Proceedings of the 15th International Conference on Computational and Mathematical Methods in Science and Engineering

Costa Ballena, Rota, Cádiz (Spain)

July 6rd-10th, 2015

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W. Sprößig (Germany), B. Wade (USA)

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Preface

Applied Mathematics is a fundamental field in which important societal challenges are considered and solved. Moreover, other areas, such as Engineering, Computer Science, Physics and Chemistry, observe the advances in applied mathematics in order to find new tools and mechanisms to be applied in their fields, improving the current methodologies and solving new challenges.

The **15th International Conference on Computational and Mathematical Methods in Science and Engineering** (CMMSE 2015), will be held at Rota, Cádiz (Spain), July 6th-10th, 2015, and it will bring together researchers from several disciplines of applied mathematics in order to present new advances in the area and share them with the rest of the scientific community. These achievements are detailed in the extended abstracts and papers accepted to the conference and will be collected in these proceedings of CMMSE 2015. The proceedings we have the pleasure to present here are comprised of four volumes- the first three correspond to the articles typeset in LaTeX and the fourth to articles typeset in Word.

We hope that during CMMSE 2015 the usual discussion about new advances and open problems provides the desirable exchange of ideas, comments and suggestions leading to the improvement and deepening of the papers presented to allow further development of the research to occur. We also hope that the developed activity narrows and renews the links between participants.

More than twenty symposia show the variety of disciplines considered in the conference, which are formed from high quality accepted papers. The first one, *high-performance computing*, considers new large-scale problems that arise in fields like bioinformatics, computational chemistry, and astrophysics. *Mathematically modeling the future Internet and developing future Internet security technology* is a self-explanatory session. The third symposium addresses analytical, numerical and computational aspects of *partial differential equations in life and materials science*. *Computational finance* is a session focusing on solving problems related to asset pricing, trading and risk analysis of financial assets that have no analytic solutions under realistic assumptions and thus require computational methods to be resolved. A forum for discussion of the growing impact of new technologies on teaching and the development of new tools to increase learning efficiency is provided in the symposium: *new educational methodologies supported by new technologies*. The symposium on *mathematical models and information-intelligent transport systems* researches in the field of flow-modelling of particles with motivated behavior in complex networks, applied to traffic flows, pedestrian flows, ecology, etc. The seventh symposium studies *computational methods for linear and nonlinear optimization* and *numerical methods for solving nonlinear problems* is given in another session. *Bio-mathematics* studies both theoretical and practical applications of population dynamics, eco-epidemiology, epidemiology of infectious diseases and molecular and antigenic evolution. The 10th symposium presents *recent methodological developments in function approximation, multiway array decompositions, ODE and PDE solutions: applications from dynamical systems to quantum and statistical dynamics*. Model problems arising in Computer Science, considering algebraic and computational (fuzzy) techniques is the main goal of *mathematical models for computer science*. The aim of the 15th symposium is to obtain a consistent description of the transition from small clusters to a liquid or solid state, which is a major challenge in computational chemistry and physics. *Hypercomplex methods in mathematics and applications* considers contributions on applications of hypercomplex algebras (quaternions, bi-complex number, Clifford algebras, etc.) to boundary value problems in mathematical physics, fluid mechanics and elasticity theory. Furthermore, general sequences, Fourier series expansions, signal processing, geometric algebras and their applications, etc., are also suitable topics. The enormous potential of fixed point theory, which is needed in mathematics, engineering, chemistry, biology, economics, computer science, and other sciences, justifies the great interest in *fixed point theory in various abstract spaces and related applications*. *Computational methods in direct and inverse (systems of) PDE's* covers general phenomena formulated as control problems or inverse problems associated with mathematical models described by partial differential equations (PDE). *Advances in the Numerical Solution of Nonlinear Time-Dependent Partial Differential Equations* is focused on the latest theory and practice for the numerical solution of nonlinear time-dependent partial differential equations and their applications. An overview of mathematical and computational research focusing on corporate or government applications and problems arising from different economic sectors is presented in the mini-symposium: *Industrial Mathematics*. Parallel implementation using hybrid architectures with accelerators, either GPUs or FPGAs, of numerical methods for solving problems within the following topics of interest: industrial mathematics, fluid mechanics, global optimization, finance, geophysical flows, computational chemistry, electromagnetism, magneto hydrodynamics, atomic physics, relativistic flows; is given in the symposium: *Numerical simulation on GPUs*.

We would like to thank the plenary speakers for their outstanding contributions to research and leadership in their respective fields, including physics, chemistry and engineering. We would also like to thank the special session organizers and scientific committee members, who have played a very important part in setting the direction of CMMSE 2015. Finally, we would like to thank the participants because, without their interest and enthusiasm, the conference would not have been possible.

We cordially welcome all participants. We hope you enjoy the conference.

Costa Ballena, Rota, Cádiz (Spain), July 4th, 2015

I. P. Hamilton, J. Vigo-Aguiar, B. Wade

CMMSE 2015 Mini-symposia

Session Title	Organizers
High Performance Computing (HPC)	CAPAP-H network
P.D.E.'S in Life and Material Sciences	Paula Oliveira & J.A. Ferreira
Computational Finance	Juan C. Reboredo
Computational Methods for Linear and Nonlinear Optimization	Maria Teresa Torres Monteiro
Numerical Methods for Solving Nonlinear Problems	Juan R. Torregrosa & A. Cordero
Bio-mathematics	Ezio Venturino & Nico Stollenwerk & Maíra Aguiar & Roberto Cavoretto
Mathematical Models for Computer Science	Jesús Medina & Manuel Ojeda-Aciego
Analytical and numerical methods for fractional differential equations	Luisa Morgado & Miguel Nobrega & Luis Ferrás
Mathematics meets Chemistry – Theoretical Models at the Nanoscale	Ian Hamilton & Peter Schwerdtfeger & Ottorino Ori & Istvan Laszlo
Hypercomplex methods in mathematics and applications	Klaus Gürlebe & Helmuth Malonek & Wolfgang Sproessing
Fixed Point Theory in various abstract spaces and related applications	Antonio F. Roldán López de Hierro & Juan Martínez Moreno
Computational methods in direct and inverse (systems of) PDE’s	Rob De Staelen
Advances in the Numerical Solution of Nonlinear Time-Dependent Partial Differential Equations	Bruce A. Wade & Abdul Q.M. Khaliq & Qin Sheng
Industrial Mathematics	Bruece Wade
Numerical simulation on GPUs	José Antonio García Rodríguez & José Manuel González Vida

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- M. Teresa de Bustos - *Universidad de Salamanca, Spain*
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CMMSE 2015 Plenary Speakers

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- Nico Stollenwerk - Lisbon University, **Portugal**

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The Holling-Tanner model considering an alternative food for predator

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Abstract

This work deals with a modified Leslie-Gower type predator-prey model where the predators have an alternative food when the quantity of prey diminish and a Hollling type II functional response is considered.

With both assumptions the model obtained has a significative difference with the known May-Holling-Tanner model, since the new equilibrium point appears, which can be a repellor or a saddle point for different set on the parameter space

We can obtain different dynamics according the parameter values and it can possible to prove that the existence of a unique postive equilibrium point which is global asymptotically stable.

Also, for other parameter constraints we prove the existence of separatrix curves on the phase plane that divide the behavior of the trajectories, which can have different $\omega - límit$ implying that solutions are highly sensitives to initial conditions.

Key words: Predator-prey model, Allee effect, separatrix, bifurcations, limit cycle, stability

MSC 2000: 92D25; 34C23; 58F14; 58F21

1 Introduction

In this presentation a predator–prey model described by an autonomous bidimensional differential equation system is analyzed considering the following aspects:

i) the functional response or predator consumption rate is hyperbolic a particular form of Holling type II, and

iii) the equation for predator is the logistic-type, as in the Leslie–Gower model [1, 18].

In this type of model, the conventional environmental carrying capacity for predators K_y is a function of the available prey quantity [1, 10, 11]. An iimportant case is the May-Holling-Tanner model [3, 15, 18] whch plays a special role in Theoretical Ecology for its interesting dynamics [15]. It was proposed by J.T. Tanner in 1975 [16] and based on the Leslie-Gower scheme [18], in which K_y is proportional to prey abundance $x = x(t)$, that is $K_y = K(x) = nx$ and the functional response is hyperbolic.

In this work, we modify the above asumption, considering that $K(x) = nx + c$, i.e. the predators have an alternative food when the quantity of prey diminish; in that case, it is said that the model is represented by a Leslie–Gower scheme and it is also known as modified Leslie–Gower model [2, 4, 13]; if $x = 0$, then $K(0) = c$, concluding that the predator is generalist since it search an alternative food.

The Leslie-Gower type predator-prey models (with $c = 0$) may present anomalies in its predictions, because it predicts that even in very low prey population density, when the consumption rate per predator is almost zero, predator population might increase, if the predator/prey ratio is very small [18]. Then, the modified May-Holling-Tanner model) here presented, studied partially in [20], can enhanced these anomalies.

On the other hand, the predator functional response or consumption function refers to the change in attacked prey density per unit of time per predator when the prey density changes [14]. In many predator-prey models is assumed the functional response grows monotonic, being an inherent assumption the more prey in the environment, the better for the predator [18].

Here, we will consider that the predator functional response is expressed by the function $h(x) = \frac{q\,x}{x\,+\,a}$, a hyperbolic functional response [5, 17] corresponding to the Holling type II [14]. The parameter a is a abruptness measure of the functional response [9]. If $a \to 0$, the curve grows quickly, while if $a \to K$, the curve grows slowly, that is, a bigger amount of prey is need to obtain $\frac{q}{2}$.

We will describe the behavior by means of a bifurcation diagram [5], depending on the parameter values and to classify the different dynamics resulting. Also, we compare the results with those obtained for the May-Holling-Tanner [15] and the model anlyzed in[2] considering Allee effect in prey [5].

2 The Model

The predator-prey model that will be analyzed is described by the autonomous bidimensional differential equations system of Kolmogorov type [8] given by

$$X_\mu : \begin{cases} \frac{dx}{dt} &= \left(r\left(1 - \frac{x}{K}\right) - \frac{qy}{x+a}\right)x \\ \frac{dy}{dt} &= s\left(1 - \frac{y}{nx+c}\right)y \end{cases} \tag{1}$$

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where $x = x(t)$ and $y = y(t)$ indicate the prey and predator population sizes respectively, (number of individuals, density or biomass); $\mu = (r, K, q, a, s, n, c) \in \mathbb{R}^7_+$ and for biological reasons $a < K$. The parameters have the following meanings:

r is the intrinsic prey growth rate,

K is the prey environmental carrying capacity,

q is the consuming maximum rate per capita of the predators (satiation rate),

a is the amount of prey to reach one-half of q (that is, it is half saturation rate),

s is intrinsic predator growth rate,

n is the food quality and it indicates how the predators turn eaten prey into new predator births,

c is the amount of alternative food available for the predators,

This last parameter indicates that the predator is generalist and that if it does not exist available prey, it has a source of alternative food. Clearly, if $c = 0$, the predator carrying capacity is $K\,(x) = nx$ and system (1) is no defined in $x = 0$.

As system (1) is of Kolmogorov type, the coordinates axis are invariable sets and the model is defined at

$$\Omega = \{(x, y) \in \mathbb{R}^2 /\ x \geq 0,\ y \geq 0\} = \mathbb{R}^+_0 \times \mathbb{R}^+_0$$

The equilibrium points of system (1) or vector field X_μ are $(K, 0)$, $(0, 0)$, $(0, c)$ and (x_e, y_e) satisfying the equation of the isoclines $y = nx + c$ and $y = \frac{r}{q}\left(1 - \frac{x}{K}\right)(x + a)$. Clearly, (x_e, y_e) can be a positive equilibrium point (equilibrium at interior of the first quadrant) or cannot exists there, depending of the sign of factor $1 - \frac{x}{K}$.

To simplify the calculus we follow the methodology used in[10, 12, 15], doing the change of variable and the time rescaling, given by the function $\varphi : \breve{\Omega} \times \mathbb{R} \longrightarrow \Omega \times \mathbb{R}$, so that

$$\varphi(u, v, \tau) = \left(Ku, Knv, \frac{\left(u + \frac{a}{K}\right)\left(u + \frac{c}{Kn}\right)}{rK}\tau\right) = (x, y, t)$$

and we have that

$$\det D\varphi(u, v, \tau) = \frac{Kn\left(u + \frac{a}{K}\right)\left(u + \frac{c}{Kn}\right)}{r} > 0.$$

Thus, φ is a diffeomorphism[6], for this reason the vector field X_μ, is topologically equivalent to the vector field $Y_\eta = \varphi \circ X_\mu$ with $Y_\eta = P(u, v)\frac{\partial}{\partial u} + Q(u, v)\frac{\partial}{\partial v}$ and the associated differential equation system is given by the polynomial system of forth degree.

$$Y_\eta : \left\{ \begin{array}{ll} \frac{du}{d\tau} & \left((1 - u)(u + A) - Qv\right) u\,(u + C) \\ \frac{dv}{d\tau} & S\,(u + C - v)(u + A)\ v \end{array} \right. \tag{2}$$

where $\eta = (A, S, C, Q) \in \Delta =]0, 1[\times \mathbb{R}^3_+$ with $A = \frac{a}{K} < 1$, $S = \frac{s}{r}$, $Q = \frac{nq}{r}$ and $C = \frac{c}{Kn}$. System (2) is defined in

$$\breve{\Omega} = \{(u, v) \in \mathbb{R}^2 /\ u \geq 0,\ v \geq 0\}.$$

The equilibrium points (2) or singularities of vector field Y_η are $(1, 0)$, $(0, 0)$, $(0, C)$ and the points lie in the intersection of the curves

$$v = \frac{1}{Q}(1 - u)(u + A) \text{ and } v = u + C.$$

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Then, the abscise u is solution of the second degree equation:

$$u^2 - (1 - A - Q)\,u + (CQ - A) = 0 \tag{3}$$

2.1 Positive equilibrium points

Considering the Descartes signs rule and according to the sign of the factors $1 - A - Q$ and $CQ - A$, equation (3) has two, one or none positive roots.

1) If $1 - A - Q > 0$ and $CQ - A > 0$, the solutions of equation (3) are:

$u_1 = \frac{1}{2}\left(1 - A - Q - \sqrt{\Delta}\right)$ and

$u_2 = \frac{1}{2}\left(1 - A - Q + \sqrt{\Delta}\right)$

with $\Delta = (1 - A - Q)^2 - 4\,(CQ - A)$.

Then, we have three possibilities:

1.1 There are not equilibrium points at interior of the first quadrant, if and only if, $\Delta < 0$.

1.2 There are two equilibrium points at interior of the first quadrant, if and only if, $\Delta > 0$, $(u_1, u_1 + C)$ and $(u_2, u_2 + C)$ and $u_1 < u_2$.

1.3 There is a unique equilibrium point at interior of the first quadrant, if and only if, $\Delta = 0$. In this case, the points coincide, i.e.,

$(u_1, u_1 + C) = (u_2, u_2 + C) = (E, E + C)$

with $E = \frac{1-A-Q}{2}$.

2) If $1 - A - Q > 0$ and $CQ - A < 0$, or $1 - A - Q < 0$ and $CQ - A < 0$, the solutions of equation (3) are $u_1 < 0 < u_2$. In this case, a unique equilibrium point at interior of the first quadrant exists:

$(u_2, u_2 + C) = (L, L + C)$

with $L = \frac{1}{2}\left(1 - A - Q + \sqrt{\Delta}\right)$.

3) If $1 - A - Q = 0$ and $CQ - A < 0$, equation (3) has two solutions, one positive and other negative. Then, is one critical point at interior of the first quadrant $(F, F + C)$ with $F = \sqrt{A - C\,(1 - A)}$ and $A - C\,(1 - A) > 0$.

4) If $1 - A - Q > 0$ and $CQ - A = 0$, equation (3) has two solutions

$u_1 = 0$ and $u_2 = \ G = 1 - A - Q = 1 - A - \frac{A}{C}$.

Then, $\left(\frac{C-A-AC}{C}, \frac{(C-A)(C+1)}{C}\right)$ is the unique equilibrium point at interior of the first quadrant.

5) Equation (3) does not have solutions and then, there are not equilibrium points at interior of the first quadrant, if and only if, $1 - A - Q = 0$ and $CQ - A > 0$, or $1 - A - Q < 0$ and $CQ - A = 0$, or $1 - A - Q < 0$ and $CQ - A > 0$.

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The above classification implies the study of different cases in this class of systems, according to the quantity of the equilibrium points.

3 Main results

For system (2) we have the following results:

Lemma 1 *The set* $\tilde{\Gamma} = \left\{(u,v) \in \tilde{\Omega} / \ 0 \leq u \leq 1, \ v \geq 0\right\}$ *is an invariant region*

Proof. Clearly the $u - axis$ and the $v - axis$ are invariant sets because the system is a Kolmogorov type.

If $u = 1$, we have

$\frac{du}{d\tau} = -Qv\,(1 + C) < 0$

and whatever it is the sign of

$\frac{dv}{d\tau} = S\,(1 + A)\,(\ 1 + C - \ v)\ v$

the trajectories enter and remain in the region $\tilde{\Gamma}$. ■

Lemma 2 *The solutions are bounded*

Proof. Using the Poincaré compactification [6, 7] we make a change of variables and a time rescaling given by the function $\theta : \tilde{\Omega} \times \mathbb{R} \longrightarrow \breve{\Omega} \times \mathbb{R}$.

So that:

$\theta\,(X, Y, T) = \left(\frac{X}{Y}, \frac{1}{Y}, Y^3 T\right) = (u, v, \tau)$

Doing a large algebraic work we get the system:

$$\bar{U}_\eta : \begin{cases} \frac{dX}{dT} &= -X\left(X^3 + (AY - Y + CY + SY)\,X^2 + a_1 X + a_2\right) \\ \frac{dY}{dT} &= -SY^2\left(X^2 + (AY + CY - 1)\,X + AY\,(CY - 1)\right) \end{cases}$$

with:

$a_1 = MY^2 - CY^2 - AY^2 + SY^2 + ACY^2 - AMY^2 - CMY^2$

$a_2 = QY^2 - SY^2 - ACY^3 + AMY^3 + CMY^3 + ASY^3 + CSY^3 - ACMY^3$

The Jacobian matrix[3] in the new system is

$$D\bar{U}_\eta(X, Y) = \begin{pmatrix} -\left(4X^3 + b_1 X^2 + b_2 X + b_3\right) & -DJ\,(1,2) \\ -SY^2\,(2X + AY + CY - 1) & -DJ\,(2,2) \end{pmatrix}$$

with:

$DJ\,(1,2) = X\left((A + C + S - 1)\,X^2 + b_4 X + b_5\right)$

$DJ\,(2,2) = SY\left(2X^2 - 3AY - 2X + 4ACY^2 + 3AXY + 3CXY\right)$

and

$b_1 = 3AY - 3Y + 3CY + 3SY$

$b_2 = 2QY - 2CY^2 - 2AY^2 - 2SY + 2ACY^2 + 2ASY^2 + 2CSY^2$

$b_3 = CQY^2 - ASY^2 - ACY^3 + ACSY^3$

$b_4 = Q - S - 2AY - 2CY + 2ACY + 2ASY + 2CSY$

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$b_5 = 2CQY - 2ASY - 3ACY^2 + 3ACSY^2$

Evaluating the Jacobian matrix $D\bar{U}_\eta(X,Y)$ in the point $(0,0)$ we obtain

$$D\bar{U}_\eta(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

To desingularize the origin in the vector field $\bar{U}_\eta$ we apply the blowing-up method [7]. By the change of variables $X = rw$ and $Y = w$ and the time rescaling given by $\zeta = w^2 T$ [19], the following system is obtained:

$$\breve{U}_\eta : \begin{cases} \frac{dr}{d\zeta} &= r\left(-CQ - Qr + ACw - ACrw\right) \\ \frac{dw}{d\zeta} &= Sw\left(A + r - rw - ACw - Arw - Crw\right) \end{cases}$$

The Jacobian matrix of the system is:

$$D\breve{U}_\eta(r,w) = \begin{pmatrix} ACw - 2Qr - CQ - 2ACrw & -ACr(r-1) \\ -Sw(w + Aw + Cw - 1) & \breve{U}(2,2) \end{pmatrix}$$

with $\breve{U}(2,2) = -S(2rw - r - A + 2ACw + 2Arw + 2Crw)$

Evaluating the matrix $D\breve{U}_\eta(r,w)$ in the point $(0,0)$ we obtain:

$$D\breve{U}_\eta(0,0) = \begin{pmatrix} -CQ & 0 \\ 0 & AS \end{pmatrix}$$

from here we obtain:

$$\det D\breve{U}_\eta(0,0) = -CAQS < 0$$

Therefore, we have that $(0,0)$ is a hyperbolic saddle point of the vector field $\breve{U}$ and a non-hyperbolic saddle point of the vector fieldof $\bar{U}$; then, the point $(0,\infty)$ is a saddle point in the vector field Y_η. Then, the trajectories of system (2) are bounded. ■

To determine the nature of the equilibrium points we must obtain the Jacobian matrix of system (2), that is:

$$DY_\eta(u,v) = \begin{pmatrix} -4u^3 + 3c_1u^2 + 2c_2u + c_3 & -Qu(u+C) \\ Sv(A + C + 2u - v) & S(A+u)(C + u - 2v) \end{pmatrix}$$

with

$c_1 = 1 - C - A$

$c_2 = (A - C(A-1) - Qv)$

$c_3 = C(A - Qv)$

3.1 Nature of equilibrium points over the axis

Lemma 3 *For all* $\eta = (A,S,C,Q) \in]0,1[\times\mathbb{R}_+^3$ *the singularity* $(1,0)$ *is a saddle point.*

Proof. Evaluating the Jacobian matrix in the point we obtain:

$$DY_\eta(1,0) = (C+1)\begin{pmatrix} -(A+1) & -Q \\ 0 & S(A+1) \end{pmatrix}$$

Then, we obtain that the

$$\det DY_\eta(1,0) = -S(A+1)^2 < 0.$$

Therefore, $(1,0)$ is saddle point. ■

Lemma 4 *The point $P_0 = (0,0)$ is a repellor point for any parameter value.*

Proof. As the Jacobian matrix in the origin is

$$DY_\eta(0,0) = \begin{pmatrix} AC & 0 \\ 0 & ACS \end{pmatrix}$$

Then, it has that the $\det DY_\eta(0,0) = A^2C^2S > 0$ and $\mathrm{tr} DY_\eta(0,0) = AC(1+S) > 0$

Therefore, the equilibrium $(0,0)$ is a repellor point. ■

Lemma 5 *The equilibrium $P_C = (0, C)$ is*

i) a saddle point, if and only if, $CQ - A < 0$.

ii) an attractor point, if and only if, $CQ - A > 0$.

iii) a non hyperbolic attractor point, if and only if, $CQ - A = 0$.

Proof. The Jacobian matrix evaluated in the point is

$$DY_\eta(0,C) = C \begin{pmatrix} A - QC & 0 \\ AS & -AS \end{pmatrix}$$

Then, we obtain that the $\det DY_\eta(0,C) = AS(CQ - A)$ and $\mathrm{tr} DY_\eta(0,C) = A - QC - AS$.

Therefore, the point $(0, C)$ is

i) a saddle point, if and only if, $CQ - A < 0$, because $\det DY_\eta(0,C) < 0$.

ii) an attractor point, if and only if, $CQ - A > 0$, since $\det DY_\eta(0,C) > 0$ and $\mathrm{tr} DY_\eta(0,C) < 0$.

iii) If $CQ - A = 0$, then, we obtain that $\det DY_\eta(0,C) = 0$ and $\mathrm{tr} DY_\eta(0,C) < 0$.

In this case, the Jacobian matrix has a proper zero value.

Applying the Central Manifold Theorem, let $u = a_2v^2 + a_3v^3 + a_4v^4 + \ldots + a_nv^n$, then, $\frac{du}{dv} = \frac{\frac{du}{d\tau}}{\frac{dv}{d\tau}}$.

After an algebraic work we obtain $a_2 = a_3 = a_4 = \ldots = a_{20} = 0$. This means that the tangent curve in $v = C$ approach to the axis v. Therefore, we have that it is an attractor point not hyperbolic. ■

3.2 Nature of positive equilibrium points

The positives singularities lie at the straight line $v = u + C$,

Let $h(u) = (1-u)(u+A)$, then

$$h(u) - (u+C)Q = 0$$

and we have the Jacobian matrix is:

$$DY_\eta(u, u+C) = (u+C) \begin{pmatrix} h'(u)u & -Qu \\ S(u+A) & -S(u+A) \end{pmatrix}$$

with $h'(u) = (1 - 2u - A)$. So,

$$\det DY_\eta(u, u+C) = S(u+A)(Q - h'(u))u$$

$$\mathrm{tr} DY_\eta(u, u+C) = h'(u)u - S(u+A)$$

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a) If $Q > h'(u)$ implies that $\det DY_\lambda(u,u) > 0$ and the behavior of singularity depends on the sign of the $\mathrm{tr} DY_\lambda(u,u)$

b) If $Q < h'(u)$, then $\det DY_\lambda(u,u) < 0$ and (u,u) is a saddle point.

c) If $Q = h'(u)$, then the two equilibrium points coincide.

Theorem 6 *The equilibrium point* $(u_1, u_1 + C)$ *is a saddle point, if* $h'(u_1) \neq \frac{S(u_1+A)}{u_1}$.

Proof. As the Jacobian matrix is

$$DY_\eta(u_1, u_1 + C) = (u_1 + C)\begin{pmatrix} h'(u_1)\, u_1 & -Qu_1 \\ S(u_1 + A) & -S(u_1 + A) \end{pmatrix}.$$

Thus, $\det DY_\eta(u_1, u_1 + C) = S(u_1 + A)(Q - h'(u_1))\, u_1$

with $Q - h'(u_1) = -\sqrt{(1 - A - Q)^2 - 4(CQ - A)} < 0$

Therefore, the equilibrium $(u_1, u_1 + C)$ is saddle point. ■

Theorem 7 *The equilibrium point* $(u_2, u_2 + C)$ *is*

i) an attractor point. if $h'(u_2) < \frac{S(u_2+A)}{u_2}$

ii) a repellor surrounded by a limit cycle, if $h'(u_2) > \frac{S(u_2+A)}{u_2}$

iii) weak focus if $h'(u_2) = \frac{S(u_2+A)}{u_2}$

Proof. As the Jacobian matrix is

$$DY_\eta(u_2, u_2 + C) = (u_2 + C)\begin{pmatrix} h'(u_2)\, u_2 & -Qu_2 \\ S(u_2 + A) & -S(u_2 + A) \end{pmatrix}$$

then, $\det DY_\eta(u_2, u_2 + C) = S(u_2 + A)(Q - h'(u_2))\, u_2$

with $Q - h'(u_2) = \sqrt{(A + Q - 1)^2 - 4(CQ - A)} > 0$

and the behavior depends on

$$\mathrm{tr}(DY_\eta(u_2, u_2 + C)) = h'(u_2)\, u_2 - S(u_2 + A)$$

i) If $h'(u_2) < \frac{S(u_2+A)}{u_2}$, then $\mathrm{tr} DY_\eta(u_2, u_2 + C) < 0$.

Therefore, the point $(u_2, u_2 + C)$ is an attractor.

ii) If $h'(u_2) > \frac{S(u_2+A)}{u_2}$, then $\mathrm{tr} DY_\eta(u_2, u_2 + C) > 0$, therefore the point $(u_2, u_2 + C)$ is a repellor and by Poincaré-Bendixon Theorem, is surrounded by a limit cycle.

iii) When $h'(u_2) = \frac{S(u_2+A)}{u_2}$, then $\mathrm{tr} DY_\eta(u_2, u_2 + C) = 0$, and the point $(u_2, u_2 + C)$ is a weak focus, and its weakness must be determined. ■

Theorem 8 *The equilibrium point* $(E, E + C)$ *is*

i) a saddle-node attractor if $2AS > (1 - A - Q)(Q - S)$

ii) a saddle-node repellor if $2AS > (1 - A - Q)(Q - S)$

iii) a cusp point if $2AS = (1 - A - Q)(Q - S)$

Proof. In the equilibrium point $(E, E + C) = \left(\frac{1-A-Q}{2}, \frac{1-A-Q+2C}{2}\right)$ the Jacobian matrix is

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$DY_\eta(E, E+C) = c_1 \begin{pmatrix} y1 & -y1 \\ y2 & -y2 \end{pmatrix}$
with $c_1 = 1 - A - Q$, $y1 = \frac{1}{4} Q c_1 (c_1 + 2C)$, $y2 = \frac{1}{4} S (c_1 + 2A)(c_1 + 2C)$.
Then, $\det DY_\eta(E, E+C) = 0$, and

$\operatorname{tr} DY_\eta(E, E+C) = (1 - A - Q)(Q - S) - 2AS$

a) If $2AS > (1 - A - Q)(Q - S)$, the point is a saddle-node attractor.
b) If $2AS < (1 - A - Q)(Q - S)$, the point is a saddle-node repellor.
c) If $2AS = (1 - A - Q)(Q - S)$, the Jacobian matrix is
$DY_\eta(E, E+C) = \frac{1}{4} Q (1 - A - Q + 2C)(1 - A - Q) \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$
whose Jordan form matrix [3] is $J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
and we have the Bogdanov-Takens bifurcation [21], and the point $(E, E+C)$ is a cusp point. ■

Theorem 9 *The equilibrium point* $(L, L+C)$ *with*
$L = \frac{1}{2}\left(1 - A - Q + \sqrt{(A + Q - 1)^2 - 4(CQ - A)}\right)$ *is:*
i) a local attractor point, if and only if, $(1 - 2L - A) L < S(A + L)$,
ii) a repellor point surrounded by a limit cycle, if and only if, $(1 - 2L - A) L > S(A + L)$,
iii) a weak focus, if and only if, $(1 - 2L - A) L = S(A + L)$.

Proof. The Jacobian matrix is
$DY_\eta(L, L+C) = (L + C) \begin{pmatrix} (1 - 2L - A) L & -QL \\ S(L + A) & -S(L + A) \end{pmatrix}$
then, $\det DY_\eta(L, L+C) = LS(L + A)(2L + A - 1 + Q) > 0$,
and the behavior depends on

$\operatorname{tr} DY_\eta(L, L+C) = (1 - 2L - A) L - S(L + A)$

i) If $(1 - 2L - A) L < S(L + A)$, then $\operatorname{tr} DY_\eta(L, L+C) < 0$.
Therefore, the point $(L, L+C)$ is an attractor.
ii) If $(1 - 2L - A) L > S(L + A)$, then $\operatorname{tr} DY_\eta(L, L+C) > 0$.
Therefore, the point $(L, L+C)$ is surrounded by a limit cycle as consequence of Poincaré-Bendixon Theorem.
iii) If $(1 - 2L - A) L = S(L + A)$, then $\operatorname{tr} DY_\eta(L, L+C) = 0$.
Thus, the point $(L, L+C)$ is a weak focus, and the weakness must be determined. ■

Theorem 10 *The equilibrium point* $(F, F+C)$
with $F = \sqrt{A - CQ}$ *is:*
i) a local attractor point, if and only if, $(1 - 2F - A) F < S(F + A)$,
ii) a repellor point surrounded by a limit cycle, if and only if, $(1 - 2F - A) F > S(F + A)$,
iii) a weak focus, if and only if, $(1 - 2F - A) F = S(F + A)$.

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Proof. Evaluating the Jacobian matrix it has

$$DY_\eta(F,F+C)=(F+C)\begin{pmatrix}(1-2F-A)F & -QF\\ S(F+A) & -S(F+A)\end{pmatrix}$$

then, $\det DY_\eta(F,F+C)=FS(F+A)(2F+A+Q-1)>0$,

and the behavior depends on

$$\operatorname{tr}DY_\eta(F,F+C)=(1-2F-A)F-S(F+A)$$

i) If $(1-2F-A)F<S(F+A)$, then $\operatorname{tr}DY_\eta(F,F+C)<0$.

Therefore, the point $(F,F+C)$ is an attractor.

ii) If $(1-2F-A)F>S(F+A)$, then $\operatorname{tr}DY_\lambda(F,F+C)>0$.

Therefore, the point $(F,F+C)$ is a repellor, surrounded by a limit cycle, by Poincaré-Bendixon theorem.

iii) If $(1-2F-A)=S(F+A)$, then $\operatorname{tr}DY_\eta(F,F+C)=0$.

So, the point $(F,F+C)$ is a weak focus and its weakness must be determined ■

4 Conclusions

In this work, a modified Leslie-Gower predator-prey model [4, 13] was studied. By means a diffeomorphism we analyzed a topologically equivalent system depending on four parameters establishing the local stability of the equilibrium points and we have that the points $(0,0)$ and $(1,0)$ are always a repellor and a saddle point respectively, for all parameter values.

We show that the dynamics of the model in which the predators has an alternative food to low densities of prey, differs of the May Holling-Tanner model [3, 15], since the model here studied can have one, two or none positive equilibrium points at interior of the first quadrant with a more varied dynamic; meanwhile, the May-Holling-Tanner has a unique equilibrium point and it has not a cusp point.

The model here studied has a behavior more closed to the model studied in [2], where the Allee effect provokes a similar dynamics by the existence of two positive equilibriuem points.

An important result is the existence of two positive equilibrium points, being one of them always is a saddle point. The other can be an attractor, a repellor or a weak focus, depending of the trace of its Jacobian matrix. Also, both equilibrium points can collapse and we obtain a cusp point (Bogdanov-Takens bifurcation) [21].

When, two equilibrium point exist at interior of the first quadrant, the point $(0,C)$ is an attractor determining a separatrix curve which divides the phase plane in two regions. The trajectories having initial conditions above this curve have the point $(0,C)$ as their $\omega-limit$, meanwhile those that lie below the separatrix can have a positive equilibrium point or a limit cycle as their $\omega-limit$.

This implies that there exists a great possibility of the population of prey can go to extinction, although the ratio prey-predator is high (many prey and little predator); however,

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the populations can coexist for a same set of parameter values for which the population of prey is depleted

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