breve{A}_{11}+\breve{I}_{13}\breve{A}_{12}+\breve{I}_{14}\breve{A}_{13})\breve{I}_{12}$, $\breve{D}=(\breve{I}_{12}\breve{A}_{21}+\breve{I}_{13}\breve{A}_{22}+\breve{I}_{14}\breve{A}_{23})\breve{I}_{13}, \breve{E}=(\breve{I}_{12}\breve{A}_{31}+\breve{I}_{13}\breve{A}_{32}+\breve{I}_{14}\breve{A}_{33})\breve{I}_{14}$, with $\breve{A}_{11}=\breve{I}_{33}\breve{I}_{44}-\breve{I}_{34}\breve{I}_{34}$, $\breve{A}_{12}=\breve{I}_{24}\breve{I}_{34}-\breve{I}_{23}\breve{I}_{44}$, $\breve{A}_{13}=\breve{I}_{23}\breve{I}_{34}-\breve{I}_{24}\breve{I}_{33}$, $\breve{A}_{21}=\breve{I}_{24}\breve{I}_{34}-\breve{I}_{23}\breve{I}_{44}$, $\breve{A}_{22}=\breve{I}_{22}\breve{I}_{44}-\breve{I}_{24}\breve{I}_{24}$, $\breve{A}_{23}=\breve{I}_{23}\breve{I}_{24}-\breve{I}_{22}\breve{I}_{34}$, $\breve{A}_{31}=\breve{I}_{23}\breve{I}_{34}-\breve{I}_{24}\breve{I}_{33}$, $\breve{A}_{32}=\breve{I}_{23}\breve{I}_{24}-\breve{I}_{22}\breve{I}_{34}$, $\breve{A}_{33}=\breve{I}_{22}\breve{I}_{33}-\breve{I}_{23}\breve{I}_{23}$. In Model I,

$$
\begin{aligned}
\bar{I}_{11}=&\frac{k_2 r\bar{\theta}(1-\bar{\eta})+r\bar{\eta}}{\bar{\pi}_1}+\frac{k_2 r(\bar{\eta}+\bar{\theta}-\bar{\eta}\bar{\theta})^2}{1-\bar{\pi}_1(\bar{\eta}+\bar{\theta}-\bar{\eta}\bar{\theta})}+\frac{r(1-k_2)\bar{\eta}^2}{1-\bar{\pi}_1\bar{\eta}}, \bar{I}_{12}=\bar{I}_{11},\\
\bar{I}_{13}=&\frac{k_2 r(1-\bar{\theta})}{1-\bar{\pi}_1(\bar{\eta}+\bar{\theta}-\bar{\eta}\bar{\theta})}+\frac{r(1-k_2)}{1-\bar{\pi}_1\bar{\eta}}, \bar{I}_{14}=\frac{k_2 r(1-\bar{\eta})}{1-\bar{\pi}_1(\bar{\eta}+\bar{\theta}-\bar{\eta}\bar{\theta})},\\
\bar{I}_{22}=&\frac{(k_1+k_2 r)(\bar{\eta}+\bar{\theta}-\bar{\eta}\bar{\theta})+(1-k_1+r-k_2 r)\bar{\eta}}{\bar{\pi}_1}+\frac{(k_1+k_2 r)(\bar{\eta}+\bar{\theta}-\bar{\eta}\bar{\theta})^2}{1-\bar{\pi}_1(\bar{\eta}+\bar{\theta}-\bar{\eta}\bar{\theta})}\\
&+\frac{(1-k_1+r-k_2 r)\bar{\eta}^2}{1-\bar{\pi}_1\bar{\eta}},\\
\bar{I}_{23}=&\frac{(k_1+k_2 r)(1-\bar{\theta})}{1-\bar{\pi}_1(\bar{\eta}+\bar{\theta}-\bar{\eta}\bar{\theta})}+\frac{1-k_1+r-k_2 r}{1-\bar{\pi}_1\bar{\eta}}, \bar{I}_{24}=\frac{(k_1+k_2 r)(1-\bar{\eta})}{1-\bar{\pi}_1(\bar{\eta}+\bar{\theta}-\bar{\eta}\bar{\theta})},\\
\bar{I}_{33}=&\frac{(1+r)\bar{\pi}_1}{\bar{\eta}}+\frac{(k_1+k_2 r)\bar{\pi}_1\bar{\theta}}{1-\bar{\eta}}+\frac{(k_1+k_2 r)(1-\theta)^2\bar{\pi}_1^2}{1-\bar{\pi}_1(\bar{\eta}+\bar{\theta}-\bar{\eta}\bar{\theta})}+\frac{(1-k_1+r-k_2 r)\bar{\pi}_1^2}{1-\bar{\pi}_1\bar{\eta}},\\
\bar{I}_{34}=&-\frac{(k_1+k_2 r)\bar{\pi}_1(1-\bar{\pi}_1)}{1-\bar{\pi}_1(\bar{\eta}+\bar{\theta}-\bar{\eta}\bar{\theta})}, \bar{I}_{44}=\frac{(k_1+k_2 r)\bar{\pi}_1}{\bar{\theta}}+\frac{(k_1+k_2 r)\bar{\pi}_1\bar{\eta}}{1-\bar{\theta}}+\frac{(k_1+k_2 r)\bar{\pi}_1^2(1-\bar{\eta})^2}{1-\bar{\pi}_1(\bar{\eta}+\bar{\theta}-\bar{\eta}\bar{\theta})},
\end{aligned}
$$

and

$$
\begin{aligned}
\breve{I}_{11}=&\frac{k_2 r\theta(1-\eta)+r\eta}{\delta_1+\pi_1}+\frac{k_2 r(\eta+\theta-\eta\theta)^2}{1-(\pi_1+\delta_1)(\eta+\theta-\eta\theta)}+\frac{r(1-k_2)\eta^2}{1-(\pi_1+\delta_1)\eta}, \breve{I}_{12}=\breve{I}_{11}\\
\breve{I}_{13}=&\frac{k_2 r(1-\theta)}{1-(\pi_1+\delta_1)(\eta+\theta-\eta\theta)}+\frac{r(1-k_2)}{1-(\pi_1+\delta_1)\eta}, \breve{I}_{14}=\frac{k_2 r(1-\eta)}{1-(\pi_1+\delta_1)(\eta+\theta-\eta\theta)}\\
\breve{I}_{22}=&\breve{I}_{11}+\frac{k_1(\eta+\theta-\eta\theta)+(1-k_1)\eta}{\pi_1}+\frac{k_1(\eta+\theta-\eta\theta)^2}{1-\pi_1(\eta+\theta-\eta\theta)}+\frac{(1-k_1)\eta^2}{1-\pi_1\eta}\\
\breve{I}_{23}=&\frac{k_1(1-\theta)}{1-\pi_1(\eta+\theta-\eta\theta)}+\frac{k_2 r(1-\theta)}{1-(\pi_1+\delta_1)(\eta+\theta-\eta\theta)}+\frac{1-k_1}{1-\pi_1\eta}+\frac{r(1-k_2)}{1-(\pi_1+\delta_1)\eta}\\
\breve{I}_{24}=&\frac{k_1(1-\eta)}{1-\pi_1(\eta+\theta-\eta\theta)}+\frac{k_2 r(1-\eta)}{1-(\pi_1+\delta_1)(\eta+\theta-\eta\theta)}\\
\breve{I}_{33}=&\frac{\pi_1+r(\pi_1+\delta_1)}{\eta}+\frac{k_1\pi_1\theta+k_2 r(\pi_1+\delta_1)\theta}{1-\eta}+\frac{k_1(1-\theta)^2\pi_1^2}{1-\pi_1(\eta+\theta-\eta\theta)}\\
&+\frac{k_2 r(1-\theta)^2(\pi_1+\delta_1)^2}{1-(\pi_1+\delta_1)(\eta+\theta-\eta\theta)}+\frac{(1-k_1)\pi_1^2}{1-\pi_1\eta}+\frac{r(1-k_2)(\pi_1+\delta_1)^2}{1-(\pi_1+\delta_1)\eta}\\
\breve{I}_{34}=&-\frac{k_1\pi_1(1-\pi_1)}{1-\pi_1(\eta+\theta-\eta\theta)}-\frac{k_2 r(\pi_1+\delta_1)(1-\pi_1-\delta_1)}{1-(\pi_1+\delta_1)(\eta+\theta-\eta\theta)}\\
\breve{I}_{44}=&\frac{k_1\pi_1+k_2 r(\pi_1+\delta_1)}{\theta}+\frac{k_1\pi_1\eta+k_2 r(\pi_1+\delta_1)\eta}{1-\theta}+\frac{k_1\pi_1^2(1-\eta)^2}{1-\pi_1(\eta+\theta-\eta\theta)}\\
&+\frac{k_2 r(\pi_1+\delta_1)^2(1-\eta)^2}{1-(\pi_1+\delta_1)(\eta+\theta-\eta\theta)}
\end{aligned}
$$

In Model II,

$$\begin{aligned}
\bar{I}_{11} =& \frac{k_2 r + r(1-k_2)\bar{\eta}}{\bar{\pi}_1} + \frac{k_2 r}{1-\bar{\pi}_1} + \frac{r(1-k_2)\bar{\eta}^2}{1-\bar{\pi}_1\bar{\eta}}, \bar{I}_{12} = \bar{I}_{11}, \bar{I}_{13} = \frac{r(1-k_2)}{1-\bar{\pi}_1\bar{\eta}}, \bar{I}_{14} = 0,\\
\bar{I}_{22} =& \bar{I}_{11} + \frac{k_1 + (1-k_1)\bar{\eta}}{\bar{\pi}_1} + \frac{k_1}{1-\bar{\pi}_1} + \frac{(1-k_1)\bar{\eta}^2}{1-\bar{\pi}_1\bar{\eta}}, \bar{I}_{23} = \frac{1-k_1+r(1-k_2)}{1-\bar{\pi}_1\bar{\eta}}, \bar{I}_{24} = 0,\\
\bar{I}_{33} =& \frac{(1-k_1+r(1-k_2))\bar{\pi}_1}{\bar{\eta}} + \frac{(k_1+k_2 r)\bar{\pi}_1}{1-\bar{\eta}} + \frac{(k_1+k_2 r)\bar{\pi}_1}{\bar{\eta}+\bar{\theta}-1} + \frac{(1-k_1+r(1-k_2))\bar{\pi}_1^2}{1-\bar{\pi}_1\bar{\eta}},\\
\bar{I}_{34} =& \frac{(k_1+k_2 r)\bar{\pi}_1}{\bar{\eta}+\bar{\theta}-1}, \bar{I}_{44} = \frac{(k_1+k_2 r)\bar{\pi}_1}{1-\bar{\theta}} + \frac{(k_1+k_2 r)\bar{\pi}_1}{\bar{\eta}+\bar{\theta}-1},
\end{aligned}$$

and

$$\begin{aligned}
\check{I}_{11} =& \frac{k_2 r + r(1-k_2)\eta}{\pi_1+\delta_1} + \frac{k_2 r}{1-\pi_1-\delta_1} + \frac{r(1-k_2)\eta^2}{1-(\pi_1+\delta_1)\eta}, \check{I}_{12} = \check{I}_{11}, \check{I}_{13} = \frac{r(1-k_2)}{1-(\pi_1+\delta_1)\eta}, \check{I}_{14} = 0\\
\check{I}_{22} =& \check{I}_{11} + \frac{k_1+(1-k_1)\eta}{\pi_1} + \frac{k_1}{1-\pi_1} + \frac{(1-k_1)\eta^2}{1-\pi_1\eta}, \check{I}_{23} = \frac{1-k_1}{1-\pi_1\eta} + \frac{r(1-k_2)}{1-(\pi_1+\delta_1)\eta}, \check{I}_{24} = 0\\
\check{I}_{33} =& \frac{(1-k_1)\pi_1 + r(1-k_2)(\pi_1+\delta_1)}{\eta} + \frac{k_1\pi_1 + k_2 r(\pi_1+\delta_1)}{1-\eta} + \frac{(1-k_1)\pi_1^2}{1-\pi_1\eta} + \frac{r(1-k_2)(\pi_1+\delta_1)^2}{1-(\pi_1+\delta_1)\eta}\\
&+ \frac{k_1\pi_1 + k_2 r(\pi_1+\delta_1)}{\eta+\theta-1}\\
\check{I}_{34} =& \frac{k_1\pi_1 + k_2 r(\pi_1+\delta_1)}{\eta+\theta-1}, \check{I}_{44} = \frac{k_1\pi_1 + k_2 r(\pi_1+\delta_1)}{1-\theta} + \frac{k_1\pi_1 + k_2 r(\pi_1+\delta_1)}{\eta+\theta-1}.
\end{aligned}$$

C. The derivation of the validated ratio k_1 based on Wald test statistic T_{w2}

According to Equation (16), the validated ratio k_1 based on Wald test statistic T_{w2} is obtained by solving the following equation:

$$N_1\delta_1^4 + z_{1-\alpha/2}^4 v_0^2 + z_{1-\beta}^4 v_1^2 - 2z_{1-\alpha/2}^2 z_{1-\beta}^2 v_0 v_1 - 2N_1\delta_1^2(z_{1-\alpha/2}^2 v_0 + z_{1-\beta}^2 v_1) = 0,$$

where $v_1 = A/k_1 + B$, $v_0 = C/k_1 + D$, and

$$\begin{aligned}
C &= \frac{\bar{\bar{\pi}}_1(1-\bar{\bar{\eta}})[1-\bar{\bar{\pi}}_1\bar{\bar{\eta}} - \bar{\bar{\theta}}(1-\bar{\bar{\eta}})]}{\bar{\bar{\eta}}\bar{\bar{\theta}}(1-\bar{\bar{\pi}}_1\bar{\bar{\eta}})} \cdot \frac{1+r}{r}, D = \frac{\bar{\bar{\pi}}_1(1-\bar{\bar{\pi}}_1)^2\bar{\bar{\eta}}}{1-\bar{\bar{\pi}}_1\bar{\bar{\eta}}} \cdot \frac{1+r}{r},\\
\bar{\bar{\pi}}_1 &= \frac{\overline{h}_1\overline{h}_4 + \overline{h}_2\overline{h}_3}{(1+r)\overline{h}_4}, \bar{\bar{\eta}} = \frac{\overline{h}_1\overline{h}_4}{\overline{h}_1\overline{h}_4 + \overline{h}_2\overline{h}_3}, \bar{\bar{\theta}} = \theta,\\
\overline{h}_1 &= \pi_1\eta + (\pi_1+\delta_1)\eta\cdot r, \overline{h}_2 = 1-\pi_1\eta + [1-(\pi_1+\delta_1)\eta]\cdot r,\\
\overline{h}_3 &= \pi_1(1-\eta) + (\pi_1+\delta_1)(1-\eta)\cdot rs, \overline{h}_4 = 1-\pi_1\eta + [1-(\pi_1+\delta_1)\eta]\cdot rs
\end{aligned}$$

in Model I; and

$$\begin{aligned}
C &= \frac{\bar{\bar{\pi}}_1(1-\bar{\bar{\pi}}_1)(1-\bar{\bar{\eta}})}{1-\bar{\bar{\pi}}_1\bar{\bar{\eta}}} \cdot (1+\frac{1}{rs}), D = \frac{\bar{\bar{\pi}}_1(1-\bar{\bar{\pi}}_1)^2\bar{\bar{\eta}}}{1-\bar{\bar{\pi}}_1\bar{\bar{\eta}}} \cdot \frac{1+r}{r},\\
\bar{\bar{\pi}}_1 &= \frac{\overline{h}_1\overline{h}_2 + \overline{h}_3\overline{h}_4}{(1+r)\overline{h}_1}, \bar{\bar{\eta}} = \frac{\overline{h}_1\overline{h}_2}{\overline{h}_1\overline{h}_2 + \overline{h}_3\overline{h}_4},\\
\overline{h}_1 &= 1-\pi_1\eta + [1-(\pi_1+\delta_1)\eta]s\cdot r, \overline{h}_2 = \pi_1\eta + (\pi_1+\delta_1)\eta\cdot r,\\
\overline{h}_3 &= \pi_1(1-\eta) + (\pi_1+\delta_1)(1-\eta)\cdot rs, \overline{h}_4 = 1-\pi_1\eta + [1-(\pi_1+\delta_1)\eta]\cdot rs
\end{aligned}$$

in Model II, respectively. It is easily shown that k_1 is the solution of the following equation:

$$Gk_1^2 - 2Fk_1 + E = 0,$$

where

$$\begin{aligned}
E &= (z_{1-\alpha/2}^2 C - z_{1-\beta}^2 A)^2, \\
F &= (z_{1-\alpha/2}^2 C - z_{1-\beta}^2 A)(z_{1-\beta}^2 B - z_{1-\alpha/2}^2 D) + N_1 \delta_1^2 (z_{1-\alpha/2}^2 C + z_{1-\beta}^2 A), \\
G &= N_1^2 \delta_1^4 + (z_{1-\alpha/2}^2 D - z_{1-\beta}^2 B) - 2N_1 \delta_1^2 (z_{1-\beta}^2 B + z_{1-\alpha/2}^2 D).
\end{aligned}$$

Therefore, the validated ratio k_1 based on Wald test statistic T_{w2} is given by

$$k_1 = \frac{E}{F - \sqrt{F^2 - EG}}.$$