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Dynamics of Endogenous Economic Growth Theory and Related Issues: A Case Study of the 'Romer Model'

by

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Abstract

Analysis of the transitional dynamics of endogenous growth models with more than a single state variable has been a somewhat neglected area. No doubt this has been due to analytical difficulties associated with the non-linear differential equation systems. While some 'characterisations' of the dynamics have been made, complete numerical analyses have been rare. Instead, the tendency has been for models to be examined in terms of their steady-state growth paths; and for their responses to exogenous economic shocks to be analysed by comparing their pre- and post-shock steady-state equilibria, a so-called comparative statics approach. This thesis seeks to contribute towards closing the gap in the analysis of transitional dynamics. It also seeks to examine the dynamics of an important model of endogenous growth, both to illustrate the techniques of dynamic analysis, and to advance examination of the economics of the model beyond that of comparative statics.*

Various (approximate) techniques for analysing the dynamics of the model are developed: First it is linearised and a 'closed form' solution obtained. Next its phase-space is calculated and examined qualitatively. Then an abridged 'Solowian-Romer' model is developed and its simpler dynamic system is examined by both numerical integration and quantitative phase space analysis. This abridged model is shown to be simply a particular parameterisation of the full model.

Numerical integration techniques for dynamic analysis of the full model are then examined. 'Shooting' is found not to work. Three other techniques are considered: 'time elimination', 'eigenvector-backwards integration' and 'finite differences'. The latter, when implemented via a software package known as GEMPACK, is shown to be clearly superior. This technique is applied to the dynamic analysis of a variety of simulated exogenous shocks to the model. Numerical results are scrutinised and the underlying economic mechanisms identified and explained. Comparative statics analysis is shown to conceal a great deal of transitory change, both in terms of relative magnitudes and in terms of persistence over time. Transitional adjustment is found to be characterised by relatively large discontinuous jumps in many variables, and by half-lives of adjustment of the order of two decades.

A social optimum solution to the model is derived and compared with the sub-optimal market solution, revealing a significant degree of 'forgone welfare'. The imperfections of the market model are examined and identified as the causes of specific differences between the two models. From this a variety of subsidisation schemes capable of converting the steady-state market outcomes into the socially optimum ones are devised. Subsidies are then implemented, in various different ways, as shocks to the model and the resulting adjustment paths from the market solution steady-state to the socially optimum one are computed and analysed.

Finally, the implications for economic policy arising from the characteristics of adjustment paths are identified and discussed. In particular, once long-term policy is set, a case is made for a greater focus on the economic management of short–run adjustment.

* Paul M. Romer (1990), "Endogenous Technological Change", *Journal of Political Economy*, Vol. 98, No. 5, Part 2

Affirmation of original and individual work

This thesis contains no material that has been accepted for the award of any other degree or diploma in any university or institution. Also, except where due reference is made the thesis contains no material previously published or written by another person. Finally, I affirm that the work is all my own. I therefore have no acknowledgement to make for any help or for work carried out by any other person.

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Gordon W. Schmidt

April 2001

To Susie,
who encouraged me to begin,
and who sustained me throughout.

Chapter 1

1 Introduction

1.1 Background

This dissertation is concerned with the dynamics of endogenous economic growth theory. In particular, it involves a detailed examination of the transitional dynamics (that is the movement from some current state towards a steady-state equilibrium) of the endogenous growth model from Paul Romer's 1990 Journal of Political Economy article: "Endogenous Technological Change" (Romer, 1990b).

The initial idea for a PhD topic was to examine the 'new growth theory' literature with a view to incorporating its fundamentals in a large computable general equilibrium model of the Australian economy: the 'MONASH Model', being developed at the Centre of Policy Studies at Monash University by Peter Dixon and his colleagues. However, it gradually became apparent that this was not a viable task, at least not for a PhD topic. As a result, such work remains as a possible area for further and future research (see Section 6.3).

Nevertheless, it was apparent that the field offered more than ample scope for identifying a suitable alternative. Despite there having been a great deal written on the subject of endogenous growth by the mid-1990s, and despite its fundamentally dynamic nature, it seemed that very little work had fully addressed the dynamics of the growth systems. Most work was concerned with solving the dynamic systems to establish their *balanced growth equilibria*, and then undertaking a form of *comparative statics* analysis whereby the impact of economic shocks or policy changes were assessed by comparing the pre- and post-shock equilibria.[1] For example, see Arrow (1962b), Sheshinski (1967), Lucas (1988), Romer (1990a and 1990b), Jones and Manuelli (1990), and Rebelo (1991).

Although generally a more difficult problem, the basic differential equations of any growth model can also be solved to establish the paths towards equilibrium from arbitrary starting points.[2] These are the *transitional dynamics* of a growth system. The issues here seemed interesting and important, and there seemed to be something of a gap in the literature.[3] For these reasons the "dynamics of endogenous economic growth" was selected as the topic for the PhD dissertation. Motivation for this choice was also provided by certain views expressed in the literature. For example:

[1] Typically, the dynamic equilibrium of a growth model is a *balanced growth path*, where the asymptotic growth rates of key variables are identical and constant. Since this means that the ratios of many variables are asymptotically constant it is also referred to as a *steady-state equilibrium*.

[2] Actually, it is rarely possible to obtain analytic solutions; but numerical solutions can be obtained by a number of techniques (see Chapters 3 and 4).

[3] Mulligan and Sala-i-Martin (1993) also identified this gap in the study of transitional dynamics, noting that: "Due to its analytical difficulty…these transitional dynamics are always left unexplained". But their paper, which contributed significantly towards filling the gap, was not discovered until much later.

In respect of his endogenous growth model that is studied here, Paul Romer commented that:

> "...Studying this kind of model is feasible in principle, but it would require explicit attention to dynamic paths along which growth rates vary, a kind of analysis much harder to undertake than balanced growth analysis."(Romer, 1990a; p.346).

and

> "By focussing on only balanced growth paths, the analysis neglects the transient dynamics....One should be able to study convergence to the balanced growth....by using the tools used for studying the Solow and Uzawa models, but this analysis is not attempted here." (Romer, 1990b; p.S90).

and also

> "Any intervention designed to move an economy from one balanced growth path to another must consider the transition dynamics along the way, and an explicit analysis of these dynamics is beyond the scope of this paper."(Romer, 1990b; p.S97).

Having also developed a two-sector endogenous growth model with human and physical capital, Lucas only conjectured about the dynamics of the system, noting that:

> "The dynamics of this system are not as well understood as those of the one-good model." (Lucas, 1988; p.25).

And Chiang, commenting on Romer's (1990b) model noted that:

> "Since there are four differential equations, the system cannot be analysed with a phase diagram. And solving the system explicitly for its dynamics is not simple." (Chiang, 1992; p.273).

In addition, a comment by Mulligan and Sala-i-Martin that would have been equally motivating but for the fact that it was not discovered until some time after the relevant work had already been achieved was:

> "I invite (dare) the reader to draw a phase diagram for the Lucas model....and for the more general two sector models...." (Mulligan and Sala-i-Martin, 1991; footnote p.11).

Having decided to study the dynamics of endogenous growth, one option would have been to select a technique by which dynamic paths could be calculated and then to apply this to the dynamic systems of a number of different endogenous growth models, comparing the results. However, a considerable amount of effort went into understanding and developing a full dynamic version of the model due to Romer (1990b),[4] and into learning about the techniques of dynamic analysis and applying them to that model. As a result, the dissertation has focussed upon a detailed examination of the dynamics of the Romer (1990b) model, the analysis being conducted through the use

[4] Particularly into establishing the necessity for, and then identifying a 'second so-called *transversality condition*'; and into calibrating the model to fit the Australian economy (see Chapter 2, especially Sections 2.2.5, 2.3 and 2.4).

of a variety of techniques.[5] Since the results from these are all approximations of one sort or another, the performances of the different techniques are compared (Appendix 4.4).

The remainder of this chapter is structured as follows: First, a short history of the development of economic growth theory up to Romer's (1990b) model is presented in Section 1.2. The principal aim here is to provide some perspective for the dissertation in general; and more particularly, for where the model studied stands in relation to the history of economic thought on growth theory. Finally, an outline or summary of the dissertation is provided in Section 1.3.

1.2 A short history of economic growth theory

Research activity in the theory of long-run growth has, ironically, tended to follow a path more akin to the fluctuations of a long business cycle (or a short Krondratieff one) than the smooth advancement of the steady-state growth process itself. The current episode of research activity, the 'endogenous growth era', began some 15 years ago with the publication of Paul Romer's influential 1986 Journal of Political Economy article "Increasing Returns and Long-Run Growth". This was preceded by about 15 years of relative inactivity that in turn followed the active 'neoclassical growth era', again of about 15 years, whose beginning in 1956 was marked by the seminal contributions of Solow (1956) and Swan (1956). Prior to this were the 'Keynesian' growth model contributions of Harrod (1939) and Domar (1946) and before these, the pioneering and precocious work of Ramsey (1928).

1.2.1 Classical, Keynesian, and neoclassical growth theory

As an essential and central issue of economics, growth theory is as old as the discipline itself, many of the fundamental ideas coming from the classical economists of the 18th and 19th centuries: Adam Smith, Thomas Malthus, David Ricardo and Alfred Marshall. Much later, the ideas of Young (1928), Schumpeter (1934 and 1942), and Knight (1925 and 1944) played significant roles. Moreover, some of the concepts accepted today as central elements of growth theory had lengthy gestations, with many different economists contributing to their development. For example, Romer (1986a and 1986b) traced the development of the idea of externalities and increasing returns as an essential element of modern growth theory from:

- Adam Smith's (1776) story of the pin factory and his concept of the division of labour;
- through Marshall's (1890) notion of economies that are external to firms;
- Young's (1928) model of growth based on specialisation and externalities;

[5] This model is of considerable importance. It represents an extension of what was the seminal work on the *new endogenous growth theory* (Romer, 1986a); and it captures both of the two main issues long held to be the driving forces of economic growth: Namely, increasing returns due to specialisation, and externalities from the accumulation of knowledge. Some work was also done on the model due to Lucas (1988), sufficient to indicate that it could also be analysed by all the techniques employed here (including the development of a *Solowian-Lucas* version of the Solowian-Romer model – see Section 3.4.). Otherwise, work on different models has to be considered an opportunity for future research.

- Knight's (1925) objections to the many (indeed spurious) industrial specialisation and organisational examples of such externalities;[6]
- Mead's (1952) formalisation of the concept of true 'technological external economies';
- Arrow's (1962b) 'learning–by-doing' growth model; and
- Chipman's (1970) demonstration that a perfectly competitive general equilibrium is not inconsistent with increasing returns and (true) externalities.

In this way a formal growth model based upon a version of Marshall's concept of external economies was finally developed and rigorously justified. As a result, it became widely recognised that as an input to production, the output of research and development, or knowledge, leads to increasing returns and spillovers that cannot be internalised by the firms or individuals who generate it.[7]

The chronology of the development of modern growth theory begins here with the qualitative model of Young (1928), who described a model of growth driven by increasing returns. Following Adam Smith, Young took the degree of specialisation to be both encouraged and limited by the 'extent of the market'.[8] However, he argued that the limiting effect would be overcome by the increased demand generated by rising incomes. Thus, Young described how a virtuous circle of growth could arise: with increased specialisation improving productive efficiency and allowing output to grow faster than inputs; thereby generating growth, raising per capita incomes and expanding demand; and so increasing the 'extent of the market' to permit further specialisation for the next round of the growth cycle. Young also argued, following Marshall, that such growth could be consistent with a competitive equilibrium because the increasing returns from specialisation would take the form of improvements in industrial organisation as a whole, and so be external to individual firms. While this view of external economies was later discredited (following Knight, 1925; Meade 1952; and Chipman, 1970), it turned out that Young's insight into specialisation as an engine of growth (regarded as 'fundamental' by Kaldor),[9] relied on a different type of externality. But one that had to wait nearly 60 years before it was formalised by Romer (1986b and 1987).

It was also in 1928 that Frank Ramsey's classic work on optimal savings appeared. In purely technical terms Ramsey's treatment of intertemporal optimsation on the part of consumers must rank as among the most influential of contributions, not only to modern growth theory, but to modern economics in general. Though Ramsey operated in a world of the 'calculus of variations' and its 'Euler equations' the principles and conditions he introduced to economists now pervade the discipline to such as extent that, as Barro and Sala-i-Martin (1995) point out, it is difficult to discuss whole topics without them. In terms of its influence on growth theory, Barro and Sala-i-Martin

[6] According to Romer (1986a), Knight had been a student of Young's.
[7] With the addition of Romer's (1986b and 1987) models, the above path also leads to the modern recognition of 'growth based on increasing returns due to specialisation'.
[8] The idea appears to be that with a relatively small market it may not be 'worth' paying the fixed costs associated with an increase in specialisation, such as the cost of equipment for some new production technique; but that with a relatively large market it may become 'worthwhile'. Note that this requires a capacity to recover such fixed costs – by charging a price greater than the marginal costs of the newly made products.
[9] See Kaldor (1981) and Sandilands (1990).

(1995) have described Ramsey's work as "the starting point for modern growth theory"; and Hahn (1990) has opined that "Ramsey led the way to a study of optimal growth". But Ramsey's work was ahead of its time. Almost 40 years was to pass before it would suddenly take-off with the work of Cass (1965), Koopmans (1965), and Shell and his contributors (1967a), newly couched in the terms of 'optimal control theory', the 'Hamiltonian', and the 'Maximum principle'.

Following Ramsey there was little development of growth theory, at least in terms of formal modelling, until the work of Harrod (1939) and Domer (1946). Of course this is not to say that thinking about economic growth ceased. On the contrary, the work of economists such as Schumpeter (1934 and 1942) and Knight (1944) introduced ideas that are fundamental to modern theory. For example, Schumpeter's (1934) process of 'creative destruction', whereby new inventions render existing capital obsolete and result in the demise of its producers, is the basis for the modern 'vertical innovation' or 'quality ladder' models.[10] And his contention that research and development is an ordinary economic activity, undertaken intentionally and in response to profit incentives, and that this implies a sub-optimal equilibrium characterised by market power (Schumpeter, 1942), are central and explicit propositions to Romer's (1990a and b) models, and similarly underlie virtually all of the subsequent endogenous growth theory. Also, the idea that diminishing returns to capital might be avoided if a broad enough concept of capital was considered, in particular, one including 'human capital', goes back to Knight (1944). This idea forms the basis of Rebelo's (1991) so-called AK-model of endogenous growth.[11]

Around 1940 a model was developed that is notable now for the fact that it contributed little to the subsequent development of growth theory. This was the Keynesian styled model proposed (independently) by Harrod (1939) and Domar (1946). Using a fixed proportions production function (so no substitution was allowed between labour and capital), the Harrod-Domar model suggested that balanced growth was only possible on a 'knife edge' (when four exogenous parameters satisfied an equality relationship).[12] Otherwise the model led to perpetually increasing unemployment of either labour or capital. No doubt the backdrop of the Great Depression influenced the conclusion of Harrod and Domar – that capitalist economies were inherently unstable –and the credence this received at the time. Nevertheless, the implausibility of the assumptions seem spectacular when viewed from today: Why should consumers continue to save at a constant rate in the face of ever increasing idle capacity?

After another period of relative inactivity, the models of Solow (1956) and Swan (1956) rescued growth theory from the 'knife edge' notion of the Harrod-Domar model and began a revival of interest in the subject. On the consumption side the Solow-Swan model was (mostly) rudimentary. With the main specification comprising an exogenously constant savings rate, consumption was also basically exogenous. The key feature of the model was on its production side. Introduction of the neoclassical

[10] For example: Segerstrom, Anant and Dinopoulis (1990); Grossman and Helpman (1991a, b and d); Aghion and Howitt (1992 and 1998); and Barro and Sala-i-Martin (1995).
[11] The name refers to the form of the aggregate production function: $Y=AK$, where A is a positive constant reflecting the level of technology, and K is a broad concept of capital including human capital.
[12] With the production function $Y=\min.(AK,BL)$, for A and B given constants; depreciation at the constant rate δ; an exogenously constant savings rate of s; and exogenous population growth at the constant rate n, the condition that had to be satisfied was: $sA=n+\delta$.

production function (with constant returns to scale, diminishing returns to individual factors, and substitution possibilities between them)[13], allowed full employment of both labour and capital, and formalised the idea that savings and investment could lead to rising labour productivity and growth. However, diminishing returns to capital meant that in the absence of an exogenously specified rate of technological progress, per capita growth would eventually cease.[14]

While this conclusion of zero long-run growth accorded with the *'increasing population-limited natural resources-limits to growth'* ideas of Malthus (1798) and Ricardo (1817), it flew in the face of the empirical evidence. It was therefore avoided in the Solow-Swan model by the addition of an exogenous rate of technological progress. In particular, a level of technology term A(t) that grew at a specified rate was added to the production function. For technical reasons this technological progress was specified as 'labour augmenting technical change': Y=F[K, A(t).L], and A was interpreted as the number of *efficiency units* per unit of labour (see Barro and Sala-i-Martin, 1995, p.32).

In the model without technological progress, transitional growth was possible between steady-states with zero long-run growth, the equilibria being distinguished by differing levels of their (constant) capital per head. Little was changed in this respect by the inclusion of exogenous technological progress. In either case, the speed of the transitional dynamics at any time is inversely proportional to the divergence of capital per head from its steady-state level. This property led to the empirically testable concept of *conditional convergence* among economies, based upon the fact that those with lower levels of per capita income relative to their steady-state values should tend to exhibit faster per capita growth. Econometric analysis of cross-sectional data on countries and other regions have confirmed this convergence hypothesis and so lent empirical support to the Solow-Swan model (see Mankiw, Romer and Weil, 1992). Nevertheless, the fact remained that this was a growth model that did not explain long-run growth. Nor did it allow any consumer sovereignty over the important choice of the consumption-savings trade-off. As a result, the model was impossible to evaluate in an efficiency sense since it lacked any measure of consumer welfare.

Research proceeded in three directions. One was empirically based. Another addressed the problem of exogenous savings. And the third confronted the problem of exogenous long-term growth itself. Empirical research sought to explain differences in growth rates across countries according to the convergence proposition; to measure speeds of transitional dynamics; and most significantly, to measure the contribution of technological progress to economic growth – the so-called *growth accounting* research.[15]

[13] See Barro and Sala-i-Martin (1995, p.16) for a formal specification.
[14] With diminishing returns to capital, less and less extra output is generated from extra units of capital. And since a constant proportion of each unit of output is saved to invest in new capital, marginal investment and output are eventually reduced to zero and growth stops.
[15] 'Growth accounting' seeks to measure the contribution of technological change to economic growth as the *residual* from output growth left 'unexplained' by the growth of inputs. Early work such as that of Abromovitz (1956), Kendrick (1956), and Solow (1957) put the contribution at around 50 per cent. A vast body of subsequent work, with improved measurement and techniques, gradually whittled this away (for example: Denison, 1962 and 1967; Jorgenson and Griliches, 1967; Kendrick, 1976; and Jorgenson, Gollop and Fraumeni; 1987); the last of these estimating the contribution from productivity growth at about 25 per cent. Aghion and Howitt (1998) also report growth accounting exercises by Young (1995b) and Jorgenson (1995) that "portray technological progress as an unimportant source of economic growth

The models of Cass (1965) and Koopmans (1965) overcame the problem of exogenous savings by at last resurrecting Ramsey's (1928) approach of intertemporal optimisation by consumers. The time path of savings was thus determined endogenously by utility maximising consumers who traded-off current for future consumption according to the constraint of a neoclassical production function and their rate of time preferences, or discount rate. Along an optimal path capital would increase whenever its net marginal product exceeded the discount rate, and decrease whenever it fell short of it. With a neoclassical production function this prevents sustained per capita growth since diminishing returns to capital would eventually drive its net marginal product below the discount rate. Exogenous technological progress was added to the model in exactly the same way as for the Solow-Swan model.

Transitional growth was along a single trajectory in per capita *consumption-capital space*. This is a very common result for applications of dynamic optimisation and models that exhibit it are said to possess the property of 'saddle-path stability'. It is currently regarded as a most propitious outcome in that it allows unique transition paths to be identified. But when it was first discovered it was regarded more as a property of 'saddle-path instability', something like the Harrod-Domar 'knife edge', which had the potential to bring down the whole theory (see Hahn, 1966).

Overall, the effect of the Cass-Koopmans neoclassical growth model was to introduce optimum savings and a welfare measure, and to enrich the transitional dynamics of the Solow-Swan model, while preserving the empirically supported property of conditional convergence. However, long-run per capita growth continued to depend on exogenously specified technical change.

It was clear that a relevant and fully specified model of economic growth must incorporate some means for the **endogenous** determination of knowledge creation and technical advancement. But there were difficulties in embedding a theory, or process of technical change in the neoclassical framework; and such a framework continued to seem appropriate. The fundamental problem is that when technology, or knowledge is included as a factor of production, it leads to increasing returns, which are incompatible with the notion of perfect competition.[16] As a result, income from production would not be sufficient to pay all the factors (labour, capital and technology) at the value of their marginal products. But on the other hand, if existing technological knowledge were to be paid at its current marginal cost of production (zero), there would be no incentive for the private creation of new knowledge. Inclusion of technology in the Solow-Swan model avoided these problems, but at the cost of specifying technology as both publicly provided and exogenous.

relative to capital accumulation", but go on to dispute this. Romer (1987a) showed how the presence of increasing returns in production (in the manner of his 1986a and 1987b models) meant that a conventional growth accounting approach could be expected to yield a residual – suggesting that increasing returns are a fact of the real world.
[16] Since technology, or knowledge, is a non-rival good, that is its use by any one agent in no way precludes its use by any other, it can be used repeatedly at no (extra) cost. For example, once purchased a blueprint or design for a product may be used indefinitely in producing the product. Now, given the existing level of technology and a neoclassical type of production function, since it is not necessary to replicate the (non-rival) technology, replication of the other (rival) factors generates increased output with constant returns to scale. Thus, if technology has any productive value as it grows, a production function that includes it must show increasing returns in all its factors taken together. See Romer (1990b and 1990c) or Section 2.1 here for a more detailed discussion.

Neoclassical growth theory effectively came to an end not long after the work of Cass and Koopmans in 1965. Since the remaining problem with the theory was the exogeneity of technological progress, research focussed on this issue and a tentative beginning was made on what was later to be termed 'endogenous growth theory'. Over a period of about 10 years commencing with the work of Arrow (1962b), a series models emerged for which technological progress was an endogenous outcome. However, all these models had some undesirable feature or other, including the fact that per capita growth was not actually sustainable.

1.2.2 Early (neoclassical) endogenous growth theory

Arrow (1962b) constructed a vintage capital model in which knowledge and hence technological progress were generated endogenously and through private economic activities. In his model knowledge was created as an unintended by-product of *learning-by-doing* in the course of private investment. This knowledge then spilled over as an external benefit to the economy as a whole. Thus, initial investment and *learning* by any one producer raised the productivity of others; and the state of knowledge at any time was equated with cumulative investment in the economy up to that time. Growth was driven by increasing returns, which arose because the (productive) new knowledge came as a by-product and was publicly available. Because the increasing returns were external to individual firms the equilibrium was competitive. However, although it derived from private activity, aggregate knowledge was still formally treated as a public good in the model. It was completely non-excludable. The spillovers could not be prevented, not even partially, and the generation of knowledge received no market compensaton.[17]

In addition, while growth in Arrow's model was endogenous in the sense that it would respond to changes in savings behaviour, it was only fully worked out at the industry level and for fixed proportions production functions. This meant that the marginal product of capital was diminishing given a fixed labour supply, and the rate of growth of output was therefore limited by that of labour. In an aggregate economy interpretation of the model, which involved no dynamic optimisation, the savings function is constant. Thus, as in the Solow-Swan model, the rates of growth of the variables in the system are neither sustainable nor depend on savings behaviour; however their levels do.

Arrow's model was extended by Levhari (1966a and 1966b) and Sheshinski (1967). Levhari (1966b) simply replaced Arrow's fixed proportions production function with a linearly homogeneous one, but maintained the vintage capital technology, and so reproduced much the same results as Arrow. Sheshinski (1967) replaced both the fixed proportions production function and the vintage capital notion of embodied technological change. Instead he employed a standard neoclassical production function with (disembodied) labour-augmenting technical change.[18] This made it possible to conduct the analysis in terms of standard dynamic optimisation techniques. In the

[17] And as Romer (1990b) has pointed out, the results of Dasgupta and Stiglitz (1988) meant that knowledge had to be completely non-excludable or a competitive equilibrium would not be sustainable.

[18] But this technological progress was not exogenous as in the similar Solow-Swan formulation because it continued to depend, as in Arrow (1962b), on cumulated investment that was publicly available, this also continued to be the source of increasing returns and growth with a competitive equilibrium.

Sheshinski model once the diminishing returns to capital cause its marginal product to fall below the discount rate the incentive for investment vanishes and growth stops. Thus, while they could generate growth endogenously, the models of Arrow (1962b), Levhari (1966b) and Sheshinski (1967) all faced the shortcomings of having technological progress arising only incidentally, and also unsustainably.

Shell (1966 and 1967b) modelled the generation of knowledge as an intentional output of those who produce it. But they were not private economic agents motivated by profit. Instead, knowledge arose from a separate research sector financed by Government taxes. This sector drew resources from the market economy but its size was exogenous. Knowledge entered the market economy as a pure public good in a production function that was neoclassical in terms of capital and labour. Thus, as in the previous endogenous growth models, growth was again driven by increasing returns, a competitive equilibrium ensued, and accumulation eventually ceased due to diminishing returns. Shell's main contribution from these models was his emphasis that the creation of knowledge requires the diversion of resources from other economic activities.

Uzawa (1965) showed how sustained growth could be achieved in a neoclassical model with labour augmenting technical progress produced with diminishing returns by labour inputs in a separate education sector. The resulting efficiency units of labour were interpreted as human capital and so both physical and human capital are accumulated in the model; or in effect, a single broad measure of capital is accumulated. Although there are no increasing returns, the fact that the accumulation of human capital per worker is linear in itself ($\dot{A}(t) = A(t)\phi[1 - l(t)]$, where $l(t)$ is the fraction of the total workforce devoted to production), combined with the constant returns in production allows both human capital and physical capital to grow at the same asymptotic rate: $g=\phi(1-l^*)$, where l^* maximises consumer welfare. Thus, Uzawa's model generated unbounded per capita growth endogenously. When consumer welfare was maximised according to a linear utility function the optimal path was one for which investment was specialised in either human or physical capital ($l(t)= 0$ or 1) until l^* was achieved. But the model was limited to the description of optimal accumulation paths and did not resolve the economics of how factors that made A grow would be compensated.

As work on the neoclassical model came to an end, and not long after the early attempts to develop a theoretically and empirically justifiable model of endogenous long-run growth, interest in growth theory began to wane. Barro and Sala-i-Martin (1995) expressed the view that it became "excessively technical and steadily lost contact with empirical applications". It is possible that the advent of the new 'optimal control theory' and the 'Maximum principle' of Pontryagin et al. (1962) contributed to such a state of affairs. The new techniques were certainly regarded as highly technical and esoteric at the time and it is possible there was some 'over concentration' on this.[19] Perhaps the lack of real success of the early endogenous growth models also contributed to waning interest in growth theory. In any event, research into growth theory became relatively inactive for some 15 years from about 1970, attention turning to short-term fluctuations such as the incorporation of rational expectations into business-cycle models (Barro and Sala-i-Martin, 1995).

[19] Dixt (1990) remarked that " the early enthusiasts in this field had to grapple with a new and difficult technique, and sometimes neglected the economics of their models, thereby earning a reprimand....from Hahn's (1968) review of Shell (1967a).... for an *"unseemly haste to get down to the Hamiltonian"* ".

1.2.3 Modern endogenous growth theory

Romer (1994) has emphasised that both empirical and theoretical streams of work have led to the development of modern endogenous growth theory. Empirical work on convergence initially posed a problem for the neoclassical model. But when the model was augmented with a broad definition of capital that included the human as well as the physical variety, it could be made to fit the data (Mankiw, Romer and Weil, 1992). But other models with endogenous growth and which could also explain the data on convergence had also emerged. For example, Romer (1987a) employed a model with knowledge spillovers from capital investment (like that of Arrow, 1962b), and with negative externalities from increases in the labour supply, arguing that such increases could reduce incentives for labour-saving capital. And Barro and Sala-i-Martin (1992) devised a technology diffusion model to explain differences in the stocks of technology that slowly converged over time. The multiplicity of models that fitted the limited data created further unease and dissatisfaction with the neoclassical model. This reinforced the search for a more comprehensive theory that not only fitted the data, but also explicitly recognised the special nature of knowledge or technology as a *rival* input to production that necessarily precluded perfect competition.

The work of Romer (1986a) began a new research boom that became known as 'endogenous growth theory'. After all the false starts of the 1960's research, Romer (1986a) showed that a competitive equilibrium with externalities was possible in a model with long-term growth in per capita income driven by the endogenous accumulation of knowledge by private, profit maximising firms. New knowledge was the product of a diminishing returns research-technology (necessary to ensure that consumption does not diverge - and proved in Romer, 1986c); and its creation by any one firm produced external benefits for all others (essential for the existence of an equilibrium). But the crucial feature of the model, necessary to generate unbounded growth, was increasing marginal productivity of knowledge. This is what distinguished Romer's model from those of Arrow (1962b) and Sheshinski (1967). The production for a firm i could be taken as $Y_i = A.F(A_i, K_i, L_i)$, where A_i is firm specific knowledge; K_i and L_i are its inputs of capital and labour; and $A = \Sigma A_i$ is the publicly available aggregate stock of knowledge. While output Y_i exhibited overall increasing returns to scale, in order to ensure the existence of a competitive equilibrium F(.) had to be linearly homogeneous of degree not greater than one. The model employed dynamic optimisation on the part of consumers by taking the path of A as given, maximising utility subject to the technology it implies, and later substituting the equilibrium condition $A = \Sigma A_i$.[20]

But Romer realised that the model failed to deal properly with the firm specific inputs of knowledge (Romer, 1990a). As noted before, the function F(.) had to exhibit non-increasing returns to scale. But the non-rival nature of the input A_i means that with A and A_i fixed, F(.) should show constant returns to scale in K_i and L_i, and therefore increasing returns in A_i, K_i and L_i. Also, to make the dynamics tractable Romer ended up assuming that the firm specific technological input, the excludable portion of firms' research, was employed in fixed proportion to capital. As a result the model turns out to

[20] Romer regarded this demonstration of how 'analysis of this kind of sub-optimal equilibrium can proceed even though the equations describing the equilibrium cannot be derived from any stationary maximisation problem' as one of the main contributions of his paper. Also see Romer (1989a).

be equivalent to one like that of Arrow (1962b) where learning-by doing is incidental to the production of capital. These problems were corrected in Romer (1990b).

Young's (1928) qualitative growth model based on increasing returns from specialisation was eventually formalised by Romer (1986b and 1987). Romer captured the productive efficiencies of specialisation by employing a production function in which output is increasing in the total number A, of specialised intermediate inputs.[21] Each type of specialised input was produced out of a 'primary resource' (Z) in fixed supply, with a cost function that involved fixed costs. Thus, market power, and therefore imperfect competition, was necessary to recoup these fixed costs. The extent of variety A was a matter of choice, but one that was limited by the fixed costs and the fixed supply of Z. While the production of final output was perfectly competitive, it turns out that there is a single producer for each type of specialised intermediate, and the equilibrium in that sector is one of monopolistic competition. Then, the equilibrium is calculated by having the monopolists maximise profits by setting marginal cost equal to marginal revenue in the normal manner. This generates an equilibrium level A_E, which depends on capital, and a production function that exhibits increasing returns to scale in its other factors (capital and labour) given A_E. While the model has no true technological externality it behaves as if it does. In the spirit of Scitovsky (1954), Romer interprets it as having a *pecuniary externality*. The key feature of this model is the fact that it shows how a decentralised market equilibrium in the presence of increasing returns (already shown to necessarily accompany the input of a non-rival input like technology or knowledge) can be supported by imperfect competition.

Lucas (1988) also produced a highly influential work. He proposed human capital accumulation as an alternative to technological progress as the source of sustained growth. Following Uzawa (1965), human capital (h) was produced in a separate education sector, but depended on the time people spent being educated rather than on labour inputs. Of course in both models the accumulation of the growth factor required resources to be withdrawn from production. As was the case in Uzawa (1965), the accumulation of human capital was characterised by constant returns to its existing stock ($\dot{h}(t) = \delta h(t)[1 - u(t)]$, where [1-u(t)] was the fraction of time spent in education). In combination with constant returns to scale in production[22] this generated sustained endogenous growth in per capita income at the rate $g = \delta(1-u^*)$, where u^* maximised consumers' intertemporal welfare. Because consumers' CEIS utility function was more complex than Uzawa's linear one, richer transitional dynamics are generated. But these were not worked out.

In a separate model Lucas (1988) also investigated human capital accumulation from learning-by-doing. The model was highly stylised (for example there was no physical capital). There were two consumption goods produced and human capital was accumulated according to the proportion of labour employed in the production of each good. Simply being engaged in production made people more and more productive thereby generating growth endogenously. The transition in this model is between the mix of consumption goods.

[21] In addition to the source references of Romer, further details of the production function may be found in Appendix 2.1 here; and further discussion of the modelling issues in Section 5.1.

[22] Lucas (1988) actually introduced an external effect from human capital (its average level) into production. But this is not necessary for endogenous growth.

Rebelo (1990 and 1991) studied two models capable of generating endogenous growth. In the first, the so-called 'AK-model', the production function was simply Y=AK, with K given the interpretation of a broad range of capital including human capital. This led directly to endogenous per capita growth; but there were no transitional dynamics. In the second model Rebelo disaggregated the broad measure K into physical and human capital a in model similar to that of Lucas (1988), but where physical capital was introduced as an input into the accumulation of human capital, while maintaining constant returns of $\hat{h}$ with respect to stock of human and physical capital together. This allowed Rebelo to conduct some policy analysis. In particular, he showed how a change in income tax could affect the steady state growth rate (unlike the case in Lucas, 1988).

Jones and Manuelli (1990) overcame the lack of transitional dynamics from the AK-model by considering a production function that was a simple additive combination of the 'AK' and neoclassical ones. Specifically, it took the form $Y=F(K, L)=AK+\Omega(K,L)$. Where the AK part of F(.) generated endogenous growth, and the Ω(.) part, which was neoclassical, produced the transitional effects.

This early endogenous growth theory, much of which built upon the work of the 1960s, growth was sustained because diminishing returns were avoided for a broad composite of factors that could be accumulated.[23] But these models did not come to grips with the problem of having a rival factor such as knowledge or technology as a productive input and so did not represent a proper theory of technological change. While externalities were sometimes present, their role was only to help avoid diminishing returns. The exception was the model of Romer (1986b and 1987), as already discussed.

Romer (1990b) expanded upon this earlier work. The production function continued to be increasing in the variety of specialised intermediate inputs, and these were given the interpretation of specialised capital equipment constructed out of forgone consumption, or general-purpose capital. In effect, an alternative use was allowed for the resource Z of Romer (1986b and 1987b). Also, a separate research sector was introduced to produce designs for the specialised equipment. So now the extent of variety was determined by the resources diverted from the production of goods and into the research sector. Research also generated external benefits. As already stated, it is this model that is the focus of the dissertation. Consequently it is described in detail later on (in Chapter 2).

1.3 Structure and summary of the dissertation

Overall the dissertation comprises six chapters. In addition to the current introductory chapter there are four main or substantive chapters, and a brief final chapter offering some concluding remarks. A summary and an indication of the structure of the remainder of the dissertation is as follows:

[23] Rebelo (1991), King and Rebelo (1990), and Jones and Manuelli (1990) exploited the idea (recognised by Solow, 1956) that per capita long run growth would be sustainable as long as the returns to capital were bounded from below. In neoclassical terms where $g = s\,f(k)/k-(n+\delta)$, $f(k)/k$ has to be bounded above $(n+\delta)/s$.

1.3.1 Chapter 2: Development of the Dynamic System

Chapter 2 describes the model, defining it in a fully dynamic context. Importantly, it is also formally derived as the result of dynamic optimisation problems faced by its economic agents. From a brief discussion of the public finance concepts of *rivalry* and *excludability*,[24] Section 2.1 sets up the underlying premise of Romer's model. Namely, that growth is driven by the intentional accumulation of knowledge, which is non-rival and partially excludable; and that a competitive, price-taking market equilibrium is therefore not possible. Rather, the market equilibrium is supported by monopolistic competition among the manufacturers of technological goods, who exploit market power over their customers.

Description of the model

A detailed description of all the component parts of the model is given in Section 2.2. Four factors (capital, labour, human capital and technology), and four sectors (research, capital goods manufacture, final output production and consumption) are identified and the economic behaviour of agents in each sector are described in detail. A diagrammatic representation of the model is offered to assist with its assimilation (Figure 2.1).

The exposition here mostly follows that of Romer (1990a and 1990b), although the model is generalised slightly by introducing capital depreciation; and many issues are expanded and elaborated upon in an effort to clarify them and to facilitate a thorough understanding of the model. However, the exposition here does depart from that of Romer in one significant respect. Namely, problems of dynamic optimisation that are fundamental to the model are made explicit, and certain 'first-order necessary conditions' (the so-called *transversality conditions*) are shown to be essential for the rigorous determination of economically feasible growth paths of the model. While Romer simply posits the existence of a balanced growth equilibrium with a constant 'price of technology',[25] by formally solving the two dynamic optimising problems faced by economic agents of the model and utilising their transversality conditions, here the dynamic equilibrium is derived mathematically. One of these problems, 'utility maximisation by consumers', is an explicit part of the model. The other however, is only implied, concealed by certain simplifying aspects of the model specification. Details of the Section 2.2 material are as follows:

- Section 2.2.2 describes the research sector: its output of new 'designs' being produced from the input of human capital and the existing stock of designs in such a way that a differential equation for technological change is provided directly.
- Savings and capital accumulation are described in Section 2.2.3. Here the output of research is explicitly recognised as contributing to both gross output and savings, and allowance is made for capital depreciation. A second differential equation, for the growth of capital, is furnished.
- Section 2.2.4 covers the (competitive) final output sector. The production technology employs human capital, ordinary labour, and capital. Its unusual feature is that the input of capital is disaggregated into all the different types of specialised

[24] Rivalry describes the extent to which the use of a particular good by one agent precludes its use by another. Excludability refers to the extent to which the owner of a good can prevent its use by others.

[25] His (correct) intuition no doubt coming from long experience with such problems if not from formal derivation elsewhere.

equipment that exist at any date, with each type having an additively separable impact on output. This is scrutinised in some detail in an accompanying appendix.

- The production of specialised capital goods based upon the designs from the research sector is the subject of Section 2.2.5. Monopolistic behaviour on the part of capital goods producers is first derived according to the (time independent) approach adopted by Romer (1990b). This leads to a discussion of the nature of capital in the model and reveals the fact that these producers actually face an implicit dynamic maximisation problem. It is argued that this problem is important because the transversality condition that its solution must necessarily satisfy is required to properly establish the dynamics of the system. The problem is then formalised and solved, via the *Maximum Principle*, yielding the desired transversality condition as well as the same results (for monopoly output, prices and profits) as the time independent approach.
- Section 2.2.6 defines the price of a design as the present value of all future monopoly profits from the specialised capital goods associated with the design. This generates a third differential equation - for the time path of the price of technology.
- The allocation of human capital between the research and final output sectors in such a way that wages are equated in the two sectors is derived in Section 2.2.7.
- Finally, consumer behaviour is specified in Section 2.2.8. The standard problem of determining the level of resources that are to be devoted to the production of goods for current consumption on the one hand, and the level to be used for capital formation and future production of consumption goods on the other, is addressed. It is solved as another formal dynamic maximisation problem, the discounted sum of all future consumer utility being maximised subject to the economy-wide income constraint. The solution yields a fourth differential equation – one for the growth rate of consumption – as well as a second transversality condition.

Two appendices support the material of Section 2.2. Appendix 2.1 scrutinises the production technology; describes how it generates the increasing returns necessary for unbounded growth; indicates the distribution of factor incomes; reveals a fundamental departure of the competitive model, namely, that general-purpose capital is under-valued; and discusses the somewhat peculiar nature of capital in the model. Appendix 2.2 provides a theoretical background to dynamic optimisation, with a focus on economic reasoning rather than mathematical rigour; and sets up and solves the two specific problems of the model.

The dynamic system

Derivation of the complete dynamic system is completed in Section 2.3. The significance of the optimum control problems that underlie the model, and of the 'first-order necessary conditions' known as transversality conditions that their solutions must satisfy, is addressed in Section 2.3.1. It is shown that such conditions, by imposing constraints on the asymptotic behaviour of the system, either directly or indirectly also provide 'boundary conditions', which are necessary to integrate the system of differential equations and so establish the dynamics. The necessity for a 'second' transversality condition (in addition to that arising from the consumer utility maximisation problem) is also demonstrated here. In Section 2.3.2 the system of equations derived in Section 2.2 are condensed into the overall dynamic model. A system of four linked first-order differential equations in four variables (the level of technology $A(t)$; the capital stock $K(t)$; the price of technology $p_A(t)$; and the flow of consumption $C(t)$), together with the two transversality conditions are shown to

completely describe the dynamic model. The variables (including the transversality conditions) are transformed in Section 2.3.3 and the system reduced to one of only three linked differential equations. In this way its dynamic equilibrium may be represented as a *stationary* or *steady-state equilibrium* rather than one of *balanced growth*[26]. The transversality conditions are used to derive this steady-state analytically.

There are three appendices associated with Section 2.3. Appendix 2.3 provides further explanation of the meaning and economic interpretation of transversality conditions; and Appendix 2.4 emphasises their essential nature by demonstrating that if they are ignored and only the differential equations are considered, the Romer model produces two dynamic equilibria or steady-states. Finally, in Appendix 2.5 some supplementary variables, such as savings rates, capital-output ratios and factor shares of income are derived for use in later economic analyses with the model.

Calibration and sensitivity

In the remaining section of Chapter 2, Section 2.4, the model is calibrated by establishing an annual time dimension, and by setting the values of its parameters and exogenous variables to levels commensurate with various macroeconomic magnitudes of the Australian economy. This set is termed the benchmark parameter set and it is used widely throughout the dissertation as a set of initial conditions. The sensitivity of the steady-state to changes in these *benchmark* parameter values is then assessed.

Calibration of the model is necessary to ensure that its exogenous or input parameters generate endogenous (or output) variables whose magnitudes conform to actual quantitative economic measures. Of course the input parameters themselves must also satisfy any definitional or theoretical constraints and must also conform to relevant empirical data. The broad method adopted to effect the calibration is described in Section 2.4.1. Two stages go to make up the calibration process. The first is the identification of constraints, consistent with the empirical data, for both the parameters and the endogenous variables that they generate. This is undertaken in Sections 2.4.1.1 and 2.4.1.2. The second stage is the determination of actual parameter values from within these ranges. Section 2.4.1.3 describes the process adopted. Briefly, it was one of *endogenising the parameters*. Empirically quantifiable variables (including some new ones functionally related to the parameters) were set exogenously and the model used to calculate consistent parameter values endogenously, all of these subject to their constraints.

Finally, in Section 2.4.2, the sensitivity of the model's steady-state to changes in these *benchmark* parameter values is assessed. This is precisely the form of comparative statics analysis undertaken by Romer (1990a and 1990b). Some interesting results are identified and discussed, including a somewhat controversial relation between the growth and interest rates noted by Romer (1990b). Appendix 2.6 presents the transitional dynamics resulting from changes to parameters that produce no permanent effects on many variables. The aim being to demonstrate that dynamic analysis reveals many significant and persistent, though transitory, changes concealed by comparative statics analysis.

[26] Contrary to the earlier quote from Chiang (p. 3), this also allows the phase-space of the system to be visualised as a three dimensional figure (see Chapter 3, Section 3.3).

1.3.2 Chapter 3: Dynamic Behaviour of the System: Approximations

Analysis of the dynamic behaviour of the Romer model is commenced in Chapter 3. Because the dynamics of the complete (non-linear) model cannot be solved analytically, numerical methods or other approximations must be employed. Applying numerical methods to the full non-linear model is the subject of Chapter 4. Here the dynamics are investigated by the application of three simpler 'analytic characterisations' of the model. These are referred to as the *linearised model; phase-space analysis;* and the development of a simplified abridgment of the model termed the *Solowian-Romer model.*

Linearisation of the model

Section 3.2 deals with the linear approximation to the model. In Section 3.2.1 the model is linearised by first-order Taylor series expansions about its steady-state; and an analytical or *closed form* solution (always possible for linear systems) is derived.[27] This solution furnishes explicit and independent functions for the evolution of the dynamic variables over time. Computation of the eigenvalues of the matrix of coefficients of the linearised dynamic equations confirms a theoretical proposition that the system exhibits the property of *saddle-path stability* about its steady-state.[28] Three appendices support the preceding analysis:

- Appendix 3.1 contains the algebra of the linearisation of the model;
- Appendix 3.2 presents the solution of a (first-order) system of linear equations, and derives formulae for computing the (constrained) initial values necessary for equilibrium and for the complete solution or saddle-path of the system; and
- Appendix 3.3 shows how to calculate the eigenvalues and eigenvectors required for the solution of the linearised system, and then computes numerical values for the eigenvalues for a wide range of parameter values.

A simulation of the transient dynamics of the linearised model is undertaken in Section 3.2.2. The system is taken to be initially at its benchmark steady-state equilibrium, when a sudden and sustained shock is applied to a parameter reflecting the income share of capital (γ). Calculations made according to the analytic solution of Section 3.2.1 then furnish the paths taken by economic variables as they adjust to the new steady-state equilibrium. Both initial jumps as well as the subsequent smooth adjustments are calculated and presented graphically. The shock analysed here is considered to come as a surprise to the market, which also believes it to be permanent (or at least indefinite). Accordingly, such shocks are labelled as *unanticipated* and *sustained.* It is also possible to analyse the transient dynamics of the linearised model in response to *anticipated* shocks and to *temporary* shocks. Appendix 3.4 provides a derivation of the method and applies it to two additional simulations: an 'anticipated and sustained rise in the

[27] As a form of verification of an issue covered in Chapter 2, this solution demonstrates that for the system to attain its steady-state equilibrium certain constraints on its initial values must be met; and that these constraints result from the satisfaction of the transversality conditions associated with the underlying dynamic optimisation problems to the system.

[28] The dynamic behaviour and stability of the system nearby its steady-state equilibrium is shown to depend on the signs of these eigenvalues and upon whether these are real or complex numbers.

productivity of researchers', represented by the parameter ζ; and 'an unanticipated but temporary rise in the income of ordinary labour', achieved via the related parameter α).[29]

Section 3.2.3 examines the rate at which the linearised model converges towards its steady-state. Two measures are defined and evaluated. The *convergence coefficient* measures the 'proportional rate at which the gap between the system's current position and its steady-state equilibrium is closed'; and the *half-life* measures 'the time taken for half of the gap between the system's current position and its steady-state equilibrium to be closed'. The convergence coefficient is shown to be equal to the negative eigenvalue of the system. Details of the work here are provided in Appendix 3.5, which derives a log-linearisation of the model and proves that the 'linear' and the 'log-linear' convergence coefficients are asymptotically equal.

Phase-space analysis

An analysis of the *phase-space* of the Romer model is undertaken in Section 3.3, the object being to present a visualisation of the dynamics of the system. This involves breaking up the vector space of the dynamic system into a number of regions, each of which defines, for all principal dynamic variables, whether any individual variable must always increase or must always decrease within the region. The boundaries of these regions are the so-called *phase surfaces*, each one representing the locus of points for which a different dynamic variable is constant over the vector space. These are derived, calculated and three-dimensional graphs of the phase-space are plotted in Section 3.3.1, where the movement of each principal dynamic variable in each region is also defined, and the (two) regions from which the saddle-path must emanate are determined. Details of the calculations, and an examination of the sensitivity of the surfaces to parameter changes, are provided in Appendix 3.6. Then in Section 3.3.2, by borrowing a procedure from Chapter 4 (the *eigenvector-backward integration* technique), the saddle-path of the model is computed for the benchmark steady-state and added to the graphical representation. While all this (hopefully) assists in promoting an understanding of the dynamics, it remains difficult to visualise precisely what transitional dynamics arise from different implementations of shocks. Thus, in Section 3.3.3 a schematic two-dimensional phase diagram is analysed qualitatively to explain the dynamics arising when the same shock is implemented in fundamentally different ways: namely, when it surprises the market, when it is anticipated, and when it is correctly foreseen as being only temporary.

A Solowian-Romer model

An analytic difficulty with the full Romer model is its dimension. As discussed above, this is demonstrated in Sections 3.3.1 and 3.3.2 in respect of its phase-space analysis. It is also true in terms of its numerical solution. To overcome the problem a truncated (but still non-linear) version of the model is developed and analysed in Section 3.4. It is termed a *Solowian-Romer model* because a simple Solow-type consumption function (with an exogenously constant savings rate) replaces the consumer optimising behaviour of the full Romer model. The supply-side however, remains exactly the same as in the full model. Because this construct reduces the dimension of the system, it

[29] As shown later (Section 4.5 of Chapter 4) each of these three simulations are proxies for shocks to actual economic variables.

greatly simplifies both its numerical solution and the analysis of its phase-space. Section 3.4.1 specifies the differential equations for the modified model, showing that the dynamic system is now reduced to one of only two-equations. Steady-state relationships are also derived here and it is shown that when the exogenous savings rate is set at the steady-state level for the full model, the equilibria for the two models (with one small exception) are identical.

This truncated version of the model seems a valuable instructive simplification of the complete model. But objections may be raised about the regressive nature of its demand side. This issue is addressed in Appendix 3.7, where it is proved that the Solowian-Romer model simply represents particular paramaterisations of the complete model. Moreover, it is also shown there that parameter configurations that generate equivalence between the two versions can be found close to the empirically based benchmark set. The question of how accurately the modified model reflects the dynamics of the full Romer model when the parameter settings are not such as to generate equivalence, is left to Chapter 4 (Section 4.5.4 and Appendix 4.4 in particular).

Numerical integration of the Solowian-Romer model is undertaken in Section 3.4.2 in order to conduct the same three simulations that were run under the linearised model: namely, 'an unanticipated and sustained rise in parameter γ'; 'an anticipated and sustained rise in parameter ζ'; and 'a temporary fall in parameter α'. The numerical integration technique used, is the fourth-order Runge-Kutta method (explained in detail in Appendix 4.1 of Chapter 4). With the system now comprising only two coupled ordinary differential equations this is not difficult. It is accomplished in a simple *Microsoft Excel* spreadsheet.

The final sections of Chapter 3 deal with the phase-space of the Solowian-Romer model. Necessary functions for the system's phase diagram are derived, computed and plotted (for the benchmark parameter set) in Section 3.4.3. The basic procedure is the same as that reported in Section 3.3 for the complete model. *Phase lines* for each of the two principal dynamic variables of the model are derived and computed;[30] the direction of motion of each variable in each region of the phase-space is worked out; the saddle-path is computed numerically from a technique explained in detail in Chapter 4; and all are plotted on a phase diagram. As before, the numerical technique requires calculation of the 'eigenvalues and eigenvectors of the dynamic system'. This is performed in Appendix 3.8 where the Solowian-Romer model is log-linearised.

The resulting two-dimensional *phase plane* diagram of the Solowian-Romer model, which is similar to the schematic phase diagram of Section 3.3.3, allows clear graphical analysis of the dynamics of the system from the changes to its phase lines and saddle-path generated by economic shocks. Three such analyses are undertaken in Section 3.4.4. These are for the same three shocks that were examined numerically in Section 3.4.2. It should be emphasised that these analyses are not merely diagrammatic. On the contrary, they are completely quantitative, all the functions and paths being plotted from numerical values.

[30] These correspond to the *phase surfaces* for each of the three variables of the full model.

1.3.3 Chapter 4: Numerical Integration

The broad subject of the Chapter is the computation of the transitional dynamics of the full (non-linear) Romer model. Since an analytical or closed form solution is not possible, numerical methods are adopted.

Application of numerical integration to the Romer system

Section 4.1 presents a general description of the fundamental approach of numerical integration and its application to the Romer model, illustrating the concepts with the simple Euler method. This material is supplemented in Appendix 4.1, which examines a selection of numerical integration methods. Their difference equation approximations are defined, their levels of accuracy relative to the 'step size' are noted, and the nature of some approximations is illustrated diagrammatically. The Euler, Gragg (or 'modified mid-point'), and the fourth order Runge-Kutta (RK4) methods are covered, as well as the Richardson's extrapolation technique for further increasing accuracy. Section 4.1 also emphasises the significance of boundary conditions and how they define a dichotomy of numerical integration problems as either *initial value problems*, which are relatively easy to solve, or *boundary value problems*, which are usually much more difficult. Solution of the dynamic Romer system is recognised as a *two-point boundary value problem*, with certain stock levels defining an initial condition on the variables, and the transversality conditions from the underlying dynamic optimisation problems defining two terminal conditions. The *shooting* method of solving such problems (whereby terminal conditions are replaced by 'guessed' initial conditions that are iteratively updated) is explained and its impracticality in respect of the Romer system is noted.[31]

Conversion to an initial value problem

With the unsuitability of *shooting* as a means of solving the two-point boundary value problem posed by the Romer system, some other method is required. Two different ways of converting the problem to an initial value one are considered in Section 4.2. The basic idea of using the steady-state to provide all the necessary initial conditions is explained, as is the problem of having all the differential equations identically equal to zero at that point.

Section 4.2.1 considers the *time elimination method* as a means of surmounting this problem. By using (any) one of the system's differential equations to divide all the others, the original independent variable, time, is replaced by the (originally dependent) variable corresponding to the 'dividing equation', and a new system of differential equations is developed. Formulae for the (non-zero) values of these equations at the steady-state are then derived and evaluated for the benchmark parameter set. Using these as initial values the new system is then solved to yield the saddle-path directly.[32] An alternative way of overcoming the problem of all the equations being zero at the steady-state is to use the eigenvectors corresponding to the negative eigenvalue to take a small step away from the steady-state. This is the basis of the *eigenvector-backward*

[31] *Shooting* was easy to use in the case of the Solowian-Romer model where there was only one terminal condition. However, for the full Romer model with its two terminal conditions, the integration could not be made to converge.
[32] Whereas solving the original set of differential equations yields the time paths for each variable, solution of the new set yields paths expressed as functions of the 'new dependent variable', which defines the saddle-path.

integration method described in Section 4.2.2; where it is shown to be preferable to the time elimination method. Appendix 4.2 supplements both Section 4.2.1 and Section 4.2.2. It provides details of the mathematical derivation of the time elimination method, and it solves the initial value problems from each method by applying the Euler, Gragg, and RK4 numerical integration techniques to them both. The performances of the integrations are analysed and compared and the RK4 approach is shown to be clearly superior.

Finite differences and GEMPACK

Another numerical method for solving two point boundary value problems of the type posed by the Romer model is described In Section 4.3. This is the so-called method of *finite differences*. The method itself is described in Section 4.3.1, where it is shown to involve the simultaneous solution of a large number of non-linear equations. This usually requires linearisation of the equations and the application of specialised matrix techniques. A set of computer programs known as GEMPACK, which offers a highly flexible and accurate means of achieving this, is then described in Section 4.3.2. The particular application to the Romer model of a finite differences method implemented via GEMPACK, is also considered here.

Specification of the solution method

Section 4.4 specifies the precise procedure by which the model is to be solved and the transient dynamics to be computed. Time elimination, eigenvector-backward integration, and the finite differences-GEMPACK methods are considered as candidates. Under each of these methods the necessary steps required to solve for the dynamics of unanticipated and anticipated shocks are examined in Sections 4.4.1 and 4.4.2 respectively. Section 4.4.3 summarises the issues, demonstrating that the finite differences-GEMPACK method is clearly superior. This section also records details of the actual specification of the GEMPACK program runs to be used. The analysis leading to this specification is undertaken in Appendix 4.3 where results from the same simulation, conducted under a variety of finite differencing configurations and of GEMPACK solution methods are compared. Examples of the two primary computer-input files necessary to run the GEMPACK programs are also recorded in this appendix.

Numerical results of simulations

Numerical evaluation of the dynamic properties of the full non-linear model is the subject of Section 4.5. The dynamic impacts of a variety of simulated economic shocks (specified as realistically as possible[33]) are computed according to the finite differences-GEMPACK method as specified in the previous section, and the results are analysed in terms of qualitative economic reasoning. The general approach to the simulations was to commence with the system in equilibrium at its benchmark steady-state, and then to perturb this via shocks to its parameters or variables. Seven simulations are reported:

- an unanticipated and sustained 10 per cent rise in the profit share of income, simulated by raising the parameter γ by 10 per cent (Section 4.5.1);
- both an unanticipated and an anticipated sustained rise of 15 per cent in the productivity of researchers, simulated by increasing parameter ζ by 15 per cent (Section 4.5.2);

[33] Of course, given the highly aggregated nature of the model and its lack of policy variables, the simulations are necessarily "stylistic".

- an unanticipated but temporary rise of 15 per cent in the ordinary labour share of wages, simulated by decreasing parameter α by 20 per cent (Section 4.5.3);
- a program of immigration designed to raise the overall level of human capital by 15 per cent over five years, simulated by a series of 20 cumulating shocks of 0.75 per cent each to the variable H(t) (Section 4.5.5);
- a sudden temporary reduction of 5 per cent in the capital stock, simulated by cutting the value of variable Ψ_0 by 5 per cent (Section 4.5.6); and
- a gradual loss, of 20 per cent over three years, in the human capital employed in research, simulated by a series of negative shocks to the variables $H_A(t)$ and H(t), cumulating to –20 per cent and –5 per cent respectively (Section 4.5.7).

For each of these simulations the transient dynamics of a variety of variables are recorded graphically. Particular attention is given to the initial jumps in variables and to the rate at which they approach their new steady-states. Tables of results summarise this information for each simulation. In addition, *a priori* explanations of the dynamic responses to each simulation, based simply upon qualitative economic reasoning, are offered as a sort of confirmation of the quantitative results. However, the difficulties in these exercises are also made apparent. For example ambiguities arise due to opposing influences on some variables, demonstrating the value of the quantitative dynamic results from the model.

These results also demonstrate the importance of dynamic analysis from another point of view. Two points are particularly significant. The first concerns that of the initial jumps in economic variables, which seem to be generally large, mostly difficult to predict, and usually to be either in the opposite direction to that of the 'new' equilibrium, or to overshoot it significantly. The second point concerns the slow rate at which the dynamic system converges towards its new equilibrium. Typically, the time taken for half the initial gap between the pre- and post- shock equilibria to be closed was around 15 to 20 years, and the time taken to close three quarters of that gap was some 30 to 35 years. Both points emphasise the greater relevance of the shorter-term dynamics over static comparisons between the existing state and the equilibrium of the distant future.[34]

The first three simulations here were also conducted previously (Chapter 3) under both the linearised and the Solowian-Romer model approximations to the full non-linear model. Section 4.5.4, supported by Appendix 4.4, compares and evaluates these different methods of analysis, concluding that while both alternatives are 'good' approximations, overall it is the Solowian-Romer model that is the more accurate.

1.3.4 Chapter 5: Economic Welfare and Policy Issues

The general-purpose of Chapter 5 is to examine the welfare sub-optimality of the free market solution of the Romer model and to propose policy measures that may correct for this. Two sources of market imperfection are evident from the construction and derivation of the model. First, capital goods producers engage in monopolistic behaviour, restricting supply and raising prices in order to recoup their fixed costs for

[34] This is true at least for the purposes of economic management, including counter cyclical policy. Nevertheless, evaluation of long term equilibria is most important for welfare considerations.

designs. As a result the prices of designs exceed their marginal costs. Second, research generates external benefits that are not excludable and which are therefore not included in the remuneration of the original researchers. It follows that the marginal social product of technology exceeds its marginal private product. A third, not so evident market distortion, termed the *specialisation divergence*, arises from the unusual production technology of the model. It causes the market outcome for the stock of designs at any date to be lower than is socially optimal. These issues are discussed throughout Section 5.1.

Comparison of the social optimum and market solutions

The social optimum solution to the model is calculated in Section 5.1. This not only allows the extent of the market sub-optimality to be measured, but also identifies the goals for policy measures designed to correct for it. The 'social planning problem' is specified and solved as a problem of *optimal control theory* (Appendix 2.2) to generate the dynamic system and its steady-state equilibrium. The sensitivity of the steady-state to changes in parameter values and the transitional dynamics resulting from one such change are then briefly examined in the same way as was done for the market system (Sections 2.4 and 4.5 respectively).

Comparisons of the social planning and free market solutions to the model are presented and analysed in Section 5.1.2. In equilibrium, the allocation of human capital to research, the interest rate and (therefore) the growth rate for the market solution are all found to be unambiguously less than their counterparts for the social optimum solution. And, for the benchmark parameters at least, the differences are shown to be stark. Somewhat counter intuitively, the relative magnitudes for the social and market levels of the capital-technology ratio Ψ and the price of technology p_A are ambiguous, depending on parameter values. Although for the benchmark set the intuitive results are obtained. The reasons for all these differences between the two solutions are explored in terms of the market distortions.

Four appendices support the work of Section 5.1. Appendix 5.1 sets up and solves the social planning solution of the model, including a log-linearisation in order to calculate the speed of convergence of the resulting dynamic system. Appendix 5.2 records the 'TABLO-input file' necessary for the social optimum dynamics to be computed via the GEMPACK software. Appendix 5.3 explores something of a side issue concerning an apparent constraint on the maximum allowable level of human capital in the social planning solution. And in Appendix 5.4 the relative magnitudes of the steady-state results for the social optimum and market solutions of the capital intensity variable Ψ and the price of technology p_A are analysed in terms of their variation with respect to different parameter values.

A subsidised market solution

The key question posed by the sub-optimality of the market solution is whether it is possible to identify some package of policy instruments that could overcome the shortcomings of the free market and achieve the welfare maximising results of the social planning solution. Demonstration of the fact that this is indeed possible via various combinations of subsidies is the subject of Section 5.2.

It is noted in Section 5.2.1 that because there are distortions in both the production and research sectors of the economy, two different subsidies (at least) could be expected to be required to remove them. Following a discussion of the individual distortions, apparently possible methods of correcting them are evaluated and a policy package is developed whereby:

- savings are encouraged by a subsidy (s_K) to the rentals received by households for each unit of general-purpose capital, K; and
- the purchase of research output is encouraged by subsidising the price of designs faced by capital goods producers (at the rate s_{AK}).[35]

These subsidies are then jointly incorporated into the market model and the dynamic system and steady-state equilibrium for this new 'subsidised market model' are derived (in the same way as before in Chapter 2). The new dynamic system is shown to resemble closely both that of the 'free market model' and that of the 'social optimum model'.

Section 5.2.2 is concerned with defining analytic expressions, in terms of the model parameters, for the as yet non-specific subsidies s_K and s_{AK}, which allow the decentralised market steady-state results to be converted to the social optimum ones. The 'optimum subsidy s_K' is determined directly from the knowledge that in the free market, capital is undervalued by the factor γ (Appendix 2.1). Then, by substituting this into steady-state expressions for the 'subsidised market model' and equating the results with the counterpart expressions from the 'social optimum model', the 'optimum subsidy s_A' is determined. The effects of the individual imposition these optimum subsidies is also examined and, because of their impacts on the growth rate, it is concluded that s_K may be thought of as correcting for static distortions, and s_{AK} as correcting for dynamic ones. In the final part of Section 5.2.2, all these analytical results are confirmed numerically with calculations based on the benchmark parameter values.

Appendix 5.5 is associated with the material covered in Section 5.2. It shows how it is possible to combine different 'production side' subsidies with a subsidy to the purchase of designs, to produce alternative policy packages that will again convert the free market steady-state outcomes to those of the social optimum. It also shows that it is not possible to replicate the social optimum steady-state through a subsidy to research wages.

Adjustment of the subsidised market model to the social optimum steady-state

Section 5.3 deals with the transitional dynamics of the subsidised market system in moving from its (sub-optimal) free market steady-state to the socially optimal equilibrium. The subsidy rates s_K and s_{AK} are both initially set at zero to generate the market equilibrium, and then raised to their 'optimum levels' to produce a dynamic system with an identical steady-state to that of the social optimum. The two-point boundary value problem is the same as that confronted in Chapter 4 and is solved in exactly the same manner.

[35] The existence of alternative policy packages, based upon subsidising the purchase of specialised capital or on the manufacture of final output are also discussed and referred to an Appendix.

Three alternative simulations of the many different ways in which the optimum subsidy levels could sensibly be implemented are conducted and analysed. In each case the quantitative model results are supported by qualitative explanations of the economic mechanisms behind them. The three simulations undertaken are:

- Section 5.3.1: where both the subsidy to the rentals received for general-purpose capital s_K, and the subsidy to the purchase of technology s_{AK}, are assumed to be instantly applied at their full 'optimum' levels with no prior announcement, and without the market anticipating their introduction in any other way;
- Section 5.3.2: where the subsidy to the rentals received for general-purpose capital s_K is applied as in the first simulation, but where implementation of the optimum subsidy to the purchase of designs s_{AK} is delayed by five years, this being announced at the time s_K is implemented; and
- Section 5.3.3, the mirror image of that of the second simulation. Now it is the designs subsidy s_{AK} which is fully implemented unannounced, and the capital rentals subsidy s_K, whose full implementation in five years time is announced in advance at the same time as the designs subsidy is being introduced.

Section 5.3.4 notes that even from these simple alternatives it is apparent that different methods of implementation of the optimum subsidies can result in significant variations of the adjustment paths by which the social welfare maximising steady-state is approached.

There are two appendices relevant to Section 5.3. As it does for the dynamic system of the social optimum, Appendix 5.2 also records the 'TABLO-input file' necessary for the subsidised market dynamics to be computed via GEMPACK. Appendix 5.6 examines the transitional dynamics of simulations in which each of the two optimum subsidies, s_K and s_{AK}, are implemented individually. Qualitative explanations of these results form the basis for similarly explaining the outcomes of the joint implementation simulations of Section 5.3. Appendix 5.6 also presents exactly the same individual simulations for the 'alternative' optimum subsidies identified in Appendix 5.5.

1.3.5 Chapter 6: Concluding Remarks

Three broad issues are covered in Chapter 6. First, some conclusions are drawn from the transitional dynamics, both in respect of the technical methods examined and employed, and of the dynamic results themselves. Second, some important policy implications to emerge from the work are identified and discussed under the headings of the 'transitional dynamics' and the 'sub-optimality of the model'. Finally, some unresolved issues and other matters for further research are identified.

Chapter 2

2 The 'Romer Model': Development of the Dynamic System

2.1 Introduction

Over a series of articles from 1986 through 1990 Paul Romer developed a model of economic growth in which the fundamentally different nature of technological innovations compared to that of most economic goods plays a central role (see Bibliography: particularly Romer, 1990b and Romer, 1990c). Technological change is the principal engine of growth. Despite the gradually decreasing magnitude of the *residual* from the vast literature on 'growth accounting', this is uncontroversial. A history of observational evidence led Kaldor (1961) to include among his famous *stylised facts* about growth, that 'output per worker rises continually and productivity growth rates show no tendency to decline'.[1] Of course capital accumulation is also a powerful force driving the growth of per capita incomes (Dowrick and Nguyen, 1989). But since new capital equipment frequently embodies new technologies, rising investment-output ratios also imply rising technology-output ratios. It is in this way that technological progress provides the incentive for continued capital accumulation.

From studies of public finance, economic goods are characterised according to two basic attributes: their degree of *rivalry*, and the degree to which they are *excludable*. Rivalry describes the extent to which the use of a particular good by one agent precludes its use by another. Excludability refers to the extent to which the owner of a good can prevent its use by others.[2] While rivalry is a function only of the technological attributes of a good, excludability depends upon both technology and the legal system (patents, copyright etc.). Most rival goods are also excludable; and conventional, privately provided economic goods are predominantly rival and excludable. At the other extreme, public goods are (by definition) both non-rival and non-excludable, and as such they cannot be privately provided.

Technology, or knowledge, has long been taken to exhibit public good characteristics. As an input to production it is largely non-rival. A new design, a set of instructions, a computer program can all be used without diminution by an indefinitely large number of agents as often as desired, and at little additional cost once the (probably high) initial costs of development have been met. Non-rivalry in the technological input necessarily introduces a so-called non-convexity into the production function, which must show increasing returns in respect of all inputs together. This can be readily demonstrated by a

[1] Further evidence of the positive relationship between growth and the accumulation of technological knowledge is provided the by Romer (1986a), and the vast literature on 'growth accounting' (see footnote 15 of Chapter 1 for a small sample). Schmookler (1966) also observed the positive relationship between growth and technological knowledge at the industry level in his comprehensive and detailed study of inventions.

[2] Arrow (1962a) refers to the degree to which they are *appropriable*, but the meaning is the same.

simple replication argument, noting that it is not necessary to replicate non-rival inputs, (Romer, 1990c). Because of this, a price-taking equilibrium cannot hold unless technology is regarded as a public good, both non-rival and completely non-excludable. Market power is necessary to achieve an equilibrium in which technology is (at least) partially excludable, an idea that goes back to Schumpeter (1942).

Non-rivalry also carries a strong implication of non-excludability through externalities. And external benefits and spillovers of knowledge are undoubtedly important outcomes from innovative activity. Arrow (1962a) notes that "no amount of legal protection can make a thoroughly appropriable commodity of something so intangible as information".[3] Nevertheless, the fact remains that much (perhaps most) innovative activity is undertaken by private agents with the expectation of economic gain.[4] The total benefits from technological improvements and the generation of knowledge must therefore be at least partially excludable. Consistent with this, growth in Romer's model is driven by the intentional (and endogenous) accumulation of knowledge, which is non-rival and partially excludable; and a decentralised market equilibrium is supported by monopolistic competition among the producers of capital.

2.2 Description of the model

2.2.1 General

Romer's (1990b) model comprises four factors: capital, labour, human capital, and technology. Capital (K) is represented by a large variety of specialised equipment available for final goods production, with the extent of the variety depending on the level of technology. The usual representation of ordinary (unskilled) labour (L) is adopted. Knowledge is separated into a rival component, human capital (H) which is embodied in people, and a non-rival technological component (A) which is independent of individuals and can be accumulated without bound on a per capita basis.

The economy's aggregate supplies of both ordinary labour and human capital are taken as exogenous and constant. Ordinary labour has no alternate use to that in the production of final output (including no labour-leisure trade-off) and so it is all devoted to this. However, human capital is employed in both research and in final output production and its allocation between these sectors is endogenous.

Technology, which grows over time with research effort, is represented as a stock of (non-rival) designs for all the different types of specialised capital equipment. The designs are excludable in terms of their **direct use** in the production of specialised

[3] It should be emphasised that externalities are included in the model because they are important in the real world, not because they are necessary to generate growth endogenously (Romer, 1987b).

[4] Private rates of return on research and development in excess of 30 per cent, and social rates of more than twice as much have commonly been estimated (for example, see Griliches, 1973; Mansfield et al., 1977; and Scherer, 1982). And at the industry level again, Schmookler (1966) has provided hundreds of detailed and specific examples for which the driving force could clearly be identified as prospective profitability.

capital: For example they are patentable so that any particular type of specialised capital good can only be produced by a firm which owns the design for it. However, in adding to the general stock of design knowledge and contributing to subsequent designs, each design makes an **indirect** contribution to production that is not excludable. Overall, technology is only partially excludable.

Formally the model comprises three supply-side sectors[5] and a consumption sector. A research sector employs human capital and the existing stock of knowledge to produce new knowledge in the form of designs for new types of specialised capital equipment. A capital goods producing sector purchases the designs and uses them with forgone consumption (or general-purpose capital, K) to produce a wide variety of specialised capital goods. These are then rented (or purchased) by a final goods producing sector, which uses them in conjunction with labour and human capital, to produce final output that can either be consumed, or saved and invested (Figure 2.1).

2.2.2 Research Sector

The research sector is highly stylised: The output of research (technological knowledge growth) is manifested totally as expansions in the range of specialised capital equipment and the designs for them, and these never become obsolete.[6] This research output is a deterministic function of its inputs, subject to none of the uncertainties present in the real world. And, the idea that the production of new knowledge is relatively intensive in its use of human capital and existing knowledge is captured by specifying these as the **only** inputs. In reality of course, particular types of capital equipment would also be essential.[7]

Thus, aggregate production of designs is taken to be a deterministic function of the research inputs of human capital and the existing total stock of design knowledge, A(t). Specifically, the rate of increase of designs is:[8]

$$\dot{A}(t) = \zeta H_A(t)A(t) \tag{2.1}$$

where ζ is a productivity parameter, and $H_A(t)$ is the human capital employed in research. This specification neatly reflects the view expressed by Arrow (1962a) that "information is not only the product of inventive activity, it is also an input – in some sense the major input apart from the talent of the inventor" (p. 618). Note that the productivity of human

[5] Actually, this is merely a convenience to facilitate understanding the flows and transfer prices involved. A variety of institutional arrangements could apply.

[6] Other types of technological progress, such as expansions in the range of consumer or final output goods, and the development of capital goods of greater quality and productivity, which consequently render their predecessors obsolete, have also been analysed in the same sort of growth context as for the current model. For example, see Grossman & Helpman (1991a, b & d).

[7] The reason (of course) is mathematical tractability. Romer (1989b) considers more realistic research technologies, but not to the analytic degree of this model. On the other hand, Rivera-Batiz & Romer (1991a) analyse a model where the technology of research is identical to that of final output production (the *lab equipment* specification). While this is tractable, it fails to capture the essential factor intensities: the existing state of knowledge plays no part in the generation of new knowledge, and research is no more intensive in its use of human capital than is the production of final output.

[8] Time derivatives of variables are indicated throughout by the 'dot notation'. Thus, dZ(t)/dt is denoted by $\dot{Z}(t)$. Partial differentiation is represented by the usual $\partial/\partial t$ notation.

capital in research is an increasing function of accumulated knowledge. As a result, the costs of producing new designs decline over time. This is the source of the externality in research.

With the price of designs given by $p_A(t)$ measured in terms of the output of the consumption good[9], the value of the output of research is:

$$R(t) = p_A(t)\dot{A}(t) \tag{2.2}$$

2.2.3 Savings and capital accumulation

In most macroeconomic models capital is viewed as being produced directly out of forgone consumption.[10] The consumption forgone is taken to represent aggregate savings, and in a closed economy this is equated with gross capital investment. In the Romer model this would represent only a narrow view of aggregate savings. A broader view would recognise that the output of goods is not the only thing produced by the economy. The research sector produces designs. These add to the general stock of technological knowledge and help to produce first capital, and then consumption goods. In order to increase future consumption, the economy as a whole 'chooses' to divert resources from the current production of goods and into the production of technology in much the same way as consumers choose to forgo some current consumption in favour of capital formation. Thus, *gross product* of the economy comprises both the output of goods and the output of the designs from the research sector:

$$GP(t) = Y(t) + R(t) \tag{2.3}$$

Taking aggregate savings as the value of gross product which is not consumed, means that under this broad view aggregate savings comprise both those made in the form of research and those in the form of capital accumulation:

$$S(t) = S_K(t) + S_R(t) = Y(t) + R(t) - C(t)$$

But savings made in the form of research amount to the total value of resources devoted to research, which of course, is also the contribution to gross product from research - $S_R(t) = R(t)$. Thus, the amount of savings available for capital formation is the value of forgone consumption out of the total output of goods, which, since the economy is closed, equals aggregate capital investment:

$$I(t) = S_K(t) = Y(t) - C(t) \tag{2.4}$$

The number of different types of specialised capital goods increases as new designs are developed. At any time t there are A(t) different types in existence, and there are $X_i(t)$ units of type i. Since the capital goods sector employs the same technology as that of

[9] All prices in the model (including interest rates) are measured in real terms: Specifically, they are expressed in units of the output good which acts as the model's numeraire.

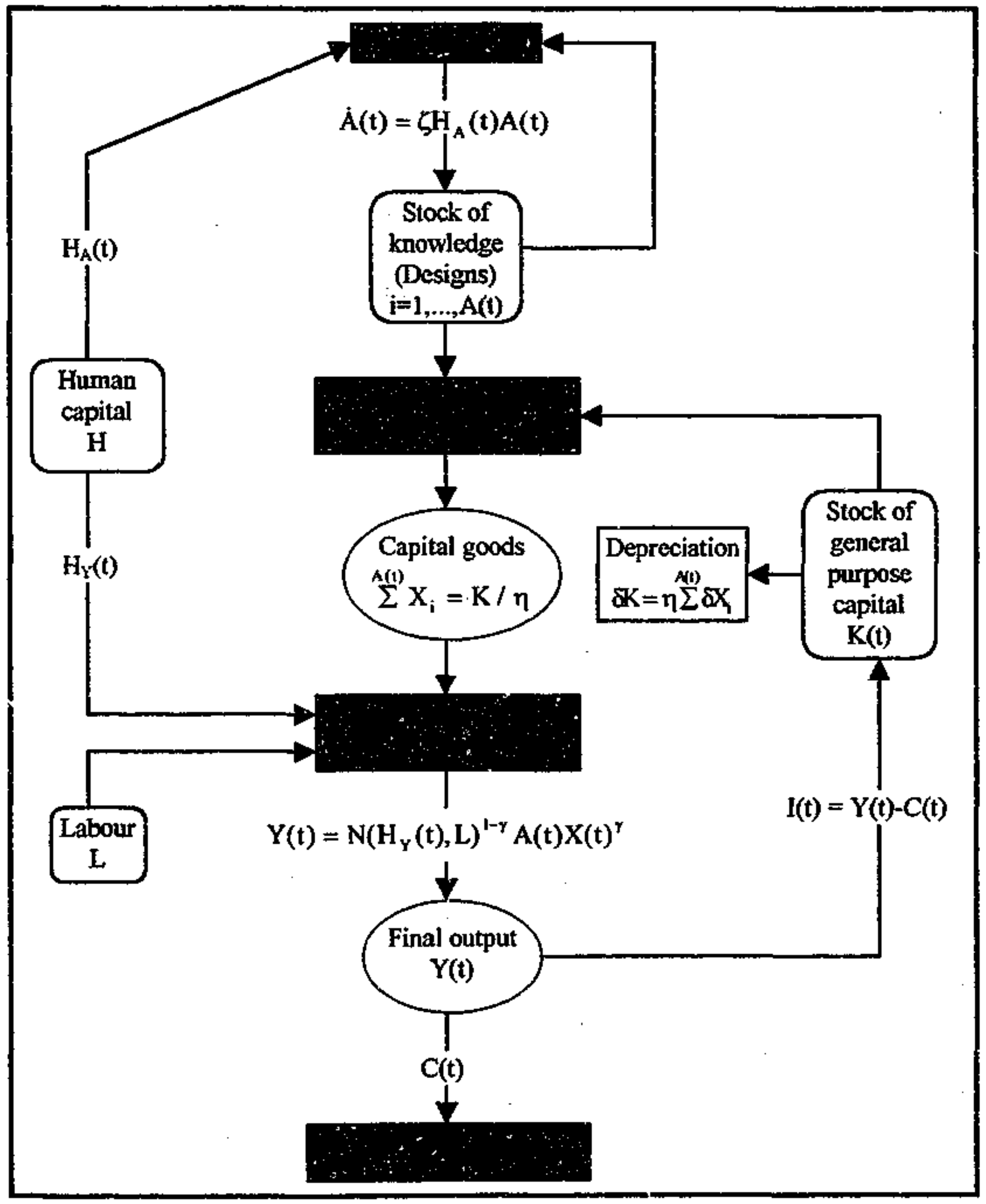
[10] *Forgone consumption* is said to be used to produce capital, but such output is not actually produced. Rather, the resources that would have been necessary to manufacture it are devoted instead to the production of capital. Given the existence of designs, the production technology for capital goods is identical with that of final output.

final output, it is possible to *exchange* consumption goods for capital goods. If it requires η units of output to produce one unit of any of the different specialised types, the aggregate capital stock at time t is given by:

$$K(t) = \eta \sum_{i=1}^{A(t)} X_i(t) \quad \text{or, ignoring indivisibilities} \quad K(t) = \eta \int_0^{A(t)} X(i,t)\,di \tag{2.5}$$

This aggregation, K(t), can be thought of simply as an accounting measure, specifying all the *potential consumption* embodied in the specialised capital equipment. It may also be thought of as an amorphous form of *general-purpose capital* that can be readily converted into any of the existing or future types of specialised equipment. Furthermore, as will become clear soon, in this model re-conversion back to general-purpose capital is also possible, and both conversion and re-conversion are costless.

Figure 2.1: Diagrammatic representation of the Romer model.



All capital goods are assumed to depreciate at the constant rate δ, so aggregate depreciation is given by:

$$D(t) = \eta\sum_{i=1}^{A(t)}\delta X_i(t) = \delta\eta\sum_{i=1}^{A(t)}X_i(t) = \delta K(t)$$

and capital accumulation at all times t, is therefore determined by the relation:

$$\dot{K}(t) = I(t) - D(t) = Y(t) - C(t) - \delta K(t) \tag{2.6}$$

2.2.4 Final output sector

Labour (L), human capital (H_Y), and physical capital in the form of specialised equipment (X_i), are the inputs to the production of final output. With capital expressed in these terms the production function exhibits diminishing returns (marginal productivities) for each input, and is linearly homogeneous with respect to all inputs together. Aggregate (economy-wide) output is given by:

$$Y(t) = N(t)^{1-\gamma}\sum_{i=1}^{A(t)}X_i(t)^{\gamma} \quad \text{or} \quad Y(t) = N(t)^{1-\gamma}\int_0^{A(t)}X(i,t)^{\gamma}\,di \tag{2.7}$$

where $N(t) = N(H_Y(t),L)$ can be regarded as a measure of the *composite labour* used in final goods production, comprising the amount of human capital, $H_Y(t)$, and the amount of ordinary labour, L. Specifying N(.) as a 'Cobb-Douglas' function:

$$N(H_Y(t),L) = H_Y(t)^{\alpha}L^{(1-\alpha)} \tag{2.8}$$

generates the production function for final output as:

$$Y(t) = H_Y(t)^{\alpha(1-\gamma)}L^{(1-\alpha)(1-\gamma)}\sum_{i=1}^{A(t)}X_i(t)^{\gamma}$$

or (2.9)

$$Y(t) = H_Y(t)^{\alpha(1-\gamma)}L^{(1-\alpha)(1-\gamma)}\int_0^{A(t)}X(i,t)^{\gamma}\,di$$

This production technology is somewhat unusual in that capital is disaggregated into all the different types of specialised equipment available at any time ($X_i(t)$, for $i = 1,\ldots\ldots,A(t)$). Thus, in addition to the usual inputs of capital and labour (the latter including human capital here), output in the model also depends on the input of designs. While the production function exhibits overall constant returns to scale in terms of the inputs of labour, human capital, and the range of specialised capital equipment, when account is taken of the fact that all of this specialised equipment is constructed out of more fundamental factors of production, namely, the accumulated stocks of designs A(t), and saved output K(t), then increasing returns become evident. It is this aspect of the production function that generates endogenous growth of output in the model. In terms of these 'more fundamental factors' the production technology exhibits Harrod neutral or

labour-augmenting technical change, which can be shown to be a necessary condition for a steady state equilibrium to exist![11] These features of the production technology are discussed in more detail in Appendix 2.1.

The final output sector is characterised by competitive price taking and constant returns to scale. Producers take rental prices $p_X(i,t)$ for each type of specialised capital equipment i as given, and choose the quantities X(i,t) to maximise profits. In aggregate the problem is, given $H_Y(t)$ and L:

$$\text{Maximise} \int_0^{A(t)}[N(H_Y(t),L)^{1-\gamma}X(i,t)^{\gamma} - p_X(i,t)X(i,t)]di$$

Since each type (i) of specialised equipment has an additively separable effect on production, the maximisation may be performed through the integral, thereby yielding the demand function:

$$p_X(i,t) = \gamma N(H_Y(t),L)^{1-\gamma}X(i,t)^{\gamma-1} = \gamma H_Y(t)^{\alpha(1-\gamma)}L^{(1-\alpha)(1-\gamma)}X(i,t)^{\gamma-1} \tag{2.10}$$

2.2.5 Capital goods producing sector

Firms in the capital goods sector convert general-purpose capital, which they rent from households, into specialised equipment according to designs they purchase from the research sector. The firms bid against one another for the sole rights to manufacture from each new design. In this way the knowledge represented by designs is excludable in terms of its direct use in the production of capital goods (through the granting of infinitely lived patents for example). Each design becomes the property of only a single firm, which produces the corresponding specialised capital and which can charge a price greater than the marginal cost of production. Indeed, to recover the *fixed costs* of their design rights they must charge more than the marginal production cost. Monopoly power is restricted however, by the free entry of the bidding process, and the rental market for specialised capital goods is one of *monopolistic competition.*

Having incurred the fixed cost of purchasing a design, the producers of specialised capital take the demand functions for their equipment arising from the final goods sector as given, and set prices to maximise the excess of their rental income over variable cost. Rental income is simply $p_X(i,t)X(i,t)$. Variable costs are the gross rentals paid to the owners (households) of general-purpose capital. These may be worked out from the rate of return on such capital. With $r_K(t)$ as the rental for each unit of general-purpose capital and $v_K(t)$ its price or *value* in terms of output, then the rate of return (net of depreciation) on such capital comprises the usual income component and a capital gain component as:

$$\text{Net ror}_K(t) = \left[\frac{r_K(t)}{v_K(t)} - \delta\right] + \frac{(1-\delta)\dot{v}_K(t)}{v_K(t)} \tag{2.11}$$

[11] For example, see Solow (1970); or Dixit (1976a).

Since $v_K(t)$ is constant at the value of unity, the capital gain term vanishes and the rate of return becomes ($r_K(t)$-δ). Then, since the production technologies in the final output and capital producing sectors are exactly the same, goods can be converted into capital one-for-one, and the interest rate on goods is also the net rate of return on capital. Alternatively, since assets can also be held as claims on loans at the interest rate r(t), arbitrage will ensure that this is equated with the net rate of return to capital. Thus, the rental rate on general-purpose capital faced by capital goods producers is determined as follows:

$$r_K(t) = r(t) + \delta \tag{2.12}$$

So variable costs amount to the sum of the opportunity interest income and the replacement costs due to depreciation at the constant rate δ, on the $\eta X(i,t)$ units of general-purpose capital or *forgone consumption* needed to produce X(i,t) units of specialised equipment. That is, variable costs are: $[r(t)+\delta]\eta X(i,t)$; and the problem of the monopolist capital goods producers is:

$$\text{Maximise} \quad \pi(i,t) = [p_X(i,t)X(i,t) - (r(t)+\delta)\eta X(i,t)]$$

First order conditions yield the optimal monopoly levels of prices, quantities and profits as follows (also see Figure 2.2):[12]

$$X(i,t) = X(t) = \left[\frac{\gamma^2 H_Y(t)^{\alpha(1-\gamma)} L^{(1-\alpha)(1-\gamma)}}{\eta[r(t)+\delta]}\right]^{\frac{1}{1-\gamma}} \tag{2.13}$$

$$p_X(i,t) = p_X(t) = [r(t)+\delta]\eta/\gamma \tag{2.14}$$

$$\pi(i,t) = \pi(t) = (1-\gamma)p_X(t)X(t) \tag{2.15}$$

Thus, it is optimal in the model for all the different types of capital goods available at any time to be used at the same level. Although each type of capital good is different, they all have the same productivity and all produce identical effects on output, including new types as they are designed.[13] It follows from equation (2.5) that this level is:

$$X(i,t) = X(t) = K(t)/\eta A(t) \tag{2.16}$$

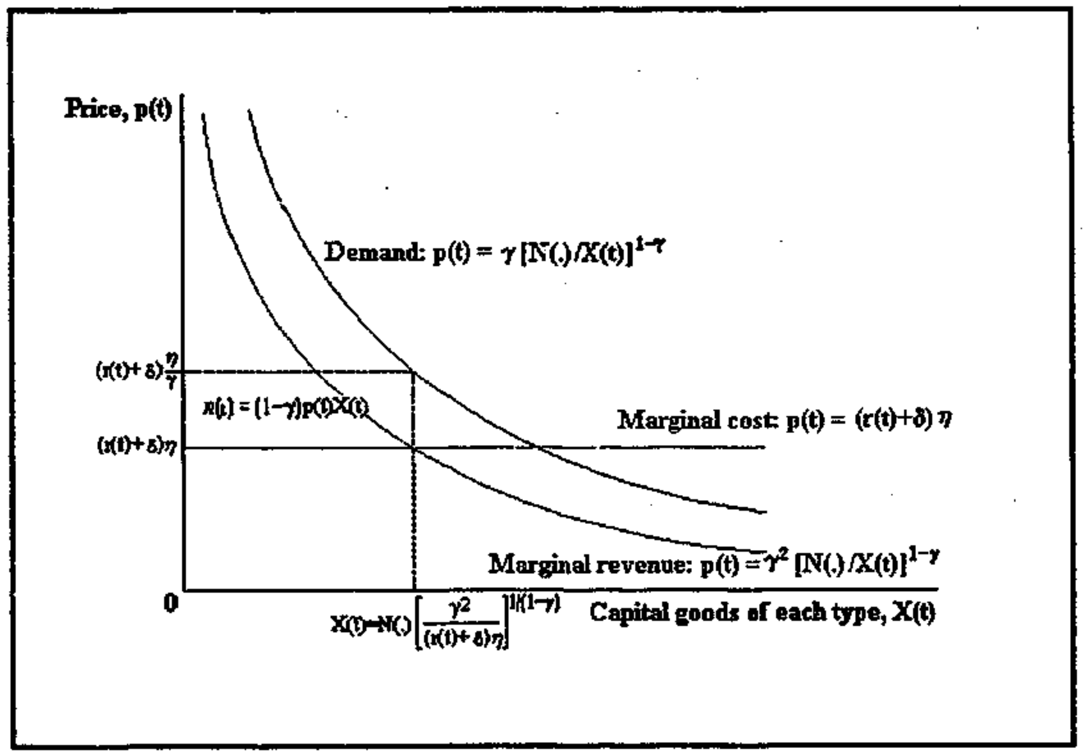
The preceding follows (more or less) the approach of Romer (1990a and b). It relies on the pliable nature of capital in the model. In particular, it depends critically on the assumption that specialised capital of any type can be costlessly *reconverted* back to general-purpose capital at all times. This sort of view of capital is not new. In fact, it is the necessary interpretation in any aggregate model of disembodied technical change where substitution possibilities exist between labour and capital. However, in a model like Romer's, where technological progress is explicitly embodied in a wide variety of specialised equipment, it is not an appealing part of the set-up (though it seems necessary for tractability).

[12] The rental price on specialised equipment, $p_X(t)$ from equation (2.14), is equal to the value of the marginal product of this type of capital. This is not the case for general-purpose capital. There the rental rate, $r_K(t)$ from equation (2.12), is less than the value of its marginal product (Appendix 2.1).

[13] Because of the usual diminishing returns in the production technology of the capital goods sector, it would otherwise be possible to increase profits by diverting resources from high to low output goods.

Updating an analogy made by Swan (1956), it is as if capital is made up of a great big box of *'lego'* which "...can be put together, taken apart, and reassembled with negligible cost or delay in a great variety of models...". At any time all the *lego* pieces are assembled, from a book of designs, into large and equal numbers of each of the many different useful models. As time goes on a few of the pieces wear out, but many more pieces become available to replace them, and some more useful designs for assembling them are also devised. If, after replacing the worn out pieces, there are not enough new pieces to construct the same number of each of the new models as there are of the existing ones, an equal number of each of the existing models are dismantled so that when these pieces are added to the new ones and used to build the new models, there will then be the same number of units for all the model types, both new and old. It must also be possible to modify all of the models so they can be operated with various combinations of labour and human capital when relative prices make such substitutions desirable. The issues associated with this view of capital are explored further in Appendix 2.1.

Figure 2.2: Monopolistic supply behaviour by the capital goods producing sector: Romer model.



d:\phd\chpt2\fig22.bitmap

In any alternative specification for which such negative investment was prohibited, or at least for which significant adjustment costs were imposed, the monopolist profit maximisation problem could not be solved independently at all points of time as before. Instead, the techniques of *dynamic optimisation* would be necessary. Of course, these techniques are equally applicable under the present *putty-putty* [14] specification of capital and so dynamic optimisation is used below to provide an alternative method of solution to that of maximising at all points of time. The motivation for this is not because it adds any extra insight to the profit maximisation solution, but because the problem is fundamentally one of dynamic optimisation, and as such its solution is required to satisfy a certain *first order necessary condition*, which ought to be made explicit. This is the so-called *transversality condition*, a constraint imposed by the problem on the terminal (or asymptotic) behaviour of the dynamic system which ensures that optimisation is achieved. The significance, and the necessity of transversality conditions in both identifying the dynamic optimum and ensuring it is attained is discussed in Section 2.3.

It is apposite to note here that when the Romer model is considered in the context of a centralised economy, in which it is possible for some social planning authority to set quantities and (shadow) prices for all goods and factors, the problem of maximising community welfare is a clearly defined dynamic optimisation one whose solution includes two transversality conditions (see Chapter 5).[15] Both of these are necessary for the (social) optimum equilibrium to be attained in that they provide the necessary boundary conditions for the transitional dynamics. Similarly, two boundary conditions are also required for the dynamics of the decentralised market solution to the model (Chapter 4), and here, short of arbitrarily imposing just the right conditions necessary to attain the market optimum, they can only be provided by transversality conditions associated with the dynamic optimisation problems of the economic agents. One is generated through the utility maximisation of consumers (Section 2.2.8); the other arises here via the profit maximisation of capital goods producers.

Formally, the problem faced by the producers of specialised capital is to choose a time path for the level of their output which maximises the discounted sum of all their future excesses of rental incomes over variable costs:[16]

$$\begin{aligned}&\text{Maximise } \int_0^\infty [p_X(i,t)X(i,t) - \eta I_X(i,t)]e^{-\int_0^t r(s)ds}\,dt\\ &\text{subject to } \dot X(i,t) = I_X(i,t) - \delta X(i,t) \qquad (2.17)\\ &\qquad X(i,0) \text{ given, and } X(i,t) \geq 0\end{aligned}$$

and where:

$I_X(i,t)$ is gross investment in $X(i,t)$; and $p_X(i,t)$ is the demand function given by equation (2.10)

[14]The terms *putty* and *clay* were used by Phelps (1963) to describe the extent of substitution possibilities between capital and labour before and after technological progress which is *embodied* in capital. Like Johansen (1959), Phelps considered substitution possibilities to be high for new investment, but zero for existing capital, the *putty-clay* case. Romer's use of the term is intended to reflect the ease with which general-purpose capital can be converted to any type of specialised capital equipment or vice versa (Romer, 1990a and 1990b).

[15] The opposite side of this technical coin is demonstrated by the 'linearised Romer model' of Chapter 3. There satisfaction of two constraints on the system's initial values are found to be required for the steady state to be approached; which in turn requires the satisfaction of two transversality conditions.

[16] Barro and Sala-i-Martin (1990) and Sala-i-Martin (1990b) adopt similar maximisation set-ups but do not draw out the full implications from the transversality conditions.

This is solved by invoking the *Maximum Principle* of Pontryagin et al. (1962), from which the first order conditions generate the identical profit maximising results for prices and quantities as obtained before when the maximisation was performed at all points of time. Details of the derivation are contained in Appendix 2.2. Importantly, the Maximum Principle also produces a transversality condition:

$$\lim_{t\to\infty} v(t) \geq 0 \quad \text{and} \quad \lim_{t\to\infty} v(t)X(t) = 0 \qquad (2.18)$$

where $v(t)$ is the shadow price of specialised capital. As mentioned above, this condition is an integral part of the dynamic optimisation process (also see Section 2.3).

2.2.6 Price of technology

Competition among capital goods producing firms to obtain the rights to any new design means that all the monopoly rents will be bid away. Thus, the research sector will be able to extract prices for its designs equal to the present value of the monopoly rents associated with each corresponding new capital good.[17] At any time t the price of designs is given by:

$$p_A(t) = \int_t^\infty \pi(\tau)e^{-\int_t^\tau r(s)ds}\,d\tau \qquad (2.19)$$

which can be solved by differentiating with respect to time:

$$\begin{aligned}\dot p_A(t) &= [-\pi(\tau)e^{-\int_t^\tau r(s)ds}]_{\tau=t} + \int_t^\infty \pi(\tau)d/dt[e^{-\int_t^\tau r(s)ds}]d\tau\\ &= -\pi(t) + \int_t^\infty \pi(\tau)[e^{-\int_t^\tau r(s)ds}\, d/dt\{-\int_t^\tau r(s)ds\}]d\tau\\ &= -\pi(t) + \int_t^\infty \pi(\tau)[e^{-\int_t^\tau r(s)ds}\{r(s)\}_{s=t}]d\tau\\ &= -\pi(t) + r(t)\int_t^\infty \pi(\tau)e^{-\int_t^\tau r(s)ds}\,d\tau\end{aligned}$$

Thus, the differential equation for the time-path of the price of technology is:

$$\dot p_A(t) = r(t)p_A(t) - \pi(t) \qquad (2.20)$$

This relation can also be seen as a condition of perfect arbitrage. Over a small period of time, (t, t+Δt) say, the interest that could be earned on an amount of money equal to the price of an asset, must be equal to the value of the alternative investment of holding the physical asset itself; that is, to the income it earns plus its capital gain over the period.

$$r_t p_t \Delta t = \pi_t \Delta t + (p_{t+\Delta t} - p_t)$$

[17]The decision of whether to incur the costs of new research and development will therefore be based upon (the expectation of) future monopoly rents exceeding such costs.

2.2.7 Allocation of human capital

As for aggregate labour (L), aggregate human capital (H) is exogenously fixed. It is also taken to be homogeneous, so there is no distinction between that employed in research and that employed in the production of final output, both of which are determined endogenously.[18] Thus:

$$H = H_A(t) + H_Y(t) \tag{2.21}$$

Moreover, the returns to human capital will be the same in each sector. In research human capital earns all the income, and its marginal (physical) product is $MP_{H_A} = \partial \dot{A}(t) / \partial H_A(t) = \zeta A(t)$. Thus its wage is:

$$w_{H_A}(t) = p_A(t)\zeta A(t) \tag{2.22}$$

That is, researchers' wages are equated to the market value of their marginal product, which in turn is based upon the market price of designs p_A. However, since this price captures only part of the of the social value of a design, the wage rate of human capital in research is actually less than the true value of its marginal product. Although some of the knowledge of designs is non-excludable, since every researcher is free to exploit **all** the knowledge, any benefits external to one researcher are captured by others. In aggregate, researchers collect all the benefits of the sale of designs. From equations (2.1) and (2.2), the total of their wages amount to the overall value of the output of research:

$$W_R(t) = w_{H_A}(t)H_A(t) = R(t) \tag{2.23}$$

Since the final output sector is competitive, human capital employed there receives the value of its marginal product. Using equations (2.9) and (2.16), wages are thus:

$$w_{H_Y}(t) = \frac{\partial Y(t)}{\partial H_Y(t)} = \alpha(1-\gamma)\eta^{-\gamma} H_Y(t)^{\alpha(1-\gamma)-1} L^{(1-\alpha)(1-\gamma)} A(t)^{1-\gamma} K(t)^{\gamma} \tag{2.24}$$

The allocation of human capital to manufacturing is then determined by equating the expressions for $w_{H_A}(t)$ and $w_{H_Y}(t)$:

$$H_Y(t) = \left[\frac{\alpha(1-\gamma)}{\zeta\eta^{\gamma}} L^{(1-\alpha)(1-\gamma)} [K(t)/A(t)]^{\gamma} p_A(t)^{-1}\right]^{\frac{1}{1-\alpha(1-\gamma)}} \tag{2.25}$$

[18]The model generates endogenous growth in technology (and hence in output per head) through research effort. However, since the aggregate supply of the input to research, human capital, is **exogenous**, much of the growth story remains unexplained - despite the fact that the share of total human capital employed in research is endogenous. This suggests a direction for further research: Namely, modifying the model to endogenise the supply of human capital, perhaps along the lines of Lucas (1988).

2.2.8 Consumption

Consumers (or households) own all the assets of the economy; and since we are dealing with a closed economy these amount to the aggregate capital stock and the accumulated value of designs.[19] They earn income from wages and interest payments on their assets of capital; and since they are competitive they take both the wage and interest rates as given. Aggregate household income equals gross product, the path of which they also take as given, along with the paths of its components: the output of goods (Y) and the output of research (R). The problem they face is to determine how much of total output to consume through the purchase of goods, and how much to save by the accumulation of additional capital. Any decision to save more, clearly means less immediate consumption is available. However, since saving also raises productive capacity it allows the possibility of greater consumption in the future. The consumption problem is to determine the appropriate trade-offs in order to maximise consumer welfare. To this end current consumers are assumed to take account of the welfare of all future consumers, and thus to maximise over an infinite time horizon.[20]

Consumer welfare is synonymous with consumer utility, which is itself an increasing function of consumption. The particular functional form employed here is the *Constant Intertemporal Elasticity of Substitution* (CIES) utility function. Since there is only one type of household or consumer in the model, it is possible to aggregate and consider only a single consumer. Thus formally, the consumption problem is 'to maximise the discounted sum of all future aggregate utility, subject to the economy-wide income constraint' That is:

$$\begin{aligned} &\text{Maximise} && \int_0^{\infty} U\{C(t)\}e^{-\rho t}dt \\ &\text{subject to} && \dot{K}(t) = W_Y(t) + r_K(t)K(t) - C(t) - \delta K(t) \\ & && K(0) \text{ given, and } K(t) \geq 0 \\ &\text{where} && U\{C(t)\} = [C(t)^{1-\sigma} - 1]/(1-\sigma) \quad \text{for } \sigma > 0 \end{aligned} \tag{2.26}$$

In this formulation $W_Y(t)$ is aggregate wages from the output of goods; $r_K(t) = r(t)+\delta$ is the rental rate on general-purpose capital; and U(.) is the instantaneous utility function (the *felicity function*), with ρ as the subjective discount rate and $1/\sigma$ the *elasticity of intertemporal substitution*. Notice that since aggregate household income corresponds to gross product, and since research wages correspond to research output, then: $W_Y(t) + r_K(t)K(t) = Y(t)$, and the economy wide income or resource constraint above corresponds to the capital accumulation relation (2.6).[21]

[19]In addition to their direct ownership claims on capital, individual households may also hold assets in the form of loans. However, in aggregate loans will be zero.

[20]Since the optimisation takes place over an infinite horizon, all future consumers are assumed to "always be governed by the same motives as regards accumulation" (Ramsey, 1928). Families or dynasties of overlapping generations, all taking account of the well-being of their progeny, are usually postulated.

[21] The economy wide income constraint is simply that aggregate income equals gross product. That is: $W(t)+r_K(t)K(t) = GP(t)$, where W(t) is aggregate wages and $r_K(t) = r(t)+\delta$ is the rental rate on general-purpose capital (equation (2.12)). Total wages are composed of those earned in the research and output

As was the case for problem (2.17), application of the *Maximum Principle* is the method employed to solve the dynamic maximisation problem (2.26). From this, first order conditions generate the following results:

$$\dot{C}(t) = \frac{1}{\sigma}[r(t) - \rho]C(t) \tag{2.27}$$

and the transversality condition:

$$\lim_{t\to\infty} \lambda(t) \geq 0 \quad \text{and} \quad \lim_{t\to\infty} \lambda(t)K(t) = 0 \tag{2.28}$$

where $\lambda(t)$ is the shadow price of (general-purpose) capital. Details of the solution are at Appendix 2.2.

The economic content of equation (2.27) can be interpreted in terms of the standard optimising condition of equating marginal benefits (or costs) from competing activities or demands. In particular, it can be seen as expressing the condition that for an optimum allocation of resources between savings and consumption, the returns from each of these should, at the margin, be equal.[22]

2.3 The dynamic system

2.3.1 Transversality conditions

The dynamic system of the Romer model is derived in part from the optimising behaviour of economic agents. As has been shown, both households (consumers) and the producers of specialised capital goods face dynamic maximisation problems. Necessary conditions furnished through the *Maximum Principle* for the solutions to these problems not only specify differential equations of the dynamic system, but also certain *transversality conditions* which impose constraints on its asymptotic behaviour (see for example Intrilligator, 1971; Chiang, 1992; or Leonard and Van Long, 1992).

Because of the particular formulations and functional forms involved in these maximisation problems it turns out that given the differential equations of the system, the transversality conditions are both **necessary and sufficient** (Appendix 2.3). For this reason it should be possible to use the transversality conditions explicitly to derive the optimum dynamic path from amongst the infinity of paths described by the differential equations alone. This is done in the following section.

It should be noted however, that this sort of procedure does not seem to be common practice in economic applications of dynamic optimisation. In most cases the dynamic system is characterised by some sort of equilibrium, either a balanced growth equilibrium

sectors, W_R and W_Y respectively, so $W = W_R + W_Y$. Then, using equations (2.3), (2.6) and (2.23), the economy wide resource constraint becomes: $W_Y(t) + r_K(t)K(t) = \dot{K}(t) + \delta K(t) + C(t)$.

[22]Written as: $r_K - \delta = \rho + \sigma\dot{C}/C$, the left hand side is seen as the marginal return to savings The right hand side can be shown to be the marginal return to consumption: (see Barro and Sala-i-Martin, 1995).

where the dynamic variables grow at constant asymptotic rates; or a steady state equilibrium where the dynamic variables are asymptotically constant. While such an equilibrium, and the path to it, are indeed usually synonymous with the dynamic optimum,[23] it is frequently simply posited as such, sometimes without explicitly checking whether necessary conditions are satisfied. In cases of multiple equilibria such an approach could go awry. Hahn (1990) indicates how easy it is to constructexamples of growth models with multiple equilibria. Also, the importance of the transversality conditions can be illustrated by the Romer model itself, since if they are ignored it can be shown that the differential equations of the model generate **two** dynamic equilibria or steady states. One of these is ruled out as a potential dynamic optimum by the transversality condition attached to the utility maximisation problem faced by consumers. While it could also have been dismissed as invalid by economic reasoning, its mathematical existence cannot be disputed (also see footnote 27 and Appendix 2.4).

On the other hand, sometimes transversality conditions can be, and are, used **directly** in the differential equations. This is possible in those (somewhat rare) cases where the dynamic system is simple enough to permit a closed-form solution - linear systems for example. Then substitution of transversality conditions can be used to evaluate the *constants of integration*,[24] which are often initial values of the *costate* variables or of price or flow type variables to which they are functionally related by the necessary conditions of the dynamic optimisation.[25] Such cases emphatically illustrate the necessity of transversality conditions in providing sufficient boundary conditions to integrate a system's differential equations to its dynamic optimum. They also raise the issues of *jumping variables* and *two-point boundary value problems*, each of which are discussed in some detail in Chapters 3 and 4, particularly in Section 4.1.

2.3.2 Condensation of the equations

From the equations specified in Section 2.2, the Romer model can be condensed into a dynamic system of four first order differential equations in the variables $A(t)$ - the stock of technology; $K(t)$ - the capital stock; $C(t)$ - the flow of consumption; and $p_A(t)$ - the price of technology. In the specification below the variables $r(t)$ and $H_Y(t)$ have been retained for notational convenience. Thus there are actually six equations and six variables. However, these 'extra' variables can readily be substituted out. The dynamic system can be condensed from the equations of Section 2.2 as follows:

- substitute the exogenous total human capital expression (2.21) into the research technology equation (2.1):

  $$\dot{A}(t) = \zeta[H - H_Y(t)]A(t)$$

[23]In fact, this can be shown generally to be the case under conditions which often characterise economic problems (Appendix 2.3).

[24]For example, see the linearised Romer model in Section 3.2 of Chapter 3 here; the dynamic models of Romer (1986b); problem 'P7(K)' in Romer (1989a); Rebelo (1990 and 1991) particularly the discussion in Sala-i-Martin (1990b); and Jones and Manuelli (1993).

[25]Like the relation between the costate variable $\lambda(t)$ and the level of aggregate consumption $C(t)$ given by equation (A2.2.29) of Appendix 2.2, from which the corresponding initial values would be related by: $\lambda(0)=C(0)^{-\sigma}$.

- use (2.13) and (2.16) first to obtain an expression for $[r(t)+\delta]$, and second, with (2.9), to generate an alternative formulation for output; then combine these results and substitute into (2.6):

$$r(t)+\delta = \gamma^2\eta^{-\gamma}H_Y(t)^{\alpha(1-\gamma)}L^{(1-\alpha)(1-\gamma)}[K(t)/A(t)]^{\gamma-1}$$

$$\begin{aligned} Y(t) &= \eta^{-\gamma}H_Y(t)^{\alpha(1-\gamma)}L^{(1-\alpha)(1-\gamma)}A(t)^{1-\gamma}K(t)^{\gamma} \\ &= \frac{r(t)+\delta}{\gamma^2}K(t) \end{aligned} \tag{2.29}$$

$$\dot{K}(t) = \frac{r(t)+\delta}{\gamma^2}K(t) - C(t) - \delta K(t)$$

- substitute the specialised capital profit maximising conditions (2.14) and (2.15) into the capital stock accounting relation (2.16); and then substitute the result into the technology pricing equation (2.20):

$$\dot{p}_A(t) = r(t)p_A(t) - \frac{1-\gamma}{\gamma}[r(t)+\delta]\frac{K(t)}{A(t)}$$

- maintain the human capital allocation equation (2.25); and the utility maximising expansion path for consumption (2.27).

At this stage the system comprises four *ordinary differential equations* in four variables (as noted before the variables r(t)and $H_Y(t)$ may be substituted out). To solve these, in the sense of generating a dynamic path, four so-called *boundary conditions* are also required. Further discussion of the meaning and significance of these, and exactly how a dynamic system such as this is solved is held over to Chapters 3 and 4. Here it is simply noted that the necessary four boundary conditions are indeed available. Two take the form of initial conditions, and two the form of transversality conditions, and they all arise from the two dynamic maximisation problems inherent in the model. Thus, the final steps in the condensation process are:

- maintain the initial condition 'K(0) given' from the consumers' utility maximisation problem (2.26), and combine it via equation (2.16) with the initial condition 'X(0) given' from the capital goods producers' profit maximisation problem (2.17) to obtain an initial condition 'A(0) given'; and

- maintain the transversality conditions (2.18) and (2.28) from the two dynamic optimisation problems.

Then, re-writing the key equations, the dynamic system for the Romer model is as follows:

$$\dot{A}(t) = \zeta[H - H_Y(t)]A(t) \tag{2.30}$$

$$\dot{K}(t) = \frac{r(t)+\delta}{\gamma^2}K(t) - C(t) - \delta K(t) \tag{2.31}$$

$$\dot{p}_A(t) = r(t)p_A(t) - \frac{1-\gamma}{\gamma}[r(t)+\delta]\frac{K(t)}{A(t)} \tag{2.32}$$

$$\dot{C}(t) = \frac{r(t)-\rho}{\sigma}C(t) \tag{2.33}$$

where

$$H_Y(t) = \Big[\frac{\alpha(1-\gamma)}{\zeta\eta^{\gamma}}L^{(1-\alpha)(1-\gamma)}[K(t)/A(t)]^{\gamma}p_A(t)^{-1}\Big]^{\frac{1}{1-\alpha(1-\gamma)}} \tag{2.34}$$

$$\begin{aligned} r(t) &= \gamma^2\eta^{-\gamma}H_Y(t)^{\alpha(1-\gamma)}L^{(1-\alpha)(1-\gamma)}[K(t)/A(t)]^{\gamma-1} - \delta \\ &= \frac{\zeta\gamma^2}{\alpha(1-\gamma)}H_Y(t)p_A(t)[K(t)/A(t)]^{-1} - \delta \end{aligned} \tag{2.35}$$

together with the four boundary conditions:

$$A(0),\ K(0) \text{ both given;} \tag{2.36}$$

$$\lim_{t\to\infty} v(t) \geq 0 \quad \text{and} \quad \lim_{t\to\infty} v(t)X(t) = 0 \tag{2.37}$$

and

$$\lim_{t\to\infty} \lambda(t) \geq 0 \quad \text{and} \quad \lim_{t\to\infty} \lambda(t)K(t) = 0 \tag{2.38}$$

2.3.3 Asymptotic dynamics and the steady state equilibrium

The asymptotic behaviour of the dynamic system is determined by the transversality conditions, (2.37) and (2.38). They require that under the system of differential equations (2.30) to (2.35) the products v(t)X(t) and λ(t)K(t) both decline towards zero as the time since their initial values were realised becomes very large. This is equivalent to requiring that in the limit, the growth rates of both v(t)X(t) and λ(t)K(t) be negative. Thus, the transversality conditions require:

$$\lim_{t\to\infty}[\dot{v}(t)/v(t) + \dot{X}(t)/X(t)] < 0$$

which, in view of equation (2.16) can be written as:

$$\lim_{t\to\infty}[\dot{v}(t)/v(t) + \dot{K}(t)/K(t) - \dot{A}(t)/A(t)] < 0 \tag{2.39}$$

and

$$\lim_{t\to\infty}[\dot{\lambda}(t)/\lambda(t) + \dot{K}(t)/K(t)] < 0 \tag{2.40}$$

This means that the four growth rate limits:

$\lim_{t\to\infty}[\dot{v}(t)/v(t)]$; $\lim_{t\to\infty}[\dot{K}(t)/K(t)]$; $\lim_{t\to\infty}[\dot{A}(t)/A(t)]$; and $\lim_{t\to\infty}[\dot{\lambda}(t)/\lambda(t)]$

must all exist. That is, they must all be constants. This is what generates the so-called *balanced growth* equilibrium for the system:

From equations (A2.2.21) and (A2.2.30) of Appendix 2.2, the growth rates of both the shadow prices, $v(t)$ and $\lambda(t)$, are equal to the negative of the interest rate, $r(t)$.[26] The transversality conditions therefore imply that the interest rate is asymptotically constant; and from (2.33) this means that the rate of growth of consumption, $\dot{C}(t)/C(t)$, is also asymptotically constant. Next, it follows from (2.30) that for the limit rate of growth of technology $\dot{A}(t)/A(t)$ to be constant, the allocation of human capital between the research and output sectors, $H_A(t)$ and $H_Y(t)$, must also be constant in the limit. Equations (2.34) and (2.35) then require the ratio of capital to technology, $K(t)/A(t)$, and the price of technology, $p_A(t)$, to be asymptotically constant. The former implies that in the limit, the stocks of capital and technology must grow at the same (constant) rate, and it then follows from (2.31) that the limiting value of the consumption-capital ratio, $C(t)/K(t)$, is constant so that consumption also grows at the same rate as capital in the limit.[27] Moreover, from the expression for output in equation (2.29), the capital-output ratio, $K(t)/Y(t)$, is asymptotically constant and the growth rate of output is also equal to that of capital.

Thus, satisfaction of the transversality conditions together with the dynamic equations ensure that the system will tend asymptotically to a *balanced growth* equilibrium in which technology, capital, consumption, and output all grow at the same constant rate.[28] This and other features of the equilibrium from the previous paragraph are summarised below:

$$\lim_{t\to\infty}[\dot{A}(t)/A(t)] = \lim_{t\to\infty}[\dot{K}(t)/K(t)] = \lim_{t\to\infty}[\dot{C}(t)/C(t)] = \lim_{t\to\infty}[\dot{Y}(t)/Y(t)] = g^M$$

$$\lim_{t\to\infty}[\dot{v}(t)/v(t)] = \lim_{t\to\infty}[\dot{\lambda}(t)/\lambda(t)] = -\lim_{t\to\infty} r(t) = -r_{ss}^M$$

$$\lim_{t\to\infty}[K(t)/A(t)] = \Psi_{ss}^M; \quad \lim_{t\to\infty}[C(t)/K(t)] = \Phi_{ss}^M; \quad \lim_{t\to\infty} p_A(t) = p_{Ass}^M$$

$$\lim_{t\to\infty} H_A(t) = H_{Ass}^M; \quad \lim_{t\to\infty} H_Y(t) = H_{Yss}^M$$

where $g^M, r_{ss}^M, \Psi_{ss}^M, \Phi_{ss}^M, p_{Ass}^M, H_{Ass}^M$ and H_{Yss}^M are all constants of the balanced growth equilibrium (and the notation is explained further shortly). Also, referring back to equation (2.38), indicates that the transversality conditions require that $g^M < r_{ss}^M$.

The equilibrium may then be characterised in terms of the model parameters and exogenous variables by using the differential equation system (2.30) to (2.35) to set the

[26]Equality of the growth rates of the two shadow prices is only to be expected: Since specialised capital is costlessly assembled out of η units of general-purpose capital then surely $v(t)=\eta\lambda(t)$. This can be confirmed mathematically by integrating (A2.2.30) with the aid of an integrating factor $e^{\bar{r}(t)t}$, realising that $\lambda(0)=1$, and comparing the result with (A2.2.14).

[27]Strictly, these deductions on the equality of growth rates depend upon the constant values of the corresponding ratios not being zero. $K/A=\eta X=0$ is ruled out since it means the whole system degenerates to the origin; and $C/K=0$ is ruled out by the transversality condition (2.38) associated with the consumer utility optimisation problem: in the limit, optimising consumers must consume all their income. Appendix 2.4 indicates that if this condition is ignored, the differential equations of the Romer system produce a second dynamic equilibrium or steady state.

[28]As noted in footnote 23, this result can be shown to apply more generally (Appendix 2.3).

appropriate rates of growth equal to one another and then solving for the balanced growth constants. However, before proceeding along these lines it is convenient to transform the variables in such a way that the system yields a stationary equilibrium or steady state rather than the balanced growth equilibrium. This can be achieved by defining the new variables $\Psi(t) = K(t)/A(t)$ and $\Phi(t) = C(t)/K(t)$,[29] then taking logs and derivatives and substituting from equations (2.30) to (2.35):

$$\dot{\Psi}(t) = [\frac{r(t)+\delta}{\gamma^2} - \Phi(t) - \delta - \zeta H + \zeta H_Y(t)]\Psi(t) \tag{2.41}$$

$$\dot{\Phi}(t) = [\frac{r(t)-\rho}{\sigma} - \frac{r(t)+\delta}{\gamma^2} + \Phi(t) + \delta]\Phi(t) \tag{2.42}$$

$$\dot{p}_A(t) = r(t)p_A(t) - \frac{1-\gamma}{\gamma}[r(t)+\delta]\Psi(t) \tag{2.43}$$

where:

$$H_Y(t) = [\frac{\alpha(1-\gamma)}{\zeta\eta^\gamma} L^{(1-\alpha)(1-\gamma)} \Psi(t)^\gamma p_A(t)^{-1}]^{\frac{1}{1-\alpha(1-\gamma)}} \tag{2.44}$$

and

$$\begin{aligned} r(t) &= \gamma^2 \eta^{-\gamma} H_Y(t)^{\alpha(1-\gamma)} L^{(1-\alpha)(1-\gamma)} \Psi(t)^{\gamma-1} - \delta \\ &= \frac{\zeta\gamma^2}{\alpha(1-\gamma)} H_Y(t) p_A(t) \Psi(t)^{-1} - \delta \end{aligned} \tag{2.45}$$

Also, from the above analysis of the transversality conditions and the asymptotic behaviour of the system, the boundary conditions (2.36) to (2.38) transform to:

$$\Psi(0) \text{ given} \tag{2.46}$$

$$\lim_{t\to\infty} \Phi(t) = \Phi_{ss}^M \tag{2.47}$$

and

$$\lim_{t\to\infty} p_A(t) = p_{Ass}^M \tag{2.48}$$

In this way the dynamic system is reduced to one of only three differential equations in three variables, all of which are asymptotically constant. Such a reduction of the system also allows its *phase-space* to be visualised as a three dimensional figure (Section 3.3).

The steady state equilibrium of this dynamic system, equations (2.41) to (2.48), which is equivalent to the balanced growth equilibrium of (2.30) to (2.38), may be computed by taking limits and setting each of (2.41) to (2.43) to zero and solving. In the notation below (which was also foreshadowed earlier in summarising the balanced growth equilibrium outcomes), the 'ss-subscripts' denote the steady state and replace the earlier *limit* concepts. The 'M-superscripts' refer to the decentralised market solution of the

[29]The variable $\Psi=K/A=\eta X$ reflects the economy's allocation of capital per design; or the total value of each type of capital (expressed, as usual, in terms of the consumption good numeraire). It is a measure of the *capital intensity of technology*. And the variable Φ measures the contemporaneous consumption available from the capital stock.

model and are used to distinguish this from certain other solutions introduced in subsequent chapters.

$$\frac{r_{ss}^{M}+\delta}{\gamma^{2}}-\Phi_{ss}^{M}-\delta-\zeta H+\zeta H_{Yss}^{M}=0 \tag{2.49}$$

$$\frac{r_{ss}^{M}-\rho}{\sigma}-\frac{r_{ss}^{M}+\delta}{\gamma^{2}}+\Phi_{ss}^{M}+\delta=0 \tag{2.50}$$

$$r_{ss}^{M}p_{Ass}^{M}-\frac{1-\gamma}{\gamma}(r_{ss}^{M}+\delta)\Psi_{ss}^{M}=0 \tag{2.51}$$

$$\begin{aligned} r_{ss}^{M}+\delta &= \gamma^{2}\eta^{-\gamma}(H_{Yss}^{M})^{\alpha(1-\gamma)}L^{(1-\alpha)(1-\gamma)}(\Psi_{ss}^{M})^{\gamma-1} \\ &= \frac{\zeta\gamma^{2}}{\alpha(1-\gamma)}H_{Yss}^{M}p_{Ass}^{M}/\Psi_{ss}^{M} \end{aligned} \tag{2.52}$$

First substitute (2.52) into (2.51) to obtain a relation between r_{ss}^{M} and H_{Yss}^{M}:

$$r_{ss}^{M}=(\gamma\zeta/\alpha)H_{Yss}^{M} \tag{2.53}$$

Then add equations (2.49) and (2.50) and substitute for r_{ss}^{M} to obtain the steady state allocation of human capital to manufacturing:

$$H_{Yss}^{M}=\frac{(\alpha\sigma/\gamma)H+\alpha\rho/\gamma\zeta}{1+\alpha\sigma/\gamma} \tag{2.54}$$

The corresponding allocation to research, and the steady state growth and interest rates can then be obtained from this by using $H=H_{Ass}^{M}+H_{Yss}^{M}$; $(\dot{A}/A)_{ss}^{M}=g^{M}=\zeta H_{Ass}^{M}$; and equation (2.53) respectively:

$$H_{Ass}^{M}=\frac{H-\alpha\rho/\gamma\zeta}{1+\alpha\sigma/\gamma} \quad \text{and} \quad g^{M}=\frac{\zeta H-\alpha\rho/\gamma}{1+\alpha\sigma/\gamma} \tag{2.55}$$

$$r_{ss}^{M}=\frac{\sigma\zeta H+\rho}{1+\alpha\sigma/\gamma} \tag{2.56}$$

Next, the steady state ratio of consumption to capital may be obtained from $(\dot{K}/K)_{ss}^{M}=g^{M}=(r_{ss}^{M}+\delta)/\gamma^{2}-\Phi_{ss}^{M}-\delta$; and the steady state levels of the capital intensity of technology and the technology price level from (2.52) and (2.51) to generate:

$$\Phi_{ss}^{M}=(r_{ss}^{M}+\delta)/\gamma^{2}-g^{M}-\delta \tag{2.57}$$

$$\Psi_{ss}^{M}=\left[\frac{\gamma^{2}H_{Yss}^{M\ \alpha(1-\gamma)}L^{(1-\alpha)(1-\gamma)}}{\eta^{\gamma}(r_{ss}^{M}+\delta)}\right]^{\frac{1}{1-\gamma}} \tag{2.58}$$

and

$$p_{Ass}^{M}=\frac{1-\gamma}{\gamma}\frac{r_{ss}^{M}+\delta}{r_{ss}^{M}}\Psi_{ss}^{M} \tag{2.59}$$

In addition to all the economic variables already explicitly identified in the dynamic system, it will prove useful to specify some extra ones. In particular, it will be useful to have results for both the dynamics and the steady state levels of the savings rate, capital-output ratio, and the factor shares of income. It is also desirable to calculate the dynamics of the growth rates. Growth rates for capital, consumption and technology are already explicit in the system; but for output Y, and gross product GP, further derivation is necessary. The behaviour of all the variables will then be studied in the various dynamic analyses undertaken later in the dissertation. The extra variables are specified and derived in Appendix 2.5.

2.4 Calibration and sensitivity

Having established the formulae for the model's steady state equilibrium, the effect of changes in its parameters on the equilibrium levels of its (endogenous) variables may be examined. This is the sort of *comparative statics* analysis undertaken by Romer (1990a and 1990b). However, it is not the main focus of the work here since it forms part of the broader analysis of the transitional dynamics between equilibria, which is the basis of Chapter 4. Nevertheless, some idea of the numerical sensitivity of the model's equilibrium to changes in its parameters is provided, and a few special cases are discussed. Of course, this requires numerical values to be assigned to these parameters. Accordingly, the model has been calibrated with a set of parameter values intended to reflect the rudiments of the relevant magnitudes for the Australian economy. This set, termed the *benchmark parameter set*, is used widely throughout the paper in various numerical simulations of the model. Here it forms the base from which the sensitivity of the equilibrium is assessed.

2.4.1 Calibration of the model

When the model is solved numerically the differential equations defined in equations (2.41) to (2.45) are converted to difference equations in which adjacent time points are separated by suitably small intervals or *step-sizes* (see Section 4.1 of Chapter 4). The conversion from continuous to discrete time introduces the step-size into the units of measurement of certain parameters and endogenous variables such as the discount, depreciation, interest and growth rates which must be specified in terms of *per some unit of time*. Prior to *calibration* the step-size has no explicit real-time dimension and it only acquires one through a specification of parameters which is consistent in terms of time. Thus, one of the aims in calibrating the model is to set it on an annual time basis. The other aim is to ensure consistency with the empirical economic evidence.

Most, but not all of the model's parameters and exogenous variables represent economic magnitudes for which at least some form of quantitative measurement is available. For the others, at least broad indications of their magnitudes are available from their definitions and from constraints arising from the theory of the model.[30] The particular numerical values assigned to all these input parameters and exogenous variables then determine the magnitudes of the model's endogenous or output variables, such as the real interest rate or the share of savings in gross output, which can also be assessed in terms of actual quantitative economic measures. Calibration of the model is intended to ensure that its input parameters, constrained as necessary to conform with empirical data, generate outputs which are also consistent with empirical observation. Since some of the parameters of the model are only loosely constrained, there is some degree of freedom amongst the parameter set. For the calibration to be 'successful' this *freedom* must provide sufficient flexibility to allow the necessary consistency between inputs, outputs and the data.

Two broad stages go to make up the calibration process. The first is the identification of constraints, consistent with the empirical data, for both the parameters and the endogenous variables that they generate. The second is the determination of actual parameter values from within these ranges. Since variations in the different parameters have vastly different effects on the endogenous variables, in terms of both the magnitudes and the directions of their change, this second stage is complicated and messy to undertake in any 'trial and error' manner. Instead, the approach adopted is one of *endogenising the parameters*. Normally it is the parameters which are exogenous and which, through the model, generate results for the endogenous variables such as the rate of economic growth and the interest rate. Here, since empirical data for the normally endogenous variables are available these roles are reversed. The empirically quantifiable variables (including some new ones functionally related to the parameters) are set exogenously and the model is used to calculate consistent parameter values endogenously, all of these subject to their constraints. Details of the method are provided after the specification of constraint ranges (in Section 2.4.1.3).

2.4.1.1 Parameter constraints

The Cobb-Douglas parameter α: This is the elasticity with respect to human capital of *composite labour*, $N = H_Y^{\alpha} L^{(1-\alpha)}$, employed in the final output sector. It is related to the shares of total income from the output of consumption goods going to human capital and to ordinary labour. Under conditions of perfect competition, which are assumed to prevail in the final output producing sector of the model, $(1-\gamma)$ is the income share to composite labour; $\alpha(1-\gamma)$ the share to human capital; and $(1-\alpha)(1-\gamma)$ the share to ordinary labour (Appendix 2.1). Thus, α represents the share to human capital of the total labour income from the production of goods.

$$\alpha = \frac{w_H H_Y}{w_L L + w_H H_Y} \tag{2.60}$$

Empirical data exist for the incomes accruing to different occupational categories, which can be used as a proxy to differentiate human capital and ordinary labour; however such

[30] For example, from their definitions the parameters ζ, ρ, and σ are constrained only by: $\zeta > 0$; $\rho \geq 0$; and $\sigma > 0$. However, it is known from the theory of the model that: $\zeta = g/H_{Ass}$; and $\rho + \sigma g = r_{ss}$. Thus, even very crude estimates of permissible values for the endogenous variables H_{Ass}, g, and r_{ss} allows the constraint ranges for these parameters to be narrowed enormously.

data cannot distinguish between human capital employed in the production of goods and that employed in research. Thus, the data can only provide a share expression similar to (2.60) with H_Y replaced by H. For this reason (2.60) is modified as follows:

$$\begin{aligned}\alpha &= \left[\frac{w_H H}{w_L L + w_H H}\right]\left[\frac{w_H H_Y}{w_H H}\right]\left[\frac{w_L L + w_H H}{w_L L + w_H H_Y}\right] \\ &= S_\alpha \left[\frac{H_Y}{H}\right]\left[\frac{(1-\alpha)(1-\gamma)Y + \alpha(1-\gamma)YH/H_Y}{(1-\gamma)Y}\right]\end{aligned}$$

where an empirical proxy is available to measure S_α, the share of human capital wages in total wages; and where the income distribution and wage rate results from Appendix 2.1 have been substituted into the third ratio of the first line in the above equation. This is then readily simplified, at the steady state:

$$\alpha = S_\alpha [1 - (1-\alpha) H_{Ass} / H] \tag{2.61}$$

Because H_{Ass} is an endogenous variable it has not been possible to obtain a direct empirical measure for the parameter α. Instead, a new endogenous variable, S_α - which can be empirically measured - has been defined, and this will prove useful in the calibration process. The next task is to obtain this statistic explicitly.

The Australian Standard Classification of Occupations (ASCO) identifies the following broad occupational groupings: *Managers and Administrators; Professionals; Associate Professionals; Tradespersons and Related Workers; Advanced Clerical and Service Workers; Intermediate Clerical, Sales and Service Workers; Intermediate Production and Transport Workers; Elementary Clerical, Sales and Service Workers;* and *Labourers and Related Workers.* Data on earnings by occupation for 1996 indicate that if the first three of these groupings are taken to define human capital, then its share of total 'wages, salaries and supplements' was 48 per cent. When the next two occupational categories are progressively included, the human capital share rises first to 62 per cent, and then to 66 per cent (Australian Bureau of Statistics, 1996). These data suggest a constraint range for this new share variable of $S_\alpha \in [0.5, 0.7]$.[31]

The exogenous variables H (human capital) and L (ordinary labour): Using these same occupational proxies for human capital, data on employment by occupation suggest a ratio of ordinary labour to human capital of about unity to two (Australian Bureau of Statistics, 1997a). On the basis of this the exogenous variables L and H have been set at H = 1 and L = 2 in the benchmark parameter set.[32]

The Cobb-Douglas parameter γ is the elasticity of output with respect to capital, and like α it is also related to income shares. In particular, and as shown in Appendix 2.1,

[31] This is precisely the range identified by Mankiw, Romer and Weil (1992) who note that "In the United States the minimum wage - roughly the return to labor without human capital - has averaged about 30 to 50 percent of the wage in manufacturing." Oddly, this 'guesstimation' appears to take no account of the relative amounts of human capital and ordinary labour.

[32] In any case, these are not critical measures in the calibration. Different values for H can be thought of as arising simply from a change in the units in which it is measured (and thus subsumed into the level of the productivity parameter ζ, see footnote 33). And the model results are largely unaffected by changes in the value of L (Table 2.3).

γ is the capital share of total income from the production of goods: $\gamma = p_X AX / Y$, or in terms of the capital share of gross product:

$$\gamma = \frac{p_X AX}{Y + p_A \dot{A}} \frac{Y + p_A \dot{A}}{Y} = S_\gamma [1 + p_A \dot{A} / Y]$$

where S_γ is the share of income to capital in gross product, which can readily be measured from national income statistics. Then, using equations (2.29) and (2.32), and evaluating at the steady state:

$$\gamma = S_\gamma [1 + \gamma(1-\gamma) g / r_{ss}] \qquad (2.62)$$

Thus, in similar vein to the case for the parameter α, since this expression contains the endogenous variables g and r_{ss}, it does not provide a direct empirical estimate of γ. However, once again another empirically measurable endogenous variable, S_γ, has been defined which will facilitate the calibration.

Australian national accounts data over the ten years 1986-87 to 1995-96, indicate that returns to capital as a share of GDP have been highly stable, averaging 44 per cent, and only varying between 43 and 45 per cent (Australian Bureau of Statistics, 1997b). Accordingly, the constraint range for this share term was set at $S_\gamma \in [0.4, 0.5]$.

The research productivity parameter, ζ (which is not dimensionless but depends, rather, on the units in which H is counted)[33] was merely constrained according to its definition as $\zeta > 0$.

The capital depreciation rate δ: Data on the 'consumption of fixed capital' from the Australian Bureau of Statistics (1997c) indicate an average aggregate rate of depreciation of capital in Australia of some five per cent per annum, varying only from 5.2 per cent to 5.5 per cent over the entire thirteen year period reported (1982-83 to 1994-95). According to the Australian Bureau of Statistics (1990), these estimates are based on the concept of 'expected economic lifetimes' and allow for 'foreseen obsolescence'. Because obsolescence is explicitly ruled out in the Romer model, the calibration process allows for the possibility of a slightly lower depreciation rate. Specifically, in defining the benchmark parameter set the model parameter δ was constrained within the range $\delta \in [0.04, 0.06]$.

The discount rate ρ; and the elasticity of intertemporal substitution, σ^{-1}: From their definitions, these parameters are constrained only as $\rho \geq 0$; and $\sigma > 0$. However, for certain technical reasons (see Chapter 5) it is desirable to have $\sigma > 1$. Nevertheless, these very wide ranges can be narrowed considerably by noting that from equation (2.33) the

[33]Chiang (1992) sees a problem with the formulation of the steady state growth rate from Romer's model. He claims that like the other parameters in the growth rate expression, his research productivity parameter, σ (equivalent to ζ here), is a *pure number*; and thus the appearance of the human capital term, S_0 (equivalent to H here), renders the growth rate a dimensioned quantity. This is false. The research productivity parameter can be seen from the technology accumulation relation to have dimension equal to the inverse of that in which human capital is measured. From this the dimensions of these two quantities cancel in the growth rate formulation and the growth rate is indeed a *pure number*.

parameters are related by $\rho + \sigma g = r_{ss}$. When the permissible ranges for g and r_{ss} (to be developed shortly from the empirical evidence) are substituted into this relation, approximate upper bounds are established as: $\rho < 0.06$ and $\sigma < 7$. In order to ensure a somewhat closer conformity of these parameters with the values used in other research, the smaller constraint ranges of $\rho \in [0.01, 0.03]$ and $\sigma \in [1.5, 5]$ were adopted for the calibration process.[34]

The cost of specialised capital, η: Somewhat arbitrarily, the cost of one unit of any type of specialised capital equipment in terms of output was taken to be $\eta = 2$. However this is of little consequence since it turns out that the steady state equilibrium of the system is largely unaffected by this parameter (see Table 2.3 and the ensuing discussion).

2.4.1.2 Endogenous variable constraints

The steady state growth rate, g: Data on multifactor productivity (labour and capital productivity combined) provides an empirical measure for the model's steady state rate of growth of technology, $(\dot{A}/A)_{ss} = g$. Calculations of the growth rate of multifactor productivity are, however, highly sensitive to the relative positions of the start and end dates in the business cycle. Consequently, such calculations should be made between dates which represent corresponding phases of the cycle. The Australian Bureau of Statistics (1997d) provides the necessary data, which indicate a long-term average annual growth rate in multifactor productivity of 1.5 per cent,[35] with the highest and lowest growth figures for individual cycles being 2.1 and 0.8 per cent respectively. Over the most recent growth cycle at the time of writing, the seven years from 1988-89 to 1995-96, multifactor productivity grew at an average annual rate of 1.2 per cent. Accordingly, the model's steady state growth rate g, was constrained within the range $g \in [0.01, 0.02]$ for the calibration process.

The steady state savings rate, $s_{B_{ss}}$: National Accounts data over the ten years 1986-87 to 1995-96 indicate that in constant price terms, consumption as a share of the expenditure measure of gross domestic product varied between 74.6 and 77.9 per cent, averaging 76.5 per cent (Australian Bureau of Statistics, 1997b). On the basis of these figures, and taking national savings as the share of output not consumed (the definition of s_B in Section 2.3.3), the constraint range for the model's savings rate was set as $s_B \in [0.22, 0.26]$.

The steady state real interest rate, r_{ss}: Like economic and productivity growth rates, estimates of average interest rates are also somewhat sensitive to the business cycle, and so should also be measured across similar phases of the cycles. Using the same growth cycle definitions as above (Australian Bureau of Statistics, 1997d), together with data on 90 day Bank bills, 10 year Treasury bonds, and the Consumer Price Index from the Reserve Bank of Australia (1997), the interest rate data shown in Table 2.1 were calculated. With the 15 year/three cycle average of 5.8 per cent, and the average of the

[34]In some empirical work with a Cass-Koopmans type neoclassical growth model applied to the United States (see Cass, 1965 and Koopmans, 1965), Barro and Sala-i-Martin (1995) used the values of $\rho = 0.02$ and $\sigma = 3$ as their 'baseline' values.

[35] The period is 31 years from 1964-65 to 1995-96 and incorporates six growth cycles.

90 day and 10 year rates ranging between 5.7 and 6.8 per cent, the calibration constraint range was set as $r_{ss} \in [0.055, 0.07]$.

Table 2.1: Real interest rates, 1981-82 to 1995-96, Australia.

Growth rate cycle[a]	90 day Bank bills	10 year Treasury bonds	Average[b]
1981-82 to 1984-85	5.64	5.84	5.74
1984-85 to 1988-89	7.45	6.06	6.75
1988-89 to 1995-96	5.59	6.10	5.85
1984-85 to 1995-96	6.16	6.13	6.15
1981-82 to 1995-96	5.80	5.81	5.81

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Source: Reserve Bank of Australia, *Bulletin*, June 1997.

Notes: a As defined by Australian Bureau of Statistics, *Australian National Accounts, Multifactor Productivity: 1995-96*, July, 1997.

b Simple averages of the previous two columns.

The steady state capital-gross product ratio, k_{GPss}, is also measured empirically from National Accounts data. Capital stock and GDP figures from the Australian Bureau of Statistics (1997b) and (1997c) respectively, indicate that over the ten years 1985-86 to 1994-95 (the most recent ten years for which data was available), the average annual value of the capital-output ratio was 2.85 with a variation about this figure of just under ±2.5 per cent. For the purpose of calibration the model's capital-gross product ratio as calculated by equation (A2.5.17) was constrained as $k_{GP} \in [2.75, 3.00]$.

The speed of convergence, also known as the convergence coefficient, β is a measure of the rate at which a dynamic system approaches its steady state equilibrium. More specifically, the coefficient refers to the rate at which the divergence of a variable from its steady state value is closed. The usual variable is the growth rate of per capita income. This concept, its definition, and calculation within the model are introduced in the following chapter (Section 3.2.3). Here the issue is simply one of endeavouring to calibrate the model in such a way that its convergence is not at odds with the available empirical evidence. Because the calculation of β within the model is rather complicated, involving all the parameters and the *complex-algebra* necessary to solve a cubic equation, no specific constraint range was set for it. Nevertheless, in accordance with the empirical evidence from Mankiw, Romer and Weil (1992), and from Barro and Sala-i-Martin (1995), a value in the range of about 2.0 to 3.5 per cent was sought. As explained below, the coefficient β was assigned a pivotal role in the determination of the benchmark parameter set.

2.4.1.3 Determining the benchmark parameter set

Specific values for the parameters were identified from within their constraint ranges through the use of a non-linear programming technique: in particular, by utilising the program *Solver* from the *Microsoft Excel 97* suite. First, a specific value for the convergence coefficient variable, β, was selected as a target, and the program used to search for a parameter set which satisfied all the constraints on both the parameters themselves and on the endogenous variables. When obtaining a solution for $\beta \le 3.5$ per

cent proved impossible, the program was then used to search for the minimum value of β consistent with the constraints. This generated a result of just under 5 per cent for the convergence coefficient, together with the *benchmark* values for the parameters and, from the steady state formulae (2.53) to (2.59), and (A2.5.16) to (A2.5.18), the corresponding *benchmark steady state* values for the endogenous variables. All these are shown in Table 2.2 below.

Table 2.2: Benchmark set parameter values and the resulting steady state equilibrium values of the endogenous variables, Romer model market solution.

Parameter values	α=0.43; γ=0.54; δ=0.04; ρ=0.01; σ=3.0; ζ=0.06; η=2.0; H=1.0; L=2.0										
Endogenous variables	Ψ_{ss}	Φ_{ss}	p_{Ass}	H_{Yss} (%)	H_{Ass} (%)	r_{ss} (%)	g (%)	s_{Nss} (%)	s_{Bss} (%)	k_{GPss}	β_{ss} (%)
Steady state values	6.48	0.27	9.45	74.41	25.59	5.61	1.54	16.80	22.10	2.84	4.93

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The value of the convergence coefficient (4.9 percent per annum) which results from the benchmark set of parameter values is somewhat greater than the empirical estimates referred to above. However, there are several reasons for not being overly concerned about this, especially in the context of the current model:

i. The empirical estimates of convergence are dependent on the particular growth models used, which are themselves limited in terms of both theory and the empirical validity of their own parameter values.
ii. While the estimates of Sato, R. (1963) for a simple Solow neoclassical model applied to the United States, also implied a convergence coefficient of only some 2.5 per cent, in a slightly modified model Sato, K. (1966) preferred an estimate of some 6 to 9 per cent for the speed of adjustment.
iii. And for a Ramsey type neoclassical model calibrated to U.S. data, King and Rebelo (1993) calculated adjustment processes that implied a convergence coefficient varying between almost 3 per cent to some 11.5 per cent, depending on whether the intertemporal substitutability of consumption parameter σ, was set at σ=10 or σ=1.
iv. As discussed in Appendix 2.1, there are no adjustment costs associated with either investment or disinvestment in the model. Inclusion of these could be expected to slow down the speed of adjustment.
v. The calculation of the convergence coefficient β in the calibration here is (necessarily) from the linearised approximation to the Romer model (Section 3.2.3). Simulations with the non-linear model (Section 4.5) indicated somewhat slower adjustments – with implied convergence coefficients of about 3.6 per cent (also see footnote 10 of Chapter 3). And finally:
vi. it is also notable that with the estimated convergence coefficient β=4.9 per cent, the magnitudes of all the other economic variables of the model, whether specified as input parameters or determined as endogenous outputs, conform closely with their quantitative measures for the Australian economy. Even the consumption-capital ratio Φ, which was **unconstrained** in the calibration process, returned a steady state value, Φ_{ss}=0.27, in exact (two figure) agreement with the empirical data over 1985-86 to 1994-95 (Australian Bureau of Statistics, 1997b and 1997c).

2.4.2 Sensitivity of the steady state to parameter values

The sensitivity of the steady state values may then be gauged by examining the impact upon them of changes in the benchmark parameter values. This has been done for each parameter in turn, by raising its value by ten per cent, while holding the values of all other parameters at benchmark (Table 2.3).

Table 2.3: Sensitivity of the *benchmark steady state* to independent 10 per cent increases in each parameter, Romer model market solution, (%).

Steady state	Parameters								
variables	α	γ	δ	ρ	σ	ζ	η	H	L
Ψ_{ss}	5.7	61.8	-8.5	-0.4	-1.9	-11.2	-10.6	-7.5	5.5
Φ_{ss}	-4.2	-17.4	3.5	0.5	2.0	6.0	[illegible]	6.0	[illegible]
p_{Ass}	8.8	26.4	-4.7	-0.7	-2.8	-14.4	-10.6	-10.8	5.5
H_{Yss}	2.8	-2.9	0.0	0.5	2.3	-0.5	[illegible]	9.5	[illegible]
H_{Ass}	-8.0	8.3	0.0	-1.5	-6.9	1.4	[illegible]	11.5	[illegible]
r_{ss}	-6.6	6.8	0.0	[illegible]	[illegible]	9.5	[illegible]	9.5	[illegible]
g	-8.0	8.3	[illegible]	[illegible]	[illegible]	11.5	[illegible]	11.5	[illegible]
s_{Nss}	1.7	19.0	2.9	-0.7	-3.1	-2.2	[illegible]	-2.2	[illegible]
s_{Bss}	0.9	13.2	2.1	-1.0	-4.2	-1.1	[illegible]	-1.1	[illegible]
k_{GPss}	4.1	16.5	-4.0	-0.2	-0.8	-5.4	[illegible]	-5.4	[illegible]
β_{ss}	-0.7	-17.0	4.4	0.3	-1.1	5.4	[illegible]	5.4	[illegible]

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The high sensitivities of most of the steady state results, and particularly those for Ψ, Φ, p_A, s_B, and β, to the output elasticity of capital parameter γ are noteworthy. Next to these many of the steady state results are most sensitive to the exogenous supply of human capital H, and to the research productivity parameter ζ. As foreshadowed in footnote 32, there is also a great deal of symmetry among the sensitivity results to changes in each of these inputs (note the appropriate highlighted cells of the table).

Further symmetry is apparent in the signs of the sensitivities to changes in the discount rate ρ and in the reciprocal of the inter-temporal elasticity of substitution σ (though for ρ the results are quite insensitive). Raising either of these *preference parameters* has the effect of reducing the equilibrium growth rate. Romer (1990b) argues that this is because increases in these parameters both tend to raise the interest rate and thereby to reduce the discounted stream of future benefits from research. He also comments that it is "...a strong and robust implication that reductions in the interest rate will speed up growth." But this is contradicted by the results in Table 2.3 where movements in the steady state growth and interest rates in response to parameter changes, except for those in ρ and σ, are in the same direction.

It is true that the steady state interest rate rises directly with both ρ and σ.[36] Nevertheless, Romer's general conclusions about the effect of a rising interest rate on the rate of growth are only partly true. Romer is referring to his equation (11) **which holds only in the steady state** and which, from the technology growth equation (2.30) in its steady state form and from (2.53) here, is equivalent to:

[36] This is seen in Table 2.3. Also, it can easily be demonstrated from equation (2.56) that both $\partial r_{ss}/\partial\rho$ and $\partial r_{ss}/\partial\sigma$ are non-negative.

$$g^M = (\dot{A}/A)^M_{ss} = \zeta H - (\alpha/\gamma) r^M_{ss} \qquad (2.63)$$

On the surface this may appear to support his proposition.[37] However, one may focus instead upon the steady state version of the consumption growth rate equation, (2.33):

$$g^M = (\dot{C}/C)^M_{ss} = \frac{r^M_{ss} - \rho}{\sigma} \qquad (2.64)$$

which now seems to show that the growth rate rises **with** the interest rate, and so suggests that when ρ and/or σ are increased, the growth rate falls **despite** rather than because the interest rate rises. It also demonstrates that anything other than changes in ρ or σ which acts to raise (lower) the interest rate will also raise (lower) the growth rate, as seen in the results in Table 2.3.

Using (2.63) to conclude that reducing the interest rate will unambiguously speed up growth is true only in a partial equilibrium sense; namely, for an exogenous interest rate. For the same reason it would be equally wrong, or again only partially right, to conclude from (2.64) that raising the interest rate would speed up growth, and perhaps to endeavour to support the contention with the argument that 'since the interest rate represents the return to savings, a higher interest rate means greater savings and therefore higher growth'.[38] The problem with either contention is that the interest rate is endogenous; and great care needs to be taken about assigning an endogenous variable as a driving force. Rather, the effects on any endogenous variable of a model are best determined and explained in terms of the fundamental parameters and exogenous variables of the model. Here, as in Table 2.3, the determination should be via the steady state equations, in particular equations (2.55) and (2.56) for the growth and interest rates.

As a final comment on these issues: while it should be emphasised that neither of the two arguments advanced above concerning the effect of 'changes in the interest rate on the rate of economic growth' are true in the steady state, it may be that the partial validity of both are revealed by the dynamics of the model as analysed in Chapter 4. For example, in the pairs of Figures (4.2) & (4.3) and (4.34) & (4.35), whenever the interest rate r(t) increases (decreases) the rate of growth of technology $g_A(t)$ falls (rises) in accordance with the *discounting of future benefits* argument, while the rate of growth of capital $g_K(t)$ rises (falls) as suggested by the *return to savings* argument. This broad behaviour is also exhibited in Figures (4.18) & (4.19), (4.25) & (4.27), and (4.41) & (4.42) for the final periods of smooth adjustment; however there are periods or instantaneous jumps characterised by contrary movements in each of these cases.

[37] Equation (2.63) indeed shows that increases in ρ or σ, which raise the steady state interest rate, will also lower the steady state rate of growth. However, for any of the parameters p= α, γ, ζ, or H in the equation, knowledge that a change 'dp' increases r_{ss} is not sufficient to determine whether g rises or falls.

[38] Equations (2.4) and (2.6) indicate how the growth rate (of capital) depend on savings.

Now consider the sensitivity of the endogenous variables to a change in the rate of capital depreciation δ. A rise in δ clearly reduces the amount of capital, and therefore output to a lesser extent, and thence consumption and savings (or investment) to even lesser degrees. [39] It is perhaps unsurprising therefore, that the capital-technology ratio Ψ_{ss} and the capital-output ratio k_{GPss} fall; and that the consumption-capital ratio Φ_{ss} and the savings-output rates s_{Nss} and s_{Bss} rise; or even (through equations (2.15), (2.16) and (2.20)) that the price of technology p_{Ass} falls. However given all these changes, it is somewhat surprising that a change in the rate at which capital depreciates has no effect on the steady state allocation of human capital H_{Ass}/H_{Yss}, or the interest rate r_{ss}, or the rate of economic growth g (Table 2.3).[40]

In broad terms the reason seems to be as follows: Once it is optimal not to let the capital stock run down (in finite time) due to depreciation, then some resources have to be devoted to replacement investment. However, the problems of optimising the usage of the remaining resources, both between research and production and between capital accumulation and consumption, do not depend on the amount of such resources. As a result the steady state allocation of human capital between research and production; and the steady state rate of growth of consumption and the interest rate which supports it are all independent of the rate at which capital depreciates. Despite this invariance of the steady state levels, these variables do undergo transitory dynamic changes in response to a shift in the rate of depreciation. And it is these changes which cause the permanent shifts in the steady state values of the other endogenous variables. Full analysis of this requires a computation of the transitional dynamics of the model, not dealt with formally until Chapter 4. Nevertheless, the results of such a computation for a rise of 25 per cent in the benchmark level of δ (from 0.04 to0.05) are presented in Appendix 2.6.

Finally, perhaps the most surprising aspect of the sensitivity results concerns the lack of any impact from changes in either the parameter η (the cost of a unit of specialised capital) or L (the exogenous supply of ordinary labour) on the steady state allocation of human capital, and hence on the steady state growth and interest rates and other endogenous variables. Of course this was already known from the fact that neither η nor L appear in the steady state formula (2.53) to (2.57). The mechanisms for these results may be seen to be due to exactly offsetting effects on the returns to human capital in research and in final output production, with the result that there is no change in its allocation between these sectors.

For example, an increase in L will raise the output of each type of specialised capital, increasing monopolists' profits and raising the prices of designs, which will in turn generate an increase in the wages of researchers. At the same time, the marginal product, and hence the wages, of human capital in the output sector will rise as a result of the increased usage of the other factors employed there (the ordinary labour L, and the specialised capital X). Using ρ_v to denote a proportional change $\dot{v}/v$ in a variable v, the mechanism explained above may be quantified via the model equations as follows:

[39] The output elasticity of capital is $\gamma < 1$; and consumption and savings are component parts of output.

[40] Of course these four variables are all linked, through the steady state versions of (2.30), (2.31), and (2.33), so that a change in any one of them would mean changes in them all. It might also be noted that the steady state factor shares of gross income are also invariant to δ (see Appendix 2.1).

$$
\begin{aligned}
(2.13) \Rightarrow\ \rho_X &= (1-\alpha)\rho_L + \alpha\rho_{HY} - \rho_{(r+\delta)}/(1-\gamma) \\
(2.10) \Rightarrow\ \rho_{pX} &= (1-\alpha)(1-\gamma)\rho_L - (1-\gamma)\rho_X + \alpha(1-\gamma)\rho_{HY} = \rho_{(r+\delta)}\text{[41]} \\
(2.14) \Rightarrow\ \rho_\pi &= \rho_{pX} + \rho_X = (1-\alpha)\rho_L = \rho_{(r+\delta)} + (1-\alpha)\rho_L + \alpha\rho_{HY} - \rho_{(r+\delta)}/(1-\gamma) \\
(2.20) \Rightarrow\ \rho_{pA} &= \rho_\pi - \rho_r = \rho_{(r+\delta)} + (1-\alpha)\rho_L + \alpha\rho_{HY} - \rho_{(r+\delta)}/(1-\gamma) - \rho_r \\
&= (1-\alpha)\rho_L + \alpha\rho_{HY} - \gamma\rho_{(r+\delta)}/(1-\gamma) - \rho_r \\
&= (1-\alpha)\rho_L - (1-\alpha)\rho_{HY} - \gamma\rho_{(r+\delta)}/(1-\gamma), \quad \text{(from equation (2.53))} \\
(2.22) \Rightarrow\ \rho_{wHA/A} &= \rho_{pA} = (1-\alpha)\rho_L - (1-\alpha)\rho_{HY} - \gamma\rho_{(r+\delta)}/(1-\gamma) \\
(2.24) \Rightarrow\ \rho_{wHY/A} &= [\alpha(1-\gamma)-1]\,\rho_{HY} + (1-\alpha)(1-\gamma)\rho_L + \gamma\rho_X \\
&= [\alpha(1-\gamma)-1+\alpha\gamma]\,\rho_{HY} + [(1-\alpha)(1-\gamma) + \gamma(1-\alpha)]\rho_L - \gamma\rho_{(r+\delta)}/(1-\gamma) \\
&= (1-\alpha)\rho_L - (1-\alpha)\rho_{HY} - \gamma\rho_{(r+\delta)}/(1-\gamma)
\end{aligned}
$$

and so:

$$\rho_{wHA/A} = \rho_{wHY/A}$$

Thus, the effect of an increase in ordinary labour on the wages to human capital in research is the same as its effect on the wages to human capital in goods production.

A decrease in η has similar effects to an increase in L. By reducing the costs of specialist capital producers it induces an expansion of their output and, despite the concomitant reduction in their prices, it raises monopoly profits. This causes an increase in the price of designs and thereby raises research wages. Meanwhile, the marginal product, and hence the wages, of human capital in the final output sector also rise due to the increased employment of specialised capital. In terms of the model's equations:

$$
\begin{aligned}
(2.13) \Rightarrow\ \rho_X &= \alpha\rho_{HY} - \rho_{(r+\delta)}/(1-\gamma) - \rho_\eta/(1-\gamma) \\
(2.10) \Rightarrow\ \rho_{pX} &= \alpha(1-\gamma)\rho_{HY} - (1-\gamma)\rho_X = \rho_{(r+\delta)} + \rho_\eta \\
(2.14) \Rightarrow\ \rho_\pi &= \rho_{pX} + \rho_X = \alpha\rho_{HY} - \gamma\rho_{(r+\delta)}/(1-\gamma) - \gamma\rho_\eta/(1-\gamma) \\
(2.20) \Rightarrow\ \rho_{pA} &= \rho_\pi - \rho_r = \alpha\rho_{HY} - \gamma\rho_{(r+\delta)}/(1-\gamma) - \gamma\rho_\eta/(1-\gamma) - \rho_r \\
&= -(1-\alpha)\rho_{HY} - \gamma\rho_{(r+\delta)}/(1-\gamma) - \gamma\rho_\eta/(1-\gamma), \quad \text{(from equation (2.53))} \\
(2.22) \Rightarrow\ \rho_{wHA/A} &= \rho_{pA} = (1-\alpha)\rho_{HY} - \gamma\rho_{(r+\delta)}/(1-\gamma) - \gamma\rho_\eta/(1-\gamma) \\
(2.24) \Rightarrow\ \rho_{wHY/A} &= [\alpha(1-\gamma)-1]\,\rho_{HY} + \gamma\rho_X \\
&= -(1-\alpha)\rho_{HY} - \gamma\rho_{(r+\delta)}/(1-\gamma) - \gamma\rho_\eta/(1-\gamma)
\end{aligned}
$$

Thus: $\rho_{wHA/A} = \rho_{wHY/A}$

And as for an increase in ordinary labour, the effect of a decrease in the production cost of specialised capital on the wages to human capital in research is also the same as its effect on the wages to human capital in goods production.

As noted by Romer (1990a) and (1990b), this precise balancing is however, no economic necessity, merely a property of the Cobb-Douglas form of the composite labour function used here. So that in truth this particular formulation of the model can shed no light on

[41] These mechanisms are also discussed in Romer (1990a and 1990b). The former argues that 'an increase in L…increases the marginal product of each of the producer durables' but from the analysis here it seems that since the interest rate r turns out to be constant, the marginal product of X, which equals its price p_X, also remains constant.

the impact of these parameters.[42] As before with the case of the rate of depreciation of capital δ, changes in η and L do generate transitional dynamics. Such dynamics, for shocks of 10 per cent in η and 20 per cent in L, computed from the machinery developed in Chapter 4, are presented in Appendix 2.6.

[42] In Romer (1990a), when a more general (CES) function is specified for composite labour, the true impact of the aggregate supply of ordinary labour L, is shown to be ambiguous - the result depending on whether human capital and ordinary labour are more or less substitutable for one another than capital is for composite labour. However, even with this CES specification, the cost of specialised capital in terms of final output η, continues to have no effect on the allocation of human capital.

Appendix 2.1

Production technology and the nature of capital

A2.1.1 Production technology

A2.1.1.1 Specialised capital, increasing returns and technical change

The production function adopted by Romer (1990a & b) has its origins in a paper by Dixit and Stiglitz (1977), who employed a similar formulation in a utility function to capture the utility of variety in consumption. A similar approach was later used by Ethier (1982) in a production function designed to exhibit the economies of scale available from specialisation. In particular, final output in Ethier's model was considered to be assembled out of a set of specialised or differentiated intermediate components, and was an increasing function of the number of these. That is, output was an increasing function of the degree of specialisation in intermediate component goods. Romer (1986b and 1987b) adapted this sort of production function to capture the old idea that a rise in the degree of specialisation in the economy would increase output and generate economic growth. In order to impose a brake on the degree of specialisation, Romer introduced certain fixed costs associated with the production of the intermediates. These were measured in terms of some 'primary resource', the supply of which was initially exogenously limited. Finally, to generate a growth model an alternative use for this resource (direct consumption) was introduced, and its accumulation was allowed. It was then interpreted as a form of general-purpose capital, and the different intermediate inputs as services from specialised forms of capital equipment.

The unusual feature of the specification of the production technology is that capital is disaggregated into all the different types of specialised equipment available at any time ($X_i(t)$, for $i = 1,....,A(t)$). With the basic form of the production function taken to be 'Cobb-Douglas', aggregate output at any time is given by:

$$Y(t) = N(t)^{(1-\gamma)} \sum_{i=1}^{A(t)} X_i(t)^{\gamma} \tag{A2.1.1}$$

where N(t) is some measure of aggregate composite labour devoted to final goods production.

In a more conventional specification of an aggregate production function differences in forms of capital equipment are ignored; or equivalently, all the specialised types of capital are aggregated together into a single capital input:[43]

$$K(t) = \sum_{i=1}^{A(t)} X_i(t) \tag{A2.1.2}$$

to generate the production function:

[43] Where the X_i need to measured in terms of some 'opportunity resource cost' in order that they can be added-up.

$$Y(t) = N(t)^{(1-\gamma)}[\sum_{i=1}^{A} X_i]^{\gamma} = N(t)^{(1-\gamma)} K(t)^{\gamma} \qquad \text{(A2.1.3)}$$

Such aggregation means that all the different types of capital goods are assumed (either explicitly or implicitly) to be perfectly substitutable.

In contrast, the approach adopted by Romer means that different types of capital are never close substitutes for one another. Instead, they have additively separable effects on output. Under this technology final output may be thought of as being produced through successive value-adding stages (i = 1,.....,A(t)) in the overall production process, with each stage providing some specific transformation or assembly operation using the economy's aggregate composite labour (N(t)), and a stage specific type of capital ($X_i(t)$).

$$Y(t) = N(t)^{1-\gamma} X_1(t)^{\gamma} + N(t)^{1-\gamma} X_2(t)^{\gamma} + \ldots + N(t)^{1-\gamma} X_{A(t)}(t)^{\gamma}$$

The key property resulting from the specification of disaggregated capital inputs is that output becomes an increasing function of the extent of the disaggregation; or more significantly, of the degree of specialisation in the production of capital goods. With $0<\gamma<1$, the function $y = x^{\gamma}$ rises at a diminishing rate as x increases ($\partial^2 y/\partial x^2<0$), so $x_1^{\gamma}+x_2^{\gamma} > (x_1+x_2)^{\gamma}$. More generally, with the total K(t) from equation (A2.1.2) representing an accounting measure of total capital rather than a single homogeneous aggregate, the greater the degree of specialisation A(t), the greater the resulting output. Comparing two different index levels of A(t), namely A_1 and A_2:

$$A_1 < A_2 \quad \Rightarrow \quad \sum_{i=1}^{A_1} X_i(t)^{\gamma} < \sum_{j=1}^{A_2} W_j(t)^{\gamma} \quad \Rightarrow \quad Y_1(t) < Y_2(t) \qquad \text{(A2.1.4)}$$

where $X_i(t)$ (i=1,...,A_1) and $W_j(t)$ (j=1,...,A_2) are the measures of each type of specialised capital in the specialisation indices $A(t) = A_1$; and $A(t) = A_2$ respectively; and where:

$$\sum_{i=1}^{A_1} X_i(t) = \sum_{j=1}^{A_2} W_j(t) = K(t),$$

and provided, of course, that $0<\gamma<1$.

This may be seen most easily for the case where, within each index the measures of each type of specialised capital are equal (as they are in the Romer model). Then $X_i(t) = K(t)/A_1$ and $W_j(t) = K(t)/A_2 \quad \forall i, j$, and equation (A2.1.4) becomes:

$$A_1 < A_2 \quad \Rightarrow \quad A_1^{1-\gamma} < A_2^{1-\gamma}$$, which is obviously true for $0<\gamma<1$.

Also, it should be noted that the conventional aggregated capital specification of equation (A2.1.3) is simply a special case, with A = 1, of the more general specification in equation (A2.1.1). Thus, **any** degree of capital specialisation (A>1) in the production function will result in greater output than for the (conventional) aggregate capital approach.

Under the disaggregated or specialised capital technology there are two distinct ways in which capital can grow. Extra units of already existing types can be made and employed,

or new types can be developed and units of these constructed and brought into use. It is clear from (A2.1.1) that diminishing returns apply to the former type of capital accumulation:

$$\frac{\partial^2 Y(t)}{\partial X_i(t)^2} = -\gamma(1-\gamma) N(t)^{(1-\gamma)} X_i(t)^{-(2-\gamma)} < 0 \quad \forall i$$

Since this is the same as raising the aggregate level of capital while holding the set of specialised types fixed, the same relation can be obtained in terms of K(t). Suppose $X_i(t)=X(t)$ $\forall i$,[44] then (A2.1.2) defines this level as $X(t)=K(t)/A(t)$ and (A2.1.1) becomes:

$$Y(t) = N(t)^{(1-\gamma)} A(t)^{(1-\gamma)} K(t)^{\gamma} \qquad \text{(A2.1.5)}$$

Then

$$\frac{\partial^2 Y(t)}{\partial K(t)^2} = -\gamma(1-\gamma) N(t)^{(1-\gamma)} A(t)^{(1-\gamma)} K(t)^{-(2-\gamma)} < 0$$

As shown above, even with aggregate capital (K) fixed, output can be increased by introducing new types, that is by raising A, and spreading the total more thinly amongst them. However, this method of expanding output also encounters diminishing returns.[45] From (A2.1.5):

$$\frac{\partial^2 Y(t)}{\partial A(t)^2} = -\gamma(1-\gamma) N(t)^{(1-\gamma)} A(t)^{-(1+\gamma)} K(t)^{\gamma} < 0$$

Thus, increases in K and increases in A both encounter diminishing returns when the other is held constant. However, when both are allowed to rise the production technology shows increasing returns. As given by equation (A2.1.1) the production function exhibits constant returns to scale in terms of labour and the specialised capital inputs. But when it is expressed in terms of the more fundamental factor of general-purpose capital or saved output, K(t), from which the specialised equipment is constructed, increasing returns become evident. It is apparent from equation (A2.1.5) that with the degree of specialisation (A) fixed, the production technology shows constant returns to scale in terms of labour (N) and capital (K). However, if these last two factors are fixed, output increases with A. Overall the production function is linearly homogeneous of degree ($2-\gamma$), which with $\gamma \in (0,1)$, is clearly greater than unity.[46]

It is also apparent from equation (A2.1.5) that the production function describes an economy whose equilibrium growth behaves just like the Solow neoclassical model, exhibiting *Harrod neutral* or *labour augmenting technical change*.[47] In particular, the production function can be seen to be *composite labour augmenting*:

---

[44] This is the case in Romer's (1990b) model, the focus of this dissertation. See equation (2.13).

[45] Moreover, in the Romer model expansion of A(t) is not costless: The range of capital types is constrained at any time by the fixed costs of research necessary for their production.

[46] In terms of the model description in the text of Chapter 2, the two forms of the production function are given by equations (2.9) and (2.29). In that setting, the 'degree of specialisation' term, A(t), is an even more fundamental factor – the output of research.

[47] For definitions and discussions of neutral technical progress see, for example, Layard and Walters (1978) or Chiang (1992). Also, note that labour augmenting technical change is a necessary condition for the model to possess a *balanced growth path*, that is, one for which the rates of growth of all

$$Y(t) = [A(t)N(t)]^{1-\gamma} K(t)^{\gamma} = Y(A(t)N(t), K(t)) \qquad \text{(A2.1.6)}$$

Combined with the usual capital accumulation equation with a constant savings rate, s, equation (A2.1.6) generates the basic Solow model equation:

$$\dot{K}(t) = sY[A(t)N(t), K(t)] - \delta K(t)$$

so that if A grew at an exogenously specified rate, the economy would converge to a 'balanced growth' path along which capital, output and consumption all grew at that same rate.

A2.1.1.2 Marginal products and factor returns

Since each type of capital input enters the production function in the 'additively separable' manner of equation (A2.1.1), the marginal product of each type is independent of the usage of any (and all) other types.[48] Also, given that the production function for final goods is linear homogeneous in terms of labour, human capital and specialised equipment; and that the sector is perfectly competitive, Euler's Theorem holds and all factors can be paid the value of their marginal products. From equation (2.9):

$$\begin{aligned} MP_{H_Y}(t) &= \partial Y(t) / \partial H_Y(t) = \alpha(1-\gamma) Y(t) / H_Y(t) \\ MP_L(t) &= \partial Y(t) / \partial L = (1-\alpha)(1-\gamma) Y(t) / L \\ MP_{X_i}(t) &= \frac{\partial Y(t)}{\partial X_i(t)} = \frac{\gamma Y(t)}{\sum X_i(t)^{\gamma}} \frac{X_i(t)^{\gamma}}{X_i(t)} \end{aligned} \qquad \text{(A2.1.7)}$$

or, since $X_i(t) = X(t) \forall i$: (from (2.16))

$$MP_X(t) = \partial Y(t) / \partial X(t) = \gamma Y(t) / A(t)X(t)$$

Then:

$$\begin{aligned} Y(t) &= H_Y(t) \frac{\partial Y(t)}{\partial H_Y(t)} + L \frac{\partial Y(t)}{\partial L} + \sum_{i=1}^{A(t)} X_i(t) \frac{\partial Y(t)}{\partial X_i(t)} \\ &= H_Y(t) w_H(t) + L w_L(t) + A(t)X(t)p_X(t) \end{aligned} \qquad \text{(A2.1.8)}$$

where $w_H(t)$ and $w_L(t)$ are the wage rates for human capital and ordinary labour respectively, and $p_X(t)$ is the rental price of specialised capital equipment. Then the distribution of total income from final goods production, Y(t), is as follows:

- income to $H_Y(t)$: $\alpha(1-\gamma)Y(t)$
- income to L: $(1-\alpha)(1-\gamma)Y(t)$
- income to A(t)X(t): $\gamma Y(t)$

---

variables are constant. See for example, Solow (1970), or Dixit (1976a) where it is proved that either capital augmenting technological progress must be zero; or if not, the production function must be of the Cobb-Douglas form, in which case the technological progress may be expressed as being purely labour augmenting anyway.

[48] Barro and Sala-i-Martin (1995) think that this is a reasonable aggregate approach for breakthrough types of technologies since these would sometimes substitute for existing technologies and sometimes complement them, thus generating some cancelling effects.

Because the final goods producing sector is competitive, it uses specialised capital up to the point where its (rental) price equals the value of its marginal product. Conversely, as monopolists, the capital goods producers do not use sufficient general-purpose capital to equate its price with its marginal product.[49] From (2.10) (or (2.14) and (2.13)), together with the production function (2.9)

$$p_X(t) = \gamma H_Y(t)^{\alpha(1-\gamma)} L^{(1-\alpha)(1-\gamma)} X(t)^{(\gamma-1)} = \gamma Y(t) / A(t)X(t)$$

whence the results (A2.1.7) above demonstrate that:

$$p_X(t) = MP_X(t) \qquad \text{(A2.1.9)}$$

Then, from (2.12), (2.13), (2.16) and the production function (A2.1.5):

$$\begin{aligned} r_K(t) = r(t) + \delta &= \gamma^2 \eta^{-\gamma} H_Y(t)^{\alpha(1-\gamma)} L^{(1-\alpha)(1-\gamma)} A(t)^{(1-\gamma)} K(t)^{(\gamma-1)} \\ &= \gamma^2 Y(t) / K(t) \end{aligned} \qquad \text{(A2.1.10)}$$

and obtaining the marginal product of general-purpose capital K(t) from equation (A2.1.5)

$$MP_K(t) = \partial Y(t) / \partial K(t) = \gamma Y(t) / K(t) \qquad \text{(A2.1.11)}$$

Thus:

$$r_K(t) = \gamma MP_K(t) < MP_K(t) \qquad \text{(A2.1.12)}$$

Of the total income from goods production that goes to specialised capital, $\gamma Y(t)$, only the total profits $A(t)\pi(t)$ actually accrue to capital goods producers. From (2.13) to (2.16) and the production function (2.9) these amount to:

$$A(t)\pi(t) = (1-\gamma)\gamma Y(t) \qquad \text{(A2.1.13)}$$

The remainder, which are the rentals paid by capital goods producers for general-purpose capital, accrue to the households that own this *forgone consumption* type of capital. From (A2.1.10) these are $r_K(t)K(t) = \gamma^2 Y(t)$.

So far the only output of the economy that has been considered has been that of goods. The production of designs by human capital in the economy's research sector is additional to this; and so *gross product* of the economy is the sum of these two outputs:

$$GP(t) = Y(t) + p_A(t)\dot{A}(t) = Y(t) + \zeta p_A(t) H_A(t) A(t) \qquad \text{(A2.1.14)}$$

Human capital in research, $H_A(t)$, earns all of the income from this sector. Thus, summarising all of the above, the economy's gross product is distributed as follows:

- human capital in the final goods production sector, $H_Y(t)$, earns $\alpha(1-\gamma)Y(t)$;
- ordinary labour, L, earns $(1-\alpha)(1-\gamma)Y(t)$;
- producers of specialised capital equipment, A(t)X(t), earn (net) $\gamma(1-\gamma)Y(t)$;
- household owners of general-purpose capital, K(t), earn $\gamma^2 Y(t)$; and
- human capital in the research sector, $H_A(t)$, earns $\zeta p_A(t) H_A(t) A(t)$

---

[49] The form of the production function, with its variety of intermediate inputs, also contributes to this divergence (Romer, 1986b and 1987b). This issue is discussed further in Chapter 5 (Section 5.1).

A2.1.2 The nature of capital

Romer (1990a and b) notes that in the model specialised capital is assumed to be *putty-putty*. This was intended to mean that it can be both costlessly produced out of forgone consumption (or out of some general-purpose capital K, which is itself produced costlessly by forgoing consumption); and that it can also be costlessly reconverted back to this medium (see footnote 14). The first aspect of this *putty* nature of capital is the standard assumption for 'one-sector' models and is plausible enough when interpreted as meaning that consumption is 'foregone' by diverting resources away from the production of final output and into that of capital goods. In the current model resources capable of producing η units of final output must be diverted in order to produce one unit of specialised capital. Allowance for costs of adjustment associated with investments of specialised capital (installation costs) could be made, but notwithstanding this the putty nature of capital creation is a reasonable and common simplifying assumption.

It is the second aspect of the putty nature of capital which is unappealing. This is equivalent to allowing any type of specialised capital to be costlessly converted to any other type, even types for which designs are yet to be invented! Such a property suggests a degree of substitutability between capital types far greater than implied by the 'additive separability' of the production function.

Romer was of course fully aware of this limitation on the nature of capital, but because his analysis was primarily concerned with the model equilibrium, where the optimal level of specialised capital is constant, he indicated that it was 'harmless'. However, the current analysis is more concerned with the **transitional dynamics between equilibria**, and the possibility of disinvestment in specialised capital along these adjustment paths certainly exists. It also exists even in comparative statics analyses of the model equilibria.[50]

It can be seen from equation (2.16) that $\dot{X}(t) < 0$ whenever $\dot{K}(t)/K(t) < \dot{A}(t)/A(t)$. That is, although the aggregate capital stock may be growing, if it is not increasing as rapidly as technology (which amounts to the designs for specialised capital equipment), there will be insufficient new general-purpose capital to produce as many units of each of the new specialised types as there are for already existing types. The stock of general-purpose capital must therefore be spread more thinly across the **entire range** of specialised types of equipment and the amount of each type (X) must decline. Thus, new specialised capital equipment will be constructed partly out of newly saved output (or new general-purpose capital or consumption foregone), and partly out of general-purpose capital *released* by the declining amounts of all existing types of specialised capital. If this disinvestment in existing types of capital were not possible some units

[50]Noting that $\eta X = \Psi$, it is apparent from Table 2.3 that, for the benchmark parameters at least, such disinvestment arises whenever the parameters δ, ρ, σ, ζ or H are raised; and whenever α, γ or L are lowered. Less obviously, it is also the case when parameter η is raised (it is easy to show from equation (2.58) that $d\Psi_{ss}/d\eta < 0$).

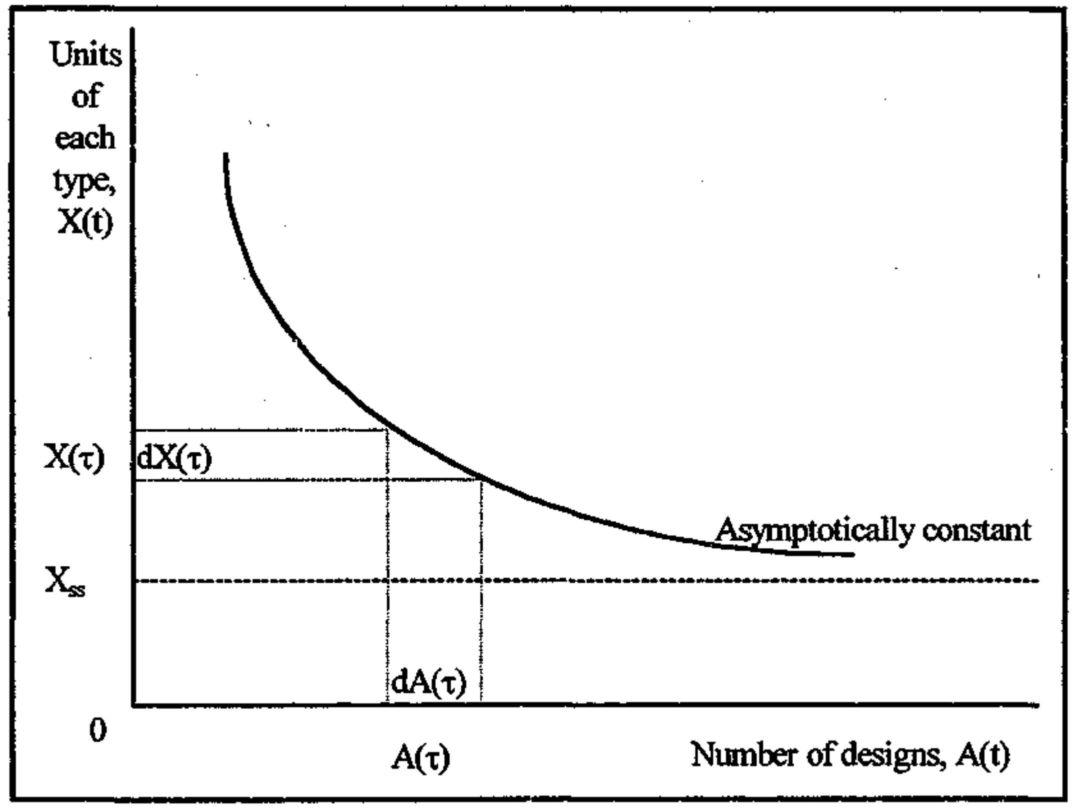
would have to be idle, and extra savings would be necessary to supply all of the general-purpose capital required for new specialised equipment.[51]

Figure A2.1.1 illustrates the situation of disinvestment in specialised capital as it currently stands in the model: As time goes on and the level of technology A(t) grows, the (uniform) number of units of each type X(t), falls asymptotically towards its steady state X_{ss}. At any point of time τ on the way there, a marginal increase in the number of types of specialised capital equipment, $dA(\tau)$, induces a fall of $dX(\tau)$ in the number of units of all the existing $A(\tau)$ types. This *releases* an amount $\eta dX(\tau)A(\tau)$ of general-purpose capital, but the new $X(\tau)$ units of the $dA(\tau)$ types require $\eta dA(\tau)X(\tau)$ general-purpose capital. The shortfall is made-up from savings of an amount $dK(\tau)$. Allowing for the fact that $dX(\tau)$ is negative generates:

$$dK(\tau) - \eta dX(\tau)A(\tau) = \eta dA(\tau)X(\tau)$$

which, upon dividing through by $\eta A(\tau)X(\tau).d\tau$, reproduces the growth relation $\dot{X}(\tau)/X(\tau) = \dot{K}(\tau)/K(\tau) - \dot{A}(\tau)/A(\tau)$.

Figure A2.1.1: Disinvestment in specialised capital equipment, Romer model.



The difficulty faced by any attempt to either prohibit disinvestment in specialised capital (other than by depreciation) or to account explicitly for costs associated with it, is that at all points of time a distinction must be drawn between existing and imminent types. That is, some sort of *vintage model* of capital creation would be necessary. Such models were developed in neoclassical growth theory when the concept of *disembodied* technical

[51]Some reduction in the aggregate extra savings requirement could be achieved by de-commissioning idle units of existing types (thereby avoiding depreciation), and by later re-commissioning them to replace the working units which did depreciate.

progress was replaced by the more realistic description of having technical progress *embodied* in capital (Phelps, 1962 and 1963; Solow, 1960 and 1962; Solow et al, 1966; and Cass and Stiglitz, 1969). Under this general approach the productivity of capital improves over time and so capital created at one time is different to that created at any other time. Hence the vintages of capital had to be explicitly considered. These models 'greatly increased the complexity of the analysis of growth' (Dixit, 1976a).[52] Nevertheless, for particular important cases such models were found to possess steady states that differed little from that of the disembodied technical progress model of economic growth, even though the transitional dynamics were different (Phelps, 1962; and Solow, 1960 and 1970).

Technological growth in the Romer model is explicitly specified as being embodied in differentiated capital goods distinguished by their vintage. However, the complexities of keeping track of the quantities of all the different types/vintages are avoided for two reasons. First, as has been noted in the text, permitting costless disinvestment of existing types of capital removes all intertemporal issues of the monopolists' profit maximisation problems, And second, because of the way the different types of capital enter the production function there are no differences in their productivity. Consequently, there is no need to trace the productivity of different vintages through time.[53] Output growth in the model arises only through the mere existence of greater specialisation and not at all through any improvements in the productivity of specialised capital, nor via any improvements new capital may engender in labour productivity. The designs for specialised capital do not "improve" over time, they merely become more numerous.

It would seem that if the modelling could be altered to dispense with the unappealing aspect of disinvestment, it might also fairly readily accommodate some desirable aspects of vintage capital models including that of more recent vintages being more productive, and that of economic obsolescence. This is all well beyond the scope of the present work, and it may prove intractable. However, it does suggest a direction for future research. Perhaps aspects of the technological progress and production technologies from both Romer (1990b) and Grossman and Helpman (1991a) could be combined? In this way technological growth would incorporate advances in both the variety and the quality of capital goods.

[52] For example, it is not generally possible to aggregate capital of different vintages; account must be taken of the capital-labour substitution possibilities both before and after the creation of a new capital vintage; and the concept of economic obsolescence must be introduced. Moreover, Dixit (1976a) wrote: "...we do not even have a general answer to the question of whether equilibrium growth paths converge to a steady state."

[53] This also means that there is no economic obsolescence.

Appendix 2.2

Dynamic maximisation problems

A2.2.1 General theory

Dynamic optimisation problems of the type faced by economic agents in the Romer model are solved by the branch of mathematics known as *optimal control theory.* The seminal and mathematically rigorous work was that of the Russians Pontryagin and his co-authors (1962).[54] Because of the abundance of economic applications, there are also many thorough treatments of optimal control theory in the economic literature. For example, see Arrow (1968), Arrow and Kurz (1970), Intrilligator (1971); Dixit (1976b); Kamien and Schwartz (1991), Chiang (1992); and Léonard and Van Long (1992). Some familiarity with such references is required for a proper understanding of the theory and its application to the Romer model. A brief and highly informal discussion is also included here. It is confined to a description of the type of problem addressed by optimal control theory and of the types of variables involved (a certain knowledge of the properties of which is necessary in understanding the transitional dynamics of the model); and to the presentation and rationalisation of the main optimisation conditions.

A typical problem of optimum control theory, is to choose the time paths of certain *control variables*, $\mathbf{x}(t)$ say, possibly subject to some constraints, in order to optimise some *objective functional*, W{.}.[55] The value of W depends upon the control variable time paths as well as on those of associated *state variables*, $\mathbf{k}(t)$,[56] which are initially 'inherited' from the past and then subsequently evolve. Usually the functional takes the form of the sum of a *pay-off* or *profit function* u(.) over the relevant time domain $t \in [0,T]$ say. At any instant this profit function depends upon the values of both the control and state variables and also possibly on time itself: $u(.) = u(\mathbf{x}(t), \mathbf{k}(t), t)$. Thus, in continuous time the objective functional is:

$$W\{\bar{\mathbf{x}},\bar{\mathbf{k}}\} = \int_0^T u(\mathbf{x}(t),\mathbf{k}(t),t)dt \tag{A2.2.1}$$

where the notation $\bar{\mathbf{x}},\bar{\mathbf{k}}$ indicates the entire time paths over $t \in [0,T]$.

The paths of the state variables are determined by their initial values and by the paths of the control variables according to a set of *equations of motion.* In continuous time these are differential equations:

[54] It appears that the work of an American mathematician, Magnus R. Hestenes, ought also be cited: According to Chiang (1992), Hestenes, had produced comparable work in 1949 in a Rand Corporation report *"A general Problem in the Calculus of Variations with Applications to Paths of Least Time"*; but this was not known to the Russians, and in fact little known anywhere.

[55] The term *functional* refers to the mapping from paths to real numbers, in contrast to the mapping from reals to reals in a normal *function.*

[56] With a total of C control variables and S state variables, $\mathbf{x}(t)$ and $\mathbf{k}(t)$ are (column) vectors of dimension (Cx1) and (Sx1) respectively.

$$\mathbf{k}(0)=\mathbf{k}_0 \text{ given,} \quad \dot{\mathbf{k}}(t)=\frac{d\mathbf{k}(t)}{dt}=\mathbf{f}(\mathbf{x}(t),\mathbf{k}(t),t) \quad \text{for } t\in[0,T] \tag{A2.2.2}$$

Because of these relations the objective functional in (A2.2.1) may be re-written as:

$$W\{\bar{\mathbf{x}},\mathbf{k}_0\}=\int_0^T u(\mathbf{x}(t),\mathbf{k}(t),t)dt \tag{A2.2.3}$$

Clearly, the values chosen for the control variables at any instant have a direct effect on the objective functional. But what makes these problems difficult is that they also have an indirect effect through their influence on future state variables via the equations of motion. Because of the dependence of state variables on initial values, Aghion and Howitt (1998) describe them as being "historically predetermined", in contrast to control variables which are subject to at least some degree of choice. A key difference between control and state variables is that while the former are only required be *piecewise continuous* (that is their paths may contain jumps), the latter must be properly continuous, although they need only be *piecewise differentiable*. For these reasons, in economic applications the state variables are usually stock-type variables such as capital, while the control variables are of the flow variety, like consumption.

Using maximisation as the form of optimisation and taking the control constraints as defining a bounded set $\mathcal{C}$, the control problem may be restated symbolically as:

$$\begin{aligned}
&\text{Maximise} && W\{\bar{\mathbf{x}},\mathbf{k}_0\}=\int_0^T u(\mathbf{x}(t),\mathbf{k}(t),t)dt \\
&\text{subject to} && \dot{\mathbf{k}}(t)=\mathbf{f}(\mathbf{x}(t),\mathbf{k}(t),t) \\
& && \mathbf{k}(0)=\mathbf{k}_0 \text{ given} \\
&\text{for all} && \mathbf{x}(t)\in\mathcal{C} \text{ and } t\in[0,T]
\end{aligned} \tag{A2.2.4}$$

The most important result of optimal control theory is a set of first-order necessary condition known as the *maximum principle* - the centrepiece of the work of Pontryagin et al (1962). First a function known as the *Hamiltonian* (which turns out to be the dynamic analogy of the more familiar static *Lagrangean* function) is defined as:

$$\mathcal{H}(\mathbf{x},\mathbf{k},\theta,t)=u(\mathbf{x},\mathbf{k},t)+[\theta(t)]^T\mathbf{f}(\mathbf{x},\mathbf{k},t) \tag{A2.2.5}$$

where a third type of dependent variable, denoted here as $\theta(t)$, has been introduced.[57] These are the so-called *costate variables*, and like the control variables they may also be discontinuous. The term $[\theta(t)]^T$ in the Hamiltonian is simply the transpose of the (column) vector of costate variables.[58] Then, the maximum principle conditions for the optimal control problem are as follows:

[57] Note that time is typically the independent variable as it is in this discussion.

[58] So the last term is $\sum_{i=1}^{S}\theta_i(t)f_i(\mathbf{x},\mathbf{k},t)$ and the Hamiltonian is a scaler.

$$\begin{aligned}
&\underset{\mathbf{x}\in\mathcal{C}}{\text{Max.}}\,\mathcal{H}(\mathbf{x},\mathbf{k},\theta,t) \quad \text{for all } t\in[0,T] \\
&\dot{\mathbf{k}}=\frac{\partial\mathcal{H}}{\partial\theta^T} \\
&\dot{\theta}^T=-\frac{\partial\mathcal{H}}{\partial\mathbf{k}} \\
&\theta(T)=0
\end{aligned} \tag{A2.2.6}$$

The first condition in (A2.2.6) is a familiar static optimisation problem, though one which must be addressed at all points of time. It is solved in the usual way: by writing the constraints on the control set in the form $\mathbf{g}[\mathbf{x}(t)]\geq 0$, attaching multipliers $\mu(t)$ to them, forming a *Lagrangean*,

$$\mathcal{L}(\mathbf{x},\mathbf{k},\theta,t,\mu)=\mathcal{H}(\mathbf{x},\mathbf{k},\theta,t,)+\mu^T\mathbf{g}(\mathbf{x})$$

differentiating with respect to $\mathbf{x}$ and μ, and applying the *Kuhn-Tucker conditions*.[59]

The second set of conditions in (A2.2.6) is merely a restatement of the equations of motion for the state variables; but the third set of conditions, which have generated equations of motion for the costate variables, are new. As a pair these two sets of differential equations are referred to as the *Hamiltonian system* or the *canonical system*. They define the dynamics of all possible time paths of the state and costate variables and their joint evolution in *state-costate space*. Since the first condition relates the control variables to the state and costate variables, the first three conditions together define all possible evolutionary paths of **all** the variables; both as individual time paths and in *control-state-costate space*.[60] That is, starting from any valid but otherwise arbitrary point, these three conditions completely define the dynamic behaviour of the system.

However, among all the possible paths only one is optimal. It is the fourth set of conditions of (A2.2.6), the so-called *transversality conditions*, which allow this optimal path to be identified. To derive any dynamic path from the canonical system of 2S first order differential equations, a corresponding 2S *boundary values* are necessary. The problem statement (A2.2.4) furnishes S of them in the form of the *initial values* $\mathbf{k}(t_0)=\mathbf{k}_{t0}$. The transversality conditions of (A2.2.6) provide the remaining S in the form of the *terminal conditions* $\theta(T)=0$ for the optimal path.

Thus, together the four parts of the maximum principle define the entire optimal path. In doing so the initial values of the costate and control variables $\theta(t_0)$ and $\mathbf{x}(t_0)$, corresponding to the given initial values of the state variables $\mathbf{k}(t_0)$, are also determined. Unless the system was already at a point precisely on the optimum path these values will involve discontinuities or jumps from their pre-existing levels. In particular, the maximum principle has made it possible to specify all the choices which need to be made for the control variables in order to optimise the objective functional.

[59] Kuhn and Tucker (1951) is the classic article on concave programming. The Kuhn-Tucker theorem and conditions are also covered in many mathematical texts; for example Intrilligator (1971), Chiang (1984), Kamien and Schwartz (1991). Since the conditions take the form $\partial\mathcal{L}/\partial\mu^T\geq 0$, $\mu^T\geq 0$, $\mu^T\partial\mathcal{L}/\partial\mu^T = 0$, they are often referred to as *complementary slackness conditions*.

[60] A set of equations of motion for the control variables may also be derived.

A general and rigorous proof of the maximum principle requires some powerful and sophisticated mathematics.[61] In keeping with the informal nature of this discussion such rigour is eschewed. Instead, a rationalisation of the maximum principle for a simple and particular case is given in terms of pure economic reasoning. This rationalisation is due to Dorfman (1969), which the following briefly paraphrases.

Dorfman analysed the case of a firm that wishes to maximise its total profits over a fixed period of time $[0,T]$. Capital $k(t)$ was the only state variable, and a single business decision variable $x(t)$, which influenced both profits and capital, was the only control variable. Profits were earned at the (instantaneous) rate $u(k, x, t)$, and the rate of change of the capital stock was $\dot{k} = f(k,x,t)$. The previous notation of this appendix was chosen to correspond with Dorfman's and as a result his control problem of choosing the optimal path $\bar{x} = x(t)\ \forall\ t \in [0,T]$ is the same as that stated in equation (A2.2.4), with the vectors simply interpreted as scalers.

By considering the profits obtainable from any time $t \in [0,T]$ as being broken into those available over a short interval "dt" (so short that there would be no change in the control $x(t)$) and those available from $(t+dt)$ onwards, Dorfman reduced the problem of optimising the entire control path to one of finding the optimal control at a single point of time. He imagined a hybrid policy whereby the (arbitrary) control $x(t)$ was followed over the interval dt, but from then on the controls optimal to the whole path were followed. This converted the functional $W\{.\}$ to an ordinary function $V[.]$ which could be differentiated. Using an asterisk superscript to denote the optimum, total profits obtainable from t are:

$$V[x(t),k(t),t] = u[x(t),k(t),t]dt + V^*[k(t+dt),(t+dt)] \qquad \text{(A2.2.7)}$$

Then, assuming an interior maximum with a concave differentiable function V, total profits from any time $t \rightarrow T$ will be maximised when $x(t)$ takes the optimal value $x^*(t)$ such that $[\partial V/\partial x(t)]_{x^*(t)} = 0$. Thus, differentiating (A2.2.7) with respect to $x(t)$, Dorfman derived the condition:

$$\frac{\partial u[x(t),k(t),t]}{\partial x(t)} + \lambda(t)\frac{\partial f[x(t),k(t),t]}{\partial x(t)} = 0 \qquad \text{(A2.2.8)}$$

where $\lambda(t) = \partial V^*[k(t), t]/\partial x(t)$ is the marginal value of capital in terms of maximum possible profits, or more succinctly, the *shadow price of capital*.[62]

Making the same assumptions about the Hamiltonian $\mathscr{H}$ as were made for V above,[63] the first condition from the maximum principle (A2.2.6) yields:

[61] Even treating time as a continuous variable, with the result that there are a continuously infinite number of control variables, raises significant formal mathematical difficulties (Dixit, 1976b). If the terminal time is extended to infinity (as is the case in the optimal control problems in this dissertation, and in economic growth theory in general), there are even more technical difficulties, particularly with the transversality conditions.

[62] The derivation uses the relations: $\partial V^*/\partial x(t) = [\partial V^*/\partial k(t+dt)][\partial k(t+dt)/\partial x(t)]$; $\partial V^*/\partial k(t+dt) = \lambda(t+dt)$; $\dot{k} = [k(t+dt)-k(t)]/dt = f[x,k,t]$; and $\dot{\lambda} = [\lambda(t+dt)-\lambda(t)]/dt$.

[63] It should be noted however, that in its general form the maximum principle does not require these assumptions. In fact $\mathscr{H}$ need not even be differentiable with respect to x.

$$\underset{x(t)}{\text{Max.}}\mathscr{H} \Rightarrow \frac{\partial \mathscr{H}}{\partial x(t)} = 0 \Rightarrow \frac{\partial u[x(t),k(t),t]}{\partial x(t)} + \theta(t)\frac{\partial f[x(t),k(t),t]}{\partial x(t)} = 0 \qquad \text{(A2.2.9)}$$

This is identical with (A2.2.8). It follows that the economic interpretation of the maximum principle costate variables $\theta(t)$ is that of shadow prices of the corresponding state variables $\mathbf{k}(t)$.[64] In view of this the Hamiltonian function (A2.2.5) may be interpreted as the sum of the rate of increase of profits and the rate of accumulation of the **value** of capital. The first term represents the direct and 'current' contribution to profits from control $x(t)$, and the second represents the indirect or 'future' profit impact of control $x(t)$. Thus, the economic interpretation of the Hamiltonian is that for all times in the optimisation period it expresses the total profit potential of any set of control variable choices.

Now consider the economic interpretation of the first maximum principle condition of (A2.2.6). Maximisation of the Hamiltonian with respect to x is now seen as simply a requirement to choose the control variables in such a way that potential profits are maximised at all times. When expressed by (A2.2.9) it becomes apparent that this involves a balancing of the marginal current profits with the marginal future costs of changing the capital stock:

$$\frac{\partial u}{\partial x} = -\theta(t)\frac{\partial f}{\partial x} \qquad \text{(A2.2.10)}$$

The next 'new' result of the maximum principle is the third condition of (A2.2.6). Expanding this in the terms of Dorfman's example produces:

$$\dot{\theta}(t) = -\frac{\partial \mathscr{H}}{\partial k} \quad \Rightarrow \quad -\dot{\theta}(t) = \frac{\partial u}{\partial k} + \theta(t)\frac{\partial f}{\partial k} \qquad \text{(A2.2.11)}$$

The left hand side of which is the rate at which the shadow price of capital depreciates, or alternatively, the marginal loss incurred by postponing the acquisition of a unit of capital. The right hand side is the sum of capital's marginal contribution to current profits plus its marginal contribution to enhancing the value of the capital stock. Thus, the condition imposed by (A2.2.11) is the economically logical one of equating, at the margin, the loss from not employing capital to the overall profit (both current and future) it could generate.

Finally, the last condition from the maximum principle of equation (A2.2.6), the transversality condition $\theta(T) = 0$, sets the terminal shadow price of capital to zero. This simply reflects the fact that in maximising over a fixed horizon any capital left over at that date exists too late to raise any profits and is therefore worthless. Some further discussion of transversality conditions is at Appendix 2.3.

[64] That the costate variables represent valuations of the state variables may also be proved more formally. Writing the maximum value function as $V^*[k_0, k(T),T]$ and differentiating with respect to the given initial k_0 and the optimal terminal $k(T)$ produces: $\partial V^*/\partial k_0 = \theta^*(0)$ and $\partial V^*/\partial k(T) = -\theta^*(T)$. So the marginal value of having more (capital) k_0 initially is $\theta^*(0)$, while the marginal cost of having more 'left over' at T is $\theta^*(T)$. This proves the result for the initial and terminal costate variables. It may also be extended for the whole optimisation period; that is $\forall\ t \in [0,T]$. See Léonard and Van Long (1992).

A2.2.2 Maximising monopoly profits

The problem faced by the producers of specialised capital is to choose a time path for the level of their output which maximises the discounted sum of all their future excesses of rental incomes over variable costs:

$$\text{Maximise} \int_0^{\infty} [p_X(i,t)X(i,t) - \eta I_X(i,t)] e^{-\int_0^t r(s)ds} dt$$

$$\text{subject to } \dot{X}(i,t) = I_X(i,t) - \delta X(i,t) \qquad \text{(A2.2.12)}$$

$$X(i,0) \text{ given, and } X(i,t) \geq 0$$

and where:

$I_X(i,t)$ is gross investment in $X(i,t)$;[65] and $p_X(i,t)$ is the demand function given by $p_X(i,t) = \gamma N(H_Y(t),L)^{1-\gamma} X(i,t)^{\gamma-1} = \gamma H_Y(t)^{\alpha(1-\gamma)} L^{(1-\alpha)(1-\gamma)} X(i,t)^{\gamma-1}$

Forming the (present valued) Hamiltonian $\mathcal{H}$ and invoking the maximum principle:

$$\mathcal{H} = [p_X(i,t)X(i,t) - \eta I_X(i,t)] e^{-\bar{r}(t)t} + v(i,t)[I_X(i,t) - \delta X(i,t)] \qquad \text{(A2.2.13)}$$

where $v(i,t)$, the dynamic multiplier of the accumulation constraint, is the costate variable associated with the state variable $X(i,t)$ and has the usual interpretation of a shadow price (here it is the *present value* shadow price of specialised capital goods, $X(i,t)$); and where $\bar{r}(t)$ is the average interest rate over $(0,t)$:

$$\bar{r}(t) = \frac{1}{t} \int_0^t r(s)ds$$

Then the first order necessary conditions for the solution of the maximisation problem (A2.2.12) are:[66]

$$\frac{\partial \mathcal{H}}{\partial I_X(i,t)} = 0 \quad \Rightarrow \quad v(i,t) = v(t) = \eta e^{-\bar{r}(t)t} \qquad \text{(A2.2.14)}$$

$$\dot{v}(i,t) = -\frac{\partial \mathcal{H}}{\partial X(i,t)} \quad \Rightarrow \quad \dot{v}(i,t) = \delta v(i,t) - \gamma^2 N(.)^{1-\gamma} X(i,t)^{\gamma-1} e^{-\bar{r}(t)t} \qquad \text{(A2.2.15)}$$

and the transversality condition:

$$\lim_{t \to \infty} v(i,t) \geq 0 \quad \text{and} \quad \lim_{t \to \infty} v(i,t)X(i,t) = 0$$
$$\text{and} \quad \lim_{t \to \infty} \mathcal{H} = 0 \qquad \text{(A2.2.16)}$$

[65] $I_X(i,t)$ is the *control* variable and $X(i,t)$ is the *state* variable of this dynamic optimisation problem.

[66] The non-negativity constraints on the state variables $X(i,t)$, were they ever to be binding, would require the formation of a Lagrangean function of the form $\mathcal{L} = \mathcal{H} + \theta \dot{X}(i,t)$ with additional 'Kuhn-Tucker complementary slackness' type conditions $\partial \mathcal{L} / \partial \theta \geq 0, \theta \geq 0, \theta \partial \mathcal{L} / \partial \theta = 0$ and $X(i,t) \geq 0$, $\theta X(i,t) = 0$ (for example see Chiang, 1992). However, since all the $X(i,t)$ turn out to be equal, and since no output is possible without some capital, $X(i,t)=X(t)$ must be strictly positive over all finite time. Hence the Lagrangean and associated additional first order conditions may be safely ignored (also see footnote 67).

Then, differentiating equation (A2.2.14) with respect to time:

$$\dot{v}(t) = -\eta \frac{d}{dt} [\int_0^t r(s)ds] e^{-\bar{r}(t)t} = -\eta r(t) e^{-\bar{r}(t)t} \qquad \text{(A2.2.17)}$$

and substituting this result together with (A2.2.14) itself into (A2.2.15), and then using the demand function $p_X(i,t)$ generates the results:

$$X(i,t) = X(t) = [\frac{\gamma^2 N(H_Y(t),L)^{1-\gamma}}{\eta(r(t)+\delta)}]^{\frac{1}{1-\gamma}} \qquad \text{(A2.2.18)}$$

$$p_X(i,t) = p_X(t) = [r(t)+\delta]\eta/\gamma \qquad \text{(A2.2.19)}$$

and

$$\lim_{t \to \infty} v(t) \geq 0 \quad \text{and} \quad \lim_{t \to \infty} v(t)X(t) = 0$$
$$\text{and} \quad \lim_{t \to \infty} \mathcal{H} = 0 \qquad \text{(A2.2.20)}$$

It may also be noted from (A2.2.17) and (A2.2.14) that:

$$\dot{v}(t) / v(t) = -r(t) \qquad \text{(A2.2.21)}$$

- **Imposing a non-negativity constraint on $I_X(i,t)$**

If the unappealing aspect of costless disinvestment in specialised capital equipment (Appendix 2.1) were addressed simply by not permitting it in the dynamic optimisation, the problem (A2.2.12) would have to include another constraint. Namely:

$$I_X(i,t) \geq 0 \qquad \text{(A2.2.22)}$$

Then, by the Kuhn-Tucker conditions the first order necessary condition (A2.2.14) would have to be replaced by a corresponding inequality and a *complementary slackness* condition:

$$\frac{\partial \mathcal{H}}{\partial I_X(i,t)} \leq 0 \quad \text{and} \quad I_X(i,t) \frac{\partial \mathcal{H}}{\partial I_X(i,t)} = 0 \qquad \text{(A2.2.23)}$$

From this it can be seen that when the non-negativity constraint (A2.2.22) is not binding ($I_X(i,t)>0$), the optimisation conditions are exactly as before and the results (A2.2.18) to (A2.2.21) stand. However, when the non-negativity constraint is binding the first order condition (A2.2.14) becomes:

$$\frac{\partial \mathcal{H}}{\partial I_X(i,t)} \leq 0 \quad \Rightarrow \quad v(i,t) = v(t) \leq \eta e^{-\bar{r}(t)t} \qquad \text{(A2.2.24)}$$

and when this is combined with the transversality condition (A2.2.16) it implies that the growth rate of $v(t)$ must be negative. Then, from first order condition (A2.2.15), which still applies:

$$v(t) < \frac{\gamma^2}{\delta} N(.)^{1-\gamma} X(t)^{\gamma-1} e^{-\bar{r}(t)t} \qquad \text{(A2.2.25)}$$

Finally, these last two results imply that when the non-negativity condition on investment in specialised capital is binding the number of units of each type of specialised capital must satisfy the inequality:

$$X(t) < \left[\frac{\gamma^2 N(H_Y(t),L)^{1-\gamma}}{\delta\eta}\right]^{\frac{1}{1-\gamma}} \qquad \text{(A2.2.26)}$$

This lacks sufficient specificity to allow the development of a system of differential equations from which defined dynamic paths can emerge. More specific modelling, as discussed in Section A2.1.2 of Appendix 2.1, would be required.

- **Imposing adjustment costs on $I_X(i,t)$**

Another method of resolving the unappealing aspect of costless disinvestment (and of costless positive investment also) might be to impose some form of adjustment costs on the processes of assembling and dismantling specialised capital. An adjustment term could be added to the unit investment cost of η. For example, the costs of investment might be defined as:

$$\text{Cost of investment} = \eta I_X(i,t)[1+\kappa(I)]$$

where $\kappa(.)$ is a non-negative function such that:

$$\kappa(0)=0;\ \kappa'(|I|)>0;\ \text{and}\ \kappa''(|I|)\le 0$$

Then this 'cost of investment term' would replace the '$\eta I_X(i,t)$' term in the integrand of equation (A2.2.12). However none of this is pursued here. Instead, it is left as an area for further research.

A2.2.3 Maximising consumer welfare

The capital stock, one of the factors determining the level of output from which consumption is possible, rises with household income (both wages and interest payments) and falls with consumption. This generates an accumulation relation for capital in the form of a first order differential equation. As such, the consumption problem is one of dynamic optimisation. Formally, the problem is 'to maximise the discounted sum of all future aggregate utility, subject to the (dynamic) economy-wide resource constraint on capital'. That is:

$$\begin{aligned}
&\text{Maximise} && \int_0^\infty U\{C(t)\}e^{-\rho t}dt \\
&\text{subject to} && \dot{K}(t) = W_Y(t) + r_K(t)K(t) - C(t) - \delta K(t) \\
& && 0 \le C(t) \le Y(t) \qquad\qquad \text{(A2.2.27)} \\
& && K(t) \ge 0 \ \text{and, } K(0) \text{ given} \\
&\text{where} && U\{C(t)\} = [C(t)^{1-\sigma} - 1]/(1-\sigma) \quad \text{for } \sigma > 0
\end{aligned}$$

In this formulation W(t) is aggregate wages from the output of goods; $r_K(t) = r(t)+\delta$ is the rental rate on general-purpose capital; and U(.) is the instantaneous utility function (the *felicity function*), with ρ as the subjective discount rate and $1/\sigma$ the *elasticity of intertemporal substitution*. Since aggregate household income amounts to gross product, and since research wages correspond to research output, then:

$$W_Y(t) + r_K(t)K(t) = Y(t)$$

and the economy wide resource constraint above corresponds to the capital accumulation relation (2.6). Also see footnote 21 in Section 2.2.8.

Then, proceeding again according to the maximum principle (Pontryagin et al, 1962), the (present valued) Hamiltonian, $\mathcal{H}$, is formed and the first order conditions for a maximum are applied:

$$\mathcal{H} = [(C(t)^{1-\sigma} - 1)/(1-\sigma)]e^{-\rho t} + \lambda(t)[W_Y(t) + r(t)K(t) - C(t)] \qquad \text{(A2.2.28)}$$

where the dynamic multiplier $\lambda(t)$, is the costate variable associated with the state variable K(t) and has the interpretation of its (present value) shadow price.[67]

$$\frac{\partial\mathcal{H}}{\partial C} = 0 \Rightarrow \qquad \lambda(t) = C(t)^{-\sigma}e^{-\rho t} \qquad \text{(A2.2.29)}$$

$$\dot{\lambda} = -\frac{\partial\mathcal{H}}{\partial K} \Rightarrow \qquad \dot{\lambda}(t)/\lambda(t) = -r(t) \qquad \text{(A2.2.30)}$$

and the transversality condition for the problem (A2.2.27) is:

$$\begin{aligned}
&\lim_{t\to\infty}\lambda(t) \ge 0 \quad \text{and} \quad \lim_{t\to\infty}\lambda(t)K(t) = 0 \\
&\text{and} \quad \lim_{t\to\infty}\mathcal{H} = 0
\end{aligned} \qquad \text{(A2.2.31)}$$

Then differentiating (A2.2.29) with respect to time and substituting (A2.2.30) generates the condition for the optimum pattern of consumption over time:

$$\dot{C}(t)/C(t) = [r(t) - \rho]/\sigma \qquad \text{(A2.2.32)}$$

[67] Rigorously, because the optimisation problem (A2.2.27) includes constraints on both the control and state variables (consumption and capital respectively) a Lagrangean function $\mathcal{L}$ incorporating the Hamiltonian together with these constraints should also be formed, with the solution to the maximisation problem, as usual, being a saddle point of $\mathcal{L}$. As before (footnote 66), this would introduce additional 'Kuhn-Tucker complementary slackness' type first order conditions which would greatly complicate the solution. Fortunately these complexities can be avoided because the general forms of the utility and production functions U(.) and Y(.) exhibit the so-called 'neoclassical' properties (in particular: $U(0)=0$, $U'>0$, $U''<0$, $U'(0)=\infty$; and $Y(0)=0$, $Y'>0$, $Y''<0$ and $Y'(\infty)<\rho+\delta<Y'(0)$) which imply that the variables automatically satisfy the constraints. For example, see Léonard and Van Long (1992, pp.146, 159); Chiang (1992, p.256) and Romer (1989a, p.79).

Appendix 2.3

Transversality conditions

Transversality conditions are constraints imposed by the particular form of an optimum control problem on the terminal, or asymptotic, behaviour of a dynamic path if such a path is not to be ruled out as a possible solution. Thus, in the first instance at least, transversality conditions are *necessary conditions*. They arise whenever the optimisation problem is not completely constrained by the exogenous specification of both the terminal time and the terminal values of all the state variables. In the absence of such a *fixed terminal point* specification there are more degrees of freedom available in the optimisation process, and so correspondingly more conditions are necessary in order to distinguish the optimal path from other admissible paths. These are the transversality conditions, so named, according to Chiang (1992), because they describe how the optimal path *transverses* the terminal time.

The particular form of a transversality condition depends on the manner in which the terminal value of its state variable is constrained in the problem statement.[68] In the case of the two dynamic optimisation problems in the Romer model, asymptotic non-negativity restrictions were imposed, the state variables in both cases being capital stocks. These took the form:

$$\lim_{t\to\infty} k(t) \geq 0 \tag{A2.3.1}$$

where k(t)=X(t) for the monopoly profits maximisation problem and k(t)=K(t) for the utility maximisation problem. Inequality constraints like these generate transversality conditions which are the continuous time, infinite horizon analogues of the Kuhn-Tucker *complementary slackness conditions* from inequality constrained static optimisation theory (Romer, 1989a). These were given in Appendix 2.2 as equations (A2.2.20) and (A2.2.31). With the above notation, and including that restriction, they take the form:

$$\lim_{t\to\infty} k(t) \geq 0\,; \quad \lim_{t\to\infty} \theta(t) \geq 0\,; \quad \text{and} \quad \lim_{t\to\infty} \theta(t)k(t) = 0 \tag{A2.3.2}$$

where θ(t) is the shadow price of capital k(t), so θ(t)=v(t) in the profits problem and θ(t)=λ(t) in the utility problem. Also, since the time horizons in these problems are infinite, the terminal times cannot be fixed and so another transversality condition is also required in each problem. Namely, in each case the limit value of the (present valued) Hamiltonian must vanish (Michel, 1982):[69]

$$\lim_{t\to\infty} \mathcal{H} = 0 \tag{A2.3.3}$$

[68]Except, of course, if it fully constrained (that is, constrained by a fixed terminal point), in which case no transversality condition is required.

[69] Notice that this condition is satisfied for both dynamic maximisation problems in the model. From equations (A2.2.13) and (A2.2.28) the first term of each *limit Hamiltonian* vanishes because of the discount factors; and the second terms vanish because, from (A2.3.2), either the costate variables are zero, or the state variables are zero. And in the latter case there can be no output and no capital accumulation $\lim_{t\to\infty} \dot{k}(t) = 0$.

These transversality conditions have intuitively appealing meanings. Following the spirit of Dorfman (1969) as discussed in Appendix 2.2, the conditions (A2.3.2) can be interpreted as stating that if capital has any value at the end of the horizon, then the optimisation process should ensure that none is 'left over'. That is:

$$\lim_{t\to\infty} \theta(t) > 0 \;\Rightarrow\; \lim_{t\to\infty} k(t) = 0$$

or, if some capital is 'left over' then it must be worthless:

$$\lim_{t\to\infty} k(t) > 0 \;\Rightarrow\; \lim_{t\to\infty} \theta(t) = 0$$

Alternatively, since θ(t)k(t) represents the value of the community's capital assets at time t, this transversality condition simply says that to maximise the total profit function, the assets should be used in such a way as to increase its value within the planning period rather than having any assets left over at the end. Also, the transversality condition (A2.3.3) may be interpreted as requiring that at the end of the planning period all the potential profit opportunities, both current and future, (recall that this was the economic interpretation of the Hamiltonian) must have been used up.

As noted above, transversality conditions are in general only *necessary* conditions for an optimum; and then they are only part of the necessary conditions from the *Maximum Principle*. However, because of the particular formulations of the maximisation problems in the model, and because the functional forms of the maximands and of the dynamic constraints in each of these problems are such that the *Hamiltonians* of the problems are jointly concave in the *state* and *control* variables, the *Maximum Principle* conditions are both ***necessary and sufficient*** (Chiang, 1992; and Léonard and Van Long, 1992). Thus, given the dynamic equations, the transversality conditions are all that is required to define the optimum.

When the transversality conditions were imposed on the differential equations of the model (Section 2.3.3) the asymptotic behaviour of the optimum turned out to be a steady-state equilibrium of the differential equations. As it happens, this result follows from the sufficiency (concavity) requirements whenever certain other conditions are satisfied. In particular, if the differential equations of a dynamic optimisation problem yield a balanced growth or steady state equilibrium with the *saddle-point* stability property (and this is shown in Chapter 3 to be the case for the Romer system), then the path leading to the equilibrium will be the optimal path provided the problem:

- is autonomous with a positive discount factor;
- has a *Hamiltonian* (or allowing for constraints, a *Lagrangean*) which is jointly concave in terms of its state and costate variables;
- has state variables which are required to be non-negative; and
- imposes non-negativity constraints on the asymptotic values of the costate variables

(Léonard and Van Long, 1992).

Appendix 2.4

A second steady state

If the transversality conditions are ignored and only the differential equations are considered, the dynamic system of the Romer model can be shown to produce **two** dynamic equilibria or steady states: Thus, consider equations (2.30) to (2.35), ignoring (2.37) and (2.38), and posit the existence of an asymptotic *balanced growth equilibrium* for the system (where, in the limit, A(t), K(t), and C(t) all grow at constant rates). Then, following similar logic to that applied in the text when the transversality conditions were used, it may be argued as follows:

From equations (2.30) and (2.33), the rates of growth of A(t) and C(t) will be constant only if $H_Y(t)$ and r(t) respectively are both constant. Then, from (2.35) and (2.34) this requires the ratio K(t)/A(t) and $p_A(t)$ also to be constant. The former, as long as it is not equal to zero,[70] implies that K(t) and A(t) must grow at the same (constant) rate, and it then follows from (2.31) that the ratio C(t)/K(t) must be constant. At this point there are two possibilities: First, that the value of this constant is strictly greater than zero, and second that it equals zero. In the first case it may safely be argued that the rate of growth of C(t) equals that of K(t), and from this the *'market determined'* steady state equilibrium denoted by equations (2.53) to (2.59) may be deduced.

In the second case however, the asymptotic rate of growth of C(t) merely has to be less than that of K(t), no matter by how small an amount. That is:

$$\lim_{t\to\infty}\frac{\dot{C}(t)}{C(t)} < \lim_{t\to\infty}\frac{\dot{K}(t)}{K(t)} \quad \Rightarrow \quad \lim_{t\to\infty}\frac{C(t)}{K(t)} = 0 \qquad \text{(A2.4.1)}$$

Then, as shown below the asymptotic versions of the differential equations (2.30) to (2.35) generate an '*alternate*' steady state equilibrium. This is denoted by 'ss-subscripts' and 'A-superscripts' to distinguish it from the 'M-superscripts' of the market determined steady state.

First, equation (2.32) is set to zero and $(r_{ss}^A + \delta)$ substituted from equation (2.35):

$$r_{ss}^A = (\gamma\zeta/\alpha)H_{Yss}^A \qquad \text{(A2.4.2)}$$

Then (2.30) and (2.31) are equated, noting that $(C/K)_{ss}^A$ is zero; and (A2.4.2) is substituted to obtain the steady state allocation of human capital to manufacturing:

$$H_{Yss}^A = \frac{\alpha\gamma H - \alpha\delta(1-\gamma^2)/\gamma\zeta}{1+\alpha\gamma} \qquad \text{(A2.4.3)}$$

[70] If K(t)/A(t) = 0, then the entire system degenerates to the origin, a rather boring static equilibrium.

The corresponding allocation to research, and the steady state growth and interest rates can then be obtained from this by using $H = H_{Ass}^A + H_{Yss}^A$; $(\dot{A}/A)_{ss}^A = g^A = \zeta H_{Ass}^A$; and equation (A2.4.2) respectively:

$$H_{Ass}^A = \frac{H + \alpha\delta(1-\gamma^2)/\gamma\zeta}{1+\alpha\gamma} \quad \text{and} \quad g^A = \frac{\zeta H + \alpha\delta(1-\gamma^2)/\gamma}{1+\alpha\gamma} \qquad \text{(A2.4.4)}$$

$$r_{ss}^A = \frac{\gamma^2\zeta H - \delta(1-\gamma^2)}{1+\alpha\gamma} \qquad \text{(A2.4.5)}$$

Finally, re-writing the steady state consumption-capital ratio condition from which this steady state was generated, and obtaining the steady state capital-technology ratio and technology price levels from (2.35) produces:

$$(C/K)_{ss}^A = \Phi_{ss}^A = 0 \qquad \text{(A2.4.6)}$$

$$(K/A)_{ss}^A = \Psi_{ss}^A = \left[\frac{\gamma^2 {H_{Yss}^A}^{\alpha(1-\gamma)} L^{(1-\alpha)(1-\gamma)}}{\eta^\gamma (r_{ss}^A + \delta)}\right]^{\frac{1}{1-\gamma}} \qquad \text{(A2.4.7)}$$

and

$$p_{Ass}^A = \frac{1-\gamma}{\gamma}\frac{r_{ss}^A + \delta}{r_{ss}^A}\Psi_{ss}^M \qquad \text{(A2.4.8)}$$

Equations (A2.4.3) to (A2.4.8) define the *alternate* steady state. They may be compared with their counterparts (2.54) to (2.59), which define the true (*market determined*) steady state of the system. It should be emphasised that this alternate steady state is **not** a valid outcome of the model. It is ruled out by the proper application of the transversality condition attached to the consumer utility optimisation problem, equation (2.38).[71] As argued in Section 2.3.3 that condition results in an asymptotically constant interest rate. Mathematically, using the steady state versions of the capital and consumption growth rate equations (2.31) and (2.33); equation (A2.4.1) implies:

$$\frac{r_{ss}^A - \rho}{\sigma} < \frac{r_{ss}^A + \delta}{\gamma^2} - \delta = 0 \quad \Rightarrow \quad r_{ss}^A < -\frac{\sigma\gamma^2}{(\sigma-\gamma^2)}\left[\frac{\delta(1-\gamma^2)}{\gamma^2} + \frac{\rho}{\sigma}\right] < 0$$

which in turn means that both capital and consumption would decline towards zero and the system would degenerate to the origin.

And qualitatively, an asymptotically constant interest rate means that in the limit, income from capital grows at the same rate as capital. This means that for consumption to be growing more slowly than capital, as required by this alternative equilibrium, it must also be growing more slowly than income - clearly not optimal behaviour for consumers seeking to maximise utility through consumption.

The purpose of including this invalid equilibrium was to illustrate the importance of transversality conditions in determining the correct dynamic solutions of optimal control problems such as this one.

[71] This was first pointed out to me by Paul Romer, who also put the 'qualitative argument' above.

Appendix 2.5

Derivation of some supplementary variables

The purpose of this appendix is to specify some additional variables to those already explicitly identified in the derivation of the dynamic system, and to derive their relationship with existing variables. Results for the savings rate, capital-output ratio, the factor shares of aggregate income, and the growth rates of output and gross product are described below.

- **savings rate**

Under a narrow view (see Section 2.2.3), the savings rate is taken simply as *"the investment share of the output of goods"*, and denoted here by s_N. From equations (2.4) and (2.29):

$$\frac{I(t)}{Y(t)} = 1 - \frac{\gamma^2}{r(t)+\delta}\frac{C(t)}{K(t)} \quad \Rightarrow \quad s_N(t) = 1 - \frac{\gamma^2}{r(t)+\delta}\Phi(t) \tag{A2.5.1}$$

Under a broader notion of output and savings the savings rate, denoted here by s_B, is defined as *"the proportion of the economy's gross product (GP) which is not consumed"*. Using equations (2.29) and (2.30), gross product is:

$$GP(t) = Y(t) + p_A(t)\dot{A}(t) = \frac{r(t)+\delta}{\gamma^2}K(t) + \zeta p_A(t)[H - H_Y(t)]A(t) \tag{A2.5.2}$$

whence:

$$\begin{aligned} s_B(t) &= 1 - \frac{C(t)}{(r(t)+\delta)K(t)/\gamma^2 + \zeta[H - H_Y(t)]p_A(t)A(t)} \\ &= 1 - \frac{\Phi(t)}{(r(t)+\delta)/\gamma^2 + \zeta(H - H_Y(t))p_A(t)\Psi(t)^{-1}} \end{aligned} \tag{A2.5.3}$$

- **capital-output ratio**

In the same way as for the saving rate, the capital-output ratio may also be defined with either the narrow output of goods measure (Y), or the broader gross product measure (GP) as denominator. Thus, using (2.29) and (A2.5.2):

$$k_Y(t) = \frac{K(t)}{Y(t)} = \frac{\gamma^2}{r(t)+\delta} \tag{A2.5.4}$$

and

$$k_{GP}(t) = \frac{K(t)}{GP(t)} = \frac{1}{[r(t)+\delta]/\gamma^2 + \zeta[H - H_Y(t)]p_A(t)\Psi(t)^{-1}} \tag{A2.5.5}$$

- **factor shares of aggregate income (gross product)**

The distribution of the economy's gross product to its factors of production was described in Appendix 2.1 (Section A2.1.1 following equation (A2.1.14)). This information, combined with the expression for output Y from equation (2.29), and that for gross product from (A2.5.2), generates the following results:

$$S_{H_A}(t) = \frac{\zeta p_A(t)H_A(t)}{[r(t)+\delta]\Psi(t)/\gamma^2 + \zeta p_A(t)H_A(t)} \tag{A2.5.6}$$

$$S_L(t) = \frac{(1-\alpha)(1-\gamma)(r(t)+\delta)}{[r(t)+\delta] + \zeta\gamma^2 p_A(t)H_A(t)/\Psi(t)} \tag{A2.5.7}$$

$$S_{H_Y}(t) = \frac{\alpha(1-\gamma)(r(t)+\delta)}{[r(t)+\delta] + \zeta\gamma^2 p_A(t)H_A(t)/\Psi(t)} \tag{A2.5.8}$$

$$S_K(t) = \frac{\gamma[r(t)+\delta]}{[r(t)+\delta] + \zeta\gamma^2 p_A(t)H_A(t)/\Psi(t)} \tag{A2.5.9}$$

- **dynamic paths of growth rates**

The dynamic paths of technology, capital, the price of technology, and consumption are already specified as part of the dynamic system. Re-writing equations (2.30) to (2.33) they are as follows:

$$g_A(t) = \zeta H_A(t) \tag{A2.5.10}$$

$$g_K(t) = \frac{r(t)+\delta}{\gamma^2} - \Phi(t) - \delta \tag{A2.5.11}$$

$$g_{pA}(t) = r(t) - \frac{1-\gamma}{\gamma}[r(t)+\delta]\Psi(t)/p_A(t) \tag{A2.5.12}$$

$$g_C(t) = \frac{r(t)-\rho}{\sigma} \tag{A2.5.13}$$

But obtaining expressions for the growth rates of output Y and gross product GP requires further derivation. Dropping the 'time' arguments temporarily to make the notation simpler: from the expression (2.29) for Y, and from the relation between r and H_Y from (say) equation (2.45):

$$g_Y = g_K + \frac{\frac{d}{dt}(r+\delta)}{r+\delta} = g_K + g_{H_Y} + g_{pA} - g_\Psi$$

and substituting $[\gamma g_\Psi - g_{pA}]/[1-\alpha(1-\gamma)]$ for g_{HY}, obtained from the relation (2.44):

$$
\begin{aligned}
g_Y &= g_K + \frac{1}{1-\alpha(1-\gamma)}[\gamma g_\Psi - g_{pA}] + g_{pA} - g_\Psi \\
&= g_K - \frac{(1-\alpha)(1-\gamma)}{1-\alpha(1-\gamma)} g_\Psi - \frac{\alpha)(1-\gamma)}{1-\alpha(1-\gamma)} g_{pA} \\
&= g_K - \frac{(1-\alpha)(1-\gamma)}{1-\alpha(1-\gamma)}[g_K - g_A] - \frac{\alpha)(1-\gamma)}{1-\alpha(1-\gamma)} g_{pA}
\end{aligned}
$$

furnishing the result:

$$
g_Y = g_K - \frac{(1-\alpha)(1-\gamma)}{1-\alpha(1-\gamma)}[g_K - g_A] - \frac{\alpha)(1-\gamma)}{1-\alpha(1-\gamma)} g_{pA} \qquad \text{(A2.5.14)}
$$

Then, using (A2.5.2), the growth rate of gross product may be written as:

$$
g_{GP} = \frac{Y}{GP} g_Y + \frac{R}{GP}[g_{pA} + g_{H_A} + g_A]
$$

Now the research share of gross product (R/GP) is simply the share of gross product distributed to the only factor employed in research, H_A, which was defined and derived earlier (equation (A2.5.6) as "S_{HA}". Using this, and noting that since $H_A = H - H_Y$, the growth rate of H_A may be expressed as: $g_{HA} = -g_{HY}(H_Y/H_A)$, generates:

$$
g_{GP} = (1 - S_{H_A}) g_Y + S_{H_A}[g_{pA} - g_{H_Y}\frac{H_Y}{H_A} + g_A]
$$

Finally, substituting an expression for g_{HY} derived from (2.44) and simplifying as before:

$$
g_{GP} = (1 - S_{H_A}) g_Y + S_{H_A}[g_{pA} - \frac{H_Y}{H_A}\frac{\gamma(g_K - g_A) - g_{pA}}{1-\alpha(1-\gamma)} + g_A] \qquad \text{(A2.5.15)}
$$

Of course the steady-state values of all these variables may be obtained simply by substituting the steady-state versions of their constituent variables. Thus, for example:

$$
s^M_{Nss} = 1 - \frac{\gamma^2}{r^M_{ss} + \delta}\Phi^M_{ss} \qquad \text{(A2.5.16)}
$$

$$
s^M_{Bss} = 1 - \frac{\Phi^M_{ss}}{(r^M_{ss} + \delta)/\gamma^2 + \zeta H^M_{Ass} p^M_{Ass} / \Psi^M_{ss}} \qquad \text{(A2.5.17)}
$$

and

$$
k^M_{GPss} = \frac{1}{[r^M_{ss} + \delta]/\gamma^2 + \zeta H^M_{Ass} p^M_{Ass} / \Psi^M_{ss}} \qquad \text{(A2.5.18)}
$$

Appendix 2.6

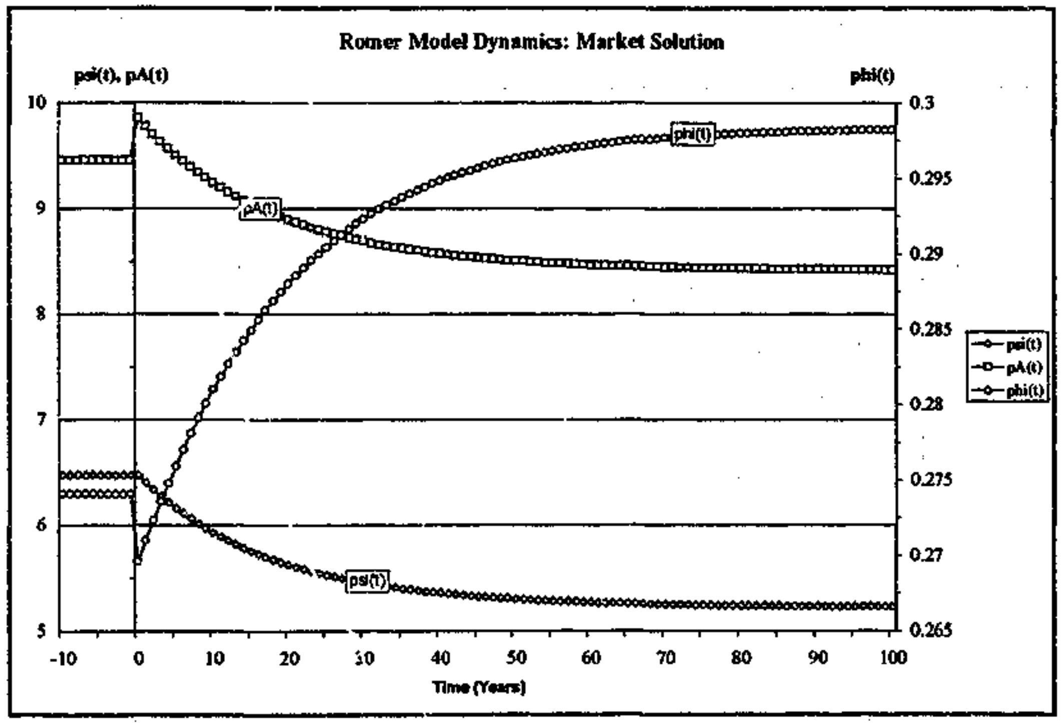
Preliminary transitional dynamics

This appendix is intended merely to provide a preliminary appreciation of the significance of identifying the adjustment paths between equilibria, rather than focussing simply upon the comparative statics of the equilibria alone. The parameters chosen for this purpose have no permanent effect on many of the model's variables. This was the point in selecting them. Comparisons of the equilibria prior to these parameter changes with the ensuing equilibria simply indicate no change to have occurred for many variables. However, the dynamic analyses presented here reveal that such comparative statics can conceal a great deal of transitory change, both in terms of relative magnitudes and in terms of persistence over time.

No attempt is made at this stage to explain either how these transitional dynamics are computed, or the meaning of the results presented. Such explanations are the subject of later chapters.

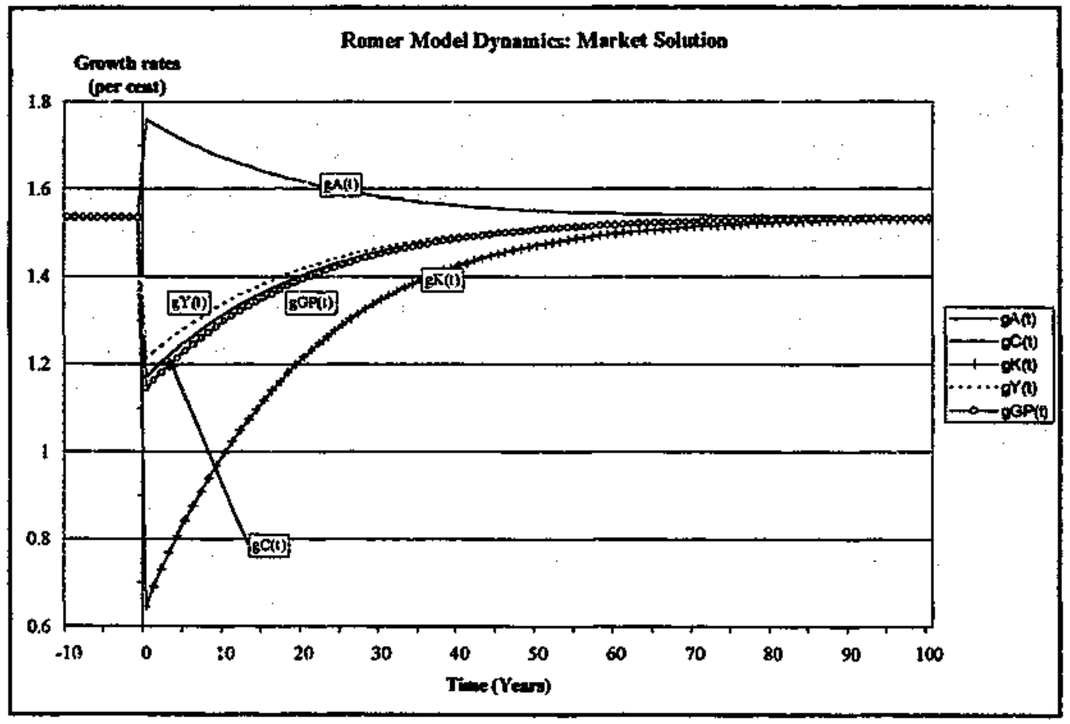
A2.6.1 A 25 per cent rise in the rate of depreciation of capital δ

Figure A2.6.1: Dynamic effects on the principal dynamic variables Ψ, Φ, and pA of a sustained 25 per cent rise in the rate of depreciation of capital (δ) from time t=0, benchmark parameter set.



d:\phd\gempack\mktrom\25d0\25d0sim.xls(psi)

Figure A2.6.2: Dynamic effects on the growth rates a sustained 25 per cent rise in the rate of depreciation of capital (δ) from time t=0, benchmark parameter set.



d:\phd\gempack\mktrom\25d0\25d0sim.xls(gA)

Figure A2.6.3: Dynamic effects on the interest rate (r) and the human capital in research (H_A) of a sustained 25 per cent rise in the rate of depreciation of capital (δ) from time t=0, benchmark parameter set.

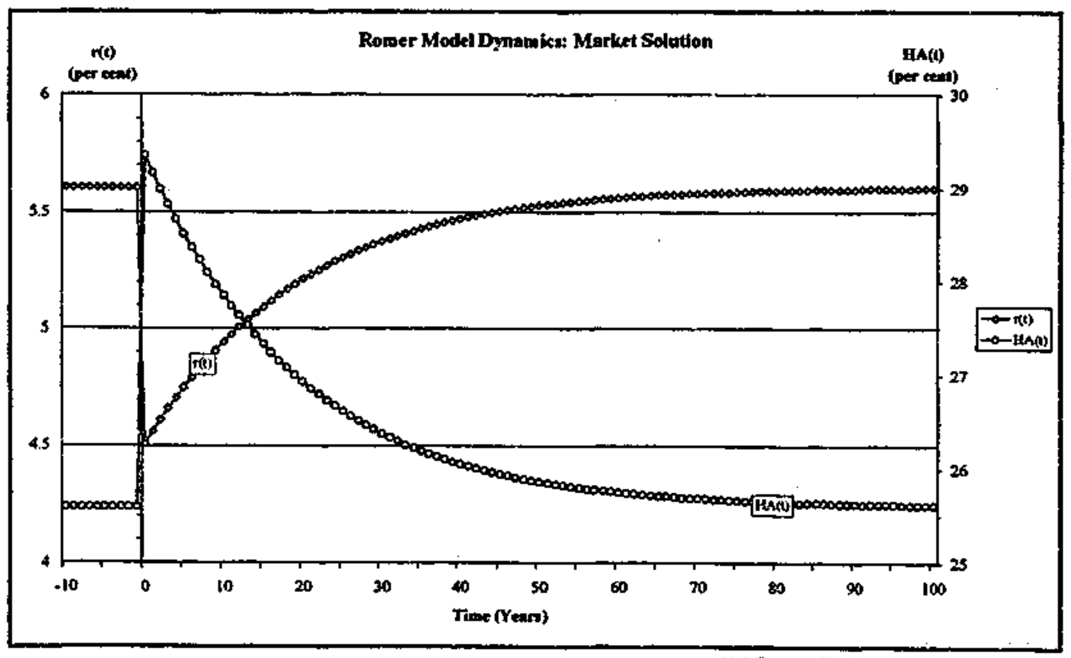


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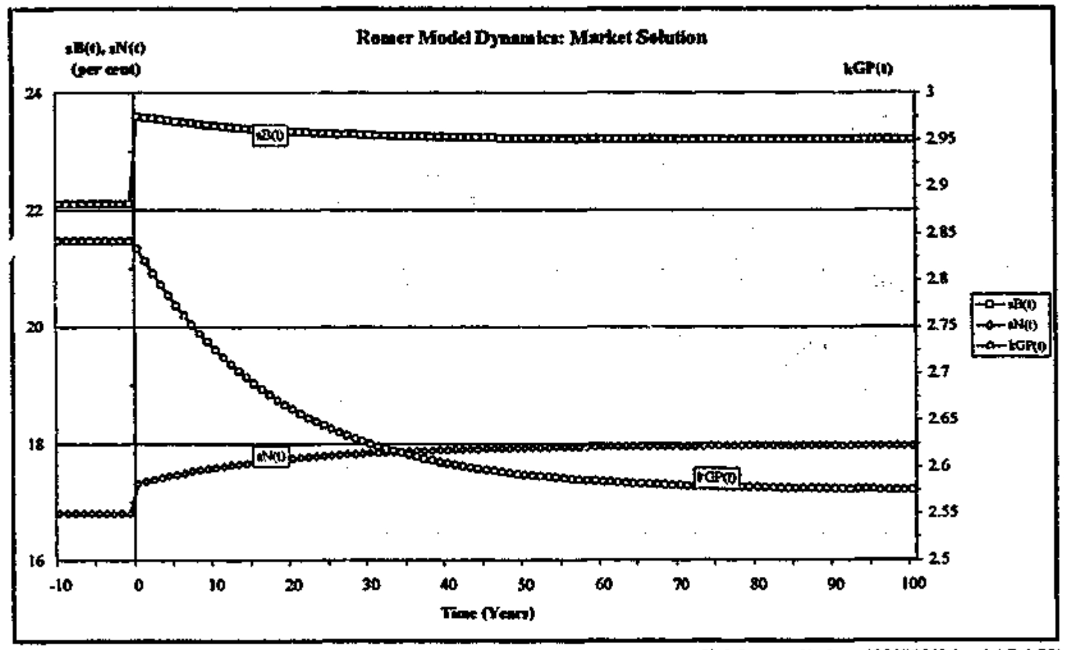
Figure A2.6.4: Dynamic effects on the savings rates (s_N &s_B) and capital-gross product ratio (k_{GP}) of a sustained 25 per cent rise in the rate of depreciation of capital (δ) from time t=0, benchmark parameter set.



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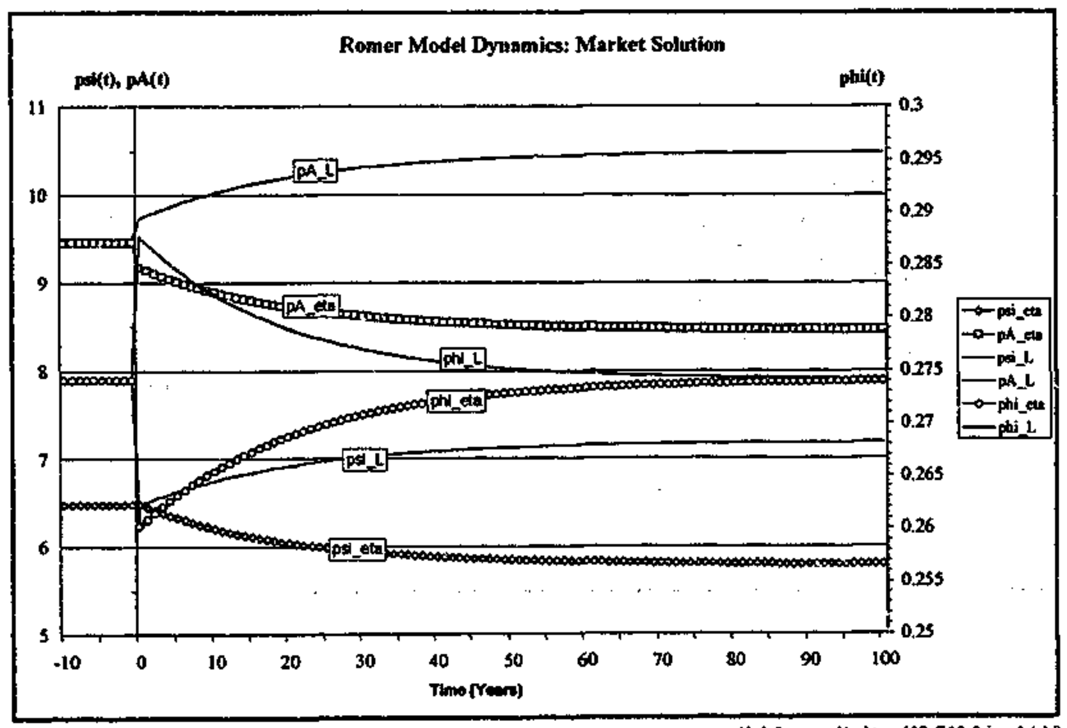
A2.6.2 A 10 per cent rise in the production cost of capital η, and a 20 per cent rise in the quantity of ordinary labour L

Figure A2.6.5: Dynamic effects on the principal dynamic variables Ψ, Φ, and p_A of a sustained 10 per cent rise in the production cost of capital (η), and a sustained 20 per cent rise in the level of ordinary labour L from time t=0, benchmark parameter set.



d:\phd\gempack\mktrom\10e0\10e0sim.xls(phi)

Figure A2.6.6: Dynamic effects on the growth rates of a sustained 10 per cent rise in the production cost of capital (η) from time t=0, benchmark parameter set.

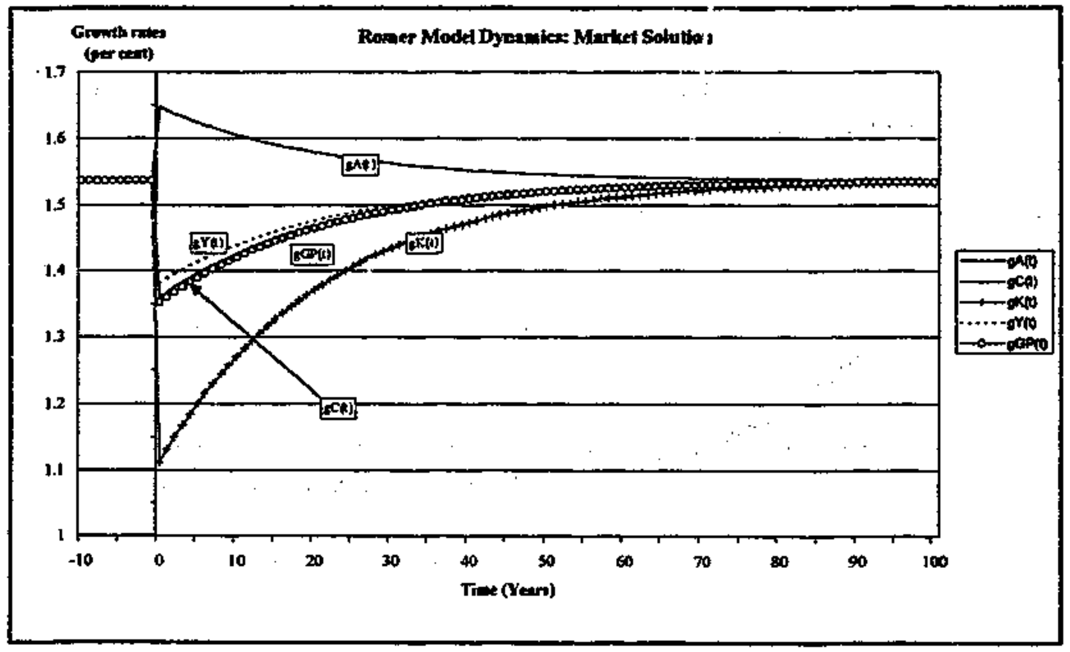


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Figure A2.6.7: Dynamic effects on the growth rates of a sustained 20 per cent rise in the level of ordinary labour L from time t=0, benchmark parameter set.

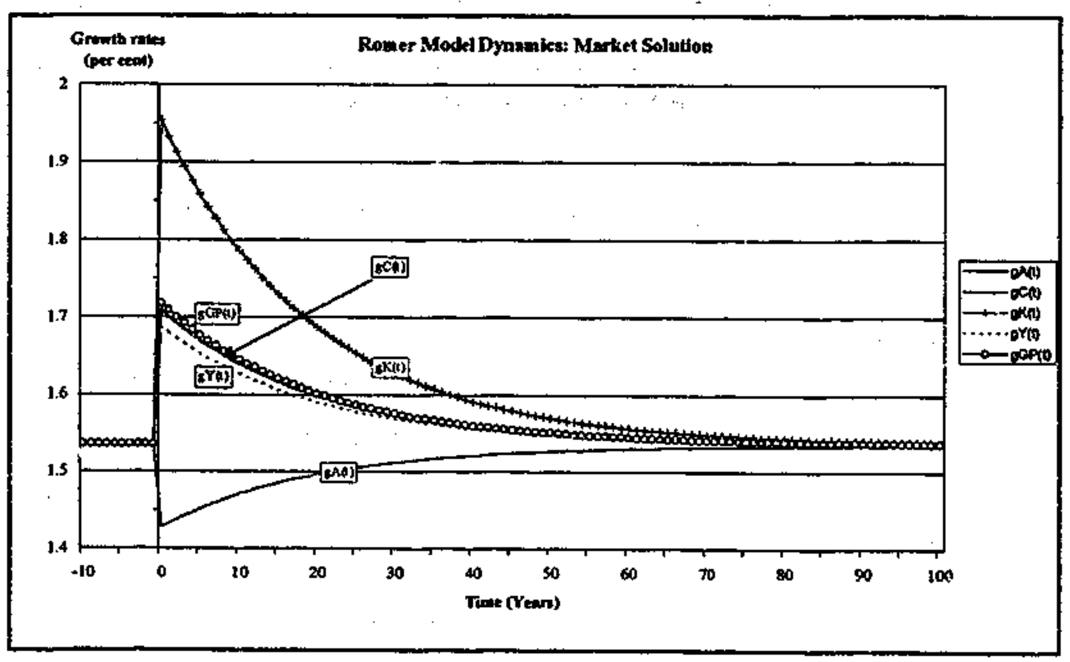

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Figure A2.6.8: Dynamic effects on r and H_A of a sustained 10 per cent rise in the production cost of capital (η), and a 20 per cent rise in the level of ordinary labour L from time t=0, benchmark parameter set.

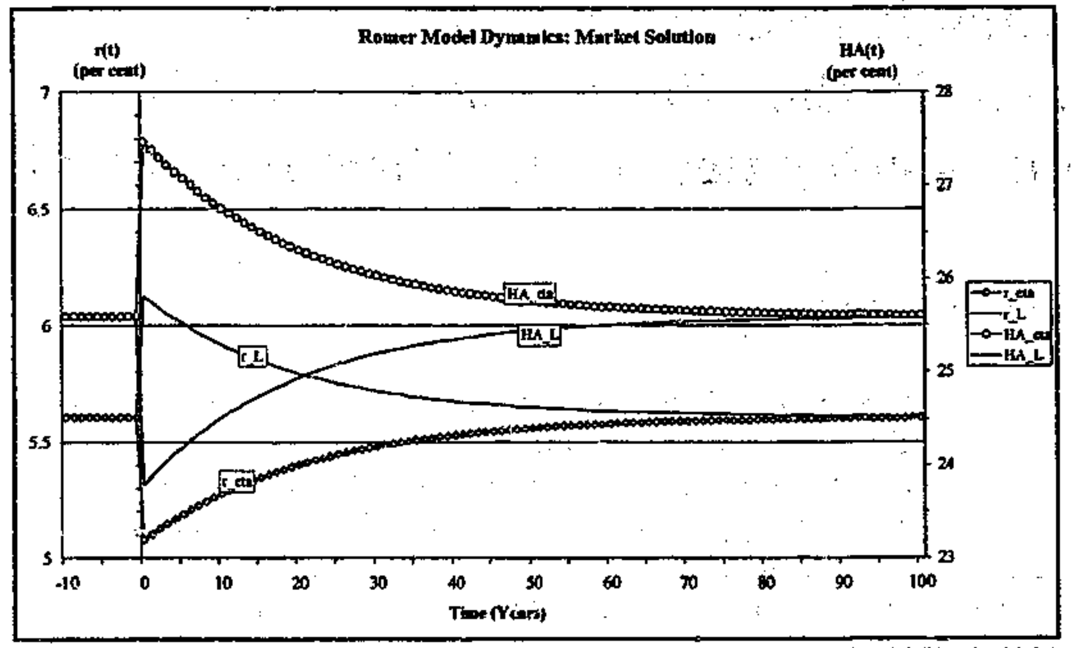

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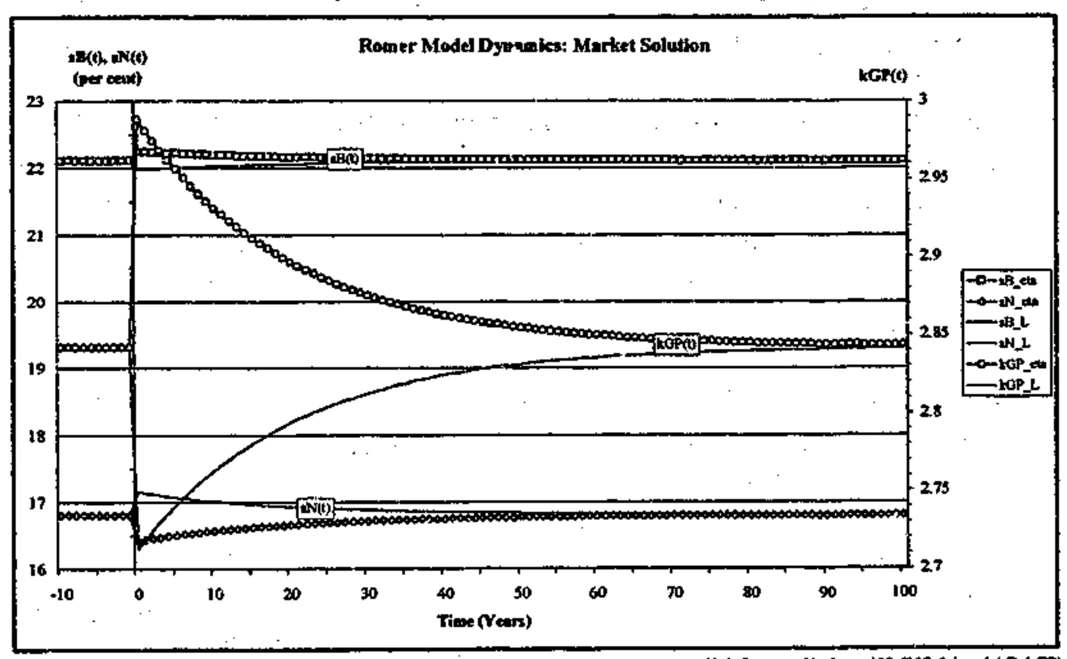
Figure A2.6.9: Dynamic effects on s_N, s_B, and k_{GP} of a sustained 10 per cent rise in the production cost of capital (η), and a 20 per cent rise in the level of ordinary labour L from time t=0, benchmark parameter set.



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Chapter 3

3 Dynamic Behaviour of the System: Linearisation, Phase-space Analysis, and Development of an Abridged Model

3.1 Introduction

With the dynamic system established and its asymptotic equilibrium or steady-state solution identified, the dynamic behaviour of the model may now be examined. Equations (2.41) to (2.43), with the variables r and H_Y substituted out via (2.44) and (2.45), provide three first-order ordinary differential equations (ODEs) in the three variables Ψ, Φ, and p_A. An analytic solution to these equations, where each variable is expressed as a function of the independent variable time, would generate three (arbitrary) *constants of integration*. Different values of these would then produce an infinity of alternative time paths for each of the variables, raising the question of how to identify, in terms of the problems (2.17) and (2.26), precisely which of these were the optimal paths. This apparent dilemma is resolved by the initial and transversality conditions (2.46) to (2.48) since their satisfaction is necessary for an optimum. By providing particular solutions on the time path (at $t = 0$ and as $t \rightarrow \infty$), they allow those particular *constants of integration* appropriate to the optimal paths to be evaluated.

In practice however, only the simplest of non-linear differential equation systems permit 'closed form' solutions (Roberts and Shipman, 1972), and the Romer system is most definitely not one of these. An analytic solution of its differential equations is not possible. Instead they must be solved by numerical methods, which present their own difficulties. For this reason a detailed investigation of the numerical solution to the dynamics of the complete Romer model is postponed until Chapter 4. Meanwhile, in this chapter the dynamic behaviour is examined by the application of three simpler 'analytic characterisations' of the model: *linearisation, phase-space analysis,* and the development of *a simplified abridgment of the model.*

First, by making linear approximations to the differential equations of the model it is possible to characterise the dynamics of the non-linear system in explicit and quantitative functional terms, a 'closed form' or analytic solution being readily obtainable from the linearised system. Second, the construction and analysis of a phase-diagram for the system can provide an illuminating picture of its dynamic behaviour. In such a delineation, plots of the differential equations are used to decompose all the possible values of a system's variables into a variety of regions in each of which the 'direction of motion' (increase or decrease) of each variable remains unchanged. While the phase-space analysis undertaken here is totally quantitative, in that all the graphical representations have been explicitly calculated from the appropriate functions, the real advantage of the method lies in the qualitative insights it provides into the dynamics of a system. Third, by replacing the full model specification of intertemporal optimising

behaviour on the part of consumers with a simple 'Solow-type' consumption function, an abridged *Solowian-Romer model* is constructed. The advantage of this over the full model is that it is much easier to integrate numerically and that its phase-space is vastly simpler, both to visualise and to analyse.

3.2 The linearised Romer model

3.2.1 Linearisation of the model and its dynamic solution

The idea behind the linearisation of a dynamic system is simply to replace the non-linear differential equations, which cannot usually be solved analytically, with linear approximations that can be. In the usual manner of differential calculus, such approximations are considered 'valid' over a *sufficiently small* domain. While this linearisation could be undertaken in the neighbourhood of **any** specific point in the entire domain of the system, it is the equilibrium or steady-state points that are of most interest and about which linearisation is usually performed. This is because the linear approximation to the system's closed form solution in the neighbourhood of its equilibrium describes the *stability* of that equilibrium.

Thus, the asymptotically stationary Romer model system specified by equations (2.41) to (2.45) was linearised by first-order Taylor series expansions about the model's steady-state, as defined by equations (2.54) to (2.59), to produce the result:

$$\dot{\mathbf{R}}(t) = \Omega_R \mathbf{R}(t) + \mathbf{v}_R \tag{3.1}$$

where $\mathbf{R}(t)$ is the column vector $(R_1(t), R_2(t), R_3(t))^T = (\Psi(t), \Phi(t), p_A(t))^T$; $\dot{\mathbf{R}}(t)$ is the corresponding vector of time derivatives; and Ω_R and $\mathbf{v}_R$ are a matrix and a vector of constant coefficients which depend upon the point about which the linearisation was performed (in this case the steady-state). Details of the linearisation and the evaluation of Ω_R and $\mathbf{v}_R$ in terms of the parameters of the model are at Appendix 3.1.

The general analytic solution to an *n*-dimensional first-order linear dynamic system like equation (3.1) is then described in Appendix 3.2, the first part of which largely follows Dixon et al. (1992). It is demonstrated there that for such a system to attain its steady-state equilibrium certain constraints on its initial values must be met. These are shown to result from the necessary satisfaction of the transversality conditions associated with the underlying dynamic optimisation problems to the system. Two alternative methods for obtaining the *unknown* initial values in terms of the *known* ones are then derived, thus allowing the complete solution to be calculated. The following particularises that appendix material to the linearised Romer model.

It turns out that the dynamic behaviour of a linear system depends on the eigenvalues or characteristic roots and eigenvectors of the system's coefficients matrix. In particular, the solution for the linearised Romer system of equation (3.1) is as follows:

$$\mathbf{R}(t) = \Gamma_R e^{\Lambda_R t} \Gamma_R^{-1} \Delta\mathbf{R}_0 + \mathbf{R}_{ss} \quad \text{for, } \Delta\mathbf{R}_0 = (\mathbf{R}_0 - \mathbf{R}_{ss}) \tag{3.2}$$

where, as described in the appendix:

- $\Gamma_R = (\gamma_{ij})$ is a (3x3) matrix whose columns are the eigenvectors $\gamma_{Ri}=(\gamma_{ji})$ of Ω ;
- $\Gamma_R^{-1} = (\gamma^{I}_{ij})$ is its inverse;
- Λ_R is a (3x3) diagonal matrix of the eigenvalues λ_{Ri}, and $e^{\Lambda_R t}$ denotes a diagonal matrix of terms $e^{\lambda_{Ri}t}$; and
- $\mathbf{R}_0$ and $\mathbf{R}_{ss}$ are (3x1) vectors of the initial values (Ψ_0, Φ_0, p_{A0}), and the steady-state levels (Ψ_{ss}, Φ_{ss}, p_{Ass}) of the dynamic variables.

As also shown in Appendix 3.2, this can be expanded from its compact matrix notation to give the solution for the kth dynamic variable as:

$$R_k(t) = \sum_i \gamma_{Rki} e^{\lambda_{Ri}t} \sum_j \gamma^{I}_{Rij} \Delta R_{0j} + R_{ssk} \tag{3.3}$$

where, for i, j, k each equal to 1, 2, 3:

- $R_k(t)$ represents the functional time paths of the dynamic variables $\Psi(t)$, $\Phi(t)$, and $p_A(t)$ respectively;
- R_{ssk} specifies the steady-states, Ψ_{ss}, Φ_{ss} and p_{Ass} of the dynamic variables;
- $\Delta R_{0j} = R_{0j} - R_{ssj}$ refers to the initial divergences of the dynamic variables from their steady-states, (Ψ_0-Ψ_{ss}), (Φ_0-Φ_{ss}), and (p_{A0}-p_{Ass});
- the λ_{Ri} and γ_{Rki} are the eigenvalues and eigenvector components respectively of the coefficients matrix Ω_R (Appendix 3.3 provides details of their calculation.); and
- the terms γ^{I}_{Rij} are derived from the eigenvectors (Appendix 3.2).

It is clear from equation (3.3) that if the first of the two terms on the right hand side vanishes over time, then the system will approach its steady-state (Ψ_{ss}, Φ_{ss}, p_{Ass}). From the presence of the exponential power terms it is also apparent that if the real part of all three characteristic roots λ_{Ri}, were negative, then the system would converge to its equilibrium from all (nearby) starting points. In such a case the equilibrium would be *locally stable*.[1] Conversely, if the real part of all three roots were positive, the system would be divergent and its equilibrium would be unstable. In this case the steady-state could only be attained if the system began there initially, with all the terms Δ_{R0j} in (3.3) equal to zero. The interesting (and most common case in economic applications at least), is where there is a mixture of both positive and negative real parts among the characteristic roots. Then the system will converge to its (local) equilibrium from some initial points, but diverge from others. This is the property of *saddle-path stability*, and if **any** path to the steady-state exists, then from Kurz (1968) this is the nature of the equilibrium that the model will exhibit.[2]

Equation (3.3) indicates that if any of the characteristic roots has a positive real component, then the only way the system can be prevented from diverging over time is for the coefficients on the corresponding exponential terms to be identically equal to zero. Since convergence is required by the transversality conditions, these conditions in

[1] A truly linear system would also converge from distant points and the system would *be globally stable*.
[2] Kurz (1968) proved the general result that if a steady-state exists for any autonomous dynamic optimisation problem with a constant positive discount factor, then the coefficients matrix of the system must have at least one characteristic root with a positive real part, so that the steady-state must be either completely unstable or exhibit saddle-path stability.

turn imply that such coefficients must indeed be zero. It follows from (3.3) that for the linearised Romer model to approach its steady-state equilibrium asymptotically it is necessary that:

$$\sum_j \gamma^{I}_{Rij} \Delta R_{0j} = \sum_j \gamma^{I}_{Rij}(R_{0j} - R_{ssj}) = 0 \tag{3.4}$$
for j = 1,2,3; and for all i such that $\lambda_{Ri} > 0$

Since the eigenvectors γ, and the steady-state values R_{ss} are basic properties of the model, such conditions amount to constraints on the set of valid initial values. Thus, in the linearised model the existence of positive real components in the eigenvalues of its coefficients matrix, and the consequent necessity for the initial value constraints (3.4), are manifestations of the transversality conditions from the underlying dynamic optimisation problems of the model. Since there are two transversality conditions associated with the Romer model (as shown in Chapter 2), two positive eigenvalues can be expected from its linearised version.

Numerical evaluation of the eigenvalues (Appendix 3.3) confirmed this correspondence between transversality conditions and the constraints on positive roots. In all economically meaningful cases there were exactly two roots with positive real parts.[3] For the most plausible values of the parameters all three eigenvalues were real (one negative and two positive). For somewhat less plausible parameter values the two real positive roots could be converted to a pair of complex conjugates with positive real components. The *benchmark parameter set* (Table 2.2 of Section 2.4), which was used as a base about which both individual and joint parameter variations were made, generated the following eigenvalues, λ_i and the corresponding eigenvectors, γ_i :

$$\begin{aligned} &\lambda_1 = -0.0493;\ \lambda_2 = 0.1899;\ \text{and}\ \lambda_3 = 0.1021 \\ &\gamma_1 = (0.6874, -0.01352, 0.7262);\ \gamma_2 = (0.8181, -0.02548, -0.5745);\ \text{and} \\ &\gamma_3 = (0.4843, -0.000768, -0.8749) \end{aligned} \tag{3.5}$$

Calculation of the eigenvalues and corresponding eigenvectors for any set of parameter values then allows the complete solution of the linear model (that is, the dynamic paths of all the variables) to be determined via equation (3.2), or (3.3), and the necessary conditions (3.4). In particular, the two initial value constraints can be evaluated numerically so as to express the initial values for the two variables associated with positive eigenvalues in terms of the initial value of the variable associated with the only negative eigenvalue.[4] Such a result defines the (linearised) saddle-path of the dynamic

[3] The system could be made to return either two real negative roots or a pair of complex conjugates with negative real parts (so that the path towards steady-state would be oscillatory), or even three positive real roots, but only with sets of parameter values which were either invalid in themselves or which generated economically meaningless outcomes (Appendix 3.3).
[4] Mathematically, the decision of which variables and which eigenvalues are to be *associated* with one another is arbitrary. Moreover, the saddle-path solution is invariant to any such allocation. The real question is: which variables are to have their initial values specified exogenously, and which are to be solved for from the constraint equations? Despite the invariance of the saddle-path to this choice, the time paths of the variables are highly dependent upon it. In some ways it is dictated by the dynamic optimisation procedure which required state variables to be continuous while control and costate variables could contain jumps (Appendix 2.2). Economic criteria are also important. Further discussion of the issues is in Section 4.1 of Chapter 4.

system. Two general methods (for an *n*-dimensional system with $m<n$ negative eigenvalues) are presented in Appendix 3.2, with the results given in equations (A3.2.12) and (A3.2.15). Application of these to the 3-dimensional linearised Romer model system produced its saddle-path solution. As shown below, the result from (A3.2.15) is by far the simpler. First, from (A3.2.12):

$$\begin{pmatrix}\Phi_0\\ p_{A0}\end{pmatrix}=\begin{pmatrix}\Phi_{ss}\\ p_{Ass}\end{pmatrix}-\begin{bmatrix}\gamma^1_{22} & \gamma^1_{23}\\ \gamma^1_{32} & \gamma^1_{33}\end{bmatrix}^{-1}\begin{pmatrix}\gamma^1_{21}\\ \gamma^1_{31}\end{pmatrix}(\Psi_0-\Psi_{ss}) \tag{3.6}$$

while from (A3.2.15):

$$\begin{pmatrix}\Phi_0\\ p_{A0}\end{pmatrix}=\begin{pmatrix}\Phi_{ss}\\ p_{Ass}\end{pmatrix}+\begin{pmatrix}\gamma_{21}\\ \gamma_{31}\end{pmatrix}\gamma_{11}^{-1}(\Psi_0-\Psi_{ss})$$

so (3.7)

$$\Phi_0=(\Phi_{ss}-\frac{\gamma_{21}}{\gamma_{11}}\Psi_{ss})+\frac{\gamma_{21}}{\gamma_{11}}\Psi_0 \quad\text{and}\quad p_{A0}=(p_{Ass}-\frac{\gamma_{31}}{\gamma_{11}}\Psi_{ss})+\frac{\gamma_{31}}{\gamma_{11}}\Psi_0$$

Numerical evaluation of equations (3.6) and (3.7) produces identical results, which may be written more simply as:

$$\Phi_0=c_1+c_2\Psi_0 \quad\text{and}\quad p_{A0}=c_3+c_4\Psi_0 \tag{3.8}$$

where c_1, c_2, c_3 and c_4 are constants determined by the parameters. For the benchmark set of parameter values: $c_1\approx0.4015$, $c_2\approx-0.0197$, $c_3\approx2.6113$, and $c_4\approx1.0564$.

By specifying a particular value for Ψ_0, equation (3.8) defines the vector R_0 in equation (3.2) or (3.3), which may then be used to calculate the dynamic paths. Alternatively, the complete linear system solution given by equation (A3.2.16) in Appendix 3.2 may be used. In any case, it is apparent that the dynamic solution takes the form:

$$\Psi(t)=C_\Psi e^{\lambda_{R1}\cdot t}+\Psi_{ss}$$
$$\Phi(t)=C_\Phi e^{\lambda_{R1}\cdot t}+\Phi_{ss}$$
$$p_A(t)=C_{p_A}e^{\lambda_{R1}\cdot t}+p_{Ass}$$

where the terms C_Ψ, C_Φ, and C_{pA} are functions of the eigenvector components and as such are constants. Moreover, as can be seen by setting the time variable to zero (t=0), these terms are equal to the initial divergences from the steady-state. That is:

$$C_\Psi=\Psi_0-\Psi_{ss}=\Delta\Psi_0;\ C_\Phi=\Phi_0-\Phi_{ss}=\Delta\Phi_0;\ \text{and}\ C_{p_A}=p_{A0}-p_{Ass}=\Delta p_A$$

So the dynamic paths are:

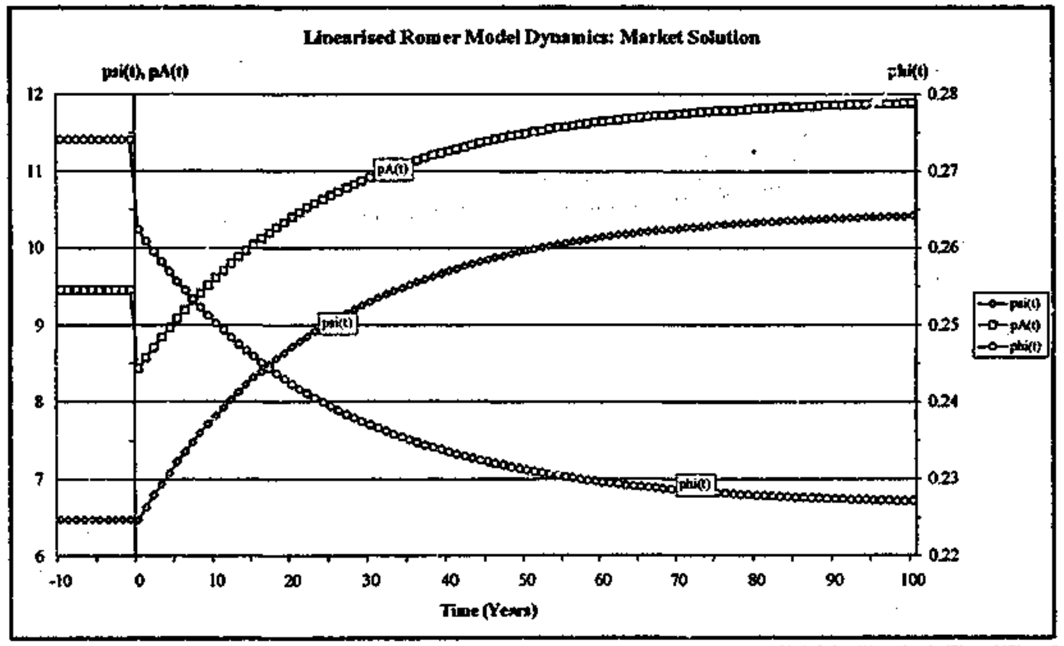
$$\begin{aligned}\Psi(t)&=\Psi_0e^{\lambda_{R1}\cdot t}+\Psi_{ss}(1-e^{\lambda_{R1}\cdot t})\\ \Phi(t)&=\Phi_0e^{\lambda_{R1}\cdot t}+\Phi_{ss}(1-e^{\lambda_{R1}\cdot t})\\ p_A(t)&=p_{A0}e^{\lambda_{R1}\cdot t}+p_{Ass}(1-e^{\lambda_{R1}\cdot t})\end{aligned} \tag{3.9}$$

3.2.2 A simulation of the transient dynamics of the linearised model

Calculations of these dynamic paths have been made with Ψ_0 set at its steady-state value under the benchmark parameter set; and the constants c_1 to c_4 in (3.8) above taking values determined by setting γ at ten per cent higher than its benchmark level from time t=0 onwards while the other parameters remained at benchmark. Thus, the calculations simulate the application to the system, in its benchmark equilibrium, of a sustained ten per cent shock to the output elasticity of capital (γ), which may also be interpreted as the profit share of income from the production of final output (Appendix 2.1). The results of the simulation are presented in Figure 3.1 to Figure 3.5 and the principal numbers involved are (approximately) as follows:

- $\Psi(-10)$ to $\Psi(0)\approx6.48$; $\Phi(-10)$ to $\Phi(-1)\approx0.274$; and $p_A(-10)$ to $p_A(-1)\approx9.45$ (all benchmark values);
- c_1, c_2, c_3, and $c_4\approx0.320$, 0.009, 2.75, and -0.877 respectively;
- $\Phi(0)\approx0.262$ and $p_A(0)\approx8.43$ (from equation (3.8)); and
- Ψ_{ss}, Φ_{ss}, and $p_{Ass}\approx10.48$, 0.227, and 11.94 respectively.

Figure 3.1: Dynamic effects on Ψ, Φ, and p_A of an unanticipated and sustained 10 per cent rise in parameter γ from time zero, benchmark parameter set.



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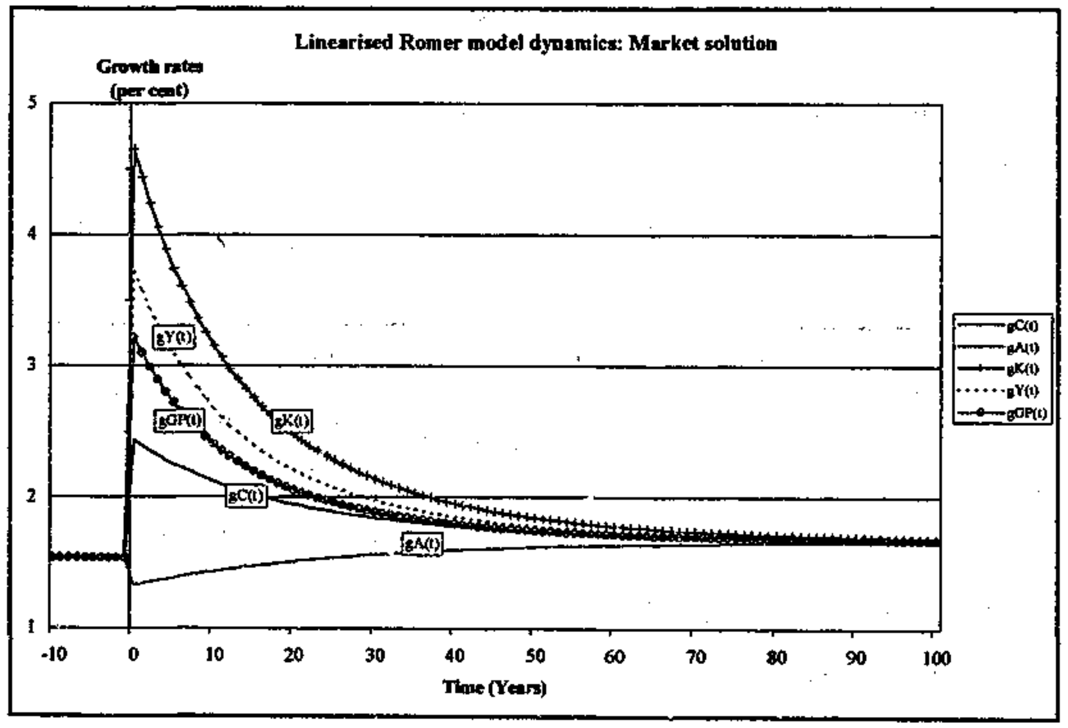
The jumps (from their previous equilibria) in the initial values of the consumption-capital ratio Φ and the price of technology p_A, which are necessary to get to the new saddle-path, are evident from Figure 3.1, as are their subsequent exponential growth paths towards the new equilibrium, according to equation (3.3) or (3.9). Eventually, when the system adjusts to its new equilibrium, the overall changes in the variables Ψ, Φ, and p_A are 61.8 per cent, -17.4 per cent, and 26.4 per cent respectively; as calculated previously in Chapter 2 (Table 2.3 of Section 2.4.2).

Interestingly, while the initial jump in Φ (-4.3 per cent) is in the same direction to its subsequent adjustment towards equilibrium, this is not the case for p_A, which initially drops by 10.8 per cent before gradually rising towards its new equilibrium! Discussion of the economics behind the directions of adjustment in these and other variables is postponed until Section 4.5 of Chapter 4 where, along with other simulations this one is repeated for the actual (non-linear) model, and where the dynamics are solved by numerical methods. Here the discussion is limited to descriptions of the variables plotted in the figures, and simply to pointing out some of the more significant results depicted therein.

Figure 3.2: **Dynamic effects on the growth rates of an unanticipated and sustained 10 per cent rise in parameter γ from time zero, benchmark parameter set.**



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While the growth rates of technology, consumption, capital, output and gross product all begin at the same (benchmark) level and all approach the same new steady-state level, thereby all registering the same overall change (of 8.3 per cent as calculated in Table 2.3), their individual adjustment paths are very different (Figure 3.2). In response to the autonomous ten per cent rise in γ, the initial jumps in these growth rates vary between a fall of 13.8 per cent in the case of g_A, to an increase of 202.6 per cent for g_K.

The changes in the growth rates of technology and consumption (g_A and g_C) directly reflect the respective changes in the allocation of human capital to research, H_A, and the interest rate, r (Figure 3.2 and Figure 3.3). In percentage terms, both the initial jumps in H_A and g_A, and their overall steady-state adjustments are identical; while for r and g_C there is a close, but not exact correspondence. These results follow from the relations:

$$g_A = \zeta H_A \quad \text{and} \quad g_C = (r - \rho)/\sigma$$

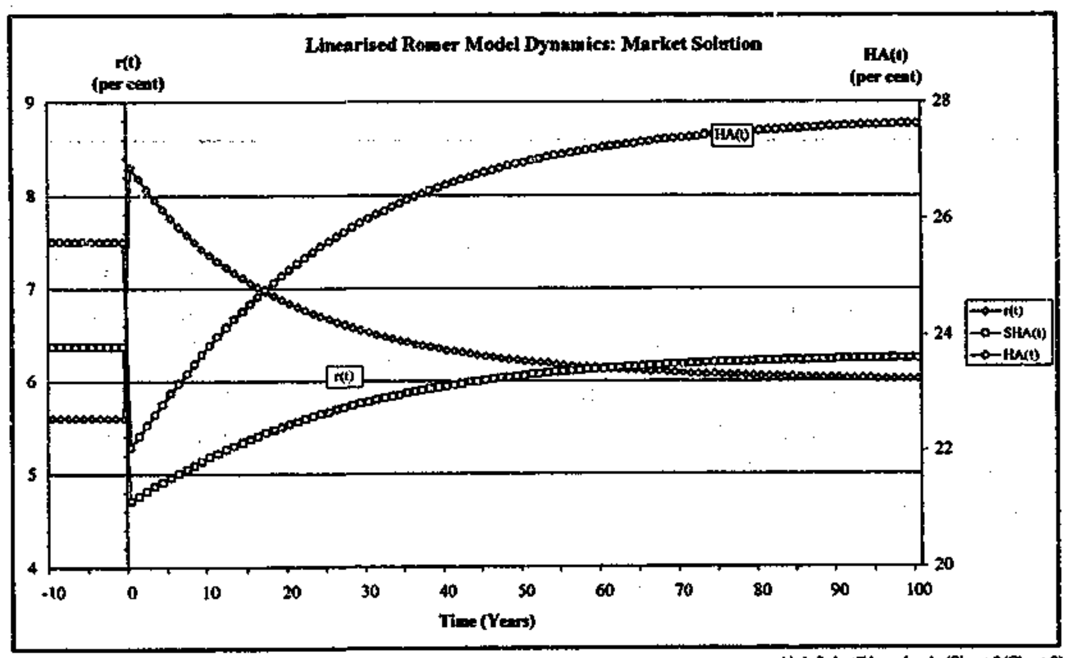
In addition to the overall savings rate s_B, both the narrow measure s_N (which is equivalent to the investment share of output, I/Y), and the research share of gross product, S_{HA}, are also measures of savings in the model. The latter measures the extent of the savings decision to devote human capital resources to research rather than directly to the output of consumables; while the former measures the proportion of output to be saved as capital rather than consumed.[5] The relationship between them is as follows (also refer back to Appendix 2.5):

Define R(t) as the total value of research, measured as usual in terms of the consumption good. Then gross product is GP(t) = Y(t) + R(t), and the savings terms are: $s_B(t) = [I(t)+R(t)]/GP(t)$; $s_N(t) = I(t)/Y(t)$; and $S_{HA}(t) = R(t)/GP(t)$. It follows that:

$$S_{H_A}(t) = s_B(t) - \frac{I(t)}{Y(t)}\left[\frac{Y(t)}{GP(t)}\right] = s_B(t) - s_N(t)[1 - S_{H_A}(t)] \qquad (3.10)$$

This relation may be confirmed visually from Figure 3.3 and Figure 3.4 - notice first that $S_{HA}(t)$ varies between about 0.045 to 0.065 over the 100 year time span so multiplication of the $[1-S_{HA}(t)]$ term by the 25 to 20 per cent variation in $s_N(t)$ indicates that the second term in (3.7) varies between about 24 per cent at t=0 to 18½ per cent at t=100; deducting these from the corresponding variation in $s_B(t)$ ($\approx$ 28.5 to 25 per cent) confirms the approximately 4½ to 6½ per cent variation in $S_{HA}(t)$.

Figure 3.3: **Dynamic effects on r and H_A of an unanticipated and sustained 10 per cent rise in parameter γ from time zero, benchmark parameter set.**



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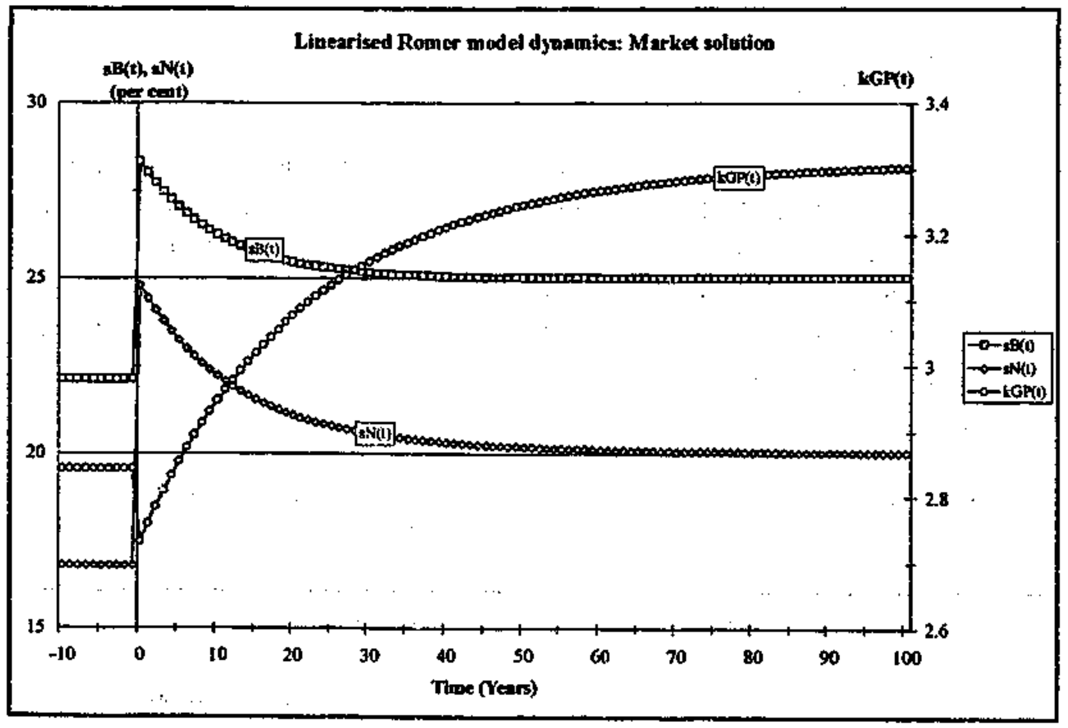
The response of both overall savings s_B, and output savings s_N=I/Y, to the autonomous ten per cent rise in the output to capital elasticity γ, are substantial initial upward jumps

[5] Or alternatively, the share of non-research resources to be devoted to capital formation rather than to the production of consumables.

(28 and 48 per cent respectively) followed by gradual declines which erode some 55 or 60 per cent of the initial increases, so that these variables finish 13 per cent and 19 per cent respectively above their pre-shock levels. In contrast, savings manifested through research effort initially fall by 26 per cent then gradually recover, but remain 1½ per cent below the pre-shock level.

Figure 3.4: Dynamic effects on s_B, s_N, and k_{GP} of an unanticipated and sustained 10 per cent rise in parameter γ from time zero, benchmark parameter set.



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The distribution of gross income (which equals gross output) between the factors of production: capital, human capital in production, ordinary labour, and human capital in research are shown in Figure 3.5. The formulations for these were derived in Appendix 2.1. The share of gross income to human capital in research is equivalent to the research share of gross output, S_{HA}. In this simulation most of the response in these variables arises in the initial jumps. Also, the overall responses in S_K, S_{HY}, and S_L are almost completely determined by the direct impact of the shock itself:

From Appendix 2.1 the shares are determined by:

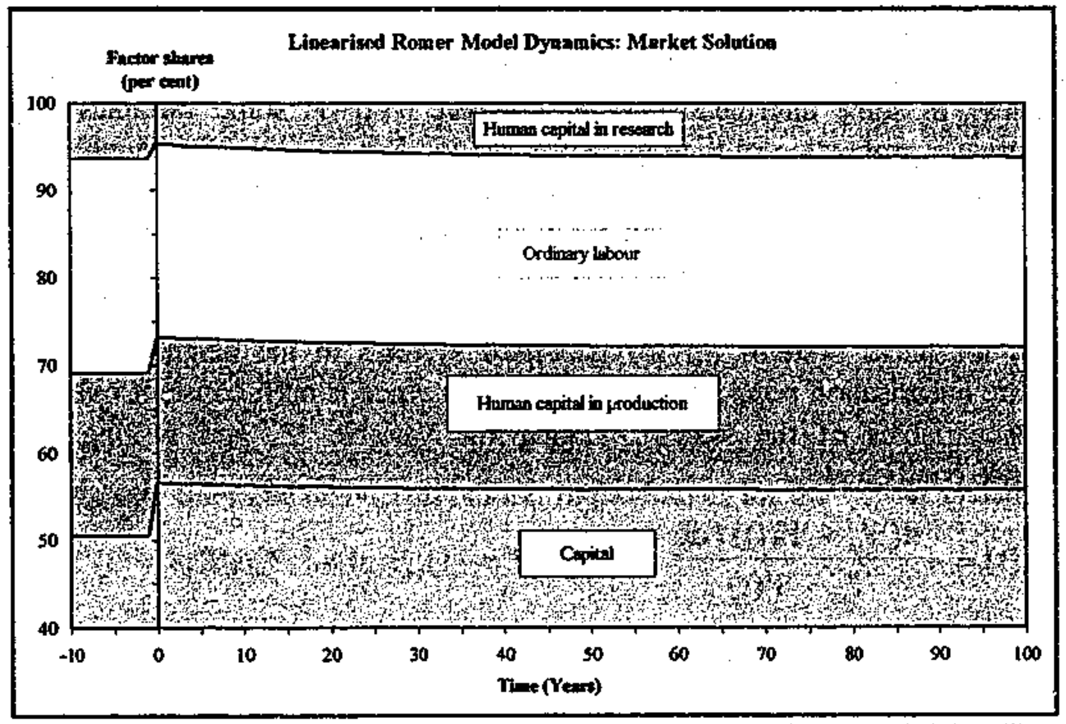
$$s_K(t) = \gamma Y(t)/GP(t); \quad s_{H_Y}(t) = \alpha(1-\gamma)Y(t)/GP(t); \text{ and}$$
$$s_L(t) = (1-\alpha)(1-\gamma)Y(t)/GP(t) \tag{3.11}$$

It follows from this that the overall percentage changes are:

$$\frac{\Delta s_K}{s_K} \approx \frac{\Delta\gamma}{\gamma} + \frac{\Delta(Y/GP)}{(Y/GP)} = 10\% + 0.1\% = 10.1\%$$
$$\frac{\Delta s_{H_Y}}{s_{H_Y}} = \frac{\Delta s_L}{s_L} \approx \frac{\Delta(1-\gamma)}{(1-\gamma)} + \frac{\Delta(Y/GP)}{(Y/GP)} = -\frac{\Delta\gamma}{\gamma}\frac{\gamma}{1-\gamma} + \frac{\Delta(Y/GP)}{(Y/GP)} \tag{3.12}$$
$$\approx -10.(0.54/0.46)\% + 0.1\% = -11.6\%$$

The actual changes in S_K, S_{HY} and S_L (to one decimal place) are 10.1, 11.7 and 11.7 per cent respectively.

Figure 3.5: Dynamic effects on the factor shares of gross income from an unanticipated and sustained 10 per cent rise in parameter γ from time zero, benchmark parameter set.



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Although somewhat more difficult than the *'unanticipated and sustained'* type of shock examined above, the linearised model also allows the analytic determination of the dynamics resulting from *'anticipated'* and *'temporary'* shocks. These types of shocks are discussed in general terms with the aid of a 'phase diagram' in Section 3.3.3. In Appendix 3.4 the analytic determination of their linearised model dynamics is derived and the method applied to two different shocks: 'an anticipated and sustained rise in the productivity of researchers' (parameter ζ); and 'an unanticipated but temporary decline in parameter α'.[6] Later in the current chapter all three of the shocks examined with the linearised model are also analysed under the Solowian-Romer model framework; and in Chapter 4, both these sets of results are compared with the corresponding dynamics of the full non-linear model obtained from numerical methods.

[6] Which is shown in Chapter 4 (Section 4.5.3) to be equivalent to a 15 per cent rise in the wages of ordinary labour.

3.2.3 Speed of convergence

The signs of the linearisation eigenvalues have been seen to be important in determining the type of convergence or stability of the dynamic system. Because the sign of one and only one of the three is negative, the system converges with saddle-path stability. As will be demonstrated in the following, the magnitude of this eigenvalue is also significant since it determines the rate at which the system approaches its steady-state; the greater the absolute value of the negative eigenvalue, the faster the convergence.

In broad terms the *speed of convergence*, or the *convergence coefficient* may be defined as the "proportional rate at which the gap between the system's current position and its steady-state equilibrium is closed." For non-linear systems this will depend upon the divergence of the 'current position' from equilibrium and upon which variable is chosen for the measurement. On the attainment of equilibrium, at time T say,[7] all variables will have closed 100 per cent of whatever gaps there were at any earlier time, $t_0<T$ say. Nevertheless, at some intermediate time t_1, where $t_0<t_1<T$, some variables may have closed a greater proportion of their gaps than others. For example, for variables x(t) and $y(t)=\log_{10}x(t)$, in the time taken to close 50 per cent of the gap between $x(t_0)=10$ and $x_{ss}=100$ - so $x(t_1)=55$; 74 per cent of the initial divergence of y(t), from $y(t_0)=1$ to its equilibrium level $y_{ss}=2$ is closed since $y(t_1)=\log_{10}55=1.74$.

However, for linear systems, or for linearised systems in the neighbourhood of the steady-state linearisation point, the speed of convergence is both constant over time and across all variables of the system. Taking the variable Ψ, and writing β as the convergence coefficient:

$$\beta(t) = -\frac{d(\Psi_{ss}-\Psi(t))/dt}{\Psi_{ss}-\Psi(t)} \tag{3.13}$$

and from the first equation of the set (3.9):

$$\Psi_{ss}-\Psi = (\Psi_{ss}-\Psi_0)e^{\lambda_{R1}\cdot t}$$

So:

$$\beta(t) = \beta = -\lambda_{R1} \tag{3.14}$$

Thus, the magnitude of the negative eigenvalue determines the speed of convergence, at least in the neighbourhood of the steady-state. As calculated in Chapter 2 (Section 2.4), the convergence coefficient for the benchmark parameter set was 0.049, or 4.9 per cent per annum. The source of this estimate was, of course, the negative eigenvalue from the linearised model coefficients matrix calculated under the benchmark parameter set and recorded in (3.5). Under the simulation considered here, this rate of convergence is slowed by some 17 per cent to 4.1 per cent a year.

Empirical measures of the speed of convergence have usually been based on the convergence of a growth rate variable, commonly income per unit of labour, or per unit

[7] Strictly, equilibrium is not actually attained, but only approached asymptotically. Nevertheless, points arbitrarily close to equilibrium are attained.

of *effective labour*,[8] in a log-linearised model[9] (see Mankiw, Romer and Weil, 1992; and Barro and Sala-i-Martin, 1995). This raises the question of whether a log-linearised version of the Romer model would generate the same convergence coefficient as the ordinarily linearised version. The prior expectation would seem to be that the results should be at least asymptotically the same. After all, the underlying model is the same and both methods of linearisation are merely approximations, valid in the neighbourhood of the steady-state. A formal proof to this effect is offered in Appendix 3.5 and, as will be seen below, the result is confirmed by numerical evaluation of the eigenvalues of the log-linear system.

Corresponding to the ordinarily linearised form (3.1), the log-linearised version of the Romer model is:

$$\mathbf{d/dt(lnR(t))} = \mathbf{\Omega_{RL}\,lnR(t)} + \mathbf{v_{RL}} \tag{3.15}$$

where $\mathbf{lnR(t)}$ is the column vector $(\ln R_1(t), \ln R_2(t), \ln R_3(t))^T = (\ln\Psi(t), \ln\Phi(t), \ln p_A(t))^T$; $\mathbf{d/dt(lnR(t))}$ is the corresponding vector of time derivatives; and $\mathbf{\Omega_{RL}}$ and $\mathbf{v_{RL}}$ are a matrix and a vector of constant coefficients. Log-linearisation of the model has been undertaken in Appendix 3.5 where the matrix $\mathbf{\Omega_{RL}}$ and the vector $\mathbf{v_{RL}}$ are calculated in terms of the parameters.

The general solution to a linear system of first-order differential equations was described in Appendix 3.2. This is directly applicable to the log-linearised system (3.15), and generates a result corresponding to equation (3.2). Moreover, the same arguments concerning the transversality conditions and the necessity for zero coefficients on terms with positive eigenvalues continue to apply, and results similar to equation (3.9) can be derived in exactly the same manner as before. In particular, for the log-linearised system:

$$\ln\Psi(t) = \ln\Psi_0 e^{\lambda_{RL1}\cdot t} + \ln\Psi_{ss}(1-e^{\lambda_{RL1}\cdot t}) \tag{3.16}$$

In these log-linearised systems the convergence coefficient, denoted here as β_L, is defined as 'the rate of decline in the growth rate with respect to the corresponding logged variable'; or equivalently, 'the proportional rate of decline in the growth rate' (see Barro and Sala-i-Martin, 1995). That is:

$$\beta_L(t) = -\frac{d(g_\Psi(t))}{d(\ln\Psi(t))} = -\frac{d(g_\Psi(t))/dt}{d(\ln\Psi(t))/dt} = -\frac{\dot{g}_\Psi(t)}{g_\Psi(t)} \tag{3.17}$$

where

$$g_\Psi(t) = \dot{\Psi}(t)/\Psi(t) = d(\ln\Psi(t))/dt$$

Applying this definition to (3.16) generates a result similar to (3.14):

$$g_\Psi(t) = \lambda_{RL1}(\ln\Psi_0 - \ln\Psi_{ss})e^{\lambda_{RL1}\cdot t}$$

and

$$\dot{g}_\Psi(t) = \lambda_{RL1}^2(\ln\Psi_0 - \ln\Psi_{ss})e^{\lambda_{RL1}\cdot t}$$

So:

[8] *Effective labour* is the product of the normal labour and the (labour augmenting) level of technology.
[9] Log-linearisation naturally produces growth rate variables since $d(\ln z(t))/dt = \dot{z}(t)/z(t)$.

$$\beta_L(t) = \beta_L = -\lambda_{RL1} \tag{3.18}$$

The result also follows from (3.16) if the convergence coefficient is defined in a corresponding way to that used for the simple linear model. Namely, as 'the proportional rate at which the gap between the current position of a logged variable and its steady-state is closed'. That is:

$$\beta_L(t) = -\frac{d(\ln\Psi_{ss} - \ln\Psi(t))/dt}{\ln\Psi_{ss} - \ln\Psi(t)} = \beta = -\lambda_{RL1} \tag{3.19}$$

Now, from (3.18) and (3.14), the convergence coefficients from the simple linearisation and the log-linearisation, despite being somewhat differently defined, will be equal if $\lambda_{RL1}=\lambda_{R1}$. This is proved in Appendix 3.5 by applying the log-linear definition of the convergence coefficient to the ordinarily linearised model and vice versa. Moreover, when the eigenvalues and eigenvectors of the log-linear coefficients matrix, Ω_{RL}, are evaluated numerically, and this is done in exactly the manner described earlier in Appendix 3.3, all the eigenvalue results turn out to be the same as for the ordinary linearisation. The eigenvectors however, are different. Results for the benchmark parameter set, which may be compared with those reported earlier in equation (3.5), are as follows:

$$\begin{aligned}&\lambda_1 = -0.0493;\ \lambda_2 = 0.1899;\ \text{and}\ \lambda_3 = 0.1021\\ &\gamma_1 = (0.7582, -0.3523, 0.5487);\ \gamma_2 = (0.7510, -0.5527, -0.3613);\ \text{and}\\ &\gamma_3 = (0.6283, -0.0235, -0.7776)\end{aligned} \tag{3.20}$$

Some comment on the effect of the different definitions of the convergence coefficients in the alternative linearisations is in order here. It turns out that in the simple linear model, the log-linear convergence coefficient is asymptotically equal to the *'true'* simple linear coefficient; and that in the log-linear model the simple linear coefficient is asymptotically equal to the *'true'* log-linear coefficient. In particular:

$$\beta_L(t) = \beta + g_\Psi(t) \quad \text{and} \quad \lim_{t\to\infty}\beta_L(t) = \beta \tag{3.21}$$

and

$$\beta(t) = \beta_L(\ln\Psi_{ss} - \ln\Psi(t))\frac{\Psi(t)}{\Psi_{ss} - \Psi(t)} \quad \text{and} \quad \lim_{t\to\infty}\beta(t) = \beta_L \tag{3.22}$$

This issue is explored in a little more detail in Appendix 3.5.

Besides the convergence coefficient β, another common measure of the speed of convergence of a dynamic system is its *half-life*, denoted by $t_{1/2}$. This is "the time taken for half of the gap between the system's current position and its steady-state equilibrium to be closed." Once again, in non-linear systems $t_{1/2}$ will depend upon the divergence of the 'current position' from equilibrium and upon which variable is chosen for the measurement; but in a linear system it is constant, both over time and across all the dynamic variables.

As before, the variable Ψ is chosen. Generalising, when a proportion p of the initial gap from its steady-state equilibrium is closed, $\Psi(t)$ will take the value:

$$\Psi(t_p) = \Psi_0 + p(\Psi_{ss} - \Psi_0) = p\Psi_{ss} + (1-p)\Psi_0$$

Then, substituting this value for $\Psi(t)$ in its dynamic equation - the first of the set (3.9):

$$\begin{aligned}&\Psi(t_p) = p\Psi_{ss} + (1-p)\Psi_0 = (\Psi_0 - \Psi_{ss})e^{\lambda_{R1}\cdot t_p} + \Psi_{ss}\\ &(1-p)(\Psi_0 - \Psi_{ss}) = (\Psi_0 - \Psi_{ss})e^{\lambda_{R1}\cdot t_p}\\ &(1-p) = e^{\lambda_{R1}\cdot t_p} \quad \Rightarrow \quad t_p = \frac{\ln(1-p)}{\lambda_{R1}}\end{aligned} \tag{3. 23}$$

Thus, setting p=1/2, for a linear dynamic system with saddle-path stability, or for a non-linear one in the neighbourhood of a steady-state, the half-life measure of the speed of convergence is given by:

$$t_{1/2} = \frac{\ln(1/2)}{\lambda_1} \approx \frac{0.693}{|\lambda_1|} \tag{3.24}$$

where λ_1 is the negative eigenvalue of the coefficients matrix (computed about the steady-state for a non-linear system).

In the case of the linearised Romer model with the benchmark parameter set, where the negative eigenvalue was λ_{R1}=-0.049, the half-life of the system is a little over 14 years; and for the simulated ten per cent increase in the output elasticity of capital (for which λ_{R1}=-0.041) this rises to 17 years. These are relatively long periods of time and they suggest that the processes of adjustment to economic shocks are slow.[10] With only half the adjustment undertaken in about 15 years, and with some three decades required to achieve three-quarters of the adjustment it is unlikely for economies ever to be in steady-state equilibrium. Some new shock will surely arise long before the adjustment to a previous one is even half complete. Of course this in turn suggests that for practical economic management it is the transient dynamics which are more important than the (eventual) long-run behaviour of the economy.

3.3 Phase-space analysis

In general, the *phase-space* of any dynamic system is simply an *n*-dimensional vector space (with n orthogonal coordinates) encompassing all the possible values which can be jointly taken by its *n* dependent variables ($x_i(t)$, for i = 1,...*n*, say). The first derivatives of each of these dependent variables with respect to the system's independent variable (t, say) over all points in this *hyperspace* define a vector field which, in principle, could be evaluated and mapped to describe the dynamics of the system over the entire phase-space. However, it is more convenient to break the phase-space up into a set of specific regions in each of which the change in any dynamic variable always has the same sign. That is, within any one of these regions any particular variable is either always increasing

[10] Somewhat lengthier adjustment periods were revealed from simulations with the non-linear model (Section 4.5), where half-lives of almost 20 years, were generated. From equation (3.24) above, this would imply a convergence coefficient of about 3.6 per cent.

or always decreasing. Also, in crossing the boundary between two of these regions there is a switch in the sign of the direction in which a specific variable changes.

Thus, analysis of the phase-space is undertaken by defining n hypersurfaces (each of dimension n-1) as the loci of points for which the first derivatives of each of the system's dependent variables with respect to its independent variable are zero: $dx_i(t)/dt = 0$, $i = 1,\ldots,n$. Then, for any specific independent variable, $i = i^*$ say, the direction of change of i^*, which is given by $dx_{i^*}(t)/dt$, switches sign in moving across the surface $dx_{i^*}(t)/dt = 0$. Moreover, in the phase-space regions defined by the intersections of these *phase surfaces* the 'directions of motion' of each of the system's dependent variables (that is the signs of dx_i/dt) can be ascertained. Any point of common intersection of all these hypersurfaces defines a steady-state of the system.

3.3.1 Phase-surfaces and phase-space regions

With only three dynamic variables, the 'stationary' form of the Romer system (as specified in equations (2.41) to (2.45)), defines a three-dimensional phase-space comprising three two-dimensional surfaces which can be illustrated diagrammatically. In turn, these surfaces define boundaries in the phase-space at which the (time) derivatives of the dynamic variables change sign. For $\Psi(t)$, $\Phi(t)$, and $p_A(t)$ all positive, the stationary dynamic system generates the surfaces *psisurf*, *phisurf* and *pAsurf* described by equations (3.25) to (3.27) respectively (see Appendix 3.6 for details).

$\dot{\Psi} \gtreqless 0$ as

$$\Phi \lesseqgtr \zeta\left[\frac{p_A}{\alpha(1-\gamma)\Psi}+1\right]\left[\frac{\alpha(1-\gamma)}{\zeta\eta^{\gamma}}L^{(1-\alpha)(1-\gamma)}\Psi^{\gamma}p_A^{-1}\right]^{\frac{1}{1-\alpha(1-\gamma)}}-(\delta+\zeta H) \tag{3.25}$$

$\dot{\Phi} \gtreqless 0$ as

$$\Phi \gtrless \frac{\zeta p_A}{\alpha(1-\gamma)\Psi}\left(1-\frac{\gamma^2}{\sigma}\right)\left[\frac{\alpha(1-\gamma)}{\zeta\eta^{\gamma}}L^{(1-\alpha)(1-\gamma)}\Psi^{\gamma}p_A^{-1}\right]^{\frac{1}{1-\alpha(1-\gamma)}}-\frac{\delta(\sigma-1)-\rho}{\sigma} \tag{3.26}$$

and

$\dot{p}_A \gtreqless 0$ as

$$\Psi \lesseqgtr \frac{\gamma\zeta p_A}{\alpha}\left[\frac{\gamma p_A}{(1-\gamma)\Psi}-1\right]\left[\frac{\alpha(1-\gamma)}{\zeta\eta^{\gamma}}L^{(1-\alpha)(1-\gamma)}\Psi^{\gamma}p_A^{-1}\right]^{\frac{1}{1-\alpha(1-\gamma)}}-\delta p_A \tag{3.27}$$

These phase-surfaces have been computed numerically and then plotted in Figure 3.6 to Figure 3.9 for the benchmark parameter set described in Section 2.4 of Chapter 2.[11] The first three of these figures show the individual surfaces, while in the fourth (Figure 3.9) all three surfaces are plotted jointly. Here the point of intersection of all three surfaces is the steady-state of the system. By re-calculating and re-plotting the phase surfaces that result from changes to the different parameters, an indication of the sensitivity of the

[11] Using the mathematical software package *"Maple"* from Waterloo Maple Software Inc., Canada; and then copying to *"Microsoft Paint"* for labelling.

whole phase-space to such variations in parameter values is obtained. The results of such calculations are presented in Appendix 3.6 in the form of further diagrams similar to Figure 3.9.

Figure 3.6: **Romer system phase surface for which $\dot{\Psi} = 0$ (ie. "psisurf"), seen from the viewpoint: angle of incidence=60 degrees, angle of declination=70 degrees; benchmark parameter set.**

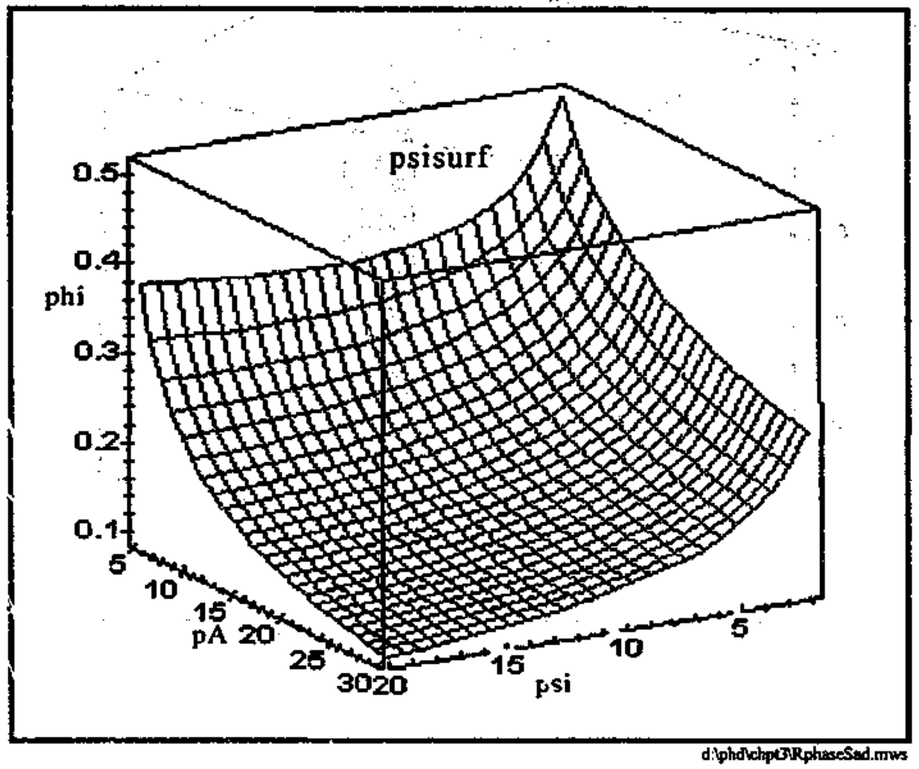

Figure 3.7: **Romer system phase surface for which $\dot{\Phi} = 0$ (ie. "phisurf"), seen from the viewpoint: angle of incidence=60 degrees, angle of declination=70 degrees; benchmark parameter set.**

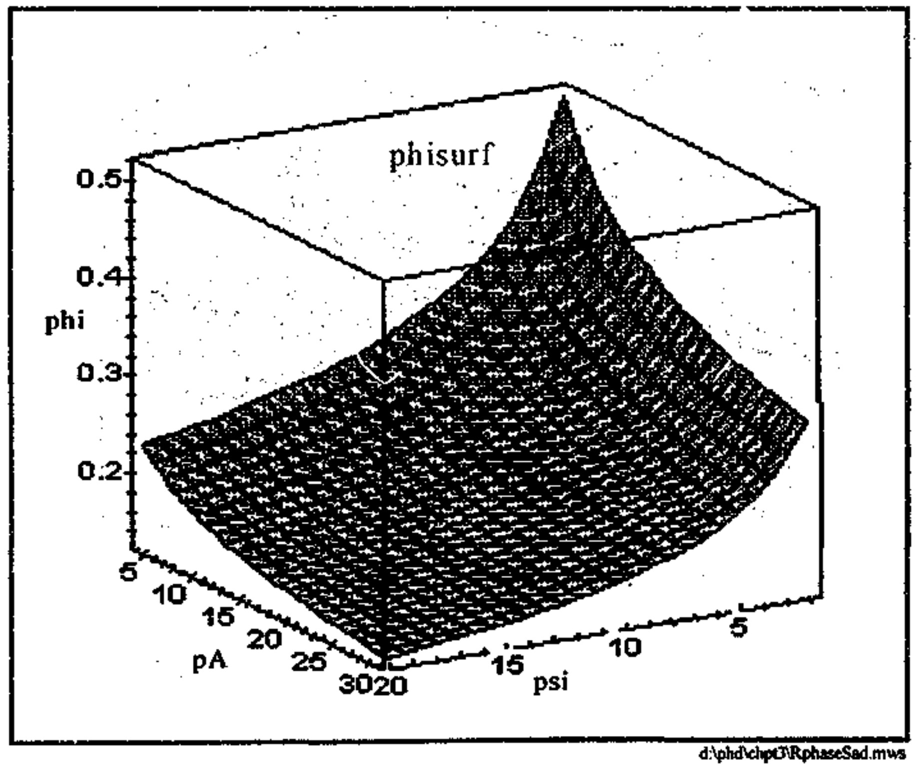

Figure 3.8: **Romer system phase surface for which $\dot{p}_A = 0$ (ie. "pAsurf"), seen from the viewpoint: angle of incidence=60 degrees, angle of declination=70 degrees; benchmark parameter set.**

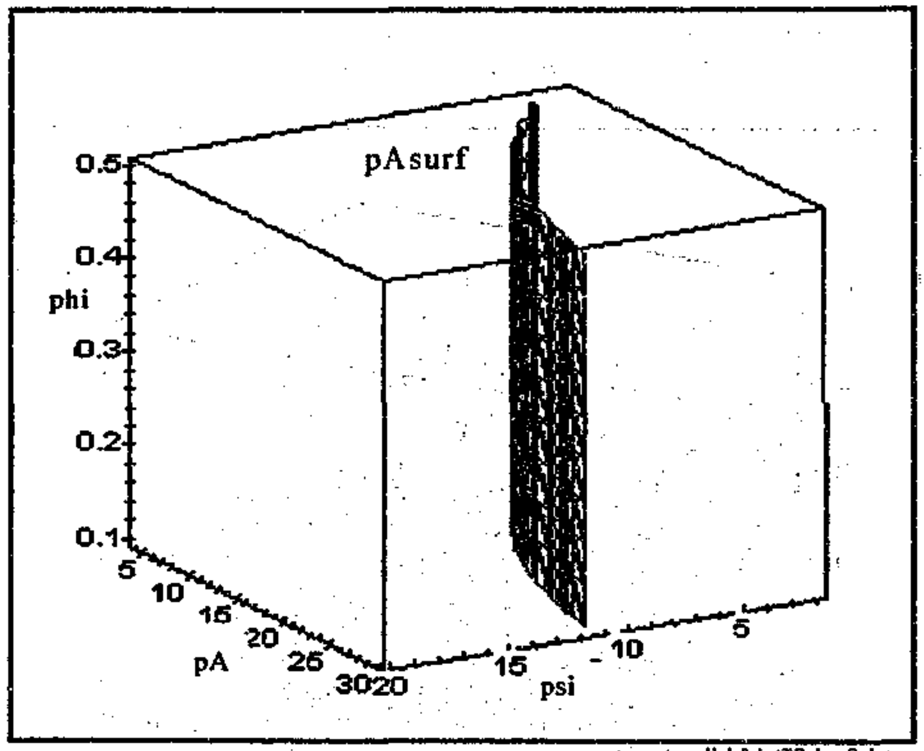

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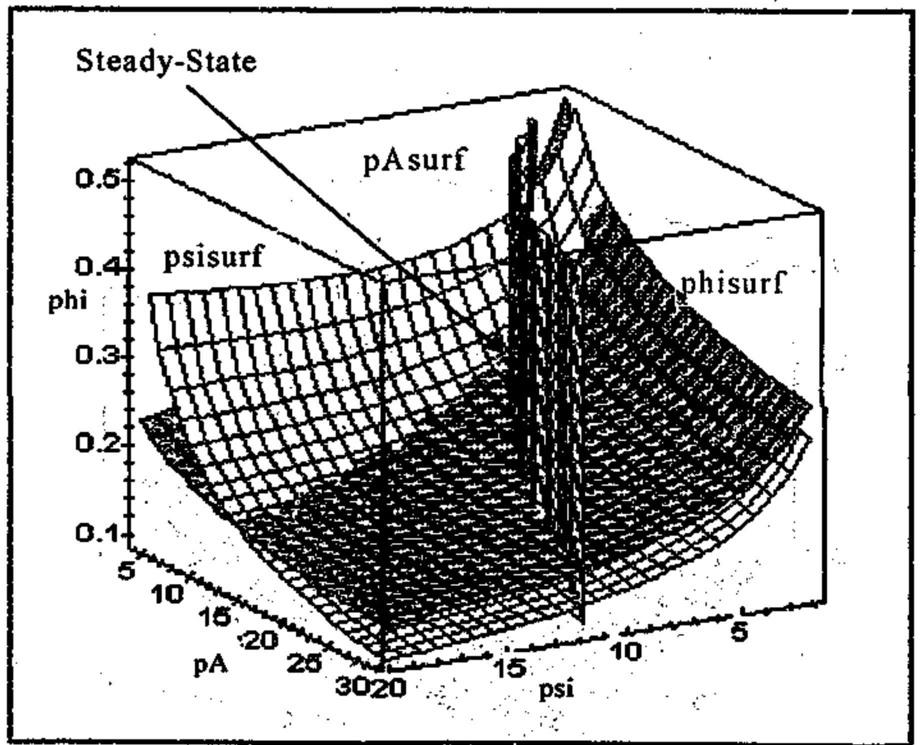
Figure 3.9: **Romer system phase-space, seen from the viewpoint: angle of incidence=60 degrees, angle of declination=70 degrees; benchmark parameter set.**



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With its three surfaces intersecting each other only once, the phase-space of the Romer model comprises eight separate regions (2^3=8). In each of these the *directions of motion* of the model's dynamic variables are uniquely specified from equations (3.25) to (3.27). This allows the identification of those regions from which convergence to the steady-state is possible. The eight regions, denoted R1 to R8, and their corresponding *directions of motion* are marked on Figure 3.10 and defined in Table 3.1, where an analysis comparing the *valid* types of motion to the motion *necessary* to reach the steady-state is undertaken. The regions from which convergence to the model's steady-state equilibrium is possible are highlighted in the table.

Table 3.1: **Romer model phase-space, valid regions from which to reach the steady-state.**

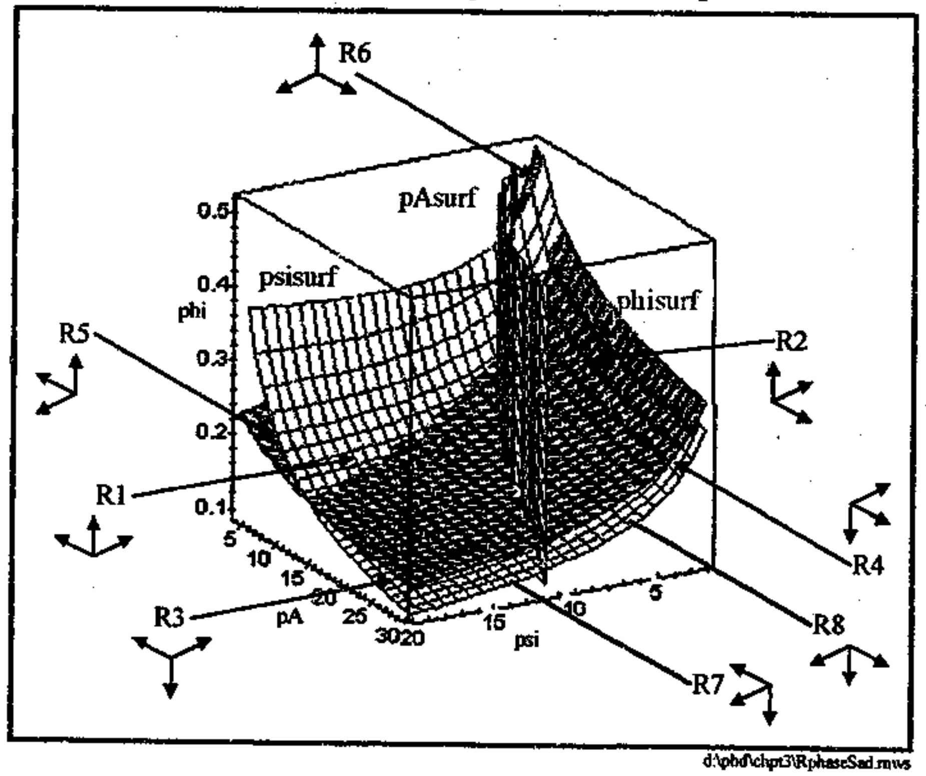
Region number	Definition of region	Dynamic motion	Validity of the region as a source to the steady-state
R1	Φ > psisurf Φ > phisurf Ψ > pAsurf	$\dot{\Psi} < 0$ $\dot{\Phi} > 0$ $\dot{p}_A < 0$	The necessary motion to reach the ss can be satisfied for $\Psi > \Psi_{ss}$, $\Phi < \Phi_{ss}$, and $p_A > p_{Ass}$ in this region $\Rightarrow$ R1 VALID
R2	Φ > psisurf Φ > phisurf Ψ < pAsurf	$\dot{\Psi} < 0$ $\dot{\Phi} > 0$ $\dot{p}_A > 0$	Since $\dot{\Phi}>0$, must have $\Phi<\Phi_{ss}$. Then it is not also possible for $\dot{p}_A>0$ in this region. $\Rightarrow$ R2 INVALID
R3	Φ> psisurf Φ < phisurf Ψ > pAsurf	$\dot{\Psi} < 0$ $\dot{\Phi} < 0$ $\dot{p}_A < 0$	To reach the ss requires $\dot{\Phi}>0$ which is not possible in this region. $\Rightarrow$ R3 INVALID
R4	Φ > psisurf Φ < phisurf Ψ < pAsurf	$\dot{\Psi} < 0$ $\dot{\Phi} < 0$ $\dot{p}_A > 0$	Satisfaction of the $\dot{\Phi}<0$ requirement would imply $\dot{\Psi}>0$. Since this is not possible here: $\Rightarrow$ R4 INVALID
R5	Φ < psisurf Φ > phisurf Ψ > pAsurf	$\dot{\Psi} > 0$ $\dot{\Phi} > 0$ $\dot{p}_A < 0$	With $\dot{\Psi}>0$, reaching the ss would require $\dot{\Phi}< 0$ which is not possible in this region. $\Rightarrow$ R5 INVALID
R6	Φ < psisurf Φ > phisurf Ψ < pAsurf	$\dot{\Psi} > 0$ $\dot{\Phi} > 0$ $\dot{p}_A > 0$	Here it is not possible to reach the ss without $\dot{\Phi}<0$. As this is not possible in this region: $\Rightarrow$ R6 INVALID
R7	Φ < psisurf Φ < phisurf Ψ > pAsurf	$\dot{\Psi} > 0$ $\dot{\Phi} < 0$ $\dot{p}_A < 0$	To reach the ss with $\dot{p}_A<0$, would also need to have $\dot{\Phi}>0$ which is impossible here $\Rightarrow$ R7 INVALID
R8	Φ < psisurf Φ < phisurf Ψ < pAsurf	$\dot{\Psi} > 0$ $\dot{\Phi} < 0$ $\dot{p}_A > 0$	Here the necessary dynamic motion allows the ss to be reached from points where $\Psi<\Psi_{ss}$, $\Phi>\Phi_{ss}$, and $p_A<p_{Ass}$ $\Rightarrow$ R8 VALID

Thus, to reach the steady-state, the dynamic variables must follow a path from either:

- the region R1 where Φ>psisurf, Φ>phisurf, and Ψ>pAsurf, and along points where $\Psi>\Psi_{ss}$, $\Phi<\Phi_{ss}$, and $p_A>p_{Ass}$; or
- the region R8 where Φ<psisurf, Φ<phisurf, and Ψ<pAsurf, and along points where $\Psi<\Psi_{ss}$, $\Phi>\Phi_{ss}$, and $p_A<p_{Ass}$.

This result could also have been inferred from the linearised model since the eigenvector associated with the negative eigenvalue specifies the direction of the saddle-path at (or in the neighbourhood of) the steady-state. For the benchmark parameter set the first and third components of this eigenvector, corresponding to the variables $\Psi(t)$ and $p_A(t)$ given the way in which the linear model was set-up, were of common sign, and different to that of the second ($\Phi(t)$) component. The interpretation of this is that at the steady-state $\Psi(t)$ and $p_A(t)$ are either both increasing while $\Phi(t)$ falls, or are both decreasing while $\Phi(t)$ rises. Since the regions R1 and R8 are the only ones in which such motion is possible, the saddle-path must lie within them. Not surprisingly, the result can also be seen to hold from the solution for the saddle-path of the linearised model. Given the signs of the eigenvector components, it can easily be seen from the saddle-path equation (3.7), that if $\Psi_0>\Psi_{ss}$ then $\Phi_0<\Phi_{ss}$ and $p_{A0}>p_{Ass}$; and if $\Psi_0<\Psi_{ss}$ then $\Phi_0>\Phi_{ss}$ and $p_{A0}<p_{Ass}$.

Figure 3.10: **Phase-space regions and their directions of motion for the Romer system, seen from the viewpoint: angle of incidence=60 degrees; angle of declination=70 degrees; benchmark parameter set.**



All this offers a qualitative description of the saddle-path of the model: Namely, that its stable arms emanate from the steady-state equilibrium point into those specific parts of the phase-space denoted here as R1 and R8. It is also possible to calculate the saddle-path quantitatively from various numerical solution methods for the differential equations of the model. Such methods are the subject of Chapter 4 and the Romer model saddle-path is calculated there in a variety of ways. However, to complete the description of the phase-space one of these calculation methods is outlined here and the results presented in phase-space diagrams.

3.3.2 The saddle-path in phase-space

As mentioned in the previous section, the eigenvector associated with the negative eigenvalue from the linearised model specifies the direction of the saddle-path in the neighbourhood of the steady-state. Thus, by taking a *small* defined step away from the steady-state into region R1, and in particular, in the direction of the eigenvector, the differential equations of the model can be integrated backwards in time from this known *initial value* to compute the 'R1-half' of the saddle-path. Similarly, by taking a corresponding step into R8 the other half of the path is calculated. Further explanation and details of the method, referred to as the *eigenvector-backwards integration method*, are contained in Section 4.2.2 of Chapter 4. In the results presented below these small steps were achieved by taking the fractions 0.05, and -0.05 of the eigenvector and adding the results to the (benchmark) steady-state.

In Figure 3.11 the benchmark saddle-path is plotted alone in phase-space, while in Figure 3.12 to Figure 3.16 its position relative to the phase surfaces is shown explicitly. Different three-dimensional views of the phase-space have been employed to assist in depicting its position relative to the phase surfaces, particularly in the difficult to see R8 region; and in particular, two-dimensional plan type views have been used to clarify the position at and nearby the steady-state (Figure 3.15 and Figure 3.16).

Figure 3.11: **Saddle-path for the Romer system, seen from the viewpoint: angle of incidence=65 degrees; angle of declination=75 degrees; benchmark parameter set.**

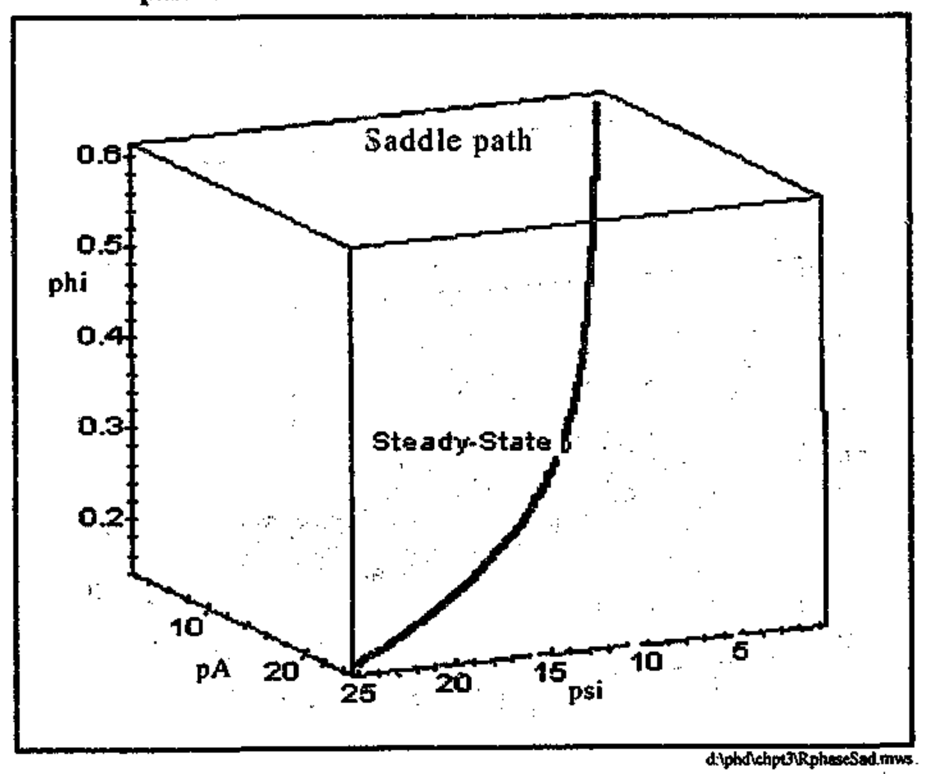


Figure 3.12: Saddle-path and phase surfaces for the Romer system, seen from the viewpoint: angle of incidence=60 degrees; angle of declination=70 degrees; benchmark parameter set.

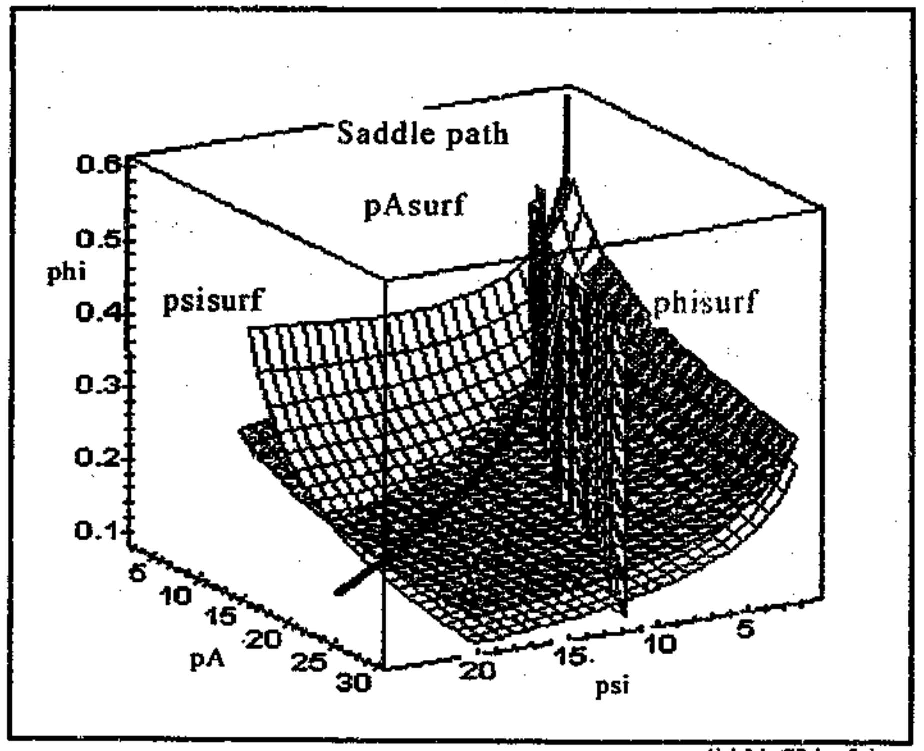


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Figure 3.13: Saddle-path and phase surfaces for the Romer system, seen from the viewpoint: angle of incidence=-70 degrees; angle of declination=70 degrees; benchmark parameter set.

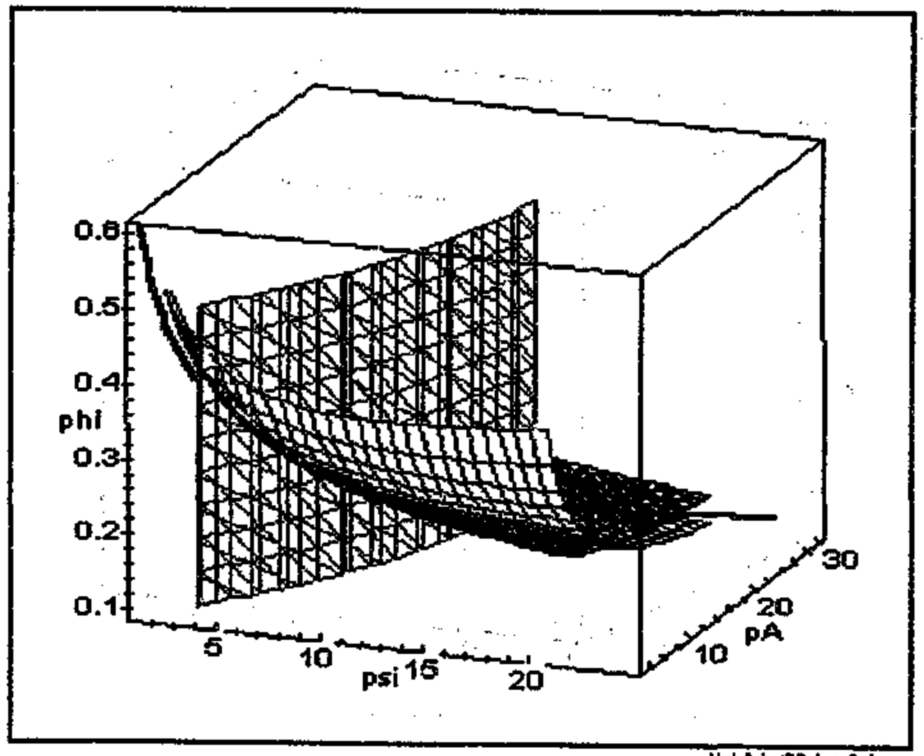

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Figure 3.14: Saddle-path and phase surfaces for the Romer system, seen from the viewpoint: angle of incidence=-110 degrees; angle of declination=50 degrees; benchmark parameter set.

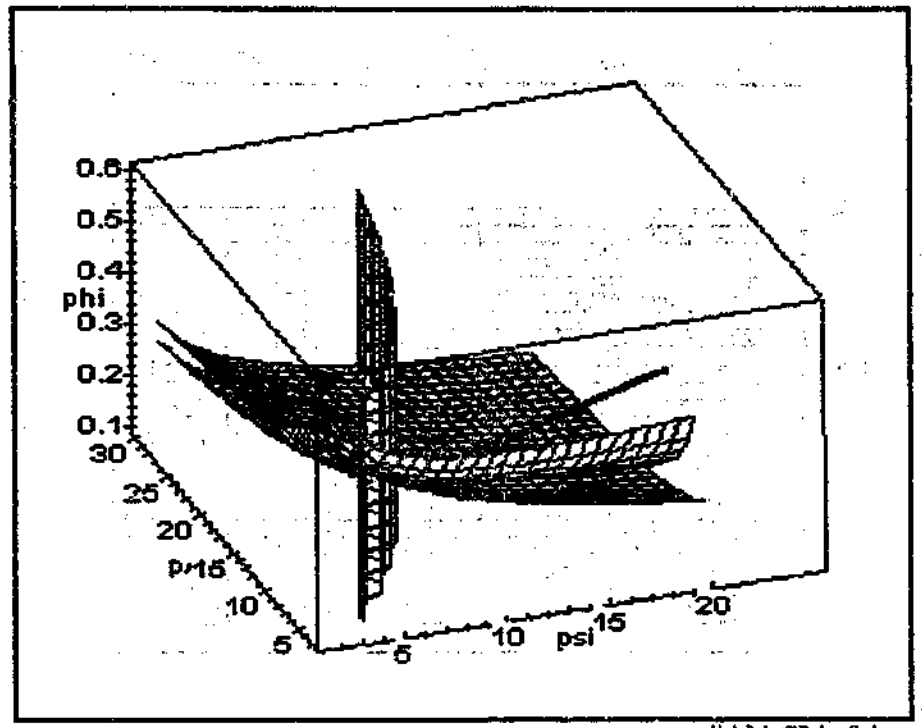

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Figure 3.15: Saddle-path and phase surfaces for the Romer system, seen from the viewpoint: angle of incidence=90 degrees; angle of declination=0 degrees; benchmark parameter set.

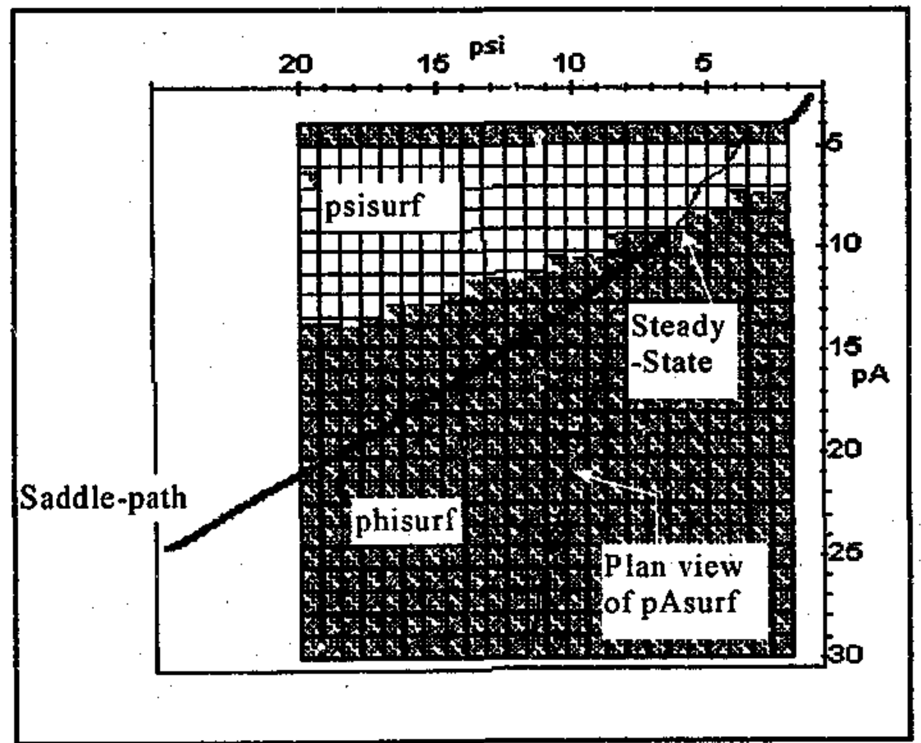

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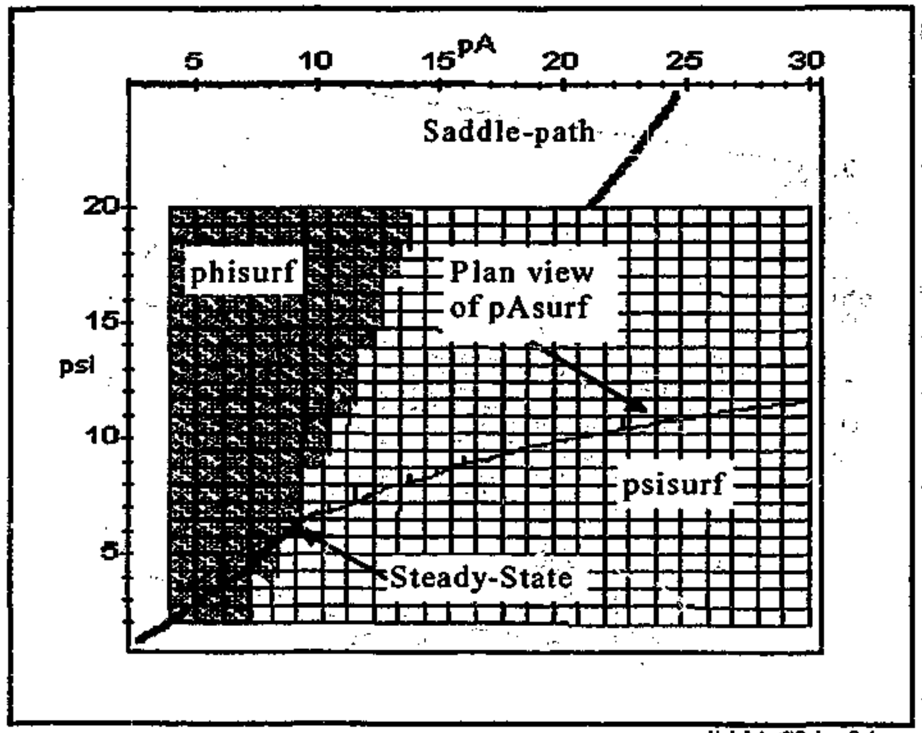
Figure 3.16: Saddle-path and phase surfaces for the Romer system, seen from the viewpoint: angle of incidence=0 degrees; angle of declination=-180 degrees; benchmark parameter set.



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Phase-space analysis is usually undertaken for two-dimensional systems. In addition to the saddle-path other 'streamlines' are usually shown on phase diagrams. These depict the evolution of the system off the saddle-path. They show (more explicitly than the 'corner arrows' of Figure 3.10) how the system can move between different sectors of the phase-space, and how it eventually diverges to economically meaningless outcomes unless it evolves along the saddle-path. Typically the approach is to determine, often only qualitatively, how the phase curves and concomitantly the saddle-path of a dynamic system are altered by some change to its parameters or exogenous variables.[12] The reaction of the system is then characterised by a jump towards the new saddle-path (typically from some prior steady-state); perhaps some further smooth adjustment to the final saddle-path; and finally, a smooth adjustment along it. Obviously, the precise nature of the transient dynamics will depend on which variable or variables are shocked. Perhaps less obviously, they also depend upon the manner in which the shock is implemented.

While such analysis of the three-dimensional phase-space described above is possible, it is complicated, messy, and difficult to visualise clearly. It would be far easier if the model could be modified and the system reduced to two dimensions. Such modification and analysis are the subject of Section 3.4; but first a variety of transitional dynamics are described in general terms with the aid of a **schematic** two dimensional phase diagram. Here the different dynamics arise from the same basic shock implemented in fundamentally different ways.

[12] It should perhaps be emphasised that all the representations of the Romer system phase-space presented here have been explicitly calculated. That is, they are all quantitative representations.

3.3.3 Schematic phase-space analysis[13]

Shocks can be either temporary or (indefinitely) sustained, but a temporary shock may be thought of as two consecutive sustained shocks, the second simply reversing the first.[14] A more fundamental dichotomy in the types of shocks that can be imposed on dynamic systems distinguishes those that surprise the market, and those that are foreseen. These are referred to as *anticipated* and *unanticipated* respectively. For a shock or policy change to be anticipated we may have in mind some sort of *perfect foresight* or *rational expectations* type of model. More prosaically, it may simply be assumed that policy changes are **announced** in advance.

Figure 3.17: Schematic two-dimensional phase diagram demonstrating the dynamics of the same shock when it is (a) unanticipated; (b) anticipated; and (c) temporary.

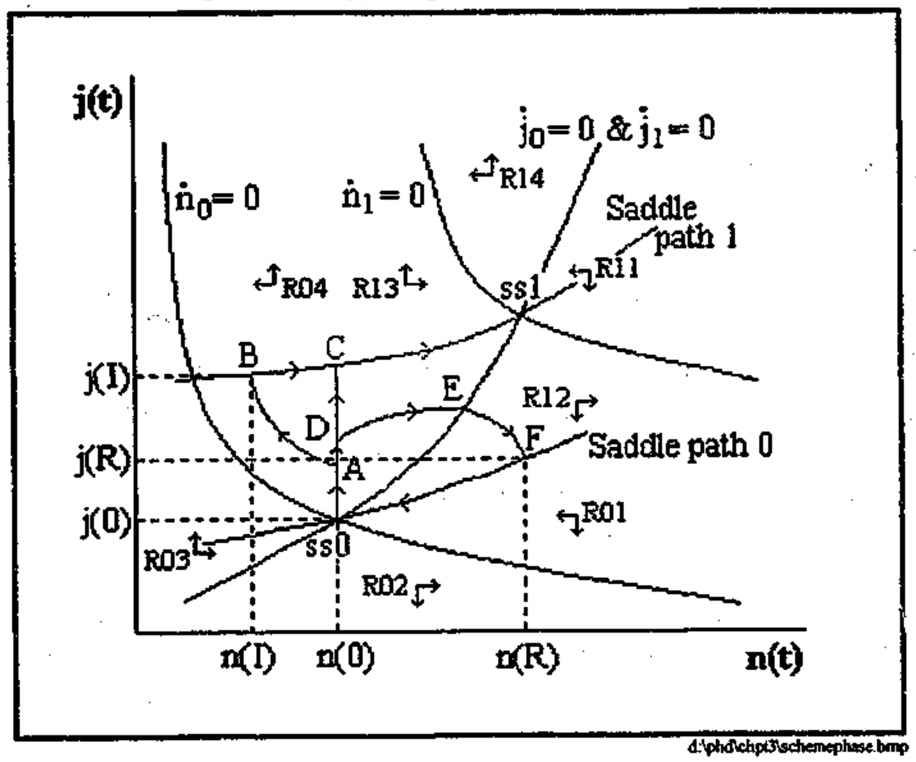

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Figure 3.17 depicts a situation where a dynamic system is initially in equilibrium at the steady-state ss0, when some specific shock is announced at time t=0 and implemented at t=I (I≥0). Obviously, when I=0 the shock is unanticipated and when I>0 it is anticipated. Other nomenclature of Figure 3.17 is as follows: n(t) is a notional variable representing the *non-jumping* variables, and j(t) is a similar abstraction for the *jumping variables*.[15] Steady-states ss0 and ss1 and the correspondingly scripted saddle-paths are associated with the pre- and post-shock differential equations respectively. The pre- and post-shock

[13] The work here was inspired by the phase-space material in Chapter 5 of Dixon et al. (1992).

[14] Of course a temporary shock can have no permanent effect in that it cannot affect the steady-state equilibrium of the system, but it can induce significant transient dynamic behaviour.

[15] The notion of jumping and non-jumping variables was encountered in the determination of initial values for variables in the linearised model (see footnote 4, Figure 3.1 and their associated text). The issues are covered in detail in Section 4.1, particularly footnotes 5 and 6 of Chapter 4 and the text that they supplement.

phase lines for the non-jumping variables, $\dot{n}_0 = 0$ and $\dot{n}_1 = 0$ respectively, are distinct from one another; but to simplify the diagram it is assumed that the shock has no effect on the phase line for the jumping variables ($\dot{j}_1 = 0$ coincides with $\dot{j}_0 = 0$).[16] And finally, the 'corner arrows' marked on the diagram and labelled Rij for i=0,1 and j=1,..,4 denote the directions of motion of the dynamic variables under both the pre- and post-shock systems; with i=0 referring to the former and i=1 to the latter. The different regions for each system are numbered j=1,..,4 beginning directly to the right of the relevant steady-state and proceeding in a clockwise direction.

When future changes are anticipated the processes of adjustment are somewhat more complex than when such changes come as a surprise; though of course the final equilibrium positions are identical. As could be expected, adjustment commences at the initial time of anticipation or announcement. Even though there is no change to the underlying dynamic equations, the announcement (or the initial anticipation) that such a change is imminent represents a shock to the system and disturbs its current path. The impending change to the dynamic equations and their new saddle-path are anticipated, and any jumping variable responds by jumping part, but not all of the way to what will be the new saddle-path. As will be demonstrated shortly, the extent of such jumps are precisely determined. While there may be further abrupt changes to the adjustment paths at the time the changes are actually realised (or the policies implemented), because it is only the announcement not the implementation which comes as a surprise, it is only at that initial time that any discontinuities in the adjustment paths can arise.[17]

Whether a shock is anticipated or not, from the point in time at which it is implemented the dynamics of the system are governed by the *post-shock* differential equations. Also, for the types of system considered here, the solution of a dynamic optimisation problem requires that it approach asymptotically the post-shock steady-state. Since the only way to achieve this is along the corresponding saddle-path, then it must be the case that the system lies on this saddle-path at the time of implementation. Of course prior to implementation it is the *pre-shock* equations which determine the dynamic behaviour. Thus, the system must evolve under the influence of these in such a way as to reach the post-shock saddle-path at precisely the time that the shock is implemented and the governing equations therefore change.

This means that in the case of an **unanticipated shock**, announced and implemented at t =0 say, the initial response must be for the jumping variables to jump instantaneously to the new saddle-path while the non-jumping variables remain constant. Then, with the original dynamic equations being replaced by the new ones at the same instant, all dynamic variables commence their smooth adjustment along the saddle-path. In terms of Figure 3.17 the adjustment path is from 'ss0 to C to ss1'.

Suppose now that the same shock as before is anticipated. In this case the previous argument means that the dynamic response immediately upon anticipation (or

[16] In the Romer model with no depreciation, this would be the case for the $\dot{p}_A = 0$ phase surface for any parameter shock other than one to γ

[17] While there can be no discontinuities associated with any of the differential equation variables at any time **after** the announcement of a shock, it is possible for other derived dynamic variables (such as H_A or r in the Romer model) to exhibit discontinuities. This is shown in Section 3.4.2.

announcement) at time t=0, must be for the jumping variables to jump to a **unique point part-way** towards what will be the new saddle-path, from ss0 to A in Figure 3.17. To understand why the jump must be part-way towards the saddle-path - and neither away from it, to it, nor beyond it - realise that there must be further adjustment governed by the pre-shock equations in the intervening period between announcement and implementation; recall that such adjustment must take the system to the post-shock saddle-path; and consider the directions of motion of the pre-shock system in the different regions of its phase-space. Also, to understand why the jump must be to a unique point, realise that from there the adjustment to the post-shock saddle-path must be made in precisely the time interval between announcement and implementation; and that any lesser jump would entail more adjustment which would be at a slower rate because the system would be closer to the pre-shock steady-state; while any greater jump would entail less adjustment at what would be a faster rate. Thus, the initial jumps at the time of announcement must be to the exact point in phase-space, part-way to the new saddle-path, from which, evolving under the influence of the pre-shock dynamic equations, the system reaches the saddle-path of the post-shock equations at the precise moment of its birth; that is, just as the policy change is implemented!

In terms of Figure 3.17 the second phase of adjustment for the anticipated shock is along the path AB. It is worth emphasising that such paths are **not** saddle-paths, and so they do not lead directly towards an equilibrium. In fact, if followed for long enough they would either diverge to infinite values or lead to other economically nonsensical zero or negative values. However, having reached the new saddle-path at B precisely as the policy change is implemented (ie. as the shock occurs), the final phase of adjustment is then the usual smooth one from B to ss1 along it. Overall, the transitional dynamics in response to the anticipated shock describe the path from 'ss0 to A to B and through C to ss1'.

Now consider the same shock being imposed only temporarily. For a **temporary shock** it is necessary to specify both whether its imposition is anticipated or not, and whether its removal is anticipated or not. Here we consider the case where its imposition comes as a surprise at time t=0, but its removal is correctly foreseen, at time t=R. In general terms the dynamic behaviour here is somewhat similar to the case of a sustained anticipated shock. Again the initial response is a jump to a precisely defined point part-way to the post-shock saddle-path, point D in Figure 3.17.[18] Next the system must evolve according to the post-shock equations along a path which takes the system back to its pre-shock saddle-path at exactly the time that this becomes relevant again with the removal of the temporary shock. This is shown as the path 'DEF' in Figure 3.17. Note that as this path crosses the $\dot{j} = 0$ locus the adjustment of the j(t) variables must change sign. This accords with the 'direction of motion' indicators for 'R13' and 'R12'. Finally, the system re-approaches its pre-shock steady-state along this saddle-path. The overall dynamic adjustment to the temporary shock follows the path 'ss0 to D, through E to F, and back on to ss0'.

[18] As for the case of the sustained anticipated shock, this can be shown to be necessary from a consideration of the governing equation system, the directions of motion in different regions of its phase-space, the amount of adjustment required, and the times at which such adjustment must be completed.

3.4 A 'Solowian-Romer' model

3.4.1 Development and specification

An analytical difficulty with the full Romer model is its dimension. In its fundamental, balanced growth form it comprises four coupled differential equations in four variables. This can neither be visualised nor represented diagrammatically. While the system can be transformed to a stationary one of three differential equations in three variables, any calculation of its dynamic paths remains a difficult *boundary value* problem. An initial value for one of the transformed variables is known, but only the asymptotic steady-state values are known for the other two. This makes numerical integration of the system difficult, particularly by the common approach of *shooting* (see Section 4.1 which deals with these issues in much greater detail). Also, as seen in Section 3.3, while the system can then be represented pictorially with a three dimensional phase-space diagram, analysis of its dynamics when the phase surfaces and saddle-path are altered is too complicated to illustrate clearly.

By specifying savings to be a fixed proportion (s) of income, the consumer optimising behaviour of the Romer system is replaced by a simple consumption of the form C = (1-s)Y. This apparently retrograde step enables the development of a modified model with a stationary dynamic system of only two variables. This is termed the *Solowian-Romer* model[19] and its dynamics can now be computed quite easily. The *two-point boundary value problem* it poses can readily be solved by the *shooting* method since an initial value for only one variable needs to be accurately calculated, as opposed to the two required in the case of the full Romer model (again, Section 4.1 should be consulted here for more details). Moreover, having only two dynamic variables means that the phase-space of the modified model can be constructed and examined in only two dimensions, greatly simplifying the graphical analysis and facilitating an understanding of the dynamics of the system. There is of course, the question of how accurately the modified model reflects the dynamics of the full model. This question is addressed in Chapter 4 after the dynamics of the Romer model itself are computed.

Specification of the Solowian-Romer model is extremely close to that of the full Romer model. Its supply side (which is what drives the 'exogeneity of growth') is **exactly** the same as that of the full model. On the demand side however, its savings-consumption split is exogenous. In particular, the savings rate is taken as exogenously constant. Thus, the Solowian-Romer model differs from its progenitor in only a single aspect. In fact, the difference might be said to be more a matter of degree than a substantive qualitative difference since it can be shown that the S-R model simply represents particular paramaterisations of the Romer model. This is an important result since it significantly reduces objections to the apparently regressive step of abandoning utility optimisation in favour of exogenous consumption, particularly as there are parameter settings close to the empirically based *benchmark* set which make the two models equivalent. The result is proved in Appendix 3.7.

[19] Simply because this was the (main) approach adopted by Solow (1956) in his celebrated growth article. Of course, Solow was well aware that this was a simplification (for example, see Samuelson and Solow (1956) for a general model of optimal saving and accumulation.

There are two options for the specification of constant savings in the S-R model. One is to take savings, and thus consumption, as a fixed proportion of gross product, GP. The other is to base the savings rate upon the final output of goods, Y. In the former case consumption would be determined by: $C(t) = (1-s_B)GP(t)$, where s_B is the exogenously constant and broadly defined propensity to save. And in the latter case it would be written as: $C(t) = (1-s_N)Y(t)$ where s_N is the exogenously constant and narrowly defined propensity to save. An argument in support of the former approach is that gross product is equivalent to the total income of consumers and it is easy to conceive of consumers deciding upon a fixed proportion $(1-s_B)$ of this to spend on goods. On the other hand, since "the amount of savings for capital formation is the value of forgone consumption out of the total output of goods" (Section 2.2.3); and since the fundamental problem faced by consumers is to decide "how much of total output to consume through the purchase of goods, and how much to save by the accumulation of capital" (Section 2.2.8), it is also easy to conceive of this being solved in a static sense by the choice of a fixed proportion $(1-s_N)$ of output. This latter approach is the one that has been adopted, though from now the savings rate will be denoted simply as "s".[20]

Thus, the specification for the Solowian-Romer model is the same as that given in Section 2.2 of Chapter 2 for the Romer model; except that equation (2.27) is replaced by the consumption function:

$$C(t) = (1-s)Y(t) \tag{3.28}$$

Also, no transversality condition (or boundary value) of the form of equation (2.28) exists in the modified model.

After condensing the model in precisely the same way as before for the Romer model, and defining the same new variable $\Psi(t)$ (Section 2.3), the dynamic system for the Solowian-Romer model is obtained as just two coupled first-order differential equations:

$$\dot{\Psi}(t) = [\frac{s(r(t)+\delta)}{\gamma^2} - \delta - \zeta H + \zeta H_Y(t)]\Psi(t) \tag{3.29}$$

$$\dot{p}_A(t) = r(t)p_A(t) - \frac{1-\gamma}{\gamma}[r(t)+\delta]\Psi(t) \tag{3.30}$$

where:

$$H_Y(t) = [\frac{\alpha(1-\gamma)}{\zeta\eta^\gamma} L^{(1-\alpha)(1-\gamma)}\Psi(t)^\gamma p_A(t)^{-1}]^{\frac{1}{1-\alpha(1-\gamma)}} \tag{3.31}$$

$$\begin{aligned} r(t) &= \gamma^2\eta^{-\gamma}H_Y(t)^{\alpha(1-\gamma)}L^{(1-\alpha)(1-\gamma)}\Psi(t)^{\gamma-1} - \delta \\ &= \frac{\zeta\gamma^2}{\alpha(1-\gamma)}H_Y(t)p_A(t)\Psi(t)^{-1} - \delta \end{aligned} \tag{3.32}$$

[20] This decision was, it must be said. also influenced by the fact that the algebra of the model turns out to be far more complicated under the choice of s_B as the exogenously constant savings rate rather than s_N. For example, while simple expressions of the parameters can be derived for the steady-state values of the variables under the choice of s_N, corresponding steady-state values under the choice of s_B involve obtaining the positive roots of quadratic functions (whose coefficients are expressions of the parameters).

The steady-state for this model can also be derived in the same manner as before (Section 2.3). Namely, by using the transversality condition (2.37) to deduce that in the steady-state the variables r, H_Y and p_A; and the growth rate $g = \dot{\Psi}/\Psi$ are all constant, and so setting equations (3.29) and (3.30) to zero and solving. The results are as follows:

$$H_{Yss}^{SR} = \frac{H + \frac{\delta}{\zeta}(1 - s/\gamma^2)}{1 + (s/\alpha\gamma)} \tag{3.33}$$

$$H_{Ass}^{SR} = \frac{\frac{s}{\alpha\gamma}H - \frac{\delta}{\zeta}(1 - s/\gamma^2)}{1 + (s/\alpha\gamma)} \quad \text{and} \quad g^{SR} = \frac{\frac{s\zeta}{\alpha\gamma}H - \delta(1 - s/\gamma^2)}{1 + (s/\alpha\gamma)} \tag{3.34}$$

$$r_{ss}^{SR} = (\gamma\zeta/\alpha)H_{Yss}^{SR} = \frac{\gamma\zeta H + \delta\gamma(1 - s/\gamma^2)}{\alpha[1 + (s/\alpha\gamma)]} \tag{3.35}$$

$$\Psi_{ss}^{SR} = \left[\frac{\gamma^2 H_{Yss}^{SR\ \alpha(1-\gamma)} L^{(1-\alpha)(1-\gamma)}}{\eta^{\gamma}(r_{ss}^{SR} + \delta)}\right]^{\frac{1}{1-\gamma}} \tag{3.36}$$

$$p_{Ass}^{SR} = \frac{1-\gamma}{\gamma}\frac{r_{ss}^{SR} + \delta}{r_{ss}^{SR}}\Psi_{ss}^{SR} \tag{3.37}$$

Evaluating these steady-state formulae with the *benchmark parameter values* (Section 2.4), including setting the exogenous savings rate at the calculated benchmark level for the full Romer model ($s = s_{Nss}^{M} \approx 0.1680$) generates **exactly** the same steady-state levels for all but one of the endogenous variables as in the Romer model itself. The exception is for the asymptotic convergence coefficients, the βs (Table 3.2, which may be compared with the corresponding Table 2.2 from Chapter 2).

Table 3.2: Benchmark set parameter values and the resulting steady-state equilibrium values of the endogenous variables, Solow-Romer model market solution.

Parameter values	α=0.43; γ=0.54; δ=0.04; s=0.1680; ζ=0.06; η=2.0; H=1.0; L=2.0										
Endogenous variables	Ψ_{ss}	Φ_{ss}	p_{Ass}	H_{Yss} (%)	H_{Ass} (%)	r_{ss} (%)	g (%)	s_{Nss} (%)	s_{Bss} (%)	k_{GPss}	β_{ss} (%)
Steady-state values	6.48	0.27	9.45	74.41	25.59	5.61	1.54	16.80	22.10	2.84	4.22

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3.4.2 Numerical integration of the Solowian-Romer system

In some ways the presentation of the Solowian-Romer model here is premature. One reason why this abridged model is valuable is because computation of its dynamic behaviour is far simpler than for the full Romer model. That is, it can be numerically integrated much more readily. However, for this to be argued cogently the difficulties with numerical integration of the full model need to be articulated. In turn, a general description of the integration problem and its application to the Romer system is required, together with details of particular solution techniques. All this seems better left

until Chapter 4, which deals specifically with numerical integration of the full Romer model. In particular, the issues are covered in Section 4.1 and Appendix 4.1. While these may need to be read before the remainder of this section, a brief outline of the issues are nevertheless presented here.

What makes numerical integration of the SR system simpler than for the full model is the reduction from a three variable system to a two variable one. The dynamic solution of the model is a *two point boundary value problem*: an initial value is available for one variable (Ψ), but only terminal values, via the transversality conditions and the resulting steady-state levels, are available for the other two variables (Φ and p_A). This means that to solve the full model by the (straightforward and commonly used) *shooting* method, initial values for Φ and p_A need to be obtained which will take the system to its required steady-state. Given the extreme sensitivity of the system to initial conditions, such values must be highly accurate and their joint determination is difficult. By reducing the system to one of only two dynamic variables, only one such initial value (for p_A) need be found. Despite the continued sensitivity of the condensed system this is easily done.

Here it is achieved in a *Microsoft Excel* spreadsheet in the following way: The continuous-time differential equations (3.29) to (3.32) are approximated by discrete-time difference equations, written in a form appropriate to whatever integration technique is being adopted. An initial value $\Psi(0)$ is given exogenously and a corresponding value for $p_A(0)$ is simply 'guessed'. The manner in which the resulting integration path diverges from the steady-state for the given parameter settings is then observed and the 'guess' amended to improve the *shot* at this steady-state, iterating the procedure until it is approached sufficiently closely. The underlying integration technique used for this was the fourth order Runge-Kutta method (Appendix 4.1), with a time step-size of 0.5, integrated over 400 steps until $p_A(200) = p_{Ass}$ (to at least 5 figures).[21] In this way the dynamic responses of the model to a variety of simulated shocks to its parameters are computed. The system is taken to be initially at equilibrium at its benchmark steady-state, which is then perturbed by some such exogenous shock(s). Thus, the immediate post-shock initial value for the (non-jumping) variable $\Psi(t)$ is given by its pre-shock steady-state level: $\Psi_{post\text{-}shock}(0) = \Psi_{pre\text{-}shock\ ss}$. In order to make the transient dynamics of the model as good an approximation as possible to those of the Romer system itself, in addition to any other parameter shocks the savings rate is also adjusted: by an amount which takes it to the steady-state level which would be achieved in the full model under the post-shock parameter settings. This ensures that both models approach the same steady-state in response to the other exogenous shocks.

3.4.2.1 An unanticipated and sustained 10 per cent rise in parameter γ

Since the change to the system is not foreseen, the shocks to both γ and to the savings rate s (the latter is a 19 per cent rise) are simply introduced from the outset, at time t=0. Also, since the changes are considered to be sustained indefinitely the shocks apply in all the difference equations, from t=0 to t=200.

[21] In Appendix 4.2 this 'RK4' method is shown to be superior to two alternative numerical integration techniques, the 'Euler' and the 'Gragg' methods, in integrating the saddle-path of the full Romer model. Also, as an indication of the sensitivity of the system, to achieve the required accuracy in $p_A(200)$ it was necessary to refine the 'guess' for the initial value $p_A(0)$ to an accuracy of 15 figures!

Figure 3.18: **Dynamic effects on Ψ, Φ, and p_A of an unanticipated and sustained 10 per cent rise in parameter γ from time zero, benchmark parameter set.**

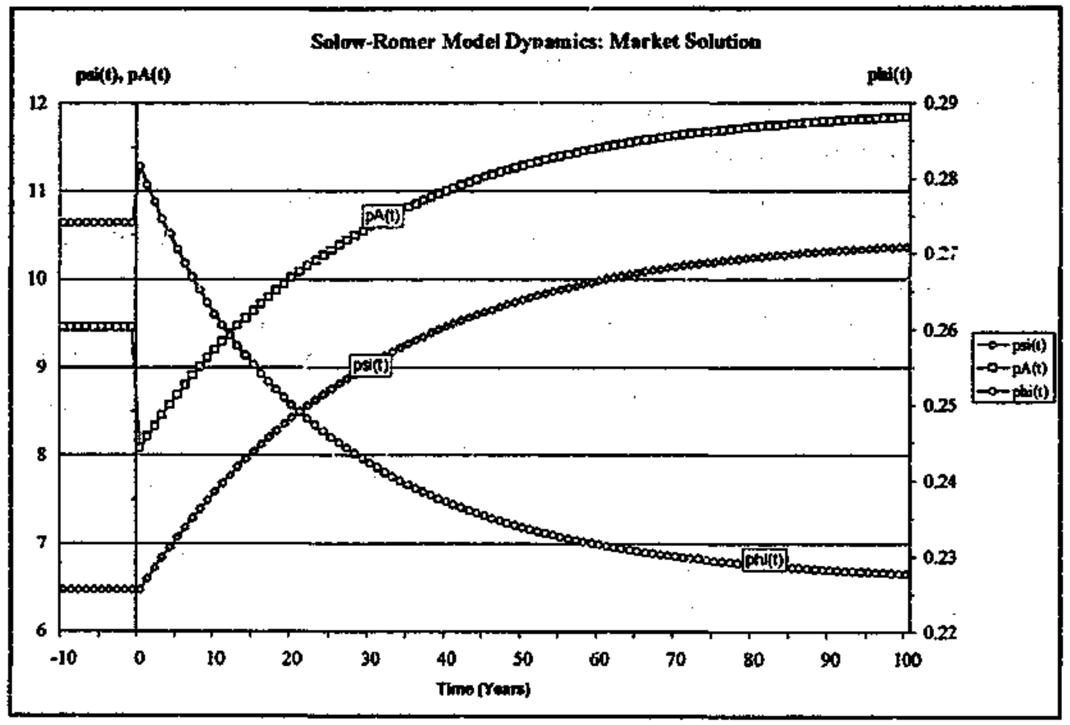

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Figure 3.19: **Dynamic effects on the growth rates of an unanticipated and sustained 10 per cent rise in parameter γ from time zero, benchmark parameter set.**

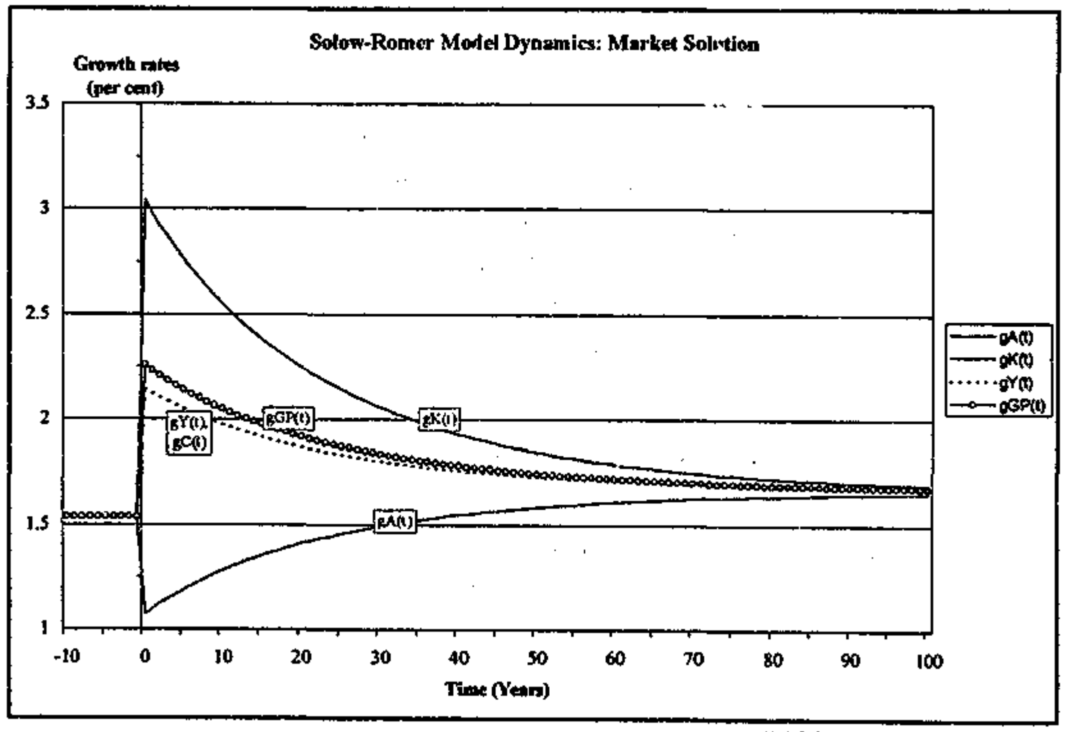

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Figure 3.20: **Dynamic effects on r and H_A of an unanticipated and sustained 10 per cent rise in parameter γ from time zero, benchmark parameter set.**

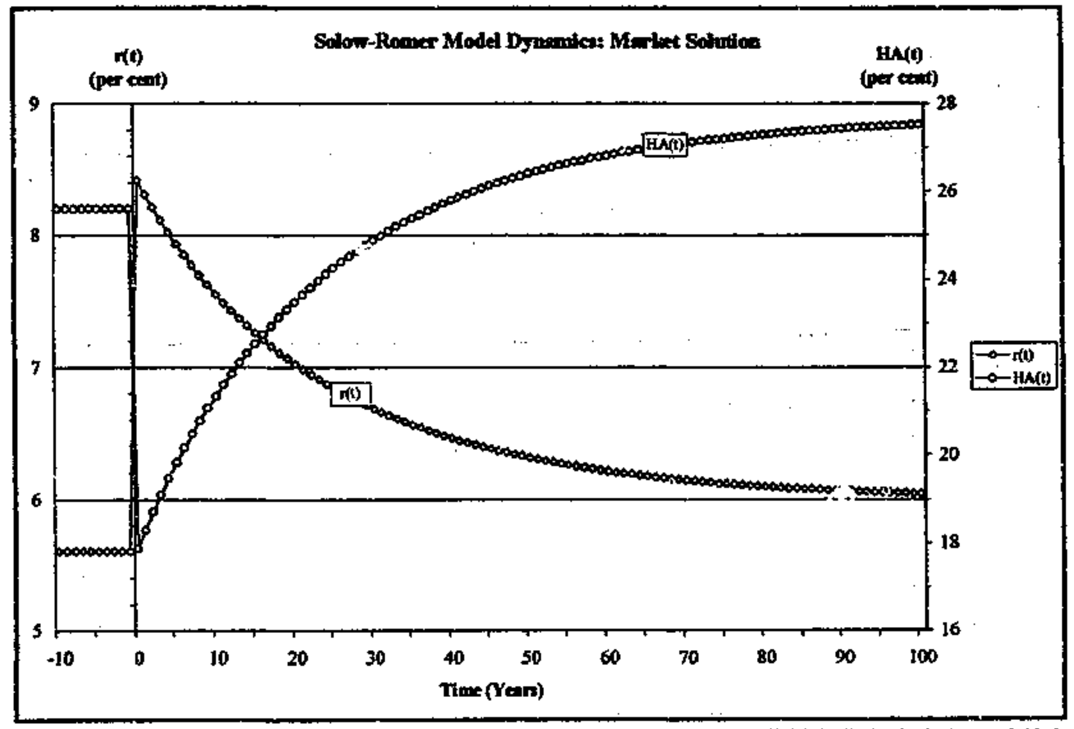

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Figure 3.21: **Dynamic effects on s_B, s_N, and k_{GP} of an unanticipated and sustained 10 per cent rise in parameter γ from time zero, benchmark parameter set.**

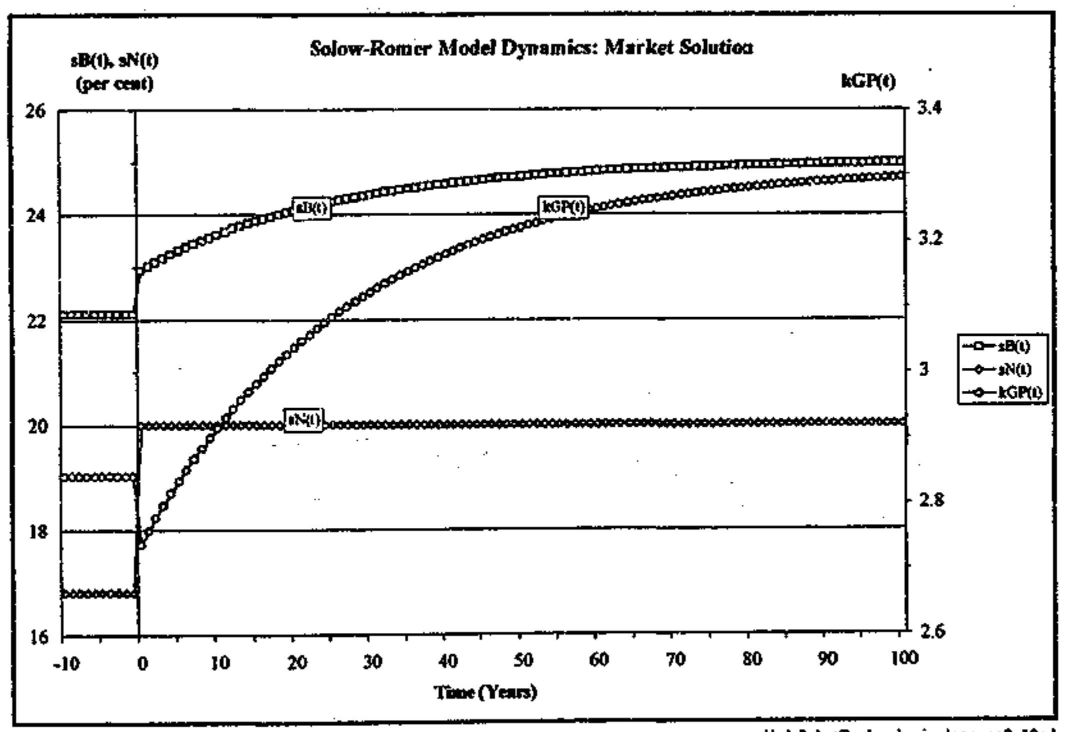

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Figure 3.22: Dynamic effects on the factor shares of gross income from an unanticipated and sustained 10 per cent rise in parameter γ from time zero, benchmark parameter set.

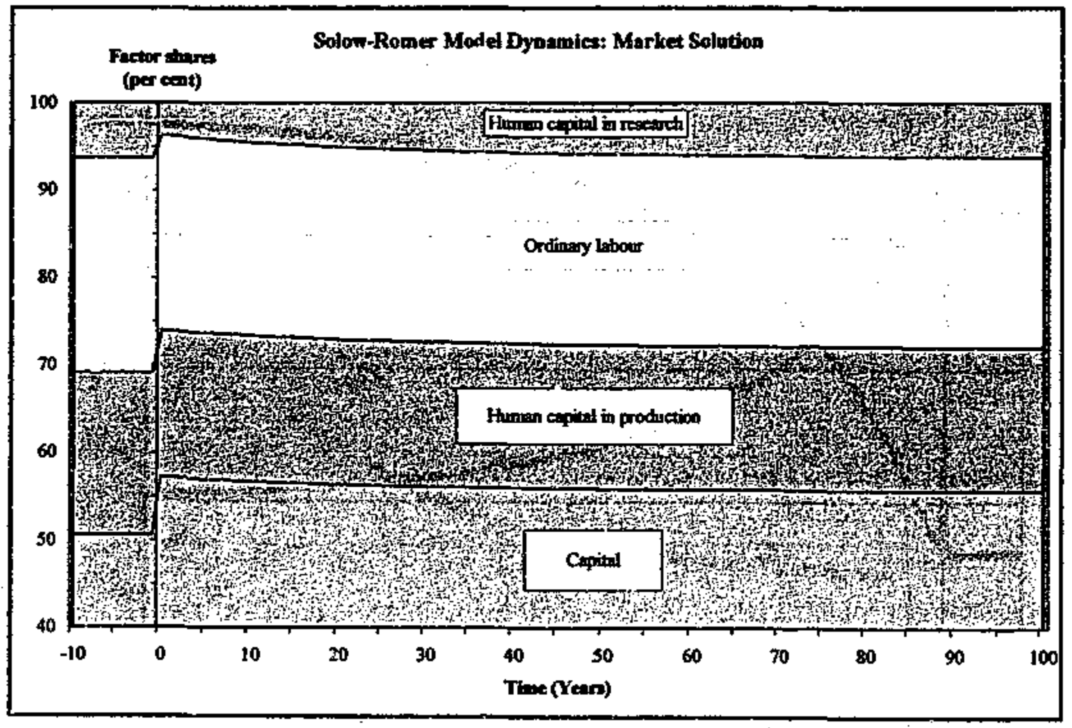

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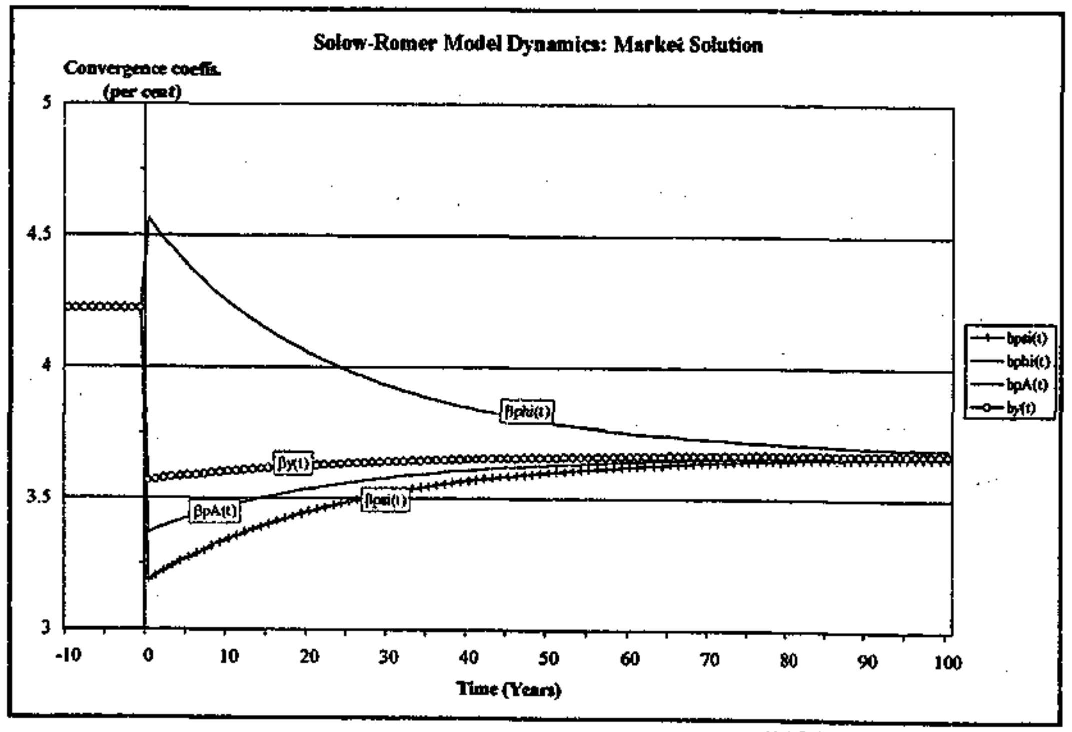
Figure 3.23: Dynamic effects on the convergence coefficients from an unanticipated and sustained 10 per cent rise in parameter γ from time zero, benchmark parameter set.



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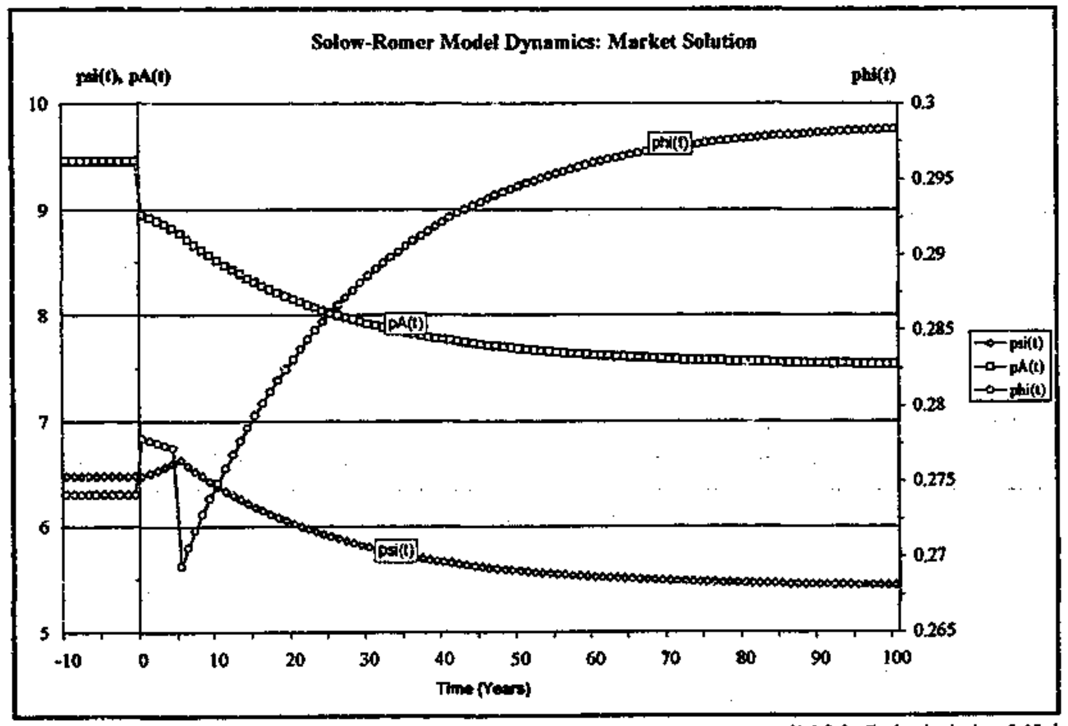
3.4.2.2 An anticipated and sustained 15 per cent rise in parameter ζ

Because the changes to the system are to be anticipated in this simulation, they must be implemented at some 'future time'. Here time t=5 has been chosen. Thus, the shocks to ζ and to s (the latter being a 3.23 per cent decline) are not introduced into the difference equations until the t=5 equation. Again, since the changes are intended to be indefinitely sustained these shocks continue to apply in all the difference equations from $t = 5$ onwards. The initial value $p_A(0)$ is then determined in the usual *shooting* manner and, since the post-shock parameter values are already written into the difference equations, the effect is as if the simulation changes to occur at time $t = 5$ are already known or anticipated at time t=0. Figure 3.24 to Figure 3.29 record the results of the simulation.

Figure 3.24: Dynamic effects on Ψ, Φ, and p_A of an anticipated and sustained 15 per cent rise in parameter ζ from time t=5, benchmark parameter set.



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In Figure 3.29 the effects of the shock to ζ may be seen to cause the convergence coefficient of the variables Ψ, Φ and y to be negative over the interval $t=(0,5)$. This is because these variables move away from their final steady-states until the shock is actually implemented (see the phase-space analysis of Section 3.4.4.2). Conversely, since the adjustment of p_A is always towards its steady-state value its convergence coefficient remains positive.

Figure 3.25: **Dynamic effects on the growth rates of an anticipated and sustained 15 per cent rise in parameter ζ from time t=5, benchmark parameter set.**

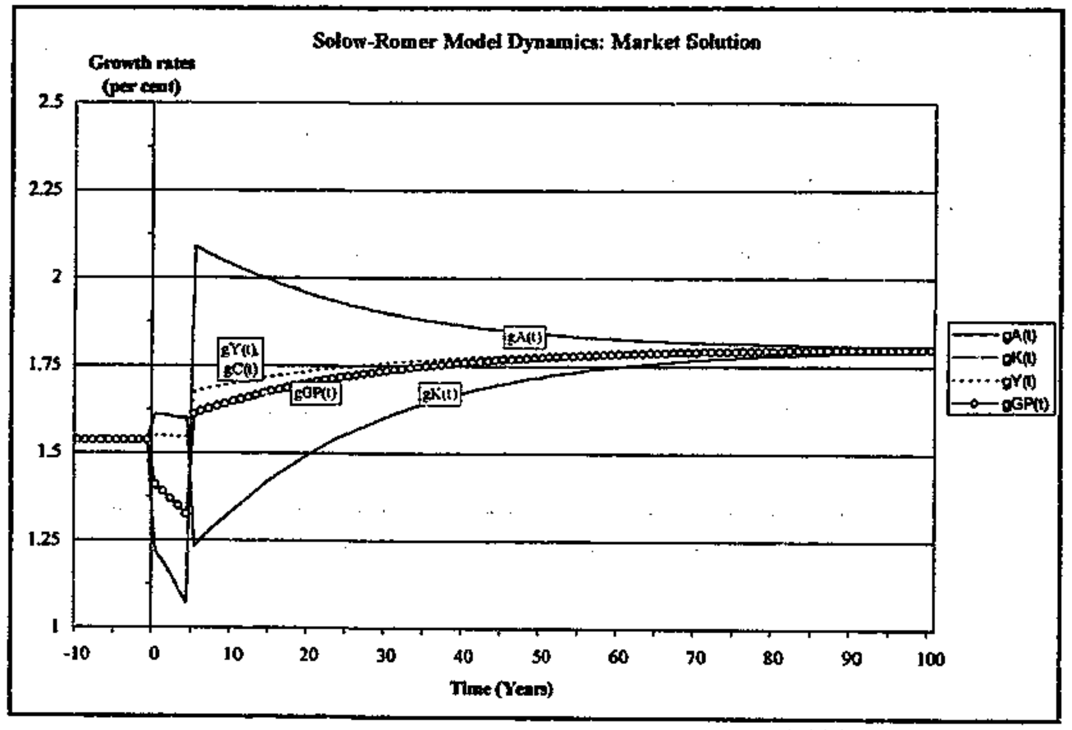

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Figure 3.26: **Dynamic effects on r and H_A of an anticipated and sustained 15 per cent rise in parameter ζ from time t=5, benchmark parameter set.**

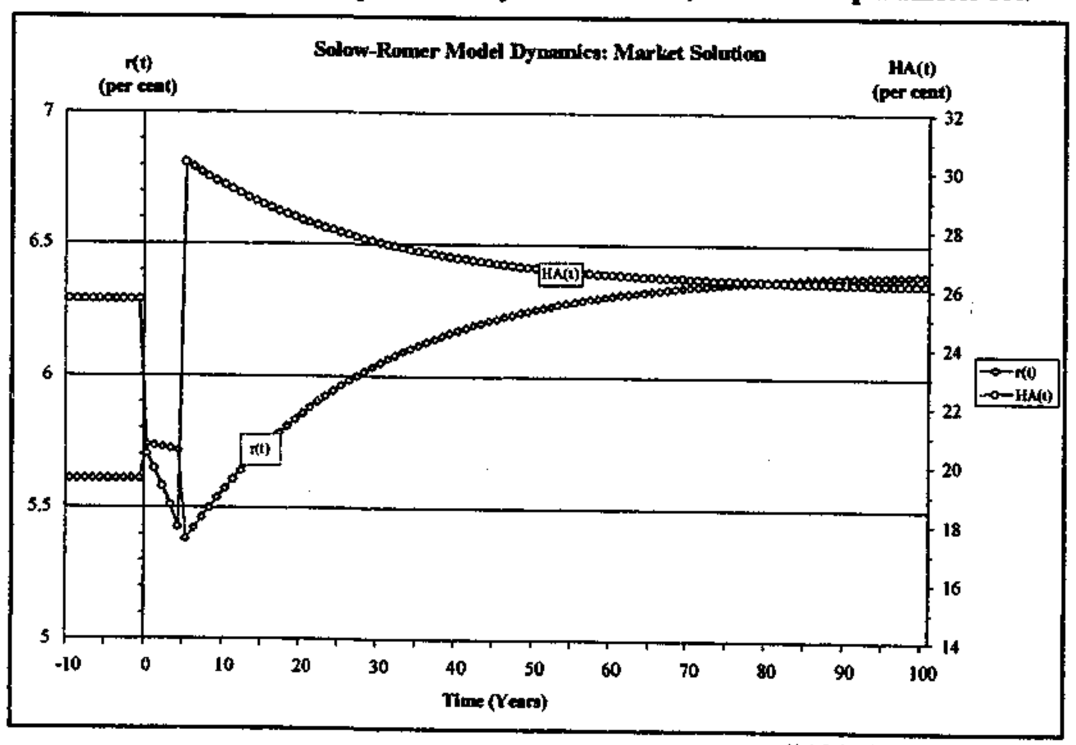

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Figure 3.27: **Dynamic effects on s_B, s_N, and k_{GP} of an anticipated and sustained 15 per cent rise in parameter ζ from time t=5, benchmark parameter set.**

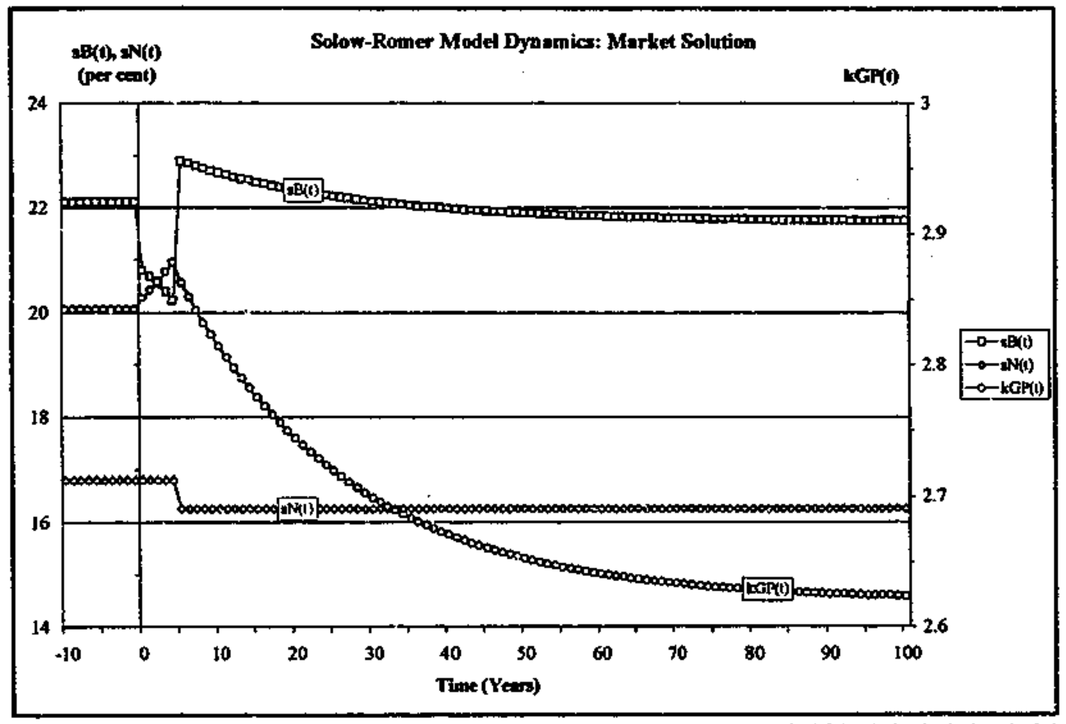

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Figure 3.28: **Dynamic effects on the factor shares of gross income from an anticipated and sustained 15 per cent rise in parameter ζ from time t=5, benchmark parameter set.**

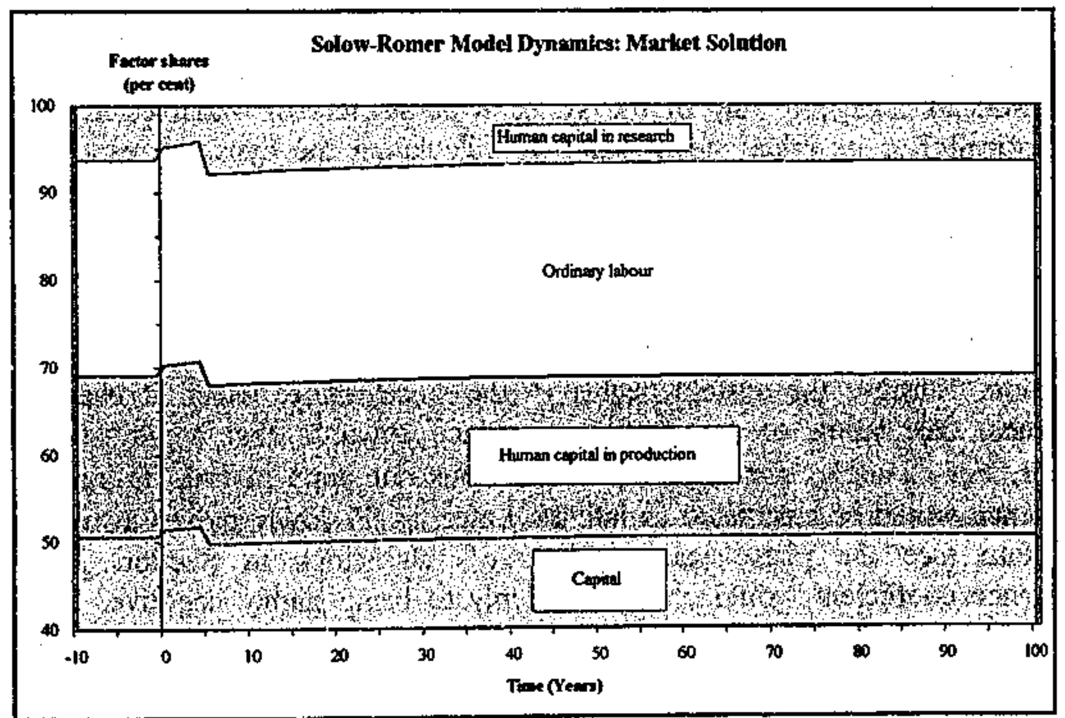

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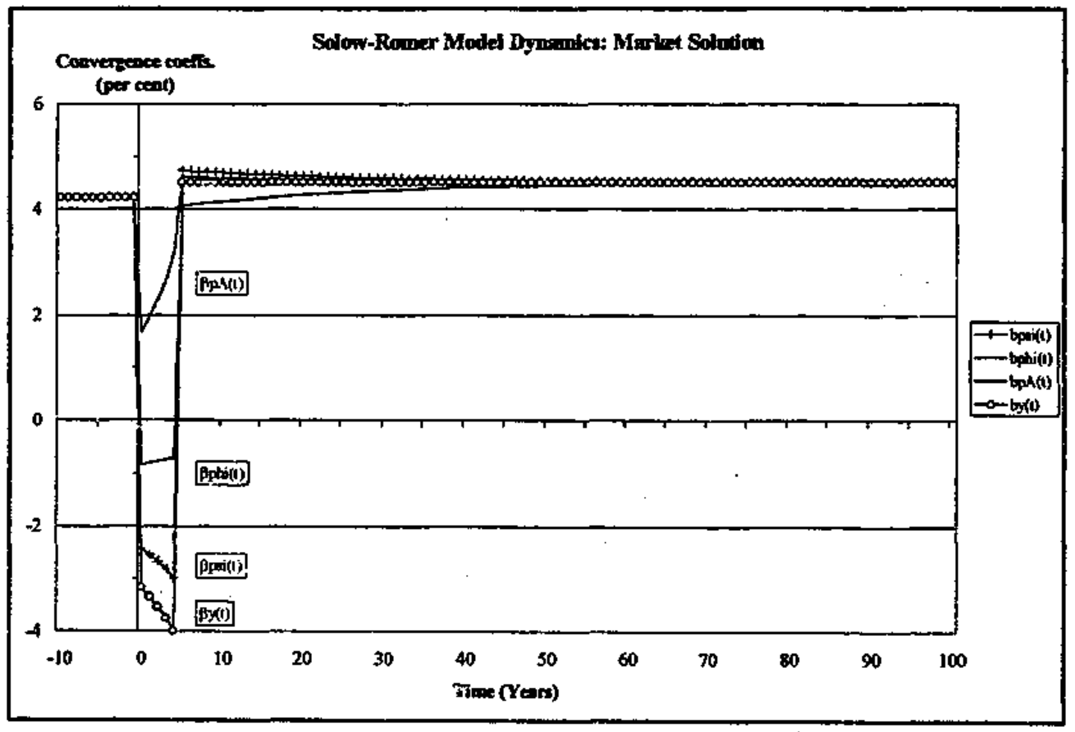
Figure 3.29: Dynamic effects on the convergence coefficients from an anticipated and sustained 15 per cent rise in parameter ζ from time t=5, benchmark parameter set.



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Note: a The negative portions of the convergence coefficients reflect temporary movements away from the new steady-states for the relevant variables.

3.4.2.3 A temporary (five year) fall of 20 per cent in parameter α

A more precise description of what is being modelled in this simulation is 'an unanticipated 20 per cent fall in α followed by its anticipated reversal five years later'. Thus, the shocks to α and to s (-3.68 per cent for the latter) are introduced to the difference equations from time t=0 and apply until their removal at time t=5. Since there is no sustained change to any of the system's parameters or variables it will return (asymptotically) to its initial steady-state. In calculating the dynamic response of the model to these shocks it is therefore this initial steady-state which is the target in the *shooting* determination of $p_A(0)$. The results are presented in Figure 3.30 to Figure 3.34 below.

The figure for changes in the convergence coefficients (Figure 3.35) requires some explanation. Because the system begins at the steady-state, which it eventually re-approaches, calculation of rates of convergence towards that steady-state causes problems. For example, since the variable Ψ does not jump at t=0, its initial rate of convergence would be calculated as infinite. Later, as Ψ slowly moves away from this steady-state (see the phase-space analysis of this shock in Section 3.4.4.3) its convergence coefficient would change suddenly to being highly negative. Because of these difficulties the convergence coefficients have been calculated differently over the two time segments of the shock. Over t=(0,5) they refer to the rate of convergence towards the intermediate (shocked) steady-state; and from then on towards the initial (and final) steady-state.

Figure 3.30: Dynamic effects on Ψ, Φ, and p_A of a temporary 20 per cent fall in parameter α from time t=0 to t=5, benchmark parameter set.

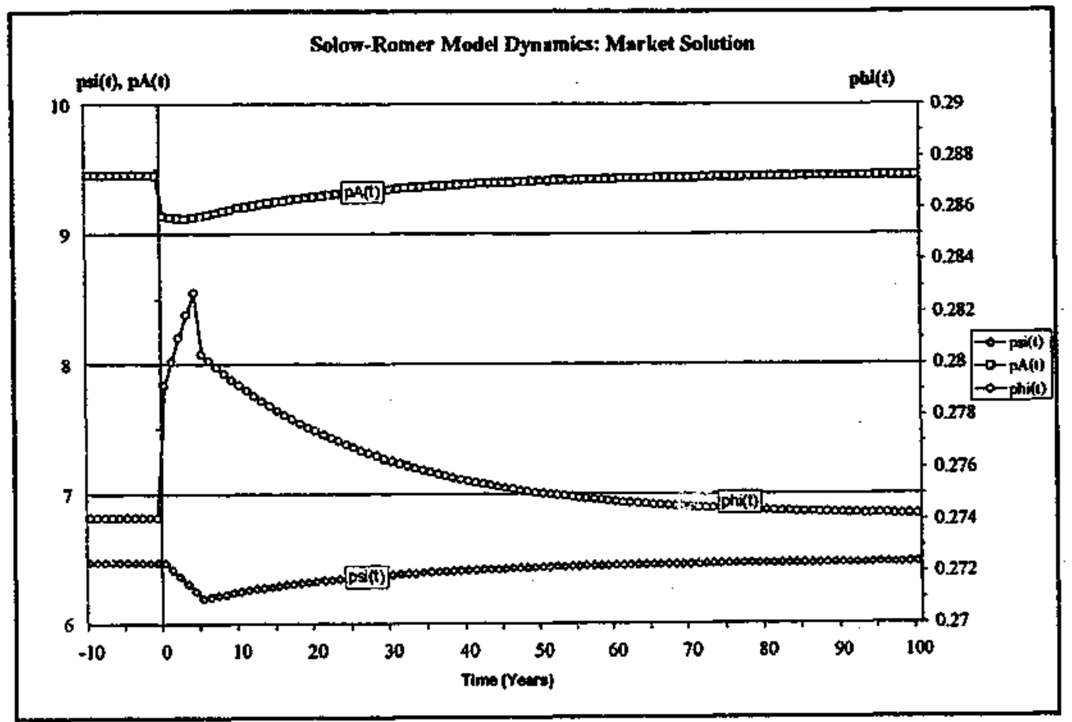

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Figure 3.31: Dynamic effects on the growth rates of a temporary 20 per cent fall in parameter α from time t=0 to t=5, benchmark parameter set.

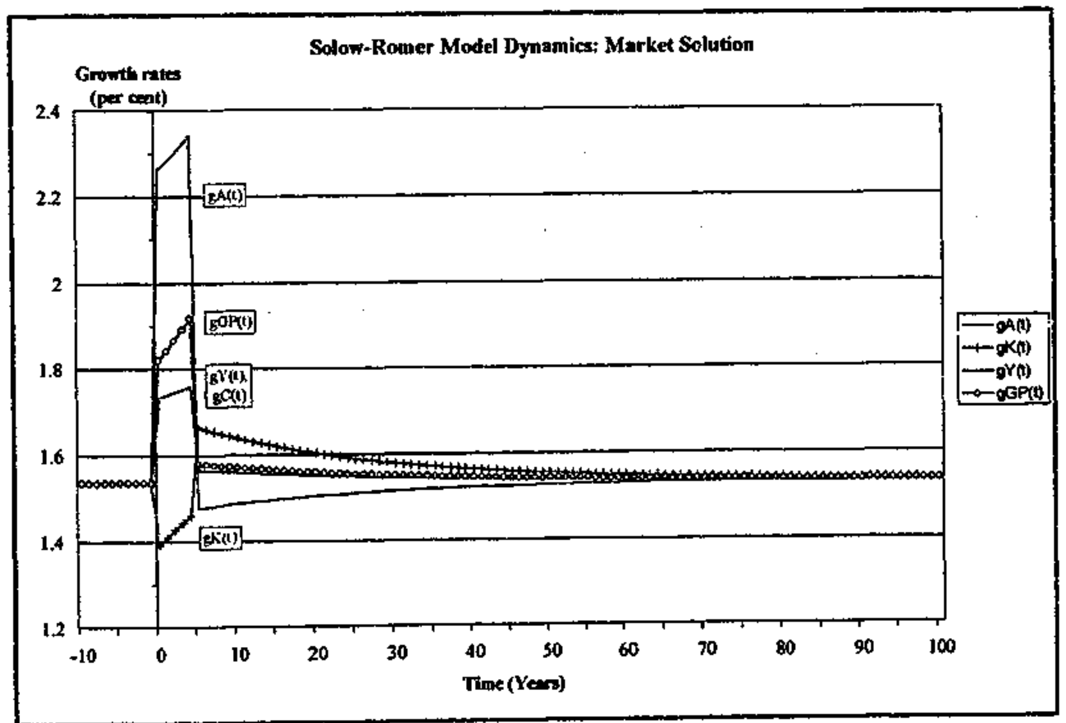

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Figure 3.32: **Dynamic effects on r and H_A of a temporary 20 per cent fall in parameter α from time t=0 to t=5, benchmark parameter set.**

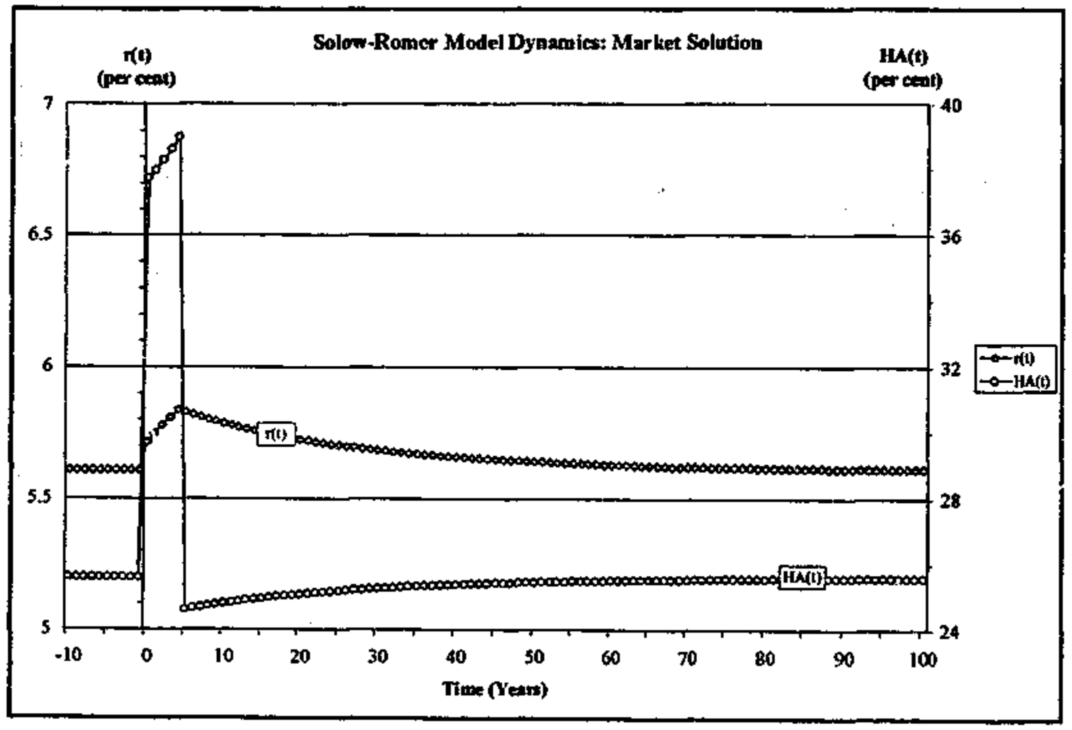

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Figure 3.33: **Dynamic effects on s_B, s_N, and k_{GP} of a temporary 20 per cent fall in parameter α from time t=0 to t=5, benchmark parameter set.**

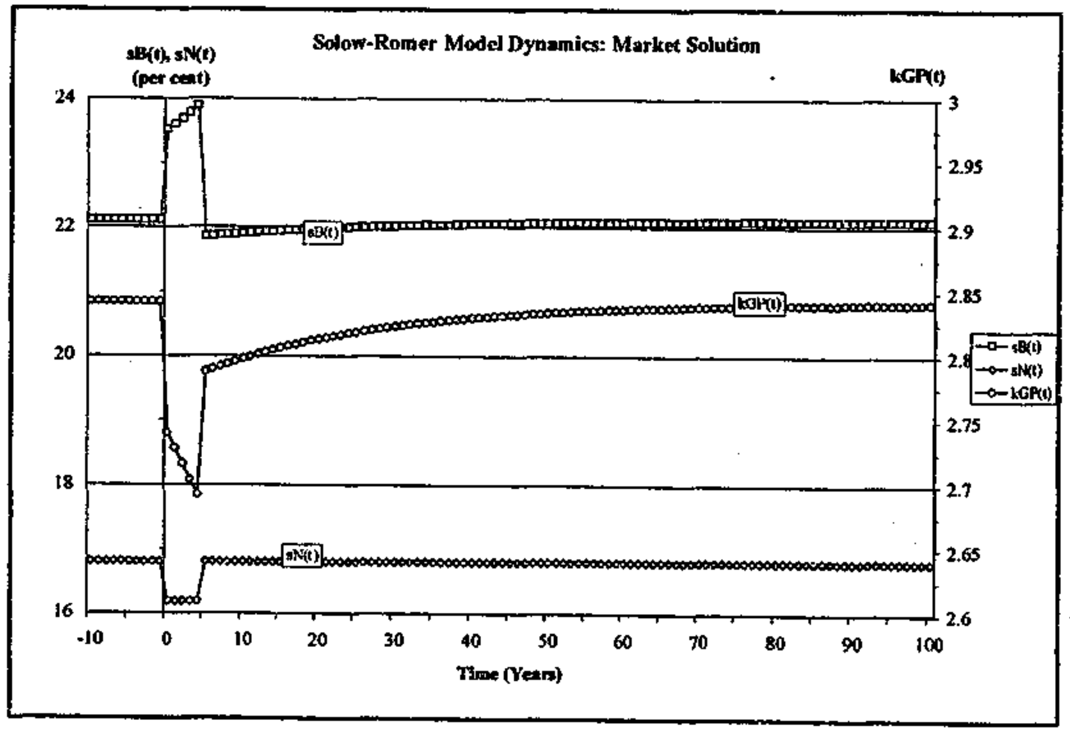

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Figure 3.34: **Dynamic effects on the factor shares of gross income from a temporary 20 per cent fall in parameter α from time t=0 to t=5, benchmark parameter set.**

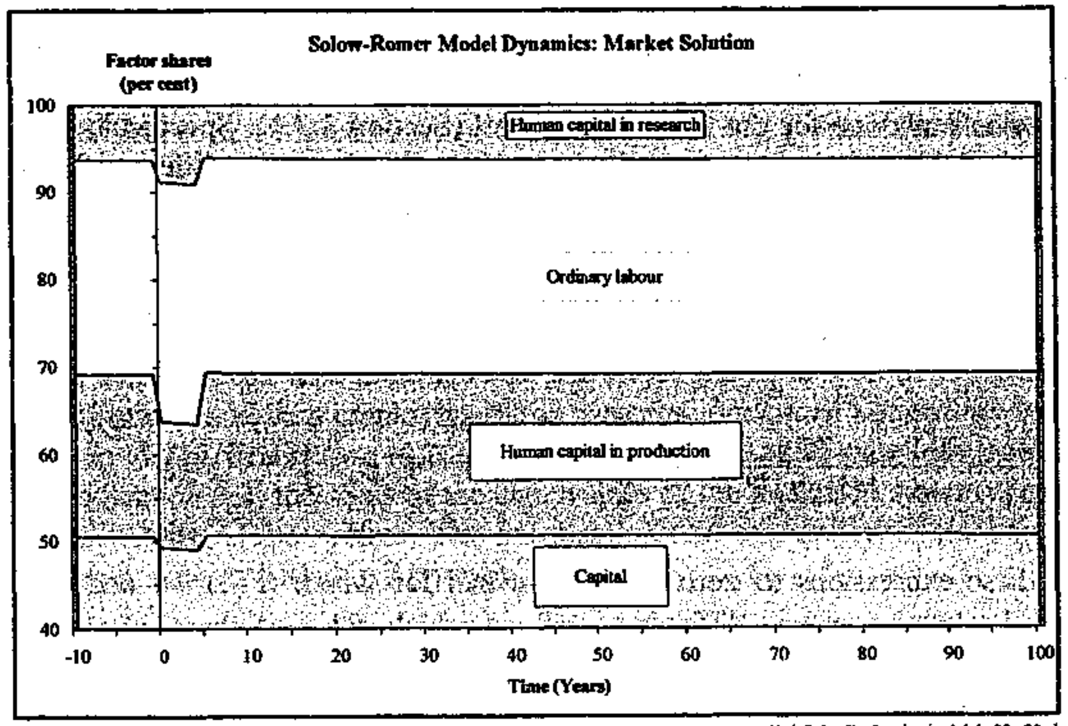

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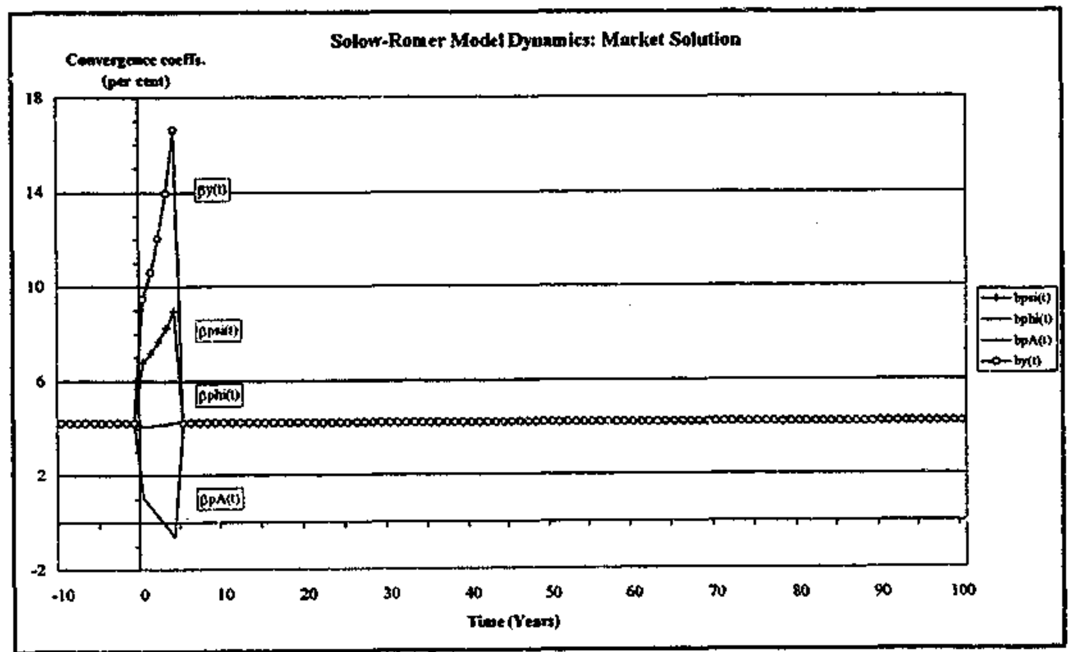
Figure 3.35: **Dynamic effects on the convergence coefficients from a temporary 20 per cent fall in parameter α from time t=0 to t=5, benchmark parameter set.**



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Note: a While the temporary shock is operating convergence is to the 'shocked ss'. After it is removed convergence reverts back to the original ss. The negative portion of βpA reflects its temporary movement away from the 'shocked ss'.

3.4.3 Phase-space of the Solowian-Romer system

Since the Solowian-Romer model comprises only **two** dynamic variables (neither of which can be negative), its phase-space is the positive quadrant of the (Ψ, p_A)-plane. Within this space the loci of points for which $\dot{\Psi}(t) = 0$ and $\dot{p}_A(t) = 0$ are obtained in the same way as was done for the full Romer model (Section 3.3.1). Specifically, equations (3.31) and (3.32) are first used to eliminate H_Y and r from the differential equations (3.29) and (3.30), for Ψ and p_A respectively, to generate:

$$\frac{\dot{\Psi}}{\Psi} = \zeta\Big[\frac{sp_A}{\alpha(1-\gamma)\Psi}+1\Big]\Big[\frac{\alpha(1-\gamma)}{\zeta\eta^{\gamma}}L^{(1-\alpha)(1-\gamma)}\Psi^{\gamma}{p_A}^{-1}\Big]^{\frac{1}{1-\alpha(1-\gamma)}} - (\delta+\zeta H) \qquad (3.38)$$

and

$$\frac{\dot{p}_A}{p_A} = \frac{\gamma\zeta p_A}{\alpha}\Big[\frac{\gamma p_A}{(1-\gamma)\Psi}-1\Big]\Big[\frac{\alpha(1-\gamma)}{\zeta\eta^{\gamma}}L^{(1-\alpha)(1-\gamma)}\Psi^{\gamma}{p_A}^{-1}\Big]^{\frac{1}{1-\alpha(1-\gamma)}} - \delta \qquad (3.39)$$

Setting these expressions to zero defines the loci for which $\dot{\Psi}(t) = 0$ and $\dot{p}_A(t) = 0$ respectively, but it remains difficult to tell on which sides of these *phase lines* the variables increase and on which sides they decrease. As in the case of the $\dot{p}_A(t) = 0$ locus in the full Romer model (which is **exactly** the same in the S-R model) this difficulty is overcome by evaluating the 'cross partial derivatives'. In particular, it is easy to show that:

$$\frac{\partial\dot{\Psi}}{\partial p_A} < 0 \quad \text{and} \quad \frac{\partial\dot{p}_A}{\partial\Psi} < 0$$

both unambiguously (see Appendix 3.6 for the $\dot{p}_A$ case).[22]

Hence, $\dot{\Psi}$ must decrease (from $\dot{\Psi}>0$, through $\dot{\Psi}=0$, to $\dot{\Psi}<0$) as p_A crosses the $\dot{\Psi}=0$ phase line; and similarly, $\dot{p}_A$ must also decrease as Ψ crosses the $\dot{p}_A = 0$ phase line. That is:

$\dot{\Psi} \gtreqless 0$ as:

$$p_A \lesseqgtr \{\zeta\Big[\frac{sp_A}{\alpha(1-\gamma)\Psi}+1\Big]\Big[\frac{\alpha(1-\gamma)}{\zeta\eta^{\gamma}}L^{(1-\alpha)(1-\gamma)}\Psi^{\gamma}{p_A}^{-1}\Big]^{\frac{1}{1-\alpha(1-\gamma)}} - (\delta+\zeta H)\}\Psi \qquad (3.40)$$

and

$\dot{p}_A \gtreqless 0$ as:

$$\Psi \lesseqgtr \frac{\gamma\zeta p_A}{\alpha}\Big[\frac{\gamma p_A}{(1-\gamma)\Psi}-1\Big]\Big[\frac{\alpha(1-\gamma)}{\zeta\eta^{\gamma}}L^{(1-\alpha)(1-\gamma)}\Psi^{\gamma}{p_A}^{-1}\Big]^{\frac{1}{1-\alpha(1-\gamma)}} - \delta p_A \qquad (3.41)$$

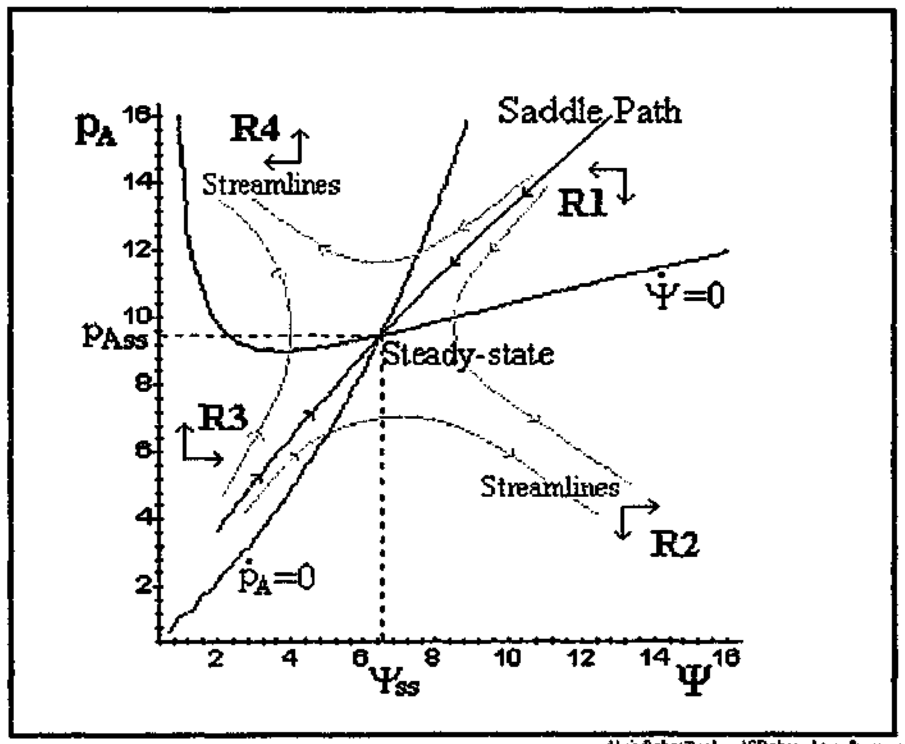
With this information the directions of change in both p_A and Ψ are known for all points in the phase-space and a phase diagram of the dynamic system may be plotted. This has

[22] The 'cross partial derivatives' were used here because, due to the *non-monotonicity* of the functions, the 'own partial derivatives' $\partial\dot{\Psi}/\partial p_A$ and $\partial\dot{p}_A/\partial\Psi$ are not of constant sign. Note that the $\dot{\Psi}=0$ curve has a clear minimum in Figure 3.36 to Figure 3.38 and Figure 3.40.

been done for the benchmark data set, **including** having the savings rate set at the benchmark steady-state level for s_N in the full (market) Romer model. That is, with the savings rate set exogenously at $s = s^M_{Nss} \approx 16.8\%$.

The results are presented in Figure 3.36, where the directions of change in p_A and Ψ in the (four) different regions of the phase-space are indicated by "corner arrows". From these it is possible to draw streamlines of the motion from any point in the phase-space.[23] Such streamlines indicate that the equilibrium (or steady-state) is one of *saddle-path stability*. The steady-state can only be reached from the regions marked R1 and R3, and then only from those points which lie on the saddle-path.[24] Here the saddle-path has been obtained in the same way as in the phase-space analysis of the full Romer model (Section 3.3). Namely, by linearising the SR model, calculating the eigenvectors from its coefficients matrix (recall that these indicate the directions of motion of the saddle-path at the steady-state), and then employing the *eigenvector-backward integration* technique (explained in detail in Section 4.2.2) to compute the saddle-path numerically. As explained in Section 3.2, the signs of the eigenvalues of the linearised system confirm the saddle-path stability of the system. The linearisation and calculation of the eigenvalues and eigenvectors are recorded in Appendix 3.8.

Figure 3.36: Phase-space and saddle-path of the Solowian-Romer model, benchmark parameter set (including savings rate of $s=s_{Nss}=0.1680$).



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[23] The streamlines must cross the $\dot{\Psi} = 0$ locus with infinite slope and the $\dot{p}_A = 0$ locus with zero slope.

[24] Even very small discrepancies from such points will cause the system to diverge.

3.4.4 Phase-space analysis of the Solowian-Romer dynamics

Having determined how to compute the phase lines and saddle-path it is now possible to analyse the dynamics of the model within its phase-space. The advantage of such analyses is that the geometric pictures produced allow clear and concise visualisations of the key elements of the dynamic responses of the model to exogenous shocks. In this way they promote understanding of the model dynamics. In order to allow comparison between the time path analysis of numerical integration and this phase-space analysis, the shocks considered here are the same as those of Section 3.4.2. Finally, it is worth emphasising here that these phase-space analyses are entirely quantitative, all the phase lines, steady-states, saddle-paths and adjustment paths having been computed numerically and then plotted graphically.[25]

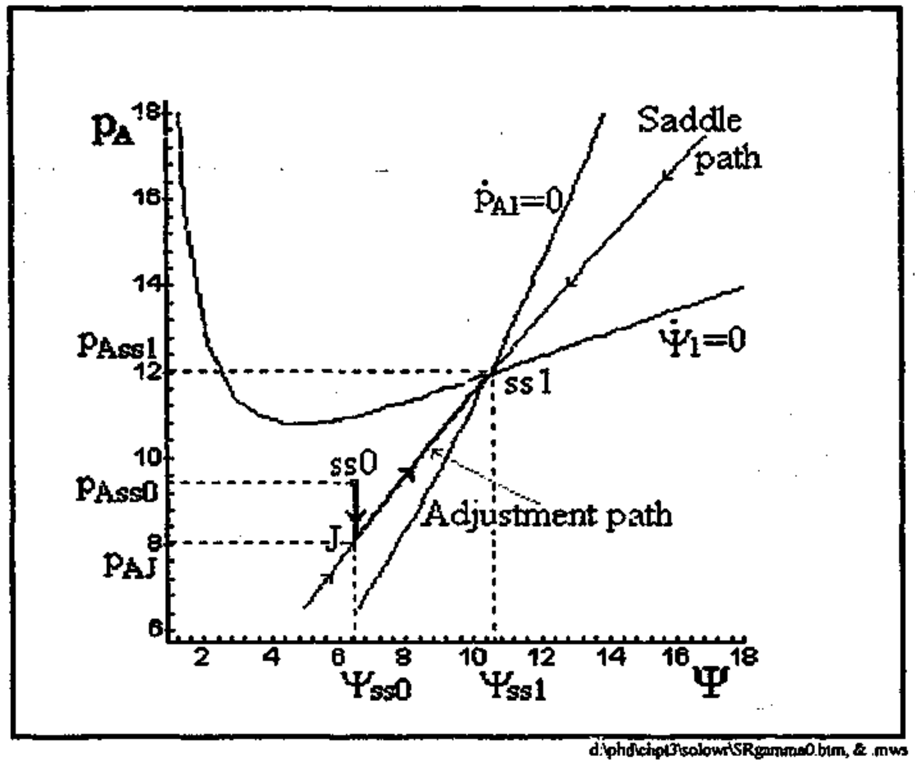
3.4.4.1 An unanticipated and sustained 10 per cent rise in parameter γ

The rise in parameter γ alters the magnitudes of the coefficients in the differential equations shifting the $\dot{\Psi}(t)=0$ locus upwards and the $\dot{p}_A(t)=0$ locus to the right, thereby moving the steady-state from ss0 to ss1 and of course, generating a new saddle-path (Figure 3.37). As required by the dynamic optimisation, the system must converge towards this new steady-state. Since the dynamics of the system from time t=0 onwards are governed by the post-shock differential equations, and since the only way for these to converge to the new steady-state is along the new saddle-path, the system must jump immediately to it. But the variable Ψ=K/A is a stock variable and is accordingly constrained not to be discontinuous. In particular, its immediate post-shock level must equal its immediate pre-shock level. This is formalised in the boundary condition "$\Psi(0)$ given" of equation (2.46). Conversely, prices readily jump discontinuously. For these reasons the jump to the post-shock saddle-path is made entirely by the remaining dynamic variable p_A.

Thus, in terms of Figure 3.37 the dynamic response of the Solowian-Romer model to the rise in γ is first for p_A to fall discontinuously from p_{Ass0} to p_{AJ} while Ψ remains constant at the level Ψ_{ss0}; and then, having instantaneously reached the saddle-path, for both variables to increase smoothly along it towards the new steady-state of the system. The entire adjustment path is along (ss0, J, ss1) as indicated in Figure 3.37. Note that although p_A ends up at a higher level than it begins at ($p_{Ass1}>p_{Ass0}$), its initial response is a sudden fall ($p_{AJ}<p_{Ass0}$). These sorts of issue are addressed in Chapter 4 when the transitional dynamics of the full model are computed.

[25] By the same procedure as used in the Section 3.3 phase-space analysis of the full Romer model (see footnote 11).

Figure 3.37: Phase-space analysis of the dynamic response of the Solowian-Romer model to an unanticipated 10% rise in parameter γ, benchmark parameter set.



3.4.4.2 An anticipated and sustained 15 per cent rise in parameter ζ

In terms of the shifts induced on the phase lines, a rise in ζ produces opposite effects to a rise in γ: Here the $\dot{\Psi}(t)=0$ phase line is shifted down and the $\dot{p}_A(t)=0$ line is shifted to the left. As a result, a new equilibrium is established at 'ss1' where both Ψ_{ss1} and p_{Ass1} are less than their initial levels at 'ss0', and a new saddle-path is generated leading to it (Figure 3.38).

With the shock to the system being anticipated, the initial dynamic response is for the *jumping variable*, p_A, to jump instantaneously to a specific point part way towards the new saddle-path from which, evolving under the influence of the pre-shock differential equations, the system reaches the saddle-path at precisely the time at which the anticipated shock is to be implemented. Then, under the post-shock equations the system adjusts smoothly along the saddle-path towards the new steady-state (recall the earlier general explanation of these sorts of dynamics in Section 3.3.3). In terms of Figure 3.38 and Figure 3.39 (the latter showing the detail of the early adjustment path) the dynamic response of the system for the shock to ζ is:

- first, for p_A to fall precipitously from ss0 to J where $p_{AJ}<p_{Ass0}$, but where Ψ remains constant at Ψ_{ss0};
- second, for the system to adjust smoothly under the pre-shock equations with Ψ increasing and p_A decreasing until the new saddle-path is reached at point A; and
- finally, for both variables to decline as the system adjusts along the saddle-path towards the new, post-shock steady-state.

Overall, the transitional dynamics of the system trace out the adjustment path (ss0,J,A,ss1) as indicated in Figure 3.38 and Figure 3.39.

Figure 3.38: Phase-space analysis of the dynamic response of the Solowian-Romer model to an anticipated 15% rise in parameter ζ, benchmark parameter set.

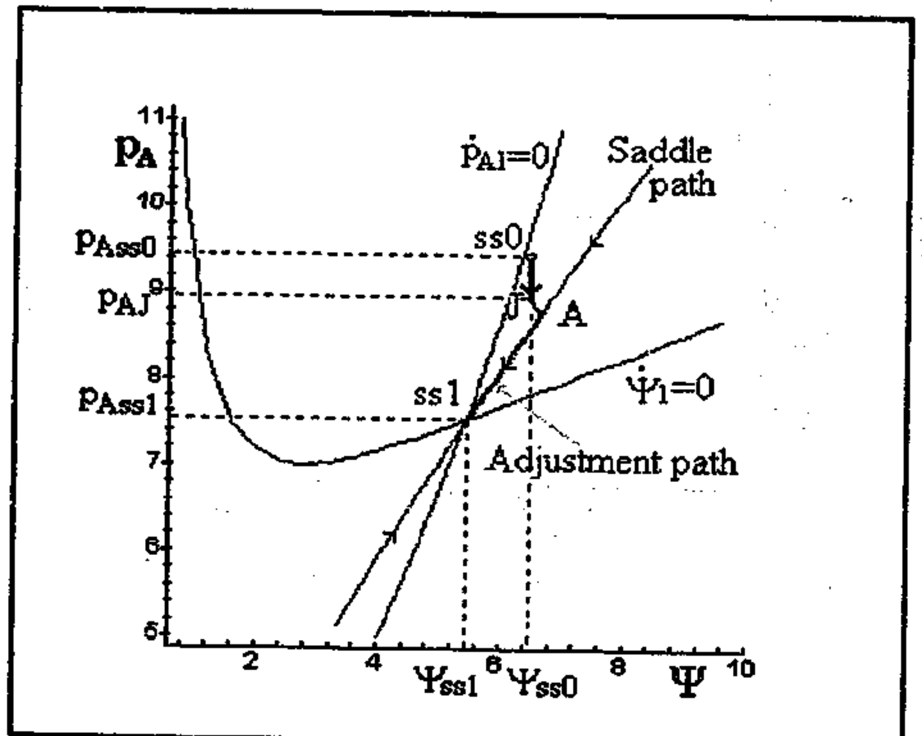


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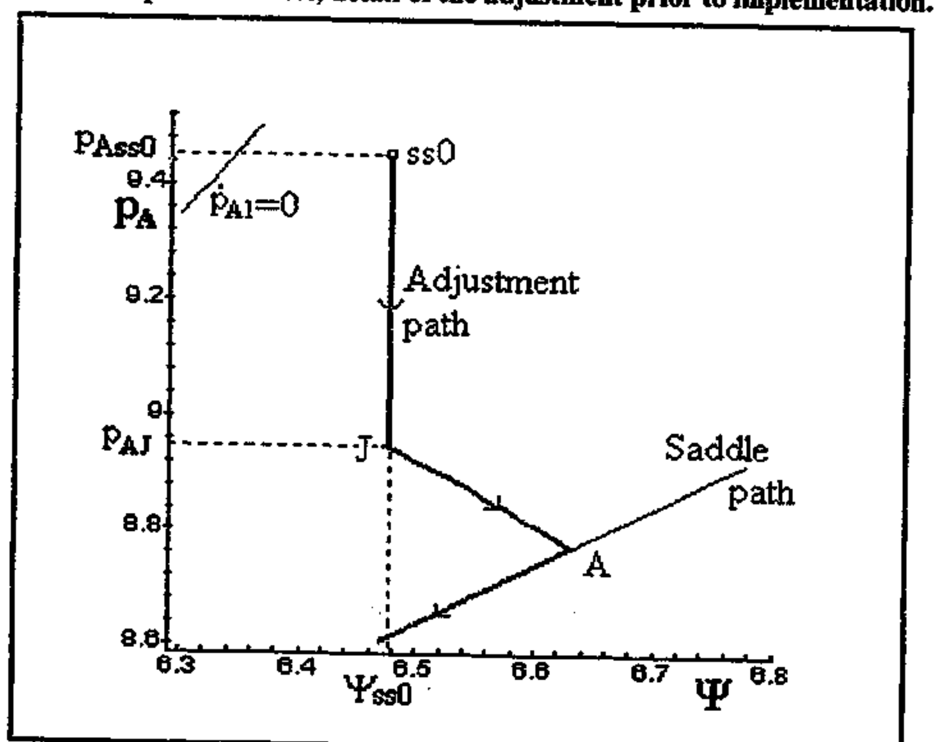
Figure 3.39: Phase-space analysis of the dynamic response of the Solowian-Romer model to an anticipated 15% rise in parameter ζ, benchmark parameter set, detail of the adjustment prior to implementation.



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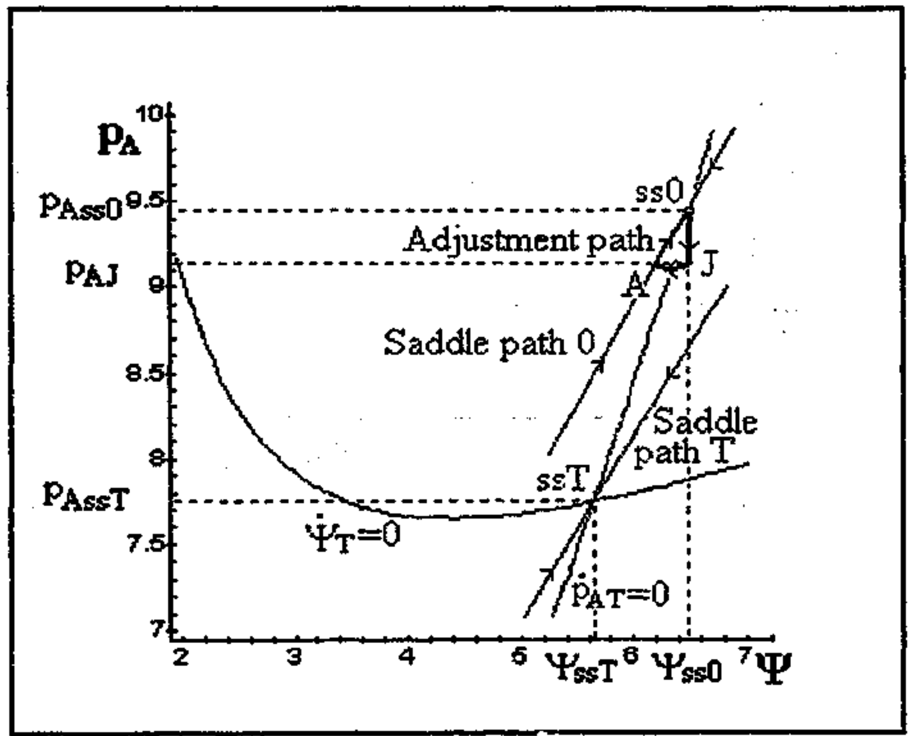
3.4.4.3 A temporary (five year) fall of 20 per cent in parameter α

It is assumed here that the implementation of the shock is unanticipated: It is not announced in advance and for no other reason is it expected. However, at the time it is implemented its duration is known. Perhaps this is announced! In any event, the 'shock' of its removal is assumed to be (correctly) anticipated.

The fall in α shifts the $\dot{\Psi}(t)=0$ phase line downwards and moves the $\dot{p}_A(t)=0$ line marginally to the right. The steady-state of this post-shock system, labelled as 'ssT' in Figure 3.40, lies below and to the left of the initial steady-state. Its saddle-path also lies below the initial steady-state. However, since the shock to α is only temporary and so can have no lasting effect, the system must eventually adjust back towards its initial steady-state along the corresponding initial saddle-path; **not** along the saddle-path for the temporary steady-state. Moreover, the system must reach this saddle-path at exactly the time at which it becomes relevant; that is, precisely when the temporary shock is reversed, for from that time the original (unshocked) differential equations once again govern its dynamics. Of course before that time the dynamics are governed by the temporary (shocked) equations.

Figure 3.40: Phase-space analysis of the dynamic response of the Solowian-Romer model to a temporary (5 years) 20% fall in parameter α, benchmark parameter set.



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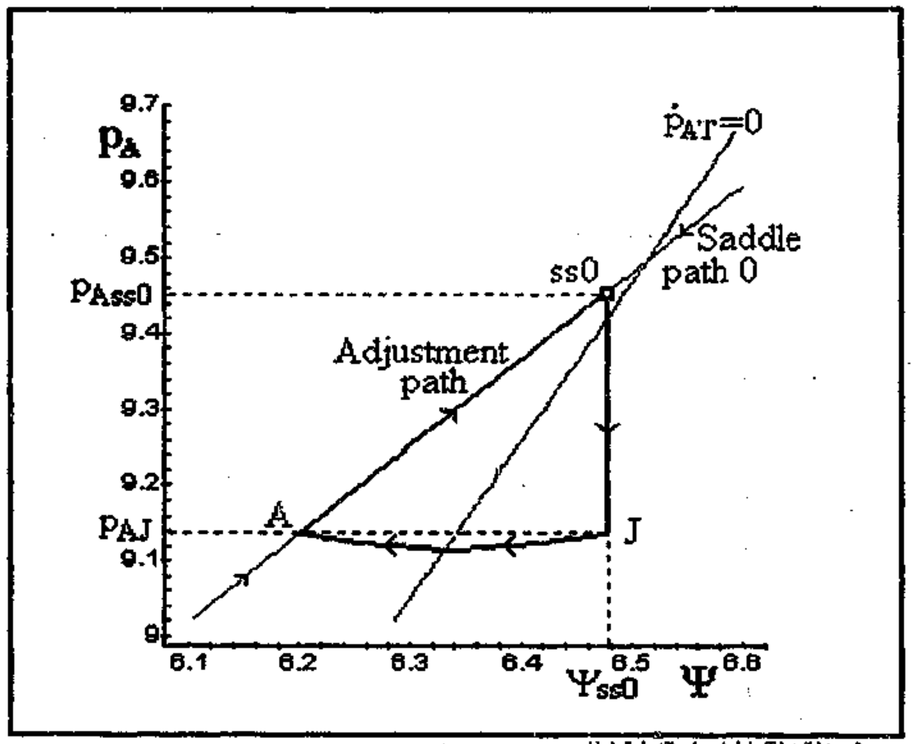
With the removal of the temporary shock to α being anticipated, the dynamic response here is similar in general terms to that of the anticipated shock to ζ discussed in the previous section (also see the explanation of these sorts of dynamics in Section 3.3.3). In terms of Figure 3.40 and Figure 3.41, the latter concentrating on a smaller part of the phase-space in order to show the adjustment path in greater detail, the full dynamic response of the system to the temporary shock to α is as follows:

- the initial response is for the *jumping variable*, p_A, to jump instantaneously from ss0 to J, a specific point part of the way towards the temporary saddle-path where $p_{AJ}<p_{Ass0}$, but where Ψ remains constant at Ψ_{ss0};
- from J the system then adjusts smoothly under the temporary-shock equations - firstly with both Ψ and p_A decreasing but later, as the system crosses the $\dot{p}_A(t) = 0$ phase line and thus changes 'regions' of the temporary phase-space, with Ψ continuing to decrease while p_A now begins to increase - until the original saddle-path is reached at point A;
- finally, from A both variables rise as the system adjusts along this saddle-path back towards its original pre-shock steady-state.

Figure 3.41: Phase-space analysis of the dynamic response of the Solowian-Romer model to a temporary (5 years) 20% fall in parameter α, benchmark parameter set, detail of the adjustment path.



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Overall, the transitional dynamics of the system trace out the adjustment path (ss0,J,A,ss0) as indicated in Figure 3.40 and Figure 3.41.

Appendix 3.1

Linearisation of the Romer model

For any vector function $f(\mathbf{z}) = f(z_1, z_2, \ldots\ldots, z_n)$, the Taylor series expansion about a point $\mathbf{z}^*$ is:[26]

$$f(\mathbf{z}) = f(\mathbf{z}^*) + \sum_{i=1}^{n} f_{z_i}(\mathbf{z}^*)\mathrm{d}z_i + \frac{1}{2!}\sum_{i=1}^{n}\sum_{j=1}^{n} f_{z_iz_j}(\mathbf{z}^*)\mathrm{d}z_i\mathrm{d}z_j + \ldots\ldots$$

where

$$\mathrm{d}z_i = z_i - z_i^*\ ;\ \ f_{z_i} = \partial f / \partial z_i\ ;\ \text{and}\ \ f_{z_iz_j} = \partial^2 f / \partial z_i \partial z_j$$

or, in complete vector notation:

$$f(\mathbf{z}) = f(\mathbf{z}^*) + (\mathbf{dz})' f_{\mathbf{z}}(\mathbf{z}^*) + \frac{1}{2!}(\mathbf{dz})' f_{\mathbf{zz'}}(\mathbf{z}^*)(\mathbf{dz}) + \ldots\ldots\ldots$$

Then, ignoring second order and higher terms (that is, assuming the $\mathrm{d}z_i$ are sufficiently small) generates the linear approximation:

$$f(\mathbf{z}) \approx f(\mathbf{z}^*) + \sum_{i=1}^{n} f_{z_i}(\mathbf{z}^*)\mathrm{d}z_i = f(\mathbf{z}^*) + \mathrm{d}f(\mathbf{z}^*) \qquad \text{(A3.1.1)}$$

This result can then be applied to the (stationary) Romer model system $\mathbf{w(t)} = (\Psi(t), \Phi(t), p_A(t))^T$, described by equations (2.41) to (2.45) about its steady-state $\mathbf{w}_{ss} = (\Psi_{ss}, \Phi_{ss}, p_{Ass})^T$, as determined by equations (2.57) to (2.62). It will prove useful first to differentiate the expressions for $H_Y(t)$ and $[r(t)+\delta]$. Thus, from (2.44):

$$\frac{\mathrm{d}H_Y}{H_{Yss}} = \frac{1}{1-\alpha(1-\gamma)}[\gamma\frac{\mathrm{d}\Psi}{\Psi_{ss}} - \frac{\mathrm{d}p_A}{p_{Ass}}] \qquad \text{(A3.1.2)}$$

and from this result and (2.45):

$$\begin{aligned}\frac{\mathrm{d}r}{(r_{ss}+\delta)} &= \frac{\mathrm{d}H_Y}{H_{Yss}} + \frac{\mathrm{d}p_A}{p_{Ass}} - \frac{\mathrm{d}\Psi}{\Psi_{ss}} \\ &= [\frac{\gamma}{1-\alpha(1-\gamma)} - 1]\frac{\mathrm{d}\Psi}{\Psi_{ss}} + [1 - \frac{1}{1-\alpha(1-\gamma)}]\frac{\mathrm{d}p_A}{p_{Ass}} \qquad \text{(A3.1.3)} \\ &= -\frac{(1-\alpha)(1-\gamma)}{1-\alpha(1-\gamma)}\frac{\mathrm{d}\Psi}{\Psi_{ss}} - \frac{\alpha(1-\gamma)}{1-\alpha(1-\gamma)}\frac{\mathrm{d}p_A}{p_{Ass}}\end{aligned}$$

Then linearising $\dot{\Psi}$ from equation (2.41):

$$\begin{aligned}\dot{\Psi}(\mathbf{w(t)}) &= \dot{\Psi}_{ss} + \mathrm{d}\dot{\Psi}(\mathbf{w}_{ss}) \\ &= \dot{\Psi}_{ss} + [\frac{r_{ss}+\delta}{\gamma^2} - \Phi_{ss} - \delta - \zeta H + \zeta H_{Yss}]\mathrm{d}\Psi + \Psi_{ss}\mathrm{d}[\frac{r+\delta}{\gamma^2} - \Phi - \delta - \zeta H + \zeta H_Y]\end{aligned}$$

[26] For example, see Leonard and Van Long (1992).

and since $\dot{\Psi}_{ss} = 0 \Rightarrow [(r_{ss}+\delta)/\gamma^2 - \Phi_{ss} - \delta - \zeta H + \zeta H_{Yss}] = 0$, and H is exogenously constant:

$$\dot{\Psi}(\mathbf{w}(t)) = [dr/\gamma^2 - d\Phi + \zeta dH_Y]\Psi_{ss} \tag{A3.1.4}$$

Substituting (A3.1.2) and (A3.1.3) into (A3.1.4), using (2.45) to write the steady-state relation: $(r_{ss}+\delta) = [\zeta\gamma^2/\alpha(1-\gamma)]H_{Yss}p_{Ass}\Psi_{ss}^{-1}$, and writing $1/[1-\alpha(1-\gamma)] = \Lambda$, yields (after some manipulation):

$$\dot{\Psi}(\mathbf{w}(t)) = \Lambda\zeta H_{Yss}[\gamma - \frac{(1-\alpha)}{\alpha}\frac{p_{Ass}}{\Psi_{ss}}]d\Psi - \Psi_{ss}d\Phi - \Lambda\zeta H_{Yss}[1+\frac{\Psi_{ss}}{p_{Ass}}]dp_A$$

and substituting $d\Psi = \Psi - \Psi_{ss}$, $d\Phi = \Phi - \Phi_{ss}$, and $dp_A = p_A - p_{Ass}$ generates the final linearised result for $\dot{\Psi}$:

$$\begin{aligned}\dot{\Psi}(t) = {} & \Lambda\zeta H_{Yss}[\gamma - \frac{(1-\alpha)}{\alpha}\frac{p_{Ass}}{\Psi_{ss}}]\Psi(t) - \Psi_{ss}\Phi(t) - \Lambda\zeta H_{Yss}[1+\frac{\Psi_{ss}}{p_{Ass}}]p_A(t) \\ & + \Lambda(1-\gamma)\zeta H_{Yss}\Psi_{ss} + \Psi_{ss}\Phi_{ss} + (\Lambda\zeta/\alpha)H_{Yss}p_{Ass}\end{aligned} \tag{A3.1.5}$$

Next, linearising $\dot{\Phi}$ from equation (2.42) about $\mathbf{w}_{ss}$:

$$\begin{aligned}\dot{\Phi}(\mathbf{w}(t)) &= \dot{\Phi}_{ss} + d\dot{\Phi}(\mathbf{w}_{ss}) \\ &= \dot{\Phi}_{ss} + [\frac{r_{ss}-\rho}{\sigma} - \frac{r_{ss}+\delta}{\gamma^2} + \Phi_{ss} + \delta]d\Phi + \Phi_{ss}d[\frac{r-\rho}{\sigma} - \frac{r+\delta}{\gamma^2} + \Phi + \delta]\end{aligned}$$

Noting that $\dot{\Phi}_{ss} = 0 \Rightarrow [\frac{r_{ss}-\rho}{\sigma} - \frac{r_{ss}+\delta}{\gamma^2} + \Phi_{ss} + \delta] = 0$, and substituting (A3.1.3) yields:

$$\begin{aligned}\dot{\Phi}(\mathbf{w}(t)) = {} & \Lambda(\frac{1}{\gamma^2} - \frac{1}{\sigma})(1-\alpha)(1-\gamma)\frac{(r_{ss}+\delta)\Phi_{ss}}{\Psi_{ss}}d\Psi + \Phi_{ss}d\Phi \\ & - \Lambda(\frac{1}{\gamma^2} - \frac{1}{\sigma})\alpha(1-\gamma)\frac{(r_{ss}+\delta)\Phi_{ss}}{p_{Ass}}dp_A\end{aligned}$$

then, with $d\Psi = \Psi - \Psi_{ss}$, $d\Phi = \Phi - \Phi_{ss}$, and $dp_A = p_A - p_{Ass}$, and using (2.43) to write the component $(1-\gamma)(r_{ss}+\delta)/p_{Ass}$ of the last term above as $\gamma r_{ss}/\Psi_{ss}$, the final linearised form of $\dot{\Phi}$ is:

$$\begin{aligned}\dot{\Phi}(t) = {} & \Lambda(\frac{1}{\gamma^2} - \frac{1}{\sigma})(1-\alpha)(1-\gamma)\frac{(r_{ss}+\delta)\Phi_{ss}}{\Psi_{ss}}\Psi(t) + \Phi_{ss}\Phi(t) \\ & - \Lambda(\frac{1}{\gamma^2} - \frac{1}{\sigma})\alpha\gamma\frac{r_{ss}\Phi_{ss}}{p_{Ass}}p_A(t) - \Phi_{ss}^2 + \Lambda(\frac{1}{\gamma^2} - \frac{1}{\sigma})(1-\gamma)(r_{ss}+\delta)\Phi_{ss}\end{aligned} \tag{A3.1.6}$$

Finally, linearising $\dot{p}_A$ from (2.43) about $\mathbf{w}_{ss}$:

$$\begin{aligned}\dot{p}_A(\mathbf{w}(t)) &= \dot{p}_{Ass} + d\dot{p}_A(\mathbf{w}_{ss}) \\ &= p_{Ass}dr + r_{ss}dp_A - \frac{1-\gamma}{\gamma}(r_{ss}+\delta)d\Psi - \frac{1-\gamma}{\gamma}\Psi_{ss}dr \\ &= \{\Lambda(1-\alpha)(1-\gamma)[\frac{1-\gamma}{\gamma} - \frac{p_{Ass}}{\Psi_{ss}}] - \frac{1-\gamma}{\gamma}\}(r_{ss}+\delta)d\Psi \\ &\quad + \{\Lambda\alpha(1-\gamma)[\frac{1-\gamma}{\gamma}\frac{\Psi_{ss}}{p_{Ass}} - 1](r_{ss}+\delta) + r_{ss}\}dp_A\end{aligned}$$

which, with $d\Psi = \Psi - \Psi_{ss}$, $d\Phi = \Phi - \Phi_{ss}$, and $dp_A = p_A - p_{Ass}$ as before; and this time using (2.43) to write $(1-\gamma)(r_{ss}+\delta)/\gamma = r_{ss}p_{Ass}/\Psi_{ss}$ becomes:

$$\begin{aligned}\dot{p}_A(\mathbf{w}(t)) = {} & \{\Lambda(1-\alpha)(1-\gamma)[r_{ss} - (r_{ss}+\delta)]\frac{p_{Ass}}{\Psi_{ss}} - \frac{r_{ss}p_{Ass}}{\Psi_{ss}}\}\Psi(t) \\ & + \{\Lambda\alpha(1-\gamma)[r_{ss} - (r_{ss}+\delta)] + r_{ss}\}p_A(t) \\ & + \Lambda(1-\alpha)(1-\gamma)\delta p_{Ass} + r_{ss}p_{Ass} + \{\Lambda\alpha(1-\gamma)\delta - r_{ss}\}p_{Ass}\end{aligned}$$

and the final linearised form of $\dot{p}_A$ is:

$$\begin{aligned}\dot{p}_A(t) = {} & -[r_{ss} + \Lambda\delta(1-\alpha)(1-\gamma)]\frac{p_{Ass}}{\Psi_{ss}}\Psi(t) + [r_{ss} - \Lambda\delta\alpha(1-\gamma)]p_A(t) \\ & + \Lambda\delta(1-\gamma)p_{Ass}\end{aligned} \tag{A3.1.7}$$

Writing (A3.1.5), (A3.1.6) and (A3.1.7) in matrix notation produces the final linearised form of the Romer model about its steady-state equilibrium:

$$\dot{\mathbf{w}}(t) = \mathbf{\Omega}_R\mathbf{w}(t) + \mathbf{v}_R \tag{A3.1.8}$$

where $\mathbf{w}(t)$ is the column vector $(\Psi(t), \Phi(t), p_A(t))^T$; $\dot{\mathbf{w}}(t)$ is the corresponding vector of time derivatives; and $\mathbf{\Omega}_R$ and $\mathbf{v}_R$ are the coefficients matrix and vector, the components of which reflect the point about which the linearisation was performed, in this case the steady-state:

$$\mathbf{\Omega}_R = \left(\begin{array}{c:c:c} \frac{\zeta H_{Yss}}{1-\alpha(1-\gamma)}[\gamma - \frac{1-\alpha}{\alpha}\frac{p_{Ass}}{\Psi_{ss}}] & -\Psi_{ss} & -\frac{\zeta H_{Yss}(1+\Psi_{ss}/p_{Ass})}{1-\alpha(1-\gamma)} \\ \hdashline [\frac{1}{\gamma^2} - \frac{1}{\sigma}]\frac{(1-\alpha)(1-\gamma)}{1-\alpha(1-\gamma)}\frac{(r_{ss}+\delta)\Phi_{ss}}{\Psi_{ss}} & \Phi_{ss} & [\frac{1}{\gamma^2} - \frac{1}{\sigma}]\frac{\alpha\gamma}{1-\alpha(1-\gamma)}\frac{r_{ss}\Phi_{ss}}{\Psi_{ss}} \\ \hdashline -[r_{ss} + \frac{\delta(1-\alpha)(1-\gamma)}{1-\alpha(1-\gamma)}]\frac{p_{Ass}}{\Psi_{ss}} & 0 & r_{ss} - \frac{\delta\alpha(1-\gamma)}{1-\alpha(1-\gamma)} \end{array}\right) \tag{A3.1.9}$$

$$
\mathbf{v}_R = \begin{pmatrix} \dfrac{\zeta H_{Yss}}{1-\alpha(1-\gamma)}[(1-\gamma)\Psi_{ss} + p_{Ass}] + \Psi_{ss}\Phi_{ss} \\ -\Phi_{ss}[\Phi_{ss} + (\dfrac{1}{\gamma^2} - \dfrac{1}{\sigma})\dfrac{(1-\gamma)}{1-\alpha(1-\gamma)}(r_{ss}+\delta)] \\ \dfrac{\delta(1-\gamma)}{1-\alpha(1-\gamma)} p_{Ass} \end{pmatrix} \tag{A3.1.10}
$$

and where, as usual (see equations (2.53) to (2.59) of Chapter 2), for the market solution to the model:

$$H_{Yss} = \frac{(\alpha\sigma/\gamma)H + \alpha\rho/\gamma\zeta]}{1+\alpha\sigma/\gamma};$$

$$r_{ss} = (\gamma\zeta/\alpha)H_{Yss};$$

$$\Phi_{ss} = (r_{ss}+\delta)/\gamma^2 - g - \delta;$$

$$g = \frac{\zeta H - \alpha\rho/\gamma}{1+\alpha\sigma/\gamma};$$

$$\Psi_{ss} = [\frac{\gamma^2 H_{Yss}{}^{\alpha(1-\gamma)} L^{(1-\alpha)(1-\gamma)}}{\eta^\gamma (r_{ss}+\delta)}]^{\frac{1}{1-\gamma}};$$

and

$$p_{Ass} = \frac{1-\gamma}{\gamma}\frac{r_{ss}+\delta}{r_{ss}}\Psi_{ss}$$

Appendix 3.2

Solution of a first-order system of linear differential equations

A3.2.1 General solution

A first-order linear system of dynamic equations may be written as:

$$\dot{\mathbf{w}}(t) = \Omega(t)\mathbf{w}(t) + \mathbf{v}(t) \tag{A3.2.1}$$

where $\mathbf{w}(t) = (w_1,...,w_n)^T$ is a column vector of the system's dynamic variables; $\dot{\mathbf{w}}(t)$ is the corresponding vector of time derivatives; and $\Omega(t)$ and $\mathbf{v}(t)$ are respectively, a matrix and a vector of coefficients. Provided such a system has an equilibrium or steady-state, Ω and $\mathbf{v}$ will be (approximately) constant in its neighbourhood. (In the case of a non-linear system which has been *linearised* - that is, approximated by a first-order Taylor series expansion - to the form (A3.2.1), Ω and $\mathbf{v}$ will have already been taken to be constant).

Now consider the solution to equation (A3.2.1).[27] From the theory of differential equations the solution comprises two parts: the *complementary function* (which is the solution to the *reduced form* or *homogeneous version* of the *complete function*), and the *particular integral* (which is simply any specific solution of the complete equation).[28] The homogeneous form of equation (A3.2.1)) is:

$$\dot{\mathbf{w}}(t) = \Omega\mathbf{w}(t) \tag{A3.2.2}$$

Following Dixon et al (1992), consider the most general case where all the eigenvalues or characteristic roots (λ_i) of Ω are distinct and non-zero. Then any vector $\mathbf{w(t)}$ may be written as a linear combination of the eigenvectors (γ_i) of Ω:

$$\mathbf{w}(t) = \Gamma\mathbf{c}(t) \tag{A3.2.3}$$

where Γ is a matrix whose columns are the eigenvectors of Ω, and $\mathbf{c(t)}$ is a vector of coefficients. Differentiating (A3.2.3) and substituting back into (A3.2.2) produces:

$$\Gamma\dot{\mathbf{c}}(t) = \Omega\Gamma\mathbf{c}(t) \tag{A3.2.4}$$

Now, the eigenvectors and eigenvalues of Ω are defined by:

$$\Omega\gamma_i = \lambda_i\gamma_i \quad \text{for i = 1,.....,n (the dimension of } \Omega)$$

[27]The *linearisation*, of a non-linear system is an approximation which is only valid in some region nearby the point of linearisation, usually a steady-state (Appendix 3.1). In this case the solution too is only valid in the neighbourhood of the steady-state.

[28]For example, see Chiang (1974).

Thus, in matrix notation: $\Omega\Gamma = \Gamma\Lambda$, where Λ is a diagonal matrix of the eigenvalues. Substituting this into (A3.2.4) and pre-multiplying by Γ^{-1} generates $\dot{\mathbf{c}}(\mathbf{t}) = \Lambda\mathbf{c}(\mathbf{t})$ which, with Λ diagonal, is simply a set of independent differential equations $\dot{c}_i(t) = \lambda_i c_i(t)$ with solutions: $c_i(t) = \xi_i e^{\lambda_i t}$, where the ξ_i are constants of integration. In matrix notation this is written as $\mathbf{c}(\mathbf{t}) = \mathbf{e}^{\Lambda t}\zeta$, where $\mathbf{e}^{\Lambda t}$ is a diagonal matrix of terms $e^{\lambda i t}$. Then, substituting back into (A3.2.3) produces the *complementary function*:

$$\mathbf{w}(\mathbf{t}) = \Gamma \mathbf{e}^{\Lambda t}\xi \tag{A3.2.5}$$

Next, the steady-state may be used as a particular integral (a specific solution) to the complete equation (A3.2.1). That is, the particular integral is taken as: $\mathbf{w}(\mathbf{t}) = \mathbf{w}_{ss}$, for $\mathbf{w}_{ss}$ a vector constant representing the steady-state. For this solution $\dot{\mathbf{w}}(\mathbf{t}) = \mathbf{0}$, and substituting produces the result for the steady-state:

$$\mathbf{w}_{ss} = -\Omega^{-1}\mathbf{v} \tag{A3.2.6}$$

Combining (A3.2.5) and (A3.2.6)), generates the following complete solution to the linear system (A3.2.1):

$$\mathbf{w}(\mathbf{t}) = \Gamma \mathbf{e}^{\Lambda t}\xi - \Omega^{-1}\mathbf{v} \tag{A3.2.7}$$

Finally, the vector of arbitrary integration constants (ξ) may be made definite by the specification of initial conditions. If these are $\mathbf{w}(\mathbf{0}) = \mathbf{w}_0$, the complete solution becomes:

$$\mathbf{w}(\mathbf{t}) = \Gamma \mathbf{e}^{\Lambda t}\Gamma^{-1}\Delta\mathbf{w}_0 + \mathbf{w}_{ss} \quad \text{where, } \Delta\mathbf{w}_0 = (\mathbf{w}_0 - \mathbf{w}_{ss}) \tag{A3.2.8}$$

A3.2.2 Necessary conditions for equilibrium

Equation (A3.2.8) may be expanded from its compact matrix notation as follows: With $\gamma_i=(\gamma_{ji})$ as the ith (column) eigenvector, the components of Γ are given by $\Gamma=(\gamma_{ij})$ for $i, j = 1,\ldots,n$. If the components of Γ^{-1} are similarly denoted as $\Gamma^{-1} = (\gamma^{1}_{ij}),\ \ i, j = 1,\ldots,n$, then the complete solution for the kth dynamic variable is obtained as:

$$w_k(t) = \sum_i \gamma_{ki} e^{\lambda_i t} \sum_j \gamma^{1}_{ij}\Delta w_{0j} + w_{ssk}$$

for

$$\Delta w_{0j} = (w_{0j} - w_{ssj}) \ \text{ and } \ i, j, k = 1,\ldots,n. \tag{A3.2.9}$$

Clearly, if the first term on the right hand side of (A3.2.9) vanishes over time, then the system will approach its steady-state, $\mathbf{w}_{ss} = (w_{1ss},\ldots\ldots, w_{nss})$. From the presence of the exponential power terms it is apparent that if the real part of every eigenvalue (λ_i) is negative, the system will converge to its steady-state equilibrium from **any** initial position. In this case the system is *globally stable*.[29] However, if **any** of the roots are positive (or more correctly, have a positive real component) the system will diverge over

[29]In the case of a linear approximation to a non-linear system, the region of convergence is only 'in the neighbourhood of the steady-state', and the equilibrium is only *locally stable*.

time unless the coefficient on the corresponding exponential term is identically equal to zero. Thus, conditions for a linear dynamic system to attain a steady-state equilibrium are that for all 'non-negative' roots, λ_i:

$$\sum_j \gamma^{1}_{ij}\Delta w_{0j} = \sum_j \gamma^{1}_{ij}(w_{0j} - w_{ssj}) = 0 \quad \text{for } j = 1,\ldots,n \tag{A3.2.10}$$

Since the eigenvectors γ, and the steady-state values w_{ss} are basic properties of the model, such conditions amount to constraints on the set of valid initial values.

If all the roots are 'positive' it follows from equation (A3.2.9) that the steady-state could only be attained if the system begins there! From any other initial position it is divergent. Thus, the equilibrium of a linear dynamic system for which all the characteristic roots or eigenvalues have positive real parts is *globally unstable*.

In systems for which there is a mixture of both positive and negative real components to the characteristic roots, convergence is possible only from some particular initial values. From all other initial values such a system will diverge. This is the property of *saddle-path stability* and it arises because of the constraints (A3.2.10). In particular, if *m* of the *n* roots have negative real components, convergence to the steady-state equilibrium requires *(n-m)* constraints. Thus, there are only *m* degrees of freedom in the choice of initial values: An arbitrary choice of initial values for any *m* of the dynamic variables fixes those necessary for the remaining *(n-m)* variables if the system is to reach its steady-state.

To this point the Appendix has largely followed Dixon et al (1992), who go onto argue that 'a unique stable path will exist if all the eigenvalues are distinct, non-zero and of mixed sign'; and that 'because the stable path is unique, features of the steady-state can be imposed on the solution as boundary conditions'. Here a somewhat different argument is advanced: one emphasising the underlying dynamic optimisation problems inherent in these types of models, and their associated transversality conditions. Also, a general method of determining the necessary boundary conditions and thus obtaining a complete solution for these linear models is developed.

There is nothing in the dynamic equations themselves that force systems towards equilibrium. However, for systems which arise from *dynamic optimisation problems* the first-order necessary conditions for an optimum, in addition to generating the dynamic equations, also produce certain constraints on the long-run dynamic behaviour of the system which ensure that it approaches its steady-state equilibrium asymptotically. These are the so-called *transversality conditions*.[30] Thus, the initial constraints (A3.2.10), imposed by insisting that a linear (or a linearised) system with some positive eigenvalues must approach a steady-state, are manifestations of these transversality conditions. In particular, if there are *(n-m)* transversality conditions associated with some dynamic optimisation problem, then there will be *(n-m)* positive eigenvalues and so *(n-m)* initial value constraints on the linearised system.

[30]They describe the manner in which the dynamic paths must cross or traverse, the boundaries of the dynamic problems (see Chiang, 1992; and Appendix 2.3 here).

Suppose now, that there are $0<m<n$ 'negative' eigenvalues.[31] Without loss of generality these may be taken to be the first m: That is, assume $\lambda_i<0$ for $i=1,\ldots,m$: and $\lambda_i \geq 0$ for $i=m+1,\ldots,n$. Then, the necessary conditions for equilibrium mean that there are (n-m) constraints of the form of equation (A3.2.10) linking the n initial values w_{0j}, $j=1,\ldots,n$. By specifying m of these, w_{0j} for $j=1,\ldots,m$ say, the constraint equations may be solved for the remaining (n-m) initial values w_{0j}, $j=m+1,\ldots,n$. From equation (A3.2.10):

$$\sum_{j=m+1}^{n}\gamma^{1}_{ij}w_{0j}=\sum_{j=1}^{n}\gamma^{1}_{ij}w_{ssj}-\sum_{j=1}^{m}\gamma^{1}_{ij}w_{0j} \quad \text{for } i=m+1,\ldots,n$$

or, in matrix notation: (A3.2.11)

$$(\Gamma^{-1})_{\bar{m},\bar{m}}\mathbf{w}_{0\bar{m}}=(\Gamma^{-1})_{\bar{m},n}\mathbf{w}_{ss}-(\Gamma^{-1})_{\bar{m},m}\mathbf{w}_{0m}$$

where:

- $(\Gamma^{-1})_{\bar{m},\bar{m}}$, $(\Gamma^{-1})_{\bar{m},n}$, and $(\Gamma^{-1})_{\bar{m},m}$ are matrices formed from Γ^{-1} by respectively: deleting the first m rows and the first m columns; deleting the first m rows and taking the first n (ie. all) of the columns; and deleting the first m rows and taking the first m columns (only); and
- $\mathbf{w}_{0\bar{m}}$ and $\mathbf{w}_{0m}$ are vectors formed from $\mathbf{w}_0$ by respectively deleting the first m elements; and then taking (only) the first m elements.

The solution to (A3.2.11) is:

$$\mathbf{w}_{0\bar{m}}=[(\Gamma^{-1})_{\bar{m},\bar{m}}]^{-1}\{(\Gamma^{-1})_{\bar{m},n}\mathbf{w}_{ss}-(\Gamma^{-1})_{\bar{m},m}\mathbf{w}_{0m}\}$$

which may be simplified by partitioning the $(\Gamma^{-1})_{\bar{m},n}$ and $\mathbf{w}_{ss}$ matrices as follows:

$$\begin{aligned}\mathbf{w}_{0\bar{m}}&=[(\Gamma^{-1})_{\bar{m},\bar{m}}]^{-1}\{[(\Gamma^{-1})_{\bar{m},m} \;\vdots\; (\Gamma^{-1})_{\bar{m},\bar{m}}]\begin{pmatrix}\mathbf{w}_{ssm}\\ \cdots\\ \mathbf{w}_{ss\bar{m}}\end{pmatrix}-(\Gamma^{-1})_{\bar{m},m}\mathbf{w}_{0m}\}\\ &=[(\Gamma^{-1})_{\bar{m},\bar{m}}]^{-1}\{(\Gamma^{-1})_{\bar{m},m}\mathbf{w}_{ssm}+(\Gamma^{-1})_{\bar{m},\bar{m}}\mathbf{w}_{ss\bar{m}}-(\Gamma^{-1})_{\bar{m},m}\mathbf{w}_{0m}\}\end{aligned}$$

Thus:

$$\mathbf{w}_{0\bar{m}}=\mathbf{w}_{ss\bar{m}}-[(\Gamma^{-1})_{\bar{m},\bar{m}}]^{-1}(\Gamma^{-1})_{\bar{m},m}\Delta\mathbf{w}_{0m} \qquad \text{(A3.2.12)}$$

Equation (A3.2.12) defines the *saddle-path* (or strictly, the *m-dimensional saddle surface*) of the linear model given by equation (A3.2.1). It specifies, for all initial values of the dynamic variables w_j for $j=1,\ldots,m$, the corresponding initial values for all other dynamic variables which are necessary for the system to converge to its steady-state equilibrium. Convergence is then along this path.

An alternative calculation of the saddle-path may be obtained from the linear model solution (A3.2.9) and the necessary conditions (A3.2.10). As before, the 'first m' of the n eigenvalues are taken to be negative, and the equations are ordered so that it is also the 'first m' of the n variables for which initial values are known. Then, invoking (A3.2.10) means that (A3.2.9) becomes:

[31] When m=0 there are n initial value constraints meaning that system is globally unstable with the steady-state equilibrium only attainable if the system begins there initially. On the other hand, when m=n there are no initial value constraints and the system is globally stable.

$$w_k(t)=\sum_{i=1}^{m}\gamma_{ki}e^{\lambda_i t}\sum_{j=1}^{n}\gamma^{1}_{ij}\Delta w_{0j}+w_{ssk} \quad \text{for } k=1,\ldots,n$$

or in matrix notation: (A3.2.13)

$$\mathbf{w}(t)=\Gamma_{n,m}e^{\Lambda_{m,m}t}(\Gamma^{-1})_{m,n}\Delta\mathbf{w}_0+\mathbf{w}_{ss}$$

where, following the earlier notation, $\Gamma_{n,m}$ is a matrix composed of the first n rows and the first m columns of Γ; $e^{\Lambda_{m,m}t}$ denotes the first m rows and columns of $e^{\Lambda t}$; and $(\Gamma^{-1})_{m,n}$ represents the first m rows and first n columns of Γ^{-1}.

Next, the left hand side of (A3.2.13) is converted to initial values by setting $t=0$. This produces the somewhat curious expression:

$$\Delta\mathbf{w}_0=\Gamma_{n,m}(\Gamma^{-1})_{m,n}\Delta\mathbf{w}_0 \qquad \text{(A3.2.14)}$$

However, by partitioning the $\Delta\mathbf{w}_0$ and $\Gamma_{n,m}$ matrices this result can be made to yield the saddle-path of the linearised model[32]:

$$\begin{pmatrix}\Delta\mathbf{w}_{0m}\\ \cdots\\ \Delta\mathbf{w}_{0\bar{m}}\end{pmatrix}=\begin{pmatrix}\Gamma_{m,m}\\ \cdots\\ \Gamma_{\bar{m},m}\end{pmatrix}(\Gamma^{-1})_{m,n}\Delta\mathbf{w}_0$$

which generates the two sets of equations:

$$\Delta\mathbf{w}_{0m}=\Gamma_{m,m}(\Gamma^{-1})_{m,n}\Delta\mathbf{w}_0 \quad \text{and} \quad \Delta\mathbf{w}_{0\bar{m}}=\Gamma_{\bar{m},m}(\Gamma^{-1})_{m,n}\Delta\mathbf{w}_0$$

then the first set is used to solve for the (nx1) vector $(\Gamma^{-1})_{m,n}\Delta\mathbf{w}_0$ in terms of the (mx1) vector of known initial divergences from equilibrium; which is then substituted into the second equation set to solve for the *unknown* initial values. As before, this defines the saddle-path. Thus:

$$\mathbf{w}_{0\bar{m}}=\mathbf{w}_{ss\bar{m}}+\Gamma_{\bar{m},m}(\Gamma_{m,m})^{-1}\Delta\mathbf{w}_{0m} \qquad \text{(A3.2.15)}$$

A3.2.3 The complete solution

Finally, the previous results allow the complete solution to a first-order linear dynamic system of the form (A3.2.1), which is also constrained to approach asymptotically its steady-state equilibrium. Specifically, the solution is obtained in terms of the system's coefficients matrix and vector, Ω and $\mathbf{v}$, and the eigenvalues and eigenvectors of Ω. First, from (A3.2.13) the $\Delta\mathbf{w}_0$ matrix is partitioned:

$$\mathbf{w}(t)=\Gamma_{n,m}e^{\Lambda_{m,m}t}(\Gamma^{-1})_{m,n}\begin{pmatrix}\Delta\mathbf{w}_{0m}\\ \cdots\\ \Delta\mathbf{w}_{0\bar{m}}\end{pmatrix}+\mathbf{w}_{ss}$$

[32] Equation (A3.2.14) is actually not as peculiar as might first appear. Writing the matrix $\Gamma_{n,m}(\Gamma^{-1})_{m,n}$ as Ξ, the expression is seen to be analogous to the eigenvector problem: 'find the vector $\Delta\mathbf{w}_0$ such that $(\Xi\text{-I})\Delta\mathbf{w}_0=\mathbf{0}$' (see Appendix 3.3). Thus $\Delta\mathbf{w}_0$ is the eigenvector of Ξ, corresponding to the only non-zero eigenvalue of unity!

then $\Delta w_{0\bar{m}}$ is substituted from either (A3.2.12) or (A3.2.15); and w_{ss} is substituted out using (A3.2.6) to give the alternative final solutions:

$$w(t) = \Gamma_{n,m} e^{\Lambda_{m,m} t} (\Gamma^{-1})_{m,n} \left(\frac{1}{-[(\Gamma^{-1})_{\bar{m},\bar{m}}]^{-1} (\Gamma^{-1})_{\bar{m},m}}\right) \Delta w_{0m} - \Omega^{-1} v$$

or (A3.2.16)

$$w(t) = \Gamma_{n,m} e^{\Lambda_{m,m} t} (\Gamma^{-1})_{m,n} \left(\frac{1}{\Gamma_{\bar{m},m} (\Gamma_{m,m})^{-1}}\right) \Delta w_{0m} - \Omega^{-1} v$$

A3.2.4 The oscillatory effect of complex-valued roots

For any complex eigenvalue, $\lambda = x + iy$, equation (A3.2.8) generates terms in the dynamic solution of the form:

$$\sum_i \gamma_{ki} e^{xt} e^{iyt} \sum_j \gamma^{I}_{ij} \Delta w_{0j}$$

which, through the use of Euler's relations, can be expressed in trigonometric terms as:

$$\sum_i \gamma_{ki} e^{xt} (\cos yt + i \sin yt) \sum_j \gamma^{I}_{ij} \Delta w_{0j}$$

Oscillatory behaviour of the dynamic solution for complex valued roots follows from the nature of these functions, provided the coefficient term is non-zero. That is, provided the real part x of the complex root λ is negative, so the constraint (A3.2.10) does not apply.

Appendix 3.3

Eigenvalues and eigenvectors of a linear dynamic system

As shown in Appendix 3.2, the solution and concomitant dynamic behaviour of a linear system such as (A3.2.1) depends upon the eigenvalues (or characteristic roots) and the eigenvectors of the system's coefficients matrix. Accordingly, to quantify the system dynamics and to determine the dynamic stability of the Romer model steady-state, these eigenvalues and eigenvectors need to be calculated for the coefficients matrix Ω_R (equation (3.1) of the text)

The eigenvectors and eigenvalues of a (square) matrix Ω are defined by γ and λ respectively, where:

$$\Omega\gamma = \lambda\gamma \quad \text{or,} \quad (\Omega - \lambda I)\gamma = 0 \qquad \text{(A3.3.1)}$$

and for this to have a non-trivial solution γ, the matrix $(\Omega-\lambda I)$ must be singular. That is:

$$|\Omega - \lambda I| = 0 \qquad \text{(A3.3.2)}$$

This is the *characteristic equation* of Ω. It is an n-degree polynomial in λ to which there are n roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ (not necessarily all distinct), called the *characteristic roots* or *eigenvalues* of Ω.

A3.3.1 Eigenvalues of a 3x3 matrix Ω

Writing $\Omega = (\omega_{ij})$ for $i, j = 1,2,3$ and evaluating (A3.3.2) algebraically, the characteristic equation is:

$$\lambda^3 - (\omega_{11} + \omega_{22} + \omega_{33})\lambda^2 + [(\omega_{11}\omega_{22} + \omega_{11}\omega_{33} + \omega_{22}\omega_{33}) - (\omega_{12}\omega_{21} + \omega_{23}\omega_{32} + \omega_{13}\omega_{31})]\lambda$$
$$-[(\omega_{11}\omega_{22}\omega_{33} - \omega_{11}\omega_{23}\omega_{32}) - (\omega_{12}\omega_{21}\omega_{33} - \omega_{12}\omega_{23}\omega_{31}) + (\omega_{13}\omega_{21}\omega_{32} - \omega_{13}\omega_{22}\omega_{31})] = 0$$

or

$$\lambda^3 - (\sum \omega_{ii})\lambda^2 + (\sum_{i \neq j}\sum \omega_{ii}\omega_{jj} - \sum_{i<j}\sum \omega_{ij}\omega_{ji})\lambda - \det(\omega_{ij}) = 0 \qquad \text{(A3.3.3)}$$

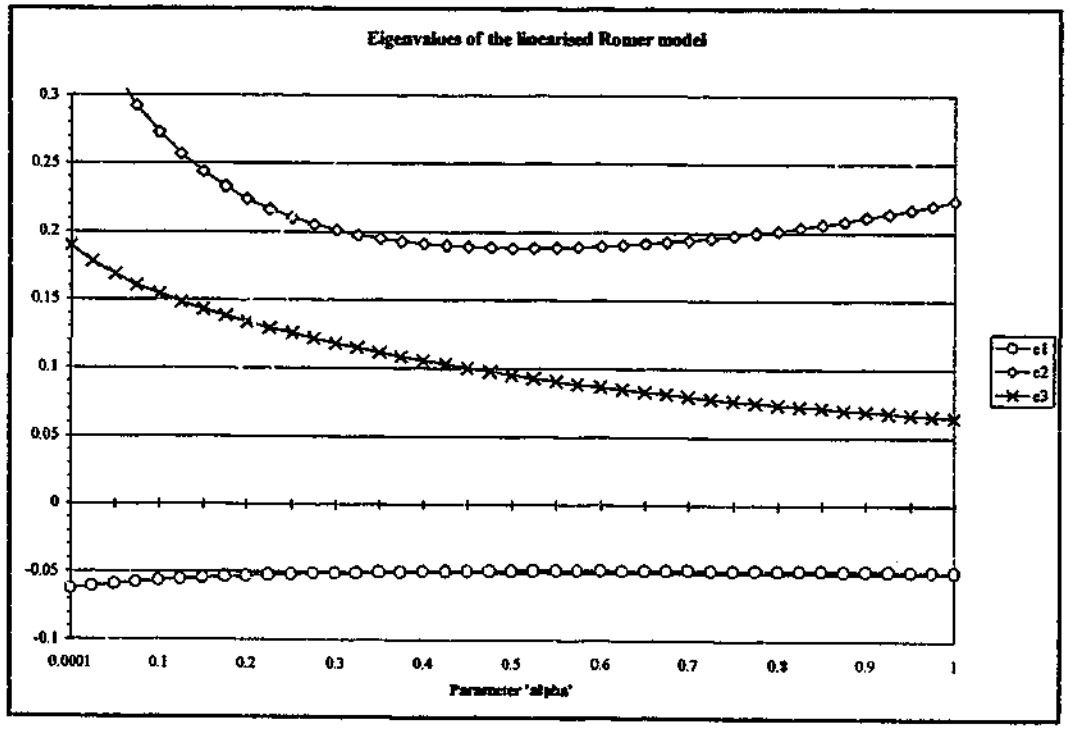
Using the results of Appendix 3.1 and a variety of numerical values for the parameters, this equation has been evaluated for the coefficient matrix Ω_R, of the linearised model, and the resulting *cubics* solved to generate the eigenvalues.[33] The results for the benchmark parameter set were: λ_1=-0.0493, λ_2=0.1899, and λ_3=0.1021; and those for the 'ten per cent rise in γ simulation' (see Section 3.2.2 of the text) were: λ_1=-0.0409, λ_2=0.1686, and λ_3=0.1021.

[33] The method adopted to solve the cubic equations (A3.3.3), in a *Microsoft Excel* spreadsheet, was that described by Press et al (1992). For an alternative method see Abromowitz and Stegun (1965).

An analysis of the variation of the eigenvalues in response to changes in the parameter values was also conducted. The results of this analysis are presented in Figure A3.3.1 to Figure A3.3.15, where the **real parts** of the eigenvalues have been plotted against each individual parameter in turn, for specified values (often the benchmark levels) of the other parameters. The notation "e1, e2, e3" in the legends of these graphs denote the real parts of the corresponding "λ_1, λ_2, λ_3" with which the eigenvalues were denoted in the text.

When the parameters were restricted to values which reflect economic realities, that is they were not *too far* from their benchmark settings, the analysis **strongly suggested** that the dynamic path and the dynamic stability of the model's equilibrium were well behaved in the sense that a single real negative eigenvalue was always returned and so the system approached its steady-state monotonically and with saddle-path stability (Figure A3.3.1 to Figure A3.3.7).[34]

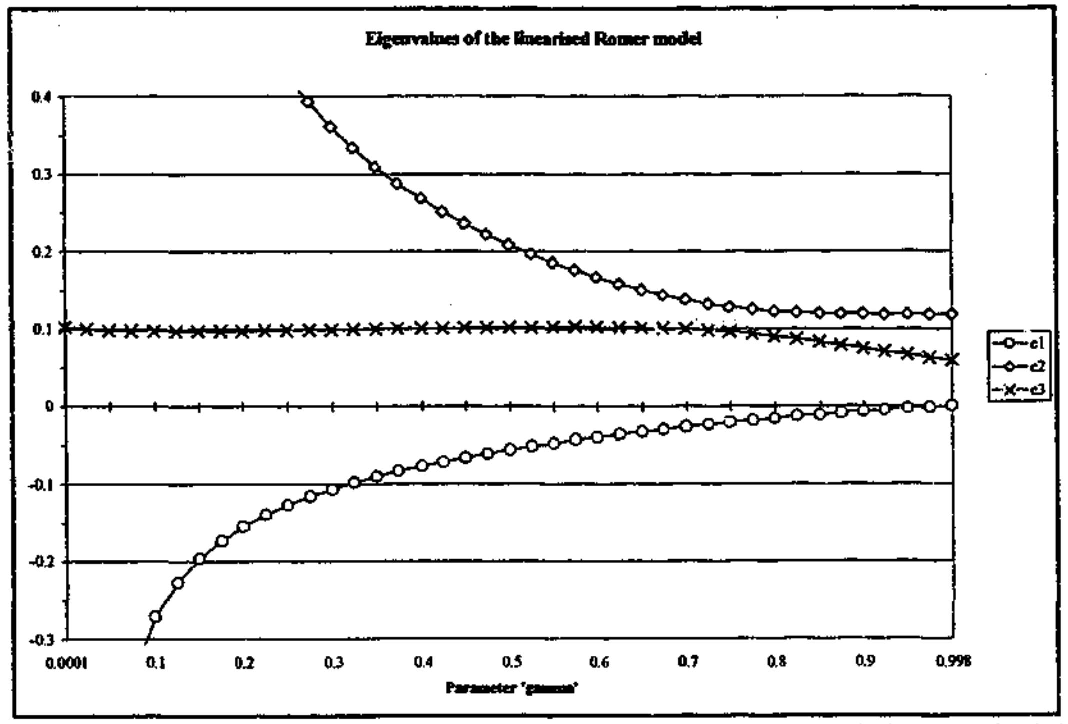
Figure A3.3.1: Variation in the eigenvalues of the linearised Romer model for changes in parameter α; all other parameters at benchmark.



d:\phd\chpt3\r_soln_eigens (Sheet 5/eigenplot1)

[34] That the implications of the analysis are only "strongly suggested", rather than (say) "confirmed", "verified" or "proved" is a reflection of two points; firstly, that it is not possible to check **all** numerical combinations of parameter values, and secondly, that in these types of dynamic systems changes in the limiting dynamic behaviour can be precipitous, fleeting and chaotic. This second point is illustrated in Figure A3.3.11 to Figure A3.3.15.Also, Hahn (1990) notes that even in the familiar econmic models with infinite horizon optimisation, a high enough discount rate can result in optimum paths that are cyclical, or indeeed chaotic.

Figure A3.3.2: Variation in the eigenvalues of the linearised Romer model for changes in parameter γ; all other parameters at benchmark.



d:\phd\chpt3\r_soln_eigens (Sheet 5/eigenplot1)

Figure A3.3.3: Variation in the eigenvalues of the linearised Romer model for changes in parameter ζ; with all other parameters at benchmark.

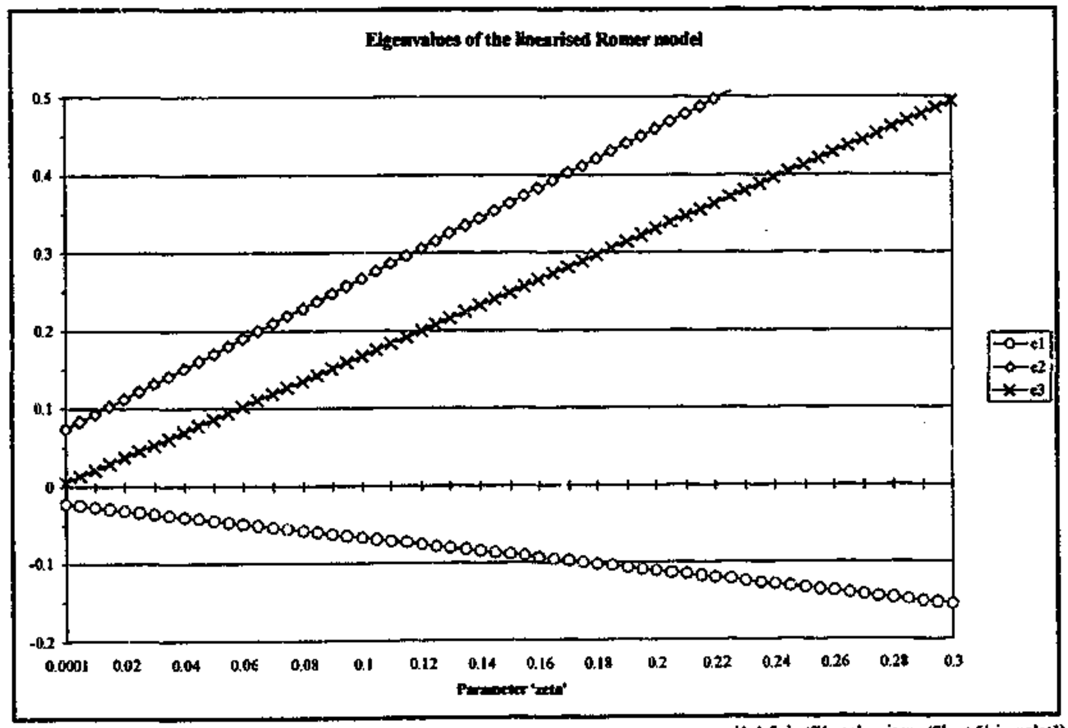


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Figure A3.3.4: Variation in the eigenvalues of the linearised Romer model for changes in parameter ρ; with all other parameters at benchmark.

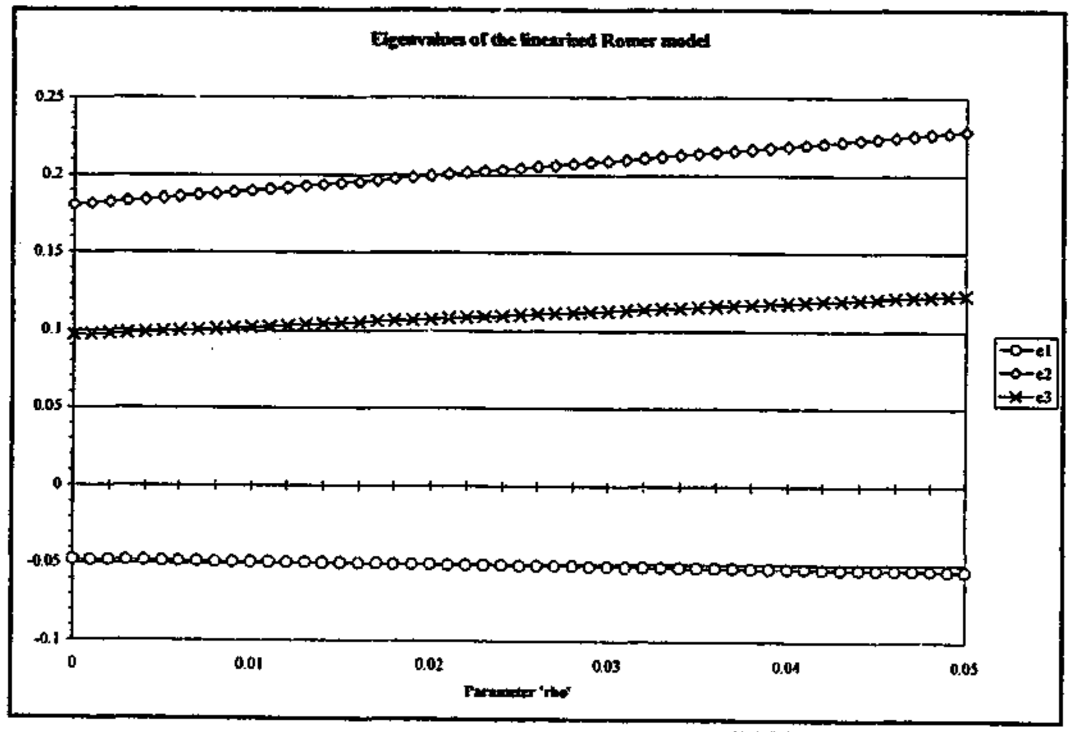

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Figure A3.3.5: Variation in the eigenvalues of the linearised Romer model for changes in parameter σ; with all other parameters at benchmark.

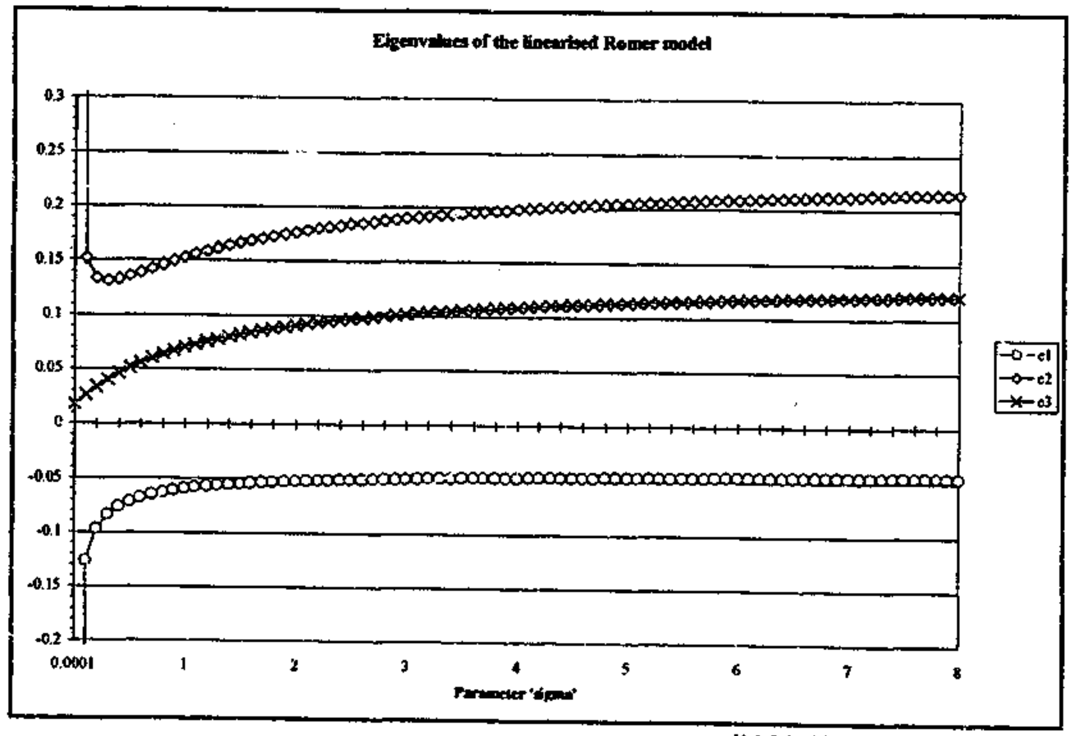

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Figure A3.3.6: Variation in the eigenvalues of the linearised Romer model for changes in parameter δ; with all other parameters at benchmark.

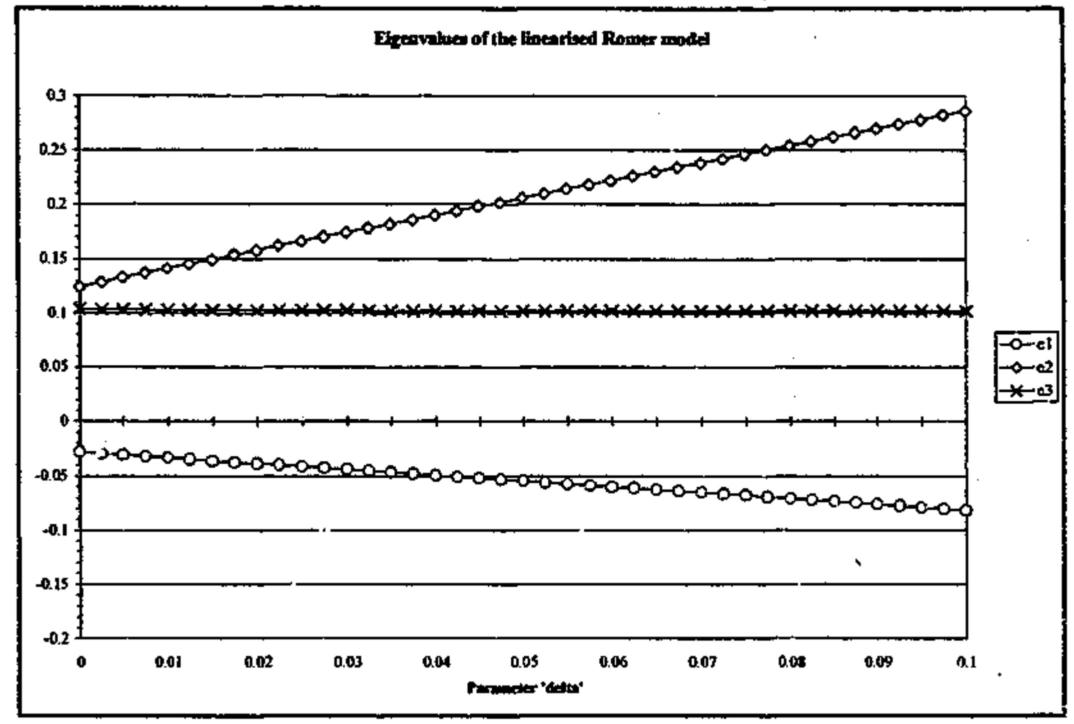

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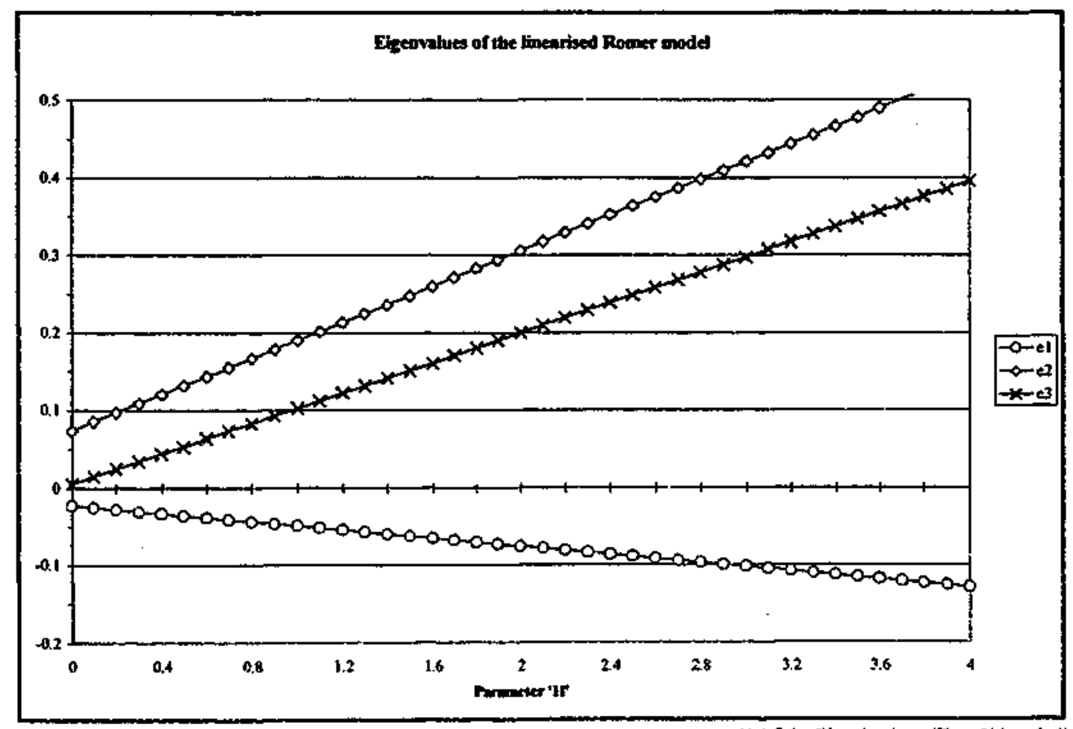
Figure A3.3.7: Variation in the eigenvalues of the linearised Romer model for changes in parameter H; with all other parameters at benchmark.



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Comparison of Figure A3.3.3 and Figure A3.3.7 confirms that the effect of variations to the exogenous variable H (the aggregate level of human capital) on the eigenvalues of the system are multiplicatively identical with those of ζ, the productivity of human capital in research (see footnotes 32 and 33 of Chapter 2). This is because in the steady-state of the system (about which the linearised model is generated) H_A is determined as some fraction of H, and ζ appears only once in the fundamental equations which generate the dynamic system, and there it multiplies H_A (equation (2.1) of the text).

Neither the parameter η nor the exogenous variable L have any effect on the eigenvalues of the system. However, as discussed in Section 2.4, this is a feature of the particular form of the production function (Cobb-Douglas) specified in the model, rather than some unambiguous outcome of the underlying economics.

In the preceding cases the two positive eigenvalues were, like the negative root, also real. A pair of complex conjugates (with positive real parts) could also be returned for certain economically allowable if somewhat implausible or unrealistic parameter settings (Figure A3.3.8 to Figure A3.3.10).

Figure A3.3.8: Variation in the eigenvalues of the linearised Romer model for changes in parameter α; with δ=0; and all other parameters at benchmark.

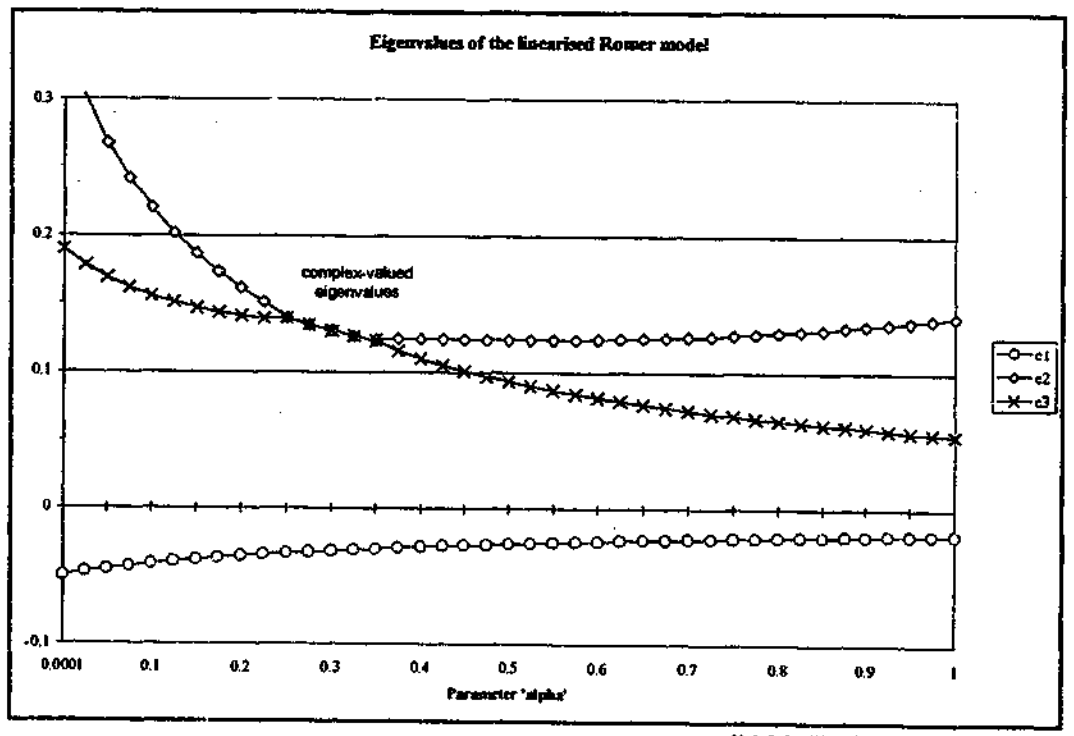

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Figure A3.3.9: Variation in the eigenvalues of the linearised Romer model for changes in parameter γ; with α=0.22; and all other parameters at benchmark.

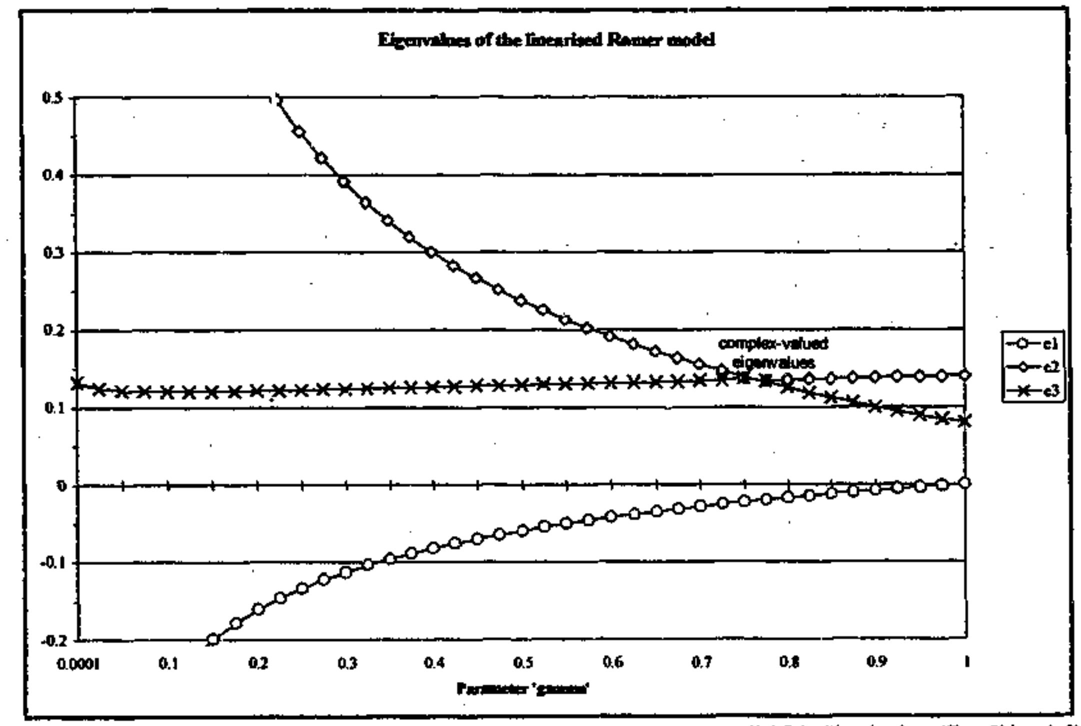

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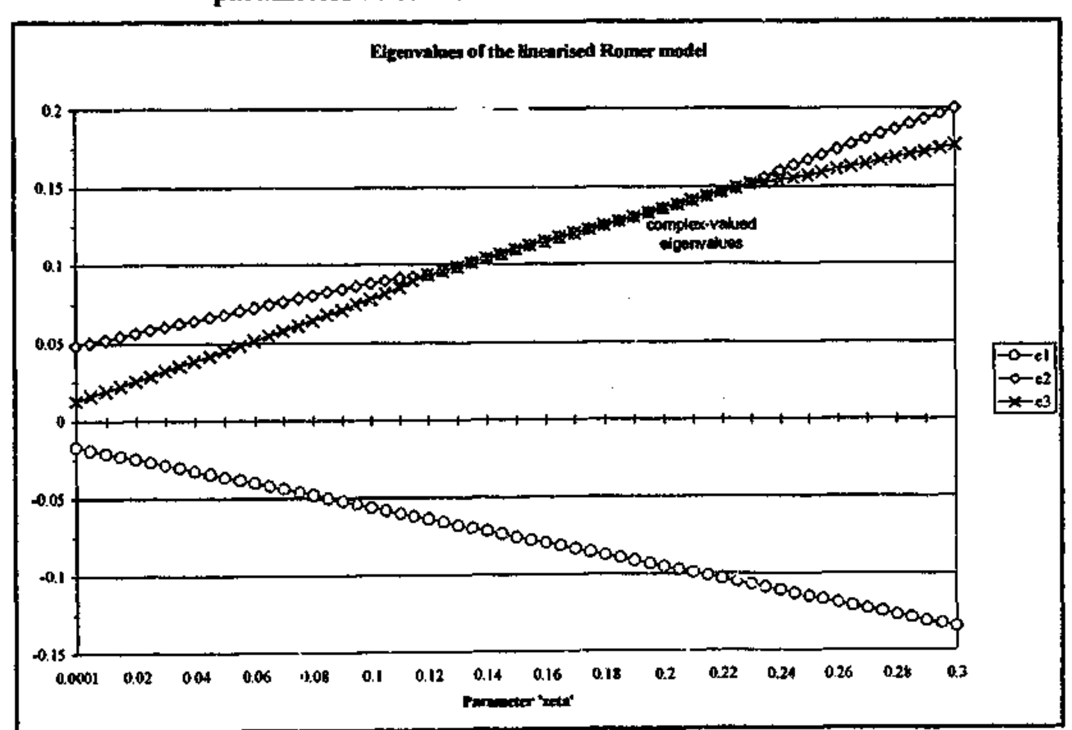
Figure A3.3.10: Variation in the eigenvalues of the linearised Romer model for changes in parameter ζ; with σ=0.5; δ=0.01; and all other parameters at benchmark.



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Mathematically, it was also possible for the system to be made to return sets of eigenvalues which, if valid, would indicate very different dynamics and stability properties to those implied by the optimisation. For example, it was possible to generate:

- three positive real roots, which would indicate a totally unstable equilibrium (Figure A3.3.11); or
- two real negative roots, so the system would follow a *saddle-surface* thereby exhibiting greater stability than for the saddle-path (Figure A3.3.12 to Figure A3.3.15); or
- a pair of complex conjugate roots with negative real parts, which would cause the steady-state to be approached along a bizarre *oscillatory saddle-surface* - see Appendix 3.2 (also Figure A3.3.12 to Figure A3.3.15)

However, all of these 'outcomes' are economically meaningless: In Figure A3.3.11 and Figure A3.3.12 the 'odd behaviour' only arises for invalid parameter values; while in other cases (Figure A3.3.13 to Figure A3.3.15) the occurrence of two negative roots, whether real or complex, was always associated with negative steady-state consumption (as well as other, merely implausible outcomes)! Nevertheless, these 'results' have been included here to illustrate the sometimes unpredictable nature and extreme sensitivity of these types of dynamic systems. It might be noted in this regard that non-linear dynamic systems are likely to exhibit such traits much more strongly, with chaotic behaviour also a possibility. Necessary conditions for chaotic behaviour of a dynamic system are that there are at least three independent dynamic variables; and that the 'equations of motion' contain at least one non-linear term coupling some of the variables (Baker and Gollub, 1990).

Figure A3.3.11: Variation in the eigenvalues of the linearised Romer model for changes in parameter γ as it exceeds unity; with all other parameters at benchmark.

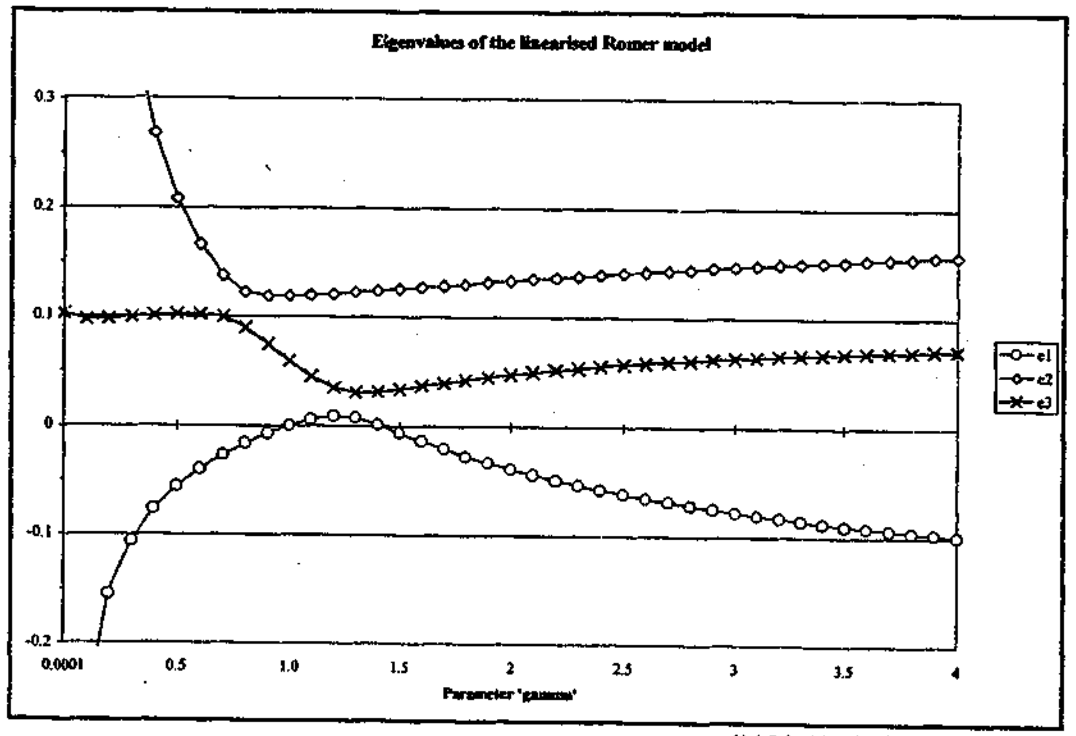

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Figure A3.3.12: Variation in the eigenvalues of the linearised Romer model for changes in parameter α as it exceeds unity; with all other parameters at benchmark.

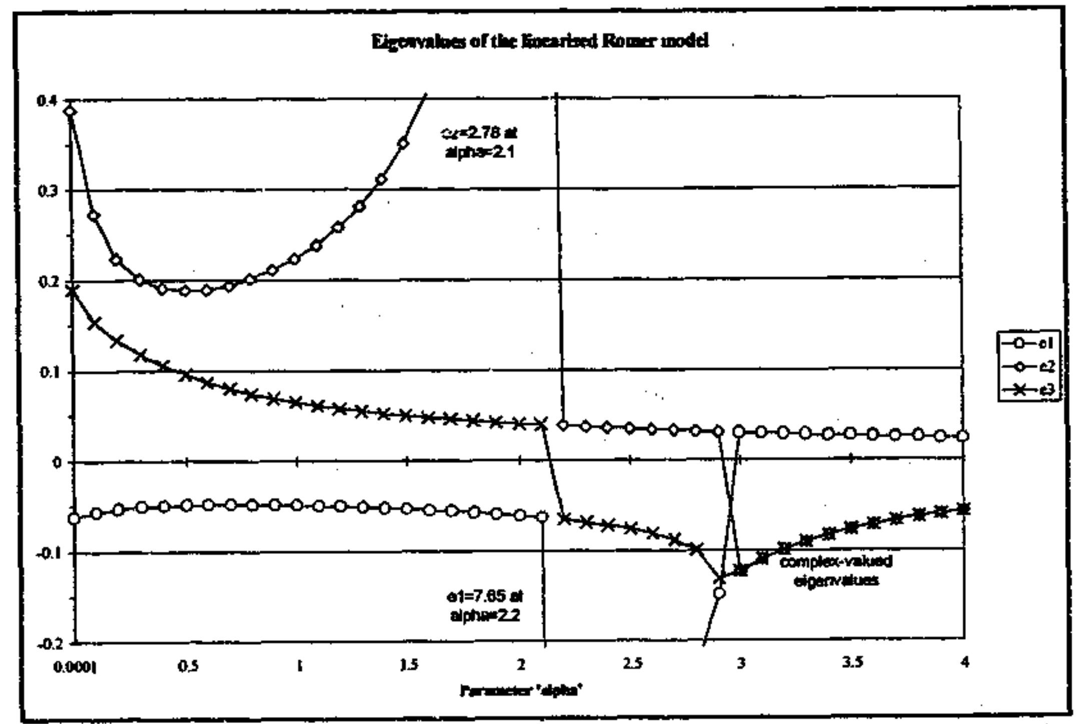

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Figure A3.3.13: Variation in the eigenvalues of the linearised Romer model in response to changes in parameter γ; with α=0.22; ζ=0.2; ρ=0.005; σ=0.3; δ=0.01; and all other parameters at benchmark.

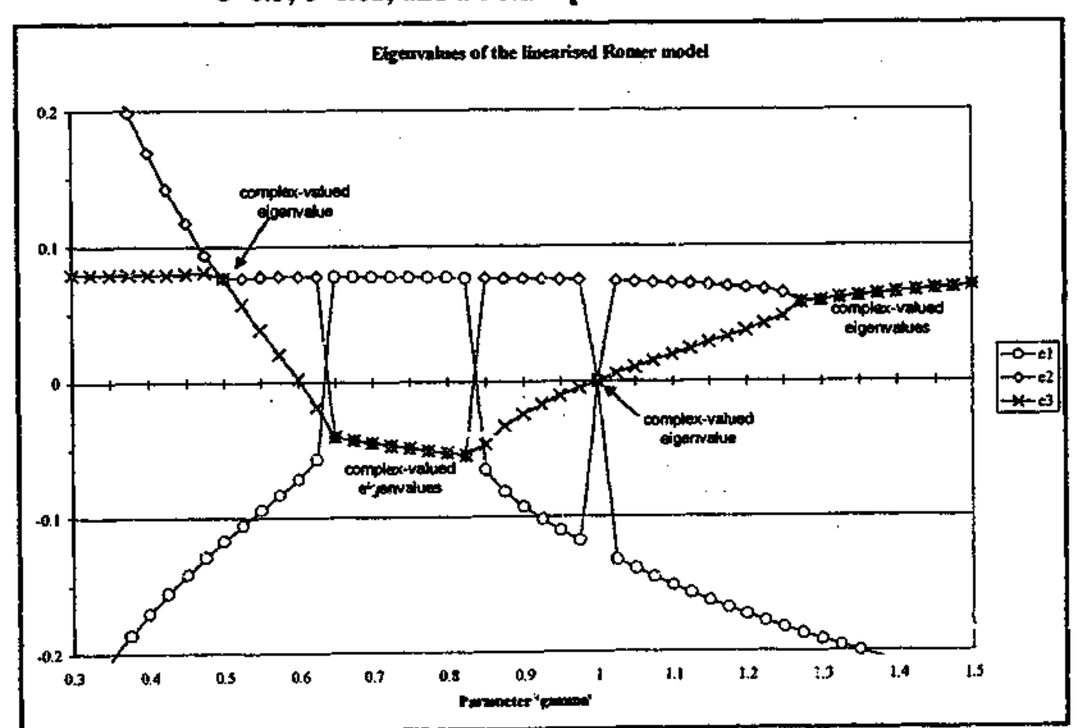

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Figure A3.3.14: Variation in the eigenvalues of the linearised Romer model for changes in parameter ρ; with α=0.2; γ=0.7; ζ=0.2; σ=0.3; δ=0.01; and all other parameters at benchmark.

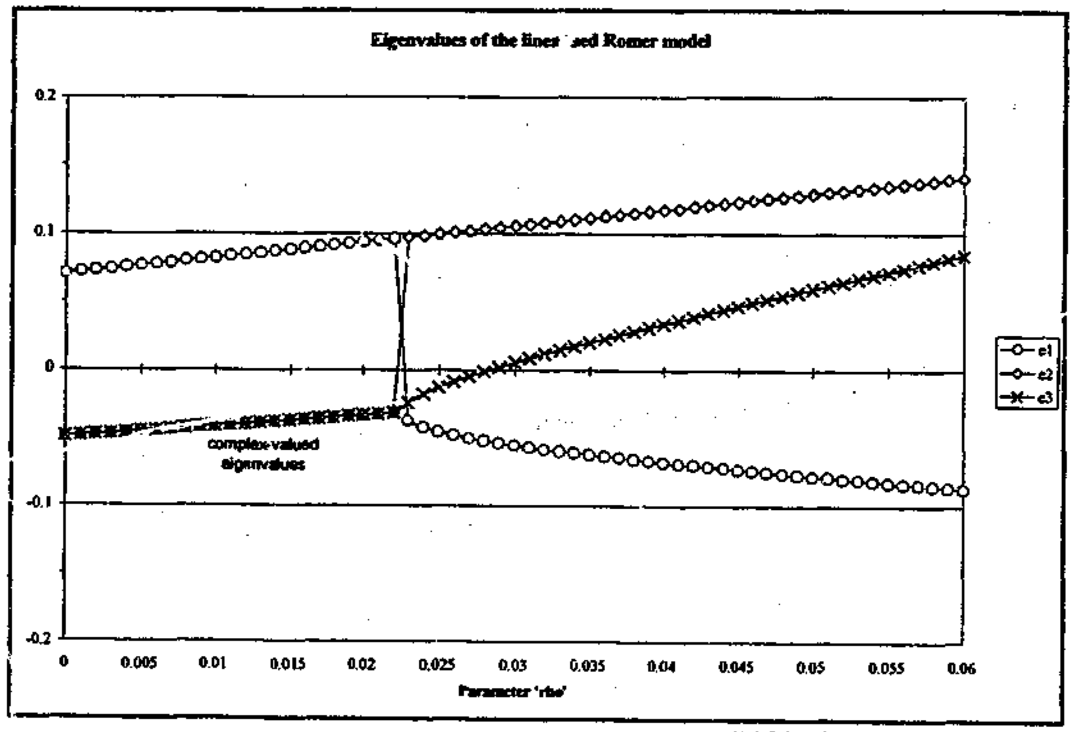

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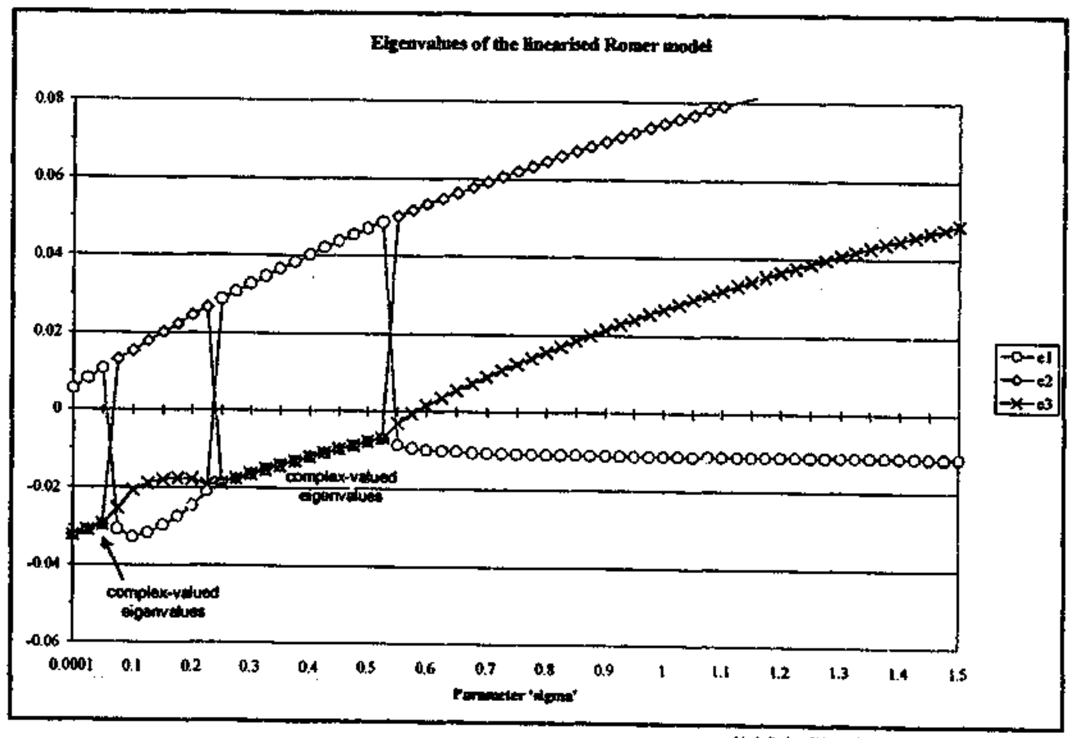
Figure A3.3.15: Variation in the eigenvalues of the linearised Romer model for changes in parameter σ; with γ=0.8; ζ=0.07; ρ=0.003; δ=0; and all other parameters at benchmark.



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A3.3.2 Eigenvectors of a 3x3 matrix Ω

Since there are three eigenvalues and three eigenvectors, equation (A3.3.1) may be written:

$$(\Omega - \lambda_i \mathbf{I})\gamma_i = \mathbf{0} \quad \text{for} \quad \gamma_i = (\gamma_{1i}, \gamma_{2i}, \gamma_{3i})^T, \quad i = 1,2,3$$

From the singularity of the matrix (Ω-λ_i**I**) and the consequent linear dependence of its rows and columns, there is only distinct information from two rows or columns. Taking (arbitrarily) the first two rows, denoting the elements of (Ω-λ_i**I**) as (a_{ij}), and imposing the usual normalisation condition to obtain specific eigenvectors generates the three equations:

$$a_{11}\gamma_{1i} + a_{12}\gamma_{2i} + a_{13}\gamma_{3i} = 0$$
$$a_{21}\gamma_{1i} + a_{22}\gamma_{2i} + a_{23}\gamma_{3i} = 0$$
$$\gamma_{1i}^2 + \gamma_{2i}^2 + \gamma_{3i}^2 = 1$$

which, as for the eigenvalues, can be solved for particular data-sets of parameter values to obtain numerical results for the eigenvectors $\gamma_i = (\gamma_{1i}, \gamma_{2i}, \gamma_{3i})^T$; and thus for the related coefficients γ_{ij}^1 (see Appendix 3.2) which appear in the initial value constraints specified by equation (3.4) in the text. As for the calculation of the eigenvalues, this was performed in a *Microsoft Excel* spreadsheet (see footnote 33).

Appendix 3.4

Anticipated shocks: Analytic determination with the linearised model

A3.4.1 Algebraic derivation

The behaviour of dynamic systems such as that of the Romer model in response to shocks which are either *'unanticipated'* or *'anticipated'* is described in detail in Section 3.3.3 of the text.[35] That material ought probably be read before the rest of this appendix since only a very much more basic explanation is offered here.

Suppose an impending shock which is to be imposed at some time t=I say, is anticipated at earlier dates. The time at which it is first anticipated will be referred to as the time of *'announcement'* and taken to be t=0. Then, from time t=0 to t=I the system will continue to evolve according to the 'pre-shock' dynamic equations, while from the instant of imposition its dynamics will be determined by the 'post-shock equations'. In particular, since the system must approach the post-shock equilibrium its dynamics from t=I must be along the post-shock saddle-path. Since the imposition comes as no surprise there will be no discontinuities in the dynamic variables ($\Psi(t)$, $\Phi(t)$, and $p_A(t)$) at that time. Thus, at t=I both the pre-shock equations and the post-shock equations must be satisfied by the same point, namely [$\Psi(I)$, $\Phi(I)$, $p_A(I)$]. However, the **announcement** of the shock does come as a surprise (to the agents whose actions are described by the dynamic system) and so discontinuities in the 'jumping variables' Φ and p_A may be expected. It is the precise extent of such jumps that allow the system, evolving under the influence of the pre-shock equations, to reach the post-shock saddle-path at time t=I, and it is these which must be determined to calculate the dynamic response of the system to the anticipated shock.

Equation (A3.2.8) from Appendix 3.2 describes the general dynamics of the linear system (A3.2.1) with arbitrary boundary conditions. That is, without the constraints on initial values necessary to ensure satisfaction of the transversality conditions and adjustment along a saddle-path towards the (optimal) equilibrium or steady-state. This formulation is appropriate to the pre-shock dynamic response to an anticipated shock. On the other hand, equation (A3.2.15) specifies those necessary constraints on initial values and is appropriate to the post-shock adjustment. However, in adapting (A3.2.15) to describe the post-shock constraints, its "0" subscript notation, which refers to initial time, needs to be interpreted as referring to time t=I. This is because the post-shock adjustment begins at t=I. Then, using "P" and "F" subscripts to denote the pre- and post-shock equations, and otherwise maintaining the notation of Appendix 3.2, in the current context equations (A3.2.8) and (A3.2.15) become:

[35] The key characteristic of such a system is that it is constrained to move towards some equilibrium with saddle-path stability. In most economic models this is the outcome of optimising behaviour on the part of economic agents (Chapter 2).

$$\mathbf{w}(t) = (\Gamma_P)e^{\Lambda_P t}(\Gamma_P)^{-1}\Delta_{Pw_0} + \mathbf{w}_{Pss} \quad \text{where, } \Delta_{Pw_0} = (\mathbf{w}_0 - \mathbf{w}_{Pss}) \qquad \text{(A3.4.1)}$$

and

$$\mathbf{w}_{\bar{m}}(I) = \mathbf{w}_{Fss\bar{m}} + (\Gamma_F)_{\bar{m},m}[(\Gamma_F)_{m,m}]^{-1}(\mathbf{w}_m(I) - \mathbf{w}_{Fssm}) \qquad \text{(A3.4.2)}$$

Partitioning (A3.4.1) into the non-jumping and jumping variables (according to the negative and the non-negative eigenvalues) generates:

$$\begin{pmatrix} \mathbf{w}_m(I) \\ \hdashline \mathbf{w}_{\bar{m}}(I) \end{pmatrix} = (\Gamma_P)e^{\Lambda_P t}(\Gamma_P)^{-1}\begin{pmatrix} \mathbf{w}_{0m} - \mathbf{w}_{Pssm} \\ \hdashline \mathbf{w}_{0\bar{m}} - \mathbf{w}_{Pss\bar{m}} \end{pmatrix} + \mathbf{w}_{Pss} \qquad \text{(A3.4.3)}$$

allowing $\mathbf{w}_{\bar{m}}(I)$ on the left to be substituted by (A3.4.2). And, given the initial values of the non-jumping variables $\mathbf{w}_{0m}$, the resulting matrix expression represents a system of n linear equations in the n variables $\mathbf{w}_m(I)$ plus $\mathbf{w}_{0\bar{m}}$, which can readily be solved for. These then represent all the 'initial values' necessary for (A3.4.1) and (A3.4.2) to be used to compute the complete dynamic response of the linear system to the anticipated shock:

- the $\mathbf{w}_0$ are first used in (A3.4.1) to compute the path over t=0 to t=I;
- then, either the $\mathbf{w}_m(I)$ can be used with (A3.4.2) to calculate the $\mathbf{w}_{\bar{m}}(I)$ also, or the first step results for t=I used to provide the $\mathbf{w}(I)$ directly; and finally
- the $\mathbf{w}(I)$ are used as initial values in a post-shock version of (A3.4.1).[36]

In terms of the three variable Romer model dynamic system (A3.4.3) and (A3.4.2) give:

$$\begin{pmatrix} \Psi(I) \\ \hdashline \Phi_{Fss} + \dfrac{\gamma_{F21}}{\gamma_{F11}}[\Psi(I) - \Psi_{Fss}] \\ \hdashline p_{A_{Fss}} + \dfrac{\gamma_{F31}}{\gamma_{F11}}[\Psi(I) - \Psi_{Fss}] \end{pmatrix} = (\Gamma_P)e^{\Lambda_P t}(\Gamma_P)^{-1}\begin{pmatrix} \Psi_0 - \Psi_{Pss} \\ \hdashline \Phi_0 - \Phi_{Pss} \\ \hdashline p_{A_0} - p_{A_{Pss}} \end{pmatrix} + \begin{pmatrix} \Psi_{Pss} \\ \hdashline \Phi_{Pss} \\ \hdashline p_{A_{Pss}} \end{pmatrix} \qquad \text{(A3.4.4)}$$

where given Ψ_0, the only unknowns are Φ_0, p_{A0} and $\Psi(I)$. To solve for these write the matrix $(\Gamma_P)e^{\Lambda_P t}(\Gamma_P)^{-1}$ as $\mathbf{M} = (m_{ij})$, expand the first term on the right hand side of (A3.4.4) accordingly, and collect all the unknowns on the left hand side. This produces:

$$\mathbf{A}\begin{pmatrix} \Psi(I) \\ \hdashline \Phi_0 \\ \hdashline p_{A_0} \end{pmatrix} = \begin{pmatrix} m_{11} \\ \hdashline m_{21} \\ \hdashline m_{31} \end{pmatrix}\Psi_0 + \mathbf{B}\begin{pmatrix} \Psi_{Pss} \\ \hdashline \Phi_{Pss} \\ \hdashline p_{A_{Pss}} \end{pmatrix} + \begin{pmatrix} 0 \\ \hdashline (\gamma_{F21}/\gamma_{F11})\Psi_{Fss} - \Phi_{Fss} \\ \hdashline (\gamma_{F31}/\gamma_{F11})\Psi_{Fss} - p_{A_{Fss}} \end{pmatrix} \qquad \text{(A3.4.5)}$$

where the matrices **A** and **B** are given by:

[36] In practice the positive eigenvalues in the matrix Λ_F are also replaced by zeros to avoid any problems with rounding errors in the computation of the constraints from (A3.4.2).

$$\mathbf{A}=\left(\begin{array}{c:c:c} 1 & -m_{12} & -m_{13} \\ \hdashline \dfrac{\gamma_{F21}}{\gamma_{F11}} & -m_{22} & -m_{23} \\ \hdashline \dfrac{\gamma_{F31}}{\gamma_{F11}} & -m_{32} & -m_{33} \end{array}\right) \text{ and } \mathbf{B}=\left(\begin{array}{c:c:c} (1-m_{11}) & -m_{12} & -m_{13} \\ \hdashline -m_{21} & (1-m_{11}) & -m_{23} \\ \hdashline -m_{31} & -m_{32} & (1-m_{11}) \end{array}\right) \quad \text{(A3.4.6)}$$

Thus, the solution of (A3.4.5) for the unknowns is:

$$\begin{pmatrix} \Psi(I) \\ \hdashline \Phi_0 \\ \hdashline p_{A_0} \end{pmatrix} = \mathbf{A}^{-1}\left[\begin{pmatrix} m_{11} \\ \hdashline m_{21} \\ \hdashline m_{31} \end{pmatrix}\Psi_0 + \mathbf{B}\begin{pmatrix} \Psi_{Pss} \\ \hdashline \Phi_{Pss} \\ \hdashline p_{A_{Pss}} \end{pmatrix} + \begin{pmatrix} 0 \\ \hdashline (\gamma_{F21}/\gamma_{F11})\Psi_{Fss} - \Phi_{Fss} \\ \hdashline (\gamma_{F31}/\gamma_{F11})\Psi_{Fss} - p_{A_{Fss}} \end{pmatrix}\right] \quad \text{(A3.4.7)}$$

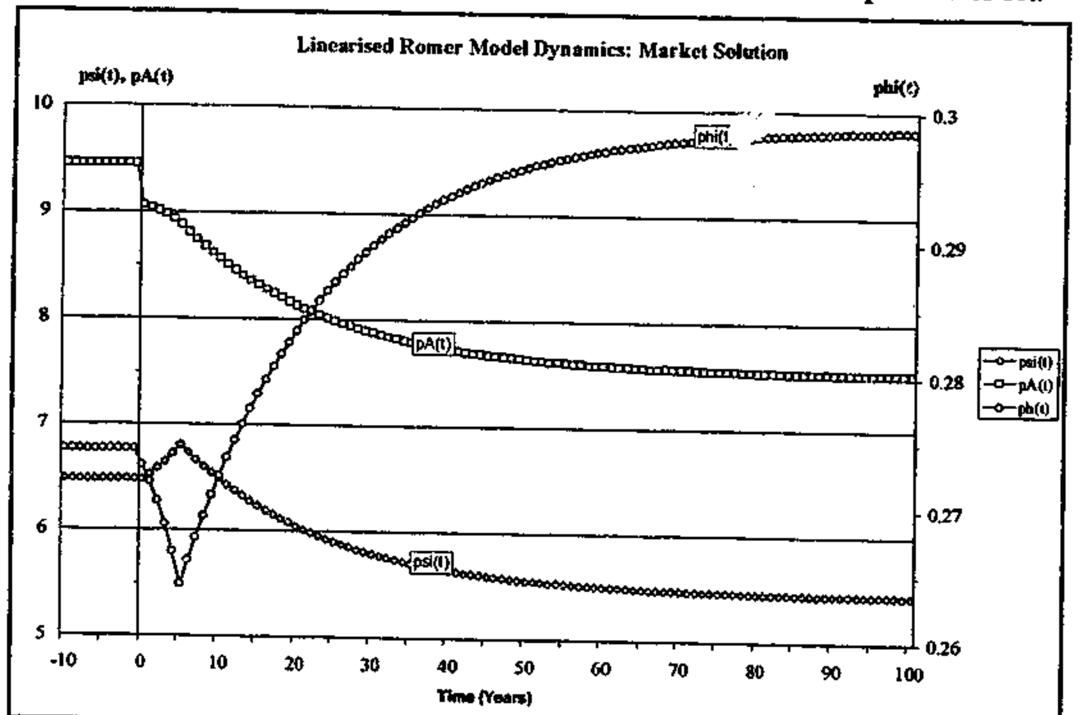
A3.4.2 Computations of adjustment paths

Equation (A3.4.7) is applied, according to the method described by the 'dot points' on the previous page, to compute the adjustment paths for two anticipated shocks. As in the simulation of the unanticipated shock to parameter γ, presented in Section 3.2.2, the system is considered to be initially in equilibrium at its benchmark steady-state.

A3.4.2.1 A sustained 15 per cent rise in parameter ζ

While the shock is not imposed until time t=5, in this simulation it is considered to be (correctly) anticipated from time t=0.

Figure A3.4.1: Dynamic effects on Ψ, Φ, and p_A of an anticipated and sustained 15% rise in parameter ζ from time t=5, benchmark parameter set.



d:\phd\chpt3\anticptd_lin.xlw (psi_z)

Figure A3.4.2: Dynamic effects on the growth rates of an anticipated and sustained 15% rise in parameter ζ from time t=5, benchmark parameter set.

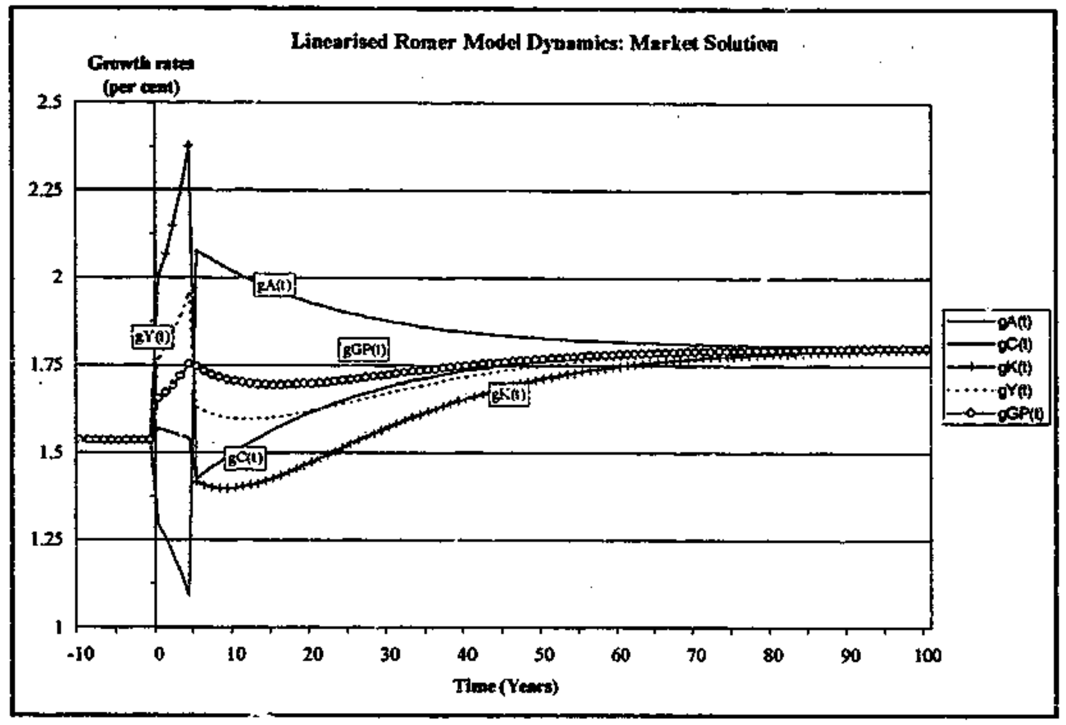


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Figure A3.4.3: Dynamic effects on the interest rate and share of human capital devoted to research of an anticipated and sustained 15% rise in parameter ζ from time t=5, benchmark parameter set.

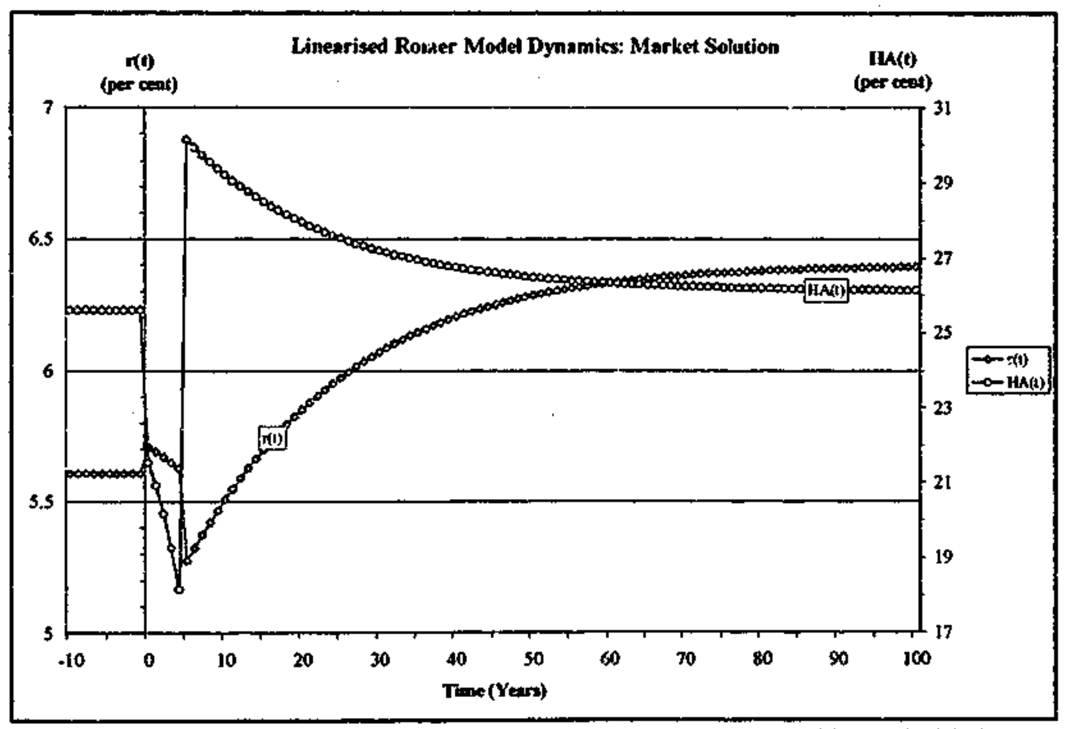

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Figure A3.4.4: Dynamic effects on the savings rates and the capital-output ratio (s_B, s_N, and k_{GP}, respectively) of an anticipated and sustained 15% rise in parameter ζ from time t=5, benchmark parameter set.

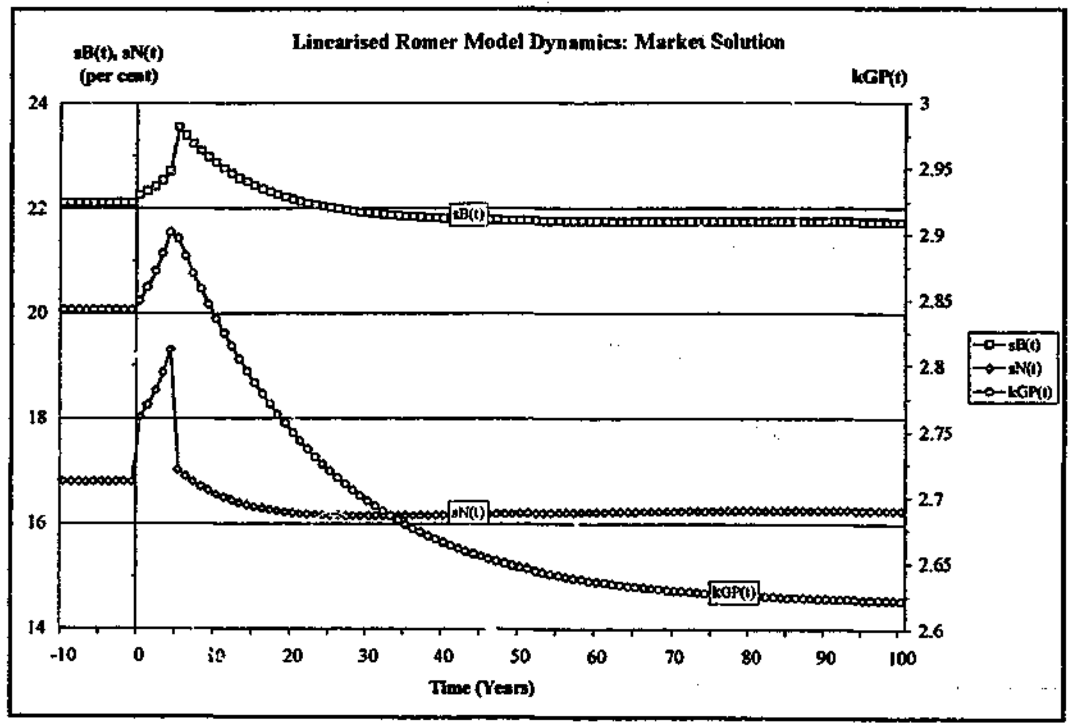

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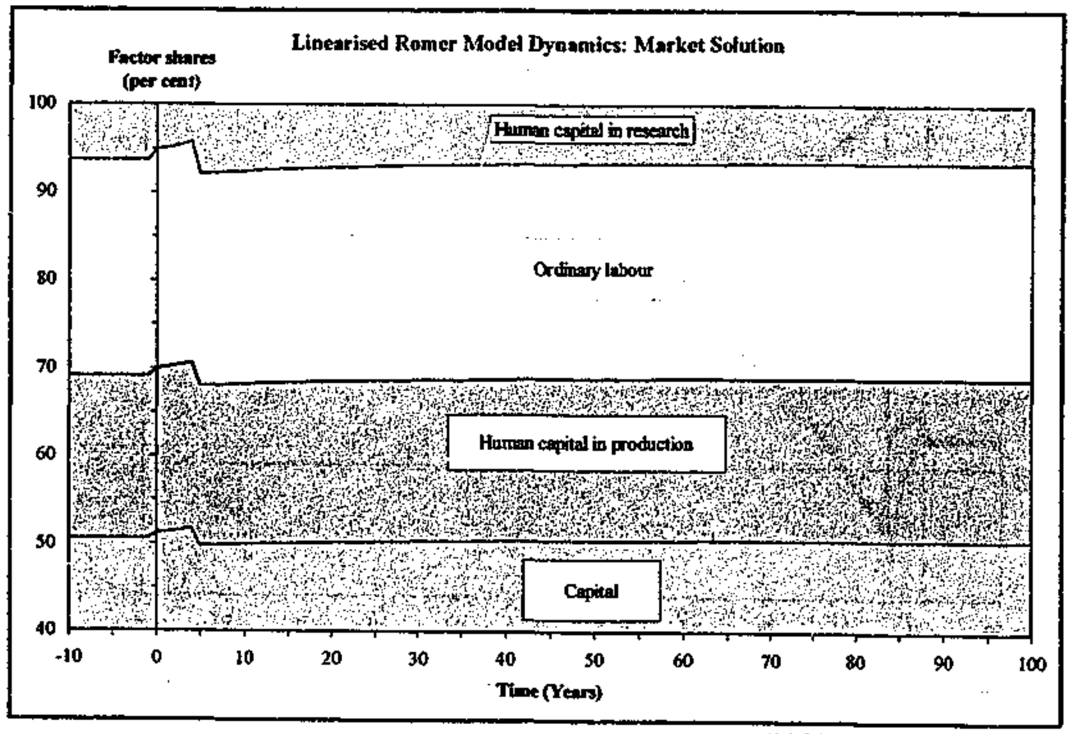
Figure A3.4.5: Dynamic effects on the factor shares of gross income from an anticipated and sustained 15% rise in parameter ζ from time t=5, benchmark parameter set.



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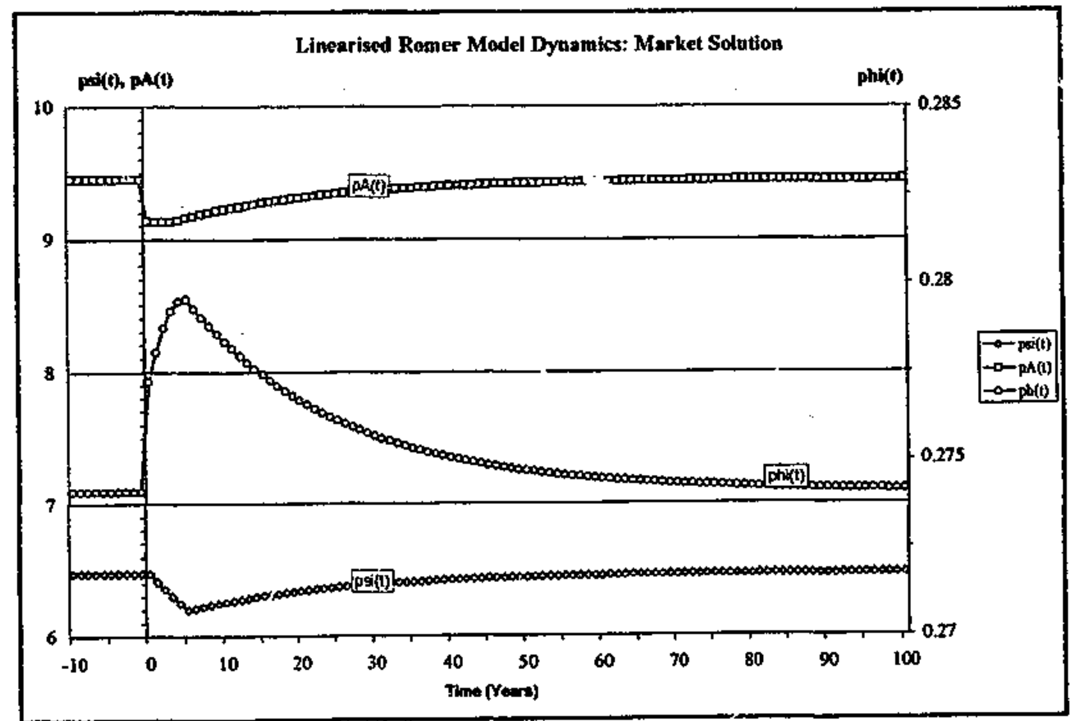
A3.4.2.2 A temporary (five year) fall of 20 per cent in parameter α.

Here the shock is implemented at time t=0 and removed at time t=5. While its implementation is unanticipated, its removal is (again correctly) anticipated from the outset. Thus, the anticipated shock simulated here is actually the increase in parameter α necessary to return it to its benchmark level.

In effect, two shocks are being simulated simultaneously. The first is an unanticipated 20 per cent reduction in the level of parameter α implemented at time t=0; and the second is a 25 per cent increase in parameter α, implemented at time t=5 but anticipated from time t=0.

Figure A3.4.6: Dynamic effects on Ψ, Φ, and p_A of a temporary 20% fall in parameter α from time t=0 to t=5, benchmark parameter set.



d:\phd\chpt3\anticptd_lin.xlw (psi_a)

Figure A3.4.7: Dynamic effects on growth rates of a temporary 20% fall in parameter α from time t=0 to t=5, benchmark parameter set.

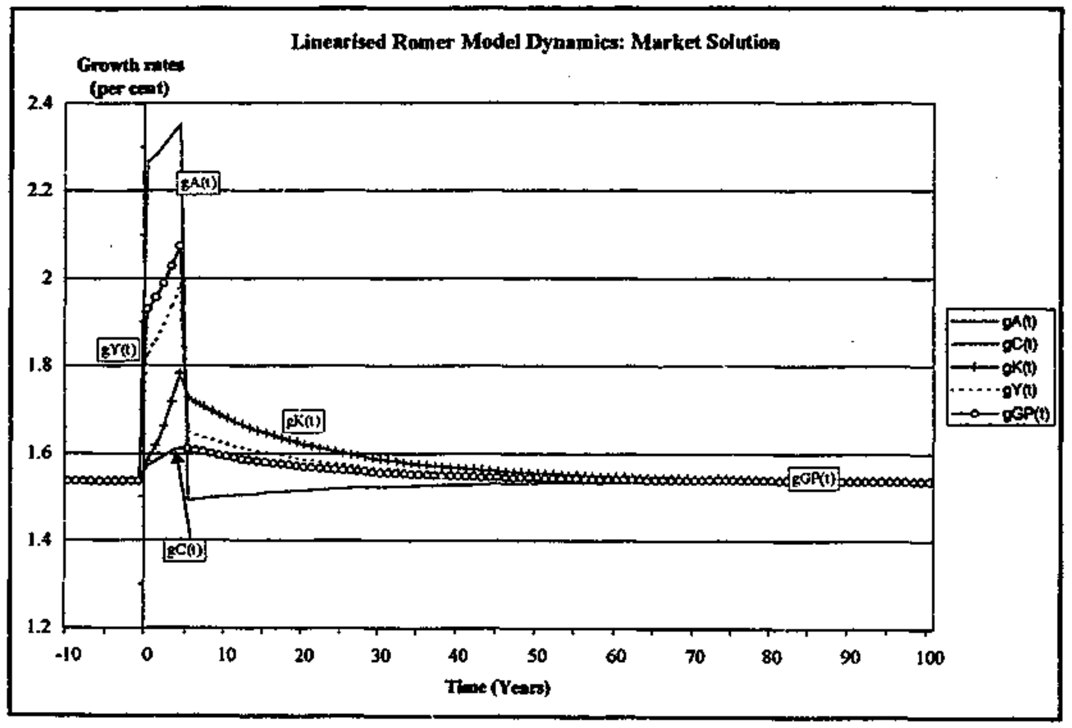


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Figure A3.4.8: Dynamic effects on the interest rate and share of human capital devoted to research of a temporary 20% fall in parameter α from time t=0 to t=5, benchmark parameter set.

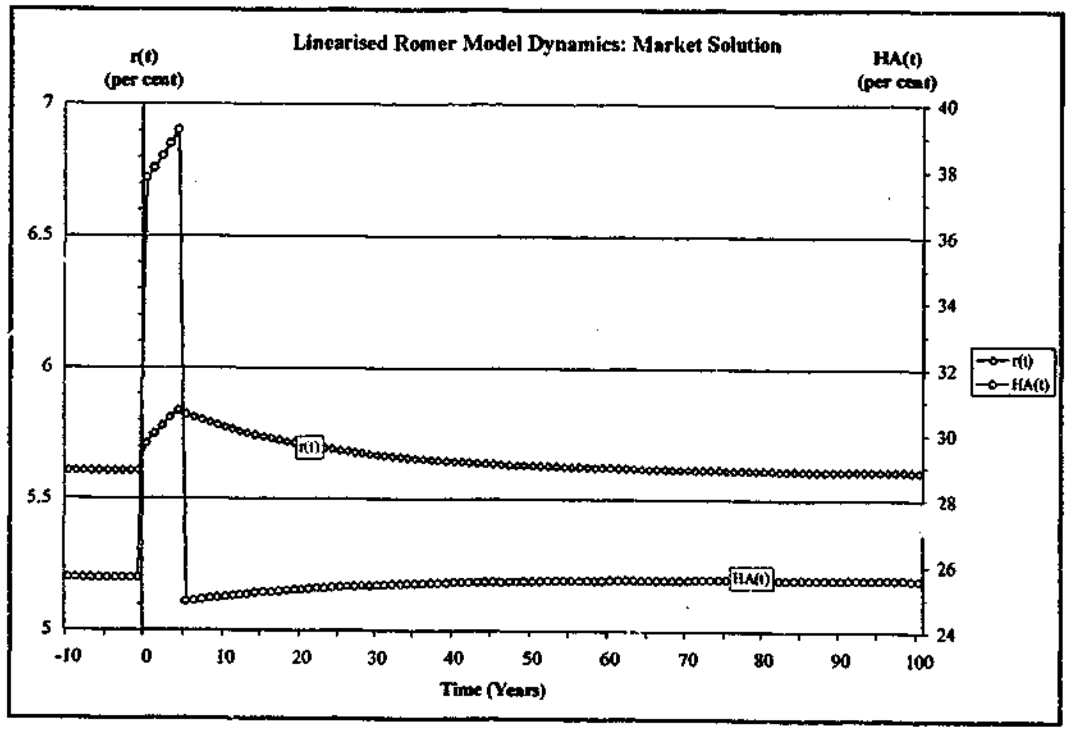

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Figure A3.4.9: Dynamic effects on the savings rates and the capital-output ratio (s_B, s_N, and k_{GP}, respectively) of a temporary 20% fall in parameter α from time t=0 to t=5, benchmark parameter set.

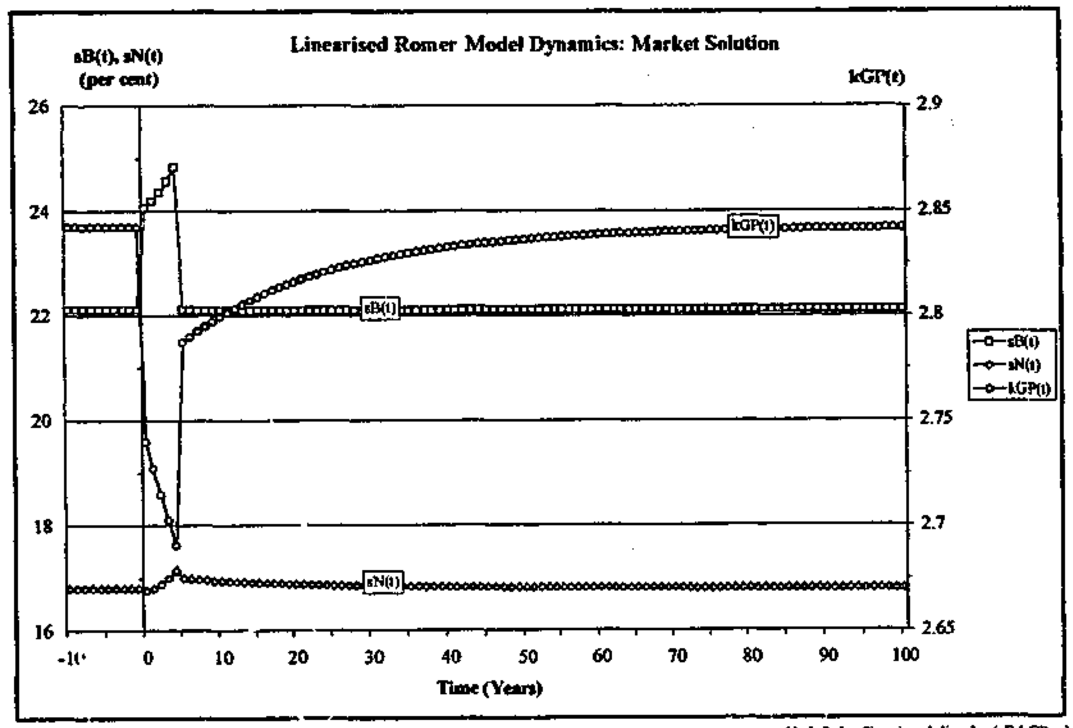

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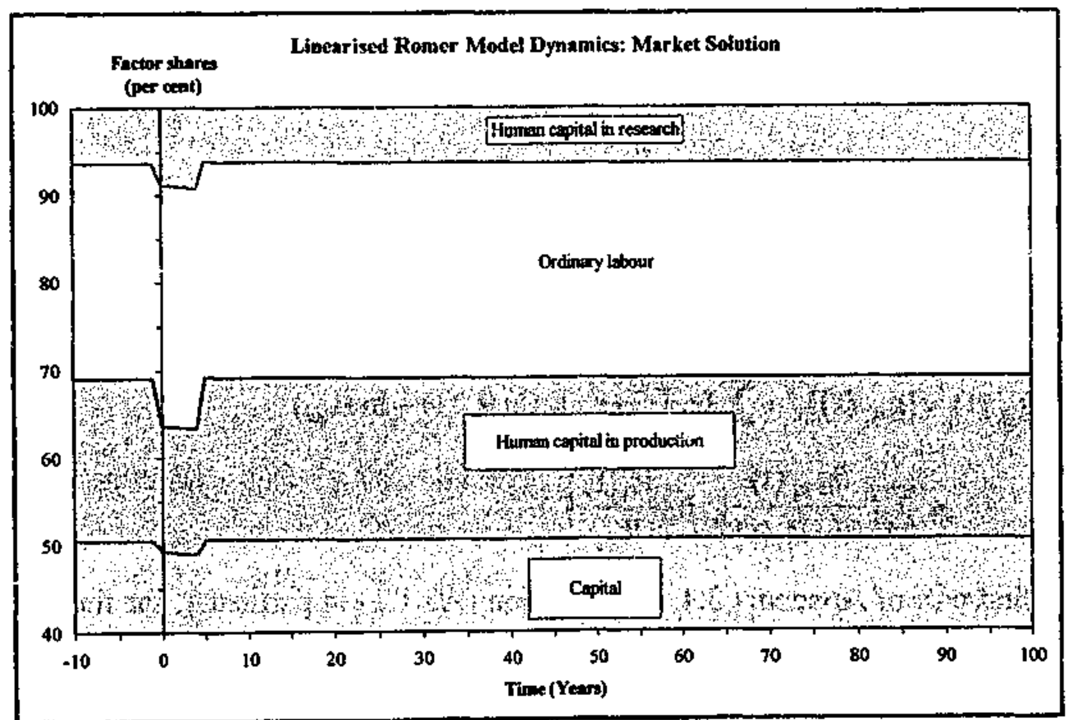
Figure A3.4.10: Dynamic effects on the factor shares of gross income of a temporary 20% fall in parameter α from time t=0 to t=5, benchmark parameter set.



d:\phd\chpt3\anticptd_lin.xlw (GPshares_a)

Appendix 3.5

Log-linearisation of the Romer model

A3.5.1 Deriving the log-linear coefficients matrix

In the ordinary linearisation of the model (Appendix 3.1), the time derivatives of the dynamic variables were expressed as (approximate) functions of the variables themselves. Analogously, here the procedure is to write the time derivatives of the logged variables as approximate functions of the logged variables. First, note that:

$$\frac{\dot{R}_i(t)}{R_i(t)} = g_{Ri}(t) = \frac{d}{dt}\ln R_i(t) \qquad \text{(A3.5.1)}$$

where i=1, 2, 3 and $R_1=\Psi$, $R_2=\Phi$, and $R_3=p_A$

Next, write equations (2.41) to (2.43) as:

$$g_\Psi(t) = \frac{1}{\gamma^2}e^{\ln[r(t)+\delta]} - e^{\ln\Phi(t)} + \zeta e^{\ln H_Y(t)} - (\delta+\zeta H) \qquad \text{(A3.5.2)}$$

$$g_\Phi(t) = (\frac{1}{\sigma}-\frac{1}{\gamma^2})e^{\ln[r(t)+\delta]} + e^{\ln\Phi(t)} + \delta(1-\frac{1}{\sigma}) - \frac{\rho}{\sigma} \qquad \text{(A3.5.3)}$$

$$g_{p_A}(t) = e^{\ln[r(t)+\delta]} - \frac{1-\gamma}{\gamma}e^{\ln[r(t)+\delta]+\ln\Psi(t)-\ln p_A(t)} - \delta \qquad \text{(A3.5.4)}$$

Then take logs of equations (2.44) and (2.45):

$$\ln H_Y(t) = \frac{\gamma}{1-\alpha(1-\gamma)}\ln\Psi - \frac{1}{1-\alpha(1-\gamma)}\ln p_A(t) + \text{ constant} \qquad \text{(A3.5.5)}$$

$$\ln[r(t)+\delta] = -\frac{(1-\alpha)(1-\gamma)}{1-\alpha(1-\gamma)}\ln\Psi - \frac{\alpha(1-\gamma)}{1-\alpha(1-\gamma)}\ln p_A(t) + \text{ constant} \qquad \text{(A3.5.6)}$$

and use these to linearise (A3.5.2) to (A3.5.4) above by first-order Taylor series expansions about the log of the steady-state. That is, obtain:

$$g_{R_i}(t) \approx [g_{R_i}(t)]_{ss} + \sum_i[\frac{\partial g_{R_i}(t)}{\partial \ln R_i(t)}]_{ss}(\ln R_i(t) - \ln R_{iss})$$

$$\approx \sum_i[\frac{\partial g_{R_i}(t)}{\partial \ln R_i(t)}]_{ss}\ln[\frac{R_i(t)}{R_{iss}}]$$

See the first part of Appendix 3.1, and equation (A3.1.1) in particular, for further details of the method.

Thus, linearising (A3.5.2), where in order to simplify the notation, the time argument is omitted from this point onwards; and the term $[1-\alpha(1-\gamma)]^{-1}$ is written as Λ:

$$g_\Psi = [\frac{-(1-\alpha)(1-\gamma)\Lambda}{\gamma^2}e^{\ln(r_{ss}+\delta)} + \gamma\zeta\Lambda e^{\ln H_{Yss}}]\ln\frac{\Psi}{\Psi_{ss}} - e^{\ln\Phi_{ss}}\ln\frac{\Phi}{\Phi_{ss}}$$
$$-[\frac{\alpha(1-\gamma)\Lambda}{\gamma^2}e^{\ln(r_{ss}+\delta)} + \zeta\Lambda e^{\ln H_{Yss}}]\ln\frac{p_A}{p_{Ass}}$$

That is:

$$g_\Psi = \Lambda[\gamma\zeta H_{Yss} - \frac{(1-\alpha)(1-\gamma)}{\gamma^2}(r_{ss}+\delta)]\ln\frac{\Psi}{\Psi_{ss}} - \Phi_{ss}\ln\frac{\Phi}{\Phi_{ss}}$$
$$-\Lambda[\frac{\alpha(1-\gamma)}{\gamma^2}(r_{ss}+\delta) + \zeta H_{Yss}]\ln\frac{p_A}{p_{Ass}} \qquad \text{(A3.5.7)}$$

Similarly, linearising (A3.5.3):

$$g_\Phi = (\frac{1}{\gamma^2}-\frac{1}{\sigma})(1-\alpha)(1-\gamma)\Lambda(r_{ss}+\delta)\ln\frac{\Psi}{\Psi_{ss}} + \Phi_{ss}\ln\frac{\Phi}{\Phi_{ss}}$$
$$+(\frac{1}{\gamma^2}-\frac{1}{\sigma})\alpha(1-\gamma)\Lambda(r_{ss}+\delta)\ln\frac{p_A}{p_{Ass}} \qquad \text{(A3.5.8)}$$

and finally, linearising (A3.5.3):

$$g_{p_A} = -(1-\alpha)(1-\gamma)\Lambda e^{\ln(r_{ss}+\delta)}\ln\frac{\Psi}{\Psi_{ss}}$$
$$-\frac{1-\gamma}{\gamma}[1-(1-\alpha)(1-\gamma)\Lambda]e^{\ln(r_{ss}+\delta)+\ln\Psi_{ss}-\ln p_{Ass}}\ln\frac{\Psi}{\Psi_{ss}}$$
$$-\alpha(1-\gamma)\Lambda e^{\ln(r_{ss}+\delta)}\ln\frac{p_A}{p_{Ass}}$$
$$+\frac{1-\gamma}{\gamma}[1+\alpha(1-\gamma)\Lambda]e^{\ln(r_{ss}+\delta)+\ln\Psi_{ss}-\ln p_{Ass}}\ln\frac{p_A}{p_{Ass}}$$

that is:

$$g_{p_A} = -[(1-\alpha)(1-\gamma)\Lambda(r_{ss}+\delta) + (1-\gamma)\Lambda(r_{ss}+\delta)\frac{\Psi_{ss}}{p_{Ass}}]\ln\frac{\Psi}{\Psi_{ss}}$$
$$-[\alpha(1-\gamma)\Lambda(r_{ss}+\delta) - \frac{1-\gamma}{\gamma}\Lambda(r_{ss}+\delta)\frac{\Psi_{ss}}{p_{Ass}}]\ln\frac{p_A}{p_{Ass}}$$

and noting that since $\dot{p}_{Ass}=0$, then from (2.43) $\frac{1-\gamma}{\gamma}(r_{ss}+\gamma)\frac{\Psi_{ss}}{p_{Ass}} = r_{ss}$, and so:

$$g_{p_A} = -\Lambda[(1-\alpha)(1-\gamma)(r_{ss}+\delta) + \gamma r_{ss}]\ln\frac{\Psi}{\Psi_{ss}}$$
$$+\Lambda[r_{ss} - \alpha(1-\gamma)(r_{ss}+\delta)]\ln\frac{p_A}{p_{Ass}} \qquad \text{(A3.5.9)}$$

Equations (A3.5.7) to (A3.5.9) express the growth rates of the dynamic variables as linear functions of the logarithms of the ratios of the variables to their steady-state levels:

$$g_R = \Omega_{RL}[\ln(R(t)/R_{ss}]$$

where $\mathbf{g_R}$; and $\mathbf{ln(R(t)/R_{ss})}$ are the column vectors: $[g_\Psi, g_\Phi, g_{pA}]^T$; and $[(\ln(\Psi(t)/\Psi_{ss}), \ln(\Phi(t)/\Phi_{ss}), \ln p_A(t)/p_{Ass})]^T$ respectively; and where $\mathbf{\Omega_{RL}}$ is a matrix of constant coefficients. Equivalently, using (A3.5.1) this may be written in terms of the logarithms of the dynamic variables and their time derivatives, in an expression exactly analogous to the ordinary linearisation of Appendix 3.1, equation (A3.1.8):

$$\mathbf{d/dt(ln(R(t))} = \mathbf{\Omega_{RL} lnR(t)} + \mathbf{v_{RL}}\text{ , where } \mathbf{v_{RL}} = -\mathbf{\Omega_{RL} lnR_{ss}} \tag{A3.5.10}$$

Here $\mathbf{lnR(t)}$ is the column vector $[\ln\Psi(t), \ln\Phi(t), \ln p_A(t)]^T$; $\mathbf{d/dt(lnR(t))}$ is the corresponding vector of time derivatives; and $\mathbf{lnR_{ss}}$ the corresponding vector of steady-states. From (A3.5.7) to (A3.5.9), the matrix $\mathbf{\Omega_{RL}}$ is as follows:

$$\Omega_{RL} = \begin{pmatrix} \dfrac{\gamma\zeta H_{Yss} - (1-\alpha)(1-\gamma)(r_{ss}+\delta)/\gamma^2}{1-\alpha(1-\gamma)} & -\Phi_{ss} & -\dfrac{\zeta H_{Yss} + \alpha(1-\gamma)(r_{ss}+\delta)/\gamma^2}{1-\alpha(1-\gamma)} \\ [\dfrac{1}{\gamma^2} - \dfrac{1}{\sigma}]\dfrac{(1-\alpha)(1-\gamma)(r_{ss}+\delta)}{1-\alpha(1-\gamma)} & \Phi_{ss} & [\dfrac{1}{\gamma^2} - \dfrac{1}{\sigma}]\dfrac{\alpha(1-\gamma)(r_{ss}+\delta)}{1-\alpha(1-\gamma)} \\ -\dfrac{(1-\alpha)(1-\gamma)(r_{ss}+\delta)+\gamma r_{ss}}{1-\alpha(1-\gamma)} & 0 & \dfrac{r_{ss} - \alpha(1-\gamma)(r_{ss}+\delta)}{1-\alpha(1-\gamma)} \end{pmatrix} \tag{A3.5.11}$$

A3.5.2 Speed of convergence in the log-linear model

Since the log-linear model (A3.5.10) takes exactly the same form as the ordinary linear one of equation (A3.1.8), its solution is also the same. In particular, the log-linear solution has its counterpart to equations (3.9) and it was from these that the *speeds of convergence* or *convergence coefficients* were defined, for each of the ordinary and the log linear models, in Section 3.2.3. For the ordinary linearisation the convergence coefficient was defined as:

$$\beta(t) \overset{\text{defn.}}{=} -\frac{d(\Psi_{ss} - \Psi(t))/dt}{\Psi_{ss} - \Psi(t)} \quad \text{for } \Psi(t) = \Psi_0 e^{\lambda_{R1}.t} + \Psi_{ss}(1 - e^{\lambda_{R1}.t})$$

so:

$$\beta(t) = \beta \overset{\text{defn.}}{=} -\lambda_{R1} \tag{A3.5.12}$$

where λ_{R1} is the negative eigenvalue of the simple linear coefficients matrix.

Similarly, for λ_{RL1} the negative eigenvalue of the log-linear coefficients matrix, the convergence coefficient for the log-linearisation was defined as:

$$\beta_L(t) \overset{\text{defn.}}{=} -\frac{\dot g_\Psi(t)}{g_\Psi(t)} \quad \text{where } \ln\Psi(t) = \ln\Psi_0 e^{\lambda_{RL1}.t} + \ln\Psi_{ss}(1 - e^{\lambda_{RL1}.t})$$

so:

$$\beta_L(t) = \beta_L \overset{\text{defn.}}{=} -\lambda_{RL1} \tag{A3.5.13}$$

Clearly, from (A3.5.12) and (A3.5.13) the convergence coefficients β and β_L will be equal if the negative eigenvalues λ_{R1} and λ_{RL1} are equal. To prove that this is indeed the case, consider applying the log-linear definition of the convergence coefficient to the solution for the ordinarily linearised model, and vice versa. That is, calculate:

$$\beta_L(t) = -\frac{\dot g_\Psi(t)}{g_\Psi(t)} \quad \text{where } \Psi(t) = \Psi_0 e^{\lambda_{R}.t} + \Psi_{ss}(1 - e^{\lambda_{R}.t}) \tag{A3.5.14}$$

and

$$\beta(t) = -\frac{d(\Psi_{ss} - \Psi(t))/dt}{\Psi_{ss} - \Psi(t)} \quad \text{where } \ln\Psi(t) = \ln\Psi_0 e^{\lambda_{R1}t} + \ln\Psi_{ss}(1 - e^{\lambda_{R1}t}) \tag{A3.5.15}$$

First, from (A3.5.14):

$$g_\Psi(t) = \dot\Psi(t)/\Psi(t) = \lambda_{R1}(\Psi_0 - \Psi_{ss})e^{\lambda_{R1}.t}/\Psi(t)$$

and

$$\dot g_\Psi(t) = \lambda_{R1}^2(\Psi_0 - \Psi_{ss})e^{\lambda_{R1}.t}/\Psi(t) - [\lambda_{R1}(\Psi_0 - \Psi_{ss})e^{\lambda_{R1}.t}/\Psi(t)]\frac{\dot\Psi(t)}{\Psi(t)}$$

$$= \lambda_{R1} g_\Psi(t) - g_\Psi(t)^2$$

So:

$$\beta_L(t) = -\lambda_{R1} + g_\Psi(t) \tag{A3.5.16}$$

and taking limits as time goes to infinity:

$$\lim_{t\to\infty}\beta_L(t) = -\lambda_{R1} + \lim_{t\to\infty} g_\Psi(t) = -\lambda_{R1} \tag{A3.5.17}$$

Comparison of (A3.5.17) and (A3.5.13) then establishes that the negative eigenvalues from the simple and the log-linearisations must be equal:

$$\lambda_{RL1} = \lambda_{R1} \tag{A3.5.18}$$

Equally, from (A3.5.15):

$$\beta(t) = \frac{\dot\Psi(t)}{\Psi(t)}\frac{\Psi(t)}{\Psi_{ss} - \Psi(t)} = \lambda_{RL1}(\ln\Psi_0 - \ln\Psi_{ss})e^{\lambda_{RL1}.t}\frac{\Psi(t)}{\Psi_{ss} - \Psi(t)}$$
$$= -\lambda_{RL1}(\ln\Psi_{ss} - \ln\Psi(t))\frac{\Psi(t)}{\Psi_{ss} - \Psi(t)} \tag{A3.5.19}$$

and using L'Hôpital's rule to obtain the limit as time goes to infinity (while noting that $\Psi(t) \to \Psi_{ss}$ as $t \to \infty$):

$$\lim_{t\to\infty}\beta(t) = -\lambda_{RL1}\lim_{\Psi\to\Psi_{ss}}\frac{d[(\ln\Psi_{ss} - \ln\Psi(t))\Psi(t)]/d\Psi}{d[\Psi_{ss} - \Psi(t)]/d\Psi}$$
$$= -\lambda_{RL1}\lim_{\Psi\to\Psi_{ss}}\frac{(\ln\Psi_{ss} - \ln\Psi(t)) - 1}{-1} = -\lambda_{RL1} \tag{A3.5.20}$$

which, when compared with (A3.5.12) again confirms the equality of the negative eigenvalues - as expressed in (A3.5.18).

Finally, the effect of the different definitions of the convergence coefficients in the alternative linearisations may be explored. It is apparent from the above equations that in the ordinary linear model, the log-linear coefficient is asymptotically equal to the *'true'* ordinary linear coefficient; and that in the log-linear model the ordinary linear coefficient is asymptotically equal to the *'true'* log-linear coefficient. In particular, from equations (A3.5.12), (A3.5.16), and (A3.5.17):

$$\beta_L(t) = \beta + g_\Psi(t) \quad \text{and} \quad \lim_{t\to\infty}\beta_L(t) = \beta \tag{A3.5.21}$$

Similarly, from (A3.5.13), (A3.5.19), and (A3.5.20):

$$\beta(t) = \beta_L(\ln\Psi_{ss} - \ln\Psi(t))\frac{\Psi(t)}{\Psi_{ss} - \Psi(t)} \quad \text{and} \quad \lim_{t\to\infty}\beta(t) = \beta_L \tag{A3.5.22}$$

The question of whether either one of these alternative definitions is 'right', or perhaps just 'better, and if so which one it is, does not seem to admit of an unambiguous or objective answer. There are simply certain relevant facts about the linear dynamic systems from which they derive. Namely, for a pair of dynamic systems, one linear in certain 'levels variables', the other linear in the logs of these, where both systems are derived from some common parent system (which may even be one of the pair):

1. For the system which is 'levels-linear', the gaps between its variables and their equilibria are diminished at a constant proportional rate (λ say), while the gaps between the 'logged variables' and their equilibria - which are, of course, the logs of the levels equilibria - are diminished at a variable proportional rate, but one which approaches the constant λ asymptotically; and
2. For the log-linear system the gaps between the logged variables and their equilibria are diminished at the same constant rate λ, while here the levels variables close their gaps to equilibrium at variable rates which again are only asymptotically equal to λ.

Appendix 3.6

Phase-space surfaces of the Romer model

The objective here is to obtain functional forms in the principal dynamic variables Ψ, Φ, and p_A (hopefully explicit) which describe the three surfaces in phase-space for which each of these variables are respectively constant; and to identify on which 'sides' of these *phase surfaces* the corresponding dynamic variables either increase or decrease over time. Then it will be possible to determine the *direction of motion* in each region of the phase-space. This is achieved from the dynamic specification of the model, as described by equations (2.41) to (2.45), simply by setting the time derivatives of each variable to zero. In particular, the variables H_Y and r are first eliminated by substituting equations (2.44) and (2.45) into (2.41) to (2.43) which are then set to zero. From this, equation (2.41) produces:

$$\frac{\dot\Psi}{\Psi} = \zeta\Big[\frac{p_A}{\alpha(1-\gamma)\Psi} + 1\Big]\Big[\frac{\alpha(1-\gamma)}{\zeta\eta^\gamma}L^{(1-\alpha)(1-\gamma)}\Psi^\gamma {p_A}^{-1}\Big]^{\frac{1}{1-\alpha(1-\gamma)}} - \Phi - (\delta + \zeta H)$$

Thus:

$$\dot\Psi \gtreqless 0 \quad \text{as:}$$

$$\Phi \lesseqgtr \zeta\Big[\frac{p_A}{\alpha(1-\gamma)\Psi} + 1\Big]\Big[\frac{\alpha(1-\gamma)}{\zeta\eta^\gamma}L^{(1-\alpha)(1-\gamma)}\Psi^\gamma {p_A}^{-1}\Big]^{\frac{1}{1-\alpha(1-\gamma)}} - (\delta + \zeta H) \tag{A3.6.1}$$

Similarly, (2.42) produces:

$$\frac{\dot\Phi}{\Phi} = \frac{\zeta p_A}{\alpha(1-\gamma)\Psi}\Big(1 - \frac{\gamma^2}{\sigma}\Big)\Big[\frac{\alpha(1-\gamma)}{\zeta\eta^\gamma}L^{(1-\alpha)(1-\gamma)}\Psi^\gamma {p_A}^{-1}\Big]^{\frac{1}{1-\alpha(1-\gamma)}} + \Phi - \frac{\delta(\sigma-1)-\rho}{\sigma}$$

So:

$$\dot\Phi \gtreqless 0 \quad \text{as:}$$

$$\Phi \gtreqless \frac{\zeta p_A}{\alpha(1-\gamma)\Psi}\Big(1 - \frac{\gamma^2}{\sigma}\Big)\Big[\frac{\alpha(1-\gamma)}{\zeta\eta^\gamma}L^{(1-\alpha)(1-\gamma)}\Psi^\gamma {p_A}^{-1}\Big]^{\frac{1}{1-\alpha(1-\gamma)}} - \frac{\delta(\sigma-1)-\rho}{\sigma} \tag{A3.6.2}$$

And lastly, from (2.43):

$$\dot p_A = \frac{\gamma^2\zeta p_A}{\alpha(1-\gamma)}\Big[\frac{p_A}{\Psi} - \frac{1-\gamma}{\gamma}\Big]\Big[\frac{\alpha(1-\gamma)}{\zeta\eta^\gamma}L^{(1-\alpha)(1-\gamma)}\Psi^\gamma {p_A}^{-1}\Big]^{\frac{1}{1-\alpha(1-\gamma)}} - \delta p_A$$

Whence:

$$\dot p_A \gtreqless 0 \quad \text{as:}$$

$$\frac{\gamma\zeta}{\alpha}\Big[\frac{\gamma p_A}{(1-\gamma)\Psi} - 1\Big]\Big[\frac{\alpha(1-\gamma)}{\zeta\eta^\gamma}L^{(1-\alpha)(1-\gamma)}\Psi^\gamma {p_A}^{-1}\Big]^{\frac{1}{1-\alpha(1-\gamma)}} - \delta \gtreqless 0$$

However, the problem with this implicit formulation is that it is not easy to tell on which 'side' of the surface $\dot p_A > 0$ and on which side $\dot p_A < 0$! To resolve this $\dot p_A$ is partially differentiated with respect to Ψ as follows:

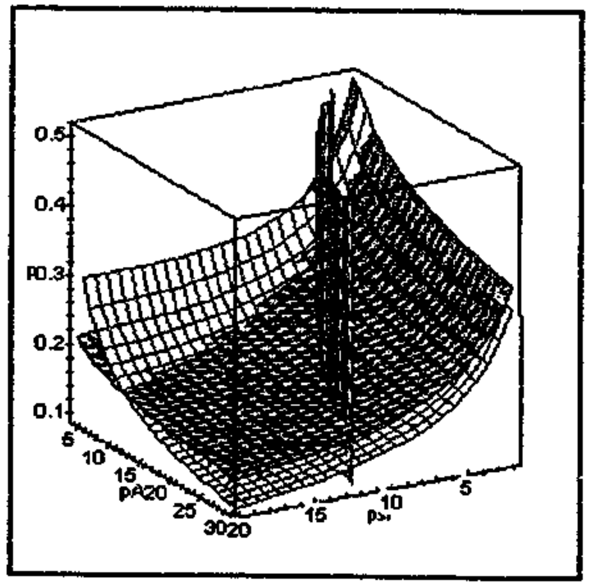
$$\frac{\partial \dot{p}_A}{\partial \Psi} = \frac{\gamma^2 \zeta p_A}{\alpha(1-\gamma)}\Big[\frac{\alpha(1-\gamma)}{\zeta\eta^\gamma} L^{(1-\alpha)(1-\gamma)}\Psi^\gamma {p_A}^{-1}\Big]^{\frac{1}{1-\alpha(1-\gamma)}}\{-\frac{p_A}{\Psi^2}$$
$$+\frac{\gamma}{1-\alpha(1-\gamma)}[\frac{p_A}{\Psi}-\frac{1-\gamma}{\gamma}]\Psi^{-1}\}$$
$$=\frac{\gamma^2\zeta p_A}{\alpha(1-\gamma)} H_Y \Psi^{-1}\{\frac{p_A}{\Psi}[\frac{\gamma}{1-\alpha(1-\gamma)}-1]-\frac{(1-\gamma)}{1-\alpha(1-\gamma)}\}$$
$$=-\frac{(r+\delta)(1-\gamma)}{1-\alpha(1-\gamma)}\{1+(1-\alpha)\frac{p_A}{\Psi}\}$$

and since the right hand side of this expression is unambiguously negative $\frac{\partial \dot{p}_A}{\partial \Psi} < 0$ and so $\dot{p}_A$ must fall as Ψ increases across the $\dot{p}_A = 0$ surface. Hence it must change from $\dot{p}_A > 0$, through $\dot{p}_A = 0$, to $\dot{p}_A < 0$. Thus

$$\dot{p}_A \gtreqless 0 \quad \text{as:}$$
$$\Psi \lesseqgtr \frac{\gamma\zeta p_A}{\alpha}[\frac{\gamma p_A}{(1-\gamma)\Psi}-1]\Big[\frac{\alpha(1-\gamma)}{\zeta\eta^\gamma} L^{(1-\alpha)(1-\gamma)}\Psi^\gamma {p_A}^{-1}\Big]^{\frac{1}{1-\alpha(1-\gamma)}} - \delta p_A \tag{A3.6.3}$$

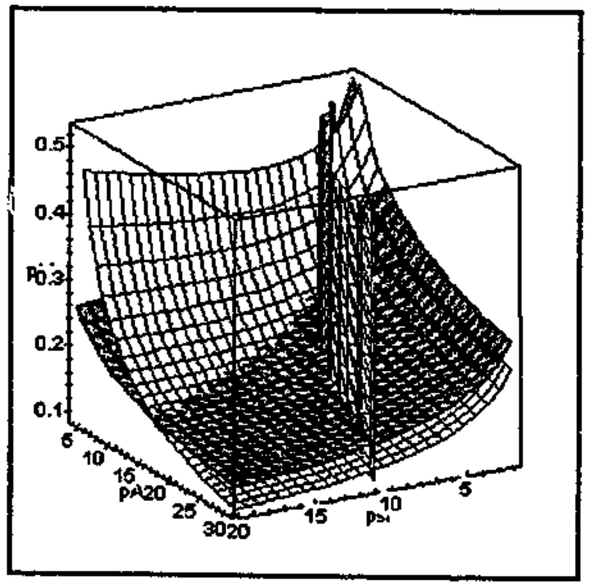
These phase surfaces may now be plotted and the *directions of motion* of trajectories in different regions of the phase-space assessed. This is done in Section 3.3 of the text for the benchmark parameter set. Here the surfaces are plotted for a range of alternative parameter settings in order to provide an idea of their sensitivity. Specifically, each parameter in turn is both reduced and raised by 20 per cent of its benchmark value while holding all the other parameters at their benchmark levels.

Figure A3.6.1: Romer system phase-space for α=0.34 (-20%); other parameters at benchmark



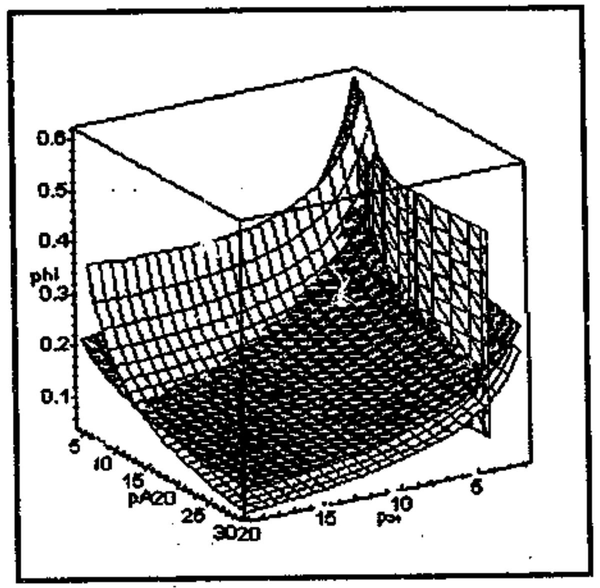
Steady-state: Ψ_{ss}= 5.60; Φ_{ss}= 0.304; p_{Ass}= 7.67
H_{Yss}= 69.0%; g = 1.86%= ; r = 6.58%

Figure A3.6.2: Romer system phase-space for α=0.52 (+20%); other parameters at benchmark



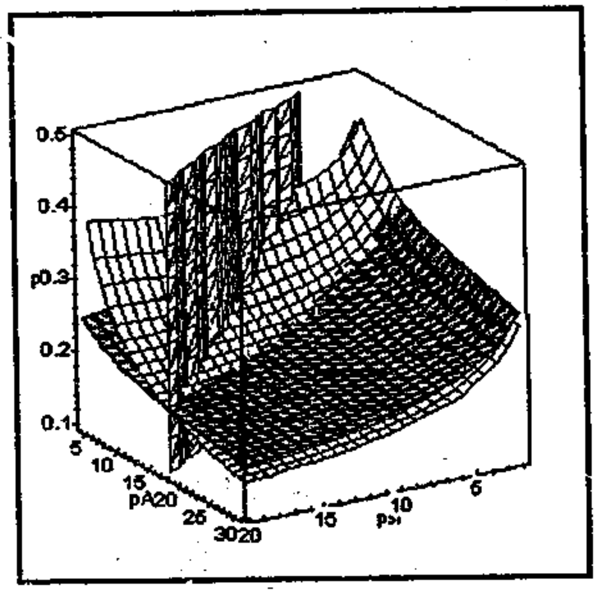
Steady-state: Ψ_{ss}= 7.21; Φ_{ss}= 0.252; p_{Ass}= 11.18
H_{Yss}= 78.4; g = 1.30; r = 4.89

Figure A3.6.3: Romer system phase-space for γ=0.43 (-20%); other parameters at benchmark



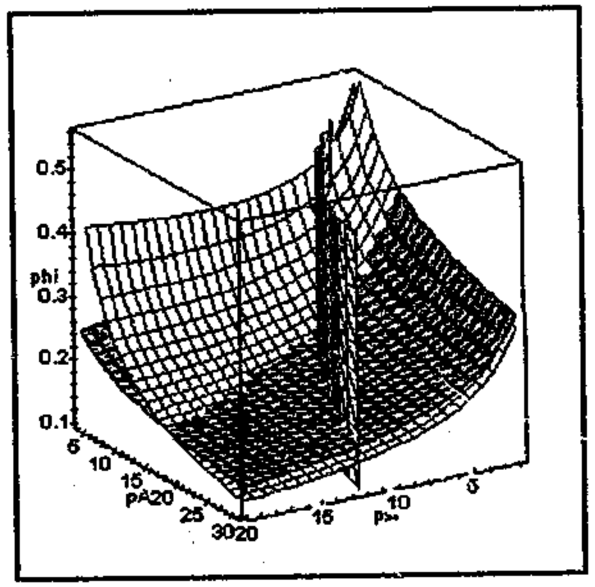
Steady-state: Ψ_{ss}= 2.96; Φ_{ss}= 0.421; p_{Ass}= 7.22
H_{Yss}= 79.2%; g = 1.25%; r = 4.75%

Figure A3.6.4: Romer system phase-space for γ=0.65 (+20%); other parameters at benchmark



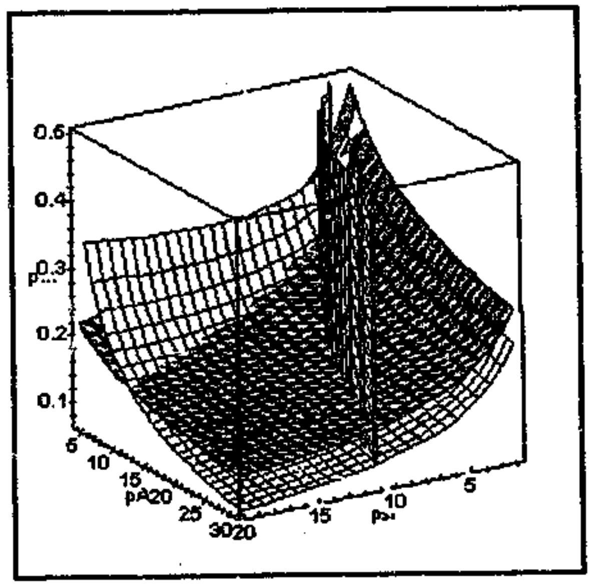
Steady-state: Ψ_{ss}= 19.49; Φ_{ss}= 0.187; p_{Ass}= 17.09
H_{Yss}= 70.2%; g = 1.79%; r = 6.37%

Figure A3.6.5: Romer system phase-space for ζ=0.048 (-20%); other parameters at benchmark



Steady-state: Ψ_{ss}= 8.40; Φ_{ss}= 0.241; p_{Ass}= 13.46
H_{Yss}= 75.4%; g = 1.18%; r = 4.54%

Figure A3.6.6: Romer system phase-space for ζ=0.072 (+20%); other parameters at benchmark



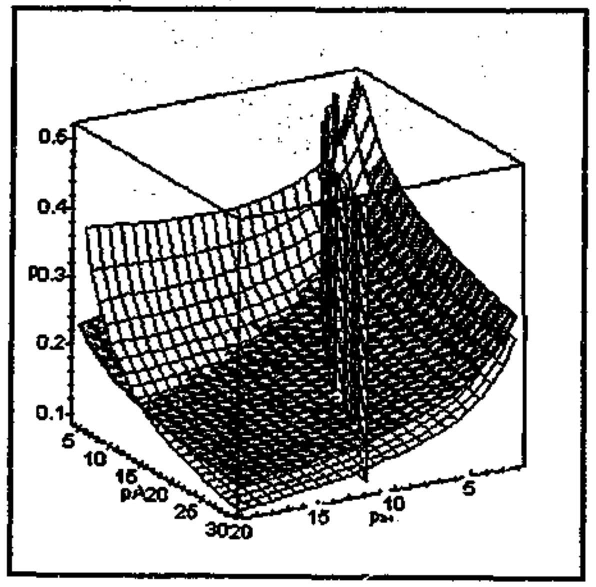
Steady-state: Ψ_{ss}= 5.14; Φ_{ss}= 0.307; p_{Ass}= 7.00
H_{Yss}= 73.8%; g = 1.89%; r = 6.67%

All the phase surfaces are affected by changes to the three parameters α, γ and ζ, and while they retain their general shapes in response to the ±20 per cent changes in these parameters, there are significant changes in their positions and their points of intersection (equivalent to changes in the steady-state of the system). Corresponding changes in their saddle and adjustment paths would result. The impact of variations to the parameter γ

can be seen to be by far the greatest, particularly on *pAsurf* (Figure A3.6.1 to Figure A3.6.6).

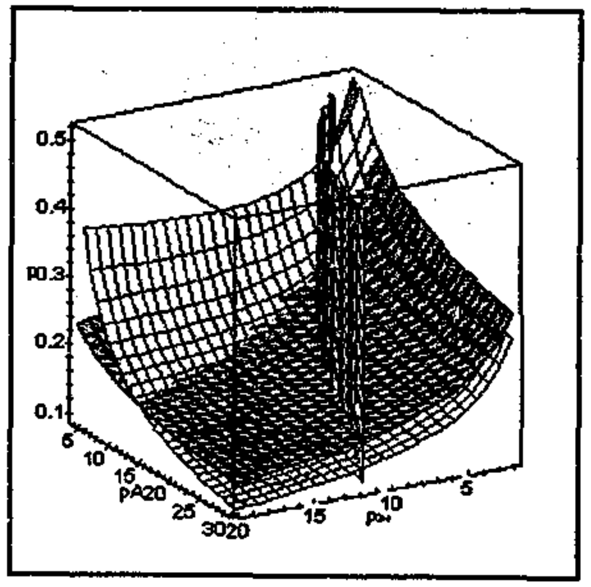
Conversely, the parameters ρ and σ affect only *phisurf*, and then not to a great degree. As a result, neither the shapes nor positions of the surfaces are very responsive to variations in these parameters (Figure A3.6.7 to Figure A3.6.10).

<u>Figure A3.6.7:</u> Romer system phase-space for ρ=0.008 (-20%); other parameters at benchmark



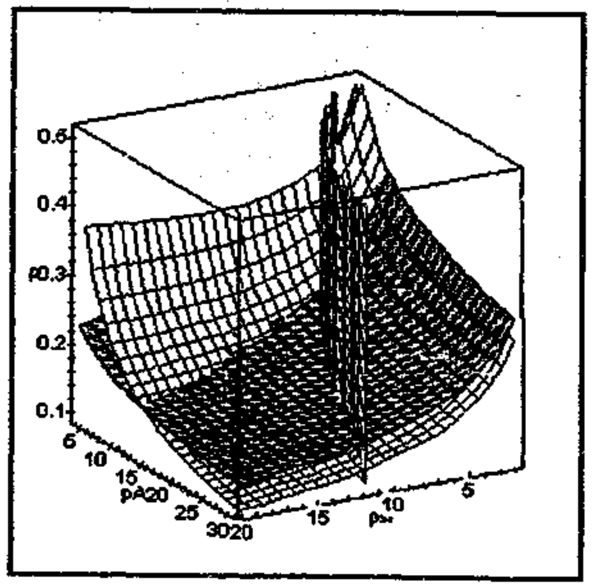
Steady-state: Ψ_{ss}= 6.53; Φ_{ss}= 0.272; p_{Ass}= 9.58
H_{Yss}= 73.6%; g = 1.58%; r = 5.55%

<u>Figure A3.6.8:</u> Romer system phase-space for ρ=0.072 (+20%); other parameters at benchmark



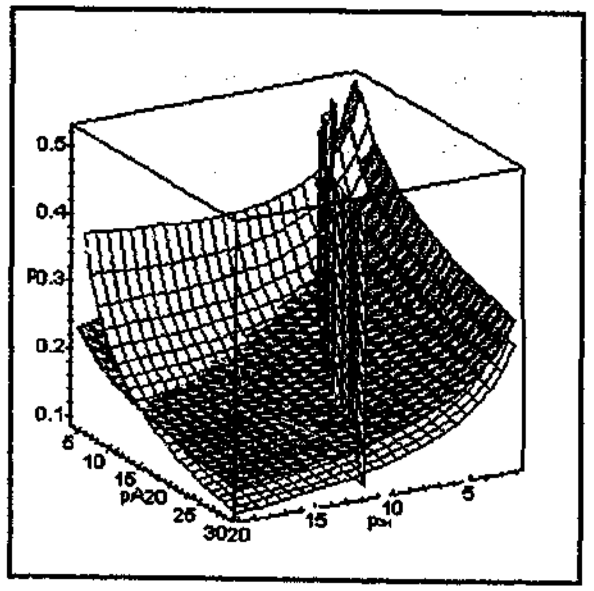
Steady-state: Ψ_{ss}= 6.42; Φ_{ss}= 0.277; p_{Ass}= 9.33
H_{Yss}= 75.2%; g = 1.49%; r = 5.67%

<u>Figure A3.6.9:</u> Romer system phase-space for σ=2.4 (-20%); other parameters at benchmark



Steady-state: Ψ_{ss}= 6.79; Φ_{ss}= 0.261; p_{Ass}= 10.16
H_{Yss}= 70.2%; g = 1.79%; r = 5.29%

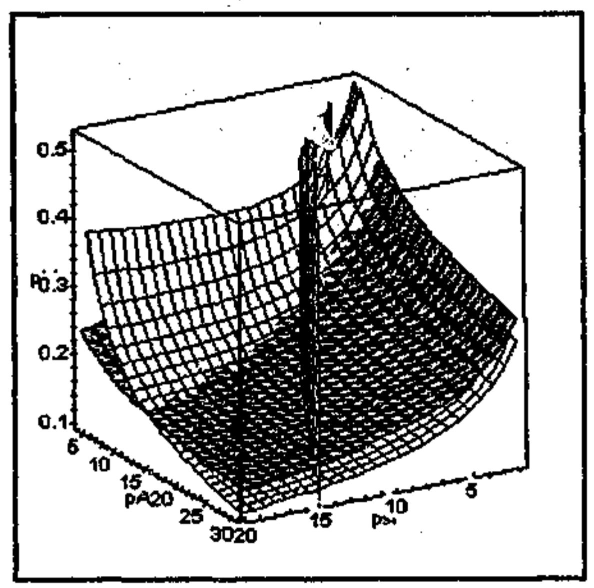
<u>Figure A3.6.10:</u> Romer system phase-space for σ=3.6 (+20%); other parameters at benchmark



Steady-state: Ψ_{ss}= 6.25; Φ_{ss}= 0.284; p_{Ass}= 8.97
H_{Yss}= 77.6%; g = 1.35%; r = 5.84%

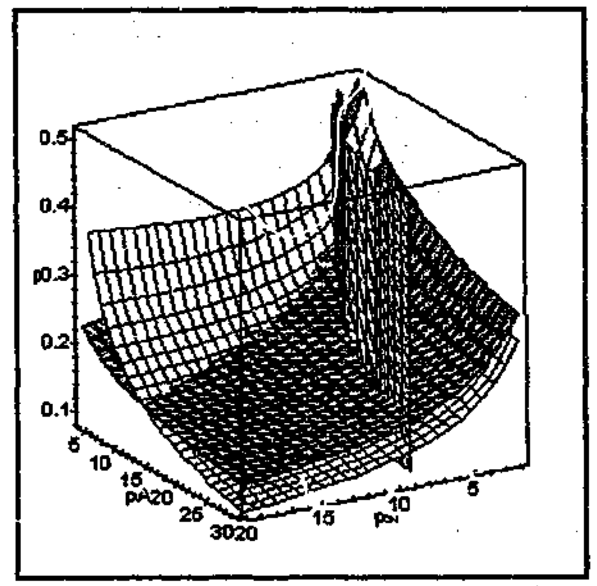
Changes to parameter δ again affect all three phase surfaces. The impacts on *psisurf* and *phisurf* are small, but that on pAsurf is quite significant. The resulting change to the steady-state are small but not insignificant. As discussed in Section 2.4.2 of Chapter 2, changes to δ have no impact on the allocation of human capital nor on the growth or interest rates (Figure A3.6.11 and Figure A3.6.12).

<u>Figure A3.6.11:</u> Romer system phase-space for δ=0.032 (-20%); other parameters at benchmark



Steady-state: Ψ_{ss}= 7.82; Φ_{ss}= 0.255; p_{Ass}= 10.47
H_{Yss}= 74.4%; g = 1.54%; r = 5.61%

<u>Figure A3.6.12:</u> Romer system phase-space for δ=0.048 (+20%); other parameters at benchmark

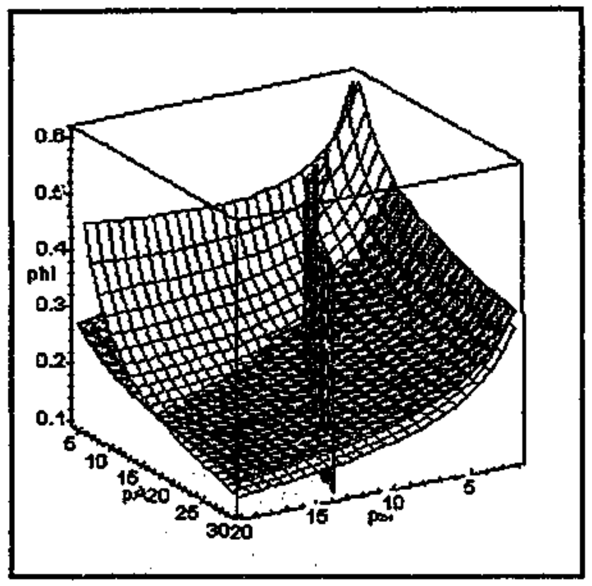


Steady-state: Ψ_{ss}= 5.44; Φ_{ss}= 0.294; p_{Ass}= 8.61
H_{Yss}= 74.4%; g = 1.54%; r = 5.61%

The magnitudes of the parameters η and L also influence the shapes and positions of all three phase surfaces, with the effects of η being somewhat the stronger of the two. However, despite the changes to the intersection points of the surfaces, neither parameter has any effect upon the steady-state values of the allocation of human capital H_{Yss}, the growth rate g, the interest rate r_{ss}, nor even on the consumption-capital ratio Φ_{ss} (Figure A3.6.13 to Figure A3.6.16).

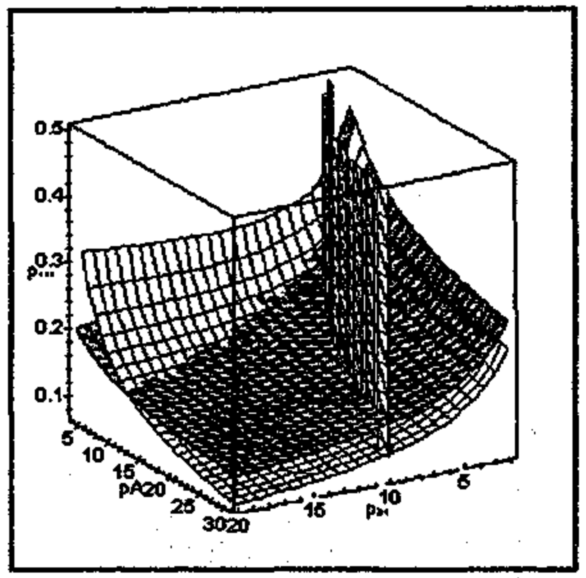
These last results could have been predicted from the steady-state equations (2.55) to (2.57) from Section 2.3.3 of Chapter 2. Also, as for δ, neither of the parameters η or L have any impact on the steady-state values of the allocation of human capital, the growth rate, or the interest rate. However, unlike the case for δ, there is no unambiguous underlying economic rationale for such results. As discussed in Section 2.4.2, they are merely the outcomes of exactly cancelling effects due to the particular form (Cobb-Douglas) of the production function of the model.

Figure A3.6.13: Romer system phase-space for η=1.6 (-20%); other params at benchmark



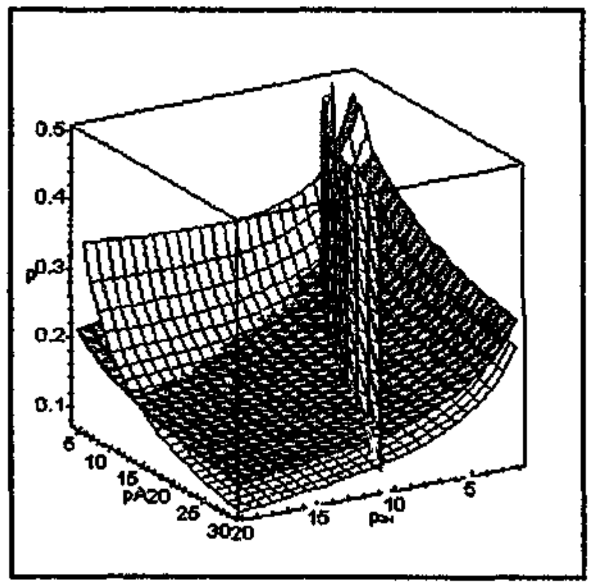
Steady-state: Ψ_{ss}= 8.42; Φ_{ss}= 0.274; p_{Ass}= 12.28
H_{Yss}= 74.4%; g = 1.54%; r = 5.61%

Figure A3.6.14: Romer system phase-space for η=2.4 (+20%); other params at benchmark



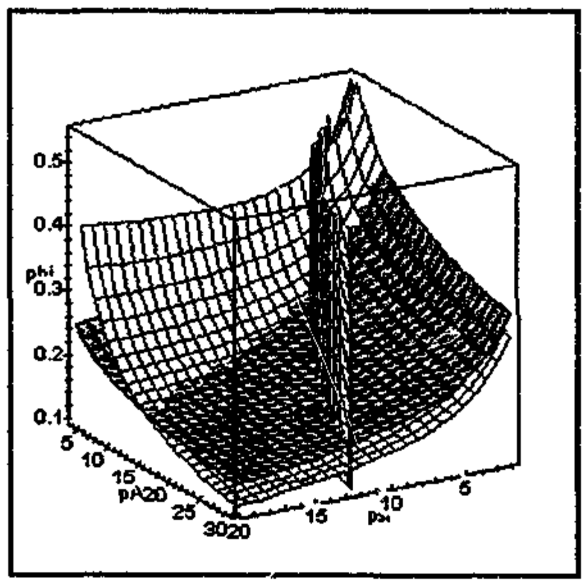
Steady-state: Ψ_{ss}= 5.23; Φ_{ss}= 0.274; p_{Ass}= 7.63
H_{Yss}= 74.4%; g = 1.54%; r = 5.61%

Figure A3.6.15: Romer system phase-space for L=1.6 (-20%); other parameters at benchmark



Steady-state: Ψ_{ss}= 5.70; Φ_{ss}= 0.274; p_{Ass}= 8.32
H_{Yss}= 74.4%; g = 1.54%; r = 5.61%

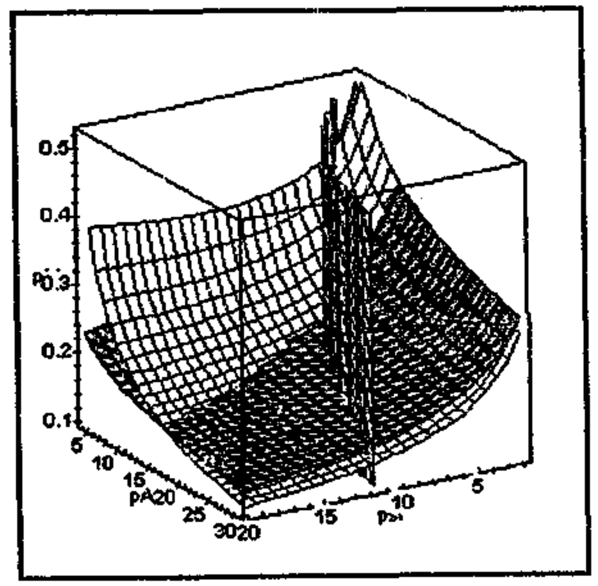
Figure A3.6.16: Romer system phase-space for L=2.4 (+20%); other parameters at benchmark



Steady-state: Ψ_{ss}= 7.19; Φ_{ss}= 0.274; p_{Ass}= 10.49
H_{Yss}= 74.4%; g = 1.54%; r = 5.61%

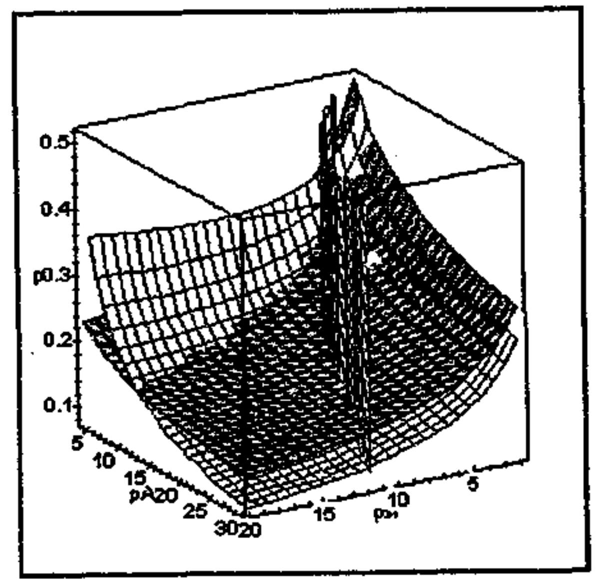
The phase surface *psisurf* is the only one upon which parameter H has any effect. Nevertheless, the ±20 per cent changes to H produce significant impacts on the points of intersection/steady-state outcomes (Figure A3.6.17 and Figure A3.6.18). In fact, the impacts upon the steady-state consumption-capital ratios (Φ_{ss}), the growth rates (g), and the interest rates (r_{ss}), but not the allocation of human capital (H_{Yss} or H_{Ass}), are identical with those produced by the same changes to parameter ζ. Once again these results could have been predicted from the steady-state equations (2.55) to (2.57) of Section 2.3.3 in Chapter 2.

Figure A3.6.17: Romer system phase-space for H=0.8 (-20%); other parameters at benchmark



Steady-state: Ψ_{ss}= 7.63; Φ_{ss}= 0.241; p_{Ass}= 12.23
H_{Yss}= 60.3%; g = 1.18%; r = 4.54%

Figure A3.6.18: Romer system phase-space for H=1.2 (+20%); other parameters at benchmark



Steady-state: Ψ_{ss}= 5.56; Φ_{ss}= 0.307; p_{Ass}= 7.57
H_{Yss}= 88.5%; g = 1.89%; r = 6.67%

Appendix 3.7

Equivalence of the Solowian-Romer model and the full Romer model

Kurz (1968) introduced the concept of an "inverse optimum problem" whereby, given some specified consumption-investment function, the aim was to find a class of objective functions (along with any concomitant parameter restrictions), which when optimised would reproduce the given function as an optimal outcome. He solved such a problem for the Solow growth model proving, in terms of the model set-up and notation of this dissertation (see Section 2.2), that the intertemporal optimisation of a '*Constant Intertemporal Elasticity of Substitution* (CIES) utility function

$$U\{C(t)\} = [C(t)^{1-\sigma} - 1]/(1-\sigma)$$

would generate the constant savings rate Solow consumption function: $C(t) = (1-s)Y(t)$ provided $\sigma = 1/s$, and $(\delta+\rho) = \gamma\delta/s$ where δ is the rate of depreciation, ρ is the discount factor and γ is the output elasticity of capital in the production function.

The objective of this appendix is to extend Kurz' results by showing that the full Romer model can generate constant transitional savings subject to certain restrictions on the parameters. In this way the Solowian-Romer model will be shown to be merely a particular parameterisation of the full model.

In the full Romer model savings for capital formation are given by equation (A2.5.1) as:

$$s(t) = 1 - \frac{\gamma^2}{r(t)+\delta}\Phi(t) \tag{A3.7.1}$$

So:

$$\frac{\frac{d}{dt}[1-s(t)]}{[1-s(t)]} = \frac{\dot{\Phi}(t)}{\Phi(t)} - \frac{\frac{d}{dt}[r(t)+\delta]}{[r(t)+\delta]}$$

Thus, for transitional savings to be constant, that is $\dot{s}(t) = 0$, it must be the case that:

$$\frac{\dot{\Phi}(t)}{\Phi(t)} = \frac{\frac{d}{dt}[r(t)+\delta]}{[r(t)+\delta]}$$

The right hand side of this expression has already been evaluated in terms of $\dot{\Psi}(t)/\Psi(t)$ and $\dot{p}_A(t)/p_A(t)$ in equation (A3.1.3) of Appendix 3.1. Using this result and substituting for $\dot{\Psi}(t)/\Psi(t)$, $\dot{\Phi}(t)/\Phi(t)$ and $\dot{p}_A(t)/p_A(t)$ from equations (2.41), (2.42) and (2.43), and dropping the time (t) argument generates:

$$\frac{r-\rho}{\sigma} - \frac{r+\delta}{\gamma^2} + \Phi + \delta =$$
$$-\frac{1-\gamma}{1-\alpha(1-\gamma)}\{(1-\alpha)[\frac{r+\delta}{\gamma^2} - \Phi - \delta - \zeta H + \zeta H_Y] + \alpha[r - \frac{1-\gamma}{\gamma}(r+\delta)\frac{\Psi}{p_A}]\}$$

Then, using (A3.7.1) where now s(t) = constant = s, to substitute for Φ; using equation (2.43) to express the very last term in the above expression as $(\zeta\gamma H_Y/\alpha)$; and collecting together the coefficients of H_Y and $(r+\delta)$ gives:

$$\zeta(1-\gamma)(1-\alpha-\gamma)H_Y + [\frac{1-\alpha(1-\gamma)}{\sigma} - \frac{s}{\gamma} + \alpha(1-\gamma)](r+\delta)$$
$$-[\frac{1-\alpha(1-\gamma)}{\sigma}(\delta+\rho) + \delta[\alpha(1-\gamma)-\gamma] + \zeta(1-\alpha)(1-\gamma)H] = 0 \tag{A3.7.2}$$

For the Romer model to display constant transitional dynamics this expression must always hold. Clearly it can only be satisfied at all times if all three of the coefficient terms are zero. Thus, the conditions for constant transitional dynamics in the Romer model are:

$$\alpha + \gamma = 1\,;$$
$$\sigma = \frac{\gamma[1-\alpha(1-\gamma)]}{s-\alpha\gamma(1-\gamma)}\ ;\ \text{and} \tag{A3.7.3}$$
$$\frac{1-\alpha(1-\gamma)}{\sigma}(\delta+\rho) + \delta[\alpha(1-\gamma)-\gamma] + \zeta(1-\alpha)(1-\gamma) = 0$$

Notice that if $\gamma = 1$ in (A3.7.2) the conditions for constant transitional growth become $\sigma = 1/s$, and $(\delta+\rho) = \delta\gamma/s$; precisely the conditions derived by Kurz (1968) for the 'inverse optimum problem' defined by the Solow model.[37]

While satisfaction of the conditions (A3.7.3) ensures that transitional savings will be constant, and while they limit the feasible parameter sets from which this is possible,[38] it remains a matter of trial and error to identify such a set. However, the search is made simple by the fact that there is an additional relation that the parameters must satisfy. Namely, if the saving rate is to be constant then it must always be equal to its steady-state level. Hence, substituting Φ_{ss} and r_{ss} from equations (2.55), (2.56) and (2.57) into (A3.7.1) above generates the steady-state savings formula:

$$s = \frac{\gamma^2[\zeta H - \alpha\rho/\gamma + \delta + \alpha\delta\sigma/\gamma]}{\sigma\zeta H + \delta + \rho + \alpha\delta\sigma/\gamma} \tag{A3.7.4}$$

while the constant transitional savings rate condition derived from (A3.7.3) is:

$$s = \frac{\gamma[1-\alpha(1-\gamma)]}{\sigma} + \alpha\gamma(1-\gamma) \tag{A3.7.5}$$

Equating these two expressions for the savings rate will allow one of the parameters to be expressed in terms of all the others; here σ is arbitrarily chosen to be the 'dependent variable'. As will be seen in the algebra below, since it is easy to bring the other two conditions from (A3.7.3) into the derivation, the resulting formula for σ will ensure constant transitional growth for arbitrarily chosen feasible values of the other parameters. It will then only remain to choose these values so as to ensure that σ itself is feasible

[37] This is probably due to the fact that when $\gamma=1$ capital is the only factor input to production, similar to the output per head formulation used by Solow.

[38] Remember that apart from empirical plausibility, the parameters are constrained so as not to be economically nonsensical. Thus $0 \le \alpha,\gamma \le 1$; $\zeta,\sigma,\eta,H,L>0$; and $\delta,\rho \ge 0$ (also see Section 2.4).

(σ>0), and that all parameters and steady-state results are economically plausible. Thus, equating (A3.7.4) and (A3.7.5); multiplying through by σ and by the denominator of (A3.7.4); and collecting together the coefficients of σ^2 and σ produces:

$$\alpha[(1-\gamma)(\zeta H+\alpha\delta/\gamma)-\delta]\sigma^2+[1-\alpha(1-\gamma)](\delta+\rho)=$$
$$[\gamma\zeta H-\alpha\rho+\delta\gamma-(\zeta H+\alpha\delta/\gamma)\{1-\alpha(1-\gamma)\}-\alpha(1-\gamma)(\delta+\rho)]\sigma$$

$$\alpha[(1-\gamma)(\zeta H+\alpha\delta/\gamma)-\delta]\sigma^2+[1-\alpha(1-\gamma)](\delta+\rho)=$$
$$[\delta\{\gamma-\alpha(1-\gamma)\}-(1-\alpha)(1-\gamma)\zeta H-\alpha\rho(2-\gamma)-(\alpha\delta/\gamma)\{1-\alpha(1-\gamma)\}]\sigma$$

Using the third of the conditions from (A3.7.3) this simplifies to:

$$\alpha[(1-\gamma)(\zeta H+\alpha\delta/\gamma)-\delta]\sigma^2=-[\alpha\rho(2-\gamma)+(\alpha\delta/\gamma)\{1-\alpha(1-\gamma)\}]\sigma$$

or

$$\sigma=\frac{\rho(2-\gamma)+(\delta/\gamma)\{1-\alpha(1-\gamma)\}}{\delta-(1-\gamma)(\zeta H+\alpha\delta/\gamma)}$$

and finally, using the first of the conditions from (A3.7.3) produces:

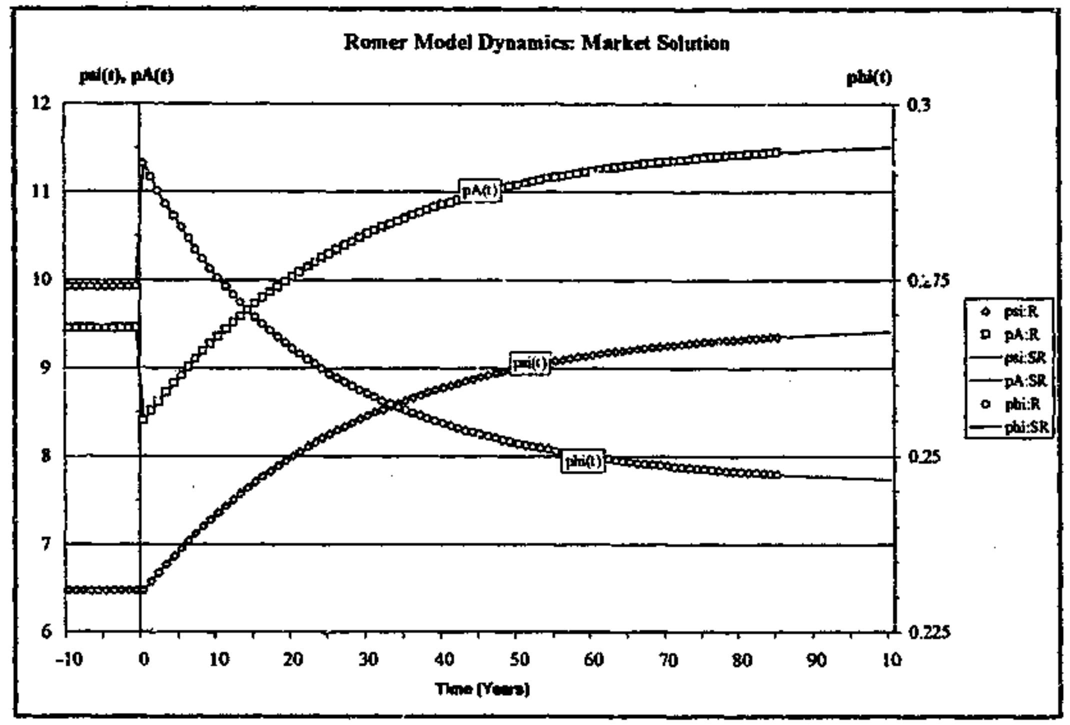
$$\sigma=\frac{(1+\alpha)(\delta+\rho)}{\delta-\alpha[\zeta H+\alpha\delta/(1-\alpha)]} \tag{A3.7.6}$$

Since σ must be positive the choice of α, δ, ζ and H must be such that the denominator of (A3.7.6) is also positive. Otherwise the choice of the parameters is restricted only by their individual feasibility constraints. One of many parameter sets which satisfy (A3.7.3) and (A3.7.6), and which may therefore be expected to generate constant transitional growth in the full Romer model, is the following:

constant s_N0 = (A3.7.7)
{α=0.4; γ=0.6; ζ=0.05; δ=0.05; ρ=0.01; σ=5.04; η=2; H=1; and L=2}

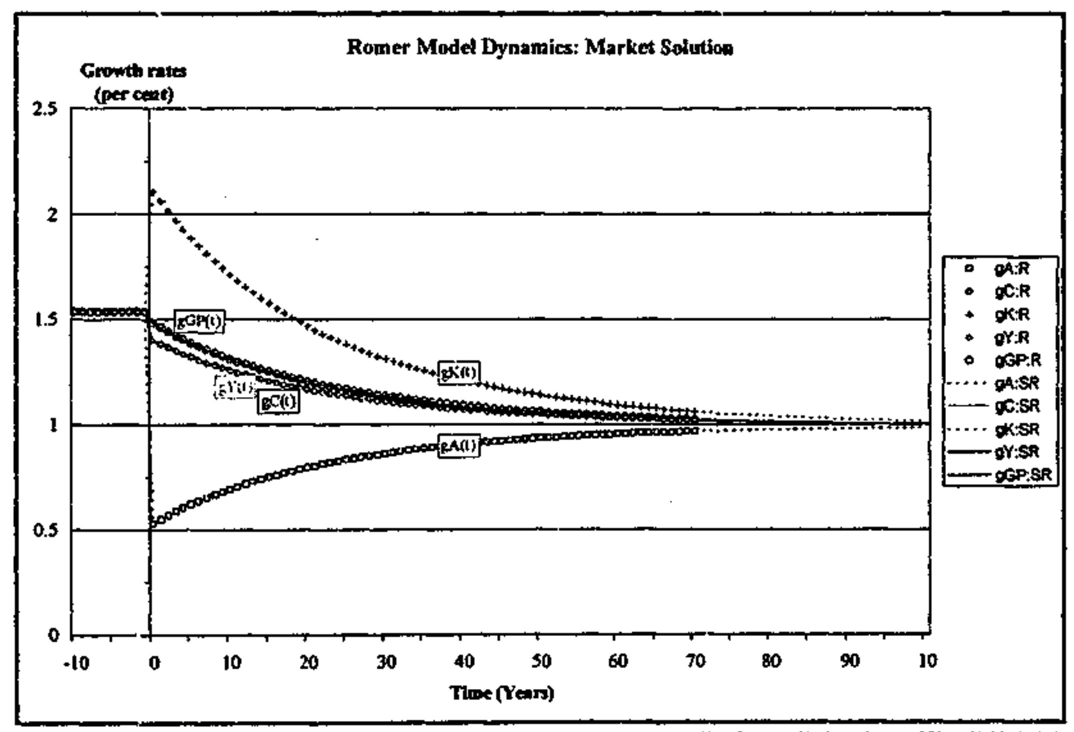
To demonstrate that these settings do indeed generate constant transitional growth, a simulation shifting the Romer model from its benchmark parameter set to the set *constant s_N0* has been conducted. The system was taken to be initially at its benchmark steady-state, when at time t = 0 all its parameters were instantly converted to their corresponding levels in the set *constant s_N0*. The dynamics of this simulation were computed by the numerical integration technique of *finite differences*, implemented via the GEMPACK modelling software (see Sections 4.3 and 4.4 of Chapter 4 for details). Furthermore, to demonstrate the dynamic equivalence of the Solowian-Romer and the full Romer models this simulation has also been run on the former. In this case the consumer preference parameters ρ and σ are 'replaced' by the exogenous savings rate parameter s, which has been moved from the full model benchmark steady-state level $s^M_{N_{ss}}$(benchmark) = 0.1680, to its post-simulation steady-state $s^M_{N_{ss}}$(constant s_{N0}) = 0.1960. Here the dynamics of the simulation were computed by a fourth order Runge-Kutta integration procedure using the technique of *shooting*. (details may be found in Section 3.4.1 of the current chapter, and Section 4.1 and Appendix 4.1 of Chapter 4. The results of both simulations are presented in Figure A3.7.1 to Figure A3.7.6, where data points towards the end of the full model series have been omitted in order to reveal the Solowian-Romer series.

Figure A3.7.1: Dynamic effects on Ψ, Φ, and p_A of a simulation exhibiting constant transitional savings $s_N(t)$=s, 'benchmark' and 'constant s_N0' parameter sets, Romer <u>and</u> Solowian-Romer models.



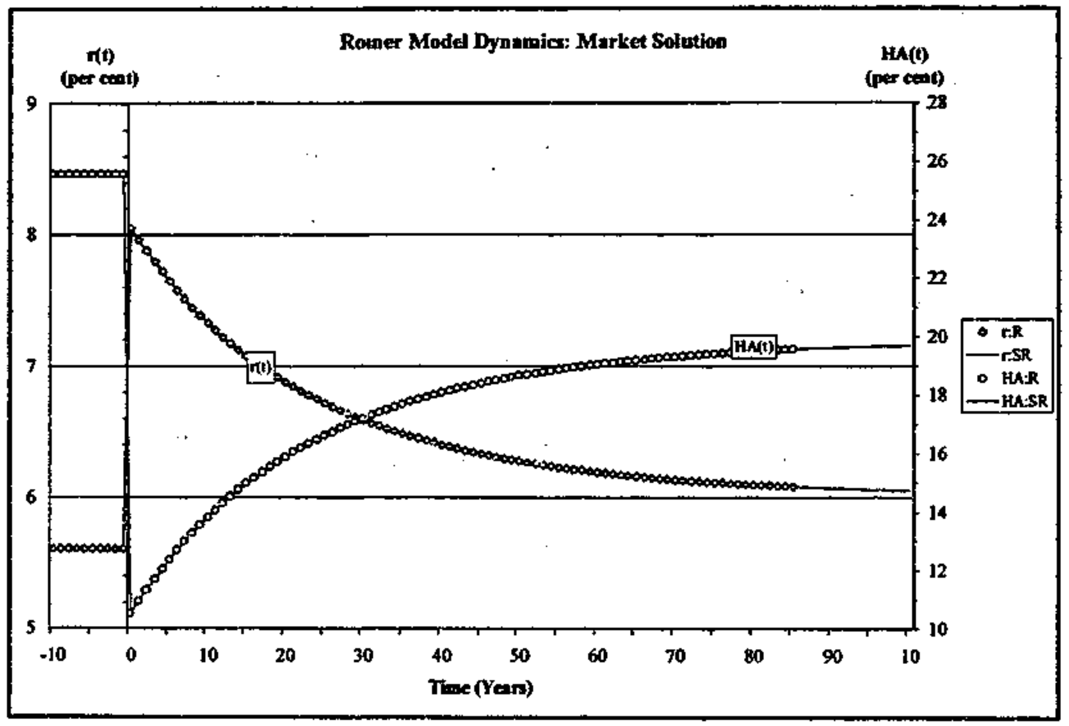
d:\phd\gempack\mktrom\const sN\bench\sN1sim(psi)

Figure A3.7.2: Dynamic effects on the growth rates of a simulation exhibiting constant transitional savings $s_N(t)$=s, 'benchmark' and 'constant s_N0' parameter sets, Romer <u>and</u> Solowian-Romer models.



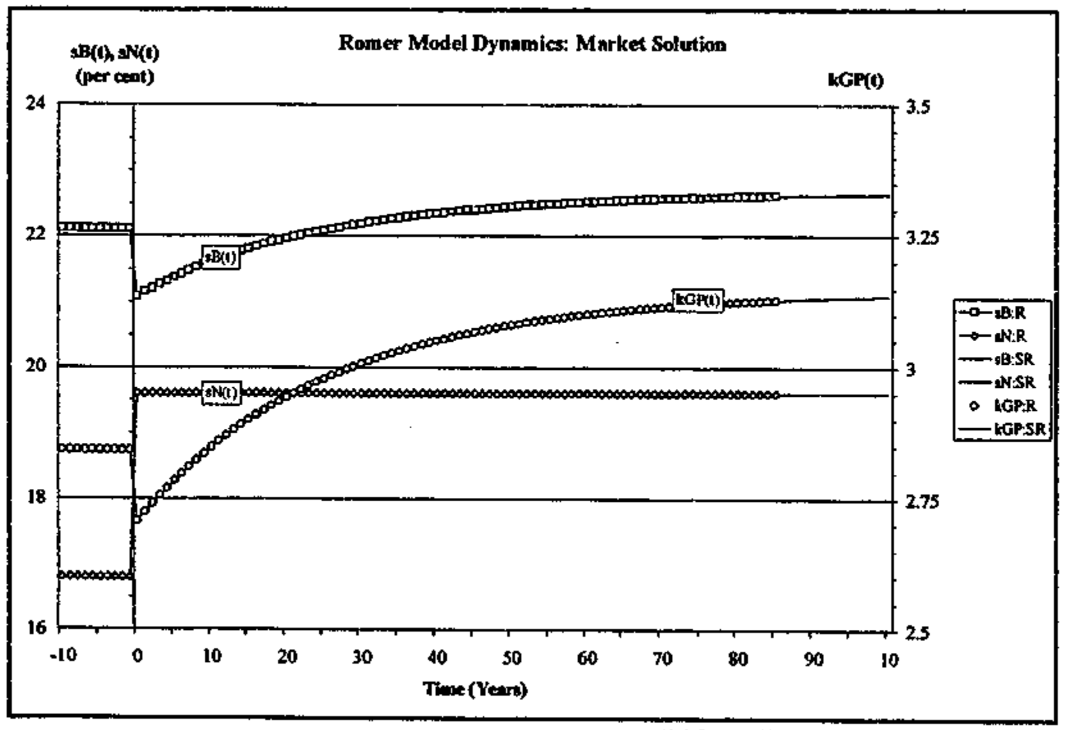
d:\phd\gempack\mktrom\const sN\bench\sN1sim(gA)

Figure A3.7.3: Dynamic effects on r, and H_A of a simulation exhibiting constant transitional savings $s_N(t)$=s, 'benchmark' and 'constant s_N0' parameter sets, Romer and Solowian-Romer models.



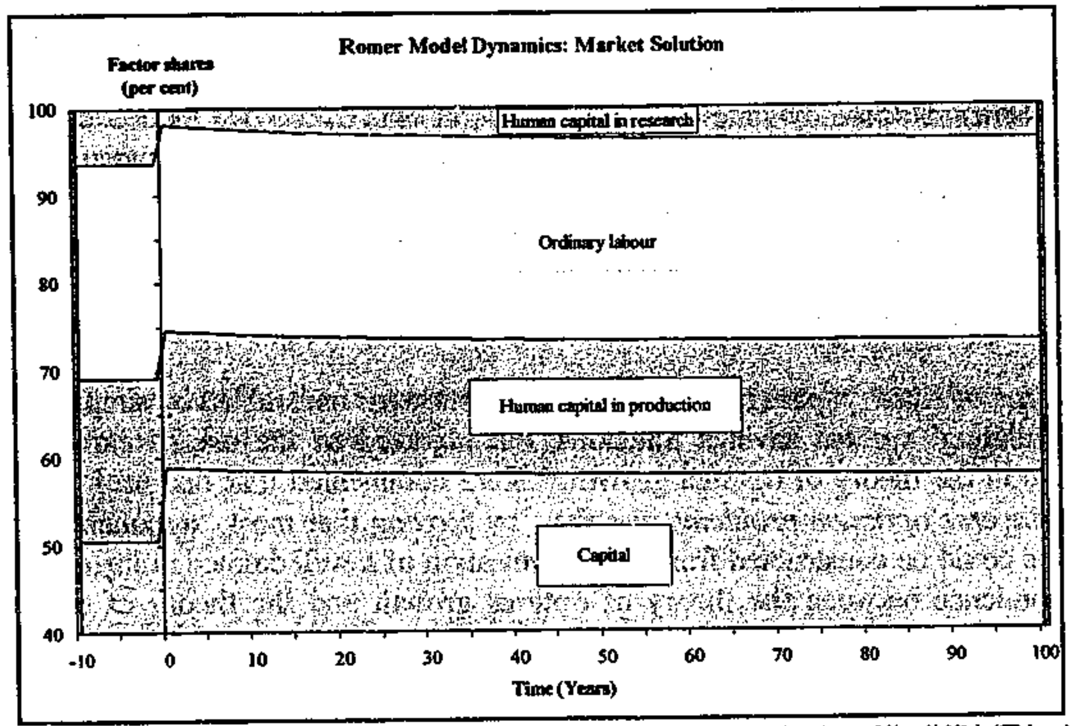
d:\phd\gempack\mktrom\const sN\bench\sN1sim(r,HA)

Figure A3.7.4: Dynamic effects on s_B, s_N and k_{GP} of a simulation exhibiting constant transitional savings $s_N(t)$=s, 'benchmark' and 'constant s_N0' parameter sets, Romer and Solowian-Romer models.



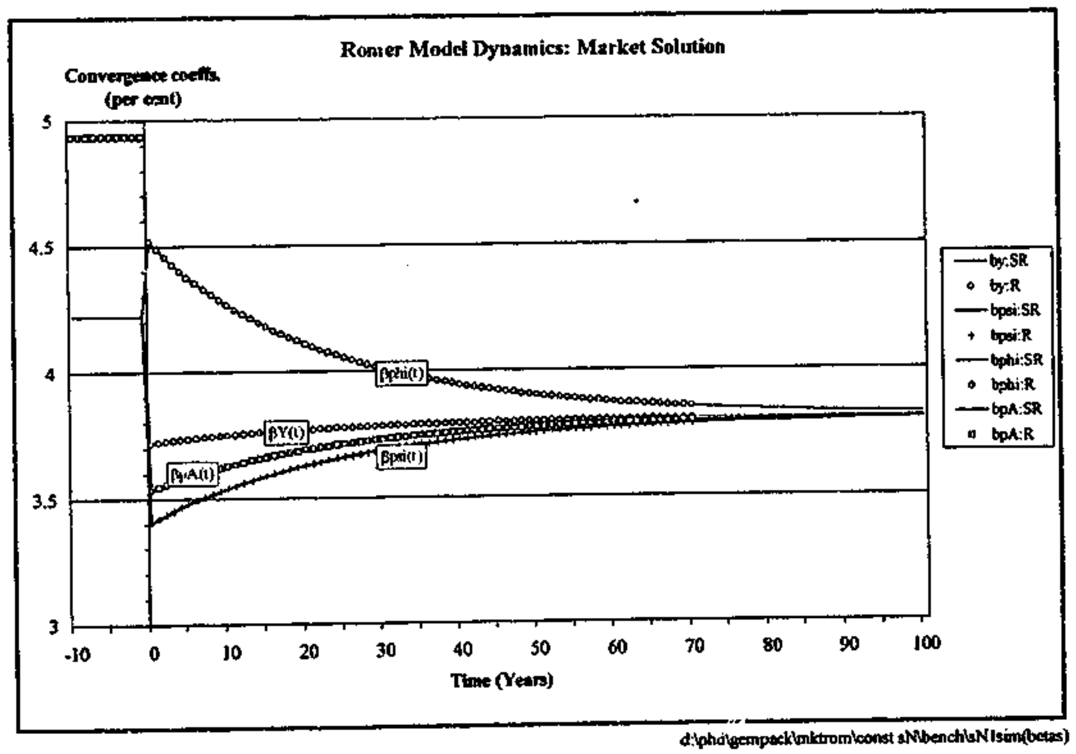
d:\phd\gempack\mktrom\const sN\bench\sN1sim(sB,kGP)

Figure A3.7.5: Dynamic effects on the factor shares of gross income of a simulation exhibiting constant transitional savings $s_N(t)$=s, 'benchmark' and 'constant s_N0' parameter sets, Romer and Solowian-Romer models.



d:\phd\gempack\mktrom\const sN\bench\sN1sim(GPshares)

Figure A3.7.6: Dynamic effects on the convergence coefficients of a simulation exhibiting constant transitional savings $s_N(t)$=s, 'benchmark' and 'constant s_N0' parameter sets, Romer and Solowian-Romer models.



d:\phd\gempack\mktrom\const sN\bench\sN1sim(betas)

Three things are apparent from these results. First, the results in Figure A3.7.4 confirm that as expected, the transitional savings rate s_N of the Romer model, despite its infinite horizon utility maximising consumers, is indeed constant for the parameter configuration *constant s_N0*. Second, since the results for the Solowian-Romer model all coincide with those for the Romer model (the plotted adjustment paths are all indistinguishable), then the two models are confirmed as equivalent for the parameter settings. Third, the fact that the benchmark steady-state convergence coefficients of the two models differ, while those for the *constant s_N0* parameter set are the same (Figure A3.7.6), suggests that the two models may be set up to generate **completely** identical steady-states only if the parameters are such as to ensure constant transitional savings in the full model. Otherwise the convergence coefficients will differ even though the steady-state values for all other variables are identical.[39]

At the time of Kurz' paper (1968) many economists were untroubled by simply asserting or 'postulating' various savings functions, but objected to the use of cardinal utility functions in the theory of optimal growth.[40] Kurz commented that one useful aspect of solving *inverse optimum problems* was that 'by proving that most "reasonable" savings functions could be constructed from the optimisation of a well defined utility function', a correspondence between the theory of optimal growth and the theory of competitive deterministic growth was established which ought to remove these objections. Nowadays the objections likely run the other way: with the theory of optimal growth being widely (if not unanimously) accepted, and the use of deterministic Solow-type savings functions regarded as regressive. Once again, proving the equivalence of the two approaches, albeit under certain parameter restrictions, ought to at least temper the objections to some degree. Here the fact that parameter configurations which generate equivalence between the Solowian-Romer and full Romer models can be found close to the empirically based benchmark set, must surely further weaken objections to the former as a didactic simplification/approximation of the latter.

[39] Further confirmation of this was obtained by examining other parameter sets, but no attempt has been made to obtain an analytic proof.

[40] It might also be noted that as recently as 1990 Frank Hahn preferred the use of a deterministic savings function in an illustrative model: "...although one is just willing to let firms peer into the indefinite future, this seems so awful an assumption for households that I prefer the dreaded *ad hoc* savings function. Moreover, that is what Solow preferred." (Hahn, 1990, p.34).

Appendix 3.8

Log-linearisation of the Solowian-Romer model

It should be noted at the outset that the procedure here is exactly the same as that for the log-linearisation of the full Romer model (Appendix 3.5). Moreover the dynamic equations of the Solowian-Romer model are also very similar to those of the full model. Nevertheless, for the sake of completeness the linearisation of the SR model is recorded here explicitly (although briefly) rather than simply stating the results and referring to the earlier appendix for the method.

The dynamic system for the Solowian-Romer model is given by equations (3.29) to (3.32) of Section 3.4 of the text. Noting that:

$$\frac{\dot{SR}_i(t)}{SR_i(t)} = g_{SRi}(t) = \frac{d}{dt}\ln SR_i(t)$$

where $i=1, 2$ and $SR_1=\Psi$, $SR_2=p_A$, these may be written in log form as:

$$g_\Psi(t) = \frac{s}{\gamma^2}e^{\ln[r(t)+\delta]} + \zeta e^{\ln H_Y(t)} - (\delta + \zeta H) \tag{A3.8.1}$$

$$g_{p_A}(t) = e^{\ln[r(t)+\delta]} - \frac{1-\gamma}{\gamma}e^{\ln[r(t)+\delta]+\ln\Psi(t)-\ln p_A(t)} - \delta \tag{A3.8.2}$$

and

$$\ln H_Y(t) = \frac{\gamma}{1-\alpha(1-\gamma)}\ln\Psi - \frac{1}{1-\alpha(1-\gamma)}\ln p_A(t) + \text{ constant} \tag{A3.8.3}$$

$$\ln[r(t)+\delta] = -\frac{(1-\alpha)(1-\gamma)}{1-\alpha(1-\gamma)}\ln\Psi - \frac{\alpha(1-\gamma)}{1-\alpha(1-\gamma)}\ln p_A(t) + \text{ constant} \tag{A3.8.4}$$

These last two expressions are then used to linearise (A3.8.1) and (A3.8.2) by first-order Taylor series expansions about the log of the steady-state by obtaining:

$$g_{SR_i}(t) \approx [g_{SR_i}(t)]_{ss} + \sum_i [\frac{\partial g_{SR_i}(t)}{\partial \ln SR_i(t)}]_{ss}(\ln SR_i(t) - \ln SR_{iss})$$
$$\approx \sum_i [\frac{\partial g_{SR_i}(t)}{\partial \ln SR_i(t)}]_{ss} \ln[\frac{SR_i(t)}{SR_{iss}}]$$

Thus, log-linearising (A3.8.1), where in order to simplify the notation the time argument is omitted from this point onwards; and the term $[1-\alpha(1-\gamma)]^{-1}$ is written as Λ:

$$g_\Psi = -[\frac{s(1-\alpha)(1-\gamma)\Lambda}{\gamma^2}e^{\ln(rss+\delta)} + \gamma\zeta\Lambda e^{\ln H_{Yss}}]\ln\frac{\Psi}{\Psi_{ss}}$$
$$-[\frac{s\alpha(1-\gamma)\Lambda}{\gamma^2}e^{\ln(rss+\delta)} + \zeta\Lambda e^{\ln H_{Yss}}]\ln\frac{p_A}{p_{Ass}}$$

That is:

$$g_\Psi = \Lambda[\gamma\zeta H_{Y_{ss}} - \frac{s(1-\alpha)(1-\gamma)}{\gamma^2}(r_{ss}+\delta)]\ln\frac{\Psi}{\Psi_{ss}} - \Lambda[\frac{s\alpha(1-\gamma)}{\gamma^2}(r_{ss}+\delta)+\zeta H_{Y_{ss}}]\ln\frac{p_A}{p_{Ass}} \tag{A3.8.5}$$

And since (A3.8.2) is identical to (A3.5.4) its linearised form may be taken directly from (A3.5.9):

$$g_{p_A} = -\Lambda[(1-\alpha)(1-\gamma)(r_{ss}+\delta)+\gamma r_{ss}]\ln\frac{\Psi}{\Psi_{ss}} + \Lambda[r_{ss}-\alpha(1-\gamma)(r_{ss}+\delta)]\ln\frac{p_A}{p_{Ass}} \tag{A3.8.6}$$

Again following Appendix 3.5, the Solowian-Romer dynamic system may be written in matrix notation as:

$$\mathbf{g}_{SR} = \Omega_{SRL}\left[\mathbf{ln(SR(t)/SR_{ss}}\right]$$

where $\mathbf{g}_{SR}$ and $\mathbf{ln(SR(t)/SR_{ss})}$ are respectively the column vectors: $[g_\Psi,\ g_{pA}]^T$; and $[(\ln(\Psi(t)/\Psi_{ss}),\ \ln p_A(t)/p_{Ass})]^T$; and where the coefficients matrix Ω_{SRL} is:

$$\Omega_{SRL} = \begin{pmatrix} \dfrac{\gamma\zeta H_{Yss} - s(1-\alpha)(1-\gamma)(r_{ss}+\delta)/\gamma^2}{1-\alpha(1-\gamma))} & -\dfrac{\zeta H_{Yss} + s\alpha(1-\gamma)(r_{ss}+\delta)/\gamma^2}{1-\alpha(1-\gamma)} \\ -\dfrac{(1-\alpha)(1-\gamma)(r_{ss}+\delta)+\gamma r_{ss}}{1-\alpha(1-\gamma)} & \dfrac{r_{ss}-\alpha(1-\gamma)(r_{ss}+\delta)}{1-\alpha(1-\gamma)} \end{pmatrix} \tag{A3.8.7}$$

Evaluation of this for the benchmark parameter set yields:

$$\Omega_{SRL} = \begin{pmatrix} 0.01996 & -0.06930 \\ -0.06914 & 0.04620 \end{pmatrix}$$

from which the eigenvalues and eigenvectors are calculated as:

$\lambda_1 = 0.10039$ and $\lambda_2 = -0.04223$

$$\gamma_1 = \begin{pmatrix} 0.61684 \\ -0.78708 \end{pmatrix} \text{ and } \gamma_2 = \begin{pmatrix} 0.78779 \\ 0.61595 \end{pmatrix}$$

As shown earlier (Section 3.2.3), the negative eigenvalue (λ_2) represents the asymptotic speed of convergence, 4.223 per cent a year; and as will be seen later (Section 4.2.2) the eigenvector associated with it (γ_2) indicates the corresponding direction of motion thereby allowing computation of the saddle-path.

Chapter 4

4 Numerical integration

4.1 General application to the Romer system

The problem of numerically solving, or integrating, a system of ordinary differential equations (ODEs) is not completely specified by the equations themselves. Any system of n first-order ODEs also requires n *boundary conditions* (analogous to constants of integration) in order to be solved.[1] Whether these boundary conditions all relate to the same value of the independent variable, or whether they relate to different values defines a dichotomy in the types of integration problems: *Initial value problems* arise when values for all the dependent variables are given at a single point, typically the initial level of the independent variable; and so-called *boundary value problems* arise when dependent variable values are specified at more than a single point. Where only two such points are involved (often the initial level and the terminal level of the independent variable) the description is one of a *two-point boundary value problem.*

Two-point boundary value problems are, under most numerical methods, considerably more difficult to solve than initial value problems.[2] While initial value problems have a given valid solution with which to begin (and so can then simply be integrated forwards to whatever value of the independent variable is required), for two-point boundary value problems a valid starting solution must first be determined.

The fundamental approach of all numerical integration, whether for initial value or boundary value problems, is that of using derivatives multiplied by *step-sizes* to add small increments to the functions to be integrated. The differential equations are replaced by difference equation approximations (constructed from Taylor series expansions of the primitive functions), and the integration is effected through iterative application of these difference equations. The simplest approach is Euler's method where all second order (and higher) step-size terms from the Taylor expansion are simply ignored. Consider the set of coupled ODEs:

$$\dot{z}_i(t) = \frac{dz_i(t)}{dt} = g_i(z_1(t),\ldots,z_n(t),t) \qquad \text{for } i = 1,\ldots,n$$

or, in vector notation: (4.1)

$$\dot{\mathbf{z}}(t) = \mathbf{g}(\mathbf{z}(t),t)$$

[1] These define the values of the dependent variables at discrete specified points (values of the independent variable). The means of definition may take complicated functional forms, and the specification of points may be anywhere in the domain of integration. However, to simplify the exposition here we shall consider the definition to be simply a numerical value; and the boundary points to be either initial or terminal ones.

[2] The exception is the method of finite differences actually used here (see Section 4.3).

Because the differential equations cannot be solved analytically, the vector function $z(t) = \int g(z(t), t)dt$ is unknown. However, with h as the step-size its Taylor series expansion is:

$$z(t+h) = z(t) + h\,\dot{z}(t) + \frac{h^2}{2!}\ddot{z}(t) + \frac{h^3}{3!}\dddot{z}(t) + \ldots$$

and for the Euler method it is approximated by the difference equation:

$$z(t+h) \approx z(t) + h\,\dot{z}(t) \tag{4.2}$$

which, with all terms of order h^2 and higher having been dropped, is accurate only to *order-h*. The Euler method of numerical integration is then to proceed iteratively from the known initial condition $z(t_0) = z_0$ to (say) $z(t_N) = z_N$ via:

$$\begin{aligned} t_{k+1} &= t_k + h_k \\ z_{k+1} &= z_k + h_k g(z_k, t_k) \\ \text{where } k &= 0,1,2,\ldots,(N\text{-}1). \end{aligned} \tag{4.3}$$

A graphical illustration of this Euler process is presented in Appendix 4.1.

Being accurate only to *order-h*, Euler's method is generally considered a somewhat crude approximation, more of conceptual significance than of practical value.[3] However, by combining Taylor series expansions at nearby points, difference equation approximations of much greater accuracy can be obtained (Dixon et al, 1992). The possible choices are numerous.[4] Two other methods are described in Appendix 4.1. First, because it is often employed in the software used to undertake the simulations reported later in Section 4.2, the modified mid-point or Gragg's method is considered. And second, because it has high order accuracy and is by far the most commonly used integrator (Birkhoff and Rota, 1969; and Press et al., 1992), the fourth-order Runge-Kutta method is briefly described. The use of variable step-sizes and of *Richardson's extrapolation* techniques as means of increasing the accuracy of numerical integration are also briefly discussed in the Appendix.

To see how the Romer model could easily be integrated forwards if it were an initial value problem consider replacing its differential equations with Euler-type difference equation approximations. The dynamic system given by equations (2.41) to (2.45) then becomes:

---

[3] Baker and Gollub (1990) note that since the errors generated by the Euler method grow rapidly, it should be avoided for extended calculations. Nevertheless, as will be seen later the simple Euler approach performs extremely well for particular applications to the Romer model. It clearly out-performs Gragg, and while not as accurate as the fourth-order Runge-Kutta method, it is far less computationally intensive and for many purposes its results would be considered just as acceptable (see Appendix 4.2).

[4] Difference equation approximations can be based upon forward, backward or central differences and can apply to 2nd and higher order derivatives as well as to 1st order ones. In addition to Gragg's method, superior and more sophisticated approaches to that of Euler include the many varieties of Runge-Kutta; the Adams-Bashford *predictor* method; and *predictor-corrector* methods such as that of Gear and of Adams-Bashford-Moulton. Vast improvements in accuracy are also sometimes possible from *extrapolation* techniques such as Richardson extrapolations and the particular Bulirsch-Stoer implementation (see for example, Boyce and DiPrima, 1969; Dixon et al, 1992; Gear, 1971; Pearson, 1991; and Press et al, 1992). The choice of method obviously depends upon truncation error and computing effort, but questions of stability and consistency of the solutions also arise (Issacson and Keller, 1966).

$$\Psi_{t+1} = \Psi_t + h[\frac{r_t + \delta}{\gamma^2} - \Phi_t - \delta - \zeta H + \zeta H_{Y_t}]\Psi_t \tag{4.4}$$

$$\Phi_{t+1} = \Phi_t + h[\frac{r_t - \rho}{\sigma} - \frac{r_t + \delta}{\gamma^2} + \Phi_t + \delta]\Phi_t \tag{4.5}$$

$$p_{A_{t+1}} = p_{A_t} + h[r_t p_{A_t} - \frac{1-\gamma}{\gamma}(r_t + \delta)\Psi_t] \tag{4.6}$$

where

$$H_{Y_t} = [\frac{\alpha(1-\gamma)}{\zeta\eta^\gamma} L^{(1-\alpha)(1-\gamma)} \Psi_t^\gamma p_{A_t}^{-1}]^{\frac{1}{1-\alpha(1-\gamma)}} \tag{4.7}$$

$$r_t = \frac{\zeta\gamma^2}{\alpha(1-\gamma)} H_{Y_t} p_{A_t} \Psi_t^{-1} - \delta \tag{4.8}$$

and, with the three initial values given as Ψ_0, Φ_0 and p_{A0} at time t=0, it can be seen that Ψ_1, Φ_1 and p_{A1} could readily be generated and so the system could be integrated forwards to obtain Ψ_2, Φ_2 and p_{A2} and so on.

However, *initial values* are known only for some of the variables. Nevertheless, as demonstrated in Chapter 2 the Romer model system of differential equations derives fundamentally from two dynamic optimisation problems where necessary conditions for optimisation not only specify the equations, but also certain *transversality conditions*, which impose constraints on the terminal behaviour of the dynamic system, ensuring that it tends asymptotically to its steady-state equilibrium. In this way, the transversality conditions supply *end values* for the remaining variables and so define the problem of solving the dynamic system as a two-point boundary value problem.

In the linearised model (Section 3.2), the transversality conditions were shown to impose certain constraints, which could be determined analytically, on the initial values of the dynamic variables. This amounted to a determination of the *policy function* or *saddle-path* of the system. In particular, with m constraints and n variables, only (n-m) initial values could be freely selected or given. For the non-linear system it is not possible to derive initial value constraints analytically. Nevertheless, such constraints still exist. In the absence of an explicit policy function or saddle-path they are imposed by fixing initial values for some variables and end values for others. Any variable whose initial value is not fixed will then have to jump discontinuously in order to take the system to the saddle-path. This raises the questions of which variables are to have their initial values fixed and which are then to be determined in order to satisfy the transversality conditions; and to what extent this choice is a free one. The maximum principle of optimal control theory, by which the dynamic maximisation problems in the model were solved, requires that state variables be continuous while control and costate variables may incorporate discontinuities (Appendix 2.2). This seems to dictate the choice; but it might be possible to swap the state and control variables in the formulations of some problems.

In any event, the choice is rarely arbitrary in economic terms. It depends on the particular problem being analysed but it is most common for initial values to be specified for stock variables like capital (which are commonly the state variables), and for end values to be specified for prices or flow variables like consumption (the costate and control variables). The reason for this is that stock variables are not **usually** discontinuous. In response to most economic shocks their immediate pre- and post-shock values will be identical.[5] On the other hand, prices and flow variables can more readily exhibit discontinuities and jump spontaneously in response to relevant shocks. Accordingly, they are sometimes referred to as *jumping variables*.[6]

In the present context, where there are two constraints imposed by transversality conditions, only one initial value may be specified freely. The obvious choice for this is the variable $\Psi = K/A$, since both K and A are stock variables. The other necessary boundary conditions are then provided by the steady-state solution to the dynamic system. In particular, equations (2.54) to (2.59) can be used to specify the *end values* Φ_{ss} and p_{Ass} expressed in terms of the model parameters and exogenous variables.

A common approach to solving two-point boundary value problems is that of *shooting*. In this method an originally 'guessed' set of initial values is iteratively updated according to how accurately the subsequent integration satisfies *the end value boundary conditions* (see Birkhoff and Rota, 1969; Dixon et al, 1992; Keller, 1968; Press et al, 1992; and Roberts and Shipman, 1972). There is no problem with the method when there is only a single 'end-value target' at which to shoot. For example, King and Rebelo (1993) calculated transition paths for one-sector neoclassical growth models by this method. Also, as seen in Section 3.4.2, the shooting method easily furnished transitional dynamics for the Solowian-Romer model. However, when *shooting* was applied to the current problem the system could not be made to converge at all.[7] In theory, for smooth varying functions (but not in the neighbourhood of a local extremum) the Newton-Raphson method, which was employed here, will converge quadratically, the error reducing by a power of two after each successive iteration, to a steady-state of the system **provided** the initial guess is "close enough". However, for a system such as the Romer one, which is extremely sensitive to initial conditions, it turns out to be extremely difficult in practice to make a sufficiently close first guess (see Roberts and Shipman, 1972 and Press et al 1992). Consequently, two other broad approaches are examined. The first is to consider conversion to an initial value problem, and the second to employ the so-called *finite differences* methods, known to be particularly useful for solving numerically sensitive two-point boundary value problems (Roberts and Shipman, 1972).

[5] Nevertheless, discontinuous jumps in stock variables can and do occur. Moreover they are not rare, often arising through natural disasters, wars, and other politically based reorganisations: The massive destruction of physical capital in World War II, particularly the atomic bombing of Japan, is perhaps the greatest example this century. Others include the Gulf war, the 1993 Los Angeles and the 1995 Kobe earthquakes, the re-unification of Germany, and the dissolution of the Soviet Union. Capital may also be considered to change discontinuously by thinking of it as utilised capital rather than as the total stock.

[6] The scope for price variables to change discontinuously is probably the stronger, although even here it is possible to envisage at least temporary restraints through general stickiness, or perhaps more likely, through various institutional rigidities such as price surveillance authorities, trade unions, minimum wage legislation, and fixed or pegged exchange rate systems like the European Monetary Union or the Hong Kong dollar. In the case of flow variables like output or consumption there may be even more constraints to discontinuous jumps. For example, with binding labour market constraints a discontinuous increase in output would not be possible without a corresponding jump in the stock variable capital! Similarly, with insufficient inventories it may be equally difficult for consumption to increase discontinuously. Finally, it should be noted that these difficulties tend to be asymmetric - discontinuous decreases not facing the same constraints as similar increases.

[7] This was the case despite the use of small step sizes (h = 0.1), allowing for a long convergence period (400 years), and employing the fourth-order Runge-Kutta integration technique.

4.2 Conversion to an initial value problem

It may be thought possible to obtain the saddle-path of a dynamic system such as that of the Romer model by solving an initial value problem where the steady-state solution to the model provides the necessary initial conditions. Up to a point this turns out to be true, but it is not as simple as might first appear. Since the steady-state values can be derived from the underlying differential equations together with the transversality conditions, they constitute a point on the saddle-path. And since the differential equations are autonomous, that is they do not explicitly involve the independent variable (time), these steady-state values correspond to a set of simultaneous values of all the dynamic variables on that path at an arbitrary but common point of time, which may be defined as initial time, t=0. Thus, the steady-state certainly provides a valid set of initial values. However, the difficulty with using such initial values is that since all of the variables are stationary at the steady-state, their time derivatives are zero and the differential equations provide no indication about how to proceed from that point and so cannot be integrated, or at least not directly. Nevertheless, there are two ways out of the dilemma.

4.2.1 Time elimination method

Mulligan and Sala-i-Martin (1991 and 1993) describe and recommend a method under which, by taking ratios of the differential equations (which must be autonomous), the original independent variable, time, is replaced by one of the dependent dynamic variables chosen arbitrarily. Despite the fact that at the steady-state the differential equation expressions for the new derivatives produce an indeterminancy (*zero÷zero*), it is possible to evaluate the limits of these by employing L'Hopital's rule. Since these limits are in general finite and non-zero they can then be used as initial values to integrate the new system. This approach is referred to as the *time elimination method*. Its great computational benefit is that it is an initial value problem rather than a boundary value problem.[8] Details of its application to the Romer system are at Appendix 4.2, while a broad description follows:

Forming the ratios $\dot{\Psi}(t)/\dot{\Phi}(t)$ and $\dot{\Psi}(t)/\dot{p}_A(t)$ from equations (2.41) to (2.43) eliminates time as the independent variable and reduces the system to one of only two linked, non-autonomous ODEs with independent variable Φ:

$$\frac{\dot{\Psi}(t)}{\dot{\Phi}(t)} = \frac{d\Psi(t)/dt}{d\Phi(t)/dt} = \frac{d\Psi(\Phi)}{d\Phi} = \frac{[(r+\delta)/\gamma^2 - \Phi + \zeta H_Y - (\delta + \zeta H)]\Psi}{[(r-\rho)/\sigma - (r+\delta)/\gamma^2 + \Phi + \delta]\Phi} \tag{4.9}$$

$$\frac{\dot{p}_A(t)}{\dot{\Phi}(t)} = \frac{dp_A(t)/dt}{d\Phi(t)/dt} = \frac{dp_A(\Phi)}{d\Phi} = \frac{rp_A - (1-\gamma)(r+\delta)\Psi/\gamma}{[(r-\rho)/\sigma - (r+\delta)/\gamma^2 + \Phi + \delta]\Phi} \tag{4.10}$$

where H_Y and r are to be substituted out in terms of Ψ, Φ, and p_A via equations (2.44) and (2.45). In geometrical terms these derivatives furnish the slopes of the saddle-path in the system's phase-space. However, since $\dot{\Psi}_{ss} = \dot{\Phi}_{ss} = \dot{p}_{Ass} = 0$, both the numerators

[8] In addition, time elimination also reduces the number of linked differential equations to be solved by one, but the new system is no longer autonomous.

and denominators in the differential equations are zero at the steady-state and these slopes cannot be evaluated there. This indeterminancy would seem to preclude integration of the system from steady-state initial values, but the difficulty can be overcome by invoking L'Hopital's rule. Denoting the new derivatives as $d\Psi(\Phi)/dt=\Psi'(\Phi)$ and $dp_A(\Phi)/d\Phi=p_A'(\Phi)$, this produces:

$$\Psi'_{ss} = \lim_{\Phi\to\Phi_{ss}} \Psi'(\Phi) = \lim_{\Phi\to\Phi_{ss}} \frac{\frac{d}{d\Phi}[(r+\delta)/\gamma^2 - \Phi + \zeta H_Y - (\delta+\zeta H)]\Psi}{\frac{d}{d\Phi}[(r-\rho)/\sigma - (r+\delta)/\gamma^2 + \Phi + \delta]\Phi} \qquad (4.11)$$

and

$$p'_{Ass} = \lim_{\Phi\to\Phi_{ss}} p'_A(\Phi) = \lim_{\Phi\to\Phi_{ss}} \frac{\frac{d}{d\Phi}[rp_A - (1-\gamma)(r+\delta)\Psi/\gamma]}{\frac{d}{d\Phi}[(r-\rho)/\sigma - (r+\delta)/\gamma^2 + \Phi + \delta]\Phi} \qquad (4.12)$$

whereupon substitution of (2.44) and (2.45) for H_Y and r, and differentiation of the above numerators and denominators generates a pair of polynomials in Ψ'_{ss} and p'_{Ass} of the form:

$$a_1(\Psi'_{ss})^2 + a_2(\Psi'_{ss}p'_{Ass}) + a_3(\Psi'_{ss}) + a_4(p'_{Ass}) + a_5 = 0 \qquad (4.13)$$

and

$$a_2(\Psi'_{ss})^2 + a_1(\Psi'_{ss}p'_{Ass}) + a_6(\Psi'_{ss}) + a_7(p'_{Ass}) = 0 \qquad (4.14)$$

in which a_1 to a_7 are functions of the parameters and steady-state values (see Appendix 4.2 for details). These equations yield three solutions. One is the desired solution for the trajectory along which the steady-state is approached, which has been referred to as "the saddle-path" but which might more descriptively have been called "the stable arm of the saddle equilibrium". The other two solutions relate to the two "unstable arms" of the saddle, trajectories which only lead **away** from the steady-state.[9]

Of course the existence of three solutions raises the question of which particular one relates to the stable arm, or saddle-path? Mulligan and Sala-i-Martin (1991) suggest that a trial and error approach is not inconvenient but that "economic intuition" will often help to identify it. The phase-space analysis of Section 3.3 more than confirms this last point, demonstrating that the signs of the slopes of the saddle-path, as well as the regions of phase-space in which it must lie, can readily be identified. Specifically, in Table 3.1 it was shown that the saddle-path must lie in either:

- region R1 where $\dot{\Psi} < 0$, $\dot{\Phi} > 0$, and $\dot{p}_A < 0$ and so both $\Psi'(\Phi)<0$ and $p'_A(\Phi)<0$; or
- in region R8 where $\dot{\Psi} > 0$, $\dot{\Phi} < 0$, and $\dot{p}_A > 0$ and again $\Psi'(\Phi)<0$ and $p'_A(\Phi)<0$.

Thus, when equations (4.13) and (4.14) are solved for the benchmark parameter set, generating the solutions:

$$\begin{aligned} &\Psi'_{ss1} = -50.8455,\ p'_{Ass1} = -53.7116; \\ &\Psi'_{ss2} = -32.1058,\ p'_{Ass2} = 22.5429; \text{ and} \\ &\Psi'_{ss3} = -630.637,\ p'_{Ass3} = 1139.28; \end{aligned} \qquad (4.15)$$

the first of these is readily identified as the solution for the saddle-path.

9 The different solutions arise because all three saddle arms satisfy the differential equations and the steady-state lies on each of them. What distinguishes the stable arm is that it also satisfies the transversality conditions.

Having evaluated Ψ'_{ss} and p'_{Ass} it can easily be seen how the 'time eliminated' differential equations (4.11) and (4.12) could be numerically integrated - for example, by an Euler process similar to that described by equations (4.3) and (4.4) to (4.8). Furthermore, as may be seen from the phase-space analysis, by making the small steps in the dependent variable positive, the R8 portion of the saddle-path is calculated; and making them negative generates the R1 portion. Computations of the saddle-path based upon the Euler, Gragg, and fourth-order Runge-Kutta methods of numerical integration are at Appendix 4.2. They reveal three principal facts:

- first, that for this application at least, the Gragg approach is highly unstable and divergent and as a result, wholly unsuitable for the problem;
- second, that for sensibly small step-sizes the Euler and RK4 methods track one another very closely for a considerable duration of the integration, and that in contrast to Gragg they are highly stable, correctly estimating the phase-space regions in which the saddle-path lies for unlimited iterations; and
- thirdly, that when the integrations are extended to distant values of the independent variable (Φ), the Euler and RK4 methods begin to diverge, with the RK4 results clearly more consistent and accurate.

Overall, taking the computing and data manipulation effort into account as well as the requirement for accuracy, the most efficient of the many integration configurations would seem to be the RK4 based on step-sizes of ±0.001.

If adjustment paths of the dynamic variables over time are required, because time has been eliminated in this system another initial value problem must be solved for the original dynamic system, where the initial values are obtained from some appropriate point on, or interpolated from, the computed saddle-path. However, in practise it turns out that integrations of these 'second' initial value problems are unstable and rapidly diverge from the saddle-path and its steady-state (Appendix 4.2).

4.2.2 Eigenvector-backwards-integration method[10]

The eigenvectors of a linearised dynamic system such as the Romer one provide a precise indication of the directions of trajectories into and out of the steady-state. That is, they measure the slopes of the saddle arms at the steady-state.[11] Mulligan and Sala-i-Martin (1991 and 1993) note that this fact can be used to identify the "correct" solution of the 'derivatives-polynomials' generated by the application of L'Hopital's rule in the time elimination method. While this is certainly true, it is also unnecessary. Given the eigenvectors of a dynamic system, time elimination is not necessary to convert the integration problem to an initial value one.

The difficulty with beginning an integration of the original dynamic system from precisely the steady-state still exists, the differential equations being zero there. But, by taking a

10 This method was used to calculate the 'benchmark saddle-path' in Section 3.3 (Figures 3.11 to 3.16).

11 It can be seen that division of the first (Ψ) and third (p_A) elements by the second (Φ) elements of the eigenvectors calculated in Section 3.2 for the benchmark parameter set (equation (3.5) in particular) reproduces, approximately, the values of the saddle-path slopes given in equation (4.15). Calculation with greater decimal accuracy shows that the correspondence is virtually exact.

small step away from the steady-state to a nearby point in the direction of the eigenvector corresponding to the negative eigenvalue, the differential equations then evaluate to non-zero values. Using the (known) coordinates of this point as initial values, the original differential equations may then be integrated **backwards in time** to generate the saddle-path. Since an eigenvector actually specifies two diametric directions,[12] the initial step away from the steady-state may be in either of these exactly opposite directions. One of these will be into region R1 of the phase-space, the other into region R8. Backwards-integration will then generate the R1 and R8 segments of the saddle-path respectively.

As for the time elimination technique, these integrations have been performed under each of the Euler, Gragg, and fourth-order Runge-Kutta methods and the results compared in Appendix 4.2. The principal conclusions from this analysis were precisely the same as those for the time elimination technique, with the superiority of the RK4 over the Euler method being even more pronounced. Overall, the most efficient of the integration configurations was determined as being an initial *eigenvector step* of $\pm 0.0007\gamma_1$, and an RK4 integration with a time step-size of -0.5, or perhaps -0.25 for the R8 arm of the saddle-path. One advantage this method has over that of time elimination is in the calculation of adjustment time paths. Since the integration here is performed over time, such paths are automatically calculated and merely require a simple change of the time origin, as opposed to the solution of another initial value problem which is necessary under the time elimination method (and which tends to be divergent).[13]

4.3 Finite differences and the GEMPACK software

Two-point boundary value problems such as that posed by the dynamic Romer model may also be solved by numerical procedures known as *finite differences* (for example, see Wilcoxen, 1985). Such methods typically involve the simultaneous solution of a large number of non-linear equations, a problem in itself. However, this can be solved flexibly and accurately via a suite of computer software known as GEMPACK.[14] The issues involved in all this are discussed in the following, both in general terms and in respect of the particular Romer model problem.

4.3.1 Method of finite differences

In common with the fundamental approach to all numerical integration techniques, the so-called *finite differences methods* involve the replacement of continuous time differential equations with discrete time finite difference equation approximations (FDEs). These are generated by the substitution of *finite difference* approximations,

[12] The information content of an eigenvector is unchanged by scalar multiplication, including by '-1'.

[13] Although unnecessary, attempting to solve a second initial value problem in the same way as for the time elimination method encounters the same sort of divergence problems (Appendix 4.1).

[14] The GEMPACK Software System was developed by the Impact Project and KPSOFT in order to solve large scale economic models, in particular, the ORANI and MONASH models. Its name is an acronym of 'General Equilibrium Modelling Package'. The approach of integrating a set of intertemporal difference equations into the Johansen (1960) solution procedure employed by GEMPACK, was due to Wilcoxen (1987). Also see Dixon et al. (1992).

constructed from Taylor series expansions in exactly the same manner discussed previously (Section 4.1), for all derivatives in the original differential equations. However, unlike other methods the resulting difference equations are not solved iteratively. In this sense methods of finite differences are not strictly numerical integration techniques.

The methods proceed by replacing the continuous domain of the set of differential equations with a finite grid of discrete points. The FDEs are then used to express the values of the primitive functions - **z**(t) from equation (4.1) and the ensuing discussion- at each of these points in terms of the values at other neighbouring points. The resulting set of FDEs is then solved **simultaneously** to yield the values of **z**(t) at every grid point, an approximation to the entire path of **z**(t) over the original domain. Since both the specified initial and terminal boundary conditions are incorporated in the FDEs, the solution is constrained to satisfy them.[15] Also, this means that under the finite differences approach there is no difference in the solution procedures for initial and two-point boundary value problems (see footnote 2).

When the original differential equations are non-linear, as they are in the Romer model, the resulting set of FDEs will be similarly non-linear. Then, because there are no general methods for solving such systems, they must be solved by some linearisation technique. This involves the approximation of the difference equations by linear functions in the neighbourhood of particular *linearisation points*. Since the set of difference equations relates to all points of the grid, the corresponding set of all the necessary linearisation points must be a complete solution path of the FDE-model over the domain of interest. Such a solution is often referred to as a *base case*, and is frequently specified as a steady-state, or constant path.

As a result of the procedures described above, a two-point boundary value problem comprising (say):

- n dynamic variables and an equal number of first-order differential equations; and
- with the correspondingly necessary n boundary conditions split as m initial conditions and (n-m) terminal conditions;

is transformed via a grid of (T+1) points: $t = 0,1,...,T$; into a problem of solving $[m + nT + (n-m)] = n(T+1)$ linear equations for the n(T+1) variables over the grid. In principle this is a simple matter of matrix inversion. The problem can be written as: Solve the equation:

$$\mathbf{Ax} = \mathbf{b} \tag{4.16}$$

for the vector **x** of dynamic variables at each grid point; where **A** is an n(T+1) x n(T+1) matrix of coefficients comprising parameter values, and the step-sizes between grid points; and **b** is a vector of parameter values, step-sizes, and boundary conditions (Dixon et al, 1992). And the solution is simply:

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

[15] Replacement of the original ODEs with an approximating set of FDEs on a grid of points spanning the range of integration also forms the basis of *Relaxation methods*. As with shooting methods, here the procedure is to begin with a *trial solution* or guess and then to improve it iteratively. As the iterations improve the solution they are said to *relax* (see Press et al, 1992).

In practice the problem of obtaining A^{-1} may not be that simple. For example, with a grid of 350 intervals applied only to each of the three dynamic variables, the matrix A would be of dimension 1053x1053.[16] However, it turns out that as a result of the finite differencing techniques that are used to generate these matrices, they have a well defined (diagonally banded) structure which allows (4.16) to be solved for **x** much more simply by non-inversion matrix techniques such as *Gaussian elimination*, or *LU decomposition* and back-substitution.[17] Such techniques are employed within the GEMPACK software.[18] Also, it should be noted here that although linearisation of the dynamic equations involves an approximation, highly accurate results can be obtained quite easily via GEMPACK. In fact theoretically, results accurate to any arbitrary level are attainable (see the reference to Harrison and Pearson (1996a), following equation (4.22) in the next sub-section.

4.3.2 GEMPACK and its application to the Romer model

Briefly, GEMPACK is:

> "...a suite of general-purpose modelling software especially suitable for general and partial equilibrium models. It can handle a wide range of economic behaviour and also contains powerful capabilities for solving intertemporal models. GEMPACK provides software for calculating accurate solutions of an economic model, starting from an algebraic representation of the equations of the model. These equations can be written as levels equations, linearised equations or a mixture of the two." (quotation from the *Abstract* of Harrison and Pearson, 1996a).[19]

In general, GEMPACK solves models of the type:

$$g_i(y_1, y_2, \ldots, y_m) = 0, \quad \text{for } i = 1,2,\ldots,n; \tag{4.17}$$

where the equations g_i may be non-linear and where there are more variables than equations (m>n), so that (m-n) of the variables must be specified exogenously. The finite difference equations for the Romer model can easily be written in this way.

For example, with the Euler differencing method the model can be written in the form of equations (4.4) to (4.8) with a time grid: t = 0,1,...,T (as in Section 4.1). In this case it comprises (T+1) discrete time-subscripted variables for each of the five continuous time variables $\Psi(t)$, $\Phi(t)$, $p_A(t)$, $H_Y(t)$, and $r(t)$. Also, in order to conduct simulations with the model by imposing shocks to parameter values, it is convenient to declare all of its nine

[16] While a 350 interval grid is actually used later (in Section 4.5) to conduct simulations on the model, the matrices whose inverses need to be found in those simulations are considerably larger than 1053x1053; namely 6305x6305 (see Section A4.3.2 of Appendix 4.3).

[17] Roberts and Shipman (1972) also mention Gauss-Seidel and Jacobi procedures - also see Press et al (1992).

[18] GEMPACK employs the Harwell Laboratory's "sparse-linear-equation-solving routines MA28 and MA48"; and the GEMPACK documentation refers to Duff (1977); and to Duff and Reid (1993). See Harrison and Pearson (1996a & 1996b).

[19] For a complete description of the GEMPACK procedures and suite of programs see Codsi and Pearson (1988) and the GEMPACK User Documentation for Release 5.2, namely: Harrison and Pearson (1993, 1994, 1996a and 1996b). Also, see Codsi, Pearson and Wilcoxen (1991) for an account of the use of GEMPACK in intertemporal modelling.

parameters (α, γ, ζ, δ, ρ, σ, η, H, and L) as variables of discrete time to the software. Then, the total number of variables confronted by GEMPACK is:

$$m = (5+9)x(T+1) = 14(T+1)$$

On the equations side, there are T *inter-temporal* equations linking the values of each of the three principal dynamic variables, Ψ_t, Φ_t, and p_{At} (viz. one equation links Ψ_1 with Ψ_0, a second links Ψ_2 with Ψ_1,..., and a T^{th} links Ψ_T with Ψ_{T-1}). In addition, there are (T+1) *atemporal* equations necessary to define each of the variables H_{Yt} and r_t at each grid point. Finally, there are two steady-state equations specifying the terminal values of the jumping variables, Φ_T and p_{AT}. Overall, the number of equations is:

$$n = 3T + 2(T+1) + 2 = 5(T+1) - 1$$

Comparing the number of variables and equations, it is clear that the number of variables that must be declared as exogenous is:

$$m-n = 9(T+1) + 1$$

Similarly, for the fourth order Runge-Kutta method of finite differencing (RK4) described in Appendix 4.1, the model includes the same (T+1) time-subscripted variables for each of the five continuous time variables and nine parameters as above. In addition, as may be seen from equation (A4.1.4), at each time point t=0,1,...,(T-1), there are four intermediate variables for each of the principal dynamic variables $\Psi(t)$, $\Phi(t)$, and $p_A(t)$. Overall the number of variables in the model is then:

$$m = (5+9)x(T+1) + 12T = 14(T+1) + 12T$$

Also, under 4^{th} order Runge-Kutta the model contains the same 2(T+1) *atemporal* equations and the same 2 terminal value boundary equations as for the case of Euler differencing. Since it also comprises 3T *intertemporal* equations linking the principal dynamic variables and 12T *atemporal* equations for the intermediate RK4 variables (equation (A4.1.4)), the total number of equations is:

$$n = 2(T+1) + 2 + 3T + 12T = 5(T+1) + 12T - 1$$

So again the number of variables that must be declared as exogenous is:

$$m-n = 9(T+1) + 1$$

Usually these would be the parameter values at all grid points, plus the initial value of the non-jumping variable, Ψ_0; and not all of these exogenous variables would be shocked in any given simulation.

Returning to the description of the manner in which GEMPACK solves the finite differences formulation of the model, the exogenous variables in (4.17) are denoted by x, and the remaining, endogenous variables by z. Then, following Dixon et al (1982) and Pearson (1991), the general type of model solved by GEMPACK may be written as:

$$g_i(z_1, z_2, \ldots, z_n; x_1, x_2, \ldots, x_{m-n}) = 0 \quad \text{for } i = 1,2,\ldots,n$$

or

$$g_i(\mathbf{z}, \mathbf{x}) = 0 \quad \text{for } i = 1,2,\ldots,n \tag{4.18}$$

where $\mathbf{z}$ and $\mathbf{x}$ are nx1 and (m-n)x1 vectors of the endogenous and exogenous variables respectively.

For such models the fundamental problem addressed by GEMPACK is a *simulation* one of the form: "Given that $\mathbf{z}=\mathbf{z}_0$ and $\mathbf{x}=\mathbf{x}_0$ is a solution of the model (4.18), find the values of the endogenous variables $\mathbf{z}=\mathbf{z}_s$ (where $\mathbf{z}_s=\mathbf{z}_0+\Delta_T\mathbf{z}$), which solve the model when the exogenous variables are shocked by $\Delta_T\mathbf{x}$ from $\mathbf{x}=\mathbf{x}_0$ to $\mathbf{x}=\mathbf{x}_s=\mathbf{x}_0+\Delta_T\mathbf{x}$".[20] One way to obtain numerical solutions to these *simulation problems* is to linearise the model equations by differentiation, make the approximation that the overall shock $\Delta_T\mathbf{x}$ to the exogenous variables can be considered as *differentially small*, and then solve the resulting linear system of equations for $\Delta_T\mathbf{z}$ by matrix algebra. Thus, from (4.18):

$$dg_i = \sum_{j=1}^{n}\frac{\partial g_i}{\partial z_j}dz_j + \sum_{k=1}^{m-n}\frac{\partial g_i}{\partial x_k}dx_k = 0 \quad \text{for } i = 1,2,\ldots,n \tag{4.19}$$

whereupon substituting $\Delta_T x_k \approx dx_k$ and $\Delta_T z_j \approx dz_j$ generates the matrix solution:

$$\Delta_T\mathbf{z} \approx -\mathbf{A}^{-1}\mathbf{B}\Delta_T\mathbf{x} \tag{4.20}$$

where

$\Delta_T\mathbf{z}$ is the nx1 vector $(\Delta_T z_1,\ldots,\Delta_T z_n)^T = (z_{s1}-z_{01},\ldots,z_{sn}-z_{0n})^T$;
$\Delta_T\mathbf{x}$ is the (m-n)x1 vector $(\Delta_T x_1,\ldots,\Delta_T x_{(m-n)})^T = (x_{s1}-x_{01},\ldots,x_{s(m-n)}-z_{0(m-n)})^T$; and
$\mathbf{A}$ and $\mathbf{B}$ are the nxn and nx(m-n) matrices of partial derivatives $\partial g_i/\partial z_j$ and $\partial g_i/\partial x_k$ respectively.

Such an approximate solution, where the entire shock is taken to be *differentially small*, is the *Johansen solution*. It may be improved by breaking the overall exogenous shocks down into smaller sub-shocks and generating an iterative solution by an Euler type of process. Indeed, as the sub-shocks are made smaller and smaller, the numerically computed solution converges to the true solution (see Dixon et al., 1982; and Johansen, 1960). This suggests a link between simulation modelling problems of the type described above, and initial value problems with differential equations which were discussed earlier in Section 4.1. Pearson (1991) has shown formally how such simulation problems can be expressed as initial value problems for a set of ODEs where the dependent variables are the endogenous variables $\mathbf{z}$; and the independent variable v, measures fractions of the overall shocks to the exogenous variables $\mathbf{x}$. For this correspondence, v is defined by:

$$\mathbf{x} = \mathbf{x}_0 + v(\mathbf{x}_s - \mathbf{x}_0) = \mathbf{x}_0 + v\Delta_T\mathbf{x} \tag{4.21}$$

so that when v = 0, $\mathbf{x} = \mathbf{x}_0$; and when v = 1, $\mathbf{x} = \mathbf{x}_s$. Using this relation and dividing equation (4.19) by "dv" produces the ODE set:

$$\frac{d\mathbf{z}}{dv} = -\mathbf{A}^{-1}\mathbf{B}\Delta_T\mathbf{x} \tag{4.22}$$

In this way, the initial value problem corresponding to the earlier simulation problem is as follows: "Given $\mathbf{z}=\mathbf{z}_0$ when v=0, and given the ODEs (4.22), find $\mathbf{z}=\mathbf{z}_s$ when v=1." This can then be solved by any of the numerical integration methods for initial value

[20] The subscript 'T' in $\Delta_T\mathbf{z}$ and $\Delta_T\mathbf{x}$ denotes that these changes are the 'total' changes in $\mathbf{z}$ and $\mathbf{x}$.

problems in exactly the way described earlier in Section 4.1 and Appendix 4.1.[21] The GEMPACK software offers the Euler, Gragg, and the midpoint (or 2nd order Runge-Kutta) methods, each of which may be extrapolated by the Richardson procedures for either two or three sets of steps (see Appendix 4.1).[22] Such methods naturally provide great improvements in accuracy over the single step Johansen solution. Theoretically, results accurate to any arbitrary level are attainable simply by taking enough steps (Harrison and Pearson, 1996a).

4.4 Specification of the solution method

The sensitivity of the Romer system steady-state to changes in its parameter values was examined in Section 2.4.2 where, after changing some parameter the resultant new steady-state was simply compared with the original (benchmark) steady-state. Such analysis is referred to as *comparative statics* and its shortcoming is that it ignores the dynamic path between equilibria, that is the *transient dynamics*.[23] Here, the object is to define precisely how to calculate these dynamics. The three candidate methods are those just discussed in Sections 4.2 and 4.3: time elimination; eigenvector-backward integration; and finite differences implemented via GEMPACK. In the following the procedures necessary, under each of these methods, to calculate the dynamic paths that result from different types of shock are examined and compared. The aim being to identify a preferred method.

4.4.1 Simulating unanticipated shocks

Time elimination technique

First, the post-shock steady-state needs to be calculated and the results substituted into the coefficients of polynomials of the form of (4.13) and (4.14). The appropriate formulae are at Appendix 4.2. These must then be solved to obtain the derivatives of the time-eliminated system at this steady-state (Ψ'_{ss} and p'_{Ass}). Next, the initial value problem for the time eliminated system must be solved to generate the post-shock saddle-path for values of the non-jumping variable (Ψ) which include its pre-shock value - that is, for either its R1 or R8 saddle-path segment as appropriate (Section 3.3.1). This numerical solution, together with the pre-shock value of Ψ, is then used to interpolate the immediate post-shock values of the jumping variables (Φ and p_A). Finally, these values of Ψ, Φ, and p_A are used as initial values to solve the original differential equations (those

[21] Note that for any small change Δv in v, where Δv corresponds to the step-size in numerical integration, the corresponding small change in $\mathbf{x}$ is found from (4.21): $\Delta\mathbf{x} = (d\mathbf{x}/dv)\Delta v = \Delta_T\mathbf{x}\Delta v$; and the resultant small change in $\mathbf{z}$ is then obtained from (4.22) as: $\Delta\mathbf{z} = (d\mathbf{z}/dv)\Delta v = -\mathbf{A}^{-1}\mathbf{B}\Delta_T\mathbf{x}\Delta v$.
[22] In practice GEMPACK is usually implemented to solve for percentage changes in the endogenous variables, **p_z** = $(\Delta\mathbf{z}/\mathbf{z})100$, given exogenous shocks which are also expressed as percentage changes, **p_x** = $(\Delta\mathbf{x}/\mathbf{x})100$. Clearly, as long as the base levels $\mathbf{z}$ and $\mathbf{x}$ are never zero, the percentage change variables can simply be calculated directly from levels changes. To calculate them directly multiply (4.19) by $\Delta v/dv$, then (4.22) becomes: **p_z** = -$\mathbf{C}^{-1}\mathbf{D}$.**p_x**, where $\mathbf{C} = ([\partial g_i/\partial z_j].z_j)$, and $\mathbf{D} = ([\partial g_i/\partial x_k].x_k)$.
[23] Of course the system need not be in equilibrium at the time of a shock, and its subsequent path need not be examined all the way to its new steady-state.

for which time is the independent variable) with the post-shock exogenous variables or parameters, to obtain the adjustment path. But note that this last integration tends to be unstable (Appendix 4.2).

Eigenvector-backwards-integration method

Here again the post-shock steady-state first needs to be calculated. Under this technique the results must then be substituted into the coefficients matrix of the linearised system (Appendix 3.1), from which the eigenvectors corresponding to the negative eigenvalue must be computed. A small fraction of these eigenvectors are then used to take a small step away from the steady-state and 'towards' the pre-shock value of the non-jumping variable, Ψ. From the resulting point, an initial value problem is solved to compute the relevant section (either R1 or R8) of the saddle-path. The immediate post-shock values of the jumping variables (Φ and p_A) are then interpolated as in the time elimination method; and finally, the time variable is re-numbered to increase from t=0 at the pre-shock value of the non-jumping variable towards the post-shock steady-state.

Finite differences and GEMPACK

Here the principal effort is that of writing two input files for the GEMPACK software. The first of these is a *TABLO Input file*, which specifies the model and its finite differences implementation. In particular, it specifies all variables and parameters; the (non-linear) dynamic equations and the method of their finite differencing; the boundary conditions; and the (pre-shock steady-state) *base case* solution.[24] TABLO Input files are model specific, so all individual simulations can be run from the one file. The second is a *GEMPACK Command file* to carry out a particular simulation. This defines the closure of the model (which variables are to be endogenous and which are to be exogenous), the shocks to the exogenous variables; and the method of multi-step solution - that is Euler, Gragg, or mid-point. Unanticipated shocks are simply applied at initial time (time zero). Although these Command files are simulation specific, only negligible alterations were necessary for the different simulations. The only other issue in obtaining results under this approach is the simple conversion of the GEMPACK output from its 'changes relative to the base case' form, to absolute levels. Details of the TABLO and Command files are at Appendix 4.3. Also see Codsi, Pearson and Wilcoxen (1991).

[24] GEMPACK subsequently converts this to a FORTRAN program, compiles and links it, and produces an executable (TABLO generated) program from it. Part of this process involves the linearisation of the model equations for which GEMPACK offers two methodological choices: *percentage change differentiation* and simple *change differentiation*. It has been found that for certain models and simulations, the former of these, unlike the latter, can produce a similar instability in the numerical results to that encountered with the Gragg method of numerical integration in Appendix 4.2. As the number of steps in the integration increases, the final results exhibit rapidly widening oscillations, making any Richardson type extrapolations (Appendix 4.1) inaccurate (Harrison and Pearson, 1994). For this reason the GEMPACK option "ACD" (always use change differentiation of levels equations), has been adopted here in producing TABLO generated programs.

4.4.2 Simulating anticipated shocks

Time elimination technique; and Eigenvector-backwards-integration method

The post-implementation adjustment path can be readily obtained from the post-shock saddle-path, calculated by either of these methods. Referring back to the schematic phase diagram Figure 3.17, this may be represented, for example, by the path from B to ss1. But the problem is that the point B is unknown. Since point A is also unknown, to calculate the adjustment which occurs before implementation - this is the path ss0 to A to B - it is necessary to solve a two point boundary value problem over the time interval from anticipation to implementation, t=0 to t=I say. In this problem the initial value of the non-jumping variable, is simply its known pre-shock value $\Psi(0)=\Psi_0$; the final point $(\Psi(I), \Phi(I), p_A(I))$, is constrained to lie on the post-shock saddle-path; and the initial values of the jumping variables, $\Phi(0)$ and $p_A(0)$, are unknown.

This sounds like a fairly standard two-point boundary value problem. Nevertheless, it presents considerable difficulty for the explicit specification of its terminal condition; that is, in the precise definition of what target to 'shoot at'. If the saddle-path were known analytically, say $\Psi = f(\Phi,p_A)$, the target would be $\{\Phi(I), p_A(I)\}$ such that $\Psi(I) = f(\Phi(I),p_A(I))$. However, the saddle-path is only known numerically, and then only at a discrete and finite number of points. As a result, some procedure for comparing and interpolating the intermediate shooting results with those from the saddle-path integration would have to be devised (and programmed) in order to sensibly update the 'guesses' for the initial values of the jumping variables and so to take another shot at the saddle-path. In fact, in the case of these anticipated shocks, knowledge of the saddle-path is of no great help in solving the two-point boundary value problem. In the absence of any such knowledge the terminal condition could readily be specified as the post-shock steady-state after a suitably large time; and the integration could proceed according to the pre-shock dynamic equations for the time interval (0,I), and thereafter according to the post-shock equations.

Finite differences and GEMPACK

Here the time of anticipation or announcement is taken as the initial time (t=0) and the shock is simply introduced into the equation set from the time of its implementation (t=I). In this way the pre-shock equations apply up to the time of implementation, and the post-shock ones apply from then on. Thus, when the entire system is solved simultaneously, adjustment in the period between announcement and implementation occurs under the influence of the pre-shock equations, and after implementation it is governed by the post-shock equations. In this way the coordinates of the points A and B in Figure 3.17 are solved for automatically.

4.4.3 The solution method: finite differences and GEMPACK

It is abundantly clear that the method of finite differences, implemented via the GEMPACK software, is far and away the best of the three methods analysed in solving for the transient dynamics of the Romer model in response to exogenous shocks. Simulation of unanticipated shocks is considerably simpler with these procedures than with either the time elimination, or the eigenvector-backwards-integration method. It is also more accurate since it avoids the need for interpolation. However, it is the simulation of anticipated shocks that most clearly establishes the superiority of the finite differences-GEMPACK method. For these type of shocks it is not otherwise possible to avoid confronting a two-point boundary value problem and some sort of shooting procedure, and a difficult one to boot, even when the post-shock saddle-path has already been numerically calculated.

Having thus chosen as the numerical solution method for the model, a finite differences approximation which is to be solved by GEMPACK, two issues remain to be resolved before simulations can commence. The first is the determination of the actual finite differencing procedure for the differential equations whose integration is sought via GEMPACK. And the second is the specification of the solution method for the initial value problem that GEMPACK must actually solve in order to implement the exogenous shocks and effect the simulations.[25] There are a number of different choices to be made for each broad issue, all involving varying degrees of trade-off between numerical precision and computational effort. These matters are analysed in Appendix 4.3 and a resultant 'standard configuration' for the finite differencing-GEMPACK approach identified. Namely:

> 4th order Runge-Kutta finite differencing with an uneven grid of 350 time intervals extending over 250 years and with step-sizes varying between 0.125 and 5 years (Appendix Table A4.3.1 contains the details); and a Gragg integration technique extrapolated from 12, 24, and 36 steps as the GEMPACK solution method for implementing exogenous shocks.

4.5 Numerical results of simulations

The dynamic properties of the full non-linear model may now be evaluated numerically. In order to furnish a degree of realism this is performed here by using the model to examine the dynamic impacts of a variety of economic shocks. Of course, given the high level of aggregation of the model and its lack of policy variables (there is no Government sector) these analyses are merely stylistic. Nevertheless, the economic effects generated are sensible and interesting. The general approach to the simulations was to commence with the system in equilibrium at its *benchmark steady-state* (Table 2.2), and then to perturb this via shocks to its parameters or variables. This initial benchmark steady-state was also used as the *base case* about which the *post-shock* solution was calculated (via GEMPACK) as percentage change variations. With the capital-technology ratio Ψ as a non-jumping (stock) variable, its initial component Ψ_0 is determined exogenously by the boundary conditions.[26] Overall seven simulations are reported:

- an unanticipated and sustained 10 per cent rise in the profit share of income, simulated by raising the parameter γ by 10 per cent (Section 4.5.1);

[25] It is perhaps somewhat ironic that having abandoned the iterative initial value methods of numerical integration in favour of the finite differences approach, it turns out that the manner in which GEMPACK solves these simulation problems is precisely by the methods of initial value problems.

[26] While Ψ_0 is not usually shocked, it is in one of the simulations that follow (Section 4.5.6).

- both an unanticipated and an anticipated sustained rise of 15 per cent in the productivity of researchers, simulated by increasing parameter ζ by 15 per cent (Section 4.5.2);
- an unanticipated but temporary rise of 15 per cent in the ordinary labour share of wages, simulated by decreasing parameter α by 20 per cent (Section 4.5.3);
- a program of immigration designed to raise the overall level of human capital by 15 per cent over five years, simulated by a series of 20 cumulating shocks of 0.75 per cent each to the variable H(t) (Section 4.5.5);
- a sudden temporary reduction of 5 per cent in the capital stock, simulated by cutting the value of variable Ψ_0 by 5 per cent (Section 4.5.6); and
- a gradual loss, of 20 per cent over three years, in the human capital employed in research, simulated by a series of negative shocks to the variables $H_A(t)$ and H(t), cumulating to –20 per cent and –5 per cent respectively (Section 4.5.7).

4.5.1 An unanticipated rise in the profit share of income

A rise in the profit share of income from goods production may be thought of, for example, as being due to some change in Government tax policy. Since this profit share is given by the Cobb-Douglas parameter γ from the production function (Appendix 2.1), a rise of say 10 per cent could be simulated with the Romer model by shocking γ by 10 per cent. The results of such a simulation are reported below in Figure 4.1 to Figure 4.6 and in Table 4.1; but before these results are examined it is instructive to think about the *a priori* effects which may be expected. That is, to try to deduce the likely effects from qualitative economic reasoning.

In this respect it helps to think about the simulation in different terms. In particular, since γ is the output elasticity of capital (γ =[dY/dK]Y/K=[dY/Y]/[dK/K] from Appendix 2.1), an increase in this parameter will raise the proportional output response from any increase in capital. In this way raising γ has the effect of increasing the productive efficiency or the efficiency of usage of capital at all levels. It should be noted that what is meant by the term *the productive efficiency of capital* is not the same as the 'productivity of capital' (in marginal or in average terms). While the marginal and average productivities of capital vary with the scale of usage or the number of units of capital, the concept of the *productive efficiency* of capital applies equally to all units and does not vary with levels of usage. Diminishing returns mean that both the average and marginal productivities of capital fall as the amount of capital rises. Raising the parameter γ also lowers both of these measures.[27] In this way raising γ is like having extra capital. Since this is also the effect produced by capital saving technical or

[27] With output given by $Y=\eta^{-\gamma}A^{1-\gamma}N^{1-\gamma}K^{\gamma}$, where $N=H_Y^{\ 1-\alpha}L^{\alpha}$ is a measure of composite labour (see Appendix 2.1), the marginal product of capital is $MP_K=\partial Y/\partial K=\gamma\eta^{-\gamma}A^{1-\gamma}N^{1-\gamma}K^{\gamma-1}=\gamma Y/K=\gamma AP_K$, and the average product of capital is $AP_K=Y/K$. Differentiating these with respect to capital demonstrates the existence of diminishing returns: $\partial MP_K/\partial K=-\gamma(1-\gamma)\eta^{-\gamma}A^{1-\gamma}N^{1-\gamma}K^{\gamma-2}<0$; and similarly $\partial AP_K/\partial K<0$.

To see how capital productivity varies with γ, use equation (2.29) to write the average and marginal products of capital as $AP_K=(r+\delta)/\gamma^2$, and $MP_K=(r+\delta)/\gamma$; and consider the interest rate at its steady-state level as given by (2.56): $r_{ss}=(\sigma\zeta H+\rho)/(1+\alpha\sigma/\gamma)$. Then differentiate with respect to γ to obtain: $\partial MP_K/\partial\gamma=(1/\gamma)\,\partial(r+\delta)/\partial\gamma - (r+\delta)/\gamma^2=(1/\gamma^2)[r_{ss}/(1+\gamma/\alpha\sigma) - (r_{ss}+\delta)]<0$; and similarly $\partial AP_K/\partial\gamma<0$

organisational change, a rise in γ can be thought of as reflecting just such an improvement, perhaps via some policy of microeconomic reform intended to raise the *productive efficiency* of capital. With this in mind the issue of gauging the *a priori* impact of raising γ may now be examined.

An increase in the productive efficiency of capital could be expected to stimulate investment as producers moved around their transformation frontiers substituting capital for labour, which takes the form of human capital (H_Y) here. This could then be expected to raise rental rates as reflected by the interest rate (r), thereby moderating the demand for capital until a new equilibrium was established. Increased investment would also raise the demand from the capital goods producing sector for new designs. In turn, this would increase the demand for human capital in research (H_A), and would push up the price of designs (p_A) until sufficient research activity were stimulated to re-establish equilibrium between the growth of capital and the growth of designs.

Although the growth rates of technology, output and consumption will all increase, these are secondary to the increase in capital which should therefore raise the capital-technology and the capital-output ratios (Ψ and k_{GP}); and lower the consumption-capital ratio (Φ). Similarly, both the narrow and broad measures of the savings rate (s_N and s_B) could be expected to rise; and finally, higher productive efficiency implies a higher rate of economic growth.

This qualitative analysis is supported by the numbers for the pre- and post-shock steady-state equilibria (Table 4.1) These numbers may also be seen respectively, as the values corresponding to the 'time = -10 to zero' and (approximately) as the 'time = 100' abscissae in Figure 4.1 to Figure 4.5. However, this analysis is simply one of *comparative statics* and could have been undertaken and confirmed in the context of the steady-state formulations of Chapter 2, no dynamic analysis of the model being necessary.[28] But this is **not** to say that the dynamics are unimportant. On the contrary, the dynamic analysis of this simulation reveals an essential limitation of the comparative statics approach. Namely, that for most variables there can be no presumption that the transition path towards the new (post-shock) equilibrium is smooth, nor even monotonic.

From previous discussions, both in the current chapter and in Chapter 3, it was known that the transition paths for most variables would not be smooth - immediate post-shock discontinuities or jumps being necessary to take the system to its new saddle-path.[29] As it turns out, these jumps are sometimes (perhaps often) in the 'opposite direction' to that of the subsequent adjustment paths towards the new equilibria so the overall transition path is not monotonic. There are two possibilities: First, that the initial jump is *perverse*, but that the subsequent adjustment *accords* with the direction of change from the comparative statics. And second, that while the initial jump takes a variable away from its pre-shock level in the direction of its post-shock steady-state, it *overshoots* this new equilibrium level with the result that the subsequent adjustment must be *perverse* in

[28] In particular see Table 2.3, which (*inter alia*) records steady-state changes for precisely this simulation.

[29] The capital-technology ratio $\Psi(t)$ is the usual exception. It is also conceivable, though unlikely, that for some particular combinations of shocks some other variable(s) will not need to jump in order for the post-shock saddle-path to be attained

terms of the comparative statics. In terms of the results in Table 4.1, the initial jumps expressed as a percentage of the total adjustment (column 5) will be negative in the first case, and greater than 100 in the second.

Table 4.1: Simulation results of an unanticipated and sustained 10 per cent rise in the capital share of income (γ) from time zero, benchmark parameter set.

Dynamic variable	Final steady-state	Total adjustment (% of initial ss)	Initial jumps as a % of: initial ss	total adjustment	½ life[a] (years)	¾ life[a] (years)
$\Psi(t)$	10.48	61.8	0.00	0.00	19(19)	37(37)
$\Phi(t)$	0.2265	-17.4	1.00	-5.75	15(16)	31(32)
$p_A(t)$	11.94	26.4	-13.0	-49.2	18(28)	36(46)
$r(t)$	5.99%	6.85	49.3	720	15(0)	31(0)
$H_A(t)$	27.7%	8.33	-23.1	-277	15(46)	31(63)
$g_{GP}(t)$	1.66%	8.33	63.1	757	15(0)	30(0)
$s_B(t)$	25.0%	13.2	9.89	74.9	15(0)	31(1)
$s_N(t)$	20.0%	19.0	25.3	133	17(0)	34(0)
$k_{GP}(t)$	3.31	16.5	-3.77	-22.9	17(22)	34(39)
$\beta_\Psi(t)$	4.09%	-17.0	-27.9	164	16(0)	33(0)

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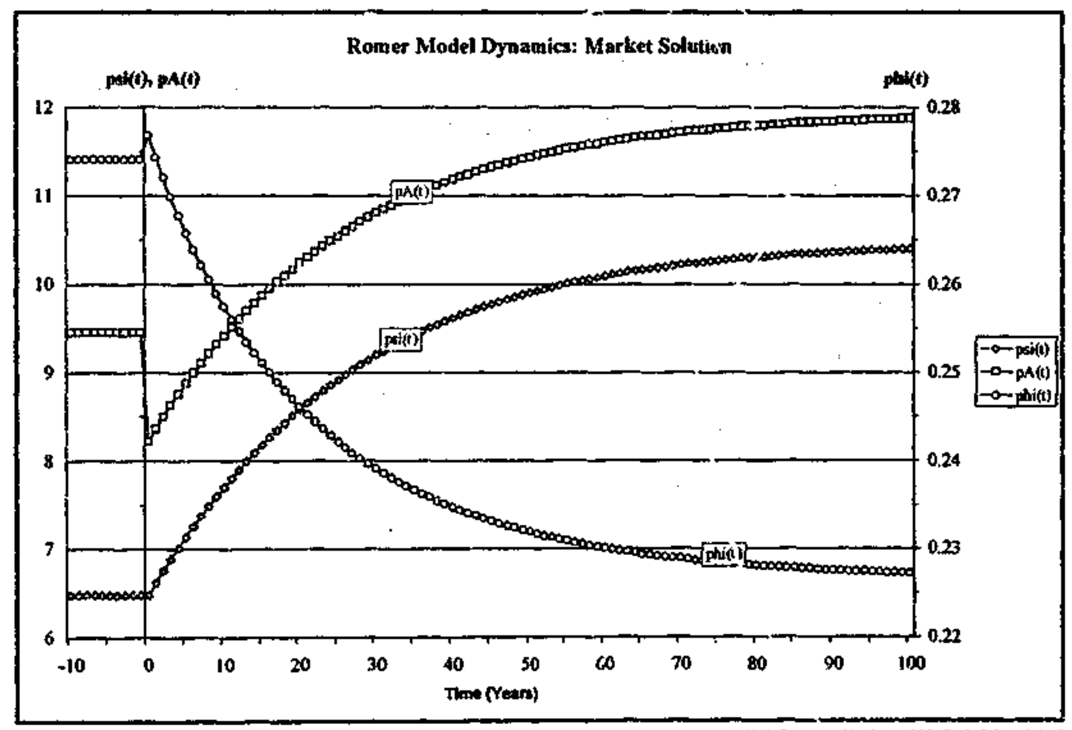
Note: a The first set of results give the time taken for ½ and ¾ of the remaining adjustment after the initial jumps; while the figures in parentheses give corresponding results from the pre-shock levels.

Figure 4.1: Dynamic effects on Ψ, Φ, and p_A of an unanticipated and sustained 10 per cent rise in the capital share of income (γ) from time zero, benchmark parameter set.



d:\phd\gempack\mkrom\10g0\10g0sim.xls(psi)

For the simulation considered here, the first of these 'non-monotonic' possibilities is exhibited by:

- the consumption-capital ratio, Φ and the price of technology, p_A (Figure 4.1);
- the growth rate of technology, g_A (Figure 4.2);
- the allocation of human capital to research, H_A (Figure 4.3);
- the capital to gross product ratio, k_{GP} (Figure 4.4); and
- the convergence coefficient for the consumption-capital ratio, β_Φ (Figure 4.6).

Similarly, the second 'non-monotonic' possibility is followed by:

- the rates of growth of capital, output, gross product and consumption, g_K, g_Y, g_{GP}, and g_C respectively (Figure 4.2);
- the interest rate, r (Figure 4.3);
- the narrow concept of savings (equivalent to the investment share of output), s_N (Figure 4.4); and
- the convergence coefficients for output per design, the price of technology and the capital-technology ratio, β_y, β_{pA}, and β_Ψ respectively (Figure 4.6).

Of all the different variables analysed in this simulation, only the capital-technology ratio Ψ, and the broad savings measure s_B, show monotonic adjustment paths. The former is due to the initial boundary condition precluding any discontinuity. Thus, only the latter exhibits an immediate post-shock jump that is in the direction of the new steady-state and does not overshoot it (Figure 4.1 and Figure 4.4 respectively).

Figure 4.2: Dynamic effects on the growth rates of an unanticipated and sustained 10 per cent rise in the capital share of income (γ) from time zero, benchmark parameter set.

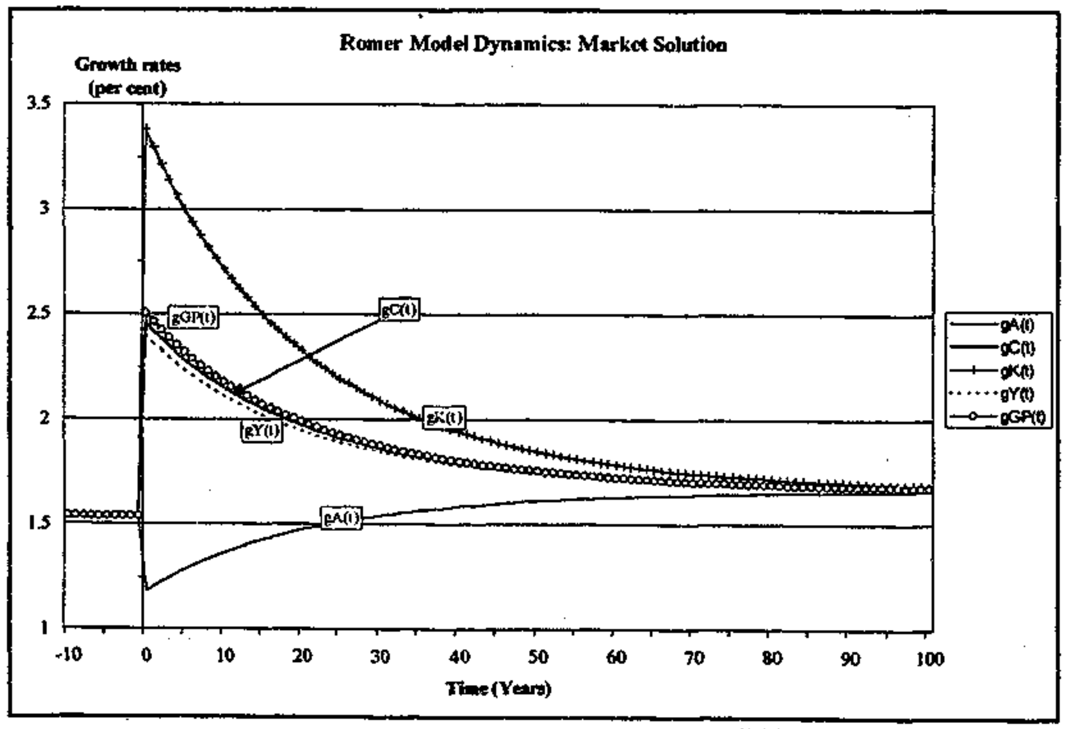


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Figure 4.3: Dynamic effects on r and H_A of an unanticipated and sustained 10 per cent rise in the capital share of income (γ) from time zero, benchmark parameter set.

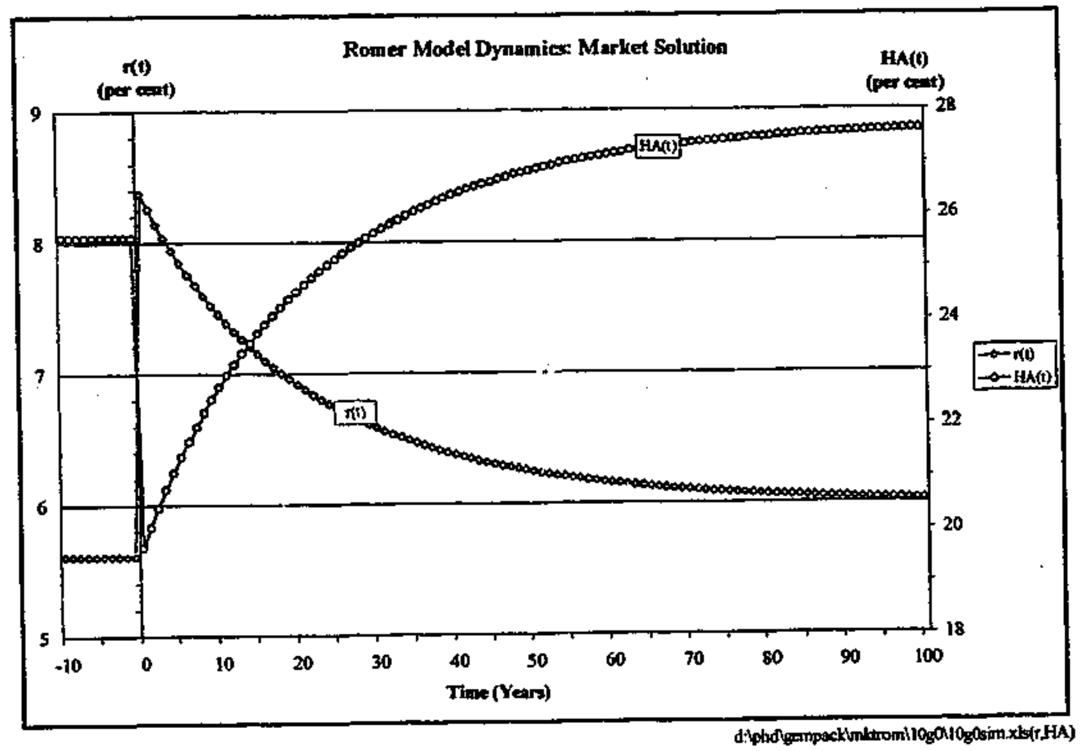

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Figure 4.4: Dynamic effects on s_B, s_N, and k_{GP} of an unanticipated and sustained 10 per cent rise in the capital share of income (γ) from time zero, benchmark parameter set.

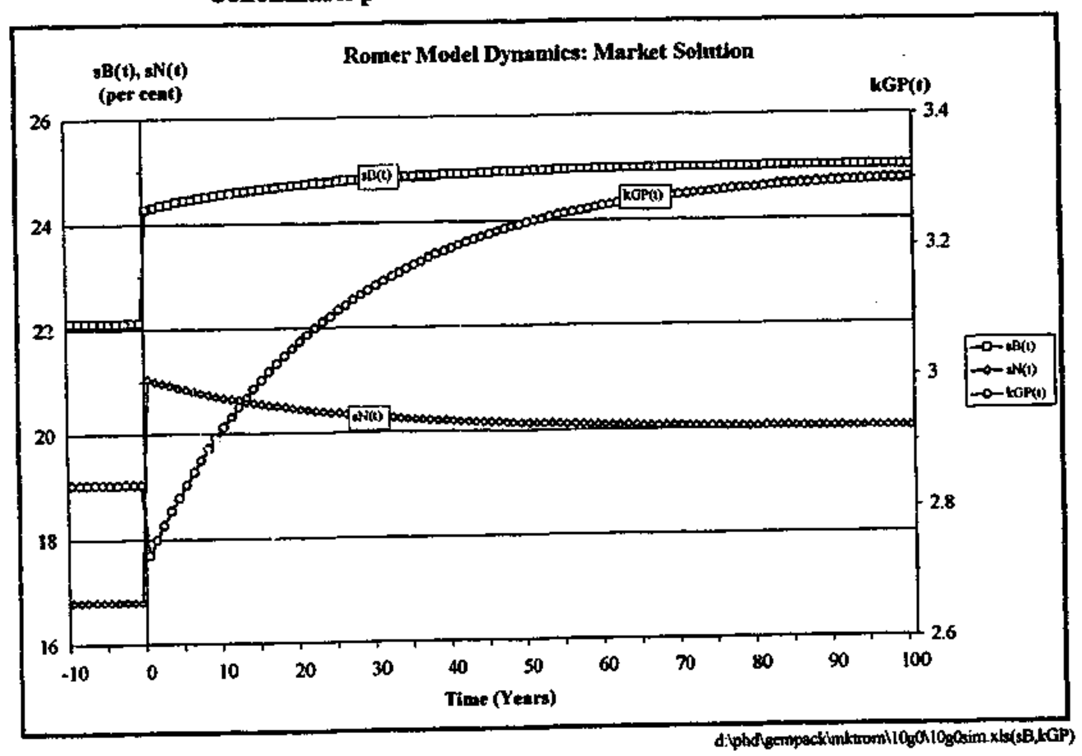

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Figure 4.5: **Dynamic effects on the factor shares of gross income from an unanticipated and sustained 10% rise in the capital share of income (γ) from time zero, benchmark parameter set.**

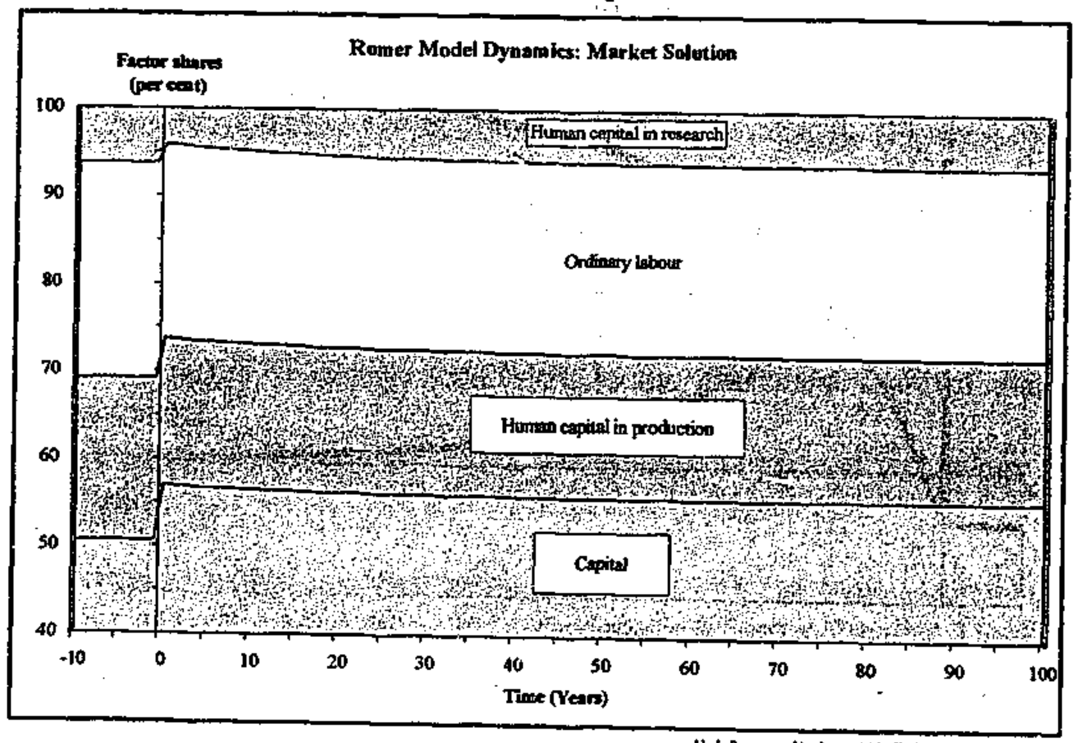

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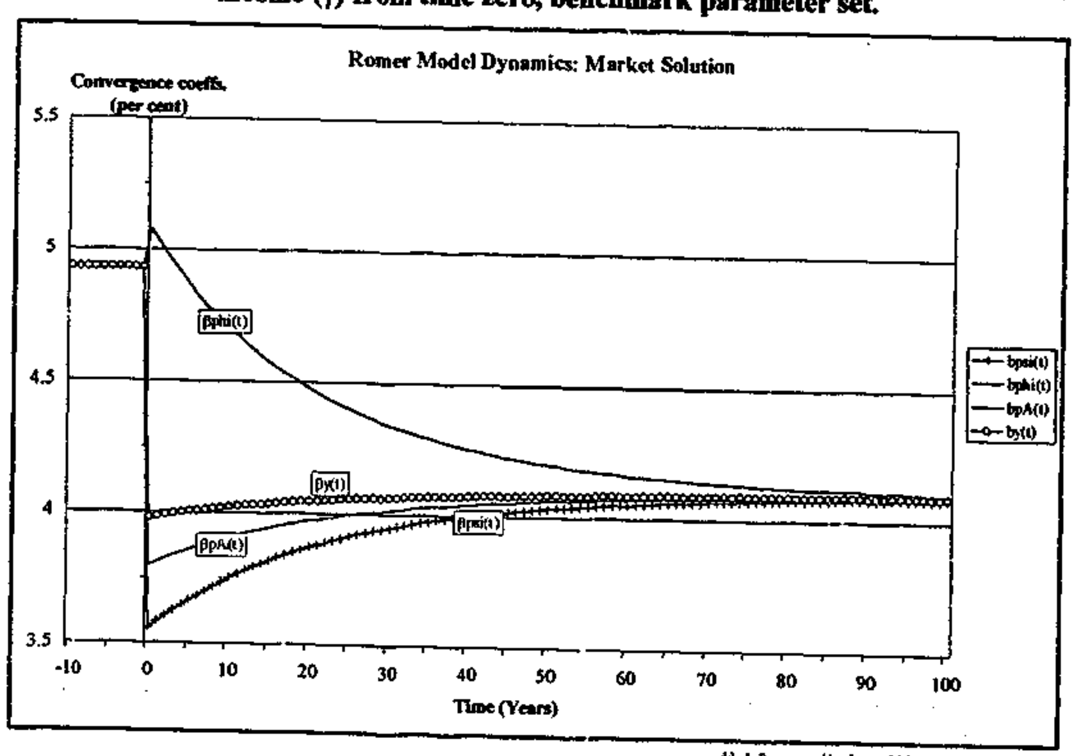
Figure 4.6: **Dynamic effects on the convergence coefficients from an unanticipated and sustained 10 per cent rise in the capital share of income (γ) from time zero, benchmark parameter set.**



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Because of this non-monotonic nature of the adjustment process, some of the shorter-term effects of economic change, whether policy induced or otherwise, may have the opposite sign to their longer-term impacts. As the model is founded upon economic agents who optimise their decision making, in a sense this is the best that can be done.[30] Nevertheless it does raise the issue of the short-term *adjustment costs* which need to be traded-off against perceived long-term benefits. Also, in a richer model with different classes of consumers (such as lenders and borrowers perhaps), there would be trade-off issues concerning the income distribution effects of economic change.

Suppose that the overall change in some variable from its initial level to its final steady-state is considered beneficial, then an immediate post-shock jump in the opposite direction to this change would presumably be considered deleterious. Or, if the overall change in a variable is thought to represent some sort of trade-off cost for other benefits of economic change, then an initial jump in this variable which significantly overshoots its new steady-state level will produce even higher trade-off costs. In such cases there will therefore be adjustment costs to be borne in the shorter term that must be taken into account in any assessment of the overall efficacy of the economic change. Moreover, as demonstrated in this simulation, *the shorter term* can persist for many years.

For example, while the eventual rise of 0.4 percentage points, or 6.8 per cent, in the interest rate may be considered a justified or worthwhile price to pay for the overall 16.5 per cent, or 0.47 point, increase in the economy's capital-output ratio (Table 4.1), such figures may not be the best basis of comparison. Two other points also seem highly relevant to the calculation. Namely:

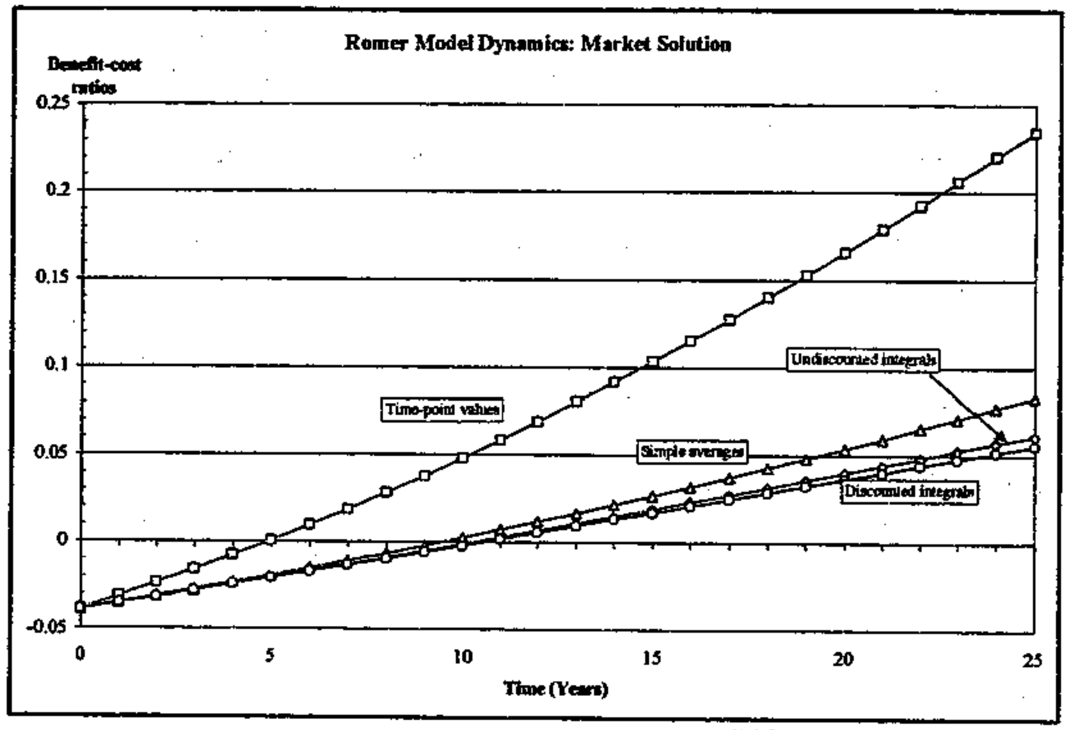
- the fact that the capital-output ratio initially falls by almost 4 per cent, taking five years to recover its pre-shock level (Figure 4.4); and
- the fact that the interest rate initially jumps by 50 per cent, and only slowly declines so that even after 20 years it remains 33 per cent above its pre-shock level (Figure 4.3)

Thus, a better idea of the relative costs and benefits may be to compare their averages, or perhaps better still, their integrals over some (finite) time horizon. When such calculations are made here, the benefit-cost ratios obtained are considerably smaller than those implied by the simple single time-point values, which are of course, asymptotic to the steady-state ratio of about $1.22 \approx 0.47/0.38$ (Figure 4.7).

All these issues emphasise the explanatory power of the dynamic model over the static one. From a comparative statics analysis of the model's steady-state formulation it was not possible to identify either the extent or the direction of the initial jumps nor, consequently, the direction of the subsequent smooth adjustment towards equilibrium. Some idea of the dynamic behaviour may be possible from more thorough qualitative economic reasoning, though in some respects it may be impossible or at least ambiguous because of opposing influences. A rationalisation of the results for this simulation might be as follows:

[30] While this 'market solution' to the model is based upon optimised behaviour by economic agents, there are certain market imperfections which mean that the solution is not a *Pareto optimum*. The economic welfare implications of this are examined in Chapter 5.

Figure 4.7: **Dynamic trade-off 'costs' of higher interest rates for the 'benefits' of a higher capital-output ratio resulting from an unanticipated and sustained 10 per cent rise in the capital share of income (γ) from time zero, benchmark parameter set.**



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Autonomously increasing the productive efficiency of capital makes capital suddenly more valuable to final goods producers and so instantaneously raises the demand for it. In the very short term, before the supply of capital can adjust, such increased demand merely raises rental prices and the rate of return on capital, and with them the interest rate (r), which therefore jumps upwards. Also, since more goods can suddenly be produced (and then consumed) with the same capital stock, the rates of growth of both output and consumption (g_Y and g_C) and the consumption-capital ratio (Φ) all rise spontaneously, while the capital-output ratio (k_{GP}) falls spontaneously. Further, since more goods can be produced with the same stock of technology, the instantaneous demand for designs falls, and with it, their price (p_A), their rate of growth (g_A), and the demand for the human capital resources (H_A) which produce them.

Later, when the supply of capital begins to adjust, the 'economic effects story' more closely resembles that already told in the comparative statics case: Greater efficiency in the usage of capital stimulates investment as producers move around their transformation frontiers substituting capital for labour. Such investment gradually satisfies the demand for extra capital, thus reducing the level of **excess demand** and with it the value of capital as reflected by its rate of return and the interest rate, which eventually decline to new equilibrium levels where all extra demand is satisfied. Since the new equilibrium demand for capital is surely greater than that before its rise in productive efficiency, the final rate of return and interest rate are also be greater than their pre-shock levels. Also, as more capital is produced and installed the consumption-capital ratio falls over time, ending up lower than its pre-shock level since more resources are devoted to capital creation than to the production of consumables at the new steady-state. Similarly, the capital-output ratio rises to a higher steady-state level. Since capital is more efficiently

employed the greater the degree of its specialisation, any rise in capital will be accompanied by increased demand from the capital goods producing sector for new designs from the research sector and consequently, increased demand by the research sector for human capital. In this way, over time the gradually increasing capital stock pushes up both the price of designs and the employment of human capital in research to higher steady-state levels.

The speed of the adjustment process in the simulation is indicated by the *half-life* and *three quarter-life* figures of Table 4.1 (columns 6 and 7).[31] Two sets of results are given. The first indicate the time taken, in years, for one half and for three quarters respectively of the gap between the **immediate post-shock and final steady-state levels** of the different variables to be closed. The second set of figures (in parentheses) use the immediate pre-shock level instead as the starting point. In this case whenever the initial jump of a variable exceeds half (three-quarters) of the total adjustment, the half-life (three quarter-life) is zero! Also, under this second definition there are wide variations in the half-life and three quarter-life results for different variables. However using the immediate post-shock level as the starting point also raises difficulties. For example, whenever the initial jump in a variable is towards, but less than half way to its final equilibrium this method of calculation 'exaggerates' the half-life adjustment time; while when the initial jump is away from the new equilibrium it 'understates' it. For these reasons it seems best to use the half-life of the non-jumping variable Ψ in any description of the speed of convergence of the system as a whole; and to use **both** the size of the initial jumps and the post-jump half-lives to describe the adjustment of particular jumping variables.

The half and three quarter-life results from this simulation indicate that the process of economic adjustment to a 10 per cent shock to the profit share of income would be quite slow. As a whole the system could be expected to take almost two decades to complete half of its total adjustment, and three to four decades before three quarters of the change had eventuated. These figures accord well with those obtained from the magnitude of the negative eigenvalue of the linearised coefficients matrix for the model as calculated in equation (3.20) of Section 3.2.3.

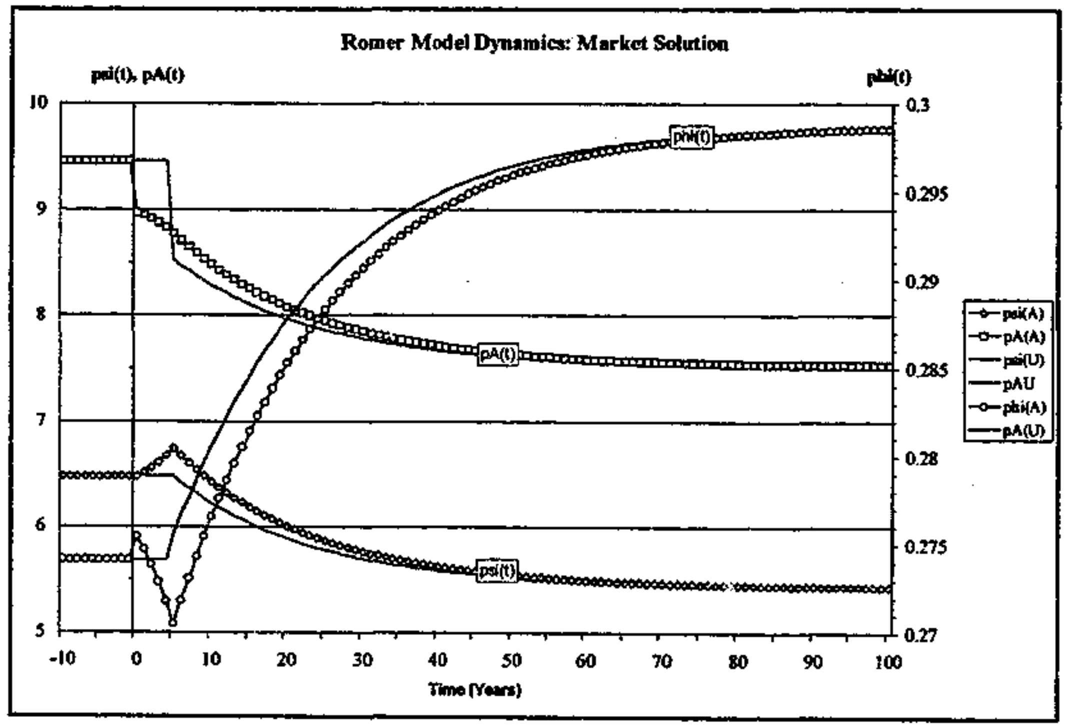
4.5.2 An anticipated rise in the productivity of researchers

Suppose that at some date the Government announces that a policy of microeconomic reform measures designed to improve productivity in the research sector is to be implemented in five years time. An economic assessment of the direct effects of the measures estimates that they will raise the productivity of researchers by some 15 per cent; and the next question is: 'what will be the broader, or overall, economic impact of the policy measures?' To answer this with the dynamic Romer model a simulation is run where the research productivity parameter ζ is increased by a factor of 1.15 from the time the measures are to be implemented (t=5 years), and the model is solved from the time of announcement (t=0) for all future time points. The results are shown in Figure

[31] The 'half-life' concept of adjustment was introduced in Section 3.2.3 in relation to the linearised model.

4.8 to Figure 4.15 and in Table 4.2 below. Unlike the previous simulation, where the shock to the capital share of income/productive efficiency of capital came as a surprise to the market, here the impending shock has been announced in advance and so is anticipated. The Figures here also show the results of an unanticipated 15 per cent rise in ζ from time t=5 years, reflecting the situation where the same policy measures are still implemented at time t=5, but where there was no prior announcement. This enables ready comparison of the effects of the anticipated shock with those of the same shock when it comes as a surprise to the market.

Figure 4.8: Dynamic effects (over 100 years) on Ψ, Φ, and p_A of both an anticipated and an unanticipated sustained 15 per cent rise in the productivity of researchers (ζ) from time t=5 years, benchmark parameter set.



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To obtain an *a priori* understanding of these results, first consider qualitatively the broad comparative statics effects that might be expected from an autonomous increase in the productivity of researchers: Such a shock would make designs cheaper to produce, thereby reducing their price p_A (Figure 4.8 and Figure 4.9), and raising the demand for them from the capital goods producing sector. More researchers would then be required in order to satisfy this increased demand, so the allocation of human capital to research H_A would rise (Figure 4.12), as would the growth rate of research output g_A (Figure 4.10).

Increased productivity is like having extra resources, so output growth will rise and so will that of consumption (Figure 4.10 and Figure 4.11). Greater consumption growth in the future suggests that less savings will be required to generate consumers' desired (and optimal) intertemporal pattern of consumption. A lower investment-output ratio s_N, could be expected to result (Figure 4.13). With relatively more resources being devoted to research and the production of output for consumption rather than for capital

generation, the capital-technology ratio Ψ can be expected to fall; the consumption-capital ratio Φ to rise; and the capital-gross product ratio k_{GP} to fall (Figure 4.8, Figure 4.9, and Figure 4.13).

Figure 4.9: Dynamic effects (over 25 years) on Ψ, Φ, and p_A of both an anticipated and an unanticipated sustained 15 per cent rise in the productivity of researchers (ζ) from time t=5 years, benchmark parameter set.

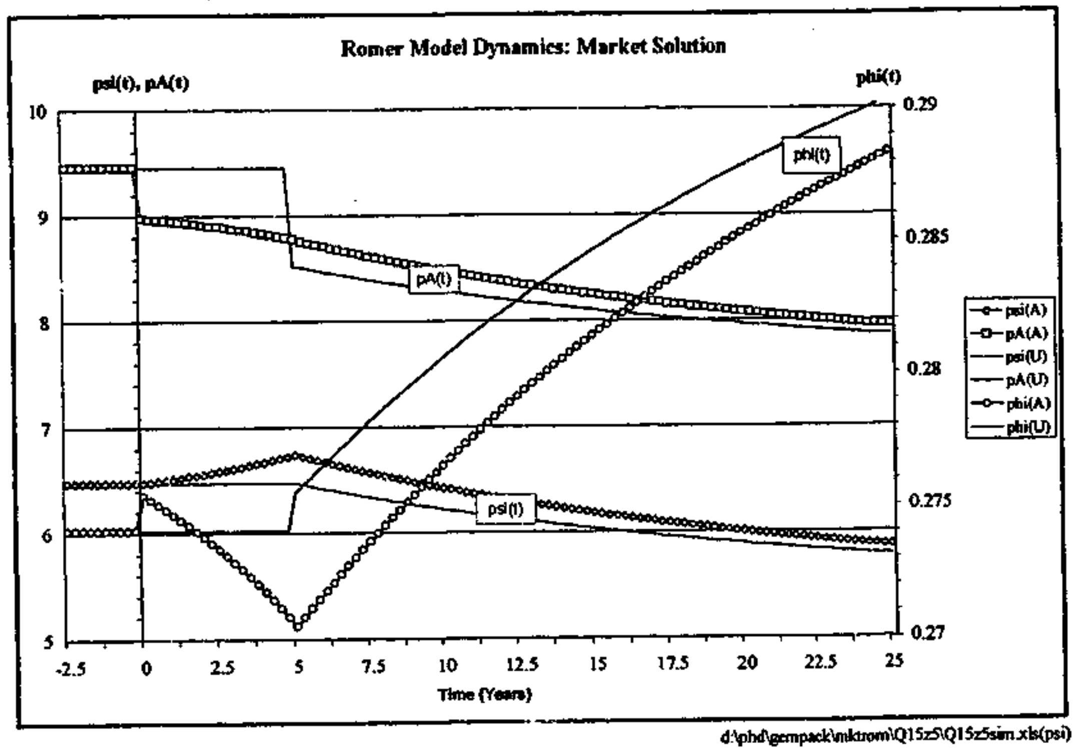

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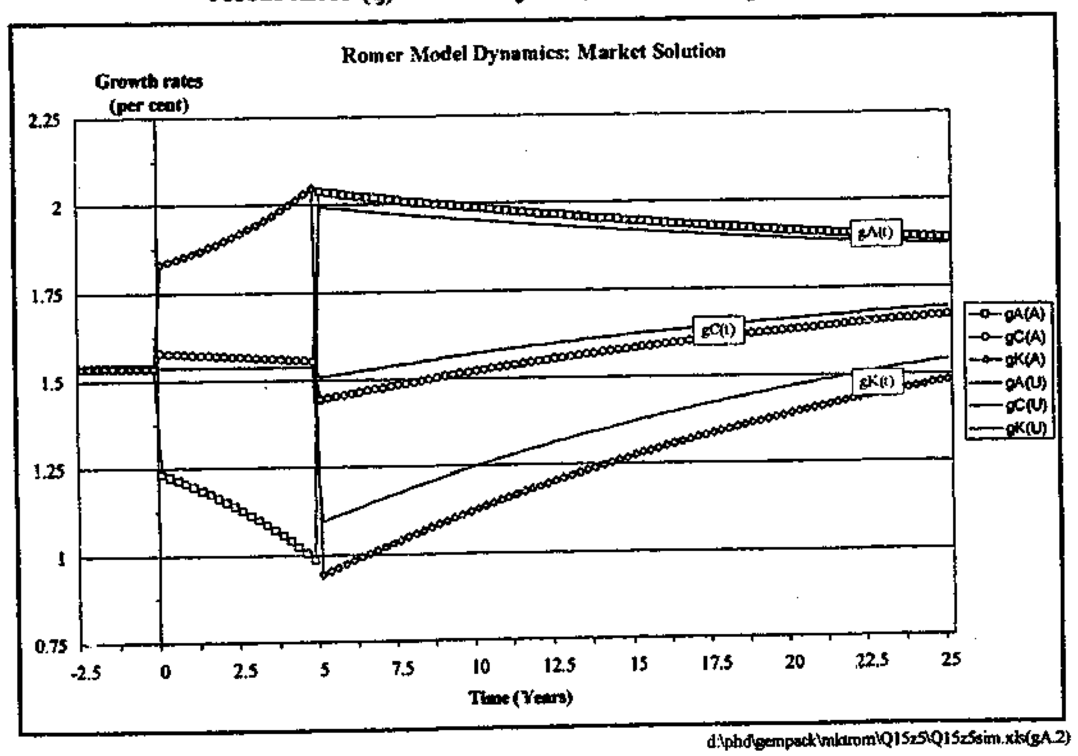
Figure 4.10: Dynamic effects on the growth rates of both an anticipated and an unanticipated sustained 15 per cent rise in the productivity of researchers (ζ) from t=5 years, benchmark parameter set.



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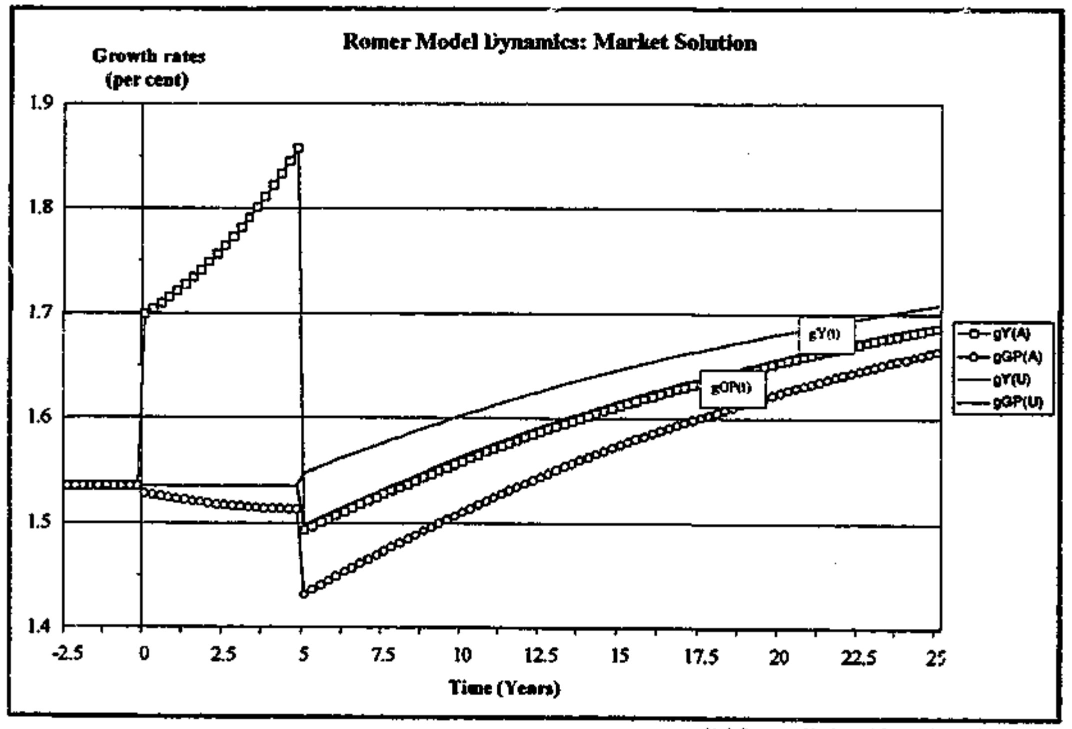
To support the higher growth of consumption (so future consumption is greater than current consumption) the return to savings and hence the interest rate r, must also rise (Figure 4.12). Finally, the increase in human capital resources devoted to research and the complementary reduction in those for the production of goods, means that the share of gross product to H_A can be expected to rise while the shares to the other factors (H_Y, L and K) can be expected to fall (Figure 4.14).

Figure 4.11: Dynamic effects on the growth rates of output and gross product (g_Y and g_{GP} respectively) of both an anticipated and an unanticipated sustained 15 per cent rise in the productivity of researchers (ζ) from t=5 years, benchmark parameter set.



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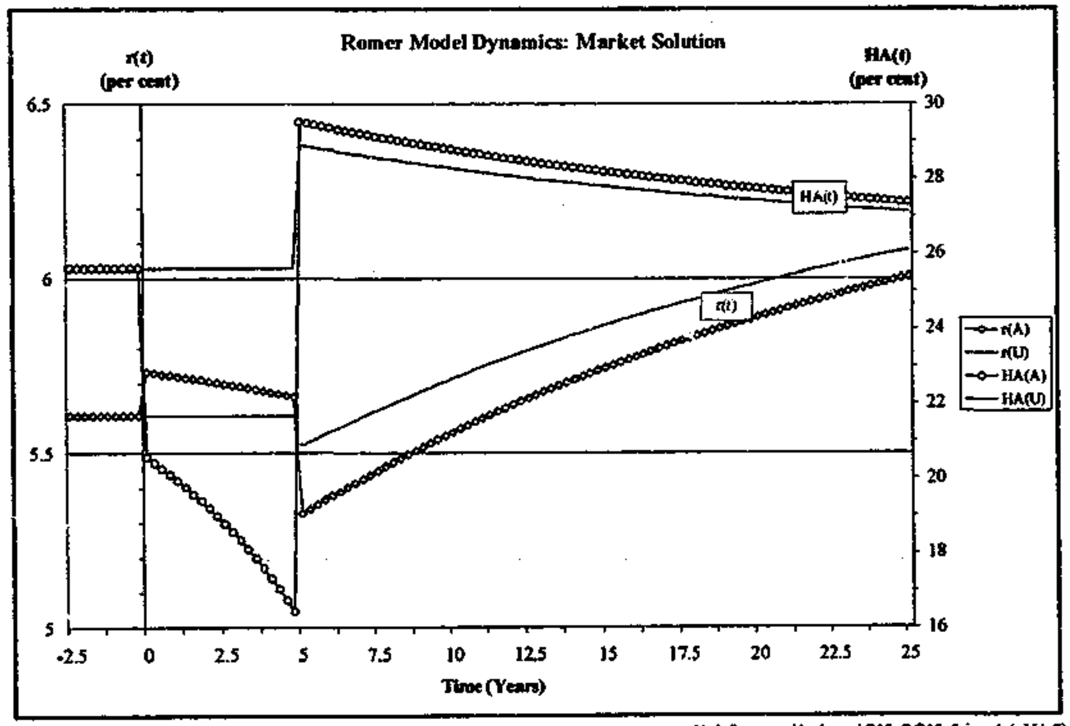
Now consider the dynamics more explicitly. In particular, consider those resulting from the **unannounced policy shock**. By suddenly and unexpectedly making designs cheaper to produce, the rise in research productivity causes their price p_A to fall immediately (Figure 4.8 and Figure 4.9). Thus, the increase in demand for designs and so for the human capital necessary to produce them is also immediate. Higher wages are offered, and since H_A is a flow not a stock variable, it responds by jumping spontaneously to a higher level (Figure 4.12). Here it helps to think about hours worked rather than numbers employed; and to suppose that at least some individuals divide their working hours between research and production.

Movements in the growth rate of designs (or technology) g_A, in response to any shock usually mirror those of H_A. Here this indirect effect via H_A is boosted by a direct response to the sudden rise in the productivity parameter ζ (see equation 2.1). Thus, the growth of designs also jumps upwards and to a relatively higher degree than does H_A (Figure 4.10). But since the number of designs, like capital, is a stock variable, it is fixed in the very short run. This is why the capital-technology (or capital-designs) ratio Ψ,

does **not** jump. Later, as the stock of designs begins to rise their price gradually falls further until this reduction eventually restrains their growth at a new equilibrium with a lower price and a higher growth rate than initially (Figure 4.8, Figure 4.9 and Figure 4.10).[32] With H_A and g_A linked inexorably by equation (2.1), after its initial jump the allocation of human capital to research declines gradually towards its new equilibrium level (Figure 4.12).

Figure 4.12: Dynamic effects on the interest rate and share of human capital devoted to research of both an anticipated and an unanticipated sustained 15 per cent rise in the productivity of researchers (ζ) from time t=5 years, benchmark parameter set.



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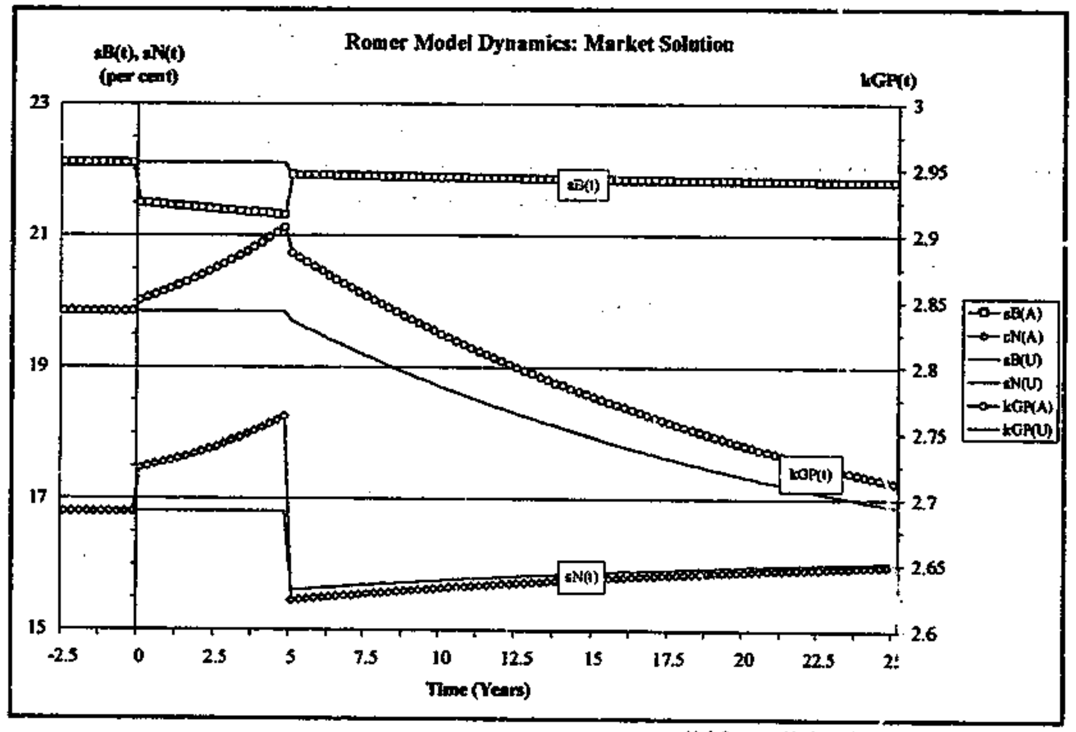
As noted before, higher research productivity implies greater output growth in the future.[33] As it turns out, optimising consumers respond to this by immediately raising their level of consumption, but lowering its instantaneous rate of growth. Later they gradually increase their consumption growth towards its new equilibrium (Figure 4.9 and Figure 4.10). To support this behaviour the return to savings, that is the interest rate, also first falls and then gradually rises to a new steady-state higher than before (Figure 4.12). The sudden transfer of human capital resources from the output sector to research (from H_Y to H_A) causes output Y to fall momentarily (recall that the other factors of production are all fixed in the very short run at least). In combination with the jump in consumption, this causes investment and thereby both the rate of growth of capital g_K and the narrow savings rate s_N to fall suddenly (Figure 4.10 and Figure 4.13). The effect

[32] Or looking at this the 'other way round': the rising demand restrains the falling prices. In any case, the two eventually equilibrate.
[33] Greater research productivity allows capital to be employed more efficiently by allowing it to be spread over more designs.

on the broad level of savings s_B however, is ambiguous. While savings in the form of research increases, those in the form of capital accumulation decline. The results show that the two influences almost balance, the latter prevailing only slightly (Figure 4.13).

Figure 4.13: Dynamic effects on s_B, s_N, and k_{GP} of both an anticipated and an unanticipated sustained 15 per cent rise in the productivity of researchers (ζ) from time t=5 years, benchmark parameter set.



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In this 'balanced growth equilibrium' model, the growth rates for all the non-stationary variables eventually converge to the same new steady-state level. Nevertheless, there are significant differences in their transient dynamics, both in terms of their initial jumps and of their subsequent smooth adjustment. Despite the sudden large upward jump in the human capital devoted to research H_A, since it must decline gradually thereafter, its instantaneous growth rate g_{HA} must be negative over the whole adjustment period, including immediately after the shock is imposed. Thus, the rate of growth of human capital employed in the output sector g_{HY} must be positive over the whole period. Now the growth rate of output g_Y depends on those of its factors (H_Y, L, K, and A). Ordinary labour L is fixed so its growth rate g_L is zero. While the rate of growth of capital g_K falls sharply initially, the sudden increases in g_{HY} and g_A are sufficient to cause a small sudden rise in g_Y which subsequently increases smoothly towards its new (higher) equilibrium as g_A and g_{HY} decline and g_K rises smoothly (Figure 4.10 and Figure 4.11).

Gross product GP is the sum of the output of goods Y and that of research R, so its growth rate g_{GP} is a weighted average of the growth rates of output and research (g_Y and g_R). The latter depends upon the rates of growth of p_A, H_A, and A (see equations 2.1 and 2.2). As has been seen, while g_A is greater than g_Y, g_{pA} and g_{HA} are negative. It turns out

that they are sufficiently negative to result in initial falls in g_R and g_{GP}, and for $g_R<g_Y$ over the whole adjustment period.[34] As a result, the growth rate of gross product falls upon implementation of the shock, before rising smoothly with $g_{GP}<g_Y$ (Figure 4.11).

Table 4.2: Simulation results of an anticipated and sustained 15 per cent rise in the productivity of researchers (ζ) from time t=5, benchmark parameter set.

Dynamic variable	Final steady-state	Total adjustment (% of initial ss)	Jumps as a % of initial ss on: Announcement	Implementation	½ life[a] (years)	¾ life[a] (years)
$\Psi(t)$	5.43	-16.2	0.00	0.00	18(22)	31(35)
$\Phi(t)$	0.2988	9.00	0.49	0.00	20(23)	33(37)
$p_A(t)$	7.51	-20.5	-5.13	0.00	18(16)	31(29)
$r(t)$	6.40%	14.2	2.24	-6.15	20(29)	33(42)
$H_A(t)$	26.10%	2.00	-19.7	47.7	19(5)	33(5)
$g_{GP}(t)$	1.80%	17.3	-0.40	-5.45	20(26)	33(39)
$s_B(t)$	21.73%	-1.69	-2.77	2.64	20(5)	34(5)
$s_N(t)$	16.26%	-3.23	3.99	-15.6	19(5)	32(5)
$k_{GP}(t)$	2.62	-7.82	0.30	-0.27	19(21)	32(34)
$\beta_\Psi(t)$	5.33%	8.08	-175	218	19(5)	32(5)

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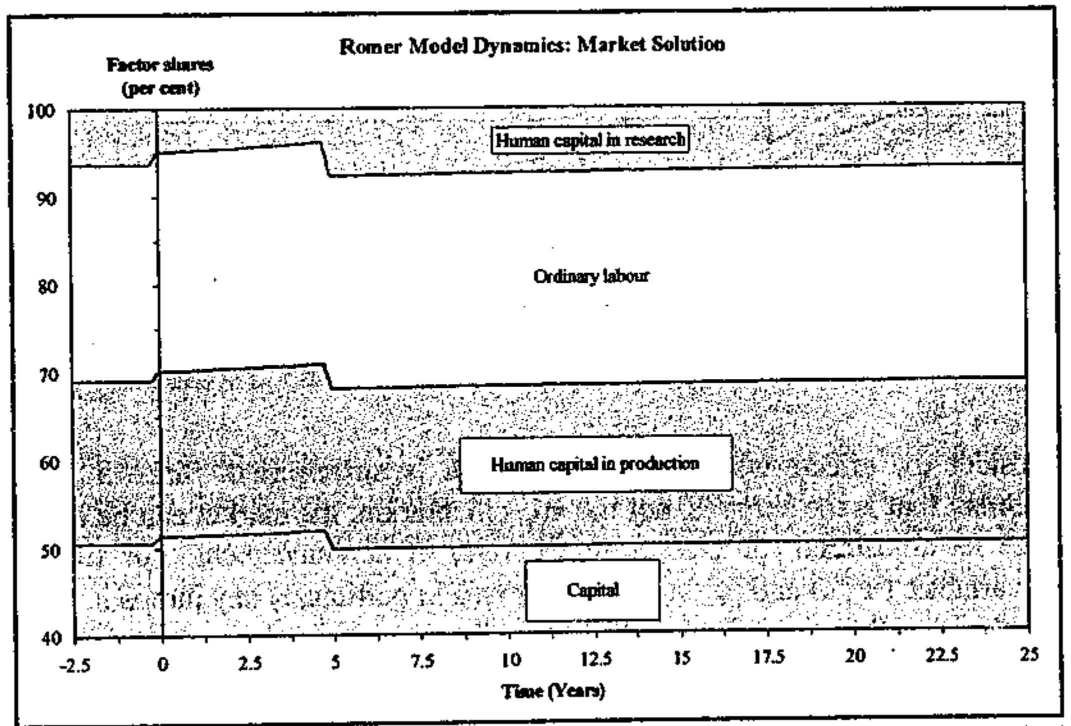
Note: a The first set of results give the total time taken (from t=0) for ½ and ¾ of the remaining adjustment after implementation; while the figures in parentheses give corresponding results after the post- announcement jumps.

Figure 4.14: Dynamic effects on the factor shares of gross income from an anticipated and sustained 15 per cent rise in the productivity of researchers (ζ) from time t=5 years, benchmark parameter set.



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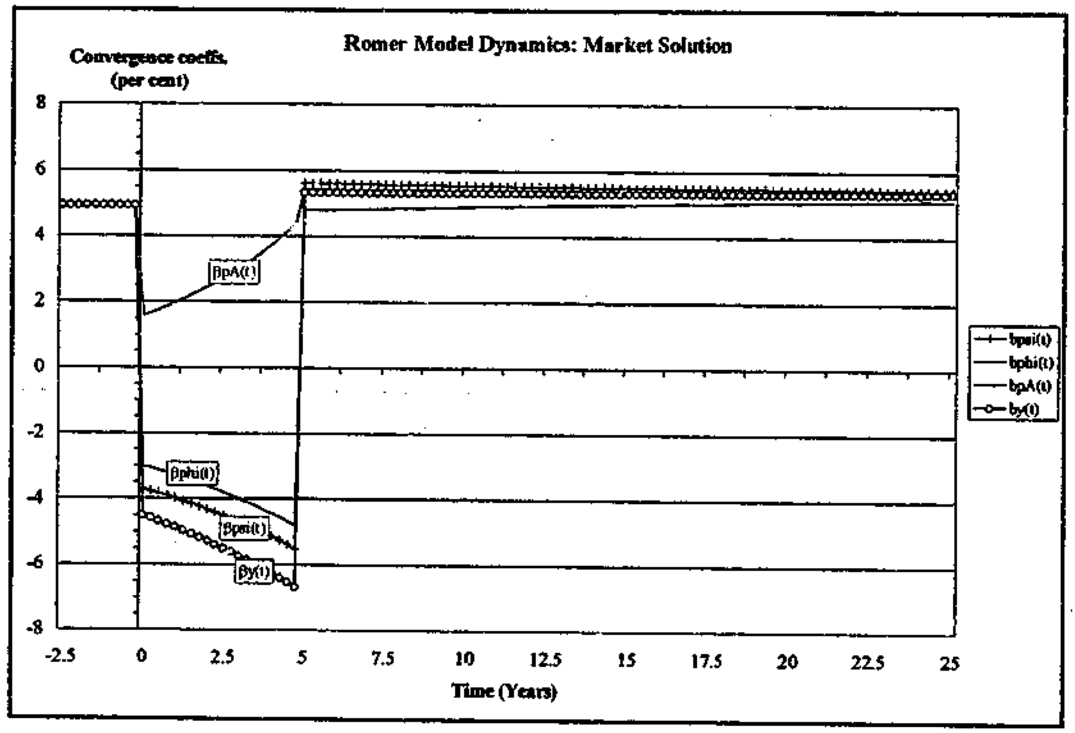
[34] Nevertheless, the magnitude of the initial jumps mean that the new equilibrium level of R/Y is greater than its initial equilibrium.

As stock variables, neither technology A nor general-purpose capital K can jump in response to the shock. Conversely, as a flow variable consumption C is free to jump. Here it responds to the unanticipated shock to ζ with a sharp but small increase. Also, over the whole adjustment period the growth rates of both technology and consumption are greater than that of capital ($g_A,g_C>g_K$). Thus, the capital-technology ratio Ψ decreases monotonically towards its new equilibrium with no initial jump; and the consumption-capital ratio Φ makes a small upwards jump before rising towards its new equilibrium (Figure 4.8 and Figure 4.9). For similar reasons the capital-gross product ratio k_{GP} decreases smoothly after making a small jump downwards (Figure 4.13).

Figure 4.15: Dynamic effects on the convergence coefficients from an anticipated and sustained 15 per cent rise in the productivity of researchers (ζ) from time t=5 years, benchmark parameter set.



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Now consider the effects when **the policy shock is anticipated**. Since it is now the announcement that is the surprise, not the implementation, the market can be expected to commence its adjustment process from the time of announcement. For the principal dynamic variables (Ψ, Φ and p_A, and the latter two in particular) any jumps in their levels will occur only at that time. They will then evolve smoothly, their motion governed by the still operative pre-implementation dynamic equations, until the shock is realised after five years. At that point the system will have just reached the post-shock saddle-path and while there may be further abrupt changes in the adjustment paths of these variables, there can be no further discontinuities. Their evolution will again proceed smoothly, this time towards their new steady-state equilibrium levels in accordance with the post-shock dynamic equations.

However, for derived dynamic variables such as the interest rate or the allocation of human capital, jumps will be likely at the time of implementation as well as at the time of announcement. This is because derived variables are functions of the main variables and the parameters, and while the main variables do not jump at implementation, any shocked parameter must jump to reflect its shock directly. Of course after implementation the derived variables adjust smoothly and asymptotically towards their post-shock steady-state levels in the same way as the principal dynamic variables.

While the steady-state equilibrium for an anticipated shock is identical to that for the same shock when it is unanticipated (see Figure 4.8 in particular), the transient dynamics differ markedly. Furthermore, since adjustment in the anticipated simulation commences when the policy change is first announced (at time t=0 here), while in the unanticipated case there is no adjustment until the change is actually implemented (at time t=5), it is during this *pre-implementation* period that the results of these simulations differ the most.

The big differences arising from the shock being anticipated lie in the fact that, for reasons yet to be explained, the post-announcement jumps tend to be in the opposite direction to the post-implementation ones, making these latter jumps much bigger than those which result when the shock is unanticipated! For most variables the immediate post-implementation and subsequent adjustment effects under the anticipated shock are qualitatively the same as those for the unanticipated shock.[35] Hence, the following discussion is confined to a 'rationalisation' of the economics underlying the pre-implementation adjustment.

Announcement of the shock raises the **prospect** of having a more productive research sector, which can generate cheaper designs. Capital goods producers respond by postponing some of their demand for designs until the shock is actually implemented and prices are reduced. Ironically, this immediate reduction in the demand for designs actually provokes a fall in their price. Similarly, it also causes sharp falls in the employment of human capital in research, and in the growth rate of technology. The mechanism is similar to that for the case when the shock was unanticipated,[36] but here the dynamic paths are not equilibrating. Falling prices only exacerbate the falling growth rate and employment of human capital which decline further until the shock is finally implemented (Figure 4.8, Figure 4.9, Figure 4.10 and Figure 4.12).

With total human capital fixed, the response of that employed in the output sector H_Y is complementary to the response of that employed in research. Thus, H_Y first rises steeply and then continues to increase more gradually until implementation. This causes output Y and its growth rate g_Y to respond similarly. Given that optimising consumers only raise

[35] The only exceptions are for g_Y and s_B, and in both these cases the jump for the unanticipated shock is only small (Figure 4.11 and Figure 4.13).

[36] A 'standard' supply-demand diagram relating the price of designs p_A and their rate of growth g_A, provides a framework for thinking about the economic mechanisms at work: Implementation of the unanticipated shock lowers the supply schedule causing p_A to fall and g_A to rise. On the other hand, announcement of the anticipated shock has the effect of (temporarily) reducing the demand schedule, again causing p_A to decline, but this time inducing g_A to fall as well. For the anticipated shock the supply schedule is not lowered until implementation. At that point the temporary fall in the demand schedule is also reversed and while p_A continues to decline (but at a different rate), g_A rises.

consumption a little, this in turn results in a large rise in investment and the growth of capital g_K, and also in the narrow savings measure or investment-output ratio s_N (Figure 4.10, Figure 4.11 and Figure 4.13).

Despite the steep rise in output Y, because of the large fall in the research component of gross product its growth rate g_{GP} falls marginally (Figure 4.11). This large fall in research is also reflected in the decline of the share of income to human capital in research and the concomitant rises in the shares to other factors (Figure 4.14); and, interpreted as a component of broad savings, it is also responsible for the fall in s_B on announcement of the shock (Figure 4.13). As before the dynamic behaviour of the interest rate reflects that of consumption growth (Figure 4.10 and Figure 4.12). Also as before, the initial jumps and transient dynamics of the capital-output ratio, the consumption-capital ratio, and the capital-gross product ratio (Ψ, Φ, and k_{GP} respectively), are readily explained by those of the growth rates of their component variables (Figure 4.8, Figure 4.9, and Figure 4.13):

- There are no jumps in Ψ (=K/A) because both K and A are stock variables for which discontinuities are not permitted. As K and A begin to adjust in the post-announcement/pre-implementation period Ψ rises because $g_K>g_A$. After implementation Ψ declines towards its new steady-state equilibrium level because $g_K<g_A$ (Figure 4.8, Figure 4.9 and Figure 4.10).
- On announcement of the shock there is a small upward jump in Φ (=C/K) reflecting just such a jump in C. Then, in the post-announcement/pre-implementation period Φ falls because $g_C<g_K$; and post-implementation it rises since then $g_C>g_K$ (Figure 4.8, Figure 4.9 and Figure 4.10).
- A small sudden fall in gross product GP=Y+R (due to a large precipitous drop in research R outweighing a steep rise in output Y), explains the slight upwards jump in k_{GP} (=K/GP) on announcement of the shock. In the post-announcement/pre-implementation period k_{GP} grows because $g_K>g_{GP}$. At implementation it falls sharply because at that point GP jumps higher (since now a steep rise in R more than compensates for a sharp drop in Y). And finally, after implementation it continues to decline since then $g_K<g_{GP}$ (Figure 4.10, Figure 4.11 and Figure 4.13).

The adjustment process for this simulation is again a lengthy one; a half of the total adjustment requiring some 20 years, and three quarters of it not being completed for over three decades (Table 4.2). This 'anticipated shock type of simulation' raises another problem for the determination of half-life etc. measures. Namely, whether they should be calculated from the time of 'announcement' or from the time of 'implementation'. As can be seen from Table 4.2, because of significant differences in the post-announcement/pre-implementation adjustment, when the former are used the half-life and three quarter-life figures for different variables are quite disparate, while when the latter are employed the results are fairly homogeneous. Convergence coefficients (see equations (3.13) and (3.17)) provide another measure of the speed of adjustment of the system towards its new steady-state equilibrium. Here the big variations, both between variables and over time, occur in the intervening period between announcement and implementation of the shock (Figure 4.15). Negative values of these coefficients indicate that the variable in question is moving away from its new steady-state. Such negativity could easily be predicted from the previous results.

Most of the preceding discussion has been something of a post-rationalisation of the dynamic results of the shock. While this indicates that it may be possible *ex ante* to describe some of the dynamics of anticipated shocks intuitively by qualitative economic reasoning, it does not seem simple. Moreover, the fact that there are opposing mechanisms and forces at work make some descriptions ambiguous. In such cases the economic outcomes cannot be discovered by *a priori* theorising alone. The empirics of the dynamic model are necessary to resolve such ambiguities and thereby to reveal the economic effects. And it is only through the dynamic vehicle that quantitative measures can be obtained. The economics of adjustment in response to shocks that are correctly foreseen certainly could not be captured, even qualitatively, with a static model. These points emphasise the power of the dynamic model.

4.5.3 A temporary rise in the ordinary labour share of wages

Consider some campaign of trade union action which, together with an accommodating wages policy, succeeds in obtaining certain benefits for ordinary labour at the expense of human capital employed in the output sector. As a result, ordinary labour's share of the total wages to these groups is estimated to increase by some 15 per cent. It is assumed that the specific measures used to achieve this change are implemented 'immediately'. That is they are not announced in advance and are not anticipated for any other reason. Conversely, it is also assumed that it is known that they are to operate only temporarily; in particular, for five years. It may be supposed either that this is announced at the time of their implementation; or that it is just correctly foreseen by the market - perhaps the changes are made under the auspices of a labour government which is correctly forecast to lose the next election to conservatives known to be hostile to them! Whatever the case, the problem is to simulate such changes using the Romer model in order to assess their broad economic implications.

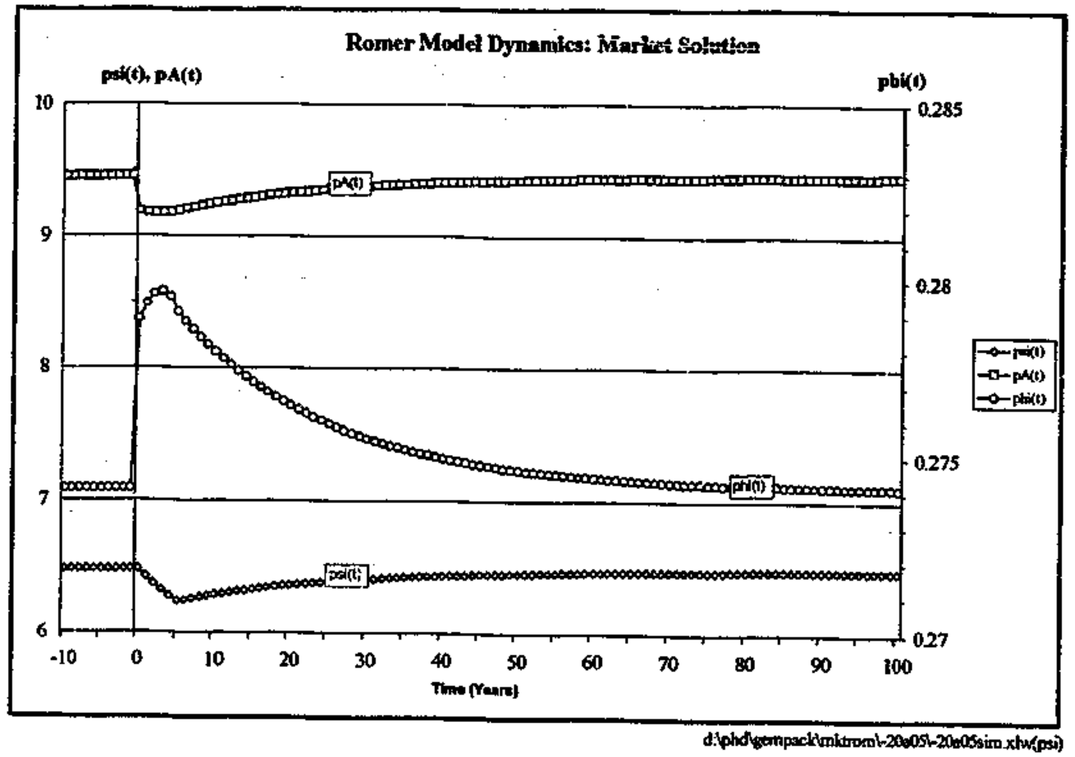
From Chapter 2 (Appendix 2.1) the wages income received by ordinary labour (L) from the manufacture of goods is $(1-\alpha)(1-\gamma)Y$, and that received by human capital (H_Y) is $\alpha(1-\gamma)Y$. The share of these wages to ordinary labour is therefore $(1-\alpha)$; and a rise in this share may be simulated by exogenously reducing the parameter α. In particular, a 15 per cent rise in $s_{WL} = (1-\alpha)$ is equivalent to a fall of some 20 per cent in α from its benchmark level.[37] Accordingly, the economic shock of a 15 per cent rise in the share of wages to ordinary labour for five years is simulated here by reducing the benchmark level of parameter α by 20 per cent from time t=0 to t=5. The results are reported below in Figure 4.16 to Figure 4.21 and in Table 4.3.

The elasticity of output Y with respect to ordinary labour L is given by $\varepsilon_{YL} = (1-\alpha)(1-\gamma)$. Thus, in addition to increasing the income of ordinary labour in the output sector, a fall in α also raises its productive efficiency. In the real world these incentives would increase its usage in producing output, drawing extra units out of overtime, unemployment or some other sector in which it was employed. In the model however,

[37] If the initial share is $s_{WL0} = (1-\alpha_0)$ and the post-shock share is $s_{WL1} = (1-\alpha_1) = 1.15.s_{WL0}$, then $\alpha_1/\alpha_0 = 1.15.- 0.15/\alpha_0$; which, with α_0 at its benchmark level of 0.43, gives $\alpha_1 \approx 0.8\alpha_0$.

ordinary labour is fully employed and has no alternative use than in the output sector. Moreover, its aggregate supply is exogenously fixed. Thus, under the confines of the model there can be no effect on ordinary labour. Nevertheless, many other effects may be expected.

Figure 4.16: **Dynamic effects (over 100 years) on Ψ, Φ, and p_A of a sustained but temporary 15 per cent rise in the ordinary labour share of wages from time t=0 to t=5 years, benchmark parameter set.**



The elasticity of output with respect to the human capital employed in producing it H_Y, is given by $\varepsilon_{YH_Y} = \alpha(1-\gamma)$; so lowering α will reduce its productive efficiency. This will induce producers to substitute physical capital K for human capital H_Y. But physical capital is fixed in the very short run. The only way to increase its supply is to produce more output from which to save (and invest); and the only way to do this is to raise the level of technology A. As a result, human capital will move from the output sector to research until the wage rates equilibrate.[38] Thus, H_A, g_A and A can all be expected to rise; and p_A and Ψ to fall (Figure 4.16 to Figure 4.19).

While the loss of human capital from the output sector to research would initially tend to reduce the output that could be produced, the greater productive efficiency of ordinary labour would tend to raise it. It turns out that the latter effect dominates with g_Y initially jumping to a higher level and continuing to rise as long as the lower α remains extant. With output growing faster, and technology even faster still, the rate of growth of gross product (which is a weighted average of Y and A) will lie between them That is: $g_Y<g_{GP}<g_A$ (Figure 4.18).

[38] Another way of looking at this is to note that since the wages of human capital in the output sector fall, it will be bid away to the research sector.

Figure 4.17: **Dynamic effects (over 25 years) on Ψ, Φ, and p_A of a sustained but temporary 15 per cent rise in the ordinary labour share of wages from time t=0 to t=5 years, benchmark parameter set.**

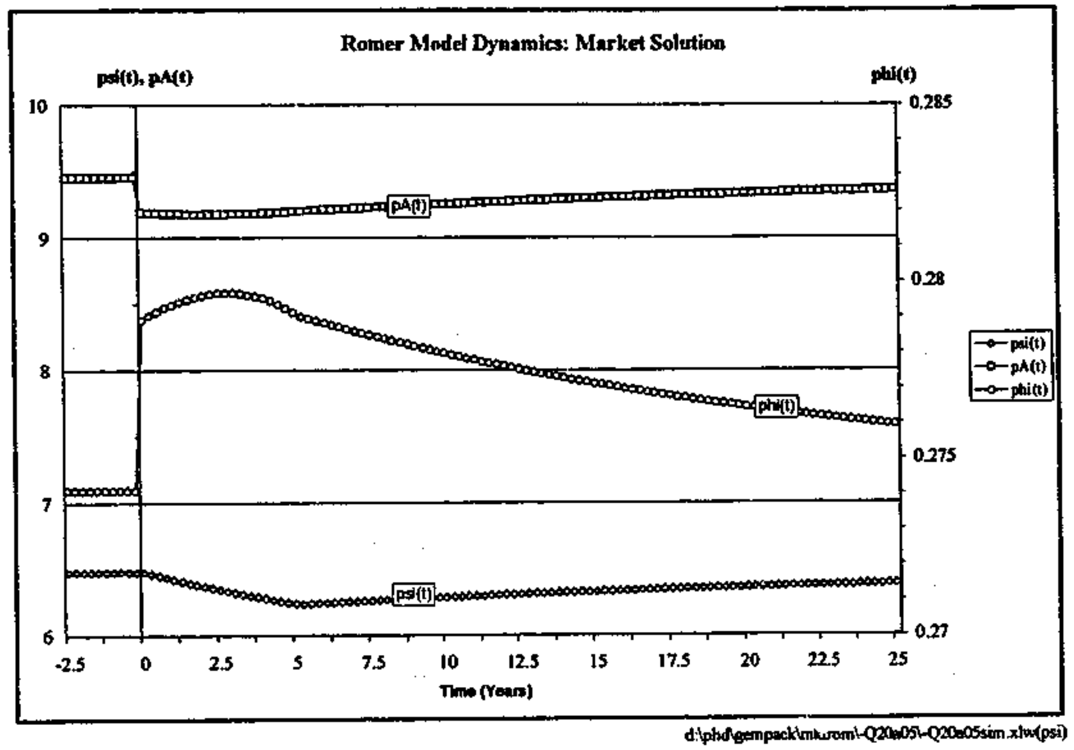


Figure 4.18: **Dynamic effects on growth rates of a sustained but temporary 15 per cent rise in the ordinary labour share of wages from time t=0 to t=5 years, benchmark parameter set.**

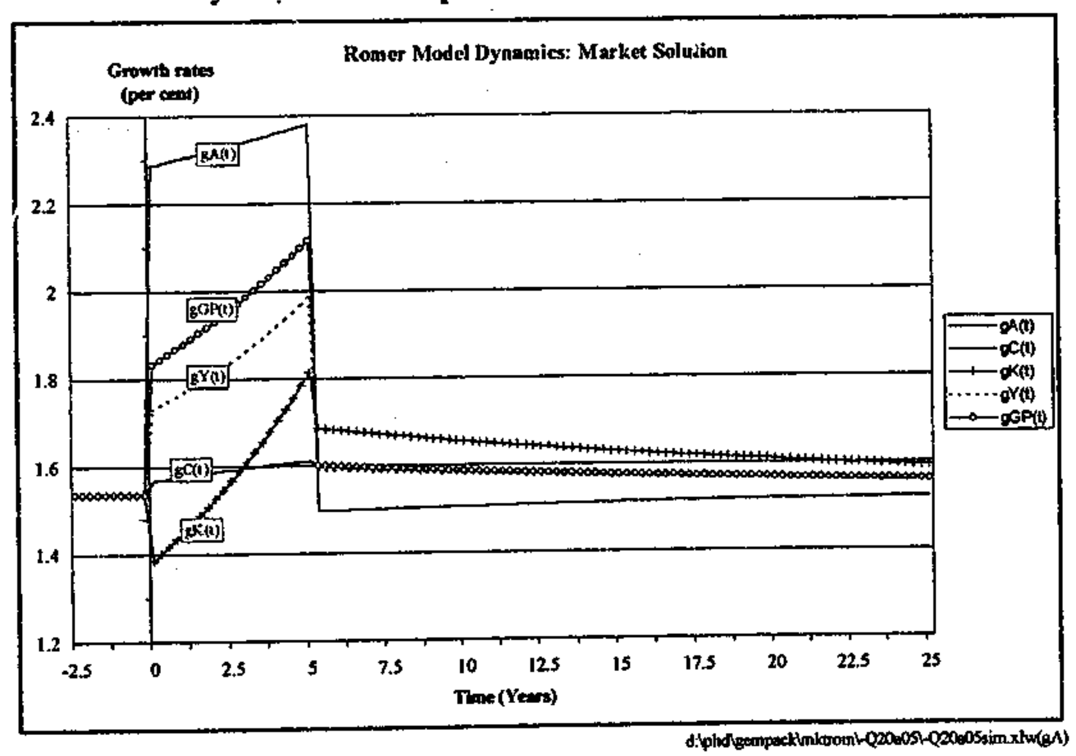
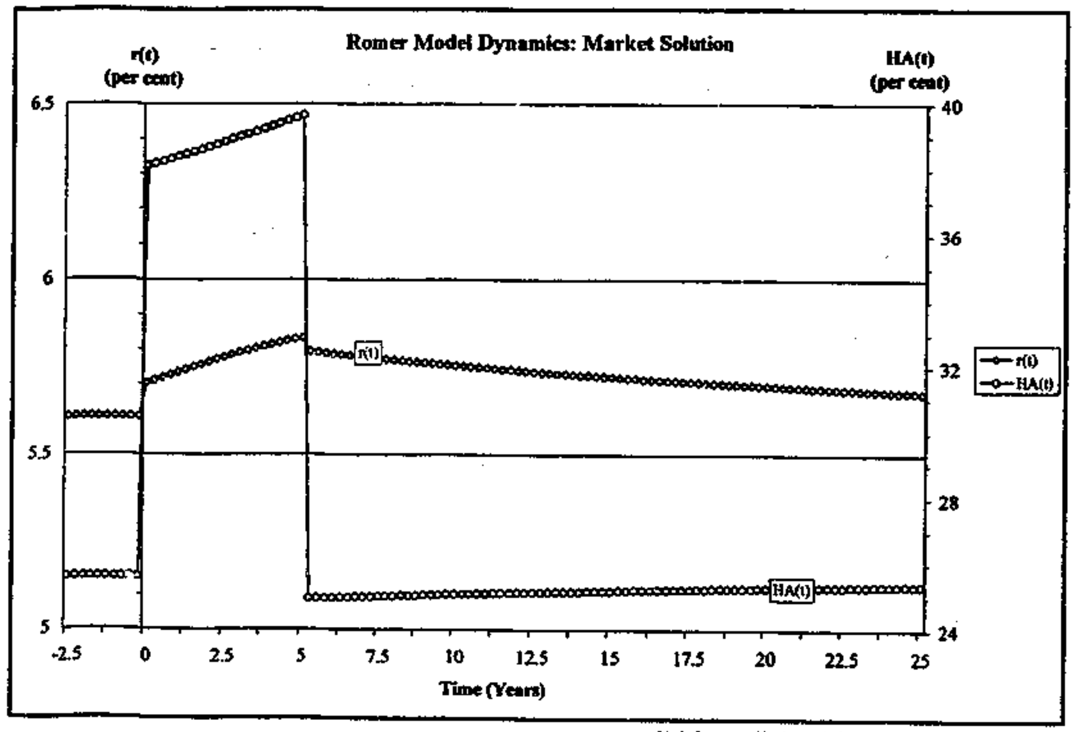


Figure 4.19: Dynamic effects on r and H_A of a sustained but temporary 15 per cent rise in the ordinary labour share of wages from time t=0 to t=5 years, benchmark parameter set.



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The results show that the optimising consumers raise the growth of consumption slightly while the lower α prevails. While this is only small, it is sufficient for the narrow savings measure s_N and the growth rate of capital g_K to fall initially. However, the growth of consumption remains below that of output and after about three years the savings rate rises by enough for g_K to equal g_C and then to exceed it (Figure 4.20 and Figure 4.18). As a result of these movements the consumption-capital ratio Φ first increases and then, after some three years, begins to fall (Figure 4.16 and Figure 4.17). To support the slightly higher growth of consumption (so future consumption is greater than current consumption) the return to savings and hence the interest rate r, must also rise slightly. Another way of looking at this is to consider the linkage between the interest rate and the return to capital; and to argue that with capital a little scarcer due to its lower growth, its return and therefore the interest rate must also rise a little. In any case this is indeed what happens in the simulation (Figure 4.19). Devoting more human capital resources to research as opposed to current production (and consumption) would be expected to result in an increase in the broad measure of savings s_B. Also, with a slower rate of growth of capital and a faster growth of gross product, the capital-gross product ratio k_{GP} must fall (Figure 4.20).

Finally, consider the factor shares of income. The shares to human capital in research, ordinary labour, human capital in goods production, and physical capital are given by: $S_{HA}=R/(R+Y)$, $S_L=(1-\alpha)(1-\gamma)Y/(R+Y)$, $S_{HY}=\alpha(1-\gamma)Y/(R+Y)$ and $S_K=\gamma Y/(R+Y)$ respectively, where $R = p_A \dot{A}$ is the output of research. Although the price of technology (p_A) declines a little, the strong growth in its quantity means that the output of research grows faster than that of goods. Thus, the share of income to human capital in research will rise, and the shares to both human and physical capital in the output sector will fall.

These effects from the relative growth rates of the outputs of research and of goods are indirect and not sufficient to overcome the direct effect of the reduction in α, so the share of income to ordinary labour rises (Figure 4.21).

Figure 4.20: Dynamic effects on s_B, s_N, and k_{GP} of a sustained but temporary 15 per cent rise in the ordinary labour share of wages from time t=0 to t=5 years, benchmark parameter set.

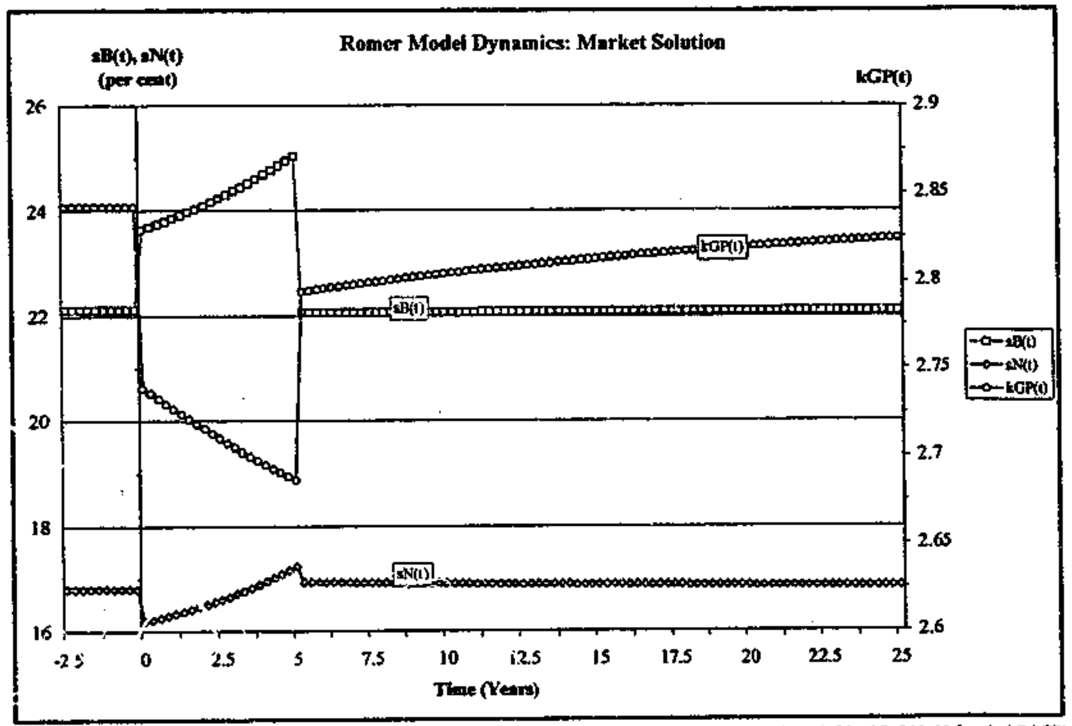

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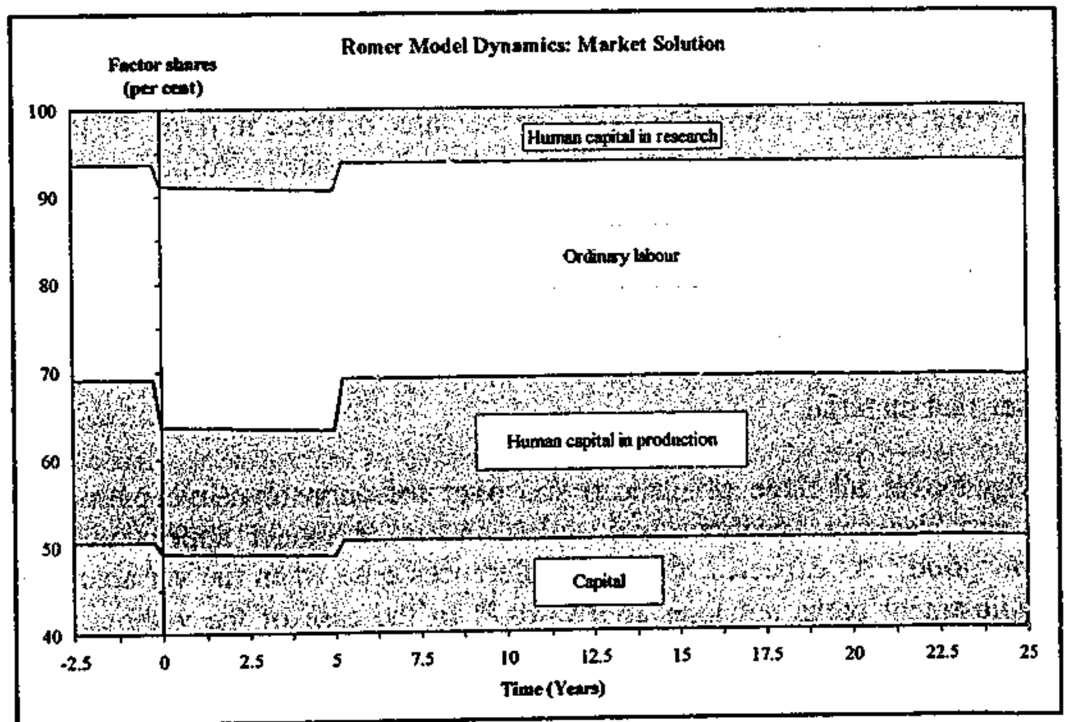
Figure 4.21: Dynamic effects on the factor shares of gross income from a sustained but temporary 15 per cent rise in the ordinary labour share of wages from time t=0 to t=5 years, benchmark parameter set.



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Of course, with the shock to α being only temporary none of these effects are lasting. Following the removal of the shock after five years, the dynamic system begins a smooth adjustment back to its initial steady-state. Once again however, this adjustment is quite a slow process - taking some 15 more years to get half way back, and some thirty more years to close three-quarters of the gap at the time of removal (Table 4.3).

Table 4.3: Simulation results of an unanticipated but known to be temporary 15 per cent rise in the ordinary labour share of wages from time t=0 to t=5, benchmark parameter set.

Dynamic variable	Final steady-state	Total adjustment (% of initial ss)	Jumps as a % of initial ss on: Imposition	Removal	½ life[a] (years)	¾ life[a] (years)
$\Psi(t)$	6.48	0.00	0.00	0.00	21(na)	35(na)
$\Phi(t)$	0.2741	0.00	1.75	0.00	20(20)	34(34)
$p_A(t)$	9.45	0.00	-2.84	0.00	21(20)	35(33)
$r(t)$	5.61%	0.00	1.71	-0.83	20(33)	34(47)
$H_A(t)$	25.60%	0.00	48.9	-57.4	20(5)	34(5)
$g_{GP}(t)$	1.54%	0.00	19.3	-33.7	20(5)	34(5)
$s_B(t)$	22.10%	0.00	6.89	-13.5	20(5)	34(5)
$s_N(t)$	16.80%	0.00	-3.70	-1.82	21(2)	35(3)
$k_{GP}(t)$	2.84	0.00	-3.65	3.85	21(5)	35(18)
$\beta_\Psi(t)$	4.93%	0.00	n.a.	399	20(na)	35(na)

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Note: a The first set of results give the total time taken (from t=0) for ½ and ¾ of the remaining adjustment after implementation; while the figures in parentheses give corresponding results after the post- announcement jumps.

4.5.4 Comparison of the linearised, Solowian- and full non-linear Romer model results

In Chapter 3 two approximations to the Romer model were developed and their transient dynamics examined. These were the 'complete' but linearised model (Section 3.2); and the truncated but non-linear Solowian-Romer model (Section 3.4). The analysis has now reached a point where it is possible to assess the value of these in terms of how closely their computed dynamics approximate those of the full non-linear Romer model. Each of these three models has now been confronted with the same three shocks: *an unanticipated and sustained 10 per cent rise in parameter γ, an anticipated and sustained 15 per cent increase in parameter ζ;* and *an unanticipated but known to be temporary fall of 20 per cent in parameter α.* Details of the results of their computed dynamics in response to these shocks are compared in Appendix 4.4. The following summarises that material.

When judged over all three simulations and over the entire dynamic paths of all the dynamic variables the linearised and Solowian-Romer model approximations seem 'reasonably good'. As expected, the main differences arise when the variables jump and in their adjustment paths not long afterwards.[39] In the case of the γ-shock simulation

[39] All three models, by their construction, correctly generate the analytic asymptotic results.

both approximations were excellent for Ψ, p_A, r, and k_{GP}. Variables g_A and H_A could be added to this list for both the ζ- and the α-shock simulations; and in the latter case so too could variables Φ, g_{GP}, and s_B. This means that for the α-shock simulation excellent approximations were achieved for almost all the dynamic variables. One reason why the 'goodness of fit' of the approximations improved from the γ-shock to the ζ-shock to the α-shock is that the total adjustments involved in these respective simulations decrease (also see Table 4.1, Table 4.2, and Table 4.3). While the linearised model tended to converge to the full model a little faster than the Solowian-Romer model, overall it was the S-R model that was the more accurate.

4.5.5 An immigration program to raise the level of human capital

Here the effects of an immigration program designed to raise the overall level of human capital in the economy by 15 per cent over five years are considered. The program is to be implemented in uniform quarterly steps with the first migrants under the scheme due to arrive at the end of the first quarter one year after its announcement. The policy is simulated by raising the benchmark level of the model's human capital variable H, by a series of 20 cumulating shocks of 0.75 per cent each, beginning five quarters from the time of announcement (that is, at time t=1.25 years). As a result, from time t=6 years onwards the level of human capital will be 1.15 times its benchmark level. Since the entire program is announced in advance of its implementation, all of the shocks are *anticipated* by the market. This means that once the shocks are written into the model's equations at the appropriate time points, the immigration policy is simulated by solving the model from the time of announcement (t=0), for all future time points.

The level of human capital devoted to research rises more than proportionally with any increase in the total level of human capital and conversely, the level devoted to the output sector rises less than proportionally.[40] Now, as noted previously in the research productivity simulation (Section 4.5.2), an increase in the productivity of researchers is like having more of them. Thus, the effects of increasing the total level of human capital H ought to be similar to those of increasing the research productivity parameter ζ; but with the human capital in research increasing even further, and with that employed in the output sector increasing instead of decreasing.[41]

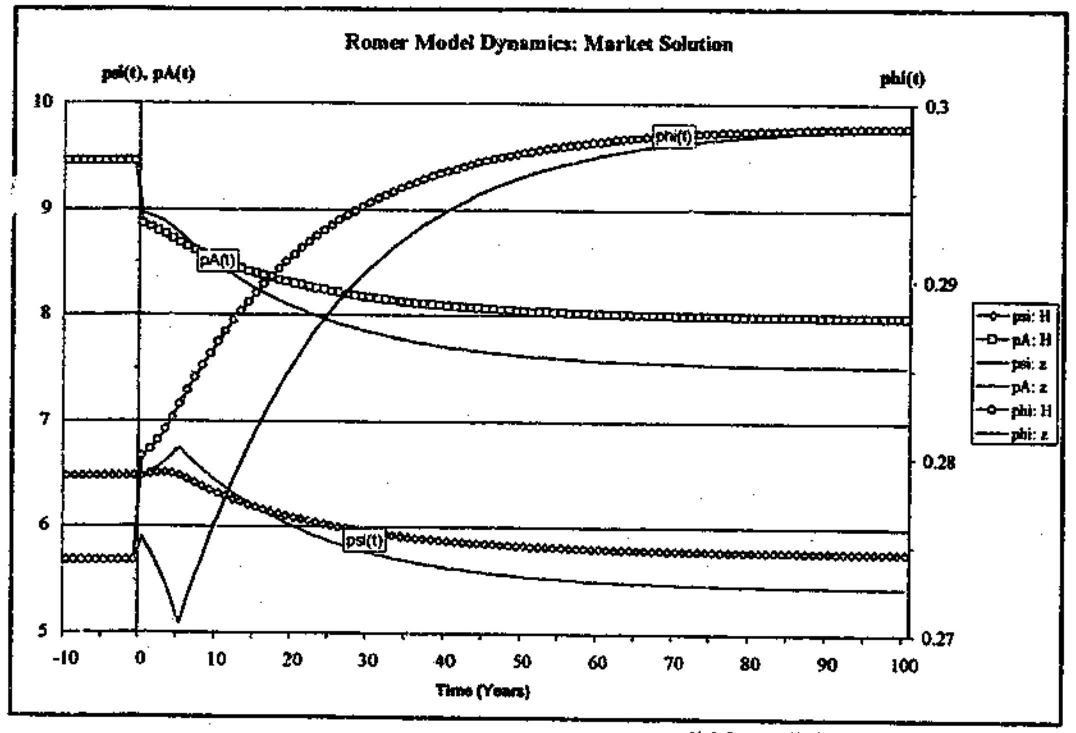
In terms of their impacts on steady-state levels the veracity of this proposition is supported by the sensitivity results of Table 2.3. For the transitional dynamics however, the connection is more difficult to see. In order to facilitate comparison, some of the results for the research productivity simulation of Section 4.5.2 have been re-plotted on

[40] This can be simply demonstrated by differentiating the formulae for the steady-state levels H_{Ass} and H_{Yss} (equations 2.54 and 2.55) with respect to H to obtain: $dH_{Ass}/H_{Ass}>dH/H$ and $dH_{Yss}/H_{Yss}<dH/H$. Also see the benchmark steady-state sensitivity results in Table 2.3 of Chapter 2.

[41] It may be noted however, that the proportional change in the **allocative share** of human capital to either the research or output sector is the same for any relative change in H as it is for an identical relative change in ζ. Once again this may be demonstrated by differentiating equations (2.54) and/or (2.55) to obtain $\frac{d(aH_A)}{aH_A}/\frac{dH}{H}=\frac{d(aH_A)}{aH_A}/\frac{d\zeta}{\zeta}$, where $aH_A=H_A/H$.

the same graphs as those recording the results for this simulation (Figure 4.22 to Figure 4.28). Other results for the current simulation are presented in Figure 4.29 and Figure 4.30, and in Table 4.4, which may be compared with the corresponding results for the research productivity simulation (Figure 4.14 and Figure 4.15, and Table 4.2).

Figure 4.22: Dynamic effects (over 100 years) on Ψ, Φ, and p_A of an anticipated and phased 15 per cent rise in the aggregate level of human capital (H), and an anticipated but sudden 15 per cent rise in research productivity (ζ), benchmark parameter set.



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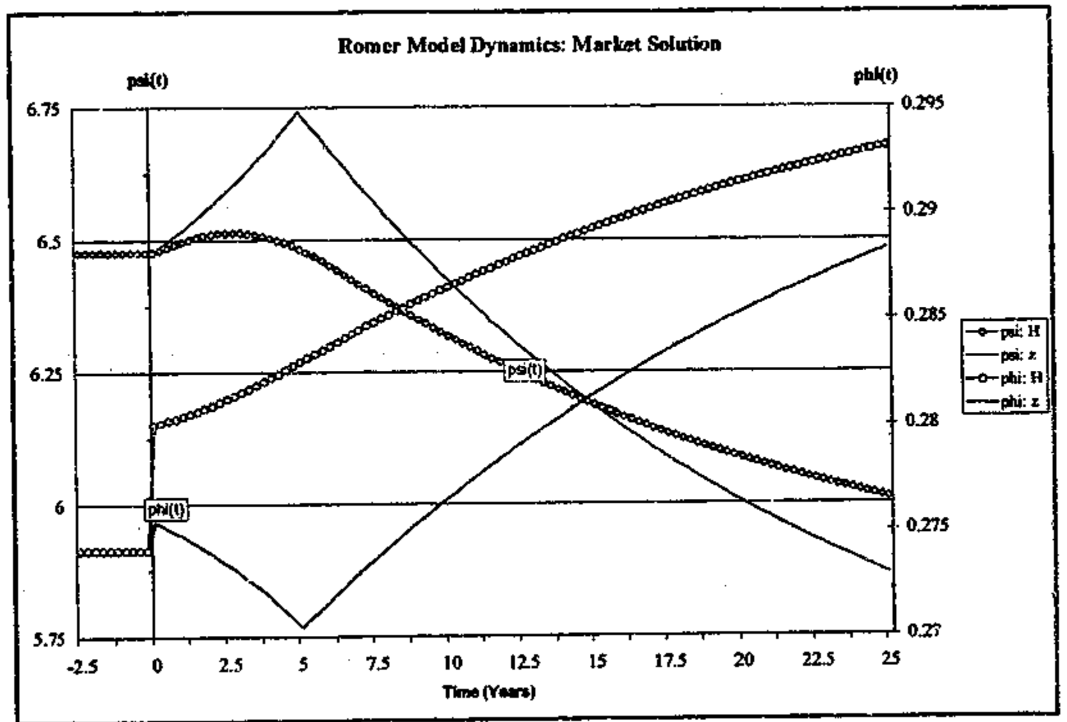
Considerable differences in the adjustment paths of the two simulations are evident. These are due largely to two factors: the differential timing of their shocks, and the variance of their impacts on H_Y. Despite the similarity of the underlying economic mechanisms of the simulations, these two factors have a profound effect in generating differences in their transitional dynamics.

Like a rise in the productivity of researchers, an increase in the level of human capital (which is fully employed at all times), allows the 'production' of more research at all price levels; or equivalently, it lowers the price of research at all output levels. Since it is also announced in advance, once again there is an incentive for capital goods producers to postpone some of their demand for designs (refer back to footnote 36). However, because the total shock to the level of human capital is phased in, while the research productivity shock all comes at the one time, the scope for such postponement of demand in the immigration simulation is very much the lesser of the two. As a result, the dynamics of the current simulation are considerably more gradual, exhibiting no significant jumps at either the beginning or the end of the implementation period. It is also the differential timing of the shocks which causes the main differences in the results to be manifested during the immigration policy implementation period and not too long

thereafter. Once the shocks are all fully implemented the adjustment paths for both simulations begin to converge. And for many variables such convergence is all the way to their (common) steady-state.

With regard to the second factor: In the productivity simulation H_Y declines, as it must, when H_A rises; while in the immigration simulation both rise. As a result of this extra productive resource more output is eventually generated and greater consumption is possible under the immigration simulation.[42]

Figure 4.23: Dynamic effects (over 25 years) on Ψ and Φ of an anticipated and phased 15 per cent rise in the aggregate level of human capital (H), and an anticipated but sudden 15 per cent rise in research productivity (ζ), benchmark parameter set.



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Interestingly, while relative variations in the growth rates of capital and designs between the simulations result in different steady-state levels of Ψ, apparently not dissimilar variations in the growth rates of capital, consumption and gross product generate steady-state levels of Φ and k_{GP} that are the same in both simulations (Figure 4.22 to Figure 4.24). However close examination of the differences in growth rates confirms that this is possible.

42 Although all growth rates are asymptotically equal, for output gY_H>gY_ζ over most of the adjustment period; and for consumption gC_H>gC_ζ at all dates (Figure 4.25 and Figure 4.26).

Figure 4.24: Dynamic effects (over 25 years) on p_A and k_{GP} of an anticipated and phased 15 per cent rise in the aggregate level of human capital (H), and an anticipated but sudden 15 per cent rise in research productivity (ζ), benchmark parameter set.

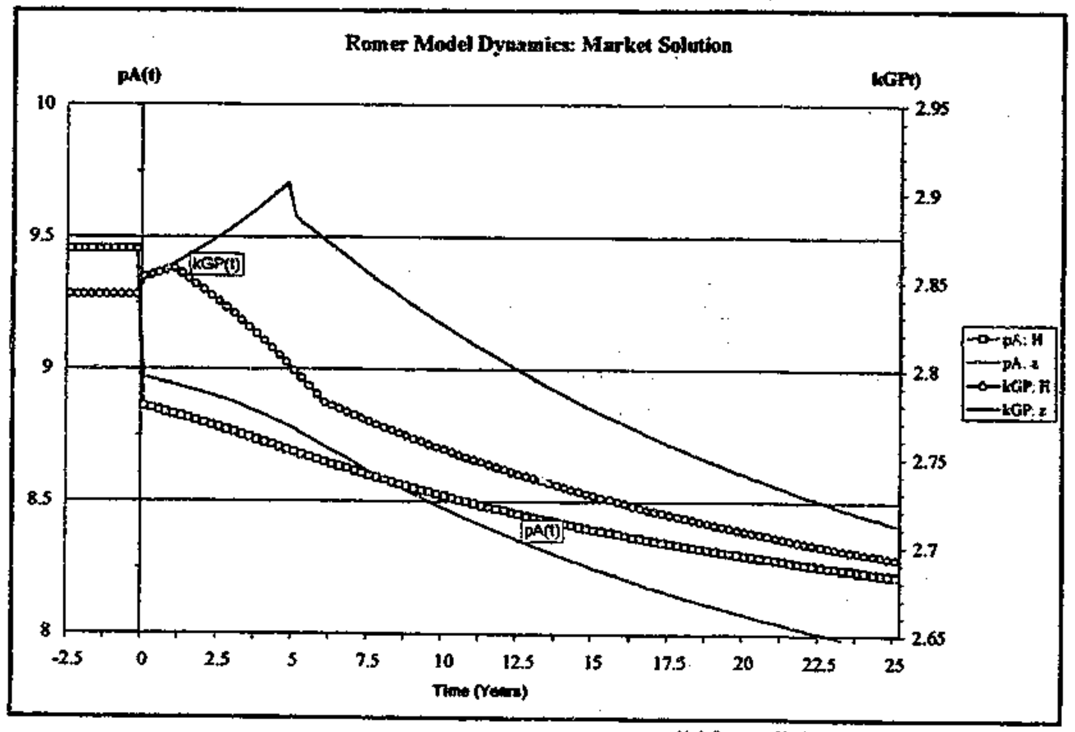

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Figure 4.25: Dynamic effects on the growth rates g_A, g_C, and g_K, of an anticipated phased 15 per cent rise in the aggregate level of human capital (H), and an anticipated but sudden 15 per cent rise in research productivity (ζ), benchmark parameter set.

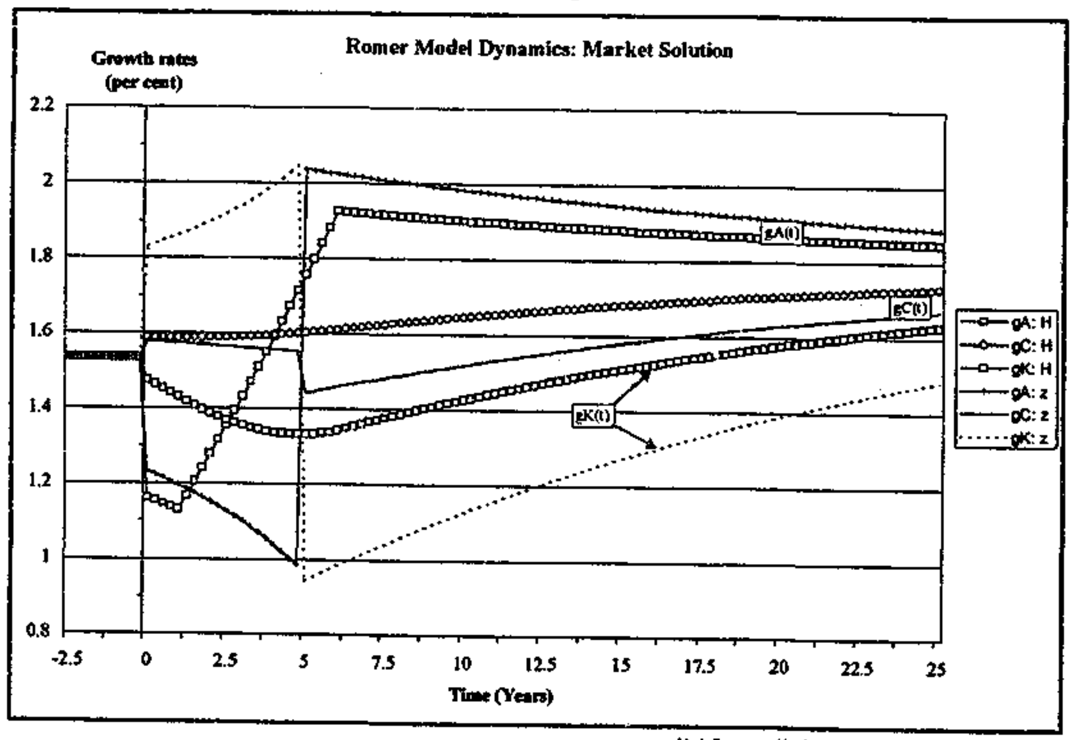

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Figure 4.26: Dynamic effects on the growth rates g_Y, and g_{GP}, of an anticipated phased 15 per cent rise in the aggregate level of human capital (H), and an anticipated but sudden 15 per cent rise in research productivity (ζ), benchmark parameter set.

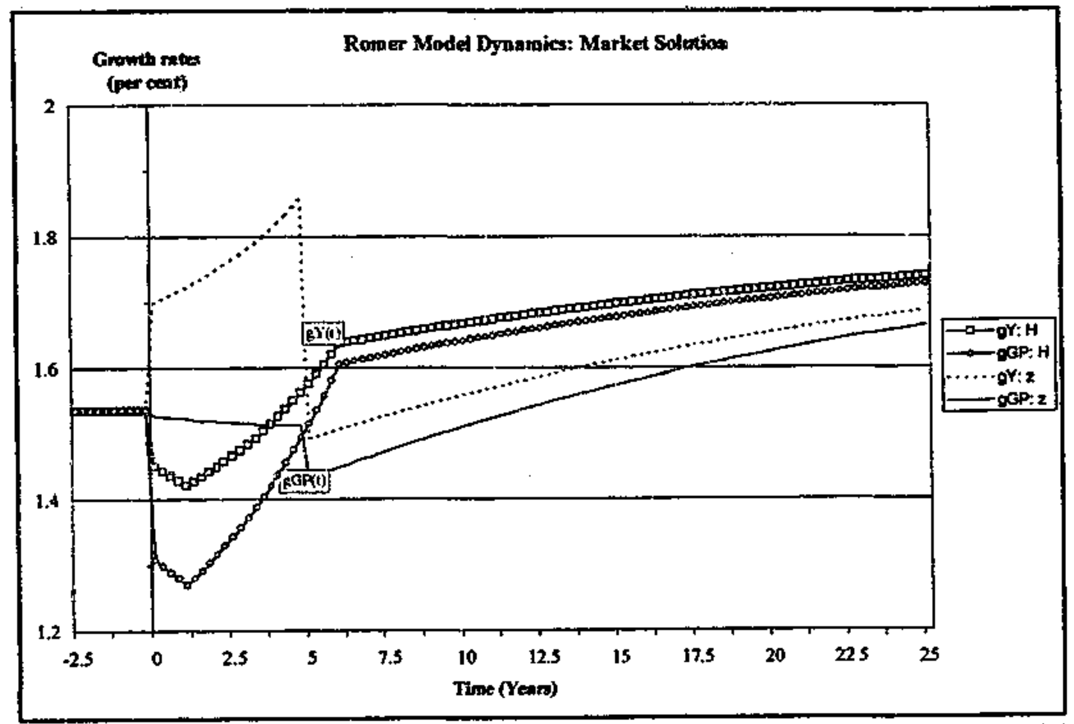

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Figure 4.27: Dynamic effects on r and H_A of an anticipated phased 15 per cent rise in the aggregate level of human capital (H), and an anticipated but sudden 15 per cent rise in research productivity (ζ), benchmark parameter set.

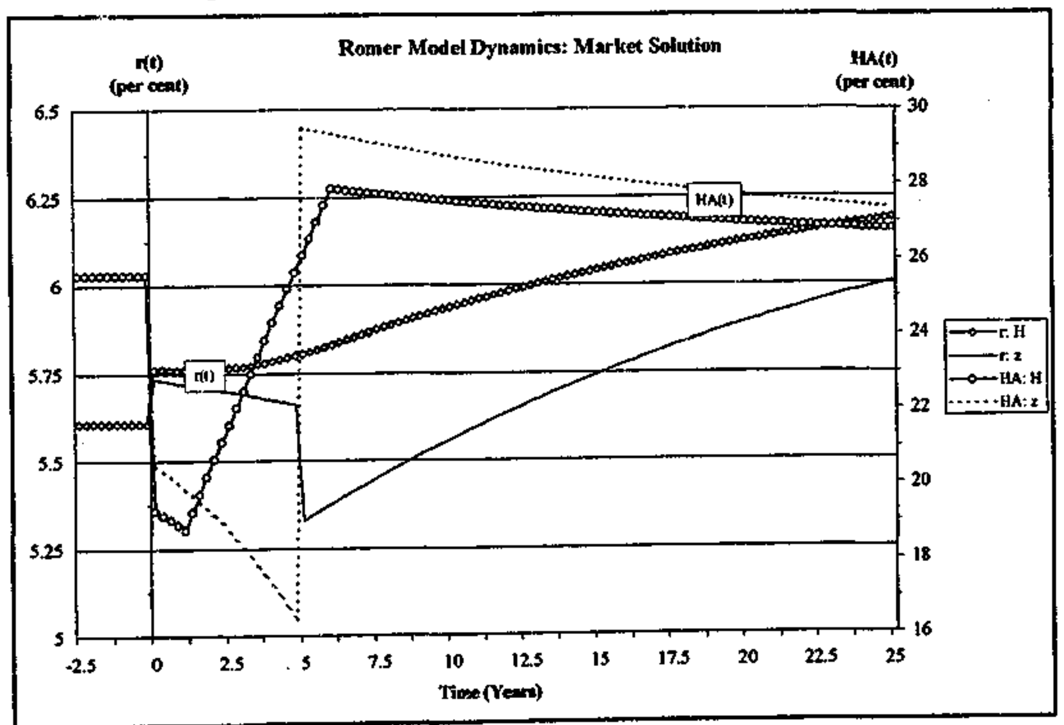

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Figure 4.28: **Dynamic effects on s_B and s_N of an anticipated phased 15 per cent rise in the aggregate level of human capital (H), and an anticipated but sudden 15 per cent rise in research productivity (ζ), benchmark parameter set.**

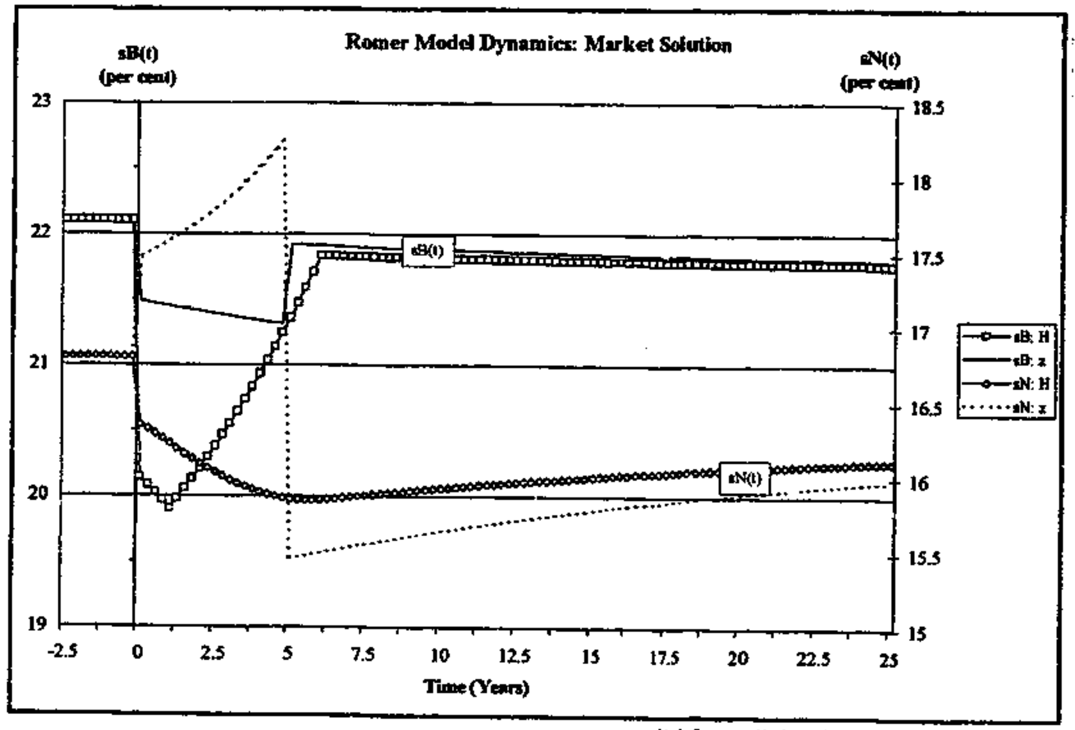

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Figure 4.29: **Dynamic effects on the factor shares of gross income from an anticipated 15 per cent rise in the aggregate level of human capital (H), implemented in 20 uniform quarterly steps from time t=1.25 years to t=6 years, benchmark parameter set.**

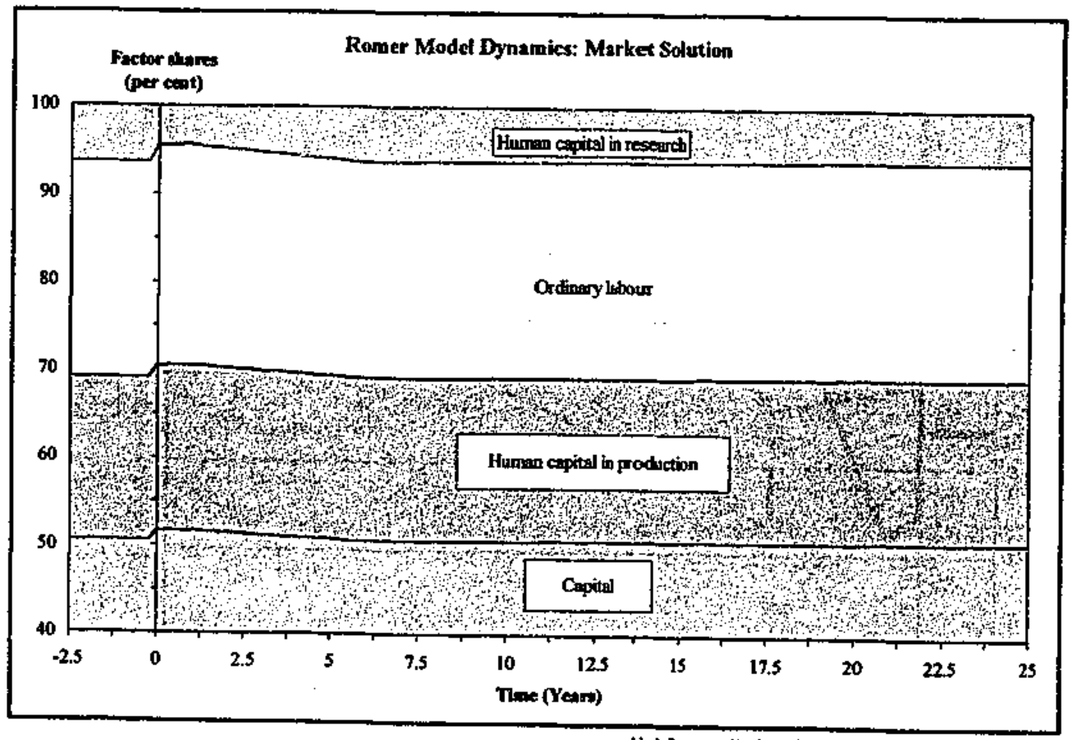

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Figure 4.30: **Dynamic effects on the convergence coefficients from an anticipated 15 per cent rise in the aggregate level of human capital (H), implemented in 20 uniform quarterly steps from time t=1.25 years to t=6 years, benchmark parameter set.**

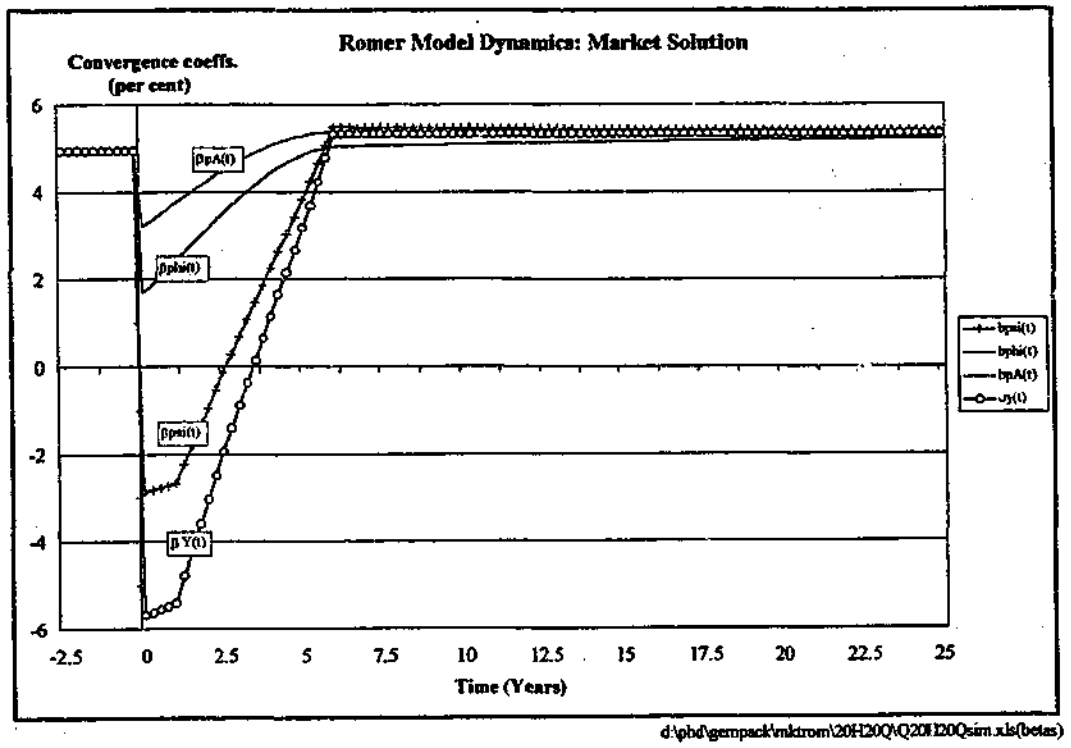

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Table 4.4: **Simulation results of an anticipated and sustained 15 per cent rise in the aggregate level of human capital (H), implemented in 20 uniform quarterly steps from time t=1.25 years to t=6 years, benchmark parameter set.**

Dynamic variable	Final steady-state	Total adjustment (% of initial ss)	Initial jumps as a % of: initial ss	total adjustment	½ life[a] (years)	¾ life[a] (years)
$\Psi(t)$	5.77	-11.0	0.00	0.00	19(19)	32(31)
$\Phi(t)$	0.2988	9.00	2.16	24.0	20(16)	33(29)
$p_A(t)$	7.98	-15.6	-6.28	40.3	19(14)	32(27)
$r(t)$	6.40%	14.2	2.76	19.4	20(18)	33(31)
$H_A(t)$	26.10%	2.00	-24.5	-1226	20(4)	33(5)
$g_{GP}(t)$	1.80%	17.3	-14.8	-85.8	20(6)	33(16)
$s_B(t)$	21.73%	-1.69	-8.89	526	20(4)	34(5)
$s_N(t)$	16.26%	-3.23	-2.68	82.9	20(0)	33(1)
$k_{GP}(t)$	2.62	-7.82	0.35	-4.52	20(14)	33(33)
$\beta_\Psi(t)$	5.33%	8.08	-158	-1959	20(4)	33(5)

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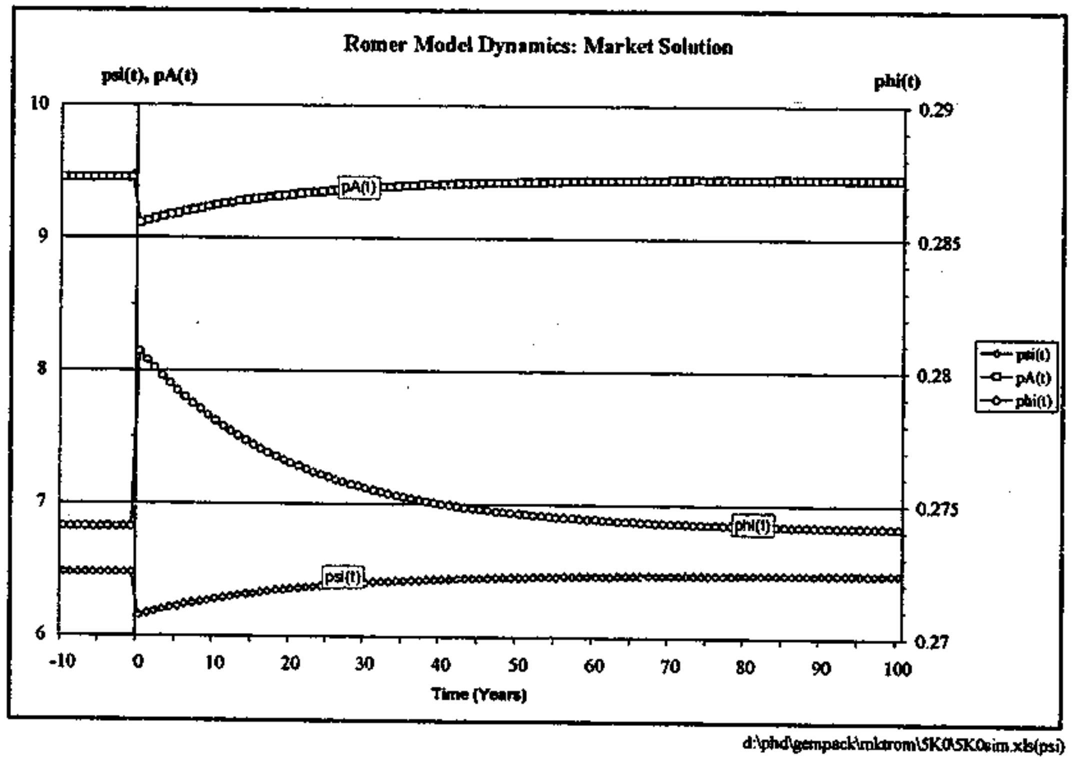
Notes: a The first set of results give the time taken for ½ and ¾ of the remaining adjustment after implementation is completed at t=6 years; while the figures in parentheses give corresponding results after the post-announcement jumps.

4.5.6 A sudden, temporary reduction of the capital stock

Here one may have in mind some sort of destructive calamitous event, such as the invasion of Kuwait, a major earthquake in Tokyo, or a nuclear strike upon Israel, which destroys a significant proportion of the capital stock of the particular country. Specifically, the simulation assumes that five per cent of an economy's capital stock is suddenly destroyed. This is modelled by imposing an unanticipated shock of minus five per cent on the initial value of the capital-technology ratio, Ψ_0. It is easily done without having to change the 'closure' of the model, since this is the one component of $\Psi(t)$ which is exogenous.[43]

The shock is a temporary one in that it does not permanently hold down the value of Ψ. Rather, it simply disturbs the existing steady-state for a time. Since there is no permanent change to any of the underlying parameters of the model, the pre-shock equilibrium continues to be optimal for the economy. Thus, the post and pre-shock steady-states are identical and once disturbed, the economy then simply re-adjusts back to an equilibrium identical in all respects to its original configuration. The results of the simulation are presented in Figure 4.31 to Figure 4.37 and Table 4.5 below. The annual long-term results of Figure 4.31 are intended to show clearly how the initial steady-state levels of the main dynamic variables are restored; while the quarterly shorter term results in the other diagrams emphasise the main period of adjustment.[44]

Figure 4.31: Dynamic effects (over 100 years) on Ψ, Φ, and p_A of an unanticipated and temporary 5 per cent fall in the initial capital stock (Ψ), benchmark parameter set.



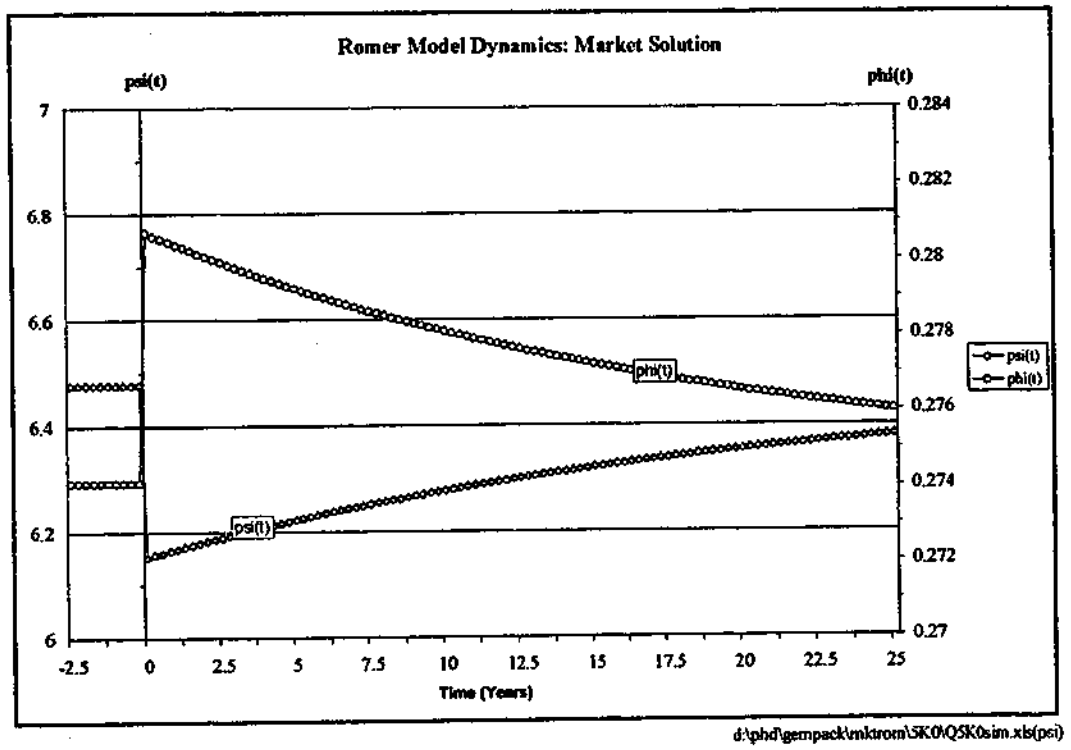
Because of the equivalence of the pre and post-shock steady-states, any *a priori* consideration/rationalisation of the impact of the shock can be restricted to explaining the directions of the initial jumps in the relevant economic variables:

[43] In this context the term 'closure' refers to the split between the endogenous and exogenous variables of a model.

[44] The 'factor shares of income' diagram has been omitted due to their lack of perceptible change.

The precipitous fall in capital, manifested via the capital-technology ratio Ψ, will cause a similar fall in the capital-gross product ratio k_{GP}, and a sudden upwards jump in the consumption-capital ratio Φ (Figure 4.31 to Figure 4.33). Restoration of the capital-technology ratio Ψ requires capital to grow faster than technology. Increased savings and investment at the expense of current consumption is one route that might be followed to achieve this. A reduction in research activity, with a transfer of human capital resources to output production is another. From the dynamic utility maximisation process underlying the model (Appendix 2.2) it turns out to be optimal for the latter of these means to be heavily preferred. Since current consumption does not have to be traded for capital formation, there is very little change to the rate of savings out of output, s_N (Figure 4.36). Instead, human capital is instantly switched from the research to the output sector and H_A falls sharply (Figure 4.35). As a result, the rate of growth of technology also falls sharply, while the growth rate of output, and with it those of consumption and capital, rise suddenly (Figure 4.34).

Figure 4.32: Dynamic effects (over 25 years) on Ψ and Φ of an unanticipated and temporary 5 per cent fall in the initial capital stock (Ψ), benchmark parameter set.



The shortage of capital and the increased demand for it drive up its price as reflected by the interest rate r (Figure 4.35). Also, with less general-purpose capital available there will be less demand for designs from specialist capital goods producers and their price (p_A) will fall (Figure 4.31 and Figure 4.33). Finally, since it happens that savings in the form of research fall slightly more than those in the form of capital formation rise, the broad savings rate out of gross product s_B, exhibits a small initial fall (Figure 4.36).

Figure 4.33: **Dynamic effects on p_A and k_{GP} of an unanticipated and temporary 5% fall in the initial capital stock (Ψ), benchmark parameter set.**

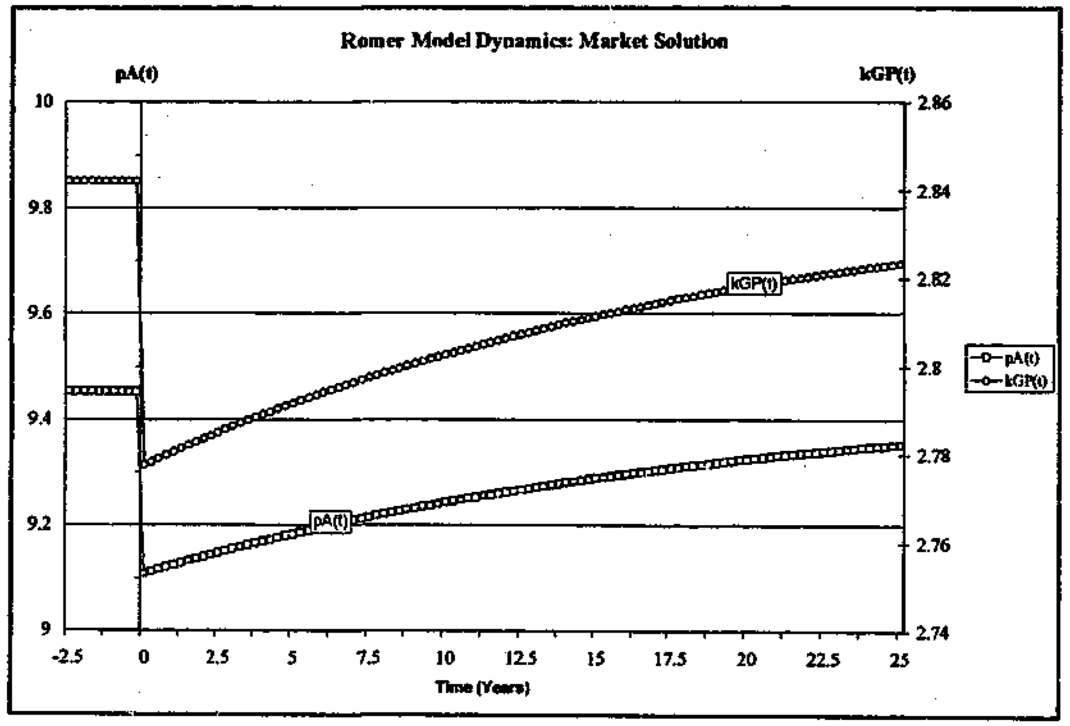


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Figure 4.34: **Dynamic effects on the growth rates, of an unanticipated and temporary 5% fall in the initial capital stock (Ψ), benchmark parameter set.**

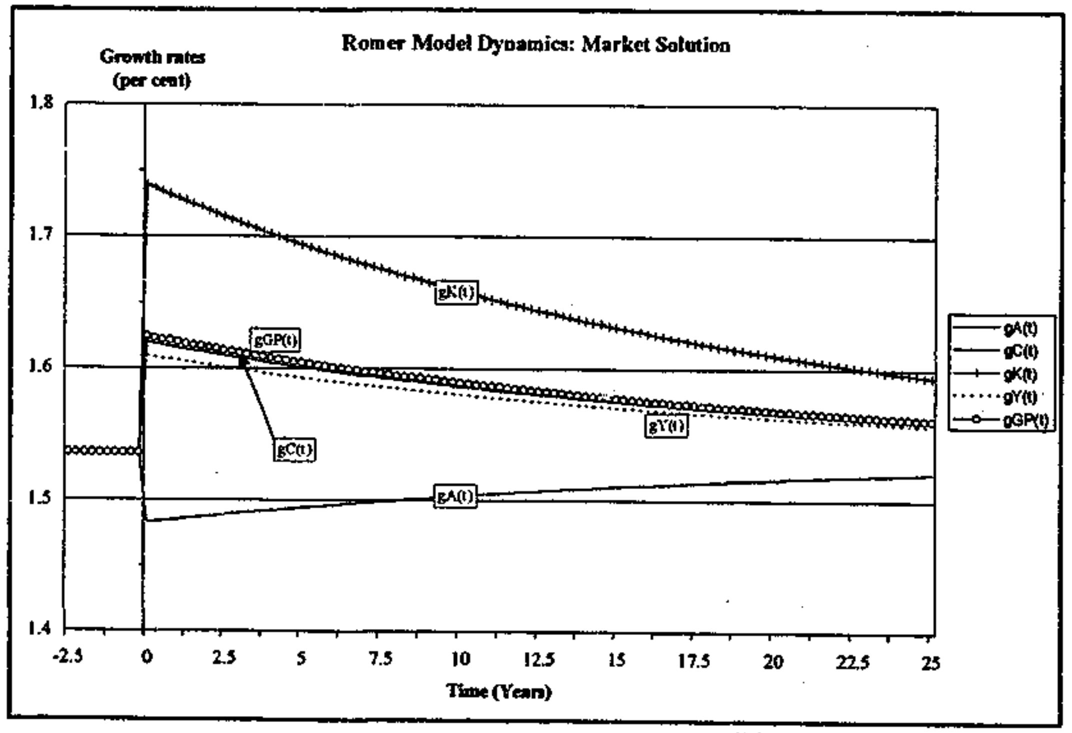

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Figure 4.35: **Dynamic effects on the interest rate and the share of human capital devoted to research, of an unanticipated and temporary 5% fall in the initial capital stock (Ψ), benchmark parameter set.**

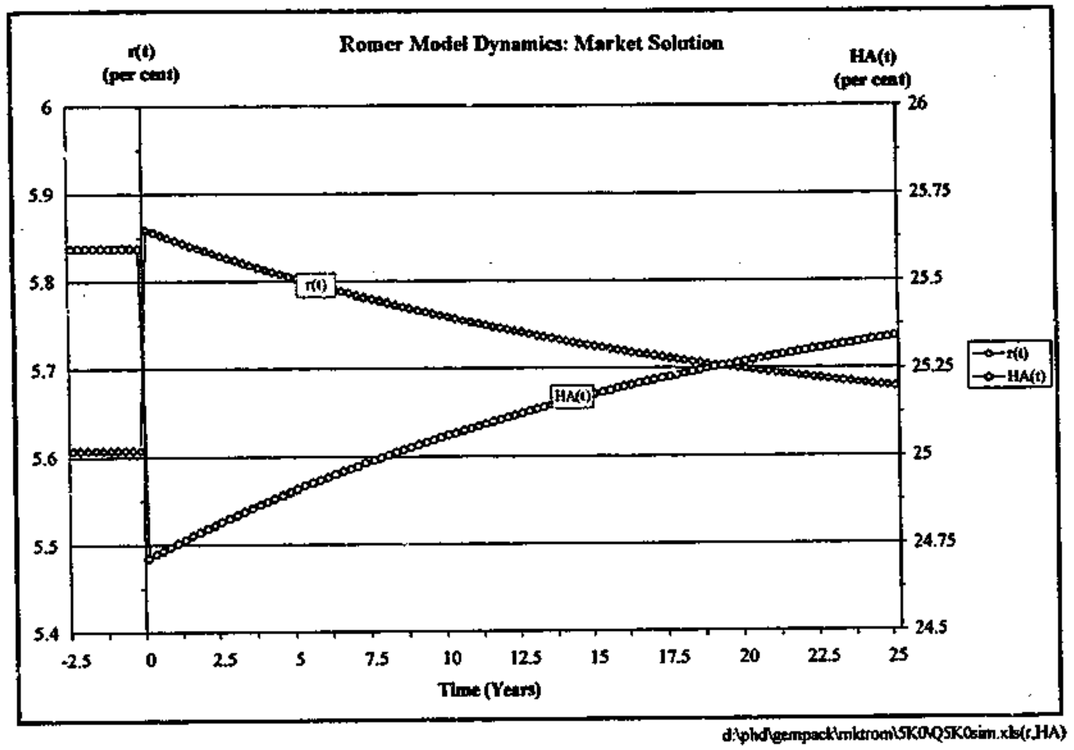

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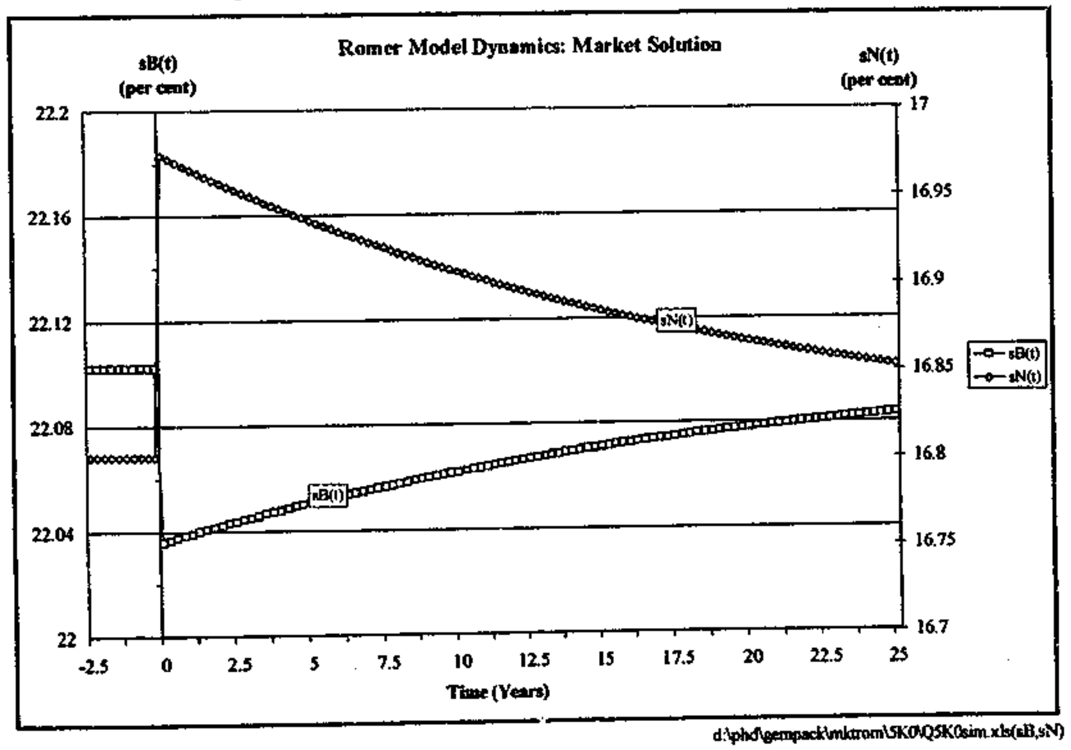
Figure 4.36: **Dynamic effects on the broad and narrow savings rates (s_B and s_N, respectively) of an unanticipated and temporary 5% fall in the initial capital stock (Ψ), benchmark parameter set.**



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This simulation suggests that while a sudden reduction in the capital stock of an economy may not permanently alter its steady-state equilibrium, such an event can be extremely traumatic, its effects remaining significant for decades. After such a shock the economic system appears likely to converge back towards its pre-shock equilibrium at a rate of about 4.9 per cent per annum. This means that half of the initial disturbance would remain after about 15 years, and a quarter would still be felt after some three decades ((Figure 4.37 and Table 4.5).

Table 4.5: Simulation results of an unanticipated and temporary 5 per cent fall in the initial capital stock (Ψ), benchmark parameter set.

Dynamic variable	Final steady-state	Total adjustment (% of initial ss)	Initial jumps as a % of: initial ss	Initial jumps as a % of: total adjustment	½ life[a] (years)	¾ life[a] (years)
$\Psi(t)$	6.48	0.00	-5.00	n.a.	15	29
$\Phi(t)$	0.2741	0.00	2.41	n.a.	14	28
$p_A(t)$	9.45	0.00	-3.65	n.a.	15	29
$r(t)$	5.61%	0.00	4.50	n.a.	15	28
$H_A(t)$	25.60%	0.00	-3.45	n.a.	14	28
$g_{GP}(t)$	1.54%	0.00	5.76	n.a.	14	28
$s_B(t)$	22.10%	0.00	-0.30	n.a.	14	28
$s_N(t)$	16.80%	0.00	1.02	n.a.	15	29
$k_{GP}(t)$	2.84	0.00	-2.27	n.a.	15	29
$\beta_\Psi(t)$	4.93%	0.00	-1.24	n.a.	14	28

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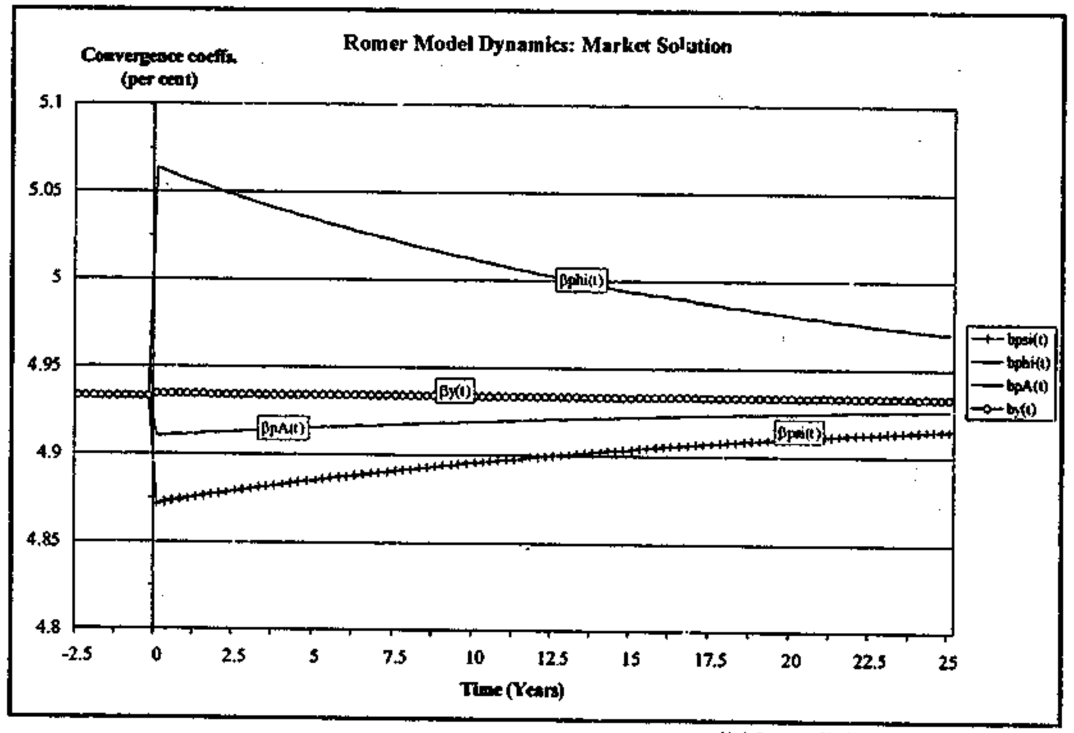
Note: a The time taken for ½ and ¾ of the adjustment after the post-announcement jumps.

Figure 4.37: Dynamic effects on the convergence coefficients from an unanticipated and temporary 5% fall in the initial capital stock (Ψ), benchmark parameter set.



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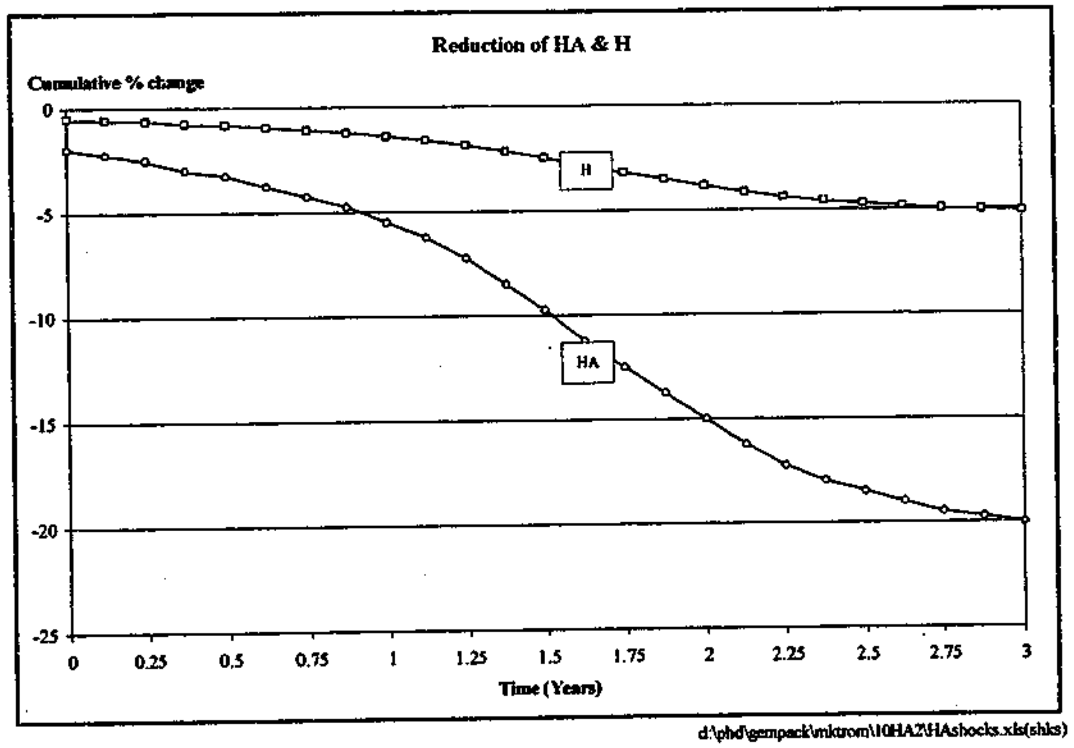
4.5.7 A gradual loss of human capital from research

Part of the received wisdom about the state of innovation and technological change in Australia is that there is a persistent and significant (net) loss of human capital, smart researchers being continually enticed overseas with higher remuneration packages and better research opportunities. The term *brain drain* is frequently trotted out by the press and elsewhere, often with some anecdotal 'support' though rarely, if ever, with any formal analytical evidence. Nevertheless, to obtain an idea of the broad economic implications of such a phenomenon it is assumed here that some sort of *brain drain* episode occurs. Or perhaps alternatively, that as a result of some rampant and diabolically mutating computer virus many researchers are infected and die! Whatever the reason, it is supposed that the level of human capital in the research sector, H_A, suddenly begins to decline, falling gradually so that after three years it is 20 per cent below its initial (benchmark steady-state) level. At that time the rate of decline has gradually been reduced to zero. The total supply of human capital, H, falls concomitantly to a cumulative level some 5.12 per cent below benchmark. The cumulative shocks specified for each of the (time) elements of both H_A and H are illustrated in Figure 4.38.

Figure 4.38: Shocks imposed on variables H_A and H in order to simulate a gradual loss of human capital in research, cumulative percentage change to elements 1 to 25 of the variables.



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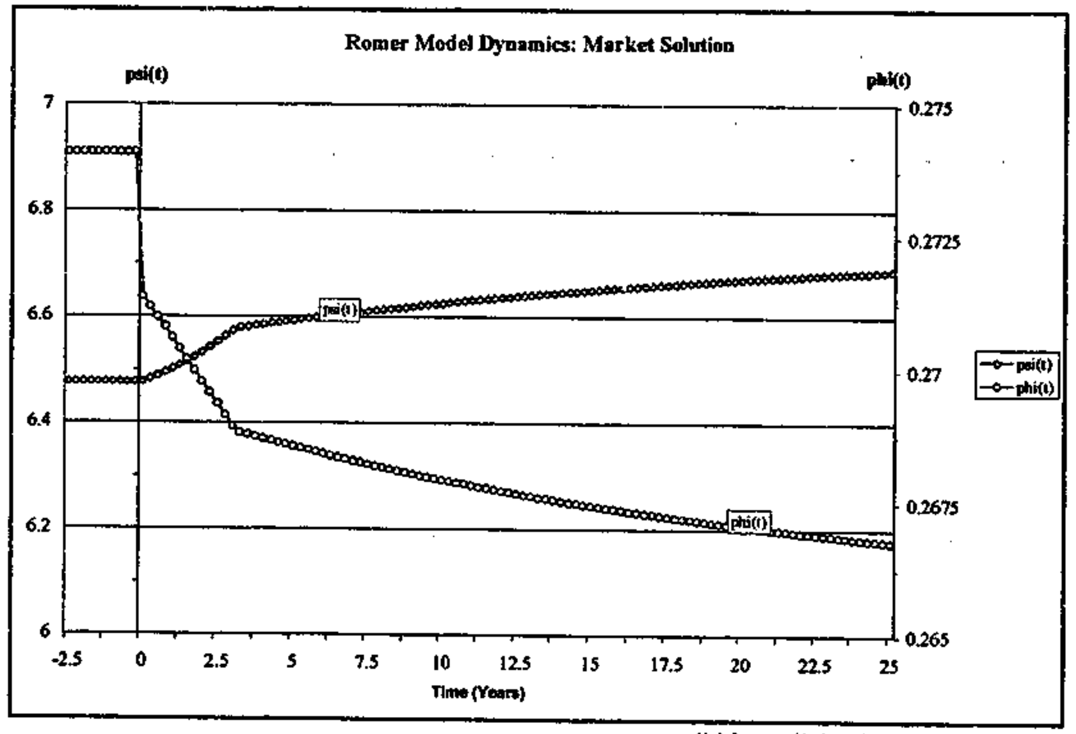
Because the allocation of overall human capital to research is usually determined *endogenously* (that is within the model), in order to impose the specified shocks the (25) particular elements of H_A were redefined as *exogenous* variables. Then, to ensure that the model continued to be soluble, the equality of the number of endogenous variables and the number of equations had to be maintained by switching 25 previously exogenous variables to endogenous ones. The 25 corresponding elements of the research productivity parameter ζ were selected for this purpose.[45] Results of the simulation are presented in Figure 4.39 to Figure 4.45 and in Table 4.6.

As a whole, the set of shocks is unexpected. However, it is assumed that once the episode begins, its progress is accurately foreseen (aka rational expectations). Mathematically, and in terms of the modelling assumptions this means that while the initial shock is unanticipated, the remainder are anticipated.

The direct effects of the exogenous reductions in researchers are shown in Figure 4.42 as the results over the first three years for H_A. (They're also directly reflected in the rate of growth of technology g_A, shown in Figure 4.41). But these reductions to researchers are not sustained. When they have ceased and the decline of H_A has been reined in, it is well below its optimum allocation. As a result, H_A first rises precipitously and later very gradually, as does the growth rate of technology.[46]

Figure 4.39: Dynamic effects on Ψ and Φ from a gradual loss of 20% of researchers (H_A), benchmark parameter set.



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Devoting human capital resources to research instead of to production is a form of savings in that some current consumption is traded off against the greater future consumption made possible by new technology. Because the exogenous decline in H_A limits the output of research, it reduces the stream of future consumption benefits from technology. To counterbalance this, more savings in the form of capital formation are necessary. That is, relatively more resources must be devoted to investment than to

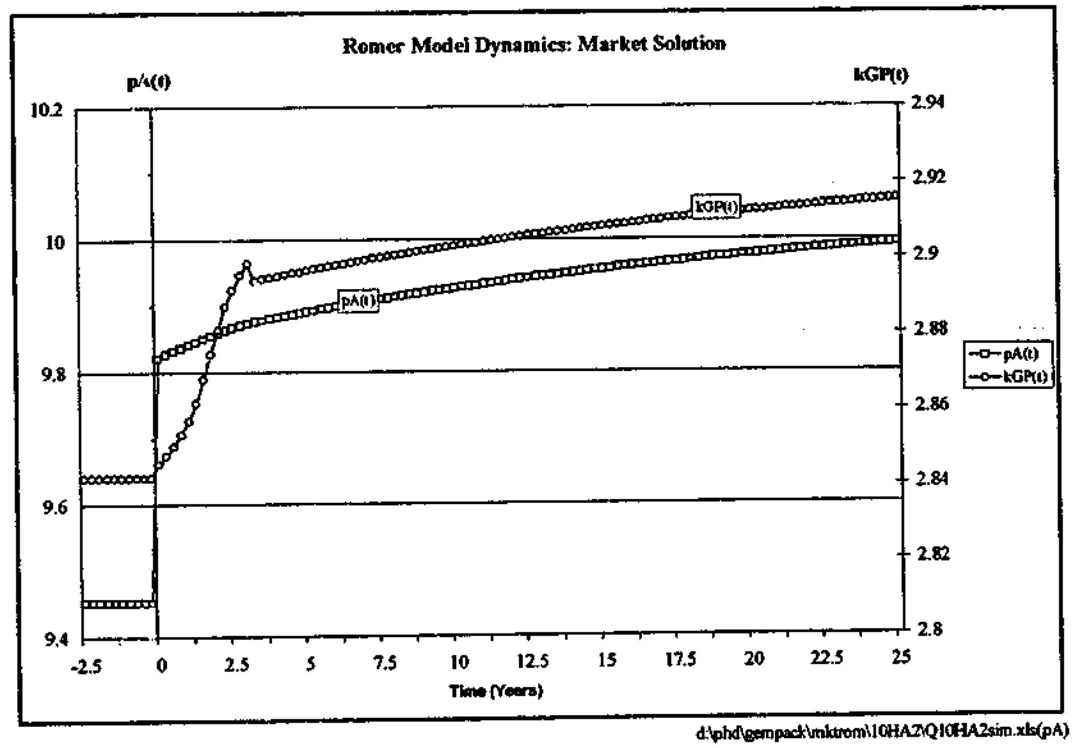
[45] The changes in these necessary to accommodate the exogenous shocks to H_A and H were then determined (within the model) to vary between -3.75 and -3.48 per cent of the benchmark level of 0.06.

[46] In terms of the modelling, at the cessation of the shocks the endogeneity of H_A and ζ are swapped and the model is suddenly free to choose the allocation of aggregate human capital to research.

consumption. The combination of the restriction of research output and this need for greater capital savings explains the directions of the initial jumps (and the subsequent adjustments while H_A continues to decline) for many of the model's economic variables. In particular:

- together they mean that the capital-technology ratio Ψ must rise (Figure 4.39);
- restricted research output creates an excess demand for designs which is only eliminated by a rise in their price p_A (Figure 4.40);
- increased capital savings means that the investment-output ratio s_N (which is the narrow definition of savings) and the rate of growth of capital g_K, must both rise (Figure 4.43 and Figure 4.41); and that the consumption-capital ratio Φ must fall (Figure 4.39);
- the greater supply of capital lowers its price as reflected by the interest rate r (Figure 4.42); and
- since it is only greater capital growth that can accelerate gross product here, the capital-gross product ratio k_{GP} must rise (Figure 4.40).

Figure 4.40: Dynamic effects on p_A and k_{GP} from a gradual loss of 20% of researchers (H_A), benchmark parameter set.



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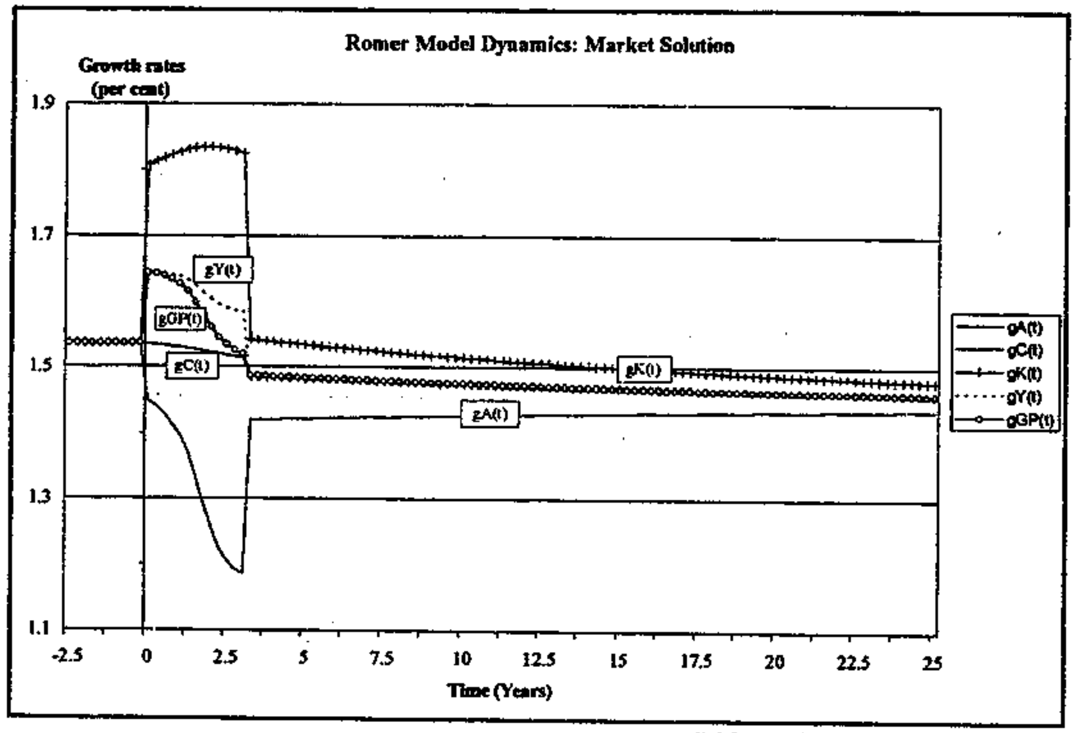
As usual, the responses of other variables are ambiguous in terms of *a priori* theorising, but are resolved by the (quantitative) model. Thus:

- the rate of accumulation of capital is initially fast enough to more than counterbalance the slower growth of designs, and so raises the rate of growth of output g_Y; but as the shocks to H_A cumulate, output growth slows. Also, though the price of designs rises, their growth rate falls sufficiently to keep the rate of growth of gross product g_{GP} below that of output (Figure 4.41);

- despite the fall of savings in the form of research, capital accumulation is sufficiently strong to cause the broad savings rate s_B to rise initially, though it then declines as the exogenous shocks build (Figure 4.43); and
- the rate of growth of consumption falls slightly, output growth not being sufficient to overcome the increased rate of savings (Figure 4.41).

Figure 4.41: Dynamic effects on the growth rates from a gradual loss of 20% of researchers (H_A), benchmark parameter set.



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Once the exogenous falls in researchers are over, more savings in the form of research are possible, and these can replace some of the capital formation savings. This explains (*inter alia*) the jumps in g_K, s_N, and k_{GP} immediately after the third year; and also the changes in the *rates of change* of Ψ, Φ, and k_{GP}. But while the exogenous reductions to H_A are only temporary, the corresponding falls in aggregate human capital H are sustained. Then, since the steady-state level of H_A changes more than proportionally with changes in H (see footnote 40), the eventual equilibrium level of human capital in research is a lower proportion of the diminished aggregate level of human capital (Figure 4.42). This continues to depress the supply of research output slightly, with the result that its price p_A rises slowly. The stream of future benefits from research are therefore permanently lower, so the savings required from capital formation will be greater in the post-shock equilibrium than in the pre-shock one. Thus, the final steady-state outcomes for Ψ, s_N, s_B and k_{GP}, must be greater than their initial counterparts; and for Φ and g (the common growth rate), and consequently for r, they must be less.

Figure 4.42: Dynamic effects on r and H_A from a gradual loss of 20% of researchers (H_A), benchmark parameter set.

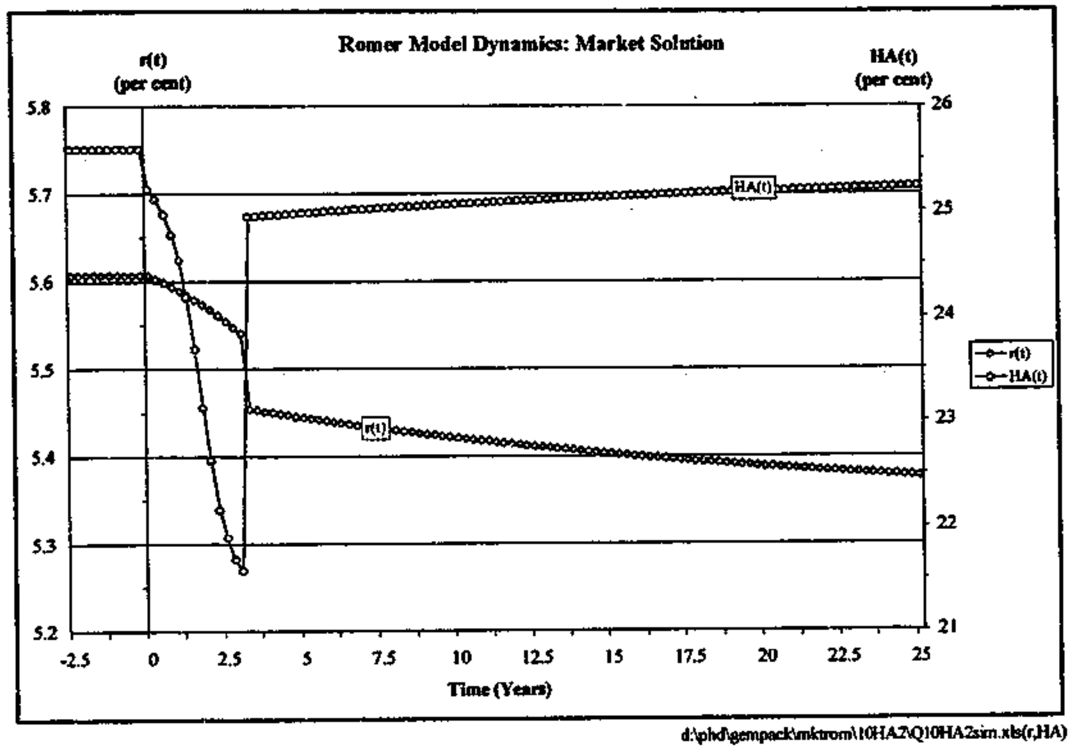

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Figure 4.43: Dynamic effects on the broad savings rate and investment share of output (s_B and s_N respectively) from a gradual loss of 20% of researchers (H_A), benchmark parameter set.

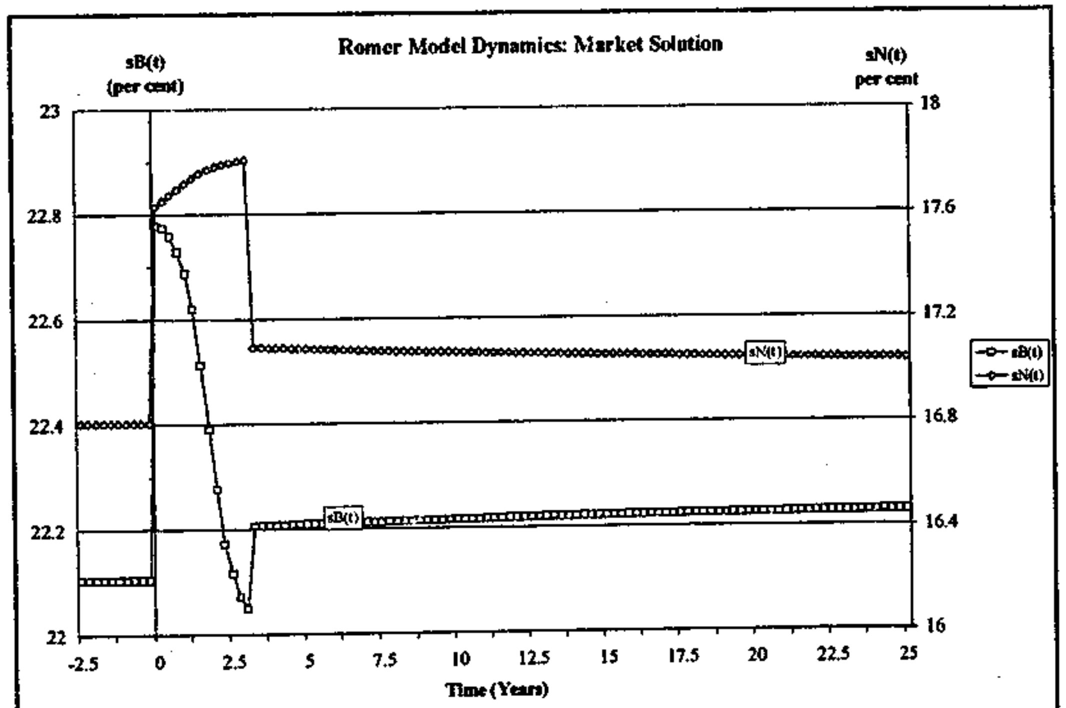

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Figure 4.44: **Dynamic effects on the factor shares of gross income from a gradual loss of 20% of researchers (H_A), benchmark parameter set.**

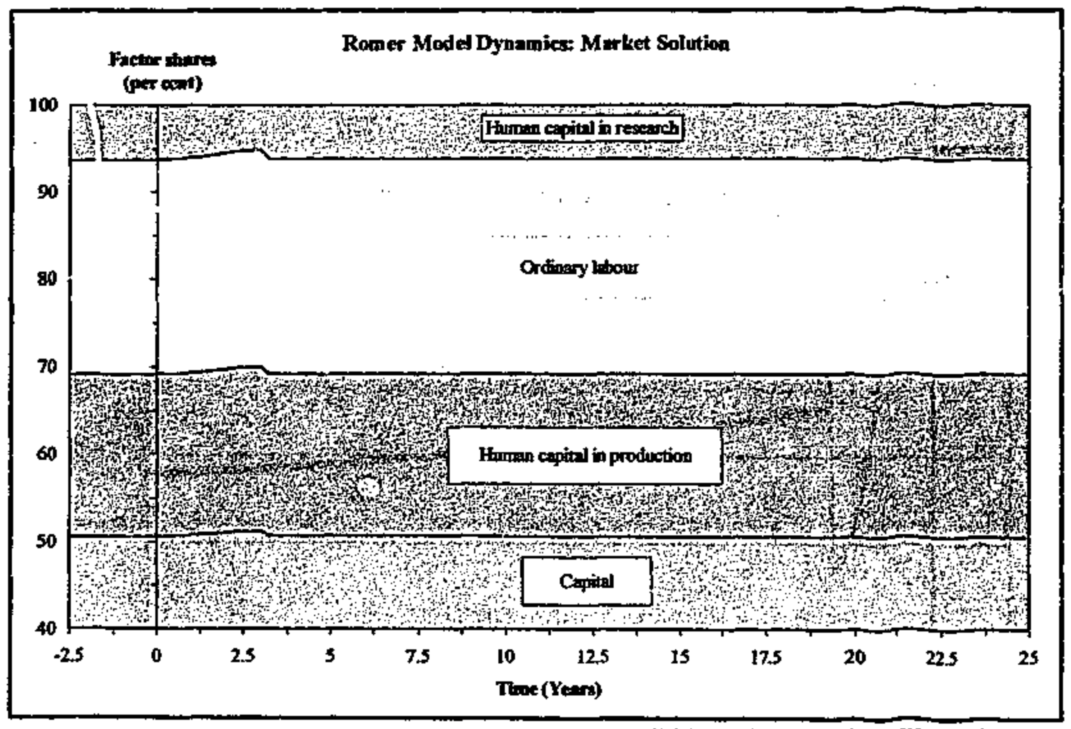

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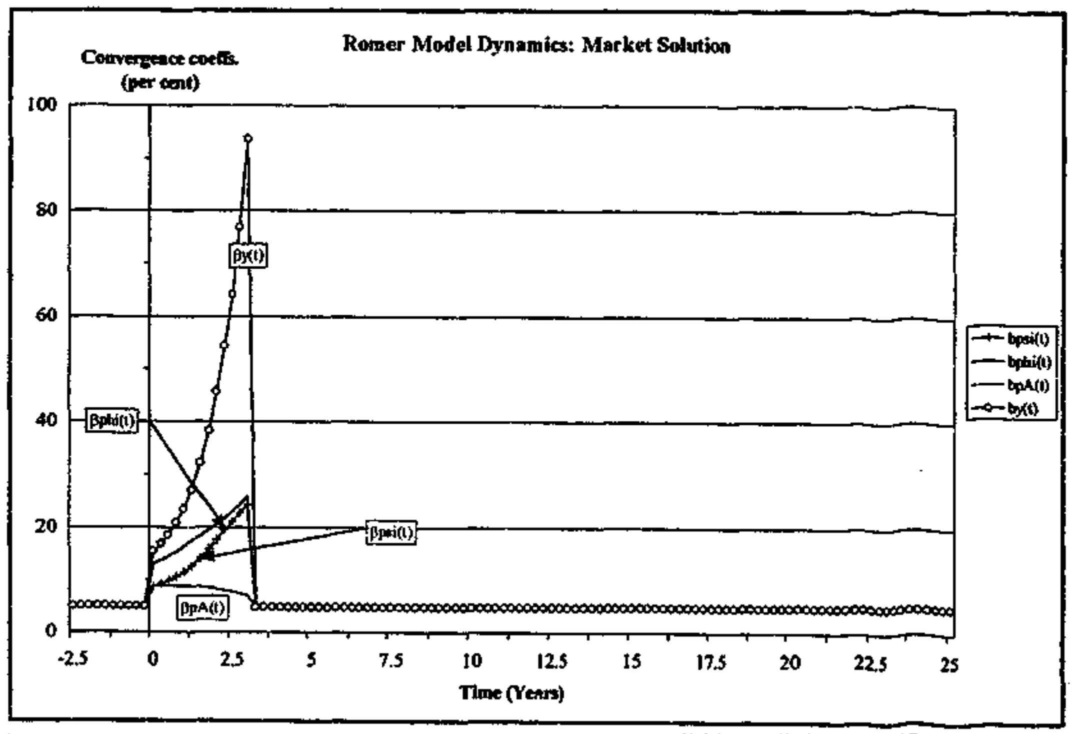
Figure 4.45: **Dynamic effects on the convergence coefficients from a gradual loss of 20% of researchers (H_A), benchmark parameter set.**



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Like all the previously analysed simulations, adjustment of the 'Romer economy' to these exogenous reductions in researchers is lengthy. After initial accelerations during the period of the shocks, the rates of convergence to the new equilibrium fall rapidly to a level slightly below their pre-shock values (Figure 4.45). This results in an adjustment period of 16 years being required to close half of the gap remaining between the position after the post-shock jumps and equilibrium; and three decades being necessary to close three quarters of that gap (Table 4.6).

Table 4.6: **Simulation results of a gradual loss of 20 per cent of researchers (H_A), benchmark parameter set.**

Dynamic variable	Final steady-state	Total adjustment (% of initial ss)	Initial jumps as a % of: initial ss	Initial jumps as a % of: total adjustment	½ life[a] (years)	¾ life[a] (years)
$\Psi(t)$	6.75	4.19	0.00	0.00	16(8)	30(23)
$\Phi(t)$	0.2657	-3.07	-0.99	32.3	16(6)	30(20)
$p_A(t)$	10.06	6.40	3.89	60.8	16(13)	30(27)
$r(t)$	5.33%	-4.85	0.00	0.00	16(4)	30(15)
$H_A(t)$	24.08%	-5.90	-2.00	33.9	16(1)	30(1)
$g_{GP}(t)$	1.44%	-5.90	7.02	-119	16(0,3)	30(0,4)
$s_B(t)$	22.24%	0.63	3.05	488	16(0,2)	30(0,2)
$s_N(t)$	17.01%	1.23	491	400	16(0,4)	30(0,4)
$k_{GP}(t)$	2.93	2.98	0.13	4.27	16(3)	30(14)
$\beta_\Psi(t)$	4.80%	-2.76	73.8	-2679	16(0,4)	31(0,4)

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Notes: a The first set of results give the time taken for ½ and ¾ of the remaining adjustment **after** implementation is completed at t=3+ years; while the figures in parentheses give corresponding results after the initial jumps at t=0.

Appendix 4.1

Numerical integration methods for initial value problems

The Euler method of numerically integrating the system of coupled ODEs:

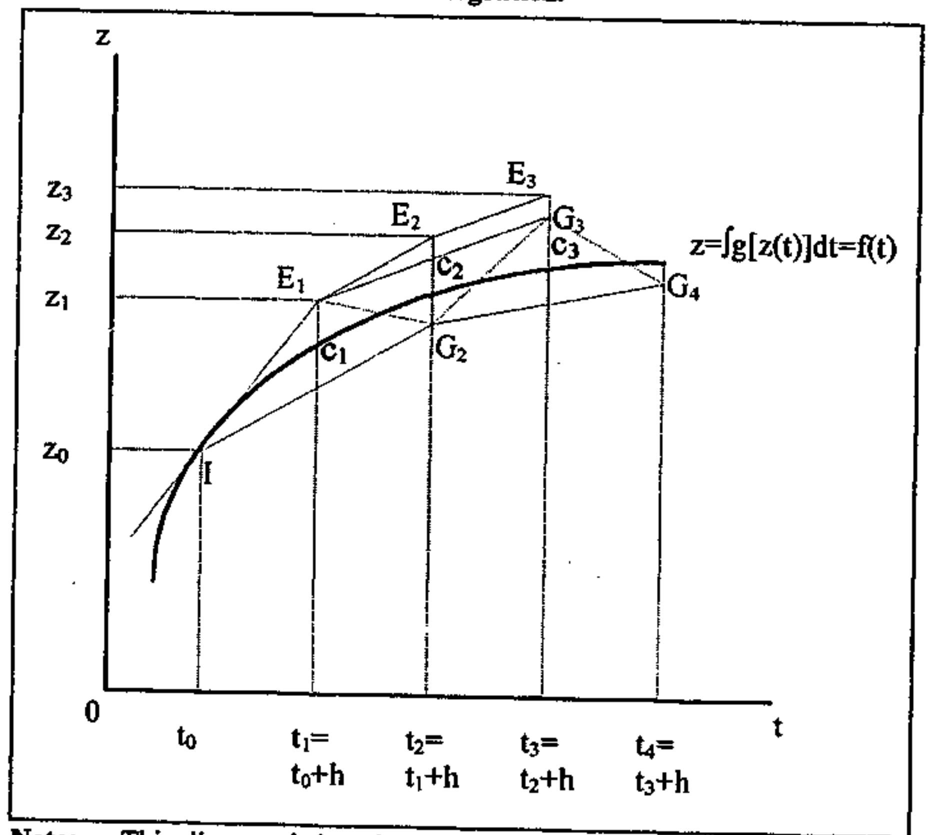
$$\dot{z}_i(t) = \frac{dz_i(t)}{dt} = g_i(z_1(t),\ldots,z_n(t),t) \qquad \text{for } i = 1,\ldots,n$$

or, in vector notation: (A4.1.1)

$$\dot{\mathbf{z}}(t) = \mathbf{g}(\mathbf{z}(t),t)$$

was described algebraically in equation (4.3). Here it is described graphically as an approximation to the actual curve for a single variable $z = f(t)$ (Figure A4.1.1). The Euler iterations step along the path I, E_1, E_2, E_3, ...*et cetera*, with each chord segment being parallel to the tangent to the curve at the independent variable coordinate corresponding to the *current point*. Thus, IE_1 is tangential at I; E_1E_2 is parallel to the tangent at c_1; E_2E_3 is parallel to the tangent at c_2 *et cetera*.

Figure A4.1.1: Illustration of the Euler and Gragg (or 'modified mid-point') methods of numerical integration.[a]



Note: a This diagram is based upon those appearing in the Gempack User Documentation (Harrison and Pearson, 1996a and 1994).

While the Euler method was based upon a Taylor series expansion about a single point, the Gragg, or modified mid-point numerical integration procedure is based upon Taylor series expansions at two points as follows:

$$z(t+h) = z(t) + h\dot{z}(t) + \frac{h^2}{2!}\ddot{z}(t) + \frac{h^3}{3!}\dddot{z}(t) + \ldots$$
$$z(t-h) = z(t) - h\dot{z}(t) + \frac{h^2}{2!}\ddot{z}(t) - \frac{h^3}{3!}\dddot{z}(t) + \ldots$$

Whereupon subtraction and the omission of all 3rd and higher order terms produces:

$$z(t+h) - z(t-h) = 2h\dot{z}(t) \qquad \text{(A4.1.2)}$$

Since it is now only 3rd and higher order terms which are dropped to form the difference equation approximation, the Gragg method of numerical integration is accurate to *order-h^2*. The integration involves first taking an Euler-step, subsequently iterating according to (A4.1.2), and finally calculating a correction, based upon the two immediately preceding uncorrected results, for the last result.[47] Thus, the Gragg integration of (A4.1.1) proceeds as follows:

$$\begin{aligned}
&t_{k+1} = t_k + h_k \quad \text{for } k = 0,\ldots,(N-1) \\
&\mathbf{y}_1 = \mathbf{z}_0 + h_0\mathbf{g}(\mathbf{z}_0,t_0) \\
&\mathbf{y}_{k+1} = \mathbf{y}_{k-1} + 2h_k\mathbf{g}(\mathbf{y}_k,t_k) \quad \text{for } k = 1,\ldots,(N-1) \qquad \text{(A4.1.3)} \\
&\mathbf{z}_1 = \mathbf{y}_1 \\
&\mathbf{z}_k = \tfrac{1}{2}[\mathbf{y}_k + \mathbf{y}_{k-1} + h_k\mathbf{g}(\mathbf{y}_k,t_k)] \quad \text{for } k = 2,\ldots,N
\end{aligned}$$

where the **y** terms are intermediate approximations.

As shown in Figure A4.1.1, the uncorrected Gragg results march along the path I, E_1, G_2, G_3, ...*et cetera* in a 'saw-tooth' manner. Like the Euler method, each chord segment is parallel to the tangent to the curve at the independent variable coordinate corresponding to the *current point*, but unlike the Euler case, the step along each chord commences from the *previous point*. Thus, IE_1 is tangential at I; IG_2 is parallel to the tangent at c_1; E_1G_3 is parallel to the tangent at c_2 *et cetera*. The corrected Gragg approximation to the actual curve $z = f(t)$ smooths out the saw-teeth of these raw, intermediate results. In Figure A4.1.1 the correction would produce the results: z_0=I; z_1=E1; z_2= a point between c_2 on the curve and the line E_1G_3 above it; z_3= a point between c_3 on the curve and G_3 above it; etc.

The fourth-order Runge-Kutta method involves four evaluations of the derivative, $\mathbf{g}(\mathbf{z}(t),t)$ in (A4.1.1), at each step: one at the current point, two at trial mid-points, and a final one at a trial end-point. The final end-point is obtained by combining all these as follows:

[47] The Gragg approach is most commonly used to calculate only an end-point rather than an entire path. Then the correction only needs to be performed once. Nevertheless 'Gragg can be used as an ODE integrator in its own right' (Press et al., 1992), correcting each point in turn. It is these corrections, as well as the first Euler step, which distinguish the Gragg approach from the unmodified mid-point method.

$$
\begin{aligned}
r_{1k} &= h_k g(z_k, t_k) \\
r_{2k} &= h_k g(z_k + r_{1k}/2, t_k + h_k/2) \\
r_{3k} &= h_k g(z_k + r_{2k}/2, t_k + h_k/2) \\
r_{4k} &= h_k g(z_k + r_{3k}, t_k + h_k) \\
z_{k+1} &= z_k + \frac{r_{1k}}{6} + \frac{r_{2k}}{3} + \frac{r_{3k}}{3} + \frac{r_{4k}}{6}
\end{aligned} \tag{A4.1.4}
$$

Fourth-order Runge-Kutta has an error of only *order-h^5*. See Birkhoff and Rota (1969) for a proof.

In addition to the use of difference equation approximations accurate to higher orders of the step size, the accuracy of numerical integration can obviously be improved by making the step sizes themselves smaller. This can be particularly important throughout regions of 'high curvature'; that is, where the derivatives are changing rapidly.[48] Uniform step sizes are often used in numerical integration, but judicious selection of non-uniform sizes, even with fewer overall steps can significantly improve accuracy.

Another, powerful means of increasing the accuracy of these numerical integration methods is through the use of so-called *extrapolation* techniques. The name arises from the idea that in integrating to some end point, z(t+H), the estimated result is itself some function of the step-size (h), say, $\hat{z}_{t+H} = \hat{z}(h)$; and that if sufficient information about this function can be discovered (from investigating estimates based on different step-sizes), then it may be possible to extrapolate the results from finite step-sizes to the limit, h=0! This is known as *Richardson's deferred approach to the limit* or simply *Richardson's extrapolation.* In practice it provides a means of combining end point estimates calculated from different step-sizes to produce an (extrapolated) estimate of higher order accuracy. Since any point on an integrated path can be considered an *end point*, it is possible to extrapolate for an entire path.

Pearson (1991) provides the formulae for the Richardson extrapolations based on step-sizes h and ½h; and also on h, ½h, and ¼h for both the Euler and Gragg methods of numerical integration. The 3-Euler solution extrapolation, which is used later in the paper, is given by:

$$
z_k^{3ER}(h) = \frac{1}{3}[8z_k^E(\tfrac{h}{4}) - 6z_k^E(\tfrac{h}{2}) + z_k^E(h)] \tag{A4.1.5}
$$

Pearson also shows that the accuracy of these extrapolations are *order-h^2* and *order-h^3* for the 2-Euler and 3-Euler combinations respectively, and *order-h^3* and *order-h^6* for the Gragg combinations.

[48] In such regions the second derivatives are (relatively) high and so difference equation approximations for which the second order Taylor series terms are dropped become relatively less accurate.

Appendix 4.2

Computing the saddle-path of the Romer model as the solution to an 'initial value' problem

A4.2.1 Time elimination method

Following the procedure suggested by Mulligan and Sala-i-Martin (1991), the 'independent variable', time (t), is eliminated from the stationary dynamic Romer system given by equations (2.41) to (2.45) by taking ratios of the differential equations. In this way the system is expressed in terms of the original 'dependent variables' only with one of these, chosen arbitrarily, as a new independent variable. In particular, choosing Φ as the new independent variable, the ratios $\dot{\Psi}(t)/\dot{\Phi}(t)$ and $\dot{\Psi}(t)/\dot{p}_A(t)$ are formed from (2.41) to (2.43) to generate the new (non-autonomous) pair of ODEs:

$$
\frac{\dot{\Psi}(t)}{\dot{\Phi}(t)} = \frac{d\Psi(t)/dt}{d\Phi(t)/dt} = \frac{d\Psi(\Phi)}{d\Phi} = \frac{[(r+\delta)/\gamma^2 - \Phi + \zeta H_Y - (\delta + \zeta H)]\Psi}{[(r-\rho)/\sigma - (r+\delta)/\gamma^2 + \Phi + \delta]\Phi} \tag{A4.2.1}
$$

$$
\frac{\dot{p}_A(t)}{\dot{\Phi}(t)} = \frac{dp_A(t)/dt}{d\Phi(t)/dt} = \frac{dp_A(\Phi)}{d\Phi} = \frac{rp_A - (1-\gamma)(r+\delta)\Psi/\gamma}{[(r-\rho)/\sigma - (r+\delta)/\gamma^2 + \Phi + \delta]\Phi} \tag{A4.2.2}
$$

whereupon substitution of equations (2.44) and (2.45) for H_Y and r allows the system to be expressed in terms of Ψ, Φ, and p_A only:

$$
\frac{d\Psi}{d\Phi} = \frac{\{\zeta[\frac{\alpha(1-\gamma)}{\zeta\eta^\gamma} L^{(1-\alpha)(1-\gamma)} \Psi^\gamma p_A^{-1}]^{\frac{1}{1-\alpha(1-\gamma)}} [1 + \frac{p_A \Psi^{-1}}{\alpha(1-\gamma)}] - \Phi - \delta - \zeta H\}\Psi}{\{(\frac{\gamma^2}{\sigma} - 1)\frac{\zeta}{\alpha(1-\gamma)}[\frac{\alpha(1-\gamma)}{\zeta\eta^\gamma} L^{(1-\alpha)(1-\gamma)} \Psi^\gamma p_A^{-1}]^{\frac{1}{1-\alpha(1-\gamma)}} p_A \Psi^{-1} + \Phi + \frac{\delta(\sigma-1)-\rho}{\sigma}\}\Phi}
$$

$$
\frac{dp_A}{d\Phi} = \frac{\{\frac{\zeta\gamma^2}{\alpha(1-\gamma)}[\frac{\alpha(1-\gamma)}{\zeta\eta^\gamma} L^{(1-\alpha)(1-\gamma)} \Psi^\gamma p_A^{-1}]^{\frac{1}{1-\alpha(1-\gamma)}} [p_A \Psi^{-1} - \frac{1-\gamma}{\gamma}] - \delta\} p_A}{\{(\frac{\gamma^2}{\sigma} - 1)\frac{\zeta}{\alpha(1-\gamma)}[\frac{\alpha(1-\gamma)}{\zeta\eta^\gamma} L^{(1-\alpha)(1-\gamma)} \Psi^\gamma p_A^{-1}]^{\frac{1}{1-\alpha(1-\gamma)}} p_A \Psi^{-1} + \Phi + \frac{\delta(\sigma-1)-\rho}{\sigma}\}\Phi} \tag{A4.2.3}
$$

In order to integrate this system as an initial value problem the two equations must be evaluated at the only point known to lie on their paths, namely at the steady-state. However, since both the numerators and denominators in the differential equations are zero at the steady-state ($\dot{\Psi}_{ss} = \dot{\Phi}_{ss} = \dot{p}_{Ass} = 0$), substitution of the steady-state values produces indeterminancies (0÷0). This difficulty is overcome in the usual way by invoking L'Hopital's rule. Working with (A4.2.1) and (A4.2.2) and denoting the new derivatives as dΨ(Φ)/dt=Ψ′(Φ) and $dp_A(\Phi)/d\Phi = p_A'(\Phi)$, this produces:

$$\Psi'_{ss} = \lim_{\Phi \to \Phi_{ss}} \Psi'(\Phi) = \lim_{\Phi \to \Phi_{ss}} \frac{\frac{d}{d\Phi}[\frac{r+\delta}{\gamma^2} - \Phi + \zeta H_Y - \delta - \zeta H)]\Psi}{\frac{d}{d\Phi}[\frac{r-\rho}{\sigma} - \frac{r+\delta}{\gamma^2} + \Phi + \delta]\Phi}$$

$$= \lim_{\Phi \to \Phi_{ss}} \frac{\Psi'(\Phi)[\frac{r+\delta}{\gamma^2} - \Phi + \zeta H_Y - \delta - \zeta H] + \Psi[\frac{1}{\gamma^2}\frac{dr}{d\Phi} - 1 + \zeta \frac{dH_Y}{d\Phi}]}{[\frac{r-\rho}{\sigma} - \frac{r+\delta}{\gamma^2} + \Phi + \delta] + \Phi[(\frac{1}{\sigma} - \frac{1}{\gamma^2})\frac{dr}{d\Phi} + 1]}$$

ie.

$$\Psi'_{ss} = \frac{\Psi_{ss}[r'_{ss}/\gamma^2 - 1 + \zeta H'_{Yss}]}{\Phi_{ss}[(\frac{1}{\sigma} - \frac{1}{\gamma^2})r'_{ss} + 1]} \tag{A4.2.4}$$

and

$$p'_{A_{ss}} = \lim_{\Phi \to \Phi_{ss}} p'_A(\Phi) = \lim_{\Phi \to \Phi_{ss}} \frac{\frac{d}{d\Phi}[rp_A - (1-\gamma)(r+\delta)\Psi/\gamma]}{\frac{d}{d\Phi}[(r-\rho)/\sigma - (r+\delta)/\gamma^2 + \Phi + \delta]\Phi}$$

$$= \lim_{\Phi \to \Phi_{ss}} \frac{rp'_A(\Phi) + p_A\frac{dr}{d\Phi} - \frac{(1-\gamma)(r+\delta)}{\gamma}\Psi'(\Phi) - \frac{1-\gamma}{\gamma}\Psi\frac{dr}{d\Phi}]}{[\frac{r-\rho}{\sigma} - \frac{r+\delta}{\gamma^2} + \Phi + \delta] + \Phi[(\frac{1}{\sigma} - \frac{1}{\gamma^2})\frac{dr}{d\Phi} + 1]}$$

so

$$p'_{A_{ss}} = \frac{r_{ss}p'_{Ass} + p_{Ass}r'_{ss} - \frac{1-\gamma}{\gamma}[(r_{ss}+\delta)\Psi'_{ss} + \Psi_{ss}r'_{ss}]}{\Phi_{ss}[(\frac{1}{\sigma} - \frac{1}{\gamma^2})r'_{ss} + 1]} \tag{A4.2.5}$$

Then, to evaluate (A4.2.4) and (A4.2.5) it is first necessary to obtain expressions for $r'_{ss} = (\frac{dr}{d\Phi})_{ss}$ and $H'_{Yss} = (\frac{dH_Y}{d\Phi})_{ss}$ in terms of Ψ'_{ss} and p'_{Ass}. Accordingly, from (2.44):

$$H'_{Yss} = \frac{[H_{Yss}^{1-\alpha(1-\gamma)}]^{\frac{1}{1-\alpha(1-\gamma)}-1}}{1-\alpha(1-\gamma)} \frac{\alpha(1-\gamma)}{\zeta\eta^\gamma} L^{(1-\alpha)(1-\gamma)}[\gamma\Psi_{ss}^{\gamma-1}\Psi'_{ss}p_{Ass}^{-1} - \Psi_{ss}^{\gamma}p_{Ass}^{-2}p'_{Ass}]$$

ie.

$$H'_{Yss} = \frac{H_{Yss}}{1-\alpha(1-\gamma)}[\gamma\frac{\Psi'_{ss}}{\Psi_{ss}} - \frac{p'_{Ass}}{p_{Ass}}] \tag{A4.2.6}$$

Which, together with (2.45) produces:

$$r'_{ss} = \frac{\zeta\gamma^2}{\alpha(1-\gamma)}[H'_{Yss}p_{Ass}\Psi_{ss}^{-1} + H_{Yss}(p'_{Ass}\Psi_{ss}^{-1} - \Psi_{ss}^{-2}\Psi'_{ss}p_{Ass})]$$

$$= \frac{\zeta\gamma^2 H_{Yss}}{\alpha(1-\gamma)}[\frac{1}{1-\alpha(1-\gamma)}(\gamma\frac{\Psi'_{ss}}{\Psi_{ss}} - \frac{p'_{Ass}}{p_{Ass}}) + p'_{Ass}\Psi_{ss}^{-1} - \Psi_{ss}^{-2}\Psi'_{ss}p_{Ass}]$$

$$= \frac{\zeta\gamma^2 H_{Yss}}{\alpha(1-\gamma)}[(\frac{\gamma}{1-\alpha(1-\gamma)} - 1)\Psi_{ss}^{-2}\Psi'_{ss}p_{Ass} + (1 - \frac{1}{1-\alpha(1-\gamma)})p'_{Ass}\Psi_{ss}^{-1}]$$

$$= -\frac{\zeta\gamma^2 H_{Yss}p_{Ass}\Psi_{ss}^{-1}}{\alpha(1-\gamma)[1-\alpha(1-\gamma)]}[(1-\alpha)(1-\gamma)\frac{\Psi'_{ss}}{\Psi_{ss}} + \alpha(1-\gamma)\frac{p'_{Ass}}{p_{Ass}}]$$

ie.

$$r'_{ss} = -\frac{(1-\gamma)(r_{ss}+\delta)}{1-\alpha(1-\gamma)}[(1-\alpha)\frac{\Psi'_{ss}}{\Psi_{ss}} + \alpha\frac{p'_{Ass}}{p_{Ass}}] \tag{A4.2.7}$$

Expressions for Ψ'_{ss} and p'_{Ass} may now be obtained from (A4.2.4) and (A4.2.5) by substituting (A4.2.6) and (A4.2.7). The algebra is messy. It results in a pair of polynomials in Ψ'_{ss} and p'_{Ass}, each including cross product terms in $\Psi'_{ss}p'_{Ass}$, expressed in terms of the known parameter and steady-state values. Specifically:

$$a_1(\Psi'_{ss})^2 + a_2(\Psi'_{ss}p'_{Ass}) + a_3(\Psi'_{ss}) + a_4(p'_{Ass}) + a_5 = 0 \tag{A4.2.8}$$

and

$$a_2(\Psi'_{ss})^2 + a_1(\Psi'_{ss}p'_{Ass}) + a_6(\Psi'_{ss}) + a_7(p'_{Ass}) = 0 \tag{A4.2.9}$$

where

$$\begin{aligned}
a_1 &= (\frac{1}{\gamma^2} - \frac{1}{\sigma})\frac{(1-\alpha)(1-\gamma)\Phi_{ss}(r_{ss}+\delta)}{\Psi_{ss}} \\
a_2 &= (\frac{1}{\gamma^2} - \frac{1}{\sigma})\frac{\alpha(1-\gamma)\Phi_{ss}(r_{ss}+\delta)}{p_{Ass}} \\
a_3 &= \Phi_{ss}[1-\alpha(1-\gamma)] - \zeta\gamma H_{Yss} + \frac{(1-\alpha)(1-\gamma)(r_{ss}+\delta)}{\gamma^2} \\
a_4 &= [\zeta H_{Yss} + \frac{\alpha(1-\gamma)(r_{ss}+\delta)}{\gamma^2}]\frac{\Psi_{ss}}{p_{Ass}} \\
a_5 &= [1-\alpha(1-\gamma)]\Psi_{ss} \\
a_6 &= \frac{1-\gamma}{\gamma}(r_{ss}+\delta)\{(1-\alpha)[\gamma\frac{p_{Ass}}{\Psi_{ss}} - (1-\gamma)] + [1-\alpha(1-\gamma)]\} \\
a_7 &= [1-\alpha(1-\gamma)](\Phi_{ss} - r_{ss}) + \alpha(1-\gamma)(r_{ss}+\delta) - \zeta\gamma H_{Yss}
\end{aligned} \tag{A4.2.10}$$

To obtain the saddle-path (A4.2.8) and (A4.2.9) must then be solved for Ψ'_{ss} and p'_{Ass}, and the results used as initial conditions to integrate the pair of differential equations (A4.2.4) and (A4.2.5). The first of these tasks was achieved relatively painlessly by the use of the mathematical software package *"Maple"*.[49] Three solutions were yielded. One is the desired solution for the trajectory along which the steady-state is approached - the "stable arm of the saddle equilibrium". The other two solutions relate to the two "unstable arms" of the saddle, trajectories that do not satisfy the transversality conditions and only lead away from the steady-state. As explained in the text, by referring to the phase-space analysis of Section 3.3 it is clear that for the 'correct' saddle-path solution, both Ψ'_{ss} and p'_{Ass} must be negative. Consequently, of the three solution pairs:

$\Psi'_{ss1} = -50.8455$, $p'_{Ass1} = -53.7116$;
$\Psi'_{ss2} = -32.1058$, $p'_{Ass2} = 22.5429$; and
$\Psi'_{ss3} = -630.637$, $p'_{Ass3} = 1139.28$;

obtained from (A4.2.8) and (A4.2.9) evaluated at the benchmark parameter values, the first is readily identified as the solution for the saddle-path. It is also apparent from the phase-space analysis that in integrating the differential equations (A4.2.3), when the small steps in the independent variable Φ are positive it is the 'R8' portion of the saddle-path that is calculated, while when they are negative it is the 'R1' portion.

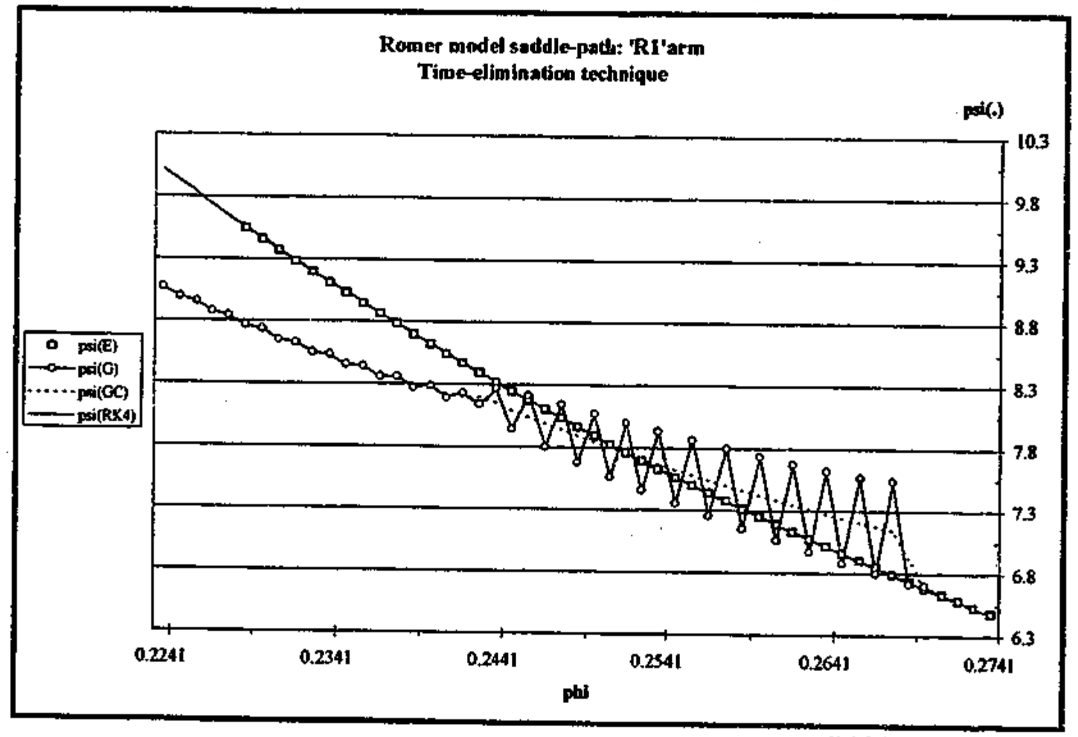
Integrations were performed under each of the Euler, Gragg, and fourth-order Runge-Kutta (RK4) methods, as explained in Section 4.1 and in Appendix 4.1. Results are

[49] From *Waterloo Maple Software Inc.*, Canada.

shown in Figure A4.2.1 to Figure A4.2.12.[50] Figure A4.2.1 to Figure A4.2.8 are presented in pairs, the first member of which depicts the 'R1' arm of the saddle-path (with the dependent variable steady-state, and its axis, to the right of the graph), and the second indicates the 'R8' arm (where now the steady-state and dependent variable axis are drawn to the left of the diagram).

Two significant observations are immediately apparent from this analysis. One is the instability of the Gragg or modified mid-point method; the other is the apparently very close correspondence of the Euler and the RK4 procedures.[51] These properties of the integrations are manifested for both the dependent variables, Ψ and p_A, and across a large range of step-sizes for the independent variable, Φ.

Figure A4.2.1: Saddle-path for the Romer model: 'R1' arm; time elimination technique; Euler, Gragg, and fourth-order Runge-Kutta methods of numerical integration; variable psi vs. phi; phi step-size = -0.001.



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The erratic behaviour of the Gragg method should not be completely unexpected. Both Gragg and the (unmodified) midpoint method are defined to be only *weakly stable*.[52] In

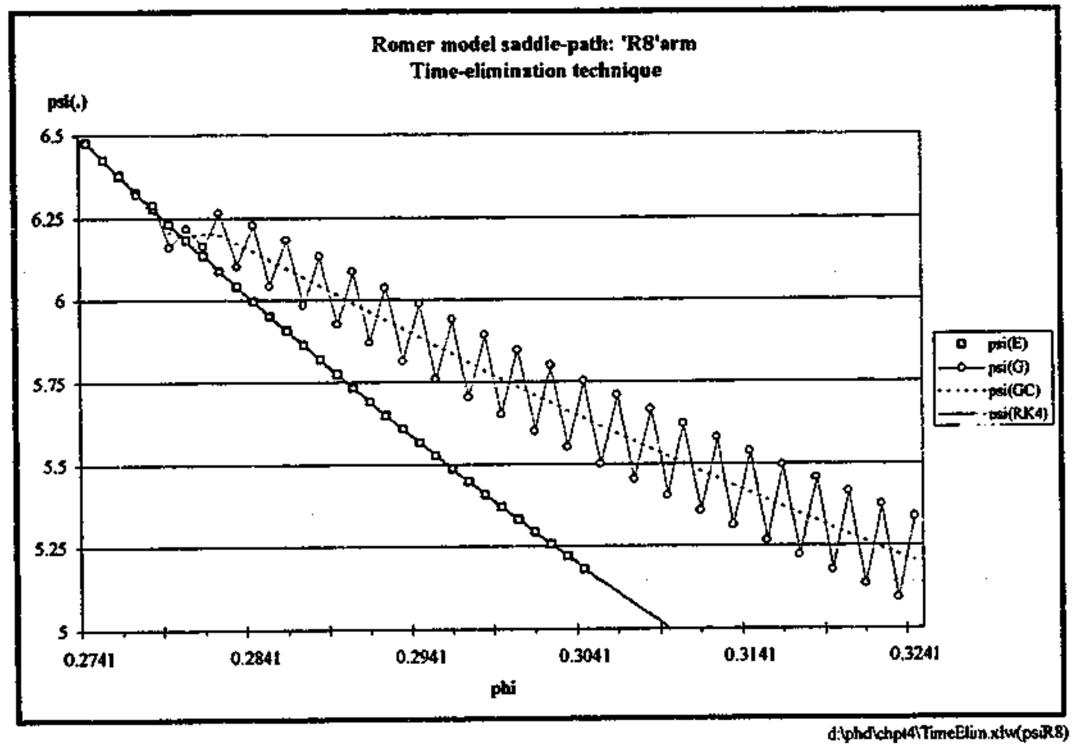
[50] Formulae for each of the methods were programmed into *Microsoft Excel* spreadsheets largely because of the quality and flexibility of the graphical output from *Excel*.

[51] Graphically the correspondence is so close that the Euler series tends to completely obscure the RK4 series. To help alleviate this problem the Euler series have been plotted as square data points with no connecting lines; and the RK4 as lines with no explicit points. Also, some of the data points from the end of the reported Euler integration have been omitted in order that, for this part of the saddle-path at least, the path plotted for the RK4 integration may be seen.

[52] See Atkinson (1989); Abramowitz and Stegun (1965), who give various numerical integration formulae and add a special caution against possible instabilities from the Gragg approach; and Harrison and Pearson (1994), who note that in some highly non-linear simulations with GEMPACK the Gragg and midpoint method results diverge rapidly.

the current application the 'saw-tooth' nature of the Gragg approach (see Figure A4.1.1 in Appendix 4.1) soon results in estimating that a point on the saddle-path lies in an invalid region of the phase-space. Since even the signs of the derivatives are then totally inappropriate to the saddle-path, and because the point still lies 'close' to the phase-surfaces, the integration tends to generate further points that bounce around in different phase-space regions. Eventually the system gets far enough away from the phase surfaces that it remains in a single region. Sometimes the oscillations then die out, as in Figure A4.2.3 for example, but of course by then the system is nowhere near the saddle-path, and the integration simply tracks whatever streamline of the phase-space it happens to fall on.

Figure A4.2.2: Saddle-path for the Romer model: 'R8' arm; time elimination technique; Euler, Gragg, and fourth-order Runge-Kutta methods of numerical integration; variable psi vs. phi; phi step-size = 0.001.



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Instability in the Gragg approach persists even when the independent variable step-size is made very small. While decreasing the step-size raises the number of integration iterations for which the estimated saddle-path points lie in the correct phase-space region, it actually seems to reduce the interval from the steady-state to the point where an invalid region is first entered! For example, with step-size 0.001 (equivalent to some 0.36 per cent of Φ_{ss}) the system first leaves the correct R8 region and enters R6 after only five iterations and when $\Phi \approx 0.2801$. When the step-size is reduced to 0.0005 (0.18 per cent of Φ_{ss}) the system departs R8 for R7 after six iterations but when $\Phi \approx 0.2776$. And when a step-size of a mere 0.00001 or only 0.0036 per cent of Φ_{ss} is employed, while the system remains in R8 for 25 iterations before entering R6, at that stage $\Phi \approx 0.2743$ only (Figure A4.2.9).

Figure A4.2.3: Saddle-path for the Romer model: 'R1' arm; time elimination technique; Euler, Gragg, and fourth-order Runge-Kutta methods of numerical integration; variable p_A vs. phi; phi step-size = -0.001.

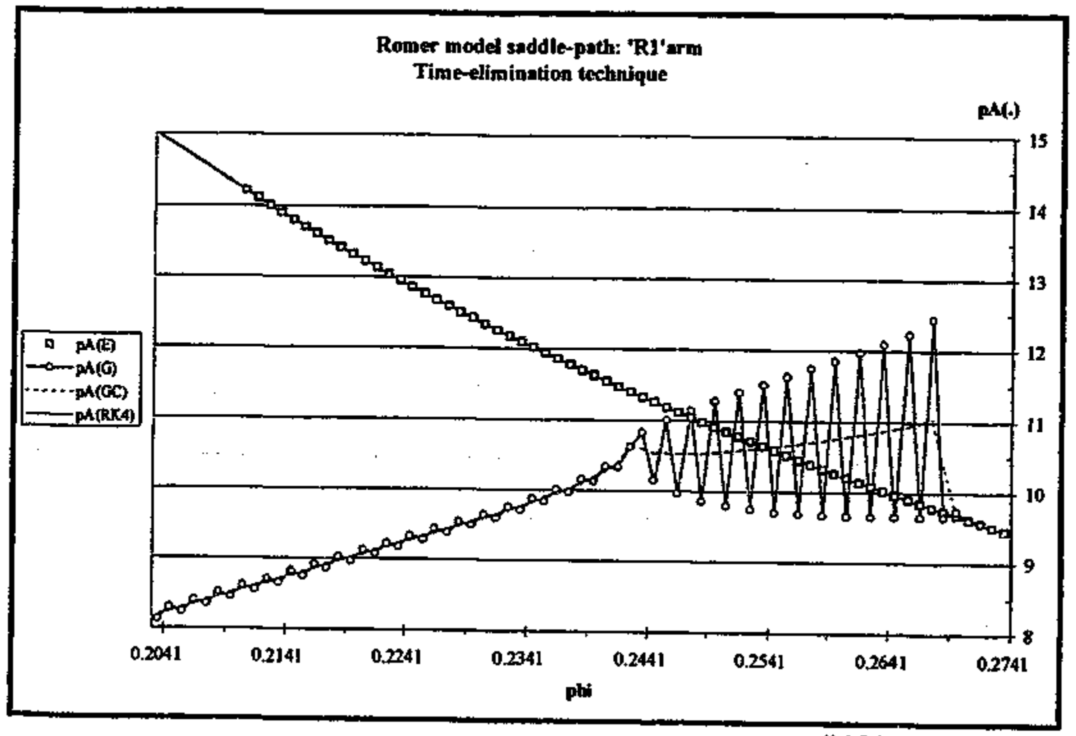

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Figure A4.2.4: Saddle-path for the Romer model: 'R8' arm; time elimination technique; Euler, Gragg, and fourth-order Runge-Kutta methods of numerical integration; variable p_A vs. phi; phi step-size = 0.001.

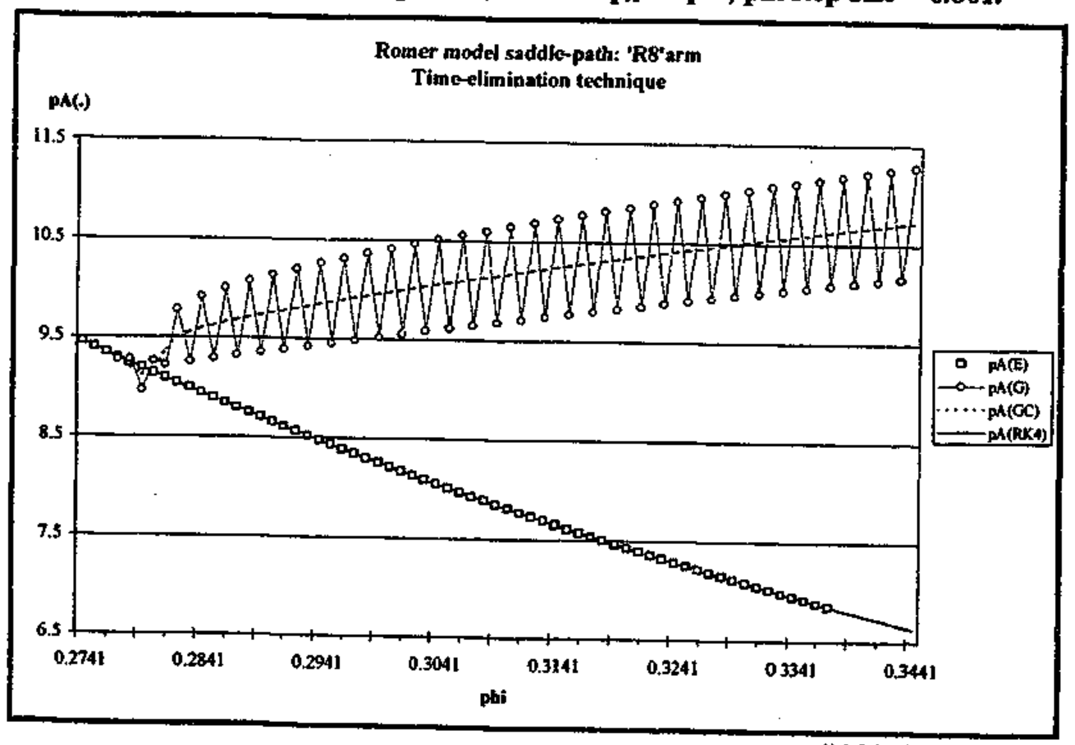

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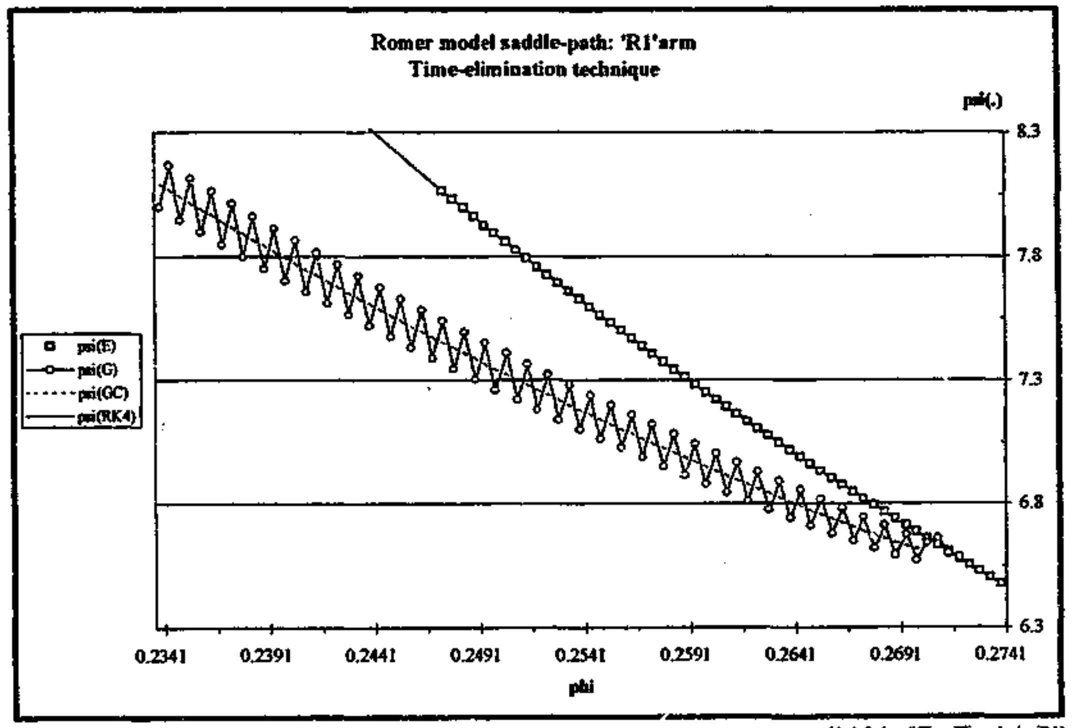
Figure A4.2.5: Saddle-path for the Romer model: 'R1' arm; time elimination technique; Euler, Gragg, and fourth-order Runge-Kutta methods of numerical integration; variable psi vs. phi; phi step-size = -0.0005.



d:\phd\chpt4\TimeElim.xls\(psiR1)

Figure A4.2.6: Saddle-path for the Romer model: 'R8' arm; time elimination technique; Euler, Gragg, and fourth-order Runge-Kutta methods of numerical integration; variable psi vs. phi; phi step-size = 0.0005.

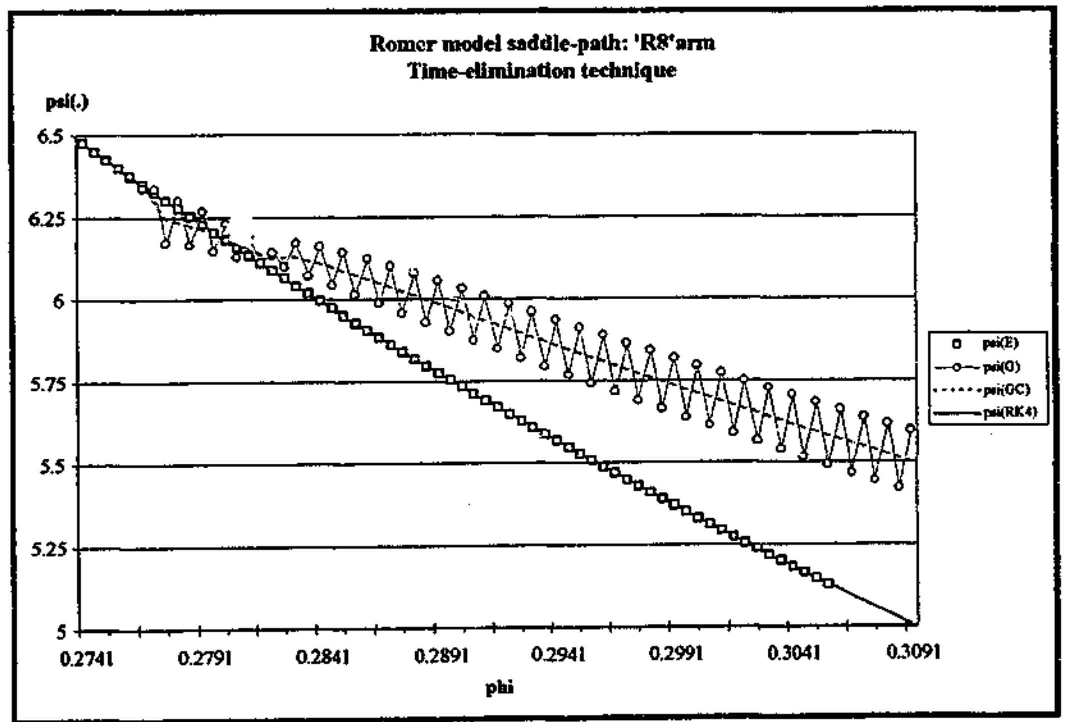


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Figure A4.2.7: Saddle-path for the Romer model: 'R1' arm; time elimination technique; Euler, Gragg, and fourth-order Runge-Kutta methods of numerical integration; variable p_A vs. phi; phi step-size = -0.0005.

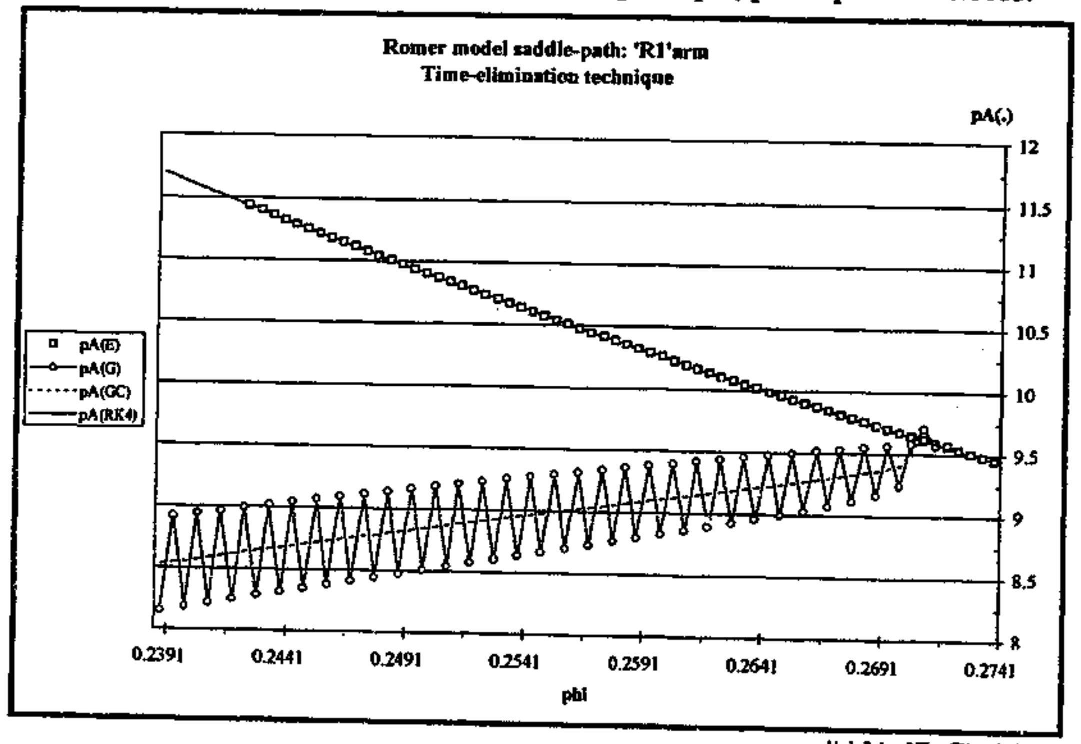

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Figure A4.2.8: Saddle-path for the Romer model: 'R8' arm; time elimination technique; Euler, Gragg, and fourth-order Runge-Kutta methods of numerical integration; variable p_A vs. phi; phi step-size = 0.0005.

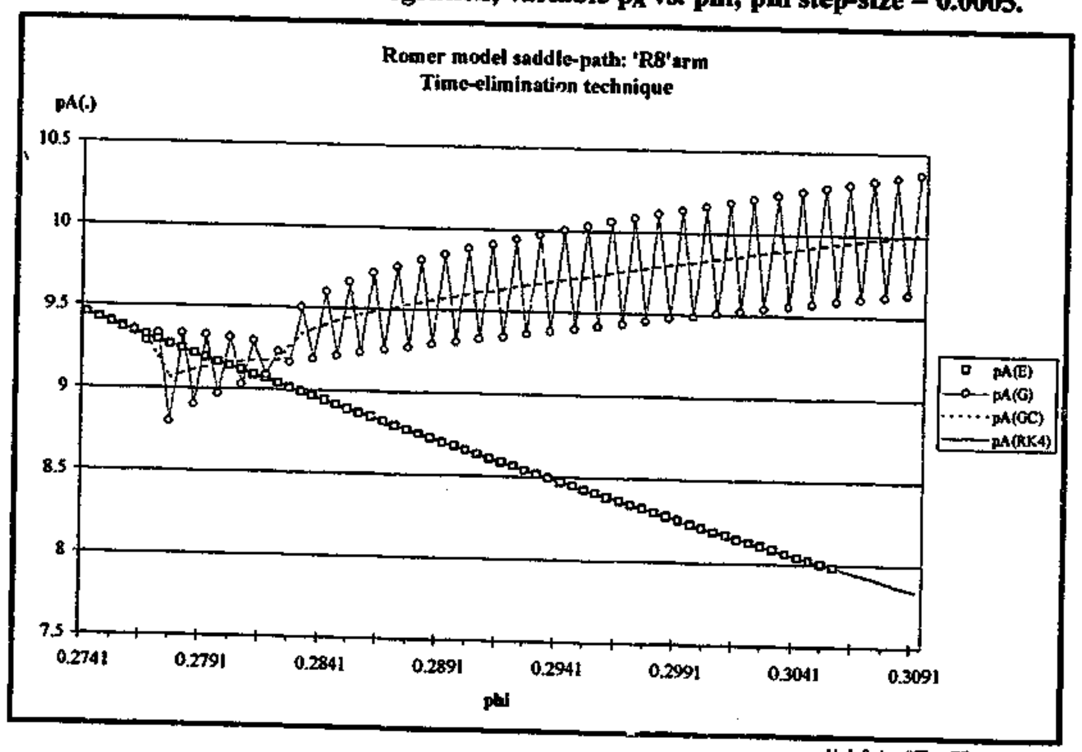

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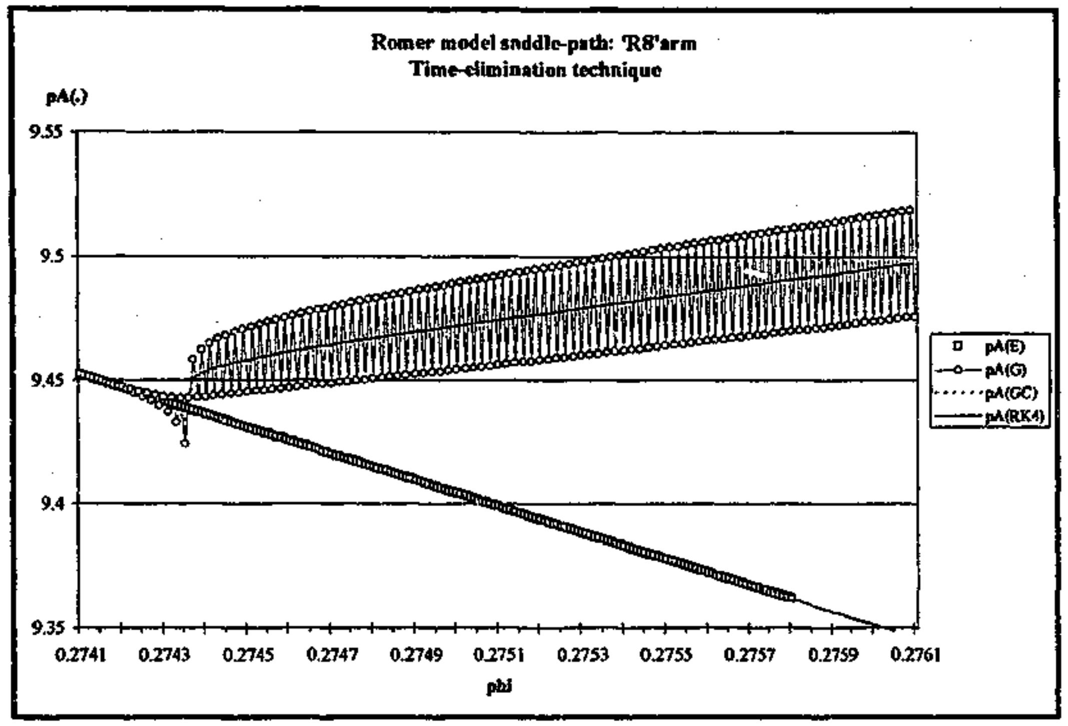
Figure A4.2.9: Saddle-path for the Romer model: 'R8' arm; time elimination technique; Euler, Gragg, and fourth-order Runge-Kutta methods of numerical integration; variable p_A vs. phi; phi step-size = 0.00001.



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In contrast to Gragg, the Euler and RK4 methods appear highly stable and the saddle-path estimates from both remain in the correct regions over indefinite iterations. Also, the fact that they both return extremely similar results suggests a degree of reliability. Their accuracy may be assessed and compared by examining the results they generate for the dependent variables, Ψ and p_A, at distant values of the independent variable (ie. relatively far from Φ_{ss}), over a range of step-sizes, and utilising the Richardson extrapolation technique (see Appendix 4.1). The calculations have been examined for step-sizes ±0.0001, ±0.0002, and ±0.0004; for ±0.0005, ±0.001, and ±0.002; and for the 3-Euler Richardson's extrapolations based upon each of these sets.[53] The 'distant' values of Φ were those reached after 1000 and 2000 iterations of the smallest step-size; namely: $\Phi \approx 0.1741$ and 0.0741 on the R1 side of the saddle-path; and $\Phi \approx 0.3741$ and 0.4741 on the R8 side.

The results are recorded in Table A4.2.1. Given the *orders of accuracy* of the differencing methods (in terms of omitted Taylor series terms - see Appendix 4.1), the most accurate calculations can be expected to be those for RK4 on the smallest step-sizes. Whether the 3-ER extrapolated results for the finest set of step-sizes are more or less accurate than those for RK4 on larger step-sizes cannot be predicted; although the calculations for the most distant Φ values suggest that the latter, for all but the biggest step-size, is probably the case. Encouragingly, the 3-ER calculations all agree with the RK4 outcomes from the smallest relevant step-size, to at least 6-figures and often to 8 or 9-figures.

[53] The "±" step-sizes confirm that the calculations cover both the R1 and R8 arms of the saddle-path.

Overall the results suggest a 'fair' degree of accuracy for the Euler method with the smallest step-sizes, showing 4 or 5-figure agreement with the RK4(±0.0001) outcomes and with the relevant 3-ER extrapolations. However, generally the Euler results lack consistency, with at best 3 or 4-figure agreement among the different step-size outcomes and sometimes only 2-figure agreement. This is all the more evident when compared with the great consistency and robustness of the RK4 results. These always show (at least) 9-figure agreement for the finer set of step-sizes, and even for largest step-size (±0.002) the results agree to at least 5-figures and usually to 7, 8, or 9-figures, with the results for the smallest step-size. Taking the computing and data manipulation effort into account, the most efficient of these integration configurations would seem to be the RK4 based on step-sizes of ±0.001. This represents about 0.36 per cent of the steady-state value $\Phi_{ss} \approx 0.2741$, and the agreement with RK4(±0.0001) only falls short of 8-figures for a single one of the calculations in Table A4.2.1.

Table A4.2.1: Assessment of the accuracy of the Euler and 4th order Runge-Kutta (RK4) methods of numerical integration for the 'Time elimination technique'.

Saddle-path region and Φ value	Step-size	Dependent variable Ψ: Euler	Dependent variable Ψ: RK4	Dependent variable pA: Euler	Dependent variable pA: RK4
Region R1	-0.0001	17.5279648	17.5323200	19.2307079	19.2316962
	-0.0002	17.5236152	17.5323200	19.2297206	19.2316962
Φ = 0.17408779	-0.0004	17.5149331	17.5323200	19.2277490	19.2316962
	3-ER on above	17.5323200	n.a.	19.2316962	n.a.
(1000 steps at	-0.0005	17.5106005	17.5323200	19.2267648	19.2316962
the smallest	-0.001	17.4890217	17.5323201	19.2218588	19.2316963
step-size)	-0.002	17.4462814	17.5323222	19.2121220	19.2316963
	3-ER on above	17.5323184	n.a.	19.2316959	n.a.
Region R8	+0.0001	3.34253737	3.34279684	5.82879095	5.82880641
	+0.0002	3.34227777	3.34279684	5.8277550	5.82880641
Φ = 0.37408779	+0.0004	3.34175815	3.34279684	5.82874460	5.82880641
	3-ER on above	3.34279684	n.a.	5.82880641	n.a.
(1000 steps at	+0.0005	3.34149814	3.34279684	5.82872916	5.82880641
the smallest	+0.001	3.34019596	3.34279684	5.82865205	5.82880642
step-size)	+0.002	3.33758114	3.34279686	5.82849825	5.82880663
	3-ER on above	3.34279681	n.a.	5.82880643	n.a.
Region R1	-0.0001	129.272661	129.505424	75.5309970	75.5765453
	-0.0002	129.040872	129.505424	75.4855759	75.5765453
Φ = 0.07408779	-0.0004	129.580187	129.505424	75.3951131	75.5765453
	3-ER on above	129.505416	n.a.	75.5765445	n.a.
(2000 steps at	-0.0005	128.351282	129.505424	75.3500703	75.5765454
the smallest	-0.001	127.220887	129.505425	75.1267192	75.5765460
step-size)	-0.002	125.028579	129.505442	74.6891109	75.5765554
	3-ER on above	129.504504	n.a.	75.5764528	n.a.
Region R8	+0.0001	2.04015237	2.04037307	4.03657968	4.03656653
	+0.0002	2.03993157	2.04037307	4.03659286	4.03656653
Φ = 0.47408779	+0.0004	2.03948970	2.04037307	4.03661931	4.03656653
	3-ER on above	2.04037307	n.a.	4.03656653	n.a.
(2000 steps at	+0.0005	2.03926862	2.04037307	4.03663257	4.03656653
the smallest	+0.001	2.03816181	2.04037307	4.03669926	4.03656653
step-size)	+0.002	2.03594106	2.04037307	4.03683464	4.03656656
	3-ER on above	2.04037306	n.a.	4.03656653	n.a.

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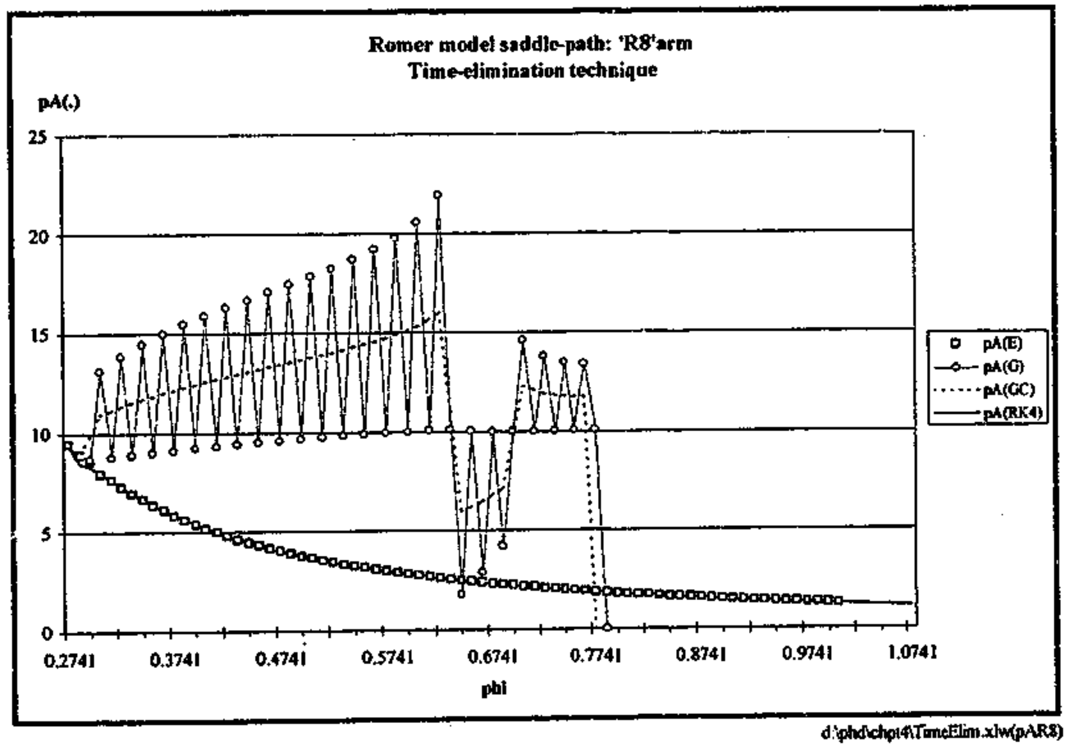
For values of the independent variable that are not so distant from the steady-state the Euler and RK4 methods of integration closely shadow one another. It is not until the step-size is made relatively large that this correspondence breaks down, and then, somewhat surprisingly, it is the RK4 method that fails. This is illustrated for the variable p_A: For step-sizes up to about -0.003 on the R1 side of the saddle-path and 0.008 on the R8 side, the series p_A(E) and p_A(RK4) continue to correspond closely and to correctly track the saddle-path. As the step-sizes are increased slightly from these levels the p_A(RK4) series begins to diverge from the p_A(E) series at the early iterations, but then to return to correspondence at later iterations. At step-sizes of -0.0038 and 0.01215 the series diverge for about a dozen iterations (Figure A4.2.11 and Figure A4.2.12).[54] Then, for only slightly larger step-sizes (no more than a 0.00001 absolute increase is necessary) the RK4 procedure breaks down completely, causing p_A to go negative at the first step. When the step-size has been increased to somewhere between -0.0039 and -0.0040 for the R1 arm, and to between 0.0130 and 0.0131 for the R8 arm, the RK4 procedure begins integrating once again; but now it is nowhere near the saddle-path. As before with the Gragg method, it merely integrates along whatever streamline it happens to be on. Throughout all this the simple Euler integration process continues its estimation of the saddle-path with great equanimity, correctly placing it in either R1 or R8 as appropriate. Moreover, it continues to do so even for step-sizes as large as -0.03 on the R1 side, and 0.12 on the R8 side. Nevertheless, as we have seen from the calculations in Table A4.2.1, for sensibly small step-sizes RK4 is clearly superior.

Figure A4.2.10 Saddle-path for the Romer model: 'R8' arm; time elimination technique; Euler, Gragg, and fourth-order Runge-Kutta methods of numerical integration; variable p_A vs. phi; phi step-size = 0.01.



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[54] This somewhat bizarre temporary divergence may also be seen from Figure A4.2.10 to arise very briefly for a step-size of 0.01.

Figure A4.2.11: Saddle-path for the Romer model: 'R1' arm; time elimination technique; Euler, Gragg, and fourth-order Runge-Kutta methods of numerical integration; variable p_A vs. phi; phi step-size = -0.0038.

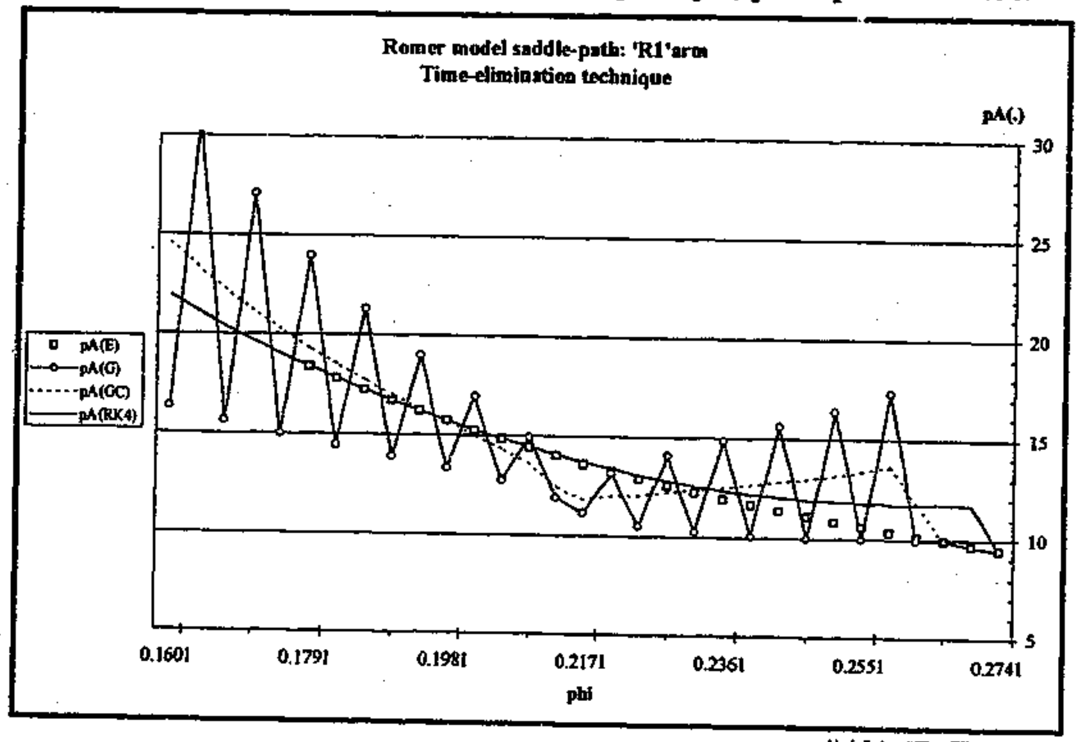

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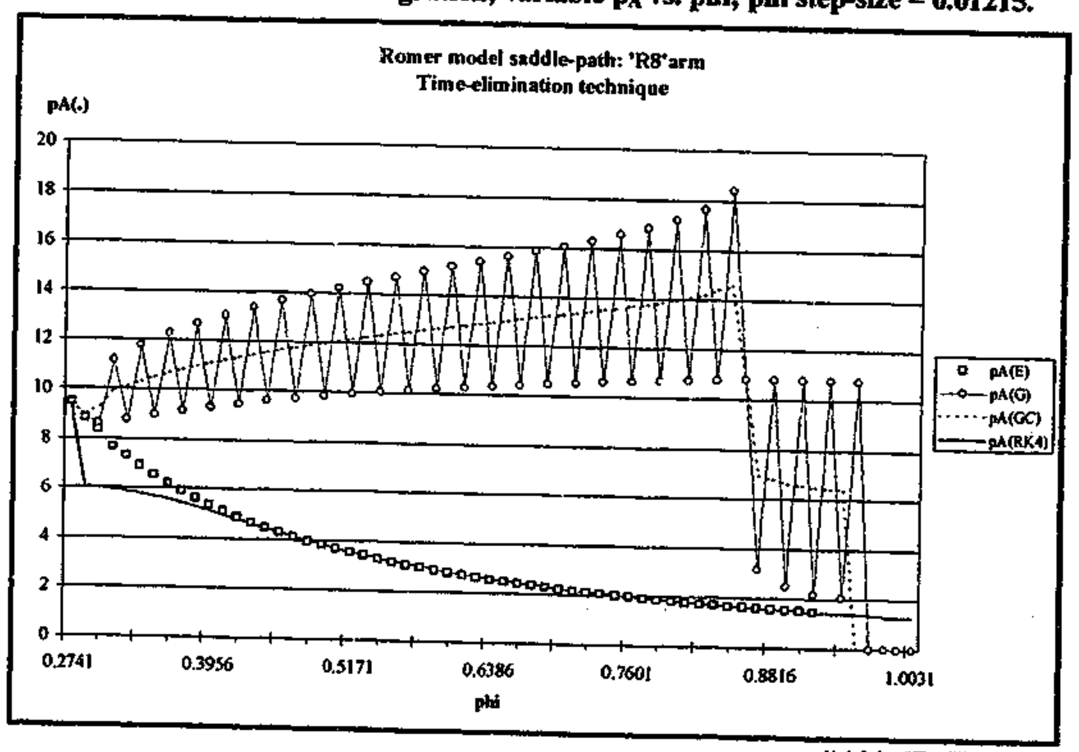
Figure A4.2.12: Saddle-path for the Romer model: 'R8' arm; time elimination technique; Euler, Gragg, and fourth-order Runge-Kutta methods of numerical integration; variable p_A vs. phi; phi step-size = 0.01215.



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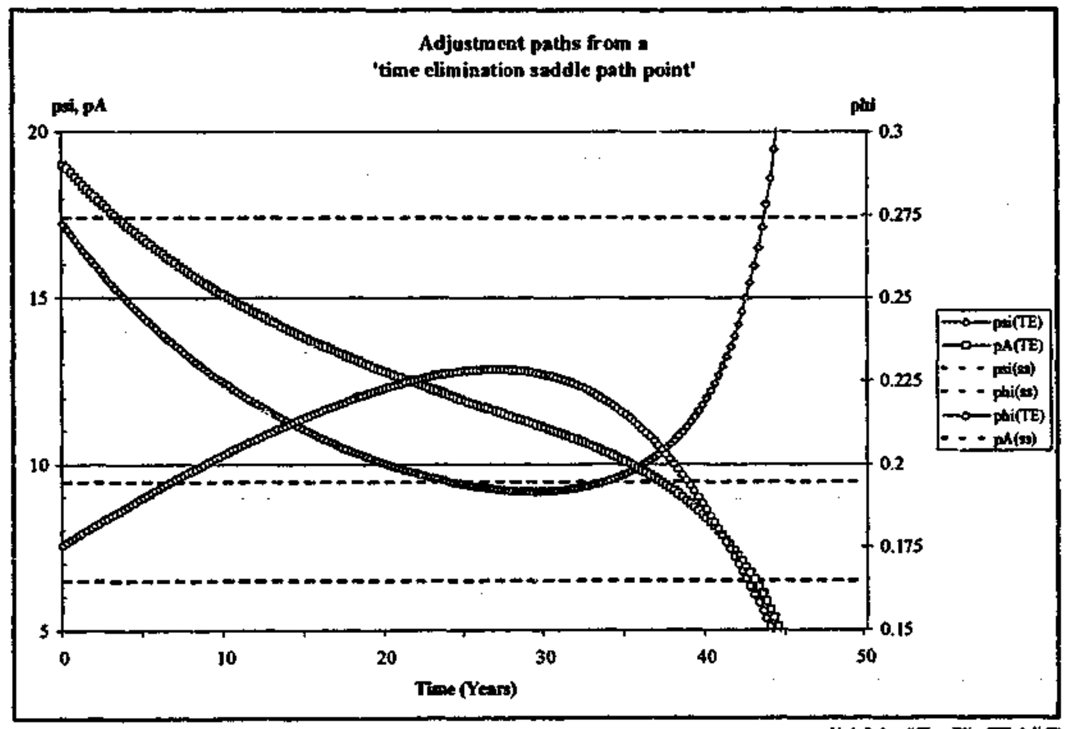
Because time has been eliminated in this system, another initial value problem must be solved in order to obtain adjustment paths of the dynamic variables over time. Specifically, initial values obtained from some appropriate point on, or interpolated from, the computed saddle-path must be used to integrate the original dynamic system, for which time is the independent variable. However, it turns out that in practice integrations of these 'second' initial value problems are unstable and tend to diverge rapidly from the saddle-path and its steady-state equilibrium. This is because the initial values are not **precisely** on the saddle-path, nor are they sufficiently close to it given the Romer system's extreme sensitivity to initial conditions. Thus the integration is along some other very nearby but nevertheless divergent streamline.

Figure A4.2.13: Adjustment paths for the Romer model: time elimination Euler generated initial values on a phi step-size =-0.0001 ('R1' arm); second Euler integration on a time step-size = 0.25.



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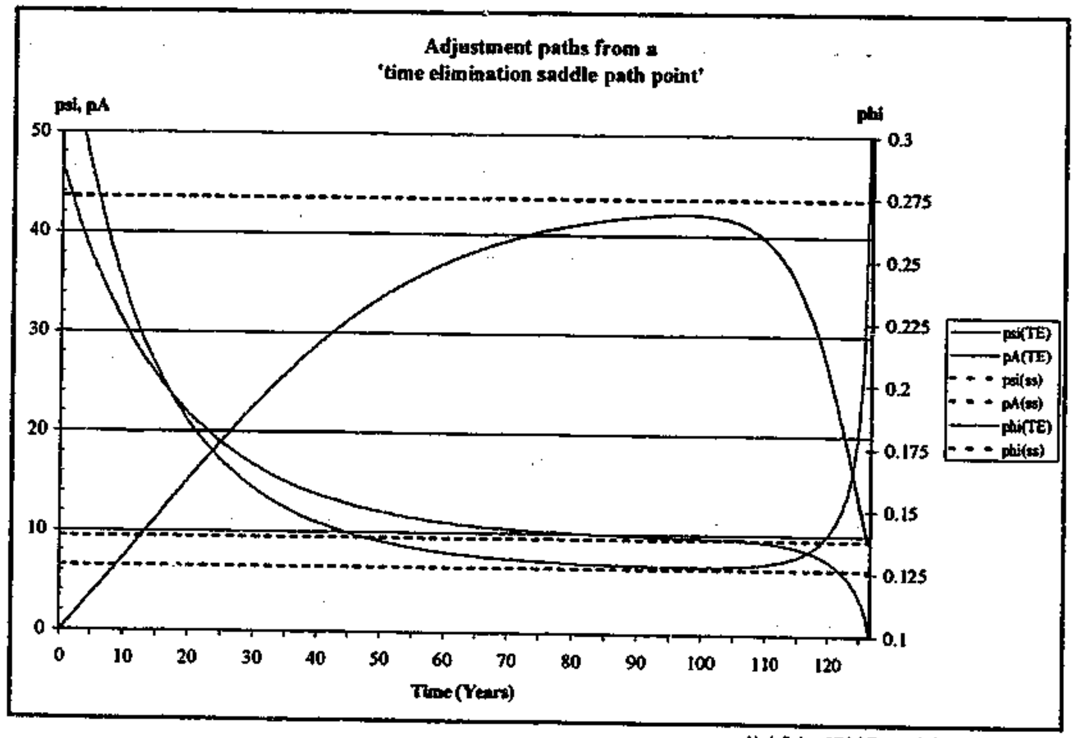
The superiority of the 4[th] order Runge-Kutta method of numerical integration over that of Euler is demonstrated quite dramatically in this context. When one of the calculated points from an Euler generated saddle-path is used to provide the initial values for an integration of the time paths of the dynamic variables on their approach towards the steady-state, this second integration rapidly becomes divergent. This is illustrated in Figure A4.2.13 for the case of what might be described as a 'medium distant' saddle-path point: ($\Psi \approx 17.53$, $p_A \approx 19.23$, $\Phi \approx 0.1741$), originally estimated by a time elimination Euler process with a (very small) phi step-size of -0.0001, and then used as initial values for a second Euler integration with a time step-size of 0.25. This second integration departs region R1 after only t=27, and it breaks down completely after t=47. Employing RK4 for the second integration improves matters only slightly, the equivalent time points to the above being t=36, and t=57 respectively.

In contrast, when the initial values for the time path integration are obtained from an RK4 generated saddle-path, the variables make far more definite and proximate approaches towards their steady-state levels. Nevertheless, these second integrations are still eventually divergent, and not after particularly long periods. Two cases of two 'very distant' RK4 generated points, ($\Psi \approx 62.86$, $p_A \approx 46.46$, $\Phi \approx 0.1000$) in region R1, and ($\Psi \approx 2.273$, $p_A \approx 4.377$, and $\Phi \approx 0.4500$) in R8, both used as initial values for ensuing RK4 time integrations, are shown below in Figure A4.2.14, and Figure A4.2.15.

Figure A4.2.14: Adjustment paths for the Romer model: time elimination RK4 generated initial values on a phi step-size =-0.0001 ('R1' arm); second RK4 integration on a time step-size = 0.25.



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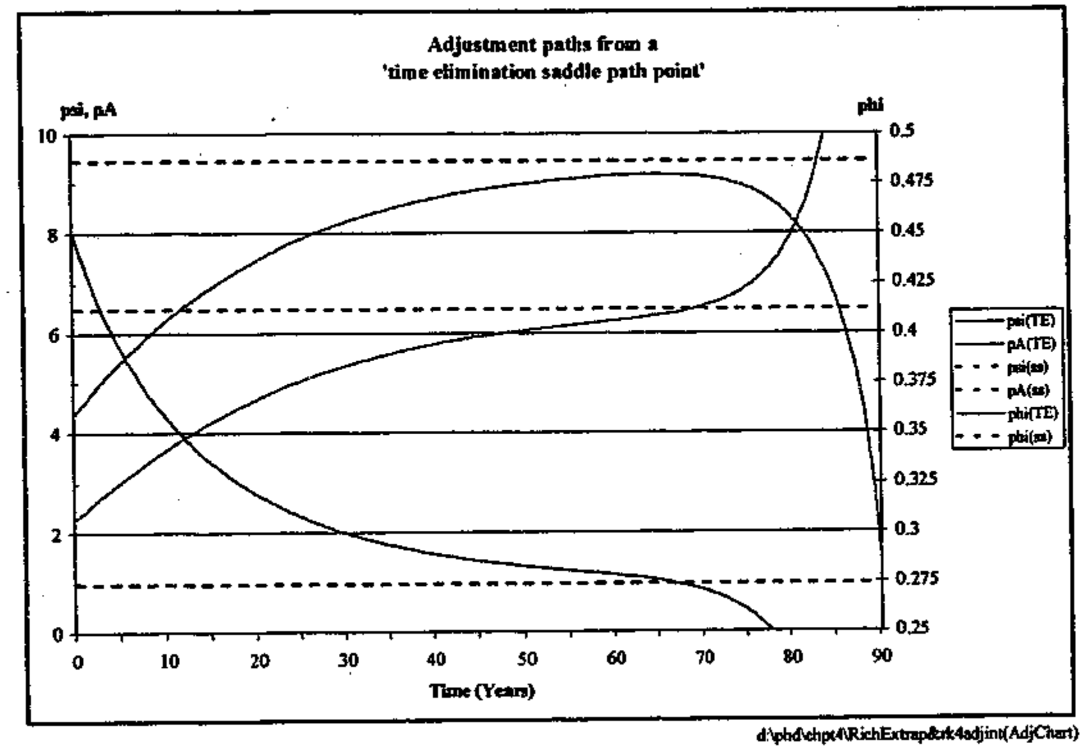
In the first case, for the R1 arm of the saddle-path the steady-state is approached in a well behaved apparently asymptotic manner for some time, the integration correctly placing the adjustment path in the R1 region of phase-space until t=96. However, from this point the divergence is rapid with the integration breaking down completely after t=126 (Figure A4.2.14). Things are not quite so well behaved for estimates of the R8 arm of the saddle-path. Here the correct region is returned only up to t=65, and the integration breaks down after t =90 (Figure A4.2.15).

Figure A4.2.15: Adjustment paths for the Romer model: time elimination RK4 generated initial values on a phi step-size =+0.0001 ('R8' arm); second RK4 integration on a time step-size = 0.25.



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A4.2.2 Eigenvector-backwards-integration method

As for the time elimination technique, integrations here were performed by the Euler, Gragg, and the 4th order Runge-Kutta methods. Moreover, the same general conclusions about their performances were found to hold. In particular, it was found firstly, that the Gragg method was again highly unstable and consequently unsuitable for the problem; secondly, that the Euler and RK4 methods approximate one another (although somewhat less so than was the case under the time elimination technique); and thirdly, that the greater consistency, stability and accuracy of the RK4 approach make it clearly superior and more efficient than Euler. These conclusions are illustrated in Figure A4.2.16 to Figure A4.2.21 and in Table A4.2.4 and will be discussed in some more detail soon. First it is necessary to discuss an additional issue that arises in this eigenvector-backwards-integration technique over that of time elimination.

This is the question of determining the magnitude of the first *small step* to be taken away from the steady-state and in the direction of the eigenvector. Of course the smaller is this first step, the more accurate the final results are likely to be; but also, the closer to zero are the time derivatives and so the slower is the progression of the integration. For this reason the integration is sensitive to the size of this initial step in the sense that changes in it generate very different values for each of the dependent variables (Ψ, p_A, and Φ) for the same (distant) value of the independent variable, t=time (Table A4.2.2).

Table A4.2.2: Sensitivity of the 'Eigenvector-backwards-integration technique' at *distant* time points, to the initial eigenvector step; Euler and 4th order Runge-Kutta (RK4) methods of numerical integration, time step-size=0.25.

Time interval & saddle arm		Eigenvector step multiple	Euler method Ψ	Euler p_A	Euler Φ	4th order Runge-Kutta method Ψ	RK4 p_A	RK4 Φ
t=-150		+0.0007	7.2730	10.279	0.2597	7.3145	10.318	0.2591
		+0.0005	7.0407	10.041	0.2637	7.0677	10.061	0.2633
t=-175		+0.0007	9.3695	12.333	0.2311	9.5474	12.498	0.2292
		+0.0005	8.4913	11.492	0.2418	8.6113	11.607	0.2403
t=-200	R1	+0.0007	18.372	19.890	0.1704	19.388	20.639	0.1665
		+0.0005	14.388	16.739	0.1928	15.020	17.247	0.1866
t=-225		+0.0007	69.713	49.989	0.0954	77.775	53.669	0.0914
		+0.0005	45.182	37.177	0.1148	49.740	39.612	0.1104
t=-250		+0.0007	465.42	176.33	0.0443	547.51	194.68	0.0420
		+0.0005	265.87	122.28	0.0552	309.72	134.43	0.0523
t=-150		-0.0007	5.7228	8.6420	0.2903	5.6896	8.6053	0.2911
		-0.0005	5.9335	8.8718	0.2855	5.9092	8.8453	0.2860
t=-175	R8	-0.0007	4.0922	6.7664	0.3398	3.9833	6.6322	0.3442
		-0.0005	4.7206	7.5117	0.3177	4.6359	7.4121	0.3205
t=-200		-0.0007	0.5248	1.1445	0.9175	0.4033	1.1592	1.0585
		-0.0005	1.5886	3.3533	0.5341	1.4155	3.0634	0.5664

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On the other hand the saddle-path itself seems relatively insensitive to changes in the eigenvector step, approximately the same points simply being reached at a later time for smaller eigenvector steps (Table A4.2.3).[55] However, as demonstrated by the problem of obtaining adjustment paths from the saddle-path computed by the time elimination technique, the dynamic Romer system is so sensitive to initial conditions that even small differences from the true saddle-path are rapidly magnified in integrating towards the steady-state. For this reason it is a moot point to say that the saddle-path is insensitive to changes in the eigenvector step, even for points whose magnitudes agree to say, five or six figures!

One thought about choosing the size of the initial eigenvector step is to make it equivalent, in terms of Φ, to the step-size found sufficiently small for accurate results under the time elimination method: that is, to a step in Φ of some ±0.001 for RK4, or of about ±0.0001 or less for unextrapolated Euler. With the relevant eigenvector having been estimated in terms of Ψ, p_A, and Φ respectively as $\gamma_1 \approx (0.6874, 0.7726, -0.0135)$, this suggests an initial step of $\pm 0.07\gamma_1$ or $\pm 0.007\gamma_1$. However, even with the smallest of these steps and for time step-sizes down to a tiny -0.1 or smaller, integration of the R8 arm of the saddle-path 'falls over' between t=158 and t=159 (Ψ having become negative) for both the Euler and RK4 methods. The 'period of successful integration' can be gradually extended by reducing the initial eigenvector step, but to get the integration beyond t=200 requires a step-size of about one-tenth of the smallest considered for the time elimination case; namely, $+0.0007\gamma_1$, and at this eigenvector step the R8 integration still only extends to time t =205.[56]

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[55] Interestingly, for the eigenvector steps and time step-size used in Table A4.2.2 and A4.2.3 (±0.0007, ±0.0005, and dt=-0.25), the time difference is approximately constant $\Delta t \approx 7$!

[56] In contrast, the integration of the R1 arm continues to proceed at t=4000 with a time step-size of -2.0!

Table A4.2.3 Sensitivity of the 'Eigenvector-backwards-integration technique' saddle-path, to the initial eigenvector step; Euler and 4th order Runge-Kutta (RK4) methods of numerical integration, time step-size=0.25.

Time interval & saddle arm	Eigenvector step multiple	Euler method Ψ	Euler p_A	Euler Φ	4th order Runge-Kutta method Ψ	RK4 p_A	RK4 Φ
t=-150	+0.0007	7.2730	10.279	0.2597	7.3145	10.318	0.25909
=-156.75	+0.0005	7.2684	10.275	0.2598	7.3085	10.315	0.25914
=-157	+0.0005	7.2785	10.285	0.2596	7.3191	10.326	0.25897
t=-175	+0.0007	9.3695	12.333	0.2311	9.5474	12.498	0.2292
=-181.75	+0.0005	9.3522	12.316	0.2313	9.5357	12.487	0.2293
=-182	+0.0005	9.3908	12.353	0.2309	9.5772	12.562	0.2288
t=-200	+0.0007	18.372	19.890	0.1704	19.388	20.639	0.1665
=-206.75 R1	+0.0005	18.291	19.828	0.1707	19.331	20.597	0.1667
=-207	+0.0005	18.471	19.966	0.1699	19.532	20.747	0.1659
t=-225	+0.0007	69.713	49.989	0.0954	77.775	53.669	0.0914
=-231.75	+0.0005	69.192	49.735	0.0957	77.397	53.493	0.0916
=-232	+0.0005	70.352	50.300	0.0950	78.742	54.119	0.0910
t=-250	+0.0007	465.42	176.33	0.0443	547.51	194.68	0.0420
=-256.75	+0.0005	461.07	175.26	0.0445	544.223	193.927	0.0421
=-257	+0.0005	470.76	177.64	0.0441	555.913	196.608	0.0418
t=-150	-.0007	5.7228	8.6420	0.2903	5.6896	8.6053	0.29113
=-156.75	-.0005	5.7269	8.6465	0.2902	5.6922	8.6083	0.29106
=-157	-.0005	5.7179	8.6366	0.2904	5.6828	8.5979	0.29129
t=-175	-.0007	4.0922	6.7664	0.3398	3.9833	6.6322	0.3442
=-181.75 R8	-.0005	4.1039	6.7807	0.3393	3.9910	6.6416	0.3439
=-182	-.0005	4.0779	6.7491	0.3404	3.9638	6.6083	0.3450
t=-200	-.0007	0.5248	1.1445	0.9175	0.4033	1.1592	1.0585
=-206.75	-.0005	0.5403	1.4781	0.9045	0.4121	1.1791	1.0470
=-207	-.0005	0.5062	1.4059	0.9340	0.3815	1.1094	1.0888

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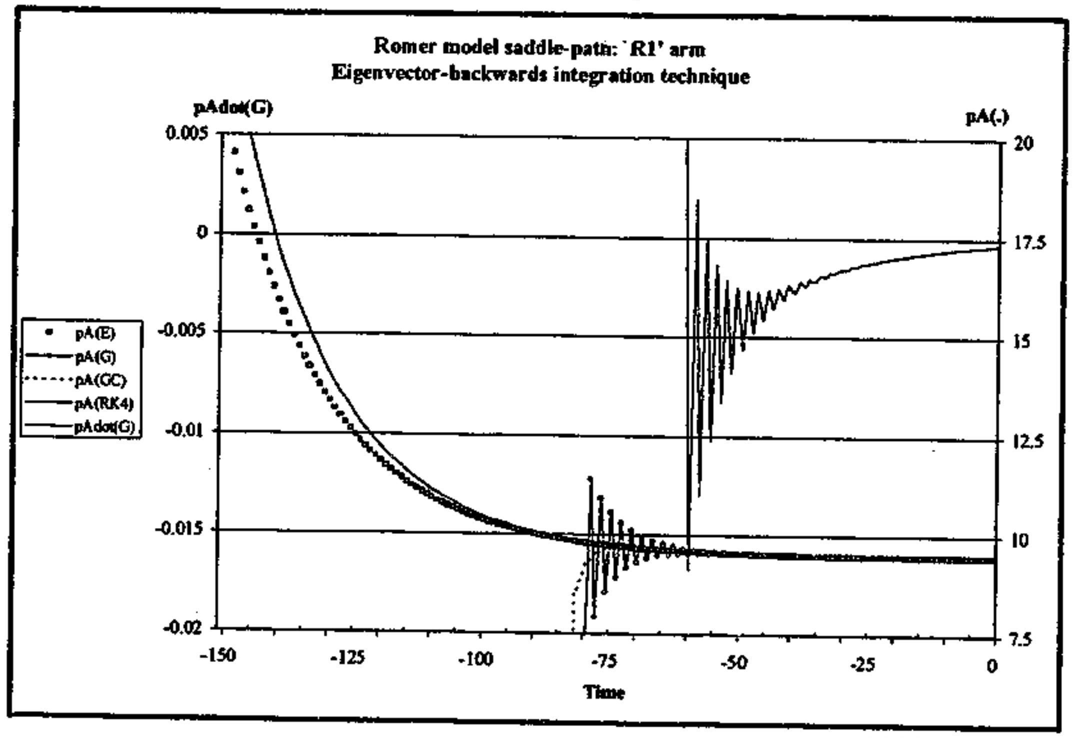
Performances of the Euler, Gragg, and RK4 integration methods are illustrated in Figure A4.2.16 to Figure A4.2.21. As was the case for the time elimination technique, the instability of the Gragg method is immediately apparent, including the oscillatory and divergent nature of its time derivative estimates. Even with the small eigenvector and time step-sizes of ±0.0007 and -0.25 respectively, for both the R1 and R8 arms of the saddle-path the Gragg integration only returns estimates in the correct region of phase-space up to t =-69, and it breaks down completely after t =-109. While the "end point-corrected Gragg" saddle-path accords closely with those for Euler and RK4 before it breaks down, this is only for a period in which the integration remains very close to the steady-state.

The explanation of how this erratic behaviour arises is exactly the same as for the time elimination technique. In the graphs presented here, a change in sign of a time derivative estimate reflects the tendency of the Gragg approach to estimate that a point on the saddle-path lies in an invalid region of the phase-space. However, a change in the sign of the plotted derivative is not necessary for such an invalid point to have been reached by the integration. One of the other derivatives not plotted may have changed sign instead.

Because of the many different combinations of the variables, saddle arms, eigenvector and time step-sizes it is possible to draw a vast number of different graphs of these integrations. However they all seem to show very much the same behaviour. Those presented here have been chosen to illustrate the general properties of the integrations for step-sizes that provide some visual clarity and which are not 'unreasonable'.

Figure A4.2.16: Saddle-path for the Romer model: 'R1' arm; eigenvector-backwards-integration technique; Euler, Gragg, and fourth-order Runge-Kutta methods of numerical integration; variable p_A; eigenvector step = $0.01\gamma_1$; time stepsize = -1.0.



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The similarity of the Euler and the 4th order Runge-Kutta integrations is evident from the diagrams. They appear to be particularly close to one another when the time step-size is small (Figure A4.2.20). Nevertheless, it turns out that when the integrations are extended further, the accuracy of the Euler results clearly fall short of those of the RK4 approach in estimating the saddle-path of the Romer model by this eigenvector- backwards-integration technique. As before for the time elimination technique, the accuracy of the Euler and RK4 here were assessed by examining their saddle-path estimates at a 'distant' integration point (in this case a time point) over a variety of time step-sizes. In this analysis the eigenvector step used was $\pm 0.0007\gamma_1$ and the integrations were performed out to time t=-200. Under the smallest step-size used of -0.125, this meant that the integration comprised 1600 iterations. Richardson extrapolations of the '3-Euler', or '3-ER' type, (see Appendix 4.1) were performed for each of the step-size groups (-0.125, -0.25, and -0.5); and (-1.0, -2.0, and -4.0). The results are presented in Table A4.2.4.

Table A4.2.4 Assessment of the accuracy of the Euler and 4th order Runge-Kutta (RK4) methods of numerical integration for the 'Eigenvector-backwards-integration technique'.

Description of the integration	Time step-size	Variable Ψ Euler	Variable Ψ RK4	Variable p_A Euler	Variable p_A RK4	Variable Φ Euler	Variable Φ RK4
	-0.125	18.8651234	19.3880093	20.2557876	20.6390020	0.16842125	0.16645538
Saddle arm	-0.250	18.3718146	19.3880093	19.8903759	20.6390020	0.17035564	0.16645538
=R1,	-0.500	17.4655281	19.3880084	19.2088275	20.6390015	0.17413001	0.16645538
eigenv' step	3-ER on above	19.3852094	n.a.	20.6376243	n.a.	0.16645538	n.a.
=+0.0007							
t=200	-1.000	15.9250158	19.3879946	18.0177826	20.6389940	0.18130311	0.16645547
(1600 steps	-2.000	13.6426998	19.3877770	16.1685060	20.6388802	0.19418854	0.16645686
at smallest	-4.000	10.9328175	19.3843559	13.8157963	20.6372510	0.21469385	0.16647877
step-size	3-ER on above	18.8255819	n.a.	20.3156739	n.a.	0.16666249	n.a.
	-0.125	0.46210380	0.40332119	1.30045187	1.15922541	0.98259782	1.05851807
Saddle arm	-0.250	0.52476979	0.40332085	1.44536321	1.15922640	0.91754475	1.05851825
=R8,	-0.500	0.66051214	0.40331513	1.74262846	1.15954273	0.81266532	1.05852043
eigenv' step	3-ER on above	0.40290792	n.a.	1.15802139	n.a.	1.03605980	n.a.
=-0.0007							
t=200	-1.000	0.96428864	0.40321579	2.34672964	1.15951230	0.66998507	1.05853265
(1600 steps	-2.000	1.63306092	0.40126086	3.49458563	1.16370664	0.51799114	1.05818470
at smallest	-4.000	2.89090007	0.35127608	5.30321366	1.21264429	0.39634144	1.06360007
step-size	3-ER on above	0.26894789	n.a.	1.03651233	n.a.	0.88275839	n.a.

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Figure A4.2.17: Saddle-path for the Romer model: 'R8' arm; eigenvector-backwards-integration technique; Euler, Gragg, and fourth-order Runge-Kutta methods of numerical integration; variable p_A; eigenvector step = $-0.01\gamma_1$; time stepsize = -1.0.

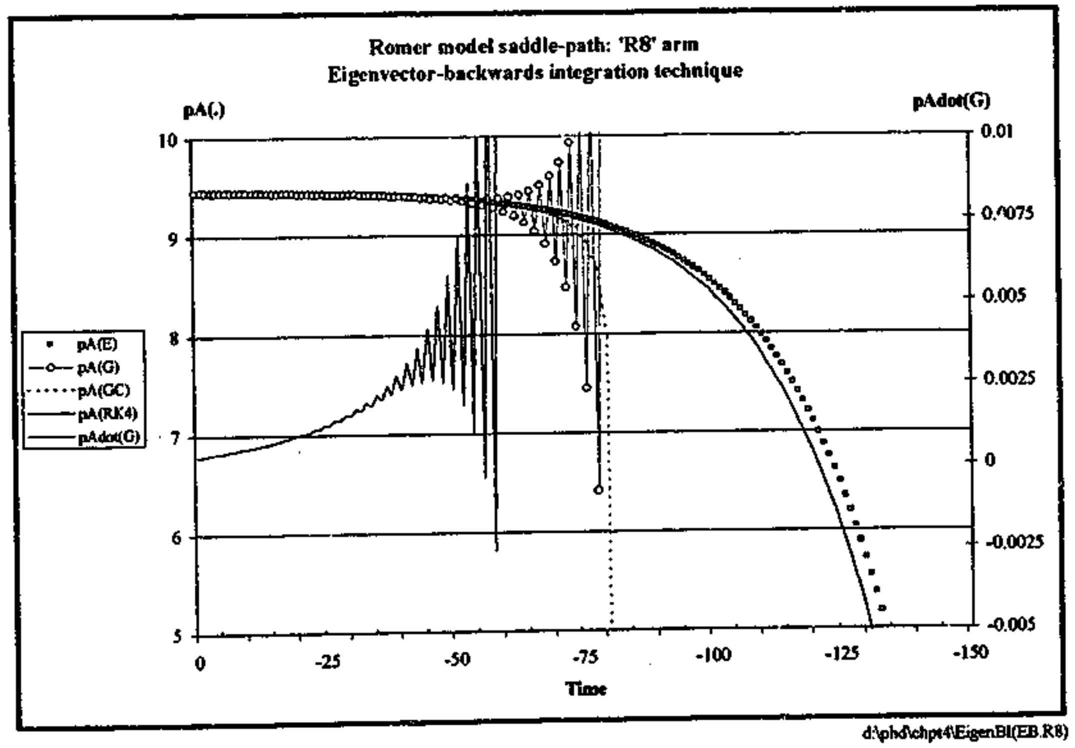

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Figure A4.2.18: Saddle-path for the Romer model: 'R1' arm; eigenvector-backwards-integration technique; Euler, Gragg, and fourth-order Runge-Kutta methods of numerical integration; variable phi; eigenvector step = $0.01\gamma_1$; time stepsize = -1.0.

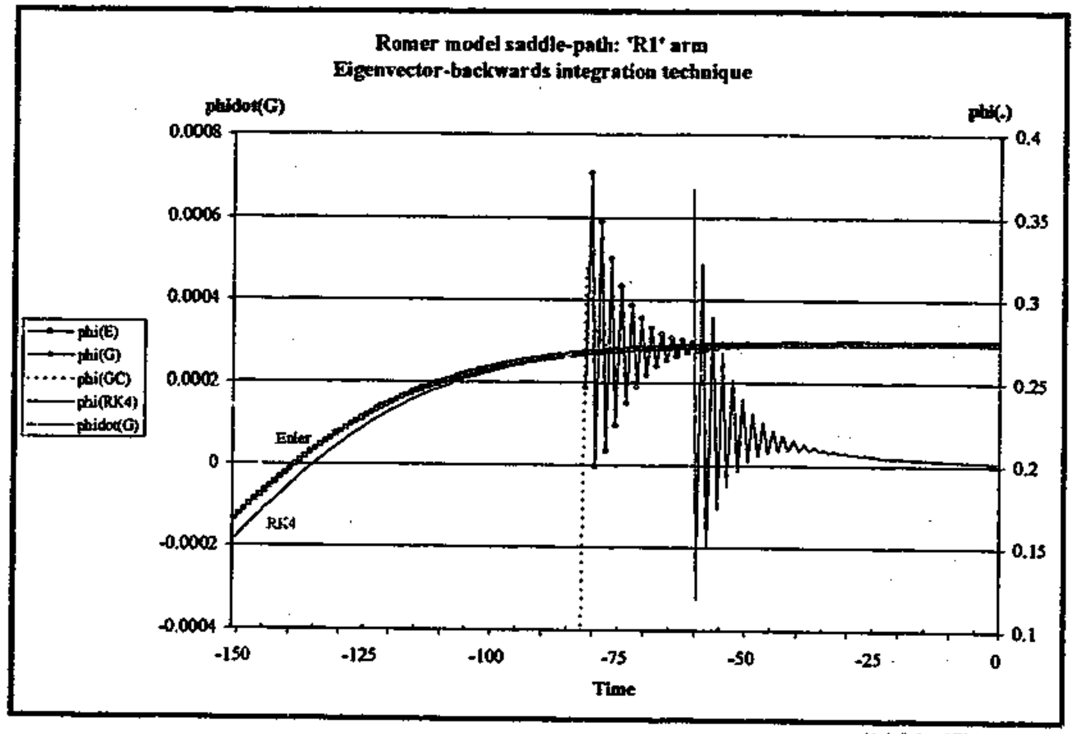

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Figure A4.2.19: Saddle-path for the Romer model: 'R8' arm; eigenvector-backwards-integration technique; Euler, Gragg, and fourth-order Runge-Kutta methods of numerical integration; variable p_A; eigenvector step = $-0.0007\gamma_1$; time stepsize = -1.0.

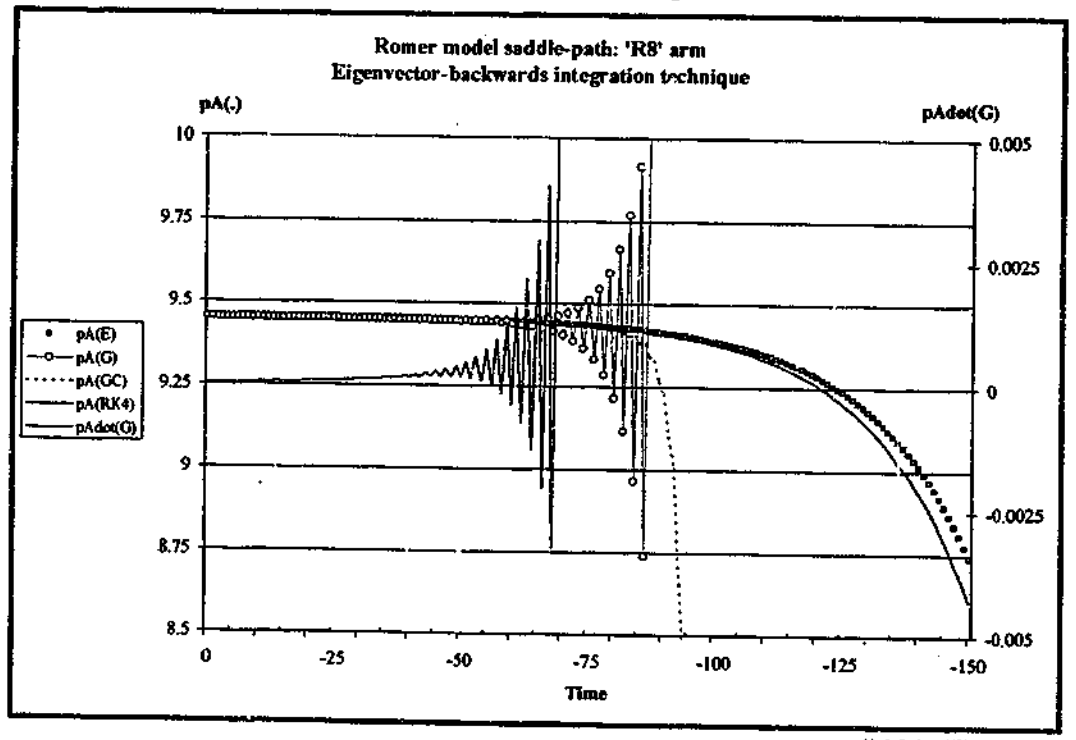

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Figure A4.2.20: Saddle-path for the Romer model: 'R1' arm; eigenvector-backwards-integration technique; Euler, Gragg, and fourth-order Runge-Kutta methods of numerical integration; variable phi; eigenvector step = $0.01\gamma_1$; time stepsize = -0.25.

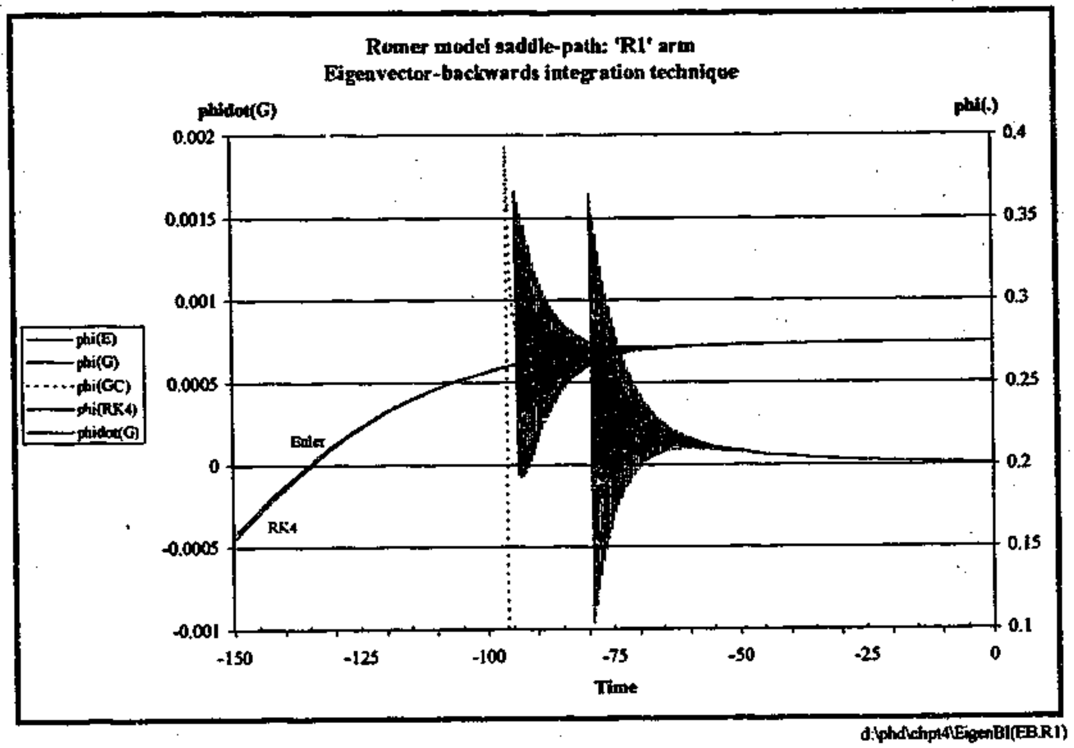

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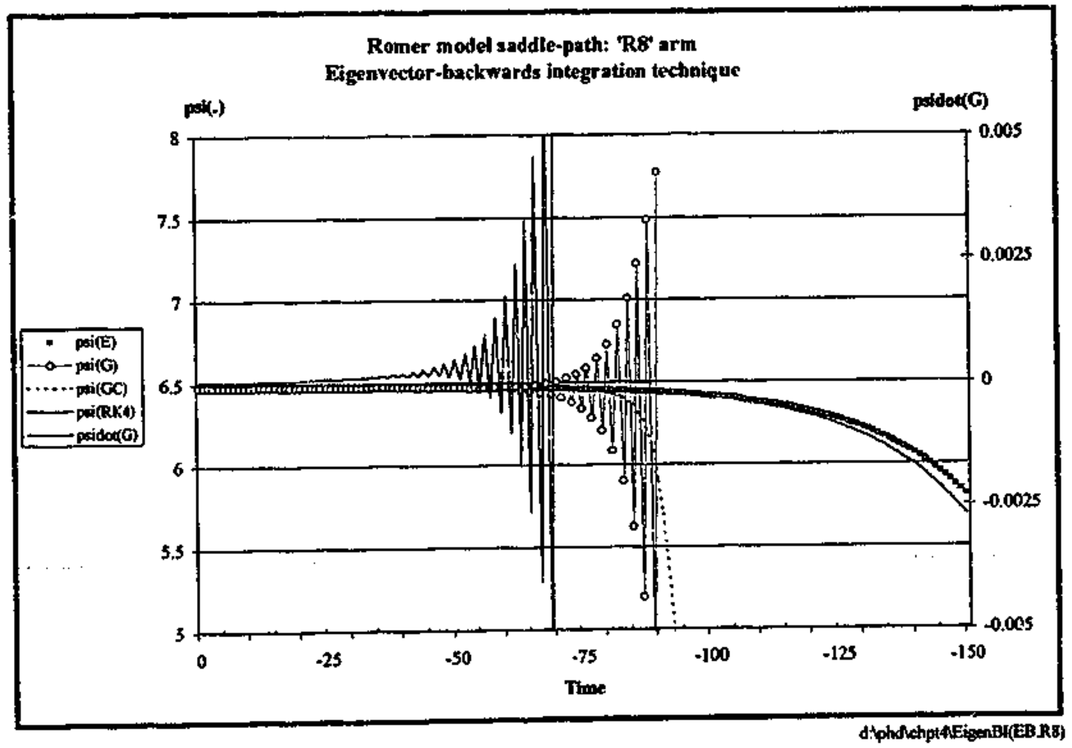
Figure A4.2.21: Saddle-path for the Romer model: 'R8' arm; eigenvector-backwards-integration technique; Euler, Gragg, and fourth-order Runge-Kutta methods of numerical integration; variable psi; eigenvector step = $-0.0007\gamma_1$; time stepsize = -1.0.



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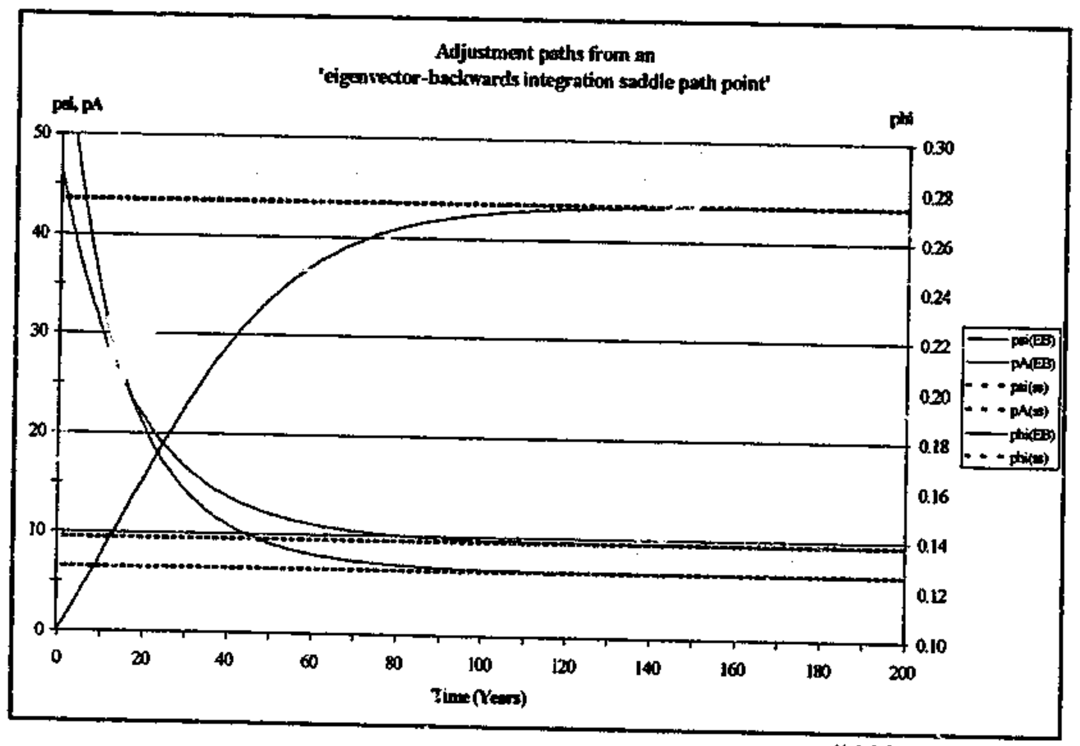
Once again, from the *orders of accuracy* of the differencing methods (Appendix 4.2) the most accurate calculations can be expected to be those for RK4 on the smallest step-sizes. Also, the 3-Euler Richardson extrapolations based upon the set of smaller step sizes can be expected to exploit any divergences between the simple Euler results and to improve them significantly. Even more so than was the case for the time elimination technique (see Table A4.2.1 and the accompanying commentary), the divergences and inconsistency of the Euler results stand in marked contrast to the great consistency of those of the RK4 method. It appears that for the Euler method only the 3-ER figures for the set of smaller step-sizes are sufficiently accurate, and that these are less accurate than the RK4 figures based on a step-size of -2.0.

Overall, it would appear that the accurate and efficient calculation of the 'benchmark' Romer model saddle-path under the eigenvector-backwards-integration technique, requires an eigenvector step of ±0.0007 or smaller and a 4[th] order Runge-Kutta procedure with a time step-size of -0.5 or smaller (or perhaps -0.25 or smaller in region R8). Furthermore, the efficiency of the integration could be improved by increasing the time step-size at the beginning of the integration where there is very little change in any of the variables (the time derivatives are very close to zero near the steady-state), and decreasing it later where the curvature is higher.

Figure A4.2.22: Adjustment paths for the Romer model: eigenvector-backwards-integration technique, RK4 method on an eigenvector step of +0.0007γ_1 ('R1' arm) and a time step-size = -0.25, simple re-ordering of the time scale.



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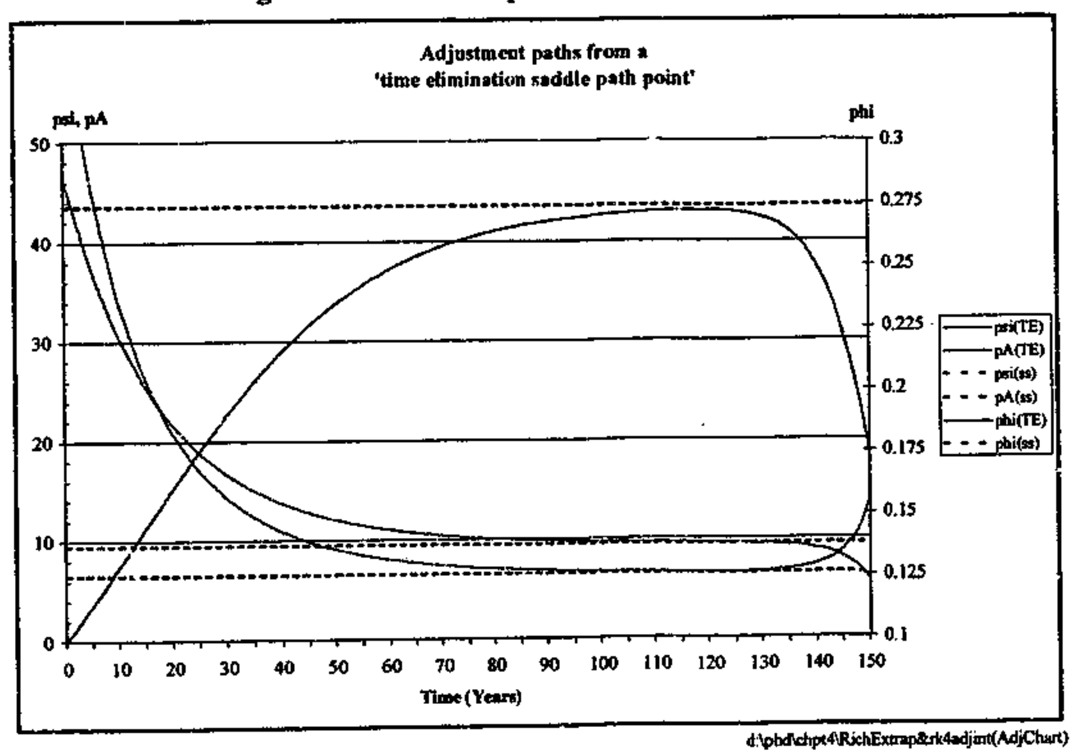
Calculation of adjustment time paths from saddle-paths generated by this eigenvector-backwards-integration technique is far simpler and more accurate than for the time elimination saddle-paths. Since the integration here is performed over time, such paths are in fact calculated automatically, merely requiring a simple re-ordering of the time

scale. In contrast, recall that the solution of another initial value problem, which tends to be divergent, is necessary under the time elimination method. Adjustment paths obtained in this manner for the 'distant point (Ψ=62.49, p_A≈46.28, Φ≈0.1002) are shown in Figure A4.2.22. These paths are directly comparable with those computed for the time elimination generated *neighbouring point* (Ψ=62.86, p_A≈46.46, Φ≈0.1000) and presented in Figure A4.2.14. The asymptotic behaviour here indicates a clearly superior estimate of the saddle-path.

Although **unnecessary** here, it is of course still possible to compute adjustment paths as the solution to a second initial value problem in the same way as was **necessary** for the time elimination method. Such calculations encounter the same sort of divergence problems and for the same reason. Namely, that given the high degree of non-linearity and consequent sensitivity of the dynamic system, the estimated initial values are simply not sufficiently close to the true saddle-path to constrain the error magnification and prevent divergence. Nevertheless, it is instructive to solve for the adjustment paths by this method in order to compare the results with those from the corresponding time elimination calculation. Figure A4.2.23 illustrates the adjustment paths computed in this way from the same initial point used in Figure A4.2.22. It is also therefore directly comparable with the time elimination calculations in Figure A4.2.14.

Figure A4.2.23: Adjustment paths for the Romer model: eigenvector-backwards-integration RK4 generated initial values on an eigenvector step of +0.0007γ_1 ('R1' arm) and a time step-size of -0.25; second RK4 integration on a time step-size = 0.25.



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The similarity is reassuring in that it suggests some reliability for both the time elimination and the eigenvector-backwards-integration techniques. Nevertheless, the adjustment paths based on initial values from the latter approach exhibit the more durable asymptotic performance (the second integration correctly places the system in region R1 up to time t =118, compared to t =96 for the time elimination initial values), and would therefore be preferred, at least for the integration settings employed. However, it may be a relatively simple matter to make the performances more similar. Perhaps a smaller Φ-step-size at the beginning of the time elimination integration, and a slightly bigger one later on would make the techniques more comparable?

Appendix 4.3

Specification of the finite differences-GEMPACK method of numerical integration

A4.3.1 General

Numerical integration of the Romer model under a finite differences approach implemented via the GEMPACK software requires the specification of two separate procedures. First is the finite differencing procedure itself. This is necessary to convert the *continuous-time differential equations* of the model into a finite set of *discrete-time difference equations*, the form in which the model is presented to GEMPACK. The second procedure that must be specified is the (multi-step) GEMPACK solution method to be used in calculating the changes to the many endogenous time differentiated finite difference variables in response to the imposition of exogenous shocks to other similar variables. The purpose of this appendix is to examine the impact of alternative configurations and to select the most appropriate for investigating the dynamic behaviour of the Romer system.

In view of the unstable and divergent performance exhibited by the Gragg or modified mid-point method in the earlier integrations of the Romer model saddle-path,[57] neither Gragg nor the mid-point method have been considered as candidates for the finite differencing to be adopted in specifying the model to GEMPACK. Rather, the choice has been restricted to the straightforward and robust, but somewhat approximate Euler method; and the more complex but reliable and accurate fourth order Runge-Kutta method. In addition to deciding upon the differencing method itself, both the time step-size and the overall time period for the difference equations must also be determined. For the solution procedure under which the effects of the exogenous shocks of simulations are calculated (shown in Section 4.3.2 to be equivalent to an initial value problem), the GEMPACK software offers the choice of the Euler, mid-point and Gragg methods, all with Richardson extrapolations from two or three multi-step applications.

Some trade-off between accuracy and computational effort is involved in choosing amongst the alternatives for each different decision, and ultimately the final selection of a composite configuration is a somewhat subjective choice, though one based upon empirical outcomes. Accordingly, a variety of alternative finite differences-GEMPACK configurations have been run for similar simulations. Their results are examined in Section A4.3.2. The model itself and the finite differencing procedure to be applied to it are specified in a *TABLO input file*; while details any particular simulation, including the exogenous shocks, the overall time horizon and time step-sizes, and the multi-step method of solution are specified via a *Command file*. These are discussed further in Section A4.3.3.

[57] Namely, the time elimination and the eigenvector-backwards-integration techniques examined in Section 4.2 and Appendix 4.2).

A4.3.2 Empirical performances on the Romer model

The simulations used were for either a decrease or an increase of 10 per cent in the capital elasticity parameter γ, in order to return it to its benchmark setting (0.54), while all other parameters were held at benchmark.[58] In this way the simulation results generated portions of either the R1 or the R8 arm of the benchmark saddle-path. The step-sizes employed were constrained by the desire to be able to report results for all complete or integral year periods up to and including *'year 100'*, and all quarterly results to *'year 25'* from the configuration eventually chosen as the 'standard'. The different configurations tested are specified in Table A4.3.1, and some diagnostic results from their corresponding simulations are recorded in Table A4.3.2.

Table A4.3.1: Specification of configurations for finite differences-GEMPACK test simulations.

Test simulation labels	Test simulation configurations: Finite differencing	Step size (years)	Total period (years)	Time intervals	GEMPACK steps and solution method	Saddle arm
E.1	Euler	1.0	250	250	2-4-6 Gragg	R1
E.2	Euler	0.25	250	1000	8-16-24 Euler	R1
RK4.1	4^{th} order Runge-Kutta	1.0	250	250	2-4-6 Gragg	R1
RK4.2	4^{th} order Runge-Kutta	0.5	250	500	8-16-24 Gragg	R1
RK4.3	4^{th} order Runge-Kutta	Uneven1[a]	250	350	8-16-24 Gragg	R1
RK4.4	4^{th} order Runge-Kutta	Uneven1[a]	250	350	12-24-36 Gragg	R1
RK4.5	4^{th} order Runge-Kutta	Uneven1[a]	250	350	12-24-36 Euler	R1
RK4.6	4^{th} order Runge-Kutta	Uneven1[a]	250	350	12-24-36 Gragg	R8
RK4.7	4^{th} order Runge-Kutta	Uneven2[b]	250	400	12-24-36 Gragg	R8
RK4.8	4^{th} order Runge-Kutta	Uneven3[c]	500	375	12-24-36 Gragg	R8
RK4.9	4^{th} order Runge-Kutta	Uneven4[d]	200	345	12-24-36 Gragg	R8
RK4.10	4^{th} order Runge-Kutta	Uneven1[a]	250	350	16-32-48 Gragg	R8
E.3	Euler	Uneven5[e]	250	625	12-24-36 Gragg	R1
E.4	Euler	Uneven5[e]	250	625	12-24-36 Gragg	R8

Notes: a 0.125 over the first 15 years; 0.25 for the next 25 years; 0.5 for the following 25 years to year 65; then 1.0 for the 35 years to year 100; 2.0 for the next 50 years; and finally, 5.0 over the last 100 years to year 250.

b 0.0625 over the first 5 years; 0.125 for the following 10 years; 0.25 over years 15 to 25; 0.5 for the 25 years to year 50; then 1.0 over the next 100 years; and 2.0 over the final 100 years.

c As for "Uneven1" but with an additional 25 steps of 10.0 from year 250 to year 500.

d As for "Uneven1" but without the last 5 steps of 10.0 from year 200 to year 250.

e 0.05 over the first 15 years; 0.1 for the next 10 years; 0.2 over years 25 to 40; 0.25 to year 50; then 0.5 for the next 15 years; 1.0 for the 35 years to year 100; 2.0 to year 150; and 5.0 for the 100 years to year 250.

The results labelled *"Time in R1 or R8"* require some explanation: They were calculated by using the coordinates of post-shock saddle-path points from the simulations as initial values for the direct integration of the model's differential equations by a 4^{th} order Runge-Kutta process with a step-size of 0.25 years. Two sets of coordinates were used: The first were those immediately after the shock, at time t=0. These define the saddle-

[58] The choice of γ was made simply because the Romer system saddle-path is more sensitive to this than to any other parameter (see Table 2.3 in Chapter 2).

path points to which the system first jumps in response to the shocks. The second set of coordinates used in these integrations were those from five periods later, at time t=5, when it turns out that about 20 per cent of the total adjustment along the saddle-path has occurred.[59] In general, the longer the system remains in its correct region of phase-space under these integrations (and hence the longer the period of asymptotic approach to the new steady-state), the closer to the true saddle-path are its initial conditions. Thus, these integrations provide some measure of the accuracy of the finite differences-GEMPACK results.

It is apparent from the diagnostic results that although GEMPACK extrapolations for Euler finite differencing can be made with a great deal of precision (**all** the results in simulations E3 and E4 achieve the *"6-figures level of machine accuracy"*), the Euler method simply does not produce very accurate estimates of the saddle-path (results E.1, E.2, E.3 and E.4). Even for the finely defined grid "Uneven5" the system never really reaches a state of asymptotic approach to the new steady-state (Figure A4.3.1 and Figure A4.3.2). In comparison to this, even the relatively crude 4^{th} order Runge-Kutta configuration RK4.1 generates better estimates.

Table A4.3.2 Some diagnostic results from the finite differences-GEMPACK test simulations.

Test simulation number	Time in R1 or R8 from: t=0	t=5	Initial jumps (% of base)[a]: phi	pA	Error in total adjustment (% of base x 10^{-3})[a]: psi	phi	pA	Accuracy of results (% of all results)[b]: 6 figures[c]	5 figures
E.1	24.75	25	-3.57497(3)	12.9021(4)	0.06(3)	0.43(2)	0.09(3)	0.5	6.0
E.2	30.75	30.75	-3.64603(5)	12.9914(6)	0.16(6)	-0.48(5)	0.09(3)	78.8	21.2
RK4.1	55.75	51	-3.66901(2)	13.0197(3)	-72.0(1)	25.1(1)	136.5(1)	0.1	3.5
RK4.2	64.5	56	-3.66917(5)	13.0203(5)	0.06(4)	-0.18(4)	0.11(4)	10.8	20.6
RK4.3	64.5	56	-3.66917(5)	13.0203(5)	0.16(4)	-0.28(4)	0.21(4)	45.8	31.7
RK4.4	64.5	56	-3.66917(6)	13.0203(6)	0.16(5)	-0.28(4)	0.31(4)	69.8	30.2
RK4.5	55.25	53	-3.66919(3)	13.0203(4)	-0.24(3)	-1.08(3)	0.21(3)	0.8	6.5
RK4.6	53.5	51.75	-0.848975(5)	-11.6679(6)	-0.06(5)	0.05(5)	-0.08(5)	87.4	12.6
RK4.7	55.5	53.25	-0.848971(5)	-11.6679(6)	-0.06(5)	0.05(5)	-0.08(5)	80.5	19.5
RK4.8	53.75	57.25	-0.848974(5)	-11.6679(6)	0.04(5)	-0.05(5)	0.12(5)	83.7	6.3
RK4.9	53.75	51.75	-0.848974(5)	-11.6679(6)	-2.26(5)	1.35(5)	-2.68(5)	82.0	18.0
RK4.10	53.75	51.75	-0.848974(5)	-11.6679(6)	-0.16(6)	0.05(5)	-0.08(5)	96.3	3.7
E.3	37.25	37.25	-3.66426(6)	13.0134(6)	0.06(6)	-0.18(6)	0.21(6)	100.0	0.0
E.4	28.25	29.75	-0.846192(5)	-11.6710(6)	-0.06(6)	0.05(6)	0.02(5)	100.0	0.0

Notes: a Figures in bracket denote the number of digits of accuracy for the extrapolated results.

b "All results" means the values of psi, phi, and pA, for every grid point in the simulation.

c With *double precision* unavailable in GEMPACK, "6 figures" is the *level of machine accuracy*.

[59] For the simulations considered here it turns out that about 20 per cent of the adjustment along the saddle-path occurs in the first 5 years; some 15 per cent in each of the next two 5 year periods; about 10 per cent in each of the following two 5 year periods; and around another 5 per cent in each of the next three sets of 5 year groups. Further adjustments of some 5 per cent of total also take place in each of the following 10 year, 15 year, and 35 year periods. Thus, in round numbers some 50 per cent of the adjustment is made in 15 years, about 75 per cent after 30 years, 90 per cent after 50 years, and over 99 per cent after 100 years. The uneven sets of grid intervals used in the finite differences-GEMPACK runs (Table A4.3.1) were defined with this pattern of adjustment in mind.

Results of the various Runge-Kutta simulations then reveal that in terms of the accuracy of both the saddle-path coordinates and the GEMPACK initial value extrapolations:

- smaller step-sizes for both the finite differencing and for the initial value problem help (RK4.1 and RK4.2);
- an uneven grid, with smaller step-sizes corresponding to ranges of greater change also helps (also see Appendix 4.1), though here apparently only with the *"figures of accuracy"*. Importantly, the gain is that considerably less overall grid intervals are required (RK4.2 and RK4.3);
- smaller steps in the GEMPACK initial value problem and extrapolation also generate more reliability in the results, though here again the accuracy of the actual estimates of the saddle-path coordinates remain unchanged (RK4.3 and RK4.4);
- using an Euler process with extrapolation for the GEMPACK initial value problem is clearly inferior to the Gragg approach (RK4.4 and RK4.5);
- integration along the R8 arm of the saddle-path is more sensitive to initial coordinates than along the R1 arm, although it generates more reliable extrapolations (RK4.4 and RK4.6);
- the gains from a finer uneven grid with more intervals (grid "Uneven2" compared to grid "Uneven1") appear to be fairly small (RK4.6 and RK4.7);
- extending the overall time horizon (the point at which the steady-state is deemed to be reached) also does not produce great improvement (RK4.6 and RK4.8);
- conversely, a reduction in the overall time horizon does not seem to cost much (RK4.6 and RK4.9); and finally
- whilst further raising the number of steps for the GEMPACK initial value problem and extrapolation improves the reliability of the results in terms of the *"figures of accuracy"*, here it produces little other improvement (RK4.6 and RK4.10).

Figure A4.3.1: Adjustment paths for the Romer model: 'Euler finite differencing-GEMPACK' integration generated initial values, simulation E.3 specification from Table A4.3.1 ('R1' arm); second RK4 integration on a time step-size = 0.25.

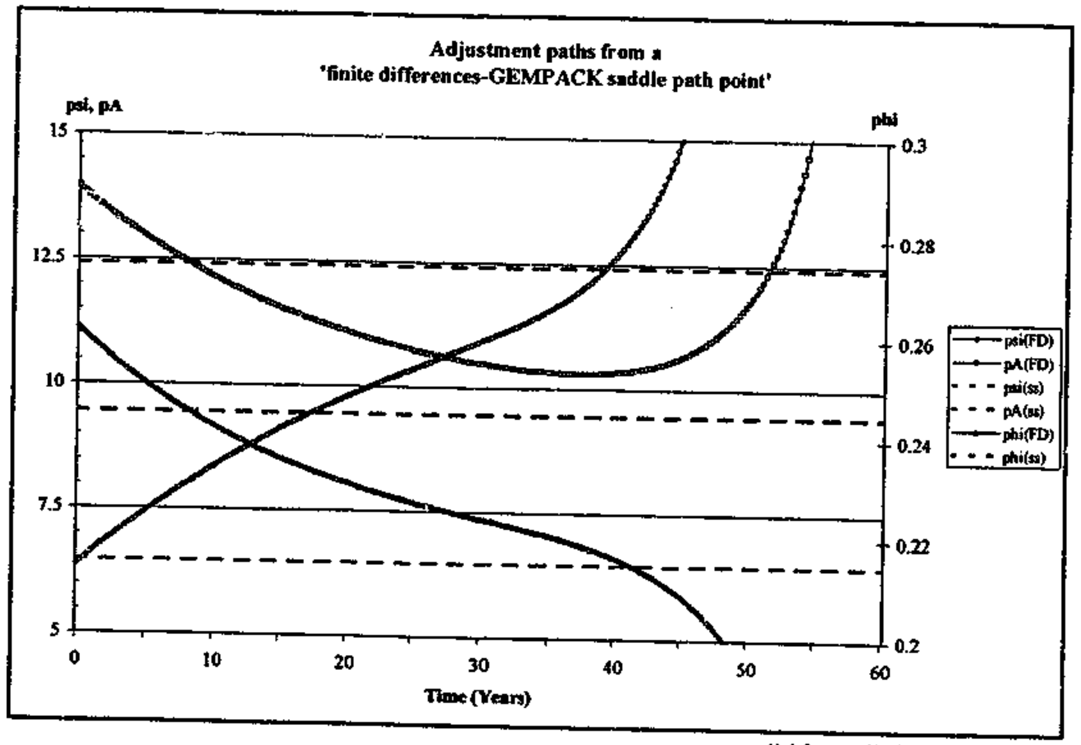

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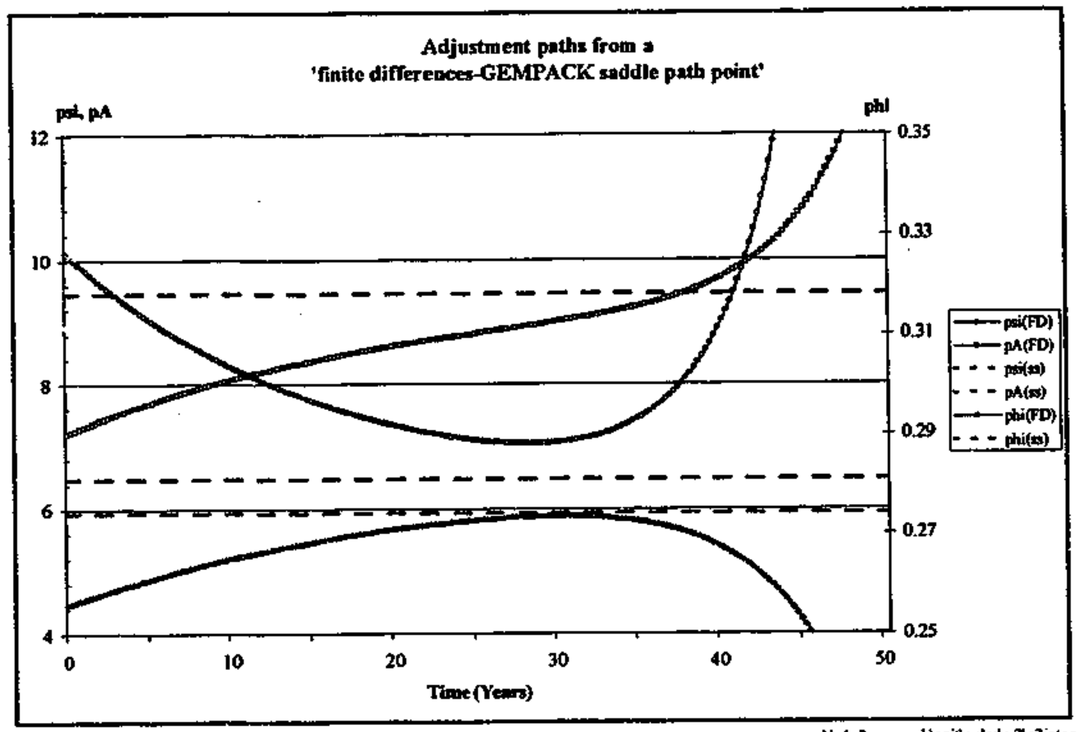
Figure A4.3.2: Adjustment paths for the Romer model: 'Euler finite differencing-GEMPACK' integration generated initial values, simulation E.4 specification from Table A4.3.1 ('R8' arm); second RK4 integration on a time step-size = 0.25.



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As a result of this analysis the finite differences-GEMPACK configuration chosen as the 'standard' for the investigation of the dynamics of the Romer model is that specified for simulations RK4.4 and RK4.6 in Table A4.3.1 above.[60] Namely:

> 4th order Runge-Kutta finite differencing with an uneven grid of 350 time intervals extending over 250 years and step-sizes varying between 0.125 and 5 years (grid "Uneven1"); and a GEMPACK solution method of Gragg, extrapolated from 12, 24, and 36 steps.

While use of the 12-24-36 step extrapolation over the 8-12-16 step one did not produce a vast improvement in the test simulations (RK4.3 and RK4.4), the former has been preferred for the standard specification since it is readily computable with the grid "Uneven1", and it allows for the possibility of discernibly greater accuracy in other simulations, particularly if larger shocks are involved. The re-integration performances of the R1 and R8 simulations based upon this standard specification are illustrated in Figure A4.3.3 and Figure A4.3.4 respectively.

It is notable that in these re-integrations of the model equations from initial points generated by the finite differences-GEMPACK approach, the best performance, in terms of the period for which the estimated saddle-path remains in its correct phase-space

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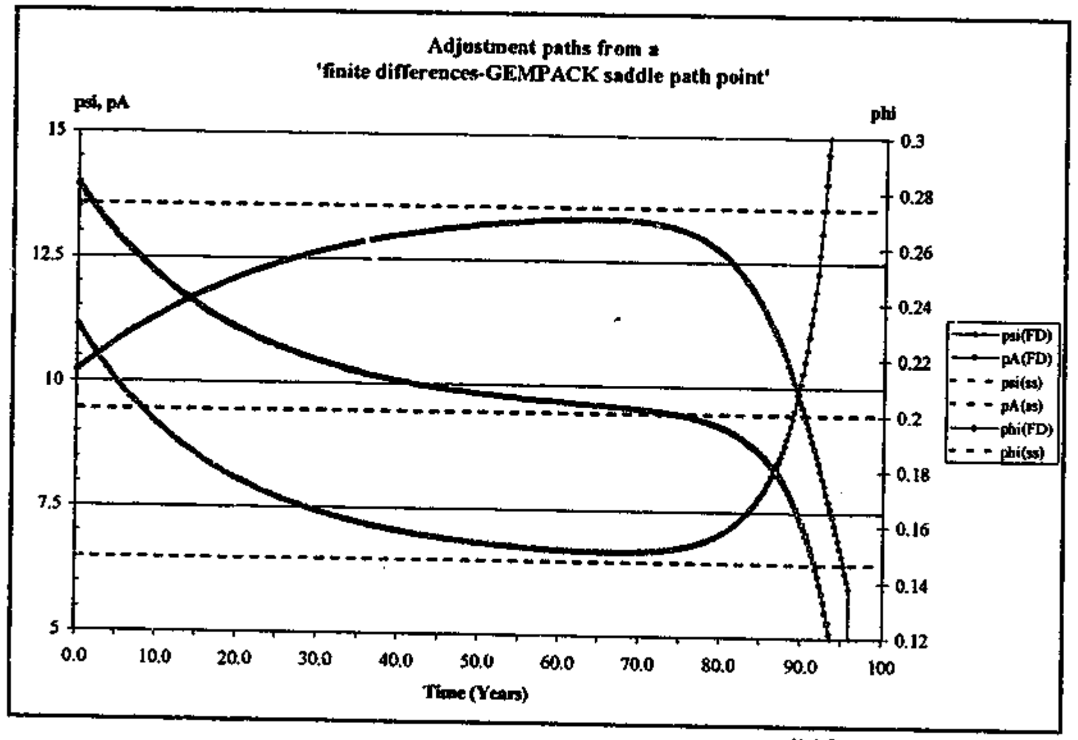
[60] For considerably bigger shocks both a finer grid and more extrapolation steps would be considered, perhaps the grid "Uneven2" and a 16-32-48 Gragg extrapolation method - computing capacity permitting.

region (R1 or R8), falls far short of that for initial points generated by the *time elimination* or the *eigenvector-backwards-integration* techniques (see Figure A4.2.15 and Figure A4.2.23 and associated text in Appendix 4.2).

The reason for this lies in the level of computing precision available from GEMPACK compared to that from Microsoft's Excel and Waterloo's Maple programs which were used in calculations with the other integration techniques. In the absence of *double precision,* the so-called level of machine accuracy for GEMPACK results is only six figures. Given the severe non-linearity and extreme sensitivity of the Romer model to initial conditions, this is simply not sufficient to define a point on the saddle-path closely enough to allow an integration from there to generate distant points which remain 'close' to the true saddle-path. Rather, such an integration tends to diverge fairly rapidly from the saddle-path.[61] Nevertheless, this is not a problem for the finite differences-GEMPACK method since, like the eigenvector-backwards-integration method, the fact that time paths are generated automatically means that all of the saddle-path points, that is all of the (Ψ, Φ, p_A)-triples, that can be calculated from them are 'more or less equally close' to the true saddle-path.

Figure A4.3.3: Adjustment paths for the Romer model: '4th order Runge-Kutta finite differencing-GEMPACK' integration generated initial values, simulation RK4.4 specification from Table A4.3.1 ('R1' arm); second RK4 integration on a time step-size = 0.25.



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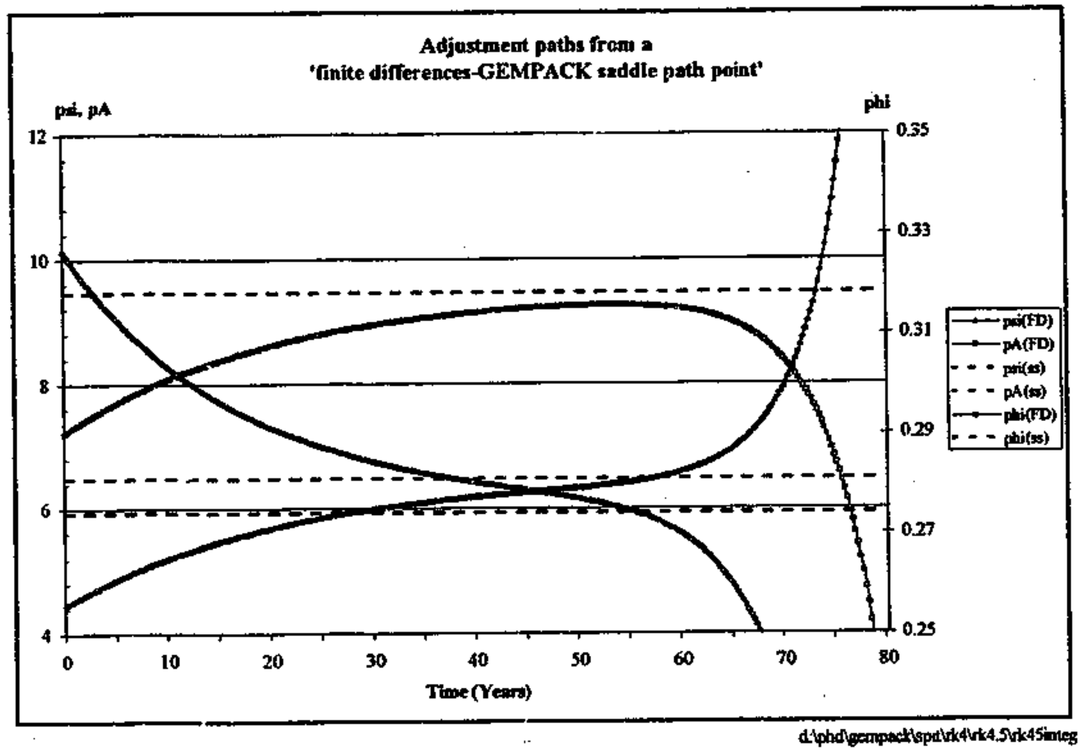
[61] It is also for this reason that the comparisons of the "time in R1 or R8", being based upon only 6-figure estimates not all of which are *accurate* to that degree, were described earlier as only providing "some measure of the accuracy of the finite-differences-GEMPACK results".

Figure A4.3.4: Adjustment paths for the Romer model: '4th order Runge-Kutta finite differencing-GEMPACK' integration generated initial values, simulation RK4.5 specification from Table A4.3.1 ('R8' arm); second RK4 integration on a time step-size = 0.25.



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Having now specified the finite differencing procedure and the grid upon which it is defined, the magnitude of the computing problem faced by GEMPACK can be calculated. In Section 4.3.2 the numbers of variables and equations in the model under fourth-order Runge-Kutta finite differencing were calculated as

$$m=14(T+1)+12T, \text{ and } n=5(T+1)+12T-1$$

respectively. Because one of the simulations undertaken in Section 4.5 involves imposing a shock to the variable H_A, it was also necessary to specify this variable explicitly (rather than continuing to write it as $(H-H_Y)$). It was then convenient to include H_A in the TABLO files for all simulations. As a result, in practice both the number of variables and the number of equations in the GEMPACK representation of the model were increased by (T+1) over the above amounts. Thus, for the uneven 350 interval grid used in the simulations the overall numbers of variables, equations, and exogenous variables were:

$$m=15(T+1)+12T = 9465;$$
$$n=6(T+1)+12T-1 = 6305 \text{ ; and}$$
$$m-n=9(T+1)+1 = 3160$$

From this and from equations (4.20) and (4.22) it can then be seen that in order to solve the simulation problems, GEMPACK must find the inverses of matrices of dimension 6305x6305 (the A^{-1} matrices); and that these must then be multiplied by matrices of dimension 6305x3160 (the **B** matrices); and then by the 3160x1 vectors of exogenous shocks.

A4.3.3 TABLO input and Command files for GEMPACK

A TABLO input file and a Command file are the two principal input files required to run the GEMPACK software (see footnotes 19 and 24 and associated text for further details and references). Here, the TABLO file defines the underlying dynamic Romer model upon which simulations are conducted. In particular, it defines all the variables, parameters and (non-linear) equations of the model and the particular finite differences technique and boundary conditions through which it is to be implemented. This involves the specification of sets for describing the grid of time points and intervals upon which the finite difference variables are defined and calculated. The number of grid points and the time step-sizes are specified in general terms by reference to generic or *logical* files containing the details (which can thereby easily be changed). In addition, the pre-simulation *base case* solution to the model, about which simulation results are reported as perturbations, is defined via instructions for the program to read certain data from another *logical* file and via specified formulae relating these data.

While a TABLO input file defines the underlying model and its finite differences implementation, details for a particular simulation on the model are then defined in a Command file. Thus, a Command file specifies the *closure* of the model (which variables are to be exogenous and which are to be endogenous), the simulation shock(s) to the exogenous variables and the multi-step method of solution of the initial value problem they pose. By defining specific data files corresponding to the generic ones referred to in the TABLO file, Command files also provide details of the grid upon which the finite differences variables and equations are constructed. Once GEMPACK has converted a TABLO input file into an executable program, that program may be run by reading an appropriate Command file. TABLO input files for the Euler and for the 4th order Runge-Kutta differencing methods are reproduced in Table A4.3.3 and Table A4.3.4, and an example Command file is presented in Table A4.3.5.

Table A4.3.3 TABLO input file specifying an Euler finite differencing method for modelling the dynamics of the decentralised market Romer model (file MKTE.TAB).

```
! ROMER MODEL DYNAMICS:  DECENTRALISED MARKET                              !
! GORDON SCHMIDT, 14 SEPTEMBER 1996.                                        !
! Solution of the "two-point boundary value problem" posed by the market solution !
! of the Romer model of endogenous growth:                                  !
!       The base case is a steady-state solution of the model, with the value of !
!       the capital/technology stock ratio equal to the desired initial level. !
!       The model is implemented in its "levels" form.                      !
!       Finite differencing is by the "Euler method".                       !

! Defaults for "levels model"                                               !
EQUATION(DEFAULT=LEVELS) ;
VARIABLE(DEFAULT=LEVELS) ;
FORMULA(DEFAULT=INITIAL) ;
```



```
COEFFICIENT(DEFAULT=PARAMETER) ;

! Number of grid intervals - representing the overall time horizon in years. To be !
!       read from a file of logical name "tperiods"                          !
FILE (TEXT) tperiods ;
COEFFICIENT (INTEGER) NINTERVAL ;
READ NINTERVAL FROM FILE tperiods ;

! Sets for describing periods                                                !
SET (INTERTEMPORAL) alltime # all time periods # MAXIMUM SIZE
2001 ( p[0] - p[NINTERVAL] ) ;
SET (INTERTEMPORAL) fwdtime # domain of fwd diffs # MAXIMUM SIZE
2000 ( p[0] - p[NINTERVAL - 1] ) ;
SET (INTERTEMPORAL) endtime # ending time # SIZE 1 ( p[NINTERVAL] ) ;
SUBSET fwdtime IS SUBSET OF alltime ;
SUBSET endtime IS SUBSET OF alltime ;

! Variables                                                                  !
VARIABLE (all,t,alltime) psi(t)     # capital/technology ratio      # ;
VARIABLE (all,t,alltime) phi(t)     # consumption/capital ratio     # ;
VARIABLE (all,t,alltime) pA(t)      # price of technology           # ;
VARIABLE (all,t,alltime) HY(t)      # human capital in mfg          # ;
VARIABLE (all,t,alltime) r(t)       # interest rate                 # ;
VARIABLE (all,t,alltime) H(t)       # total human capital           # ;
VARIABLE (all,t,alltime) L(t)       # total ordinary labour         # ;

! Parameters:  declared as variables, and as functions of time, in order to examine !
!       the dynamics of changes 'between' equilibria.                        !
VARIABLE (all,t,alltime) a(t)       # human cap productivity param  # ;
VARIABLE (all,t,alltime) g(t)       # capital productivity param    # ;
VARIABLE (all,t,alltime) z(t)       # research productivity param   # ;
VARIABLE (all,t,alltime) e(t)       # specialised cap cost param    # ;
VARIABLE (all,t,alltime) rho(t)     # consumers' discount factor    # ;
VARIABLE (all,t,alltime) s(t)       # inter-temporal substn param   # ;
VARIABLE (all,t,alltime) d(t)       # depreciation rate on capital  # ;

! Base case: established by a combination of READ statements from a file of logical !
!       name "basedata", and dependent FORMULA(INITIAL) statements.          !
FILE (TEXT)  basedata ;
```

```
READ H FROM FILE basedata ;
READ L FROM FILE basedata ;
READ a FROM FILE basedata ;
READ g FROM FILE basedata ;
READ z FROM FILE basedata ;
READ e FROM FILE basedata ;
READ rho FROM FILE basedata ;
READ s FROM FILE basedata ;
READ d FROM FILE basedata ;

FORMULA (all,t,alltime)
  HY(t) = [a(t)*s(t)/g(t)*H(t)+a(t)*rho(t)/g(t)/z(t)]/[1+a(t)*s(t)/g(t)] ;
FORMULA (all,t,alltime)
  r(t) = (g(t)*z(t)/a(t))*HY(t) ;
FORMULA (all,t,alltime)
  psi(t) = [g(t)^2/e(t)^g(t)/{r(t)+d(t)}*HY(t)^{a(t)*(1-g(t))}
  *L(t)^{(1-a(t))*(1-g(t))}]^{1/(1-g(t))};
FORMULA (all,t,alltime)
  phi(t) = {r(t)+d(t)}/g(t)^2-z(t)*{H(t)-HY(t)}-d(t) ;
FORMULA (all,t,alltime)
  pA(t) = {1/g(t)-1}*{r(t)+d(t)}/r(t)*psi(t) ;
```

```
! Time intervals: grid points, representing years, to be read from a file   !
!       of logical name "time".                                              !
```

```
FILE (TEXT) time ;
COEFFICIENT (all,t,alltime) year(t) ;
READ year FROM FILE time ;
COEFFICIENT (all,t,fwdtime) dt(t) ;
FORMULA (all,t,fwdtime) dt(t) = year(t+1) - year(t) ;
```

```
! Dynamic equations: relating variables at adjacent time points.             !
```

```
EQUATION psidot (all,t,fwdtime)
  psi(t+1) = psi(t)+dt(t)*[{r(t)+d(t)}/g(t)^2-phi(t)-d(t)-z(t)*{H(t)-HY(t)}]*psi(t) ;
EQUATION phidot (all,t,fwdtime)
  phi(t+1) = phi(t)+dt(t)*[{r(t)-rho(t)}/s(t)-{r(t)+d(t)}/g(t)^2+phi(t)+d(t)]*phi(t) ;
EQUATION pAdot (all,t,fwdtime)
  pA(t+1) = pA(t)+dt(t)*[r(t)*pA(t)-{1/g(t)-1}*{r(t)+d(t)}*psi(t)] ;
```

```
! Intraperiod equations: relationships existing over all points of time.     !
```

```
EQUATION humcapY (all,t,alltime)
  HY(t)=[a(t)*(1-g(t))/z(t)/e(t)^g(t)*L(t)^{(1-a(t))*(1-g(t))}*
        psi(t)^g(t)/pA(t)]^{1/(1-a(t)*(1-g(t)))};
EQUATION irate (all,t,alltime)
  r(t)=[z(t)*g(t)^2/a(t)/(1-g(t))]*HY(t)*pA(t)/psi(t)-d(t) ;
```



```
! Boundary conditions:                                                       !
! 1.   Initial capital/technology ratio fixed at its immediate pre-shock level: !
!      psi(0)=psi0.This is implemented simply by declaring psi(0) exogenous,  !
!      and setting its change(shock) to zero.                                !
! 2.   'Final' value of the consumption/capital ratio set at its market solution !
!      steady-state level: phi(T)=phiMss.                                    !
! 3.   'Final' value of the price of technology set at its market solution   !
!      steady- state level: pA(T)=pAMss.                                     !
EQUATION phiMss (all,t,endtime)
  phi(t) = {r(t)+d(t)}/g(t)^2-z(t)*{H(t)-HY(t)}-d(t) ;
EQUATION pAMss (all,t,endtime)
  pA(t) = {1/g(t)-1}*{r(t)+d(t)}/r(t)*psi(t) ;
```

Table A4.3.4 TABLO input file specifying a 4th order Runge-Kutta finite differencing method for modelling the dynamics of the decentralised market Romer model (file MKTRK4.TAB).

```
! ROMER MODEL DYNAMICS: DECENTRALISED MARKET                                 !
! GORDON SCHMIDT, 17 SEPTEMBER 1996.                                         !
! Solution of the "two-point boundary value problem" posed by the market solution !
! of the Romer model of endogenous growth:                                   !
!      The base case is a steady-state solution of the model, with the value of !
!      the capital/technology stock ratio equal to the desired initial level. !
!      The model is implemented in its "levels" form.                        !
!      Finite differencing is by the "4th order Runge-Kutta" method.         !
```

```
! Defaults for "levels model"                                                !
```

```
EQUATION(DEFAULT=LEVELS) ;
VARIABLE(DEFAULT=LEVELS) ;
FORMULA(DEFAULT=INITIAL) ;
COEFFICIENT(DEFAULT=PARAMETER) ;
```

```
! Number of grid intervals - representing the overall time horizon in years. To be !
!      read from a file of logical name "tperiods"                          !
```

```
FILE (TEXT) tperiods ;
COEFFICIENT (INTEGER) NINTERVAL ;
READ NINTERVAL FROM FILE tperiods ;
```

```
! Sets for describing periods                                              !
SET (INTERTEMPORAL) alltime # all time periods # MAXIMUM SIZE
2001 ( p[0] - p[NINTERVAL] ) ;
SET (INTERTEMPORAL) fwdtime # domain of fwd diffs # MAXIMUM SIZE
2000 ( p[0] - p[NINTERVAL - 1] ) ;
SET (INTERTEMPORAL) endtime # ending time # SIZE 1 ( p[NINTERVAL] ) ;
SUBSET fwdtime IS SUBSET OF alltime ;
SUBSET endtime IS SUBSET OF alltime ;
```

```
! Time intervals: grid points, representing years, to be read from a file  !
!        of logical name "time".                                            !
FILE (TEXT) time ;
COEFFICIENT (all,t,alltime) year(t) ;
READ year FROM FILE time ;
COEFFICIENT (all,t,fwdtime) dt(t) ;
FORMULA (all,t,fwdtime) dt(t) = year(t+1) - year(t) ;
```

```
! Variables                                                                 !
VARIABLE (all,t,alltime) psi(t)    # capital/technology ratio     # ;
VARIABLE (all,t,alltime) phi(t)    # consumption/capital ratio    # ;
VARIABLE (all,t,alltime) pA(t)     # price of technology          # ;
VARIABLE (all,t,alltime) HY(t)     # human capital in mfg         # ;
VARIABLE (all,t,alltime) HA(t)     # human capital in research    # ;
VARIABLE (all,t,alltime) r(t)      # interest rate                # ;
VARIABLE (all,t,alltime) H(t)      # total human capital          # ;
VARIABLE (all,t,alltime) L(t)      # total ordinary labour        # ;
```

```
! Intemediate variables: used in the RK4 finite differencing and, since their base  !
!        levels are zero, declared such that their linear equivalent                 !
!        is a 'change' variable rather than a 'percentage change' one.               !
VARIABLE (CHANGE) (all,t,fwdtime) psiR1(t) ;
VARIABLE (CHANGE) (all,t,fwdtime) phiR1(t) ;
VARIABLE (CHANGE) (all,t,fwdtime) pAR1(t) ;
VARIABLE (CHANGE) (all,t,fwdtime) psiR2(t) ;
VARIABLE (CHANGE) (all,t,fwdtime) phiR2(t) ;
VARIABLE (CHANGE) (all,t,fwdtime) pAR2(t) ;
VARIABLE (CHANGE) (all,t,fwdtime) psiR3(t) ;
VARIABLE (CHANGE) (all,t,fwdtime) phiR3(t) ;
VARIABLE (CHANGE) (all,t,fwdtime) pAR3(t) ;
VARIABLE (CHANGE) (all,t,fwdtime) psiR4(t) ;
VARIABLE (CHANGE) (all,t,fwdtime) phiR4(t) ;
```



```
VARIABLE (CHANGE) (all,t,fwdtime) pAR4(t) ;
```

```
! Parameters: declared as variables, and as functions of time, in order to examine  !
!        the dynamics of changes 'between' equilibria.                               !
VARIABLE (all,t,alltime) a(t)      # human cap productivity param   # ;
VARIABLE (all,t,alltime) g(t)      # capital productivity param     # ;
VARIABLE (all,t,alltime) z(t)      # research productivity param    # ;
VARIABLE (all,t,alltime) e(t)      # specialised cap cost param     # ;
VARIABLE (all,t,alltime) rho(t)    # consumers' discount factor     # ;
VARIABLE (all,t,alltime) s(t)      # inter-temporal substn param    # ;
VARIABLE (all,t,alltime) d(t)      # depreciation rate on capital   # ;
```

```
! Base case: established by a combination of READ statements from a file of logical  !
!        name "basedata", and dependent FORMULA(INITIAL) statements.                  !
FILE (TEXT) basedata ;
READ H FROM FILE basedata ;
READ L FROM FILE basedata ;
READ a FROM FILE basedata ;
READ g FROM FILE basedata ;
READ z FROM FILE basedata ;
READ e FROM FILE basedata ;
READ rho FROM FILE basedata ;
READ s FROM FILE basedata ;
READ d FROM FILE basedata ;

FORMULA (all,t,alltime)
 HY(t) = [a(t)*s(t)/g(t)*H(t)+a(t)*rho(t)/g(t)/z(t)]/[1+a(t)*s(t)/g(t)] ;
FORMULA (all,t,alltime)
 HA(t) = H(t)-HY(t) ;
FORMULA (all,t,alltime)
 r(t) = (g(t)*z(t)/a(t))*HY(t) ;
FORMULA (all,t,alltime)
 psi(t) = [g(t)^2/e(t)^g(t)/{r(t)+d(t)}*HY(t)^{a(t)*(1-g(t))}
 *L(t)^{(1-a(t))*(1-g(t))}]^{1/(1-g(t))};
FORMULA (all,t,alltime)
 phi(t) = {r(t)+d(t)}/g(t)^2-z(t)*HA(t)-d(t) ;
FORMULA (all,t,alltime)
 pA(t) = {1/g(t)-1}*{r(t)+d(t)}/r(t)*psi(t) ;
FORMULA (all,t,fwdtime)
 psiR1(t)=0 ;
FORMULA (all,t,fwdtime)
 phiR1(t)=0 ;
FORMULA (all,t,fwdtime)
 pAR1(t)=0 ;
FORMULA (all,t,fwdtime)
 psiR2(t)=0 ;
```

```
FORMULA  (all,t,fwdtime)
  phiR2(t)=0 ;
FORMULA  (all,t,fwdtime)
  pAR2(t)=0 ;
FORMULA  (all,t,fwdtime)
  psiR3(t)=0 ;
FORMULA  (all,t,fwdtime)
  phiR3(t)=0 ;
FORMULA  (all,t,fwdtime)
  pAR3(t)=0 ;
FORMULA  (all,t,fwdtime)
  psiR4(t)=0 ;
FORMULA  (all,t,fwdtime)
  phiR4(t)=0 ;
FORMULA  (all,t,fwdtime)
  pAR4(t)=0 ;
```

! **Dynamic equations:** relating variables at adjacent time points. !

```
EQUATION psiincr (all,t,fwdtime)
  psi(t+1)=psi(t)+psiR1(t)/6+psiR2(t)/3+psiR3(t)/3+psiR4(t)/6  ;
EQUATION phiincr (all,t,fwdtime)
  phi(t+1)=phi(t)+phiR1(t)/6+phiR2(t)/3+phiR3(t)/3+phiR4(t)/6  ;
EQUATION pAincr (all,t,fwdtime)
  pA(t+1) =pA(t)+pAR1(t)/6+pAR2(t)/3+pAR3(t)/3+pAR4(t)/6  ;

! where the RK4 variables are defined by the time derivative functions of the main   !
! dynamic variables as follows:                                                      !

EQUATION psiRK1 (all,t,fwdtime)
  psiR1(t)=dt(t)*[z(t)*{1+pA(t)/psi(t)/a(t)/(1-g(t))}*{a(t)*(1-g(t))/z(t)/e(t)^g(t)*
        L(t)^{(1-a(t))*(1-g(t))}*psi(t)^g(t)/pA(t)}^[1/(1-a(t)*(1-g(t)))]-
        phi(t)-d(t)-z(t)*H(t)]*psi(t) ;
EQUATION phiRK1 (all,t,fwdtime)
  phiR1(t)=dt(t)*[(g(t)^2/s(t)-1)*z(t)/a(t)/(1-g(t))*pA(t)/psi(t)*
        {a(t)*(1-g(t))/z(t)/e(t)^g(t)*L(t)^{(1-a(t))*(1-g(t))}*
        psi(t)^g(t)/pA(t)}^[1/(1-a(t)*(1-g(t)))]+
        phi(t)+{d(t)*(s(t)-1)-rho(t)}/s(t)]*phi(t) ;
EQUATION pARK1 (all,t,fwdtime)
  pAR1(t)=dt(t)*[z(t)*g(t)^2/a(t)/(1-g(t))*[pA(t)/psi(t)-(1-g(t))/g(t)]*
        {a(t)*(1-g(t))/z(t)/e(t)^g(t)*L(t)^{(1-a(t))*(1-g(t))}*
        psi(t)^g(t)/pA(t)}^[1/(1-a(t)*(1-g(t)))]-d(t)]*pA(t) ;

EQUATION psiRK2 (all,t,fwdtime)
  psiR2(t)=dt(t)*[z(t)*{1+[pA(t)+pAR1(t)/2]/[psi(t)+psiR1(t)/2]/a(t)/(1-g(t))}*
        {a(t)*(1-g(t))/z(t)/e(t)^g(t)*L(t)^{(1-a(t))*(1-g(t))}*
        [psi(t)+psiR1(t)/2]^g(t)/[pA(t)+pAR1(t)/2]}^
        [1/(1-a(t)*(1-g(t)))]-[phi(t)+phiR1(t)/2]-d(t)-z(t)*H(t)]*[psi(t)+psiR1(t)/2] ;
EQUATION phiRK2 (all,t,fwdtime)
```



```
  phiR2(t)=dt(t)*[(g(t)^2/s(t)-1)*z(t)/a(t)/(1-
g(t))*[pA(t)+pAR1(t)/2]/[psi(t)+psiR1(t)/2]*
        {a(t)*(1-g(t))/z(t)/e(t)^g(t)*L(t)^{(1-a(t))*(1-g(t))}*
        [psi(t)+psiR1(t)/2]^g(t)/[pA(t)+pAR1(t)/2]}^[1/(1-a(t)*(1-g(t)))]+
        [phi(t)+phiR1(t)/2]+{d(t)*(s(t)-1)-rho(t)}/s(t)]*[phi(t)+phiR1(t)/2] ;
EQUATION pARK2 (all,t,fwdtime)
  pAR2(t)=dt(t)*[z(t)*g(t)^2/a(t)/(1-g(t))*[[pA(t)+pAR1(t)/2]/[psi(t)+psiR1(t)/2]-
        (1-g(t))/g(t)]*{a(t)*(1-g(t))/z(t)/e(t)^g(t)*L(t)^{(1-a(t))*(1-g(t))}*
        [psi(t)+psiR1(t)/2]^g(t)/[pA(t)+pAR1(t)/2]}^[1/(1-a(t)*(1-g(t)))]-
        d(t)]*[pA(t)+pAR1(t)/2] ;

EQUATION psiRK3 (all,t,fwdtime)
  psiR3(t)=dt(t)*[z(t)*{1+[pA(t)+pAR2(t)/2]/[psi(t)+psiR2(t)/2]/a(t)/(1-g(t))}*
        {a(t)*(1-g(t))/z(t)/e(t)^g(t)*L(t)^{(1-a(t))*(1-g(t))}*
        [psi(t)+psiR2(t)/2]^g(t)/[pA(t)+pAR2(t)/2]}^[1/(1-a(t)*(1-g(t)))]-
        [phi(t)+phiR2(t)/2]-d(t)-z(t)*H(t)]*[psi(t)+psiR2(t)/2] ;
EQUATION phiRK3 (all,t,fwdtime)
  phiR3(t)=dt(t)*[(g(t)^2/s(t)-1)*z(t)/a(t)/(1-
g(t))*[pA(t)+pAR2(t)/2]/[psi(t)+psiR2(t)/2]*
        {a(t)*(1-g(t))/z(t)/e(t)^g(t)*L(t)^{(1-a(t))*(1-g(t))}*
        [psi(t)+psiR2(t)/2]^g(t)/[pA(t)+pAR2(t)/2]}^[1/(1-a(t)*(1-g(t)))]+
        [phi(t)+phiR2(t)/2]+{d(t)*(s(t)-1)-rho(t)}/s(t)]*[phi(t)+phiR2(t)/2] ;
EQUATION pARK3 (all,t,fwdtime)
  pAR3(t)=dt(t)*[z(t)*g(t)^2/a(t)/(1-g(t))*[[pA(t)+pAR2(t)/2]/[psi(t)+psiR2(t)/2]-
        (1-g(t))/g(t)]*{a(t)*(1-g(t))/z(t)/e(t)^g(t)*L(t)^{(1-a(t))*(1-g(t))}*
        [psi(t)+psiR2(t)/2]^g(t)/[pA(t)+pAR2(t)/2]}^[1/(1-a(t)*(1-g(t)))]-
        d(t)]*[pA(t)+pAR2(t)/2] ;

EQUATION psiRK4 (all,t,fwdtime)
  psiR4(t)=dt(t)*[z(t)*{1+[pA(t)+pAR3(t)]/[psi(t)+psiR3(t)]/a(t)/(1-g(t))}*
        {a(t)*(1-g(t))/z(t)/e(t)^g(t)*L(t)^{(1-a(t))*(1-g(t))}*
        [psi(t)+psiR3(t)]^g(t)/[pA(t)+pAR3(t)]}^[1/(1-a(t)*(1-g(t)))]-
        [phi(t)+phiR3(t)]-d(t)-z(t)*H(t)]*[psi(t)+psiR3(t)] ;
EQUATION phiRK4 (all,t,fwdtime)
  phiR4(t)=dt(t)*[(g(t)^2/s(t)-1)*z(t)/a(t)/(1-g(t))*[pA(t)+pAR3(t)]/[psi(t)+psiR3(t)]*
        {a(t)*(1-g(t))/z(t)/e(t)^g(t)*L(t)^{(1-a(t))*(1-g(t))}*
        [psi(t)+psiR3(t)]^g(t)/[pA(t)+pAR3(t)]}^[1/(1-a(t)*(1-g(t)))]+
        [phi(t)+phiR3(t)]+{d(t)*(s(t)-1)-rho(t)}/s(t)]*[phi(t)+phiR3(t)] ;
EQUATION pARK4 (all,t,fwdtime)
  pAR4(t)=dt(t)*[z(t)*g(t)^2/a(t)/(1-g(t))*[[pA(t)+pAR3(t)]/[psi(t)+psiR3(t)]-
        (1-g(t))/g(t)]*{a(t)*(1-g(t))/z(t)/e(t)^g(t)*L(t)^{(1-a(t))*(1-g(t))}*
        [psi(t)+psiR3(t)]^g(t)/[pA(t)+pAR3(t)]}^[1/(1-a(t)*(1-g(t)))]-
        d(t)]*[pA(t)+pAR3(t)] ;
```

! **Other intraperiod equations:** relationships existing over all points of time. !

```
EQUATION humcapY (all,t,alltime)
  HY(t)=[a(t)*(1-g(t))/z(t)/e(t)^g(t)*L(t)^{(1-a(t))*(1-g(t))}*
        psi(t)^g(t)/pA(t)]^{1/(1-a(t)*(1-g(t)))};
```

```
EQUATION humcapA (all,t,alltime)
 HA(t)=H(t)-HY(t) ;
EQUATION irate (all,t,alltime)
 r(t)=[z(t)*g(t)^2/a(t)/(1-g(t))]*HY(t)*pA(t)/psi(t)-d(t) ;
```

```
! Boundary conditions:                                                        !
! 1.  Initial capital/technology ratio fixed at its immediate pre-shock level:
!     psi(0)=psi0.This is implemented simply by declaring psi(0) exogenous,   !
!     and setting its change(shock) to zero.                                  !
! 2.  'Final' value of the consumption/capital ratio set at its market solution !
!     steady-state level: phi(T)=phiMss.                                      !
! 3.  'Final' value of the price of technology set at its market solution     !
!     steady- state level: pA(T)=pAMss.                                       !
```

```
EQUATION phiMss (all,t,endtime)
 phi(t) = {r(t)+d(t)}/g(t)^2-z(t)*HA(t)-d(t) ;
EQUATION pAMss (all,t,endtime)
 pA(t) = {1/g(t)-1}*{r(t)+d(t)}/r(t)*psi(t) ;
```

Table A4.3.5 GEMPACK Command file to run simulation RK4.4 as specified in Table A4.3.1

```
! GEMPACK Command file: for 'market solution Romer model dynamics'
! GORDON SCHMIDT, 11 NOVEMBER 1996:
! Solves the 'two-point boundary value problem' posed by the decentralised market
! solution of the Romer model of endogenous growth, by carrying out a multi-step
! simulation using the TABLO-generated program produced from MKTRK4.TAB.
!
!
! Input files and actions options:
BCV file = RK44 ;
NSC = yes ;
NSE = yes ;
NUD = yes ;
model = Market ;
version = 1 ;
identifier = 350 uneven intervals over 250 years ;
file basedata  = U1g10R1.dat ;
file time = U1.dat ;
file tperiods = U1tp.dat ;
updated file basedata  = U1g10R1.upd ;
!
! Note: NINTERVAL is read from the file with logical name 'tperiods'
```



```
!
! Method and steps:
method = gragg ;
steps = 12 24 36 ;
harwell parameter = 1.0 ;
!
! Output files:
solution file = RK44 ;
cumulatively-retained endogenous  p_psi  p_phi  p_pA ;
extrapolation accuracy file = yes ;
xac-retained  p_psi  p_phi  p_pA ;
!
! Closure:
exogenous  p_psi 1  p_H  p_L  p_a  p_g  p_z  p_e  p_rho  p_s  p_d ;
rest endogenous ;
!
! Shocks:
!shock p_H = uniform 10  ;
!shock p_L = uniform 10  ;
!shock p_a = uniform 10  ;
shock p_g = uniform -10  ;
!shock p_z = uniform 10  ;
!shock p_e = uniform 10  ;
!shock p_rho = uniform 10 ;
!shock p_s = uniform 10  ;
!shock p_d = uniform 10  ;
!
! Description of the simulation:
verbal description =
GENERAL: Examination of the dynamics and sensitivity of the (decentralised) market
solution of the Romer model, to changes in its parameters and its exogenous variables
(allowing some to occur AFTER time zero). Uneven 350 interval grid over a 250 year
time horizon - step-size varies from 0.1 to 10.0 (see RK44 description).
PARTICULAR: 10% fall in g from time zero (from 0.6 to benchmark of 0.54). Other
variables at benchmark data-set levels;
!
log file =RK44.log ;
```

Appendix 4.4

Comparison of the computed dynamic behaviours of the linearised Romer model, the Solowian-Romer model and the full non-linear Romer model

There being no analytic or 'closed form' solution to the dynamic Romer model of endogenous economic growth, three computable models have been used as approximations in order to calculate its transitional dynamics. These are: the Linearised Romer model (Section 3.2); a truncated version with exogenous savings, termed the Solowian-Romer model (Section 3.4); and the full non-linear model solved via the GEMPACK software (Section 4.3 and 4.4). Each of these has been confronted with the same three exogenous shocks: *an unanticipated and sustained 10 per cent rise in parameter γ; an anticipated and sustained 15 per cent increase in parameter ζ*; and *an unanticipated but known to be temporary fall of 20 per cent in parameter α*, all applied to the systems in equilibrium at their benchmark steady-state (Tables 2.2 and 3.2). The purpose of this Appendix is to compare the results of their computed transitional dynamics in response to these common shocks. Individual results for each model were presented in the Sections in which they were first analysed.[62] Here these are collected together and re-graphed in Figure A4.4.1 to Figure A4.4.26 for easy comparison.

It is worth recalling that the results for each model, including the full non-linear model, all involve different approximations. For the linearised model the results are exact computations for an approximation to the full non-linear model, and as such they are strictly only valid in the proximate neighbourhood of the linearisation point.[63] The further the system is away from this, the greater the errors in the approximation, particularly when the true system is highly non-linear as it is in the Romer model.

For the Solowian-Romer model the results are approximate computations for an approximation to the full model. Despite this 'double approximation' however, the S-R results may not be inferior to those of the linearised model. In fact they may even be better. For one thing, the S-R model maintains the significant non-linearity of the full model. And for another, whilst its approximation involves a truncation of the full model's demand side (with apparently different economics), recall that there were parameterisations for which the models were identical and that at least one of these had parameter values close to those of the empirically based benchmark parameter set (Appendix 3.7). Moreover, since direct numerical integration of the non-linear equations was used to solve the S-R model (there being no need to linearise the equations), these computations could be made highly accurate.

[62] Specifically: Section 3.2.2 and Appendix 3.4 for the linearised model; Section 3.4.2 for the Solowian-Romer model; and Sections 4.5.1 to 4.5.3 for the full Romer model with GEMPACK solution.

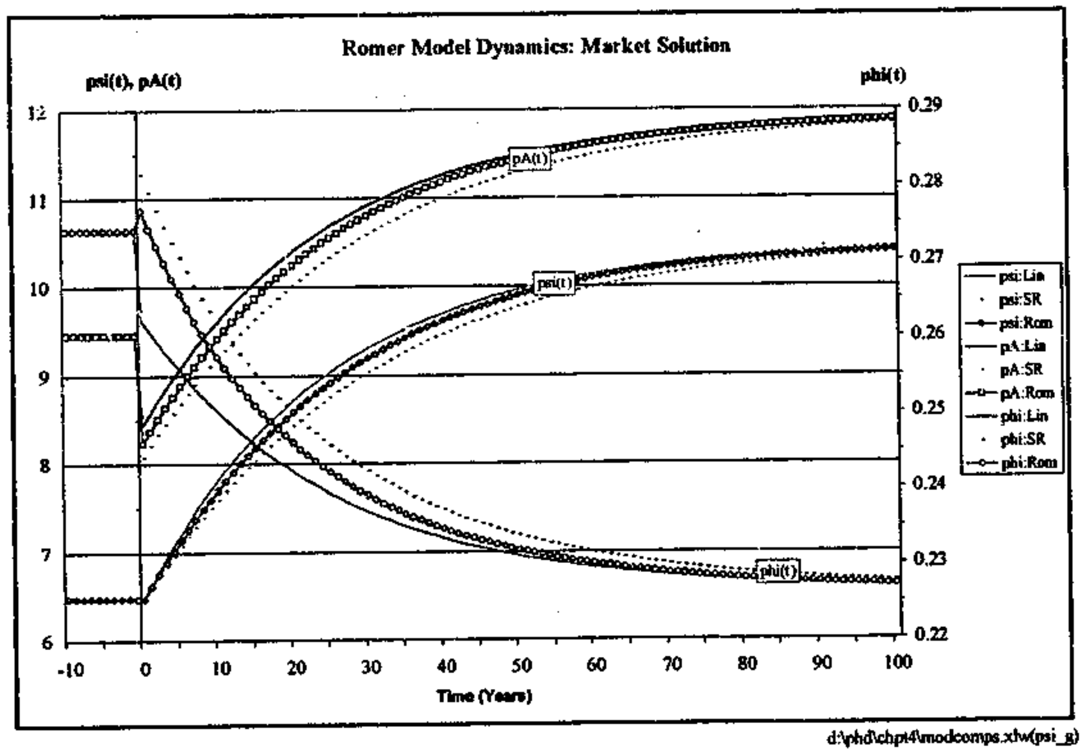
[63] For unanticipated shocks the only linearisation point is the new steady-state. In the case of anticipated shocks, however, there are two linearisation points - the steady-state prior to imposition of the anticipated shock, and that which applies after it has been imposed.

Finally, results for the full non-linear model are approximate computations for the exact model. Since the GEMPACK solution method involves linearisation of the equations, it is probably not as accurate as the direct numerical integration of the non-linear equations which is possible for the Solowian-Romer model. Nevertheless, the software facility that allows any shock to be broken into small components means that linearisation errors can be made arbitrarily small and that in the limit, it is the non-linear equations which are being solved. In practical terms this means that a very high degree of accuracy can be readily attained. Clearly these results are the most accurate of the three.

In some ways differences in the dynamics of the three 'models' are summarised by the principal dynamic variables (Ψ, Φ, and p_A). Since all the other variables are derived from these, 'modelling differences' between other variables reflect those in the principal ones.[64] However, differences in the derived dynamic variables also result from their functional forms and from the 'scaling' effect of parameter values which can, in particular, generate significant effects at the time of implementation of anticipated shocks. For these reasons comparison of the dynamics of the three 'models' has been made across the whole range of the standard variables analysed.

A4.4.1 An unanticipated and sustained 10% rise in parameter γ

Figure A4.4.1: Comparison of the linearised, S-R and full model dynamic effects on the principal dynamic variables Ψ, Φ, and p_A, of an unanticipated and sustained 10 per cent rise in parameter γ from time zero, benchmark parameter set.



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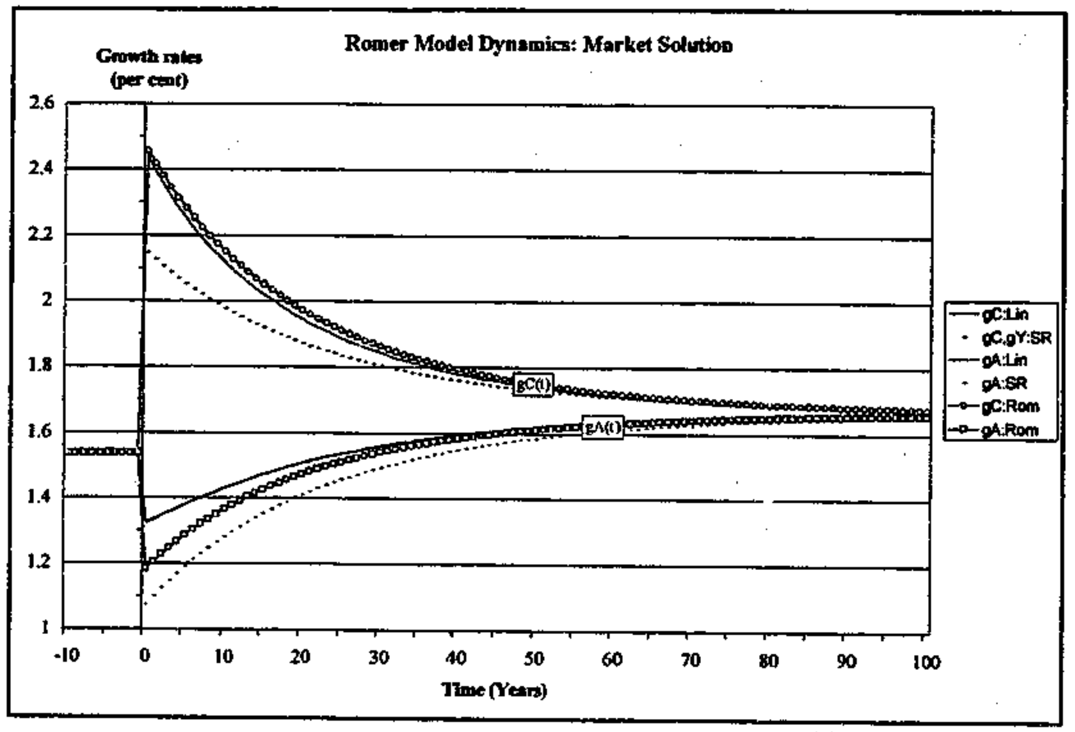
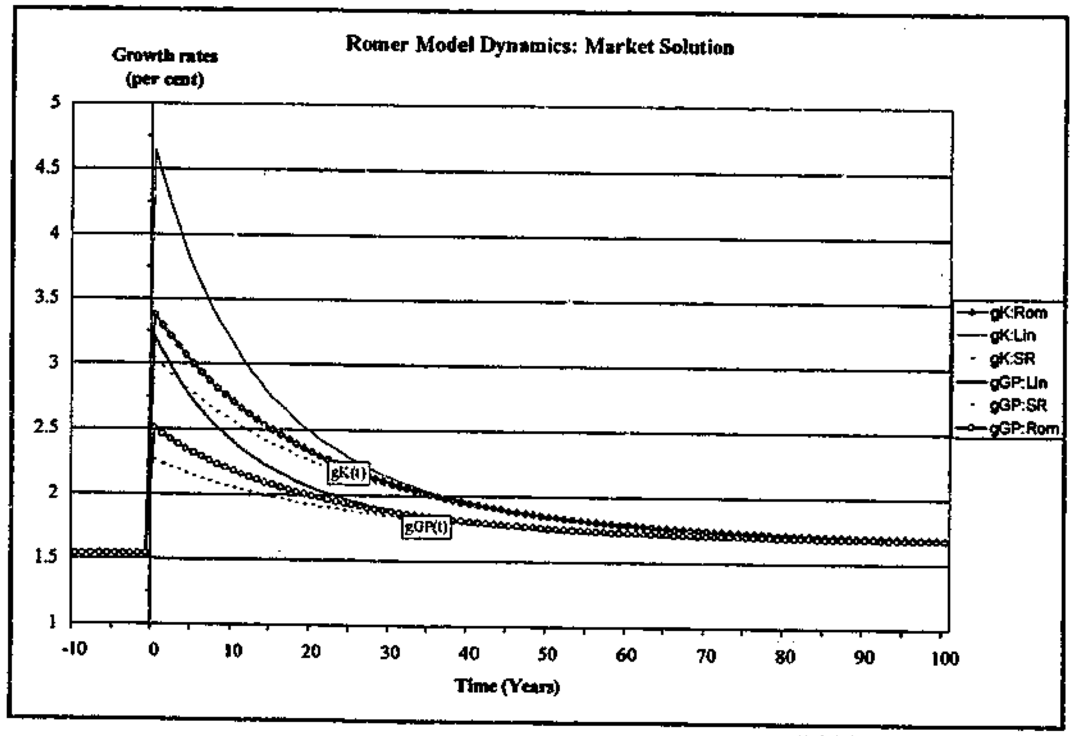
[64] In the case of the Solowian-Romer model Φ is also a derived variable.

Figure A4.4.2: Comparison of the linearised, S-R and full model dynamic effects on the growth rates g_A & g_C, of an unanticipated and sustained 10 per cent rise in parameter γ from time zero, benchmark parameter set.



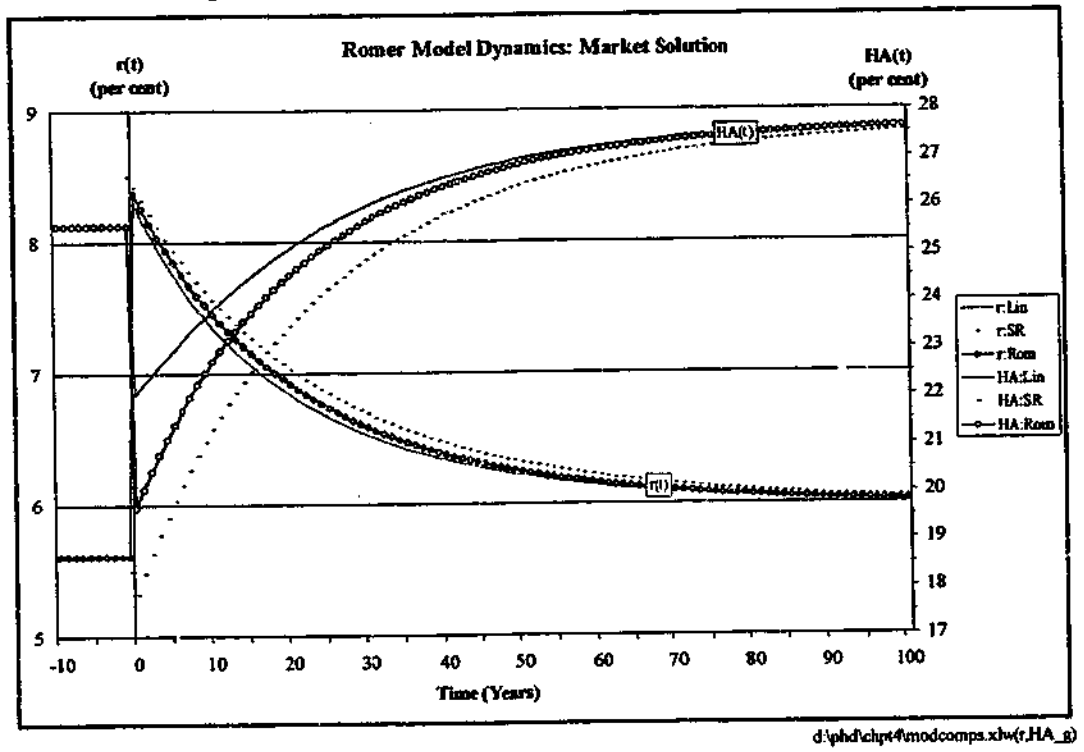
d:\phd\chpt4\modcomps.xlw(gAC_g)

Figure A4.4.3: Comparison of the linearised, S-R and full model dynamic effects on the growth rates g_K & g_{GP}, of an unanticipated and sustained 10 per cent rise in parameter γ from time zero, benchmark parameter set.



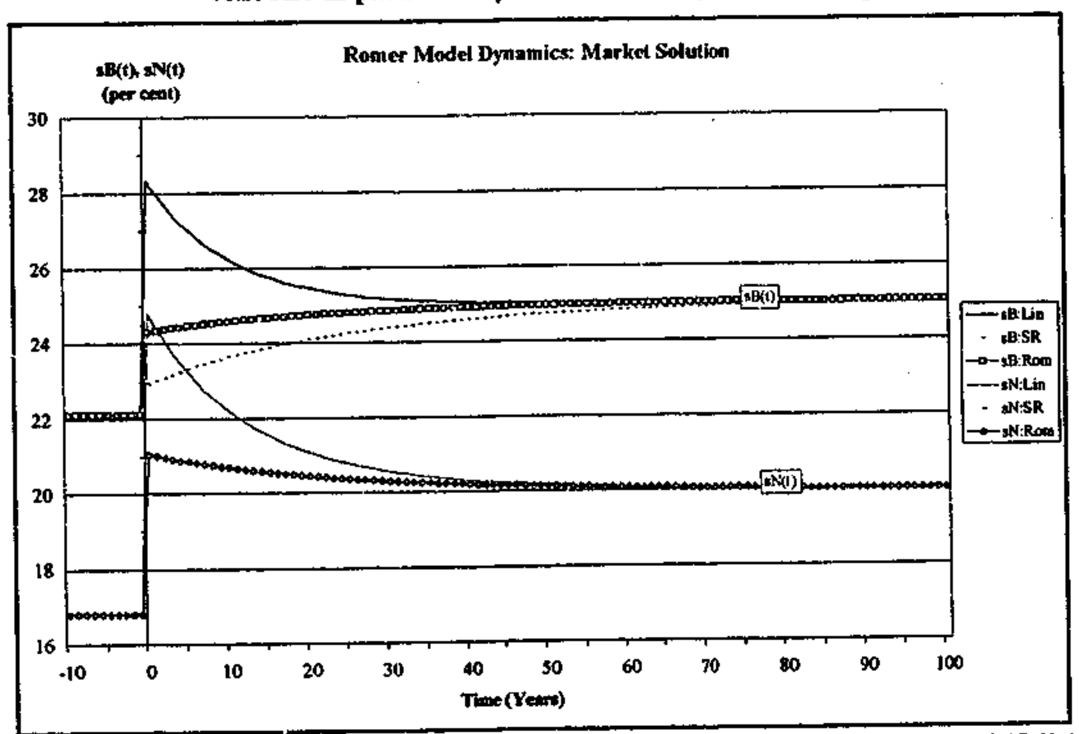
d:\phd\chpt4\modcomps.xlw(gKGP_g)

Figure A4.4.4: Comparison of the linearised, S-R and full model dynamic effects on r and H_A, of an unanticipated and sustained 10 per cent rise in parameter γ from time zero, benchmark parameter set.



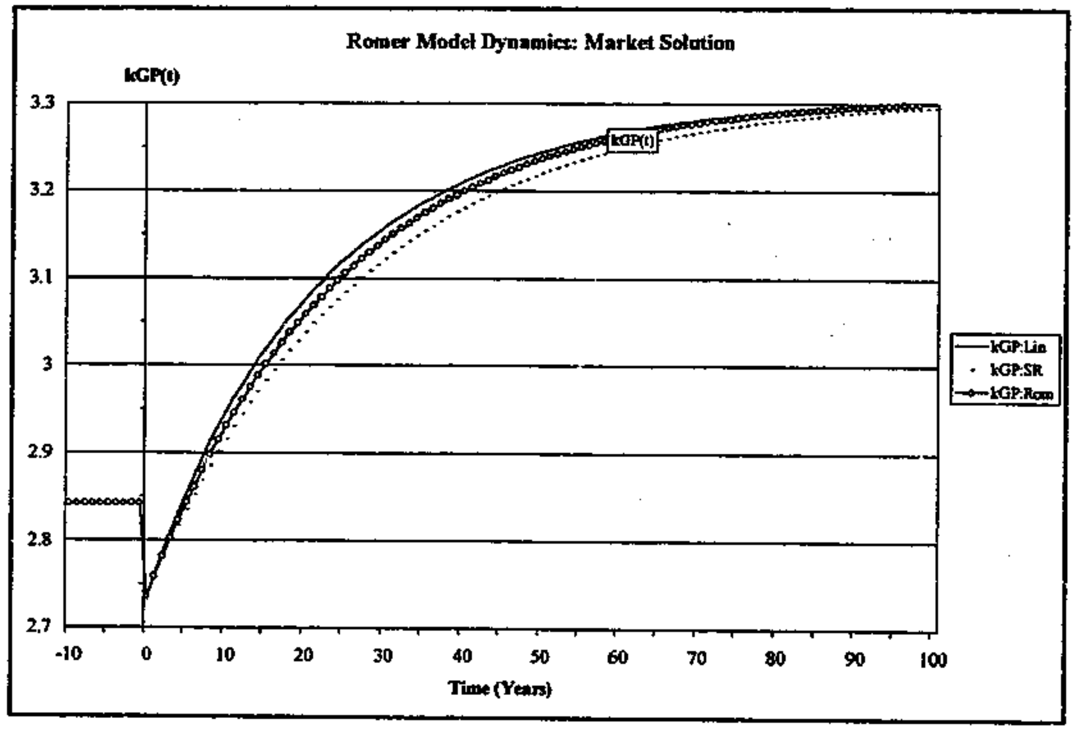
d:\phd\chpt4\modcomps.xlw(r,HA_g)

Figure A4.4.5: Comparison of the linearised, S-R and full model dynamic effects on the savings rates s_B and s_N, of an unanticipated and sustained 10 per cent rise in parameter γ from time zero, benchmark parameter set.



d:\phd\chpt4\modcomps.xlw(sB,sN_g)

Figure A4.4.6: Comparison of the linearised, S-R and full model dynamic effects on k_{GP}, of an unanticipated and sustained 10 per cent rise in parameter γ from time zero, benchmark parameter set.



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A few general points may be made in summarising the preceding results. Firstly, both the linearised and the Solowian-Romer models seem to be 'reasonably' good approximations to the full non-linear Romer model when the entire adjustment path is considered. For the variables Ψ, p_A, r, and k_{GP} both approximations were excellent; as was the linearised model approximation for g_C. Also, for all variables the discrepancy was never more than about 5 per cent after about 20 years and was usually much less. Even after only 10 years it was mostly below 10 per cent and usually not above 5 per cent. The exception in this latter case being for the linearised model result for the growth rate of capital g_K, for which the divergence was about 17 per cent (Figure A4.4.3). However, for periods less than 10 years the results were mixed. While there were many close correspondences, there were also some significant divergences.

Thus, as could have been expected the main differences arise in the initial jumps necessary to reach the saddle-paths and in the adjustments not too long thereafter. All three approaches generate the correct asymptotic results. In the case of the linearised model this is because the saddle-path is calculated analytically from the post shock steady-state. For the Solowian-Romer model it is due to the fact that the simulation also involves a shock to the exogenous savings rate s, which takes it to the final steady-state value of the savings variable s_N in the full model; that is $s_0=s_{Nss0}$ is shocked to $s_1=s_{Nss1}$, where s_{Nss0} is the original (benchmark) steady-state and s_{Nss1} is the steady-state resulting after the 10 per cent shock to γ. Finally, in the full non-linear case the correct asymptotic results are ensured because the finite differences method explicitly uses the post shock steady-state as part of its boundary conditions. All this is, of course, also true for the two (anticipated) shocks which follow in Sections A4.4.2 and A4.4.3.

In general the linearised model results tend to approach those of the full Romer model a little faster than do those of the S-R model, and in this way they are better approximations. The most significant case is for the adjustments of the allocation of human capital, H_A (Figure A4.4.4). But the linearised model results are not generally superior to those of the S-R model. They are clearly better for the growth rate of consumption, g_C (Figure A4.4.2); but they are clearly inferior for the growth rates of capital and gross product, g_K and g_{GP} (Figure A4.4.3); and also for the savings rates s_N and s_B (Figure A4.4.5).

Perhaps the most significant difference in the results occurs for the consumption-capital ratio Φ, the linearised solution producing an initial jump in the opposite direction to that generated by the finite differences non-linear solution and that of the Solowian-Romer model (Figure A4.4.1). Here, despite the fact that the linearised results converge to full model results faster than the S-R results do, it is the latter that seem the better approximation for most of the adjustment path.

A4.4.2 An anticipated and sustained 15% rise in parameter ζ

Figure A4.4.7: Comparison of the linearised, S-R and full model dynamic effects on the principal dynamic variable Ψ, of an anticipated and sustained 15 per cent rise in parameter ζ from time zero, benchmark parameter set.

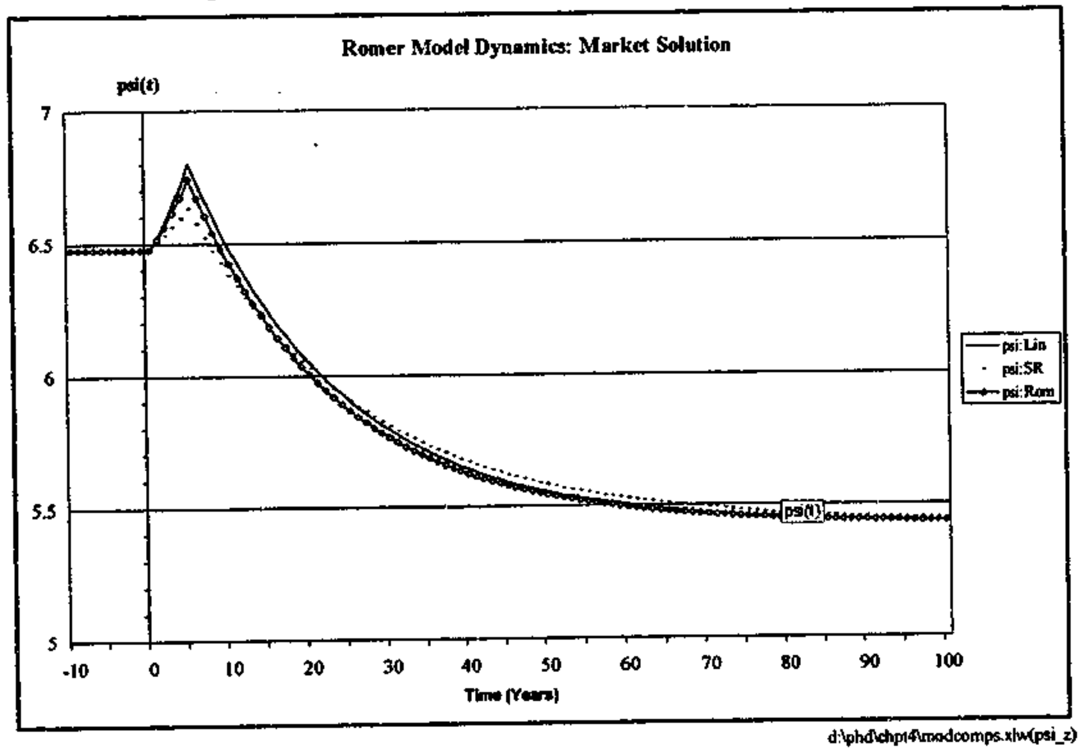

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Figure A4.4.8: Comparison of the linearised, S-R and full model dynamic effects on the principal dynamic variables Φ and p_A of an anticipated and sustained 15 per cent rise in parameter ζ from time zero, benchmark parameter set.

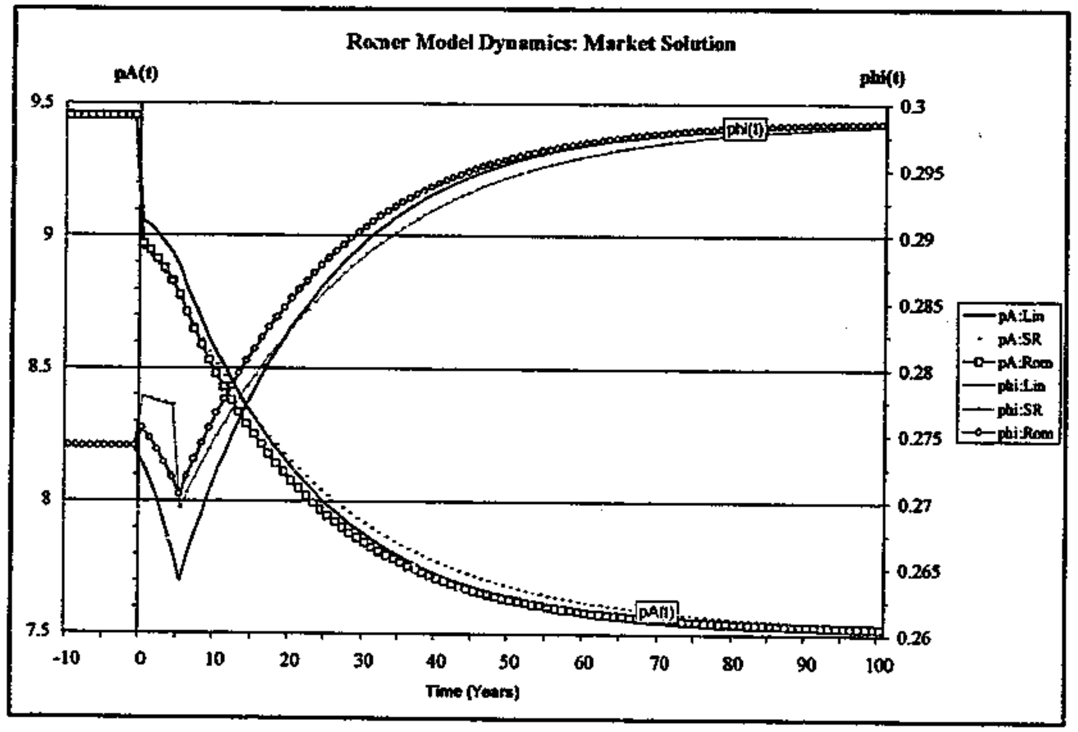

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Figure A4.4.9: Comparison of the linearised, S-R and full model dynamic effects on the growth rates g_A & g_K, of an anticipated and sustained 15 per cent rise in parameter ζ from time zero, benchmark parameter set.

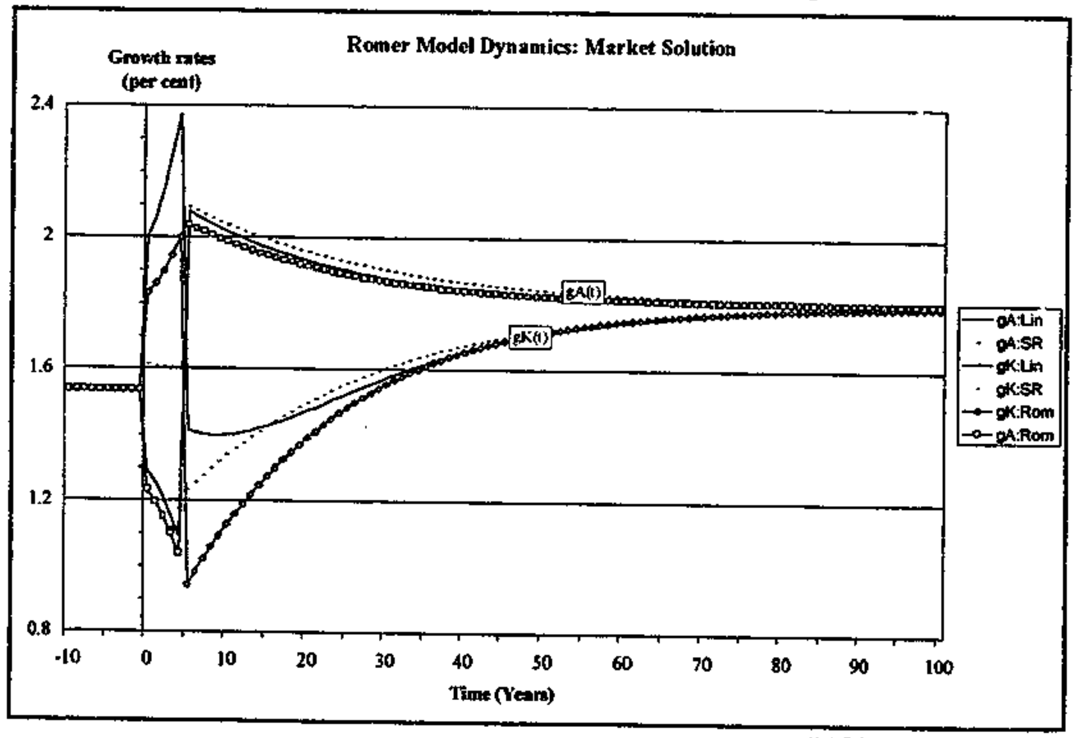

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Figure A4.4.10: Comparison of the linearised, S-R and full model dynamic effects on the growth rate g_C, of an anticipated and sustained 15 per cent rise in parameter ζ from time zero, benchmark parameter set.

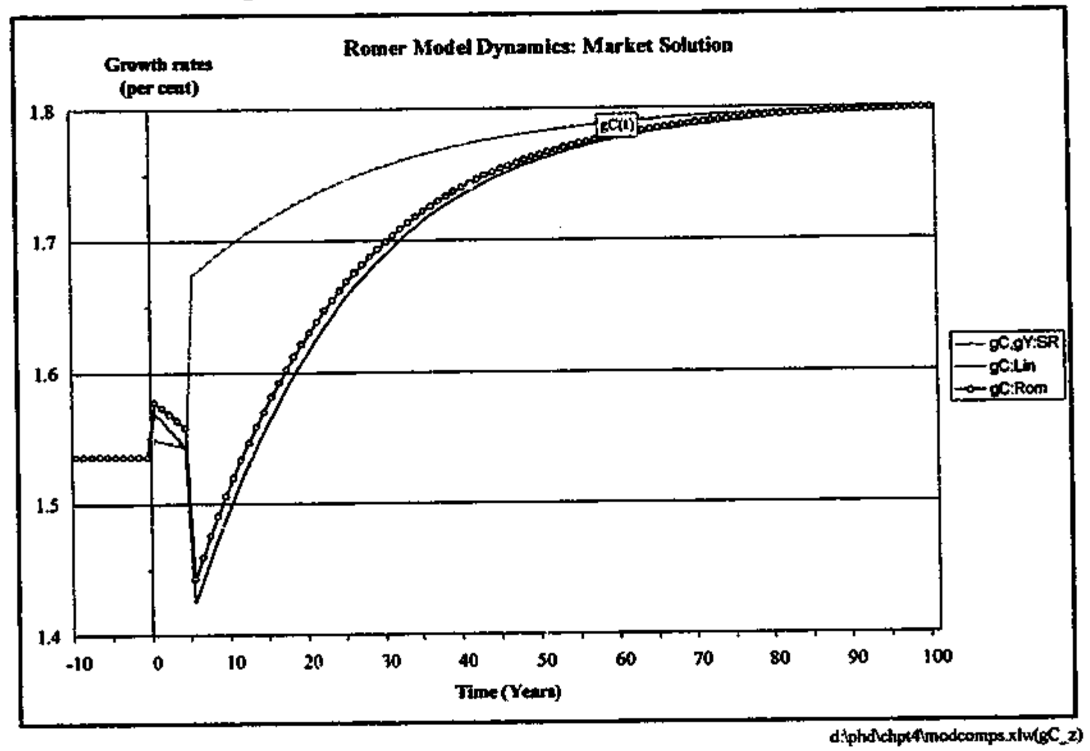

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Figure A4.4.11: Comparison of the linearised, S-R and full model dynamic effects on the growth rate of output (g_Y) of an anticipated and sustained 15 per cent rise in parameter ζ from time zero, benchmark parameter set.

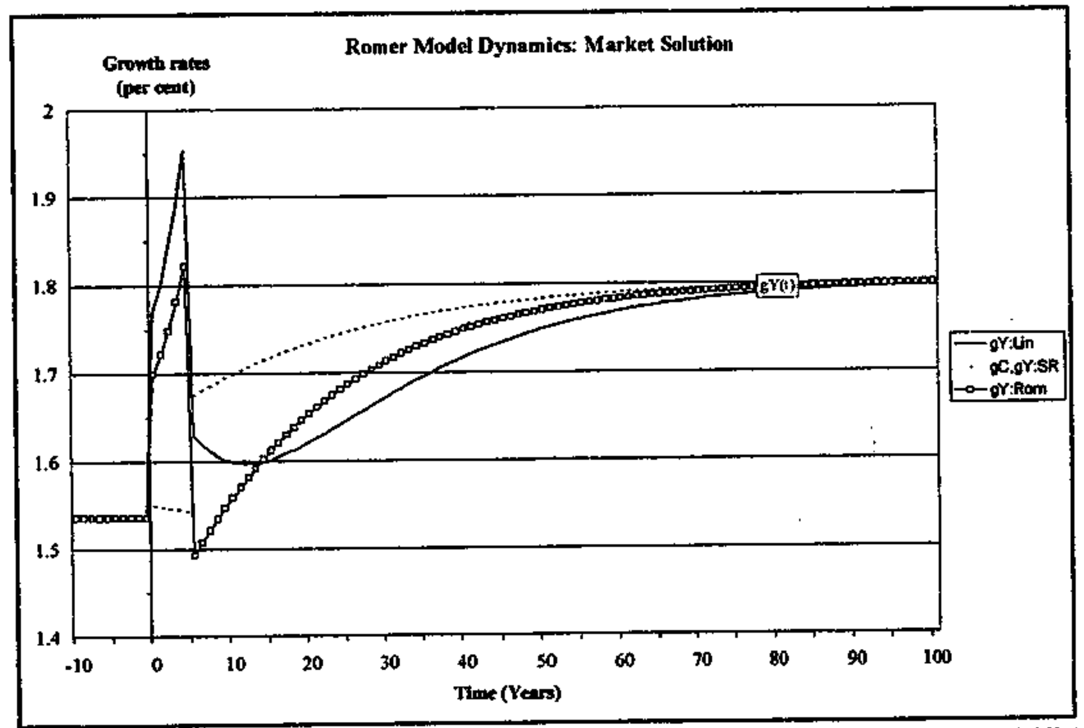

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Figure A4.4.12: Comparison of the linearised, S-R and full model dynamic effects on the growth rate g_{GP}, of an anticipated and sustained 15 per cent rise in parameter ζ from time zero, benchmark parameter set.

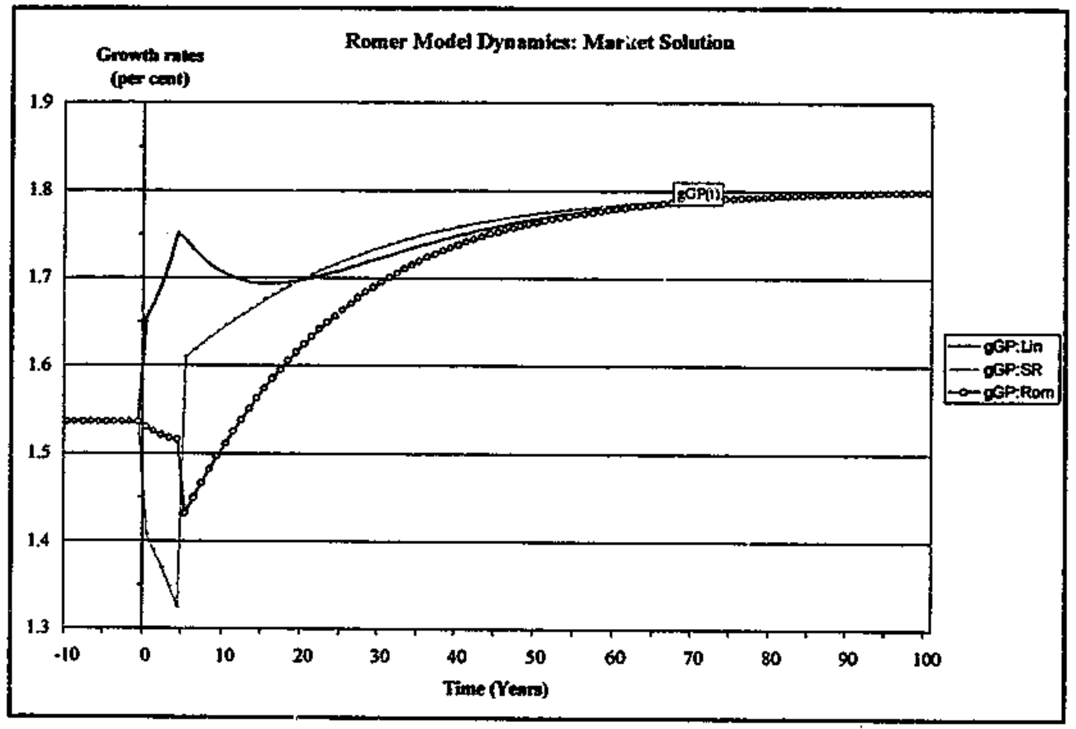

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Figure A4.4.13: Comparison of the linearised, S-R and full model dynamic effects on the interest rate (r) of an anticipated and sustained 15 per cent rise in parameter ζ from time zero, benchmark parameter set.

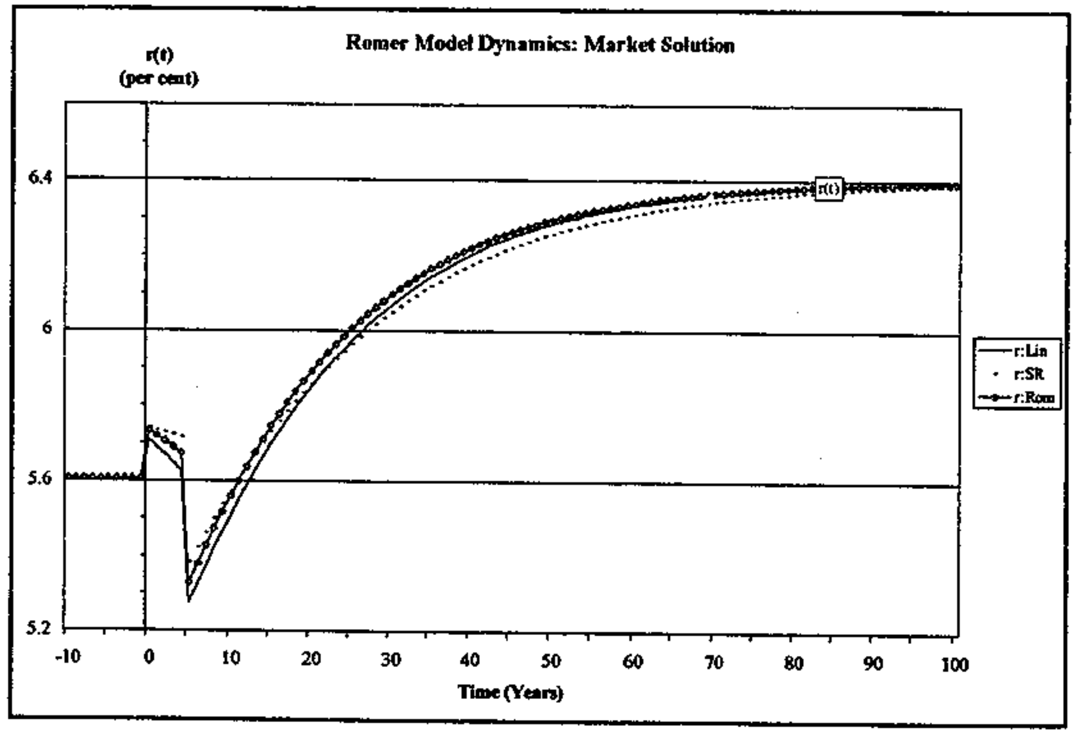

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Figure A4.4.14: Comparison of the linearised, S-R and full model dynamic effects on H_A, of an anticipated and sustained 15 per cent rise in parameter ζ from time zero, benchmark parameter set.

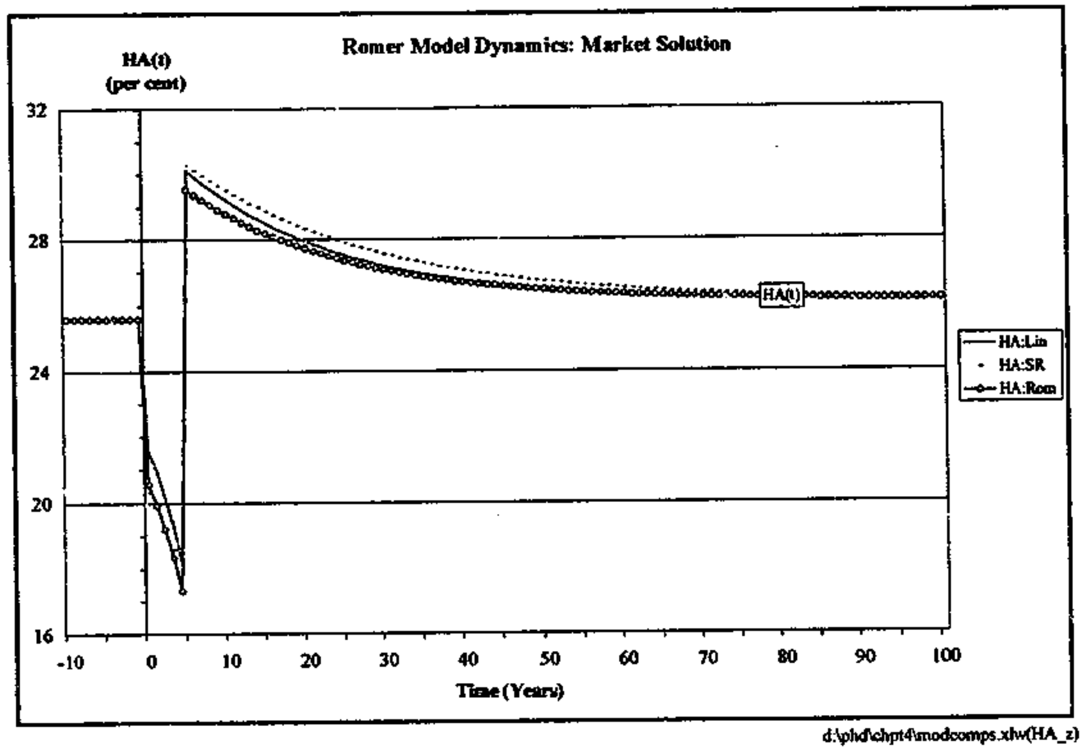

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Figure A4.4.15: Comparison of the linearised, S-R and full model dynamic effects on the savings rate s_N, of an anticipated and sustained 15 per cent rise in parameter ζ from time zero, benchmark parameter set.

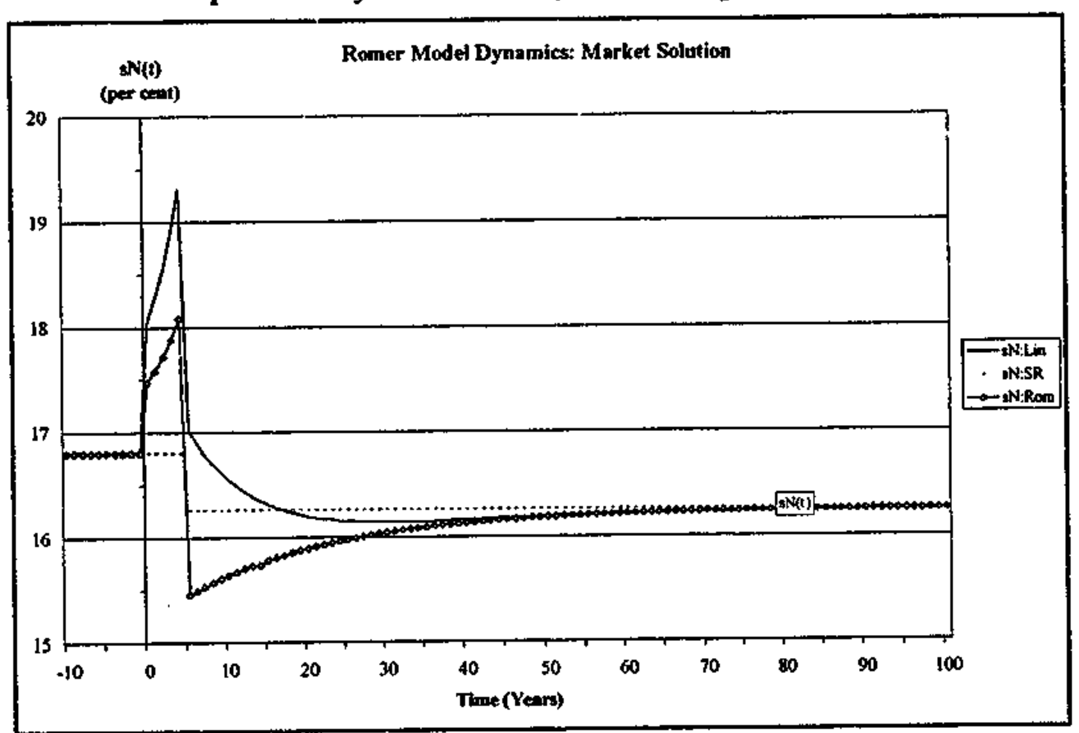

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Figure A4.4.16: Comparison of the linearised, S-R and full model dynamic effects on the savings rate s_B, of an anticipated and sustained 15 per cent rise in parameter ζ from time zero, benchmark parameter set.

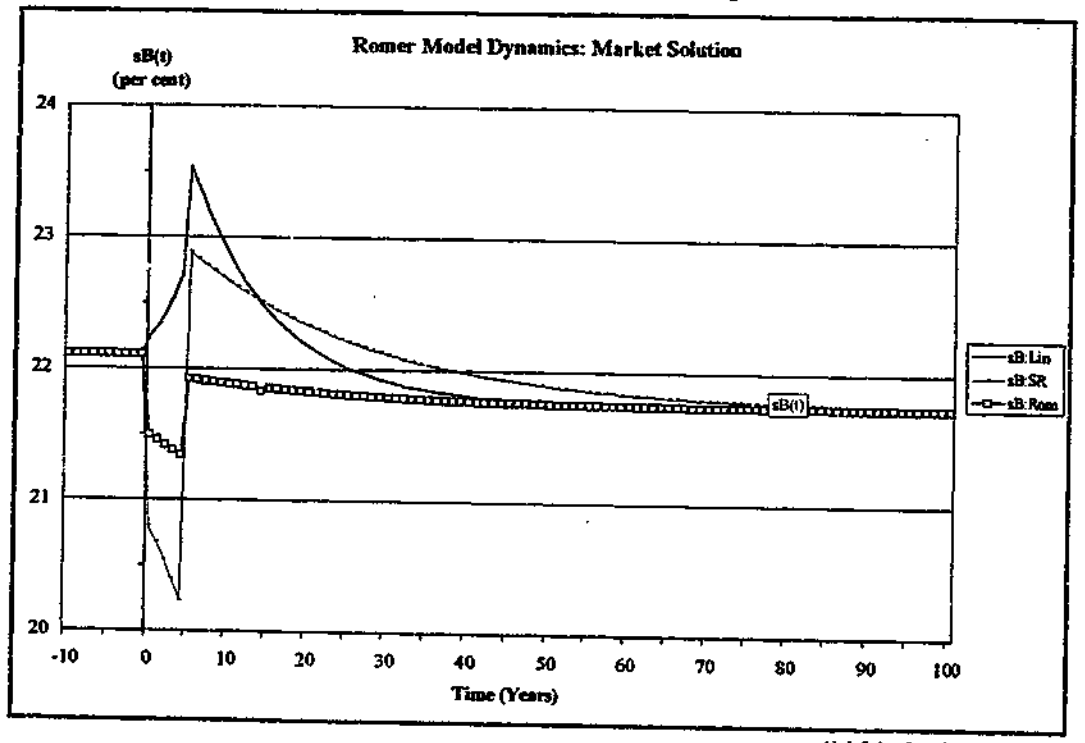

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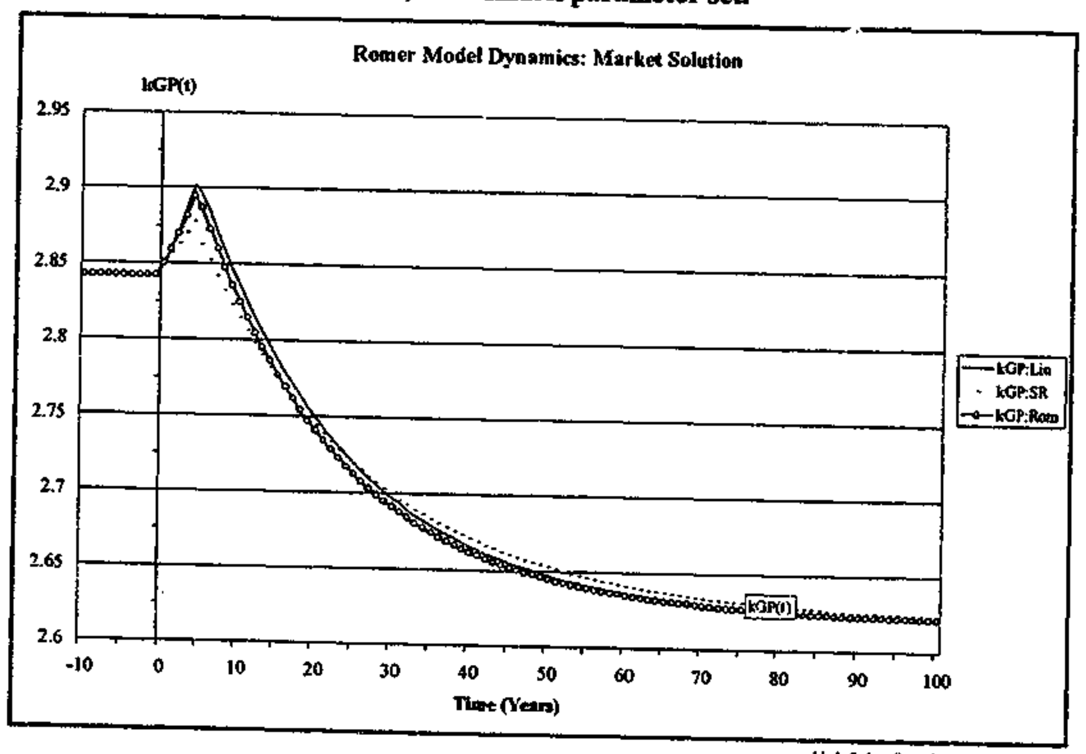
Figure A4.4.17: Comparison of the linearised, S-R and full model dynamic effects on k_{GP}, of an anticipated and sustained 15 per cent rise in parameter ζ from time zero, benchmark parameter set.



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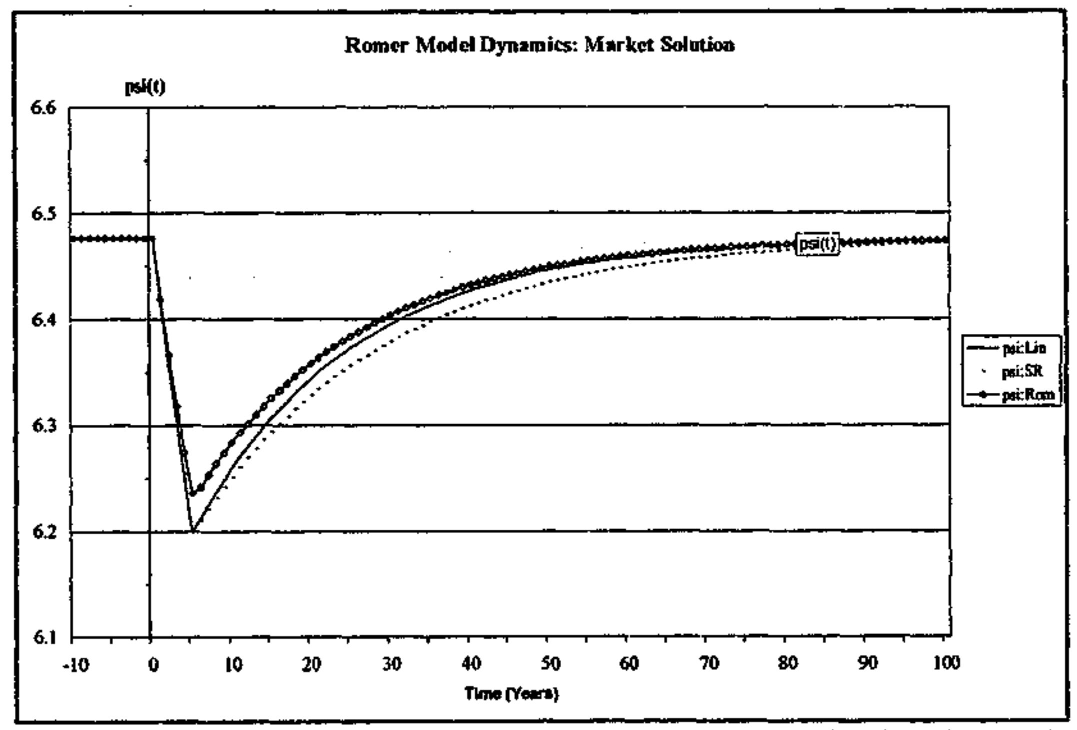
Overall, for this shock to ζ the results from the linearised and the S-R models produced even better approximations to the full Romer model than for the γ-shock.[65] Here the correspondence from both models was excellent for the variables Ψ, p_A, g_A, r, H_A and k_{GP}; as it also was for Φ in the case of the S-R model; and for g_C for the linearised one. Other general points made in respect of the γ-shock also apply here: Almost all of any discrepancies occur in the initial and 'implementation' jumps and in the adjustment soon afterwards;[66] and whether the linearised or the S-R model provides the better approximation to the full model remains ambiguous. Also, like the γ-shock, by twenty years after the ζ-shock (at time t=25 years) the discrepancies were always under 5 per cent; and ten years after it (at t=15 years) the only discrepancies greater than 10 per cent were both models' estimates in respect of g_K, where both were 12 per cent higher than the full model estimate.

A4.4.3 An unanticipated but temporary 20% fall in parameter α

Figure A4.4.18: Comparison of the linearised, S-R and full model dynamic effects on the principal dynamic variable Ψ of an unanticipated but temporary 20 per cent fall in parameter α from time zero, benchmark parameter set.



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[65] Note that the scales on the graphs need to be considered in assessing how close different estimates are to one another

[66] In the case of anticipated shocks there are now two key time points in the adjustment paths: the time from which anticipation first commences (previously referred to as the time of *announcement*), and the time at which the shock is implemented. Here these are times t=0 and t=5 years respectively. While the principal dynamic variables do not jump at implementation, the derived variables do.

Figure A4.4.19: Comparison of the linearised, S-R and full model dynamic effects on the principal dynamic variables Φ and p_A of an unanticipated but temporary 20 per cent fall in parameter α from time zero, benchmark parameter set.

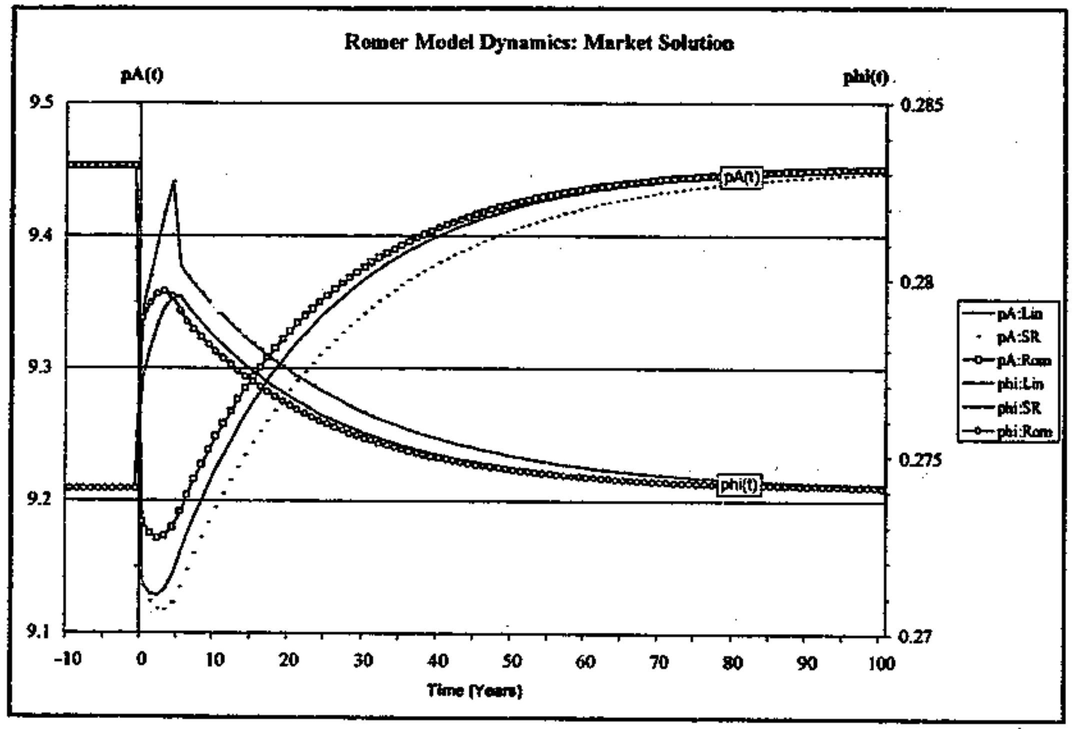

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Figure A4.4.20: Comparison of the linearised, S-R and full model dynamic effects on the growth rate g_K, of an unanticipated but temporary 20 per cent fall in parameter α from time zero, benchmark parameter set.

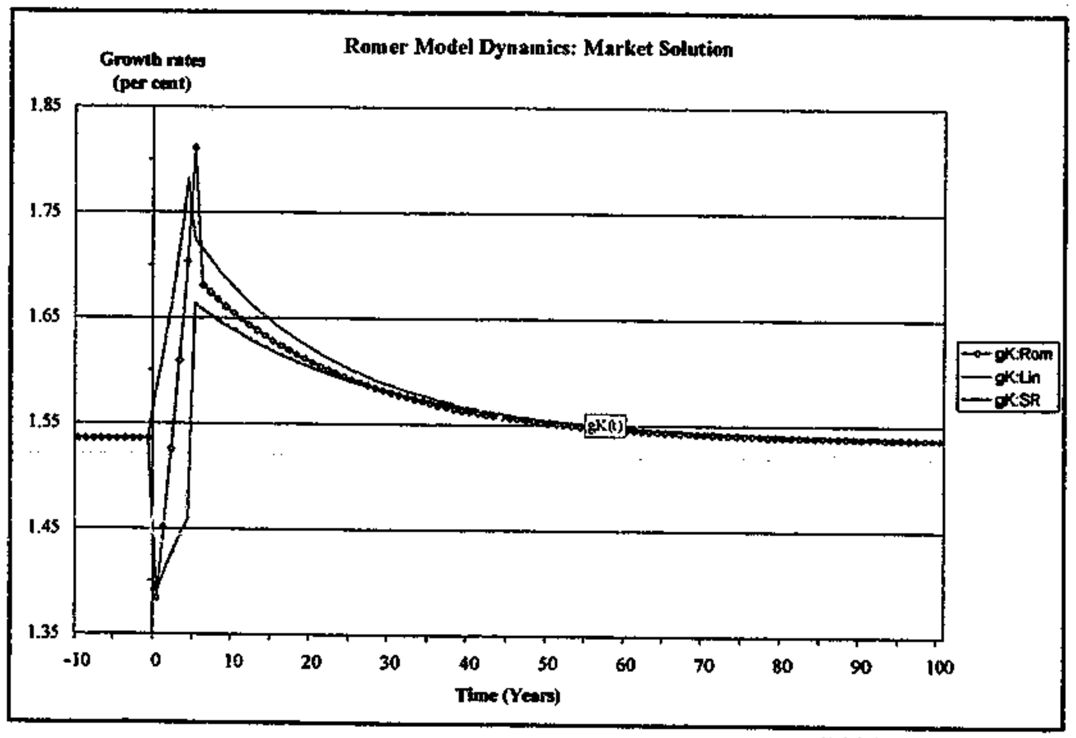

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Figure A4.4.21: Comparison of the linearised, S-R and full model dynamic effects on the growth rates g_C & g_Y, of an unanticipated but temporary 20 per cent fall in parameter α from time zero, benchmark parameter set.

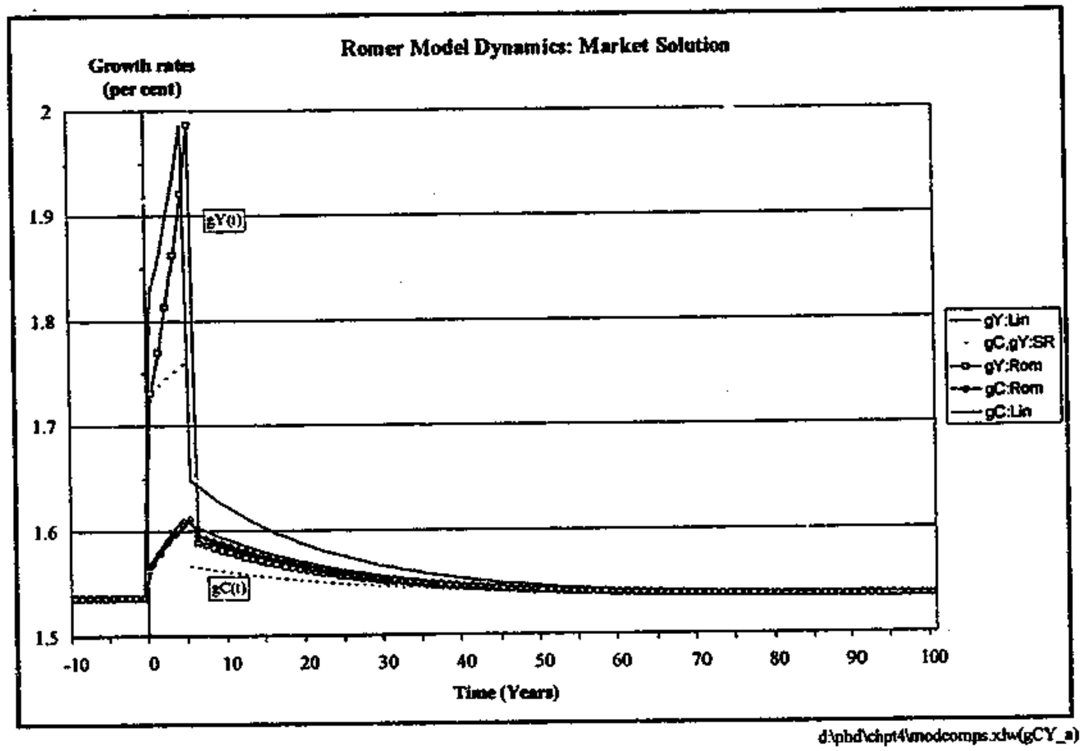

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Figure A4.4.22: Comparison of the linearised, S-R and full model dynamic effects on the growth rates g_A & g_{GP}, of an unanticipated but temp-orary 20 per cent fall in parameter α from time zero, benchmark parameter set.

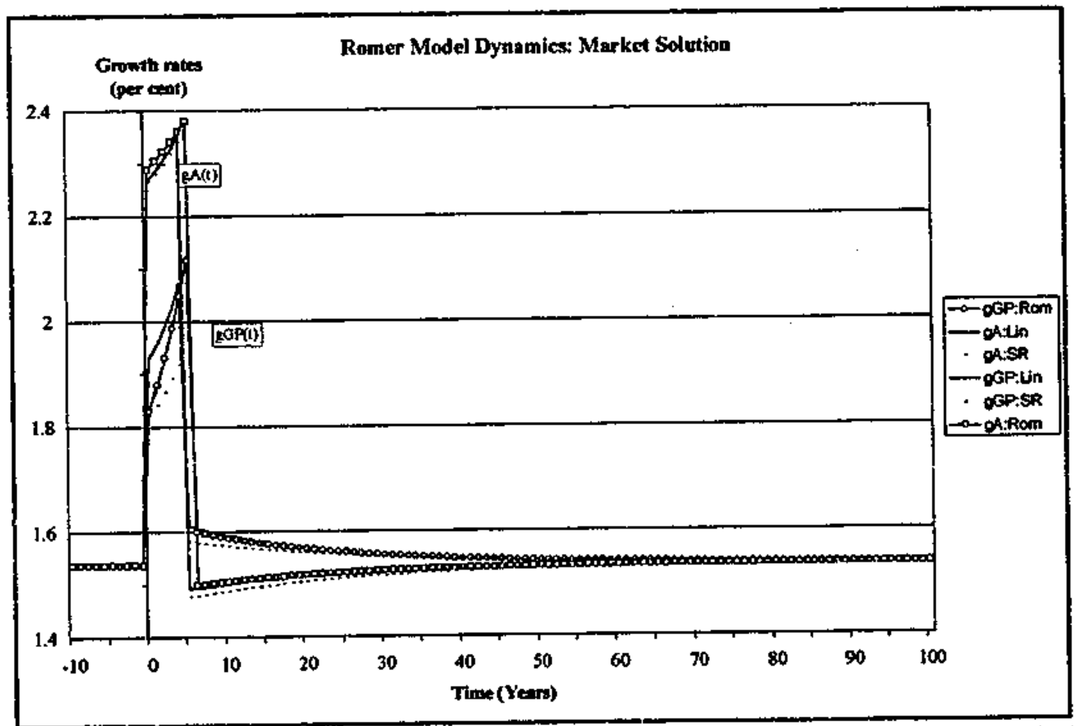

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Figure A4.4.23: Comparison of the linearised, S-R and full model dynamic effects on r & H_A, of an unanticipated but temporary 20 per cent fall in parameter α from time zero, benchmark parameter set.

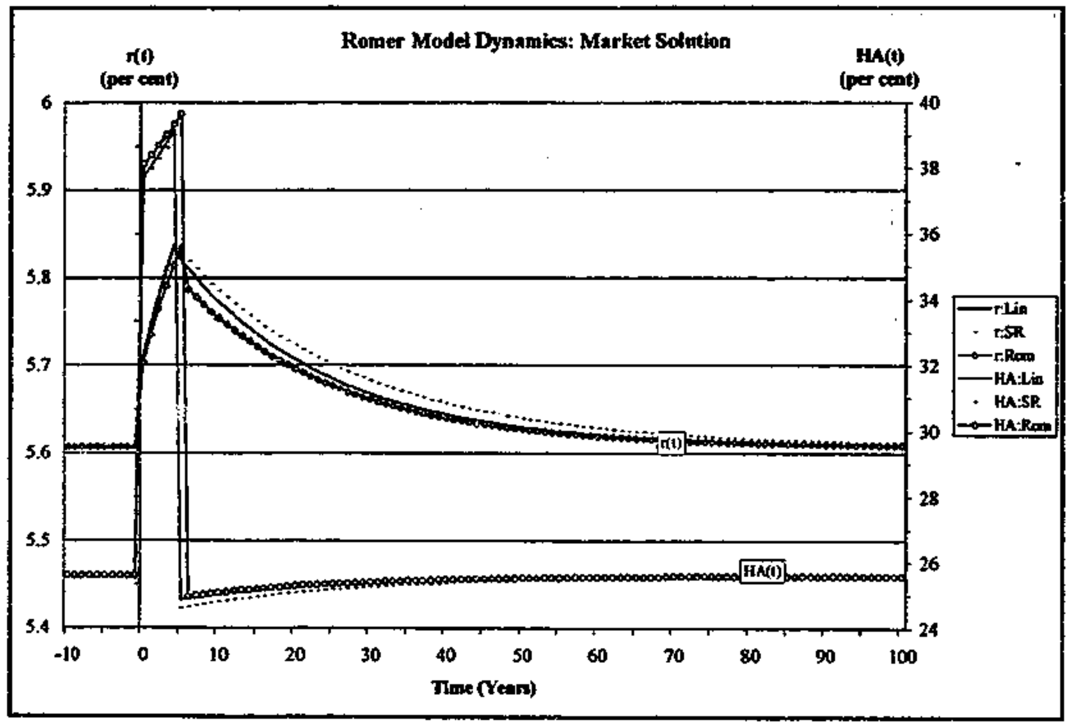

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Figure A4.4.24: Comparison of the linearised, S-R and full model dynamic effects on the savings rate s_N, of an unanticipated but temporary 20 per cent fall in parameter α from time zero, benchmark parameter set.

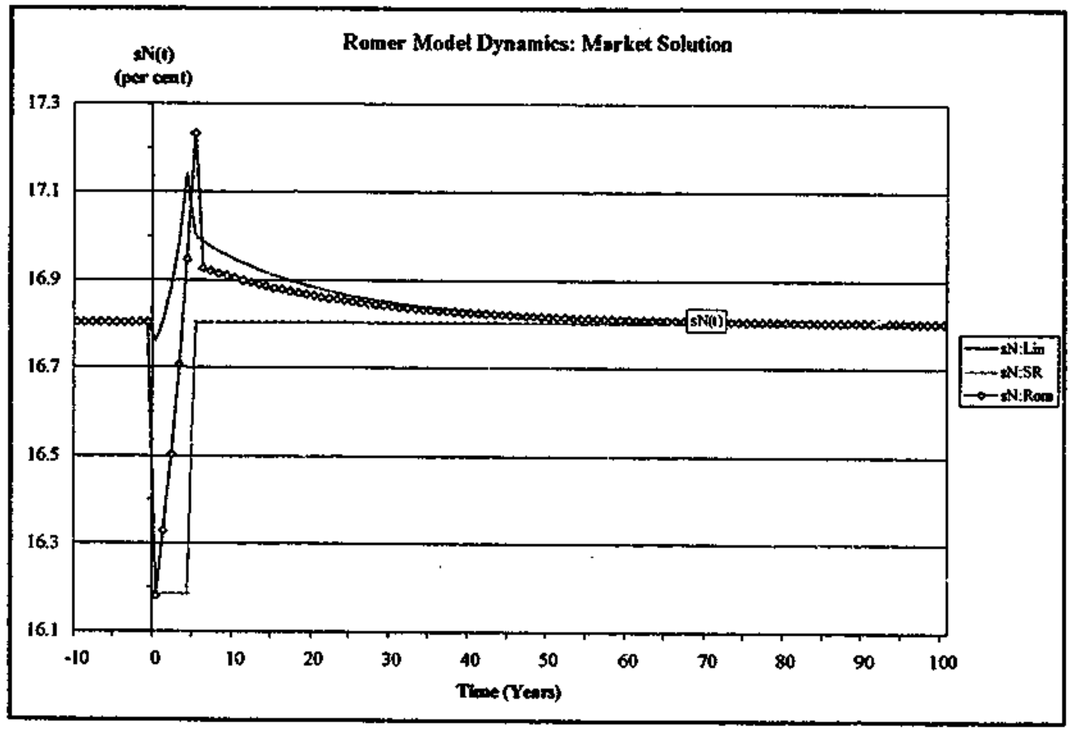

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Figure A4.4.25: Comparison of the linearised, S-R and full model dynamic effects on the savings rate s_B, of an unanticipated but temporary 20 per cent fall in parameter α from time zero, benchmark parameter set.

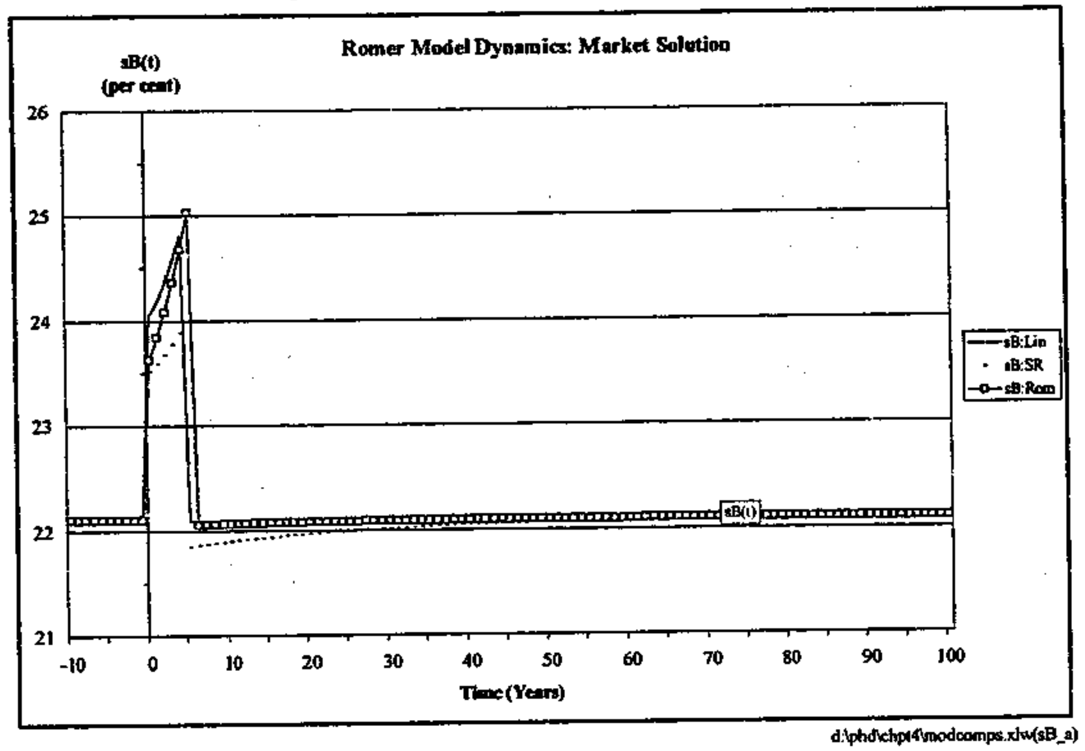

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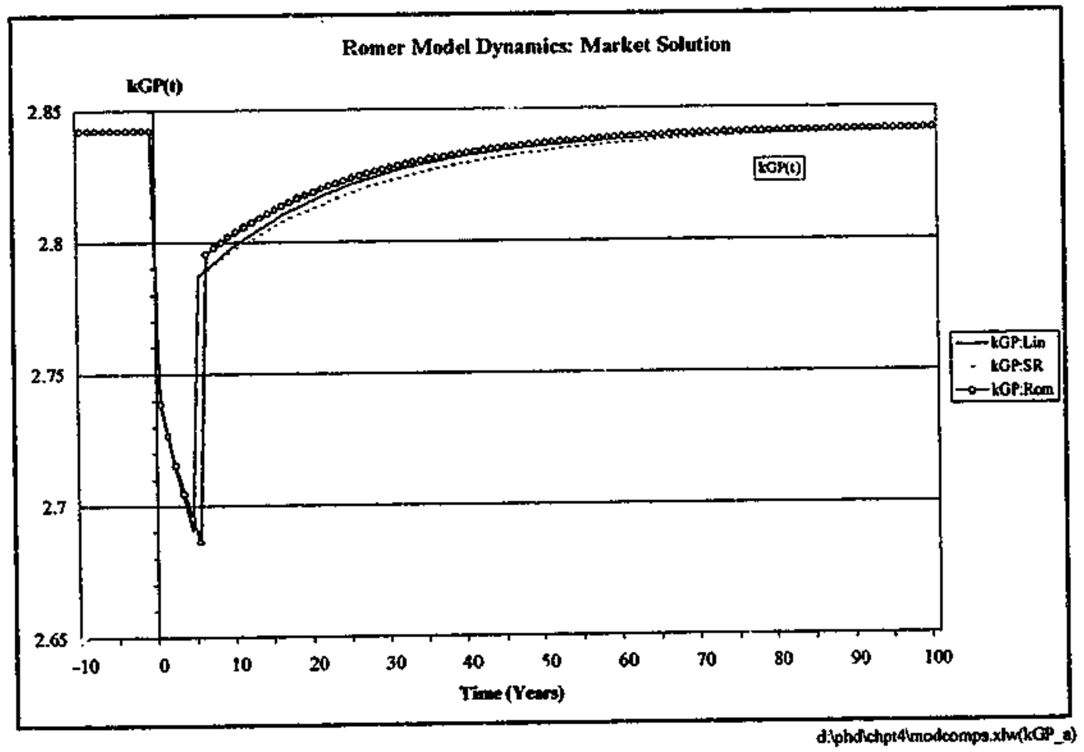
Figure A4.4.26: Comparison of the linearised, S-R and full model dynamic effects on the capital-gross product ratio (k_{GP}) of an unanticipated but temporary 20 per cent fall in parameter α from time zero, benchmark parameter set.



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The accuracy of the results from both the linearised model and the Solowian-Romer model in reflecting the dynamics computed from the full Romer model in response to the -20 per cent shock to α are all very high; significantly higher than was the case for either of the shocks examined in the previous two sections. Here both approximations are excellent over the entire adjustment paths for the variables Ψ, Φ,[67] p_A, g_A, g_{GP}, r, H_A, s_B and k_{GP}. The computed dynamics of the linearised model for g_C, g_Y, and s_N are also excellent approximations to those of the full Romer model. Moreover, from the time the shock is implemented at t=5 years both the linearised and the S-R models produce excellent approximations for all dynamic variables.

One reason that approximations from the linearised and S-R models were best for the α-shock simulation and second best for the ζ-shock simulation, is that these two involved the lowest and next lowest total adjustments. Since the shock to α was only temporary, the dynamic system eventually returned to its initial steady-state and the overall adjustment was zero. The accuracy of the linearised system in particular was always expected to deteriorate the further the system was from its steady-state linearisation point. The total adjustments for different variables under each simulation can be found in Table 4.1, Table 4.2, and Table 4.3.

[67] Even at t=5 the S-R result is only 1.2 per cent above the full model estimate for Φ; (recall that differences in the scales on the graphs need to be taken into account).

Chapter 5

5 Economic Welfare and Policy Issues

Two broad welfare issues arise from the model. First, the transition path to the steady-state will have implications for the magnitude and timing of any adjustment costs flowing from economic change. Second, the economic welfare generated in the steady-state itself can be expected to be sub-optimal and thus, perhaps, subject to policy induced improvement. Examples of the first issue, temporal adjustment costs, were encountered in the simulations undertaken in Section 4.5, where the speeds of convergence of the system towards new equilibria, and the magnitude and direction of the initial discontinuities or jumps to the new saddle-paths were found to be important. Such issues will also be met again later in this chapter; but it is the other welfare issue, that of sub-optimal equilibria and the role of policy improvement, which is the focus of this chapter.

5.1 Sub-optimality of the market solution[1]

Imperfections in the competitive economic market model have long been associated with processes of technological advance. Schumpeter (1942) was perhaps the first to regard innovation as an explicit economic activity, undertaken at private cost and with the expectation of making profits. He recognised the necessity for monopoly profits from the sale of innovations, not only in covering the sunk costs of research and development, but also in providing the fundamental incentives for undertaking these activities.[2] A difficulty with this is that once innovations arrive at the market place the welfare maximising course is for them to be sold at marginal cost, thereby suggesting a role for anti-monopoly policy. The policy problem is to find an optimum balance between the dynamic gains from innovation, which are encouraged by the prospect of monopoly profits, and the static welfare losses of the monopoly pricing and restriction of supply.

At least some degree of non-excludability, whereby not all of the economic benefits from such activity can be appropriated by the initiator, is another market imperfection usually accompanying technological advance. Here the suggested policy response is for some form of property rights, such as patents, to actually enforce a degree of monopoly power. The same natural conflict as before arises: On the one hand, the innovators would need to be able to extract all the economic value if the optimal amount of resources were to be devoted to innovative activity. On the other hand, once the community's resources have been used to generate an innovation, its competitive supply (at marginal cost) is optimal.

[1]The term "solution" refers to both the adjustment path and the steady-state behaviour of the model. It was the *market solution* which was developed in Chapter 2, and which has so far been analysed.

[2] Schumpeter's (1934) process of *creative destruction*, as its name suggests, involved new inventions replacing existing products and processes. However, such obsolescence is not a feature of the current model. This issue is discussed in the final chapter.

From the construction and derivation of the Romer model it is clear that both of these sources of distortion are present in its *market solution*. First, firms in the capital goods producing sector engage in restricting the supply of these goods through monopoly pricing, such pricing behaviour being necessary in order for these firms to recoup the fixed costs they must pay for their designs. The second distortion in the *free market* Romer economy arises from the fact that the output of research is only partially excludable. While designs for capital goods are patentable, and so excludable in terms of their direct use in the production of goods, by adding to the aggregate stock of knowledge used in the development of new designs they raise the productivity of other researchers. These indirect, or *external benefits* of research are non-excludable and are not reflected in the remuneration of the original researchers, who are therefore not compensated to the full extent of the value of the marginal social product of their output.

In addition to these monopoly distortions and research externalities, the free market equilibrium in an economy such as that described by the Romer model is also characterised by a third source of divergence from a *Pareto optimum*.[3] This may be termed a *specialisation divergence* since in general 'real-economy' terms it arises from the degree of specialisation in the supply side of the economy, along lines suggested by Adam Smith, Alfred Marshall and Allyn Young (also see Section 1.2). This is precisely what the production function used in the model seeks to capture. Thus, in terms of the model, the specialisation divergence arises because output is an increasing function of the extent of variety of capital inputs. Through an ingenious construction Romer (1986b) showed that models with such production functions behave precisely as if they had a true (technological) externality generating increasing returns.[4]

Romer (1986b and 1987b) used a model similar to the one studied here. Output was an increasing function of the extent of variety of specialised inputs, but there was no technology (such as a research sector) producing designs for generating the different types of inputs. Rather, the 'appropriate' number of types was simply chosen, either by the market or by a *social planner* seeking to maximise welfare.[5] Fixed costs involved in producing the specialised inputs from a primary resource, aggregate capital, ensured that the number of types would be finite. Thus, while the specialisation divergence seems to be simply a property of the technology, which is of course a binding constraint on markets and social planners alike, it results in a sub-optimal equilibrium since it causes the market's choice of the range of intermediates to be lower than would be chosen by a

[3] An allocation of resources and income that is preferred by at least one agent over any alternative feasible allocation, while all other agents are indifferent. See, for example, Bohm (1977) and/or Varian (1978). Here, with only a single (type of) consumer in the model, the Pareto optimum corresponds to the simple aggregate welfare maximising allocation of resources.

[4] By constructing a model of an artificial economy with a true technological externality, and which was an isomorphic transformation of the original model, Romer showed that the equilibrium quantities from the artificial economy were identical with those from the original model with specialisation.
Also, recall from Appendix 2.1 (particularly equation A2.1.5) that in terms of its fundamental factors H_Y, L, K, and A; the production function used here (from Romer, 1990b) exhibits increasing returns.

[5] A social planner is an artificial construct invested with the object of setting the economy to a Pareto optimum; and the power to achieve it through the allocation of resources and distribution of income by administrative fiat.

social planner.[6] Extending this to the current model, where the range of intermediates arises from the production of designs from a research sector, it means that the demand for designs is 'too low' and so 'too little' human capital will be allocated to research.

Further clarification of the effects of these distortions, together with quantitative measures of the extent of the sub-optimality of the market solution, may be obtained by solving the actual problem faced by a social planner concerned with maximising welfare under the constraints imposed by the Romer model technology. In the market solution to the model individual consumers have no control over the paths of either technology or aggregate capital and so treat these, together with the interest and wage rates, as given in their optimisation problems.[7] A social planner however, would recognise the relationships of both capital and labour incomes with aggregate savings (both in the form of investment and of research). In this way the expansion paths of capital, K(t) and technology A(t), would be optimised, and with them, all the other variables of the model.

5.1.1 Social planning solution to the Romer model

As a practical means of economic policy management social planning has little direct value.[8] Nevertheless, the mathematical procedures of obtaining a social planning solution to an economic model provide an extremely useful means of exploring and evaluating the efficacy of decentralised market mechanisms in a modelling environment. Such procedures can help identify the nature of any distortions and imperfections in the market model and provide estimates of the implied welfare losses. Importantly, they can also suggest practical policy measures designed to correct for the market distortions and restore the economic welfare, at least partially, towards a social optimum.

The social planning problem in the Romer economy is to maximise the discounted sum of all future aggregate utility, subject to the production and research technologies and the aggregate resource constraint on the economy. Formally, the problem is as follows:

$$
\begin{aligned}
&\text{Maximise} && \int_0^\infty U\{C(t)\}e^{-\rho t}dt \\
&\text{subject to} && \dot{K}(t) = Y(t) - C(t) - \delta K(t) \\
& && \dot{A}(t) = \zeta H_A(t)A(t) \\
& && K(0), A(0) \text{ given; } K(t), A(t) \geq 0 \qquad\qquad (5.1)\\
&\text{where} && U\{C(t)\} = [C(t)^{1-\sigma} - 1]/(1-\sigma) \quad \text{for } \sigma > 0 \\
& && Y(t) = \eta^{-\gamma} H_Y(t)^{\alpha(1-\gamma)} L^{(1-\alpha)(1-\gamma)} K(t)^{\gamma} A(t)^{(1-\gamma)} \\
& && H_A(t) + H_Y(t) = H
\end{aligned}
$$

[6] With output being an increasing function of the range of intermediates, new types generate surplus for final output producers that cannot be captured by the intermediates producers. As a result, for any given level of aggregate capital the range of intermediates is 'too small' and the number of units of each type is 'too big' from a social welfare perspective (see Romer, 1986b & 1987b).

[7] This is because these all depend on aggregate savings and thus on the savings decisions of others.

[8] The evidence from many communist regimes seems testament enough of this.

The problem is solved by the usual *Hamiltonian* techniques. This generates a solution which includes equations for the growth rates of the shadow prices of technology $\mu(t)$, and of capital $\lambda(t)$, and which also includes the ratio of these shadow prices in other dynamic equations. It is possible however, to eliminate the shadow prices completely from the system: Since μ is the price of technology measured in terms of *utils* (utility is the variable being maximised), and λ is the price of capital, or equivalently of output,[9] also measured in *utils*, then the ratio μ/λ is the price of technology measured in terms of output. Thus, by substituting $p_A(t)=\mu(t)/\lambda(t)$, as well as $\Psi(t)=K(t)/A(t)$ and $\Phi(t)=C(t)/K(t)$ as before (Section 2.3.3), a stationary dynamic system for the *social optimum solution* to the model, which is directly comparable to that for the *market solution*, is obtained. Derivation of the steady-state for this system is also the same as before in Section 2.3.3. Details of the solution are contained in Appendix 5.1. The results are as follows:

- **Stationary dynamic system for the social optimum**

$$\dot{\Psi}(t)=[\frac{r(t)+\delta}{\gamma}-\Phi(t)-\delta-\zeta H+\zeta H_Y(t)]\Psi(t) \quad (5.2)$$

$$\dot{\Phi}(t)=[\frac{r(t)-\rho}{\sigma}-\frac{r(t)+\delta}{\gamma}+\Phi(t)+\delta]\Phi(t) \quad (5.3)$$

$$\dot{p}_A(t)=[r(t)-\zeta(1/\alpha-1)H_Y(t)-\zeta H]p_A(t)$$

or[10] (5.4)

$$\dot{p}_A(t)=r(t)p_A-(1-\alpha)\frac{(1-\gamma)}{\gamma}[r(t)+\delta]\Psi(t)-\zeta H p_A(t)$$

where

$$H_Y(t)=[\frac{\alpha(1-\gamma)}{\zeta\eta^{\gamma}}L^{(1-\alpha)(1-\gamma)}\Psi(t)^{\gamma}p_A(t)^{-1}]^{\frac{1}{1-\alpha(1-\gamma)}} \quad (5.5)$$

$$\begin{aligned} r(t)&=\gamma\eta^{-\gamma}H_Y(t)^{\alpha(1-\gamma)}L^{(1-\alpha)(1-\gamma)}\Psi(t)^{\gamma-1}-\delta \\ &=\frac{\zeta\gamma}{\alpha(1-\gamma)}H_Y(t)p_A(t)\Psi(t)^{-1}-\delta \end{aligned} \quad (5.6)$$

with the boundary conditions:

$$\Psi(0)\text{ given}; \quad (5.7)$$

$$\lim_{t\to\infty}\Phi(t)=\Phi_{ss}^{O}; \quad (5.8)$$

and

$$\lim_{t\to\infty}p_A(t)=p_{Ass}^{O} \quad (5.9)$$

[9] Recall that output, capital, and consumption are all convertible 'one-for-one' according to the aggregate resource constraint for the economy: $\dot{K}=Y-C-\delta K$.

[10] The alternative formulation, obtained by substituting (5.6) in the first part of (5.4), is presented in order to facilitate comparison of the social optimum and market solution dynamic systems.

- **Steady-state for the social optimum**

$$H_{Yss}^{O}=\frac{\alpha(\sigma-1)H+\alpha\rho/\zeta}{(1-\alpha)+\alpha\sigma} \quad (5.10)$$

$$H_{Ass}^{O}=\frac{H-\alpha\rho/\zeta}{(1-\alpha)+\alpha\sigma} \quad \text{and} \quad g^{O}=\frac{\zeta H-\alpha\rho}{(1-\alpha)+\alpha\sigma} \quad (5.11)$$

$$r_{ss}^{O}=\frac{\sigma\zeta H+(1-\alpha)\rho}{(1-\alpha)+\alpha\sigma} \quad (5.12)$$

$$\Phi_{ss}^{O}=(r_{ss}^{O}+\delta)/\gamma-g^{O}-\delta \quad (5.13)$$

$$\Psi_{ss}^{O}=[\frac{\gamma H_{Yss}^{O\ \alpha(1-\gamma)}L^{(1-\alpha)(1-\gamma)}}{\eta^{\gamma}(r_{ss}^{O}+\delta)}]^{\frac{1}{1-\gamma}} \quad (5.14)$$

and

$$p_{Ass}^{O}=\frac{\alpha(1-\gamma)}{\gamma\zeta}\frac{r_{ss}^{O}+\delta}{H_{Yss}^{O}}\Psi_{ss}^{O} \quad (5.15)$$

The sensitivity of the social optimum solution steady-state to changes in parameter values may be assessed in the same manner as was done for the market solution steady-state. Namely, by raising the value of each parameter in turn by ten per cent, while holding the values of all other parameters at benchmark. The results are reported in Table 5.1. Mostly, they are similar to those obtained for the market solution (Table 2.3 of Chapter 2). For example, with one notable exception and one trivial one, the signs of the *elasticities* are the same and the magnitudes exhibit similar relativities. The greatest elasticities are in response to changes in parameters γ, ζ, and H. The 'notable exception', and perhaps the most interesting feature of the social optimum steady-state results, is that the allocation of human capital across the research and final output sectors **does not depend** on the output-elasticity of capital parameter, γ.[11] The reason for this is not pursued here. It is instead, simply noted as a possible topic for further research.

Table 5.1: Sensitivity of the benchmark steady-state to independent 10 per cent increases in each parameter, Romer model social optimum solution, (%).

Steady-state	Benchmark	Parameters increased by 10% above benchmark								
variables	ss values	α	γ	δ	ρ	σ	ζ	η	H	L
Ψ_{ss}	9.219	3.8	53.0	-5.9	-0.1	-1.3	-13.8	-10.6	-10.2	5.6
Φ_{ss}	0.189	-3.7	-12.5	1.8	0.4	3.5	7.8	[redacted]	7.8	[redacted]
p_{Ass}	15.71	5.0	22.7	-3.3	-0.7	-5.6	-15.6	-10.6	-12.1	5.6
H_{Yss}	50.09%	5.1	[redacted]	[redacted]	0.8	6.5	-0.7	[redacted]	9.2	[redacted]
H_{Ass}	49.91%	-5.2	[redacted]	[redacted]	-0.8	-6.5	0.7	[redacted]	10.8	[redacted]
r_{ss}	9.98%	-4.6	[redacted]	[redacted]	0.3	2.6	9.7	[redacted]	9.7	[redacted]
g_{ss}	2.99%	-5.2	[redacted]	[redacted]	-0.8	-6.5	10.8	[redacted]	10.8	[redacted]
s_{Nss}	27.0%	1.1	10.0	2.8	-0.5	-4.5	-2.2	[redacted]	-2.2	[redacted]
s_{Bss}	39.0%	0.5	2.8	1.6	-0.7	-5.9	-0.9	[redacted]	-0.9	[redacted]
k_{GPss}	3.226	3.6	12.2	-2.8	0.0	0.2	-6.7	[redacted]	-6.7	[redacted]

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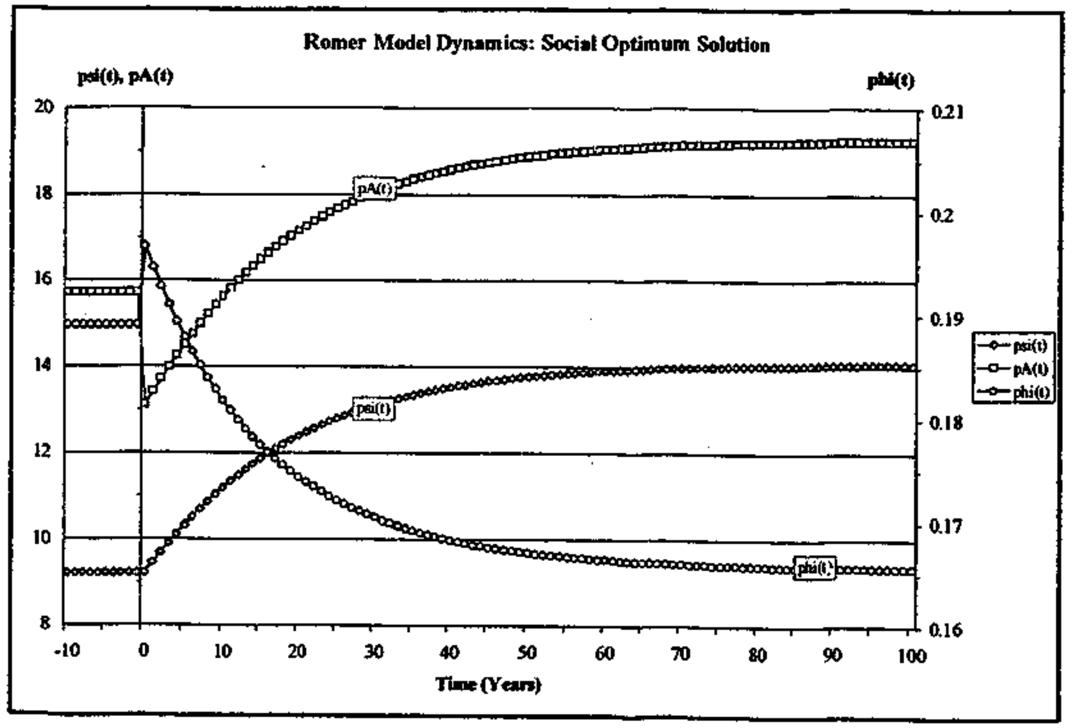
[11] This also implies (or is implied by) the similar independence of the interest and growth rates on γ.

The transitional dynamics of the social optimum solution may also be computed and examined by exactly the same numerical procedure used for the market solution. Recall that this was (principally) that of a *4^{th}-order Runge-Kutta finite differences integration* implemented via the GEMPACK software (Sections 4.4 and 4.5). It was only necessary to amend the TABLO input file to reflect the differences in the differential equations and steady-state formulae of the social optimum solution over those of the market solution. The amended file is recorded at Appendix 5.2. While many different simulations of the dynamics of this *social optimum solution* of the Romer system could of course be readily undertaken, only a single one is reported here: the standard 'unanticipated 10 per cent rise in the output elasticity of capital γ' (Figure 5.1 to Figure 5.6 and Table 5.2).

Figure 5.1: **Dynamic effects on Ψ, Φ, and p_A of an unanticipated and sustained 10 per cent rise in the output elasticity of capital (γ) from time zero, benchmark parameter set.**



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This simulation was also conducted for the *market solution* of the Romer system (Section 4.5.1).[12] Comparison of the results for the two solutions indicates that their adjustment paths for this simulation are proportionally very similar, with the adjustment of the social optimum system tending to be the smaller in overall terms, initial jumps, and the ½ and ¾ life measures (Table 4.1, Table 5.2 and the corresponding Figures). That is, the social optimum system tends to adjust to a smaller degree and to do so more rapidly,

[12] It was also carried out for the 'linearised Romer market solution' (Section 3.2.2), and for the '(non-linear) Solowian-Romer model market solution' (Section 3.4.2.1); and all three market solution results were closely scrutinised in Appendix 4.4 where they were shown to be highly similar.

its post-shock convergence coefficient (β^C=5.6%) being some 37 per cent higher than that of the decentralised market system (β^M=4.1%).[13]

Figure 5.2: **Dynamic effects on the growth rates, of an unanticipated and sustained 10% rise in the output elasticity of capital (γ) from time zero, benchmark parameter set.**

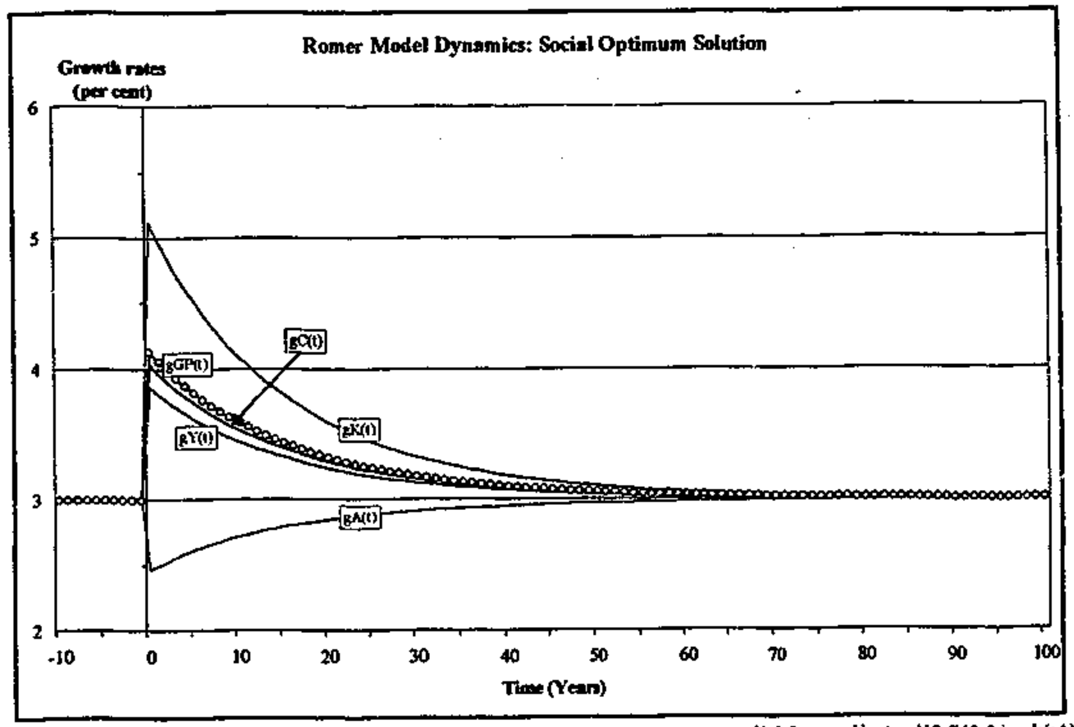

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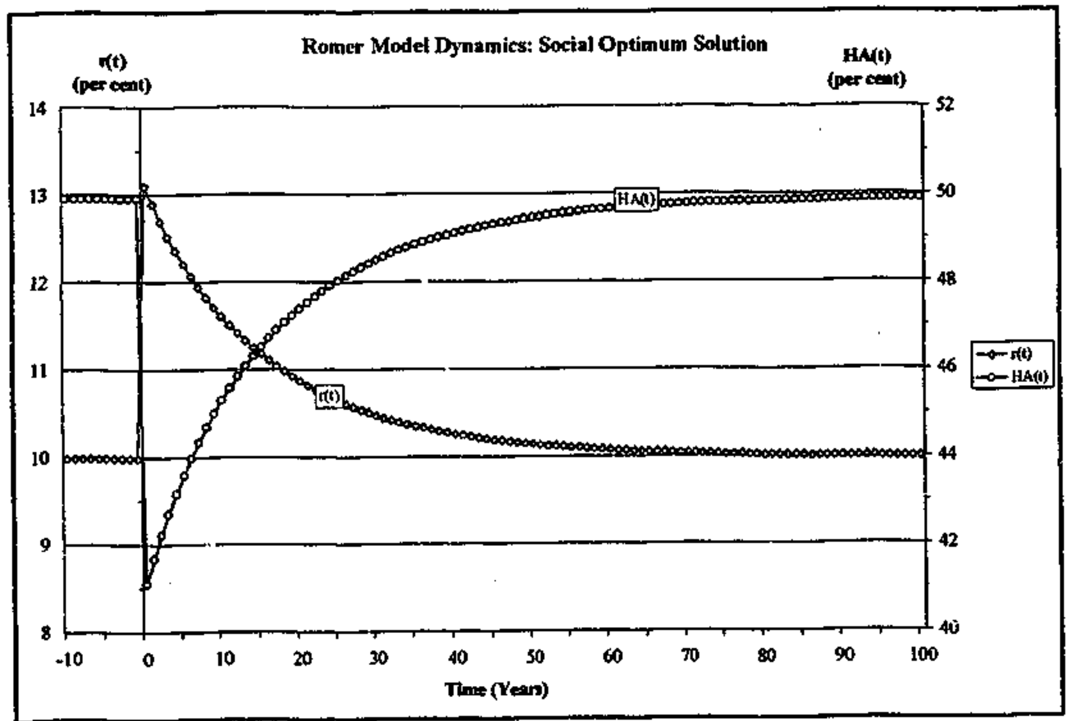
Figure 5.3: **Dynamic effects on r and H_A, of an unanticipated and sustained 10% rise in the output elasticity of capital (γ) from time zero, benchmark parameter set.**



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[13] Linearisation of the social optimum system and calculation of the negative eigenvalue of its coefficients matrix (see Chapter 3, particularly Appendices 3.1 and 3.5) is summarised in Appendix 5.1.

Figure 5.4: Dynamic effects on s_B, s_N, and k_{GP}, of an unanticipated and sustained 10% rise in the output elasticity of capital (γ) from time zero, benchmark parameter set.

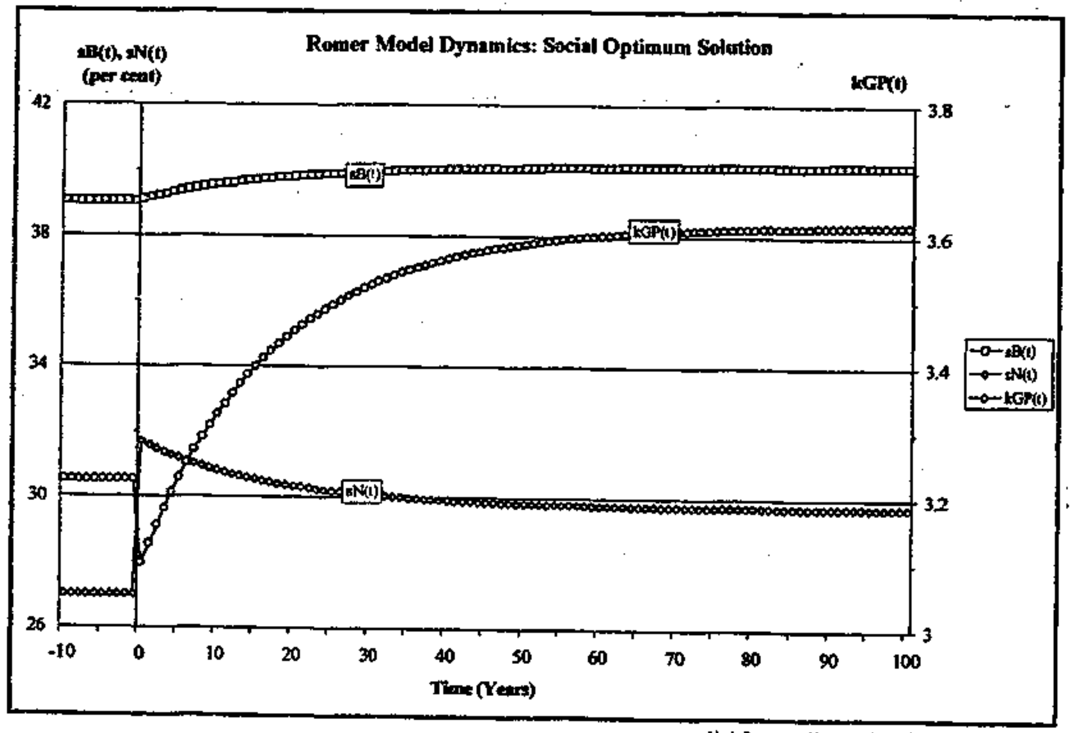

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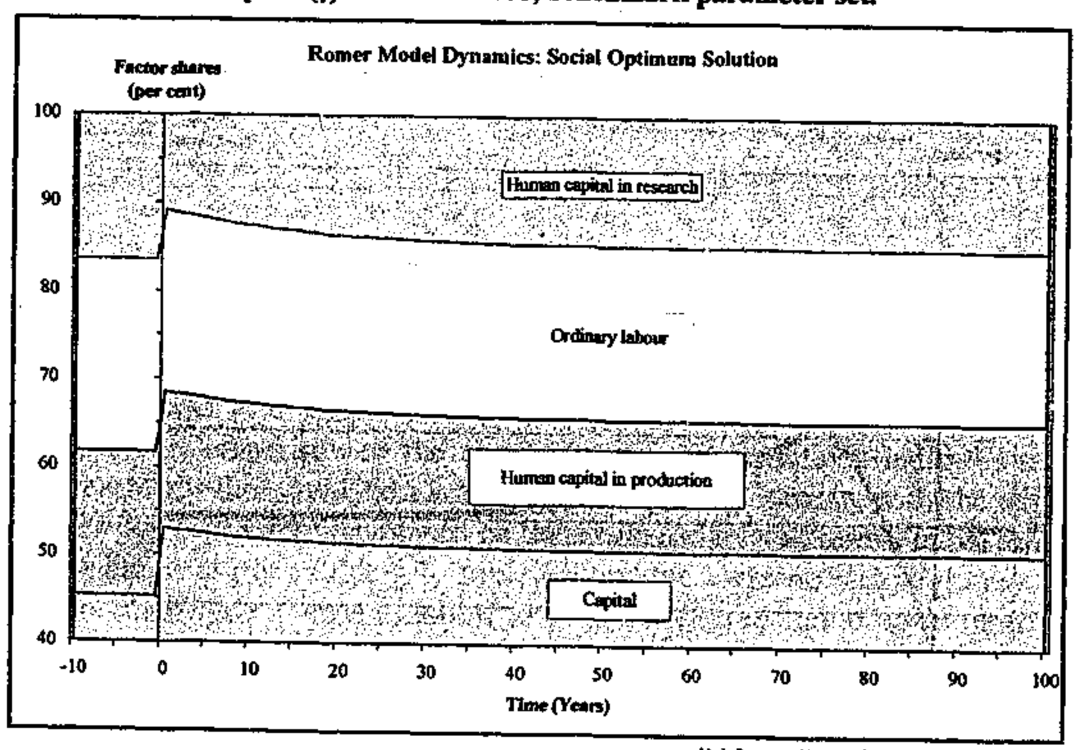
Figure 5.5: Dynamic effects on the factor shares of gross income from an unanticipated and sustained 10% rise in the output elasticity of capital (γ) from time zero, benchmark parameter set.



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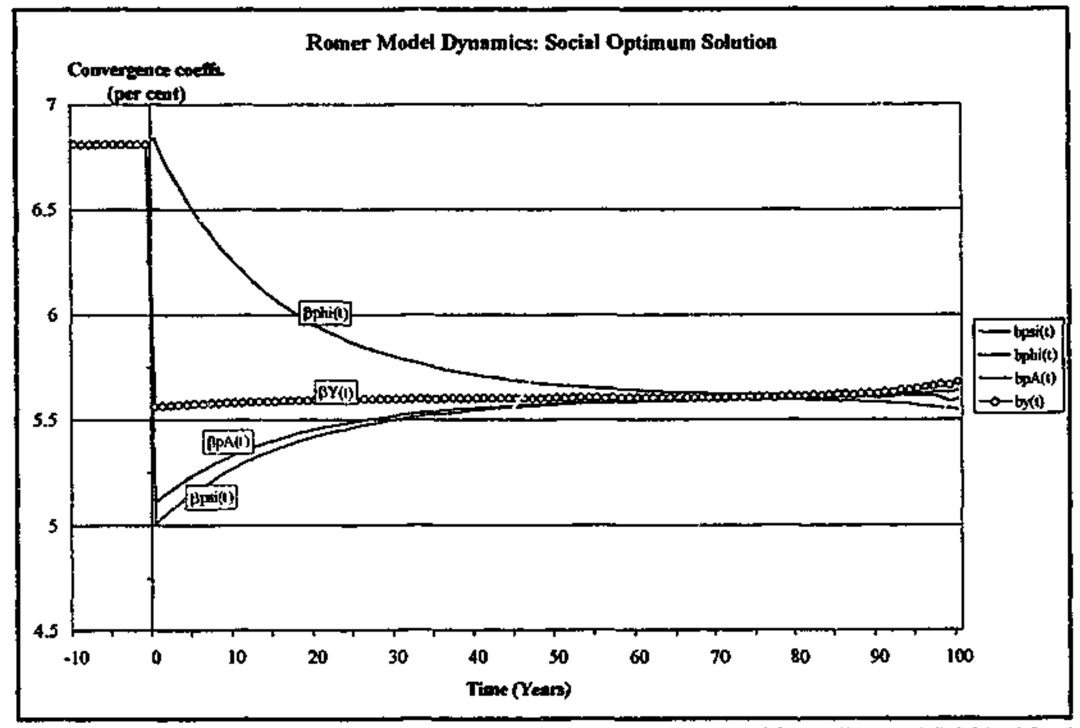
The adjustment of the growth and interest rates and the allocation of human capital for the social optimum solution further emphasise the importance of quantifying the transitional dynamics of these types of system. While there is no change at all in their pre-shock and final post-shock equilibrium levels (recall that γ does not appear in their social optimum steady-state formulations), the adjustment between them is extensive: the share of human capital to research initially falls by 18 per cent, and the interest and growth rates initially rise by almost 1/3, before adjusting back to their pre-shock levels (Figure 5.2, Figure 5.3 and Table 5.2)

Figure 5.6: Dynamic effects on the convergence coefficients from an unanticipated and sustained 10 per cent rise in the output elasticity of capital (γ) from time zero, benchmark parameter set.[a]



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Note: a The apparent non-convergence of these coefficients is due to numerical instability in their calculation near the steady-state. As may be seen from equation (3.13), the coefficients are calculated as ratios for which both the numerators and denominators approach zero.

Table 5.[illegible] Simulation results of an unanticipated and sustained 10 per cent rise in parameter γ from time zero, benchmark parameter set, social optimum solution.

Dynamic variable	Final steady-state	Total adjustment (% of initial ss)	Initial jumps as a % of: initial ss	Initial jumps as a % of: Total adjustment	½ life[a] (years)	¾ life[a] (years)
$\Psi(t)$	14.10	53.0	0.00	0.00	14(14)	27(27)
$\Phi(t)$	0.1655	-12.5	4.03	-32.3	11(16)	23(27)
$p_A(t)$	19.28	22.7	-16.6	-73.0	14(24)	26(36)
$r(t)$	9.98%	0.00	31.2	n.a.	11(∞)	23(∞)
$H_A(t)$	49.9%	0.00	-17.7	n.a.	11(∞)	23(∞)
$g_{GP}(t)$	2.99%	0.00	29.2	n.a.	10(∞)	21(∞)
$s_B(t)$	40.1%	2.82	0.17	5.97	12(11)	25(24)
$s_N(t)$	29.7%	10.0	17.3	173	12(0)	25(0)
$k_{GP}(t)$	3.62	12.2	-3.98	-32.7	13(17)	25(30)
$\beta_\Psi(t)$	5.60%	-17.7	-26.5	150	12(0)	24(0)

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Note: a The first set of results give the time taken for ½ and ¾ of the remaining adjustment after the initial jumps; while the figures in parentheses give corresponding results from the pre-shock levels.

5.1.2 Comparison of the social planning and market solutions

The dynamic systems for the market and social optimum formulations of the Romer model are perhaps more notable for their overall similarity than for their particular differences. Comparison of equation sets (2.41) to (2.45) with (5.2) to (5.6) indicates that the only differences are:

- "γ^2 terms" in the market system equations for Ψ, Φ, and r in place of the corresponding "γ terms" in the social optimum system; and
- a "(1-α)-factor" on the Ψ-term, and an additional "-$\zeta H p_A$ term" in the social optimum equation for $\dot{p}_A$.

Differences in the steady-state equations also involve extra (or absent) "γ-factors" and "(1-α)-factors".

The extent of the market solution sub-optimality may be gauged by comparison of the steady-state results above with those for the market solution (equations (2.54) to (2.59) in Section 2.3.3). Consider the ratio of the steady-state allocations of human capital to research by dividing equation (2.55) by equation (5.11):

$$\frac{H^M_{Ass}}{H^O_{Ass}} = \frac{H - \alpha\rho/\gamma\zeta}{H - \alpha\rho/\zeta} \cdot \frac{(1-\alpha)+\alpha\sigma}{1+\alpha\sigma/\gamma} \tag{5.16}$$

Since $\alpha, \gamma \in (0,1)$, both of the major ratio terms in this expression are less than unity. Hence, $H^M_{Ass} < H^O_{Ass}$. That is, less than the optimum amount of human capital is devoted to research in the decentralised market solution. The most obvious explanation for this lies with the research externality. Because the activity of research generates benefits that cannot be captured by the researchers who created them, the human capital employed there is not remunerated to the full extent of its marginal social product. Thus, from a societal perspective human capital in research is under-compensated and too little will be employed there. As will be seen shortly there are also other, less obvious, factors that reinforce this effect.

From the technology accumulation relation it follows that the steady-state growth rate of the market solution is also sub-optimal: $g^M < g^O$. Numerically, the extent of these sub-optimalities may be quantified by comparing the social optimum and market solution steady-states for the benchmark parameter set. Such a comparison indicates extremely significant differences for most variables. In particular, both the allocation of human capital to research and the growth rate for the social optimum are almost double their decentralised market solution levels (Table 5.3). It might also be noted that as a result, the social optimum share of total income to researchers is more than 2½ times its market level.

Two differences in the formulations for H^M_{Ass} and H^O_{Ass} are apparent. First, where the market formulation has a "γ-term", the optimum result effectively has a "1". Second, where the optimum allocation has a "(1-α)-term", the market formulation has a "1". Both of these differences in the formulations cause the amount of human capital allocated to research in the market determined steady-state, and hence the growth rate, to be less than optimal. Distortions arising on the production side of the economy

(monopoly pricing and the under-valuation of capital) can be shown to be responsible for the first difference noted above. By elimination, the externalities associated with research may be seen to be the cause of the second difference (Section 5.2.2).[14]

Table 5.3: Steady-state equilibrium values for the 'market' and 'social optimum' solutions to the Romer model, benchmark parameter set.

Dynamic variable	Steady-state values: Market solution	Optimum solution	Percentage difference: Optimum over Market
Ψ_{ss}	6.48	9.22	42.35
Φ_{ss}	0.2741	0.1890	-31.04
p_{Ass}	9.45	15.71	66.22
H_{Yss}	74.41%	50.09%	-32.68
H_{Ass}	25.59%	49.91%	95.02
r_{ss}	5.61%	9.98%	78.07
g	1.54%	2.99%	95.02
s_{Bss}	22.10%	39.03%	76.58
k_{GPss}	2.84	3.23	13.50
S_{HAss}	6.37	16.46%	158.47
S_{Lss}	25.55	21.90%	-10.78
S_{HYss}	18.52	16.52%	-10.78
S_{Kss}	50.56	45.11%	-10.78
β_{ss}	4.93%	6.81%	38.13

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The formula for the market determined steady-state rate of interest also differs from that of the social optimum due to "γ -" and "(1-α)-terms". Furthermore, it is also below the socially optimum level. This may be readily seen from the relations: $r^M_{ss} = \sigma g^M + \rho$ and $r^O_{ss} = \sigma g^O + \rho$, simply by noting that $g^M < g^O$ as demonstrated above. However, because it seems to make more economic sense to argue that *it is because consumers face too low a return from savings that the growth rate of capital is sub-optimal*,[15] this logic would then seem to be somewhat circular. Also, examining the ratio r^M_{ss} / r^O_{ss} in similar vein to that done for H^M_{Ass} / H^O_{Ass} above is inconclusive. Instead, the result that $r^M_{ss} < r^O_{ss}$ is proved by taking derivatives (Appendix 5.4). For the benchmark parameter set the social optimum steady-state rate of interest turns out to be some 80 per cent higher than in the free market (Table 5.3).

The marginal product of capital exceeds its rental price in the decentralised market. In particular, equation (A2.1.12) from Appendix 2.1 shows that: $r_K = \gamma MP_K < MP_K$. Thus, capital is undervalued and the return on savings is too low. It follows directly that both the interest rate and the rate of growth of capital are also too low. Also, now the

[14] The different formulation for H^O_{Ass} compared to H^M_{Ass} also raises another issue. Namely, it seems that when $\sigma<1$ a constraint on the maximum allowable level of human capital is introduced into the social planning solution. It turns out that this constraint actually derives from the dynamic maximisation of consumer utility. As such, a similar constraint also applies to the market solution. While this may represent a general deficiency of the model, it causes no problems in any of the work in this paper since all parameterisations have $\sigma>1$. The issues are discussed in detail in Appendix 5.3.

[15] In either solution to the model, anything that acts (*ceteris paribus*) to reduce the interest rate will also lower the rate of growth (see the discussion in Section 2.4.2).

relations $r_{ss}^M = \sigma g^M + \rho$ and $r_{ss}^O = \sigma g^O + \rho$ may be used 'correctly', to confirm that the growth rate in the market solution is sub-optimal.

Now consider the amounts of resources devoted to each type of capital good, $\Psi = \eta X$, under both the market and the optimal solutions. That is, consider the relative magnitudes of Ψ_{ss}^M and Ψ_{ss}^O. Because the monopolistic behaviour of capital goods manufacturers involves the elevation of price above marginal cost and the restriction of the supply of each type of capital (see Figure 2.2 of Chapter 2), the intuition may be that $\Psi_{ss}^M < \Psi_{ss}^O$ unambiguously. However, it turns out that $\Psi_{ss}^M \geq \Psi_{ss}^O$ is also possible. This may be seen by comparing the market and optimal steady-state results in equations (2.58) and (5.14) respectively. These are as follows:

$$\Psi_{ss}^M = \left[\frac{\gamma^2 H_{Yss}^{M\ \alpha(1-\gamma)} L^{(1-\alpha)(1-\gamma)}}{\eta^\gamma (r_{ss}^M + \delta)}\right]^{\frac{1}{(1-\gamma)}}$$

and (5.17)

$$\Psi_{ss}^O = \left[\frac{\gamma H_{Yss}^{O\ \alpha(1-\gamma)} L^{(1-\alpha)(1-\gamma)}}{\eta^\gamma (r_{ss}^O + \delta)}\right]^{\frac{1}{(1-\gamma)}}$$

The presence of the extra "γ-factor" in Ψ_{ss}^M tends to make $\Psi_{ss}^M < \Psi_{ss}^O$. However, the earlier results that $r_{ss}^M < r_{ss}^O$ and $H_{Yss}^M > H_{Yss}^O$ both work the other way, and the overall outcome is indeterminate.

A diagram similar to Figure 2.2 may also help to illustrate the situation. Under a social optimum regime, where (shadow) prices equal marginal costs, the output of specialised capital would be determined by the intersection of the marginal cost and demand curves rather than by equating marginal cost and marginal revenue as in the decentralised economy. But the social optimum levels of both these cost and demand schedules will differ from their free market levels. In particular, since $r_{ss}^O > r_{ss}^M$ and $H_{Yss}^O < H_{Yss}^M$, equations (2.10), (2.14) and (5.6) demonstrate that:

- MC^O: $p_X^O = (r^O + \delta)\eta > MC^M$: $p_X^M = (r^M + \delta)\eta$; and
- D^O: $p_X^O(t) = \gamma (H_Y^O)^{\alpha(1-\gamma)} L^{(1-\alpha)(1-\gamma)} (X^O)^{\gamma-1} <$
  D^M: $p_X^M(t) = \gamma (H_Y^M)^{\alpha(1-\gamma)} L^{(1-\alpha)(1-\gamma)} (X^M)^{\gamma-1}$

with the extent of the differences depending on parameter values. These are drawn in Figure 5.7, from which it can be seen that it is possible for the social optimum steady-state output of specialised capital to be greater than, equal to, or less than the corresponding level in the decentralised market.[16] That is:

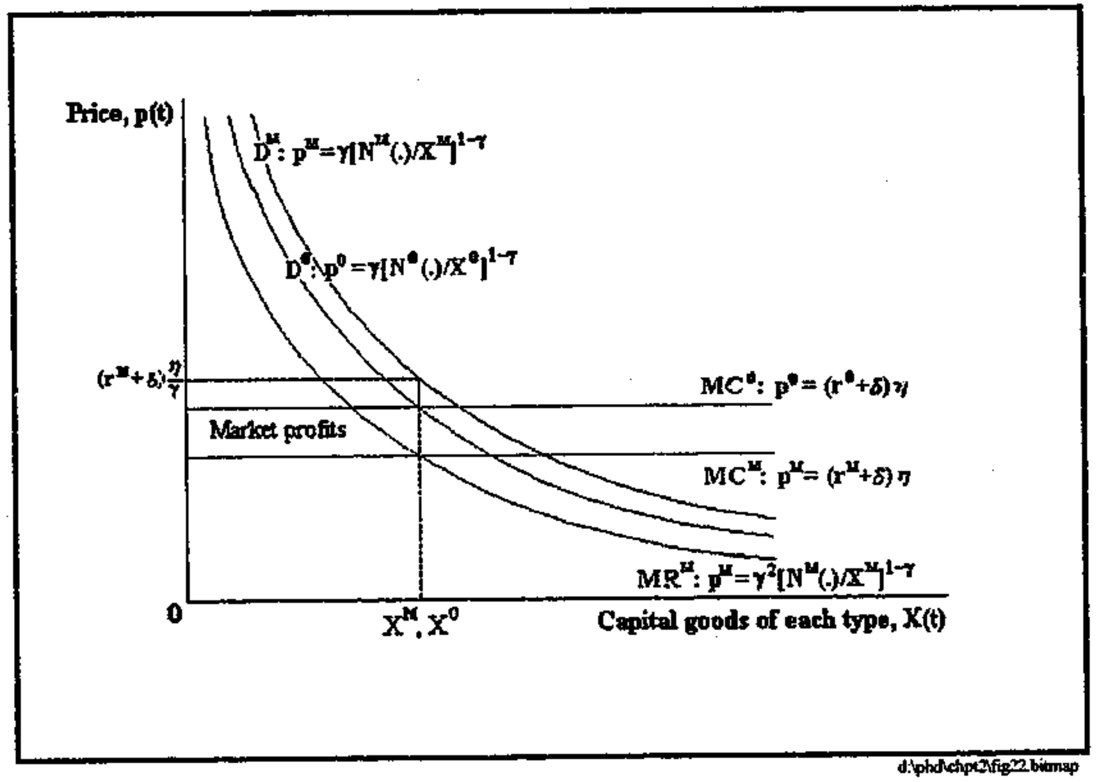
$$X_{ss}^M \gtreqless X_{ss}^O \text{ or equivalently } \Psi_{ss}^M \gtreqless \Psi_{ss}^O$$

The reason for this ambiguity is that the monopoly distortion is not the only influence causing a departure of the market system from the social optimum. The specialisation divergence arising from the increasing-returns nature of the production function is also a

[16] In Figure 5.7 they are drawn as being equal simply to avoid further cluttering the diagram.

factor, and one that has an opposite effect. For any given level of aggregate capital (K), the monopolists restriction of supply causes the number of units of each type of capital good (X) to be 'too low', and the number of different types (A) to be 'too high'. Conversely, because of the output increasing property of a greater variety of capital inputs, and the fact that final goods producers cannot control the extent of this variety, the specialistion effect causes the number of types to be 'too low', and the number of units of each type to be 'too high' (refer back to footnote 6). In the static-model versions of his 1986b and 1987b 'growth from specialisation' papers (where there are no savings and capital is simply in fixed supply), Romer shows that for precisely the 'power' type of production function used here, these opposing effects exactly balance, the decentralised market levels of the number of types and the number of units coinciding with their social optimum levels.[17] He also suggests that with the same 'power' type production function this balance is maintained when the model is made dynamic through the introduction of savings ($\dot{K} = Y - C$). Although the market solution is then sub-optimal, the reason is the under-valuation of capital, its price falling short of its marginal product due to both the monopoly and the specialisation effects.

Figure 5.7: The supply of specialised capital goods under 'decentralised market' and 'social optimum' economic regimes, Romer model.



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In a working paper on 'public finance issues in various models of economic growth', Barro and Sala-i-Martin (1990) described an extension of the Romer (1986b and 1987b) dynamic model. Increases in the number of types of specialised capital (new designs)

[17] The specialisation effect is demonstrated by the construction of a production function for which output remains an increasing function of the number of types of specialised inputs, but for which monopoly pricing is eliminated, the rental price of units of capital being equal to their marginal cost. For such a function the market and social optimum outcomes are shown to diverge (Romer, 1986b & 1987b).

were taken to be generated by 'purposive research activity'. But since these new designs were, like capital, created simply by saving output, the model remains a one-sector simplification of the two-sector Romer (1990b) model studied throughout this dissertation.[18] For this one-sector model it can be shown, as Romer (1987b) intimated, that the equilibrium level of the number of units per design (X) is the same for the market solution as it is for the social optimum. [19] However, as has been demonstrated, this is not unambiguously the case for the two-sector model studied here.

Evidently, the addition of a distinct research sector, with its externality distortion, has added to the complexity. Since the externality tends to cause the allocation of human capital to the sector to be too small, there also tends to be an insufficient number of designs at all dates. Also, as Romer (1990b) argues, the purchase of designs by capital goods producers and their mark-up of the price over marginal cost of the specialised capital the designs help to produce, 'forces a wedge between the marginal social product of the designs and the market compensation of the human capital that produces them, causing too little human capital to be devoted to research'. Of course this represents another tendency for 'too few' designs being produced by the free market at any date. While the static monopoly effect described by Romer in his earlier papers on 'growth due to specialisation' causes an opposing tendency, this has been shown (for the 'power' production function used here) to be exactly offset by the static specialisation effect. Finally, since capital is under-valued by the free market there is too little available at any date. The end result of all these effects is that at any date the free market generates less designs and less capital than are socially optimal; but that the relative magnitude of the ratio of these ($\Psi_{ss}^M / \Psi_{ss}^O$), or equivalently, of the number of specialised units per design (X_{ss}^M / X_{ss}^O), depends on parameter values. As it turns out, the intuitive result that $\Psi_{ss}^M < \Psi_{ss}^O$ holds for the benchmark parameter set (Table 5.3). Further numerical evaluation shows that this remains the case for a reasonably wide range of parameter values centred upon the benchmark set, but also identifies many of the parameter combinations for which Ψ_{ss}^O is the smaller (Appendix 5.4).

Like the resources devoted to each type of specialised capital (Ψ), the steady-state price of technology (p_{Ass}) in the free market relative to that in the social optimum turns out to be mathematically ambiguous, and numerically to depend on the model's parameter values (Appendix 5.4). Using equation (2.52) to define the market price in order to compare it directly with the social optimum price of equation (5.15), the ambiguity may be seen as:

[18] At the time, Barro and Sala-i-Martin indicated that the model had transitional dynamics which "should be similar to that in the two-sector production model", but that "the details of this transition have not been worked out". In fact, with both the interest rate and the number of units of each type of specialised capital being constants, the model has no transitional dynamics.

[19] In performing the necessary dynamic maximisation it needs to be recognised that there are two state variables: K(t) and A(t) in the notation used here; and that the economy-wide resource constraint is $Y(t) = \dot{K}(t) + C(t) + \beta \dot{A}(t)$, where β is the constant cost (in terms of output) of producing a design. The trick then is to define a share of output, $s_R(t)$ say, to be devoted to research and so to form the (current valued) Hamiltonian $\mathcal{H} = U(C)e^{-\rho t} + \lambda[(1-s_R)Y - C] + \mu[s_R Y/\beta]$ and to proceed via the Maximum Principle.

$$p_{Ass}^M = \frac{\alpha(1-\gamma)}{\gamma^2\zeta}\frac{r_{ss}^M + \delta}{H_{Yss}^M}\Psi_{ss}^M \quad \text{and} \quad p_{Ass}^O = \frac{\alpha(1-\gamma)}{\gamma\zeta}\frac{r_{ss}^O + \delta}{H_{Yss}^O}\Psi_{ss}^O \tag{5.18}$$

The results that $r_{ss}^M < r_{ss}^O$ and $H_{Yss}^M > H_{Yss}^O$, tend to make $p_{Ass}^M < p_{Ass}^O$. But with $\gamma<1$, and with $\Psi_{ss}^M > \Psi_{ss}^O$ possible, $p_{Ass}^M > p_{Ass}^O$ may also be possible. For the benchmark parameter set the market price turns out to be the lower: $p_{Ass}^M < p_{Ass}^O$ (Table 5.3). Because of the failure of the market price to reflect the contribution made by every existing design in the generation of future designs this might be seen intuitively as the expected result.[20] Nevertheless, that $p_{Ass}^M \gtrless p_{Ass}^O$ is incontrovertible.

The dramatic excess of the social optimum steady-state growth rate over that of the free market, points to the potential for a significant rise in welfare. For example, after five years of growth at the 'social optimum benchmark steady-state growth rate' annual consumption would be almost 7.5 per cent higher than at the market rate; and after ten years it would be more than 15 per cent higher. The corresponding figures for cumulative consumption are about 3.7 per cent and 7.7 per cent respectively.

5.2 Policy implications

Fundamental dilemmas are faced by policy makers concerned with the economics of research and innovation and the subsequent dispersion of the outputs of these processes into the economy at large. The usual problem is one of balance between the growth benefits from ensuring there are *sufficient* incentives for innovative activity, and the inefficiency costs from having to tolerate the associated market distortions. These issues were discussed in Section 5.1. The concern of this section may seem ambitious: It is to suggest a policy prescription, free of these balancing or compromise drawbacks, which will correct for the distortions of the market and generate the socially optimum level of economic welfare.

Specifically, identification a subsidisation policy which will convert the decentralised market economy outcomes to the socially optimum ones is sought. Such a policy must ensure that the return to savings is just sufficient to bring about the optimum rate of capital accumulation. It must also raise the incentives to researchers to the exact level required to attain the optimum allocation of human capital between the research and final output sectors. At the same time it must reduce the rentals paid for specialised capital to the marginal cost of its production, while maintaining the incentives to capital goods manufacturers that are enjoyed through monopoly pricing in the free market economy. Of course, to avoid introducing any new distortions into the economy any subsidies must be financed by lump-sum taxes.

[20] The price should correspond to the marginal benefit of designs to society. In the market economy however, it only reflects the marginal valuation (measured by the available profits) placed upon the particular type of specialised capital made directly from the design, no account being taken of the contribution any one design makes in the development of other designs. On the other hand, the shadow price of technology in the socially (and Pareto) optimal economic regime would take account of this *externality*, properly equating price with marginal benefit.

5.2.1 A subsidised market solution

Romer makes a statement that suggests it is possible to convert the free market economic outcomes to those of the social optimum simply by subsidising the research sector. In particular, he says:

> *"Within the confines of the model, the social optimum may be achieved by subsidizing the accumulation of A."* (Romer, 1990b; p.S97).

However, since there are distortions in both the production and research sectors of the economy, it seems that (at least) two different subsidies would be required: at least one to correct for the production distortions (the under-valuation of capital, and the monopoly supply of specialised goods); and another to correct for the imperfect *appropriability* of research effort.[21] These issues are investigated below.

5.2.1.1 Production-side distortions

For a similar model to the one studied here, but one with no research sector, Romer has suggested that "the only intervention needed to achieve the optimum....is a subsidy to savings" (Romer, 1987b). This was, he indicated, due to the special property of the 'power' type production function in producing an efficient static equilibrium. That is, one for which the monopoly distortion and the specialisation divergence generate exactly offsetting effects. Since the same production function is used in the current model, the balancing of these influences can be expected to continue to underlie it. While the static equilibrium may no longer be efficient, this is because of the effect of the separate research sector with its attendant externality distortion. Thus, a policy combining the subsidisation of savings with the correction of the research distortion may be expected to generate the socially optimum outcomes for the current model.

Suppose savings are encouraged via a subsidy to the rentals received by households for each unit of general-purpose capital, K. With the subsidy set at the rate $100s_K$ per cent, householders receive rentals of:

$$r_K^S(t) = (1+s_K)r_K(t) \tag{5.19}$$

by exactly the same argument as in Section 2.2.5, which led to equation (2.12), the rate of return on capital ($r_K^S(t)-\delta$), is equated to the interest rate r(t) yeilding:

$$r_K^S(t) = r(t)+\delta \qquad (5.20)^{22}$$

Producers of specialised capital continue to pay the unsubsidised rental on capital, $r_K(t)$, so their variable costs are $\{[r(t)+\delta]/(1+s_K)\}\eta X(i,t)$ Proceeding as for the development of the market model (Section 2.2.5 of Chapter 2), produces the following results, analogous to the pure market conditions in equations (2.13), (2.14) and (2.15):

[21] Barro and Sala-i-Martin (1995) also make this point about two subsidies being required.

[22] This result means that while aggregate household income now becomes $W(t)+r_K^S(t)K(t)-T(t)$ (where T(t) is the non-distortionary tax used to pay for the subsidy), the utility maximisation generates the same results as in the free market case. In particular, equation (2.27) continues to apply (refer back to Section 2.2.8 and Appendix 2.2).

$$X(i,t) = X(t) = \left[\frac{(1+s_K)\gamma^2 H_Y(t)^{\alpha(1-\gamma)} L^{(1-\alpha)(1-\gamma)}}{\eta[r(t)+\delta]}\right]^{\frac{1}{1-\gamma}} \tag{5.21}$$

$$p_X(i,t) = p_X(t) = \frac{[r(t)+\delta]\eta}{(1+s_K)\gamma} \tag{5.22}$$

$$\pi(i,t) = \pi(t) = (1-\gamma)p_X(t)X(t) \tag{5.23}$$

The key result here is equation (5.21). Together with equation (2.16) from Section 2.2.5, it determines the interest rate, r(t), in the subsidised dynamic system as:

$$r(t) = \frac{(1+s_K)\gamma^2}{\eta^\gamma} H_Y(t)^{\alpha(1-\gamma)} L^{(1-\alpha)(1-\gamma)} [K(t)/A(t)]^{\gamma-1} - \delta \tag{5.24}$$

From this result and the expression for Y(t) in the problem statement (5.1), it is also useful to express output in a manner corresponding to that in equation (2.29) as:

$$Y(t) = \frac{r(t)+\delta}{(1+s_K)\gamma^2} K(t) \tag{5.25}$$

Now, from (5.19) and from the extent of the undervaluation of capital - $r_K(t)=\gamma MP_K(t)$ from equation (A2.1.12) – it is immediately apparent that setting $s_K=(1/\gamma-1)$ will equate the subsidised rental price of capital with its marginal product, thereby removing the undervaluation distortion. Moreover, with the subsidy set at this level, it is clear from equation (5.22) that the price of specialised capital will be equated with its marginal cost, and the monopoly pricing distortion will also be removed. Finally, as is apparent from equations (5.24) and (5.6), this level of subsidy will cause the expressions for the subsidised and the social optimum interest rates to coincide. All this strongly suggests that the rate of subsidisation to savings $s_K=(1/\gamma-1)$ is indeed an 'optimal subsidy'. This will be demonstrated more rigorously in Section 5.2.2, after the research externality correction is determined and the fully *subsidised dynamic system* is developed.

As it has turned out, no extra subsidy has been necessary to correct for the monopoly pricing distortion. In fact, a subsidisation policy to correct for the mark-up of the rental rate of specialised capital over its marginal cost, is an alternative to the savings subsidy. This is true for either direct subsidisation of the rental costs faced by final output producers, or for the somewhat less direct subsidisation of final output.[23] Perhaps this could have been inferred from the outset: as the usual case of being able to generate the same price/quantity outcomes by either expanding demand via a subsidy to users, or expanding supply via a subsidy to suppliers. In any event, details of the different subsidy rates necessary for equivalence are worked out in Appendix 5.5.

5.2.1.2 Externalities in research

There are three possible candidates to correct for the distortion of having insufficient research due to the externalities it generates: subsidising the wages of researchers; subsidising the 'accumulation of research'; and subsidising the purchase of designs. The first of these approaches must be ruled out since, in combination with a subsidy to

[23] This approach was followed by Barro and Sala-i-Martin (1990 and 1995).

savings as discussed in the previous section (or to the rental costs of specialised capital goods examined in Appendix 5.5), subsidising the wages of researchers fails to replicate completely the steady-state of the social optimum (this is shown later in Table 5.5). Of course to make these calculations it was necessary to derive the modifications to the model that subsidising research wages entails. This is also reported in Appendix 5.5.

Under the second of the above subsidisation approaches it is not clear to what base the subsidy would be applied. It is also difficult to see who would receive the subsidy payments under the existing institutional set-up for the model. In order to clarify these matters an additional group of economic agents, *research entrepreneurs*, is proposed and the issues are explored in Appendix 5.5 - where it is shown that the approach is equivalent to subsidising research wages, and so similarly fails to replicate the social optimum results.

This leaves the third of the above approaches, which is examined as follows: A subsidy of $100s_{AK}$ per cent[24] to the purchase of research output means that the price of designs faced by capital goods producers would be:

$$p_A^S(t) = (1 - s_{AK})p_A(t) \tag{5.26}$$

where p_A is the unsubsidised price of designs as usual. Then, proceeding as before in Section 2.2.6: Under the monopolistic competition which characterises this segment of the economy, the price paid by capital goods producers, that is the subsidised price, is equivalent to the discounted sum of all future monopoly profits from the intermediates produced from any design. As before, this generates a dynamic equation in $p_A^S(t)$ analogous to (2.20):

$$\dot{p}_A^S(t) = r(t)p_A^S(t) - \pi(t) \Rightarrow \dot{p}_A(t) = r(t)p_A(t) - \pi(t)/(1 - s_{AK}) \tag{5.27}$$

5.2.1.3 Subsidised stationary dynamic system

With the savings subsidy s_K in place as before, the dynamic system incorporates equations (5.21) to (5.25) as well as (5.27). Otherwise the system is the same as for the decentralised market case in Section 2.2. In particular, the differential equations (2.1), (2.6) and (2.27) continue to apply, and by equating the wage rates in research and final output production, so does equation (2.25). Thus, incorporating the subsidisation results into the condensation of the equations of the model and transforming to the variables $\Psi(t)=K(t)/A(t)$ and $\Phi(t)=C(t)/K(t)$ as in Section 2.3, the stationary dynamic system for the subsidised market Romer model is:

$$\dot{\Psi}(t) = [\frac{r(t) + \delta}{(1 + s_K)\gamma^2} - \Phi(t) - \delta - \zeta H + \zeta H_Y(t)]\Psi(t) \tag{5.28}$$

$$\dot{\Phi}(t) = [\frac{r(t) - \rho}{\sigma} - \frac{r(t) + \delta}{(1 + s_K)\gamma^2} + \Phi(t) + \delta]\Phi(t) \tag{5.29}$$

[24] The 'K' sub-script is needed to distinguish this 'designs' subsidy, combined as it is with the subsidy s_K to general-purpose capital, from similar 'designs' subsidies (to be introduced later) which will be combined with different production side subsidies.

$$\dot{p}_A(t) = r(t)p_A(t) - \frac{(1-\gamma)[r(t)+\delta]}{\gamma(1+s_K)(1-s_{AK})}\Psi(t) \tag{5.30}$$

where

$$H_Y(t) = [\frac{\alpha(1-\gamma)}{\zeta\eta^\gamma} L^{(1-\alpha)(1-\gamma)}\Psi(t)^\gamma p_A(t)^{-1}]^{\frac{1}{1-\alpha(1-\gamma)}} \tag{5.31}$$

and

$$\begin{aligned} r(t) &= \frac{(1+s_K)\gamma^2}{\eta^\gamma} H_Y(t)^{\alpha(1-\gamma)} L^{(1-\alpha)(1-\gamma)}\Psi(t)^{\gamma-1} - \delta \\ &= \frac{(1+s_K)\zeta\gamma^2}{\alpha(1-\gamma)} H_Y(t)p_A(t)\Psi(t)^{-1} - \delta \end{aligned} \tag{5.32}$$

This system may be readily compared with the corresponding stationary dynamic systems for both the pure market and for the social optimum; that is, with equations (2.41) to (2.45) and (5.2) to (5.6) respectively. A high degree of similarity is apparent. In particular, it may be noted that if both subsidies are set equal to zero, the subsidised market system collapses to that of the pure market, as would be expected. Determination of subsidy values that would convert the subsidised system to that of the social optimum is not so simple. Though it is worth noting that if $(1+s_K)$ is set equal to $1/\gamma$, so the rate of subsidy is $s_K = 100(1-\gamma)/\gamma$ per cent, all of the differences apart from those in the $\dot{p}_A$-equations are eliminated). In fact, it is not possible to set constant subsidy values that make the dynamic systems equivalent.[25] Nevertheless, specific subsidy values which equate the steady-states of the two systems can be found. This is undertaken in Section 5.2.2; but first it is necessary to compute the steady-state for the subsidised system.

5.2.1.4 Steady-state of the subsidised market solution

Derivation of the steady-state for this system is exactly the same as for the pure market system (Section 2.3.3) and very similar to that for the social optimum system (Appendix 5.1). Thus, setting each of (5.28), (5.29) and (5.30) to zero and solving produces the results:[26]

$$r_{ss}^S = \frac{\gamma\zeta}{\alpha(1 - s_{AK})} H_{Yss}^S \tag{5.33}$$

$$H_{Yss}^S = \frac{(\alpha\sigma/\gamma)H + \alpha\rho/\gamma\zeta}{1/(1 - s_{AK}) + (\alpha\sigma/\gamma)} \tag{5.34}$$

$$H_{Ass}^S = \frac{H/(1 - s_{AK}) - \alpha\rho/\gamma\zeta}{1/(1 - s_{AK}) + \alpha\sigma/\gamma} \tag{5.35}$$

[25] Nevertheless, such equivalence may be possible if the subsidy to the purchases of designs was a function of time, and so followed its own adjustment path! For example, with $s_K = (1-\gamma)/\gamma$, it can be shown by equating the equations (5.4) and (5.30) that the subsidy to the purchase of designs, s_{AK} must satisfy $[1 - s_{AK}(t)] = [1 + \alpha H_A(t)/H_Y(t)]^{-1}$.

[26] For example, set (5.30) to zero and substitute (5.32) to obtain (5.33); then set (5.28) and (5.29) to zero, add them, and substitute for r from (5.33) to generate (5.34).

$$g^S = \frac{\zeta H/(1-s_{AK}) - \alpha\rho/\gamma}{1/(1-s_{AK}) + \alpha\sigma/\gamma} \tag{5.36}$$

$$r_{ss}^S = \frac{\sigma\zeta H + \rho}{1 + (1-s_{AK})(\alpha\sigma/\gamma)} \tag{5.37}$$

$$\Phi_{ss}^S = \frac{r_{ss}^S + \delta}{(1+s_K)\gamma^2} - g^S - \delta \tag{5.38}$$

$$\Psi_{ss}^S = \left[\frac{(1+s_K)\gamma^2 {H_{Yss}^S}^{\alpha(1-\gamma)} L^{(1-\alpha)(1-\gamma)}}{\eta^\gamma (r_{ss}^S + \delta)}\right]^{\frac{1}{1-\gamma}} \tag{5.39}$$

$$p_{Ass}^S = \frac{(1-\gamma)}{\gamma(1+s_K)(1-s_{AK})} \frac{r_{ss}^S + \delta}{r_{ss}^S} \Psi_{ss}^S \tag{5.40}$$

Again it may be noted that if both subsidies are set to zero, the subsidised steady-state above reduces to that of the pure market solution as in equations (2.53) to (2.59). In fact, the market solution may be considered to be a particular parameterisation of the more general *subsidised solution*, which includes the additional parameters subK and $subA_K$ based upon the subsidies s_K and s_{AK}. The market solution is then the particular one obtained by setting s_K=0 and s_{AK}=0, or more formally, setting subK=$(1+s_K)$=1 and $subA_K$=$(1-s_{AK})$=1.[27] Taking the market solution as a reference base, so the 'benchmark parameter set' now includes subK=1 and $subA_K$=1, the sensitivity of the steady-state to changes in these parameters is then assessed in Table 5.4. This assessment follows exactly what was done for the other parameters in Table 2.3 for the market solution; and in Table 5.1 for the social optimum solution.

Table 5.4: Sensitivity of the benchmark steady-state to 10 per cent increases in parameters subK and $subA_K$, subsidised market solution, (%).

Steady-state variables	Benchmark (mkt) ss values	Parameters increased by 10% SubK	$subA_K$	subK & $subA_K$
Ψ_{ss}	6.4760	23.02	10.17	35.54
Φ_{ss}	0.2741	-10.93	-4.17	-14.68
p_{Ass}	9.4524	11.84	3.10	15.30
H_{Yss}	74.41%	0.00	2.76	2.76
H_{Ass}	25.59%	0.00	-8.01	-8.01
r_{ss}	1.54%	0.00	-8.01	-8.01
g_{ss}	5.61%	0.00	-6.58	-6.58
s_{Nss}	16.80%	10.00	1.68	11.85
s_{Bss}	22.10%	7.12	-1.16	6.12
k_{GPss}	2.8421	10.00	4.70	15.17
S_{HA}	6.37%	0.00	-9.88	-9.88
S_L	24.55%	0.00	0.67	0.67
S_{HY}	18.52%	0.00	0.67	0.67
S_K	50.56%	0.00	0.67	0.67

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27 The subsidy parameters have been defined as subK=$(1+s_K)$ and $subA_K$=$(1-s_{AK})$, rather than simply s_K and s_{AK}, because it is intended to simulate the *introduction* of these subsidies (from a zero base) by raising their parameters by specified percentages.

5.2.2 Optimum rates of subsidy

As seen in Section 5.2.1.1, several of the distortions present in the free market model may be corrected by setting the variable subK= $(1+s_K)$ to the value $1/\gamma$ - so the rate of the subsidy to savings would be $100s_K = 100(1-\gamma)/\gamma$ per cent.[28] To confirm that this is indeed an 'optimal subsidy', the approach is to try to determine whether it can be combined with a specific rate of subsidy to designs, so that together they can convert all of the subsidised market steady-state outcomes to the socially optimum ones. If so, these specific subsidies will then be defined as the 'optimum rates of subsidy'.

The problem can be solved by setting s_K=$(1-\gamma)/\gamma$, and equating any of the formulae for the steady-state variables of the subsidised solution, with the formula for the corresponding steady-state variable in the social optimum solution, and solving for the subsidy level s_{AK}. In particular, if s_{AK} is to convert the subsidised steady-state to that of the social-optimum it must be valid to write equation (5.33) as:

$$subA_K = (1-s_{AK}) = \frac{\gamma\zeta}{\alpha}\frac{H_{Yss}^O}{r_{ss}^O}$$

or

$$s_{AK} = 1 - \frac{\gamma\zeta}{\alpha}\frac{H_{Yss}^O}{r_{ss}^O}$$

Substitution of the appropriate formulae for the steady-state optimum; namely, equations (5.10) and (5.12), then allows $subA_K$ and s_{AK} to be expressed in terms of the (exogenous) model parameters only:[29]

$$s_{AK} = 1 - subA_K = 1 - \gamma\frac{\zeta(\sigma-1)H + \rho}{\sigma\zeta H + (1-\alpha)\rho}$$

In summary, the decentralised market steady-state results of the Romer model can be converted to the social optimum ones through a policy of subsidising savings at the rate of $100s_K$ per cent, and the purchases of designs at the rate $100s_{AK}$ per cent, where:

$$\begin{aligned} s_K &= subK - 1 = 1/\gamma - 1 \text{ ;and} \\ s_{AK} &= 1 - subA_K = 1 - \frac{\gamma\zeta}{\alpha}\frac{H_{Yss}^O}{r_{Ass}^O} = 1 - \gamma\frac{\zeta(\sigma-1)H + \rho}{\sigma\zeta H + (1-\alpha)\rho} \end{aligned} \tag{5.41}$$

The computation of this conversion, for the benchmark parameter set, is recorded in Table 5.5. The table also contains the results of various other subsidisation schemes. For example, as mentioned before in Section 5.2.1.1, it shows that a subsidy designed to

28 In particular, it was seen that such a subsidy would equate the returns to general-purpose capital to its marginal product; equate the rental price of specialised capital to its marginal cost; and ensure that the expression for the interest rate in the subsidised system coincides with that for the social optimum.

29 It is also easy to express subA in terms of the alternative social optimum steady-state levels of H_A and H_Y as: $subA = \gamma[1 + \alpha H_{Ass}^O / H_{Yss}^O]^{-1}$, which is simply the steady-state version of the expression mentioned in footnote 25.

correct for the monopoly pricing mark-up, when similarly combined with subsidisation of the purchase of designs, also transforms the free market steady-state to that of the social optimum. This is the case whether the 'monopoly-pricing' subsidy is applied directly to the rental of specialised capital by final goods producers (subX=1-s_X), or indirectly to the sale of final output (subY=1+s_Y). Details of the model amendments for these 'alternative' subsidisation schemes are at Appendix 5.5. The optimum subsidies required are:

$$s_X = 1 - \text{subX} = (1-\gamma)$$
$$s_{AX} = 1 - \text{subA}_X = 1 - \frac{\zeta}{\alpha}\frac{H^O_{Yss}}{r^O_{Ass}} = 1 - \frac{\zeta(\sigma-1)H+\rho}{\sigma\zeta H+(1-\alpha)\rho} \qquad (5.42)$$

and

$$s_Y = \text{subY} - 1 = 1/\gamma - 1$$
$$s_{AY} = 1 - \text{subA}_Y = 1 - \frac{\zeta}{\alpha}\frac{H^O_{Yss}}{r^O_{Ass}} = 1 - \frac{\zeta(\sigma-1)H+\rho}{\sigma\zeta H+(1-\alpha)\rho} \qquad (5.43)$$

Table 5.5: Steady-state equilibrium values for the 'market', 'social optimum' and alternative 'subsidised' solutions to the Romer model, benchmark parameter set.

Steady state variables	Market, optimum and alternative subsidised solutions[a] Market solution	Optimum solution	s_K & s_{AK} subsidies[b]	s_X, s_Y & s_{AX} subsidies	s_K & s_{HK} subsidies	s_X, s_Y & s_{HX} subsidies	s_K only subsidy	s_{AK} only subsidy
Ψ_{ss}	6.4760	9.2188	9.2188	9.2188	9.2188	9.2188	24.7205	2.4150
Φ_{ss}	0.2741	0.1890	0.1890	0.1890	0.1890	0.1890	0.1225	0.4096
p_A	9.4524	15.7122	15.7122	15.7122	5.9396	2.8815	19.4844	7.6224
H_Y	74.41%	50.09%	50.09%	50.09%	50.09%	50.09%	74.41%	50.09%
H_A	25.59%	49.91%	49.91%	49.91%	49.91%	49.91%	25.59%	49.91%
g	1.54%	2.99%	2.99%	2.99%	2.99%	2.99%	1.54%	2.99%
r	5.61%	9.98%	9.98%	9.98%	9.98%	9.98%	5.61%	9.98%
s_N	16.80%	27.01%	27.01%	27.01%	27.01%	27.01%	31.12%	14.59%
s_B	22.10%	39.03%	39.03%	39.03%	32.07%	29.56%	35.50%	28.65%
k_{GP}	2.8421	3.2258	3.2258	3.2258	3.5938	3.7269	5.2631	1.7419
S_{HA}	6.37%	16.46%	16.46%	16.46%	6.93%	3.49%	6.37%	16.46%
S_L	24.55%	21.90%	21.90%	21.90%	24.40%	25.31%	24.55%	21.90%
S_{HY}	18.52%	16.52%	16.52%	16.52%	18.41%	19.09%	18.52%	16.52%
S_K	50.56%	45.11%	45.11%	45.11%	50.26%	52.12%	50.56%	45.11%

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Notes: a For the benchmark parameter set the optimum subsidy levels are approximately: s_K = 85%, s_{AK} = 62%, s_X = 46%, s_{AX} = 30%, s_Y = 85%, s_{HK} = 165%, and s_H = 43%.

b The optimum subsidies s_K and s_A are quite insensitive to changes in the benchmark parameter values: s_K changes only in response to γ (but with an elasticity of −2.2); and the largest elasticities of s_{AK} are −0.61 (with respect to γ) and −0.25 (with respect to σ), all others being less than ±0.03.

Table 5.5 also indicates (as mentioned before in Section 5.2.1.2) that it is not possible to replicate the social optimum steady-state through a subsidy to research wages, subH=1+s_H. This is true irrespective of which production-side subsidy (s_K, s_X, or s_Y) it is combined with. While it is possible to replicate the social planning allocation of human capital, the growth and interest rates, the resources devoted to each type of specialised

capital, and the consumption-capital ratio, a 'research wages' subsidy fails to reproduce the social optimum steady-state price of technology (or any other variables that depend upon it). Details of the subsidisation schemes are at Appendix 5.5.

Finally, the effects of introducing the optimum subsidies s_K and s_{AK} independently are also shown in Table 5.5. Since the steady-state formulations for H^S_{Yss}, H^S_{Ass}, r^S_{ss}, or g^S (equations (5.34) to (5.37)), contain only the subA$_K$ subsidy parameter and not subK, they all remain equal to their market levels when only the capital subsidy is introduced, whatever its level (see Table 5.4). And they are all converted to their social optimum levels when only the optimum subsidy subA$_K$ is imposed.[30] Combining this with the findings from Section 5.1.2 on the relative magnitudes of these variables:[31]

$$v^M_{ss} = v^{S_K}_{ss} < v^O_{ss} = v^{S_{AK}}_{ss} \qquad \text{for } v_{ss} = H_{Ass}, r_{ss}, g$$

and

$$H^M_{Yss} = H^{S_K}_{Yss} > H^O_{Yss} = H^{S_{AK}}_{Yss}$$

Because of the effects of these subsidies on the growth rate, s_K may be thought of as correcting for static distortions, and s_{AK} as correcting for dynamic ones. Despite this, neither of these 'partial subsidisations' can be considered to move the system from its sub-optimal state unambiguously closer to its optimum.[32] Consider the capital-technology ratio Ψ_{ss} (or equivalently, the number of specialised capital units of each design X_{ss}). Comparing the formulations for Ψ_{ss} from the market, social-optimum, and subsidised solutions (equations (2.58), (5.14), and (5.39) respectively), it is apparent that:

$$\Psi^{S_K}_{ss} > \Psi^M_{ss},\quad \Psi^{S_K}_{ss} > \Psi^O_{ss},\quad \Psi^{S_K}_{ss} > \Psi^{S_{AK}}_{ss},\quad \Psi^{S_{AK}}_{ss} < \Psi^M_{ss}, \text{ and } \Psi^{S_{AK}}_{ss} < \Psi^O_{ss}$$

Thus, either $[\Psi^{S_{AK}}_{ss} < \Psi^M_{ss} < \Psi^O_{ss} < \Psi^{S_K}_{ss}]$, or $[\Psi^{S_{AK}}_{ss} < \Psi^O_{ss} < \Psi^M_{ss} < \Psi^{S_K}_{ss}]$, which means that partial subsidisation with either s_K or s_{AK} will either over-correct for this sub-optimality of the market, or potentially worse, directly increases its divergence from the optimum. For the benchmark parameter set s_K results in Ψ_{ss} moving towards its optimum level but overshooting it, while s_{AK} moves it in the opposite direction to its optimum (Table 5.5). Thus it is the first set of inequalities in square bracket above that prevails.

These issues are of course, highly relevant to the actual implementation of policy. Chapter 6 includes some further discussion of them.

30 The steady-state income shares S_{HA}, S_L, S_{HY}, and S_K are also invariant to changes in subK since it cancels in their numerators and denominators. Similarly, these income shares also attain their social optimum levels when the optimum subA$_K$ is imposed alone. However, the steady-state levels of all the other economic variables examined throughout this paper - the principal dynamic variables Ψ, Φ, and p_A; the savings rates s_N and s_B; and the capital-output ratio k_{GP} –change from their market levels under both subK and subA$_K$ (a fact that was also apparent from Table 5.4).

31 Here 'S_K' and 'S_{AK}' super-scripts are used to denote the imposition of the optimum subsidies s_K and s_{AK} respectively.

32 The *Theory of the Second Best* (Lipsey and Lancaster, 1956) taught us that the correction of some market distortions while others remain extant may increase the divergence of an economy from its optimum. See Bhagwati (1971) for a comprehensive theoretical treatment.

5.3 Adjustment to the social optimum steady-state

Implementation of the subsidisation policy identified in the previous section will shift the system away from whatever state it was previously in, or whatever dynamic path it was following, onto a new transitional adjustment path towards the social optimum steady-state. While it is easily possible to calculate the transitional dynamics from any (arbitrary) initial position, in the analysis here the system will be considered to be initially in the decentralised market equilibrium defined by the benchmark parameter set. The subsidisation policy will then be implemented in various ways, and the resulting adjustment path computed in much the same way as was done for the various shocks to market equilibrium that were examined in Section 4.5 of Chapter 4.

For this analysis the basic dynamic system is the subsidised one given by equations (5.28) to (5.32). The subsidy rates s_K and s_A are both initially set at zero to generate the decentralised market equilibrium, and then raised to the levels given by equation (5.41) to produce a dynamic system with an identical steady-state to that of the social optimum. The two-point boundary value problem is the same as that confronted in Chapter 4 and, as there, finite differences implemented through the GEMPACK software suite is the method of solution. With the dynamic system considered to be initially in the decentralised market steady-state pertaining to the benchmark parameter set, this set can now be taken to include subK=1 and subA=1.[33] The subsidisation policy necessary to replicate the social optimum steady-state is then implemented by imposing the sustained shocks necessary to take the subsidy parameters to their optimum levels as defined by equation (5.41). For the benchmark parameter set, since $100s_K$=85 per cent; and $100s_{AK}$=62 per cent, the (overall) shock to subK is a rise of 85 per cent and to subA a reduction of 62 per cent.[34] There are, of course, a great many different ways in which these overall or 'optimum' subsidy levels could sensibly be implemented. In the following just a few of the possibilities are considered and analysed.

5.3.1 Unanticipated/single point of time imposition of the optimum subsidies

In this simulation both the subsidy to the rentals received for general-purpose capital s_K, and the subsidy to the purchase of technology or designs s_{AK}, are assumed to be instantly applied at their full 'optimum' levels with no prior announcement, and without the market anticipating their introduction in any other way. The results are shown in Figure 5.8 to Figure 5.15. In order to indicate the full adjustment of the system, the first of these

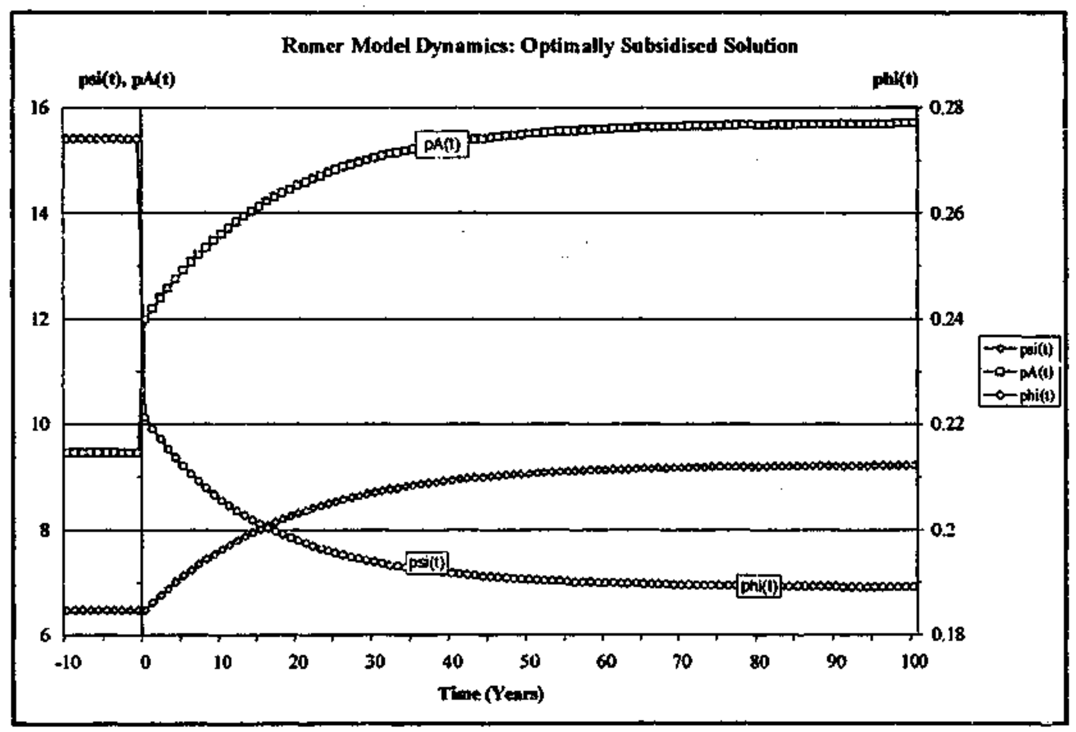
[33] The use of these parameters and their benchmark levels in preference to simply s_K=0 and s_A=0 was discussed in Section 5.2.1.4 (see footnote 27 in particular).

[34] Because these changes are quite large it was thought prudent to employ a more powerful GEMPACK configuration than used previously (see Section 4.4.3). Accordingly, the shocks were divided into 8 sub-intervals (see Harrison and Pearson 1994 & 1996b) and the results calculated from a Gragg integration technique extrapolated from 8, 16, and 24 steps across each sub-interval. In addition, the 4[th] order Runge-Kutta finite differencing grid was extended from 350 grid points covering 250 years, to 375 grid points covering 500 years by the addition of 25 steps of 10 years each. Despite all this extra power, changes to the estimated results were minimal.

graphs (for the 'principal' dynamic variables Ψ, Φ, and p_A) extends into the distant future, covering 100 years. To clarify the asymptotic behaviour of the convergence coefficients β, the last of the graphs also covers 100 years. The remainder of the graphs, by covering only 25 years, show more detail in the less distant and more policy relevant future.

The dynamic impact of introducing both subsidies is obviously a combination of the effects from each individual subsidy. Both the quantitative results and the associated qualitative explanations of the individual subsidy simulations are at Appendix 5.6. It is apparent from these results, together with those below, that the adjustment to the steady-state from the joint subsidy simulation more closely resembles that of the subsidy to savings than that to the purchase of designs.[35] This is despite the fact that application of the 'designs subsidy' alone results in many variables (namely, the allocation of human capital, the growth rates, the interest rate, and the gross income factor shares) attaining their socially optimum levels, as opposed to the 'savings subsidy' leaving them unchanged.

Figure 5.8: **Transitional dynamics of moving from the free market to the social optimum ss: effects on Ψ, Φ, and p_A (over 100 years); unanticipated introduction of the 'optimal subsidies' from time zero, benchmark parameter set.**



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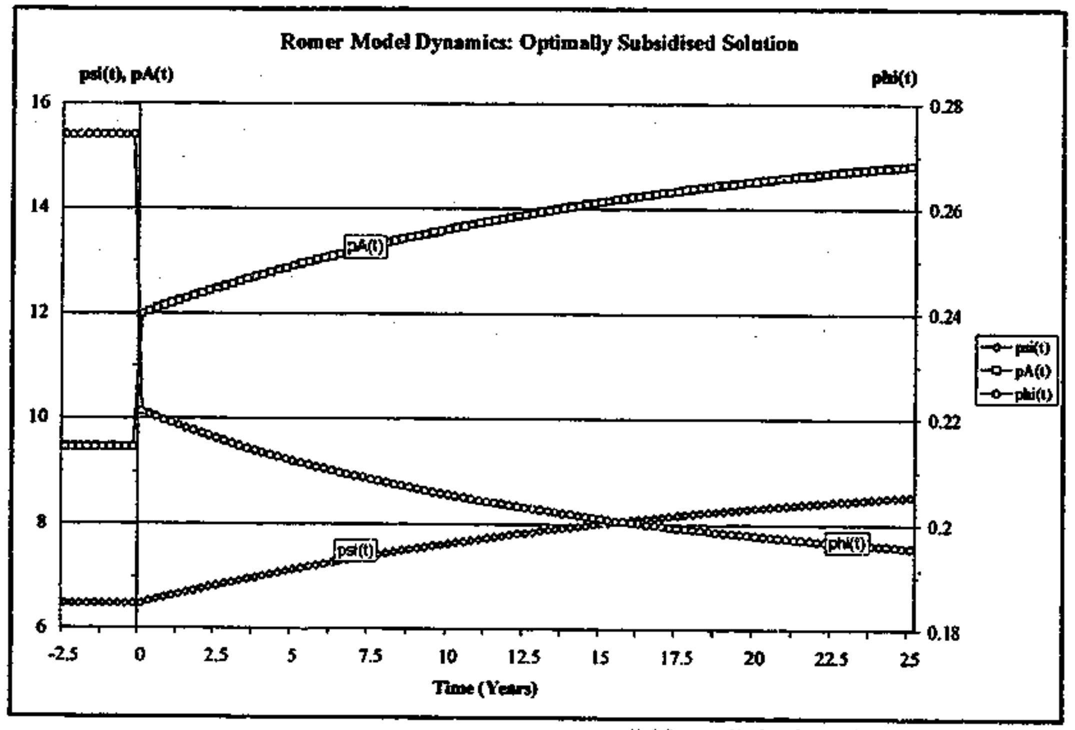
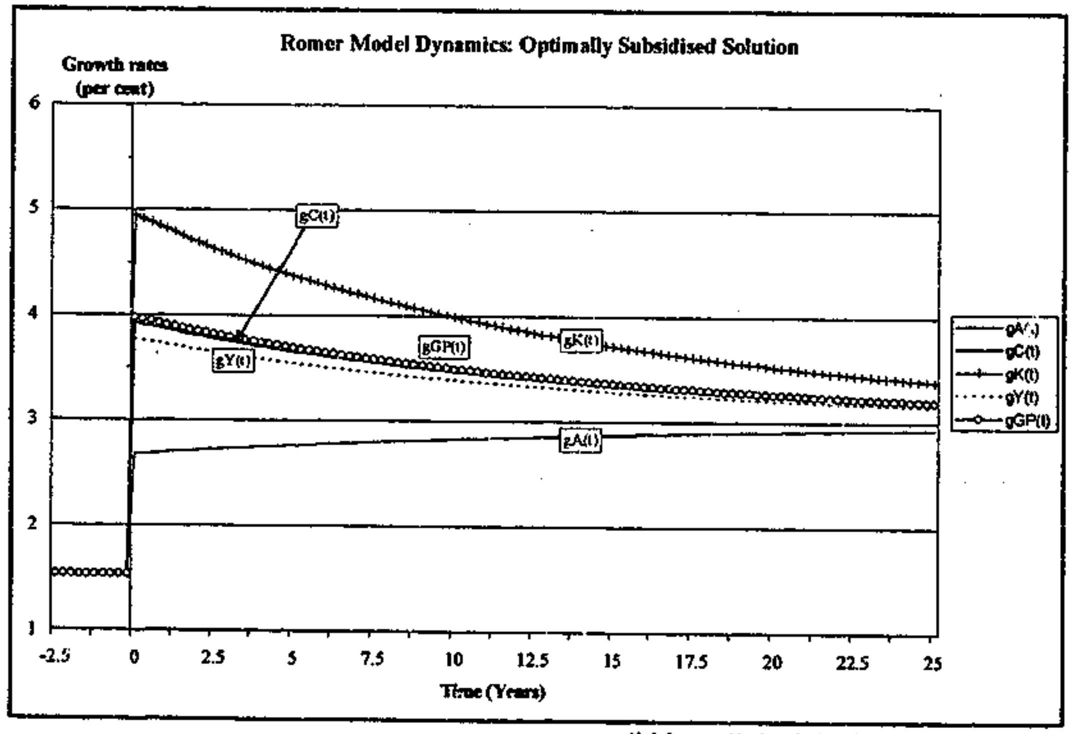
[35] The principal exceptions to this are the initial jumps in p_A, g_A, H_A, k_{GP}, and the gross income factor shares, which all have the same sign in the joint subsidy simulation as for the simulation of imposing optimum subsidy s_{AK} alone.

Figure 5.9: **Transitional dynamics of moving from the free market to the social optimum ss: effects on Ψ, Φ, and p_A (over 25 years); unanticipated introduction of the 'optimal subsidies' from time zero, benchmark parameter set.**



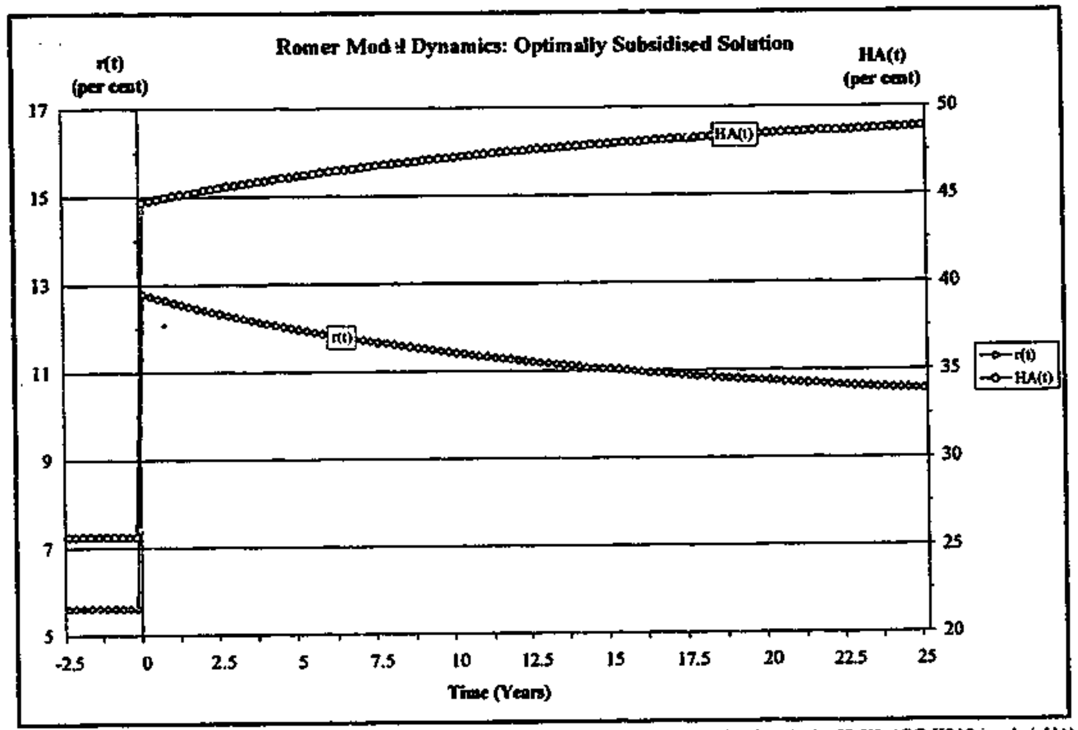
d:\phd\gempack\subrom\sub_sK\sK0sA0\QsK0A0sim.xlw(psi)

Figure 5.10: **Transitional dynamics of moving from the free market to the social optimum ss: effects on the growth rates; unanticipated introduction of the 'optimal subsidies' from time zero, benchmark parameter set.**



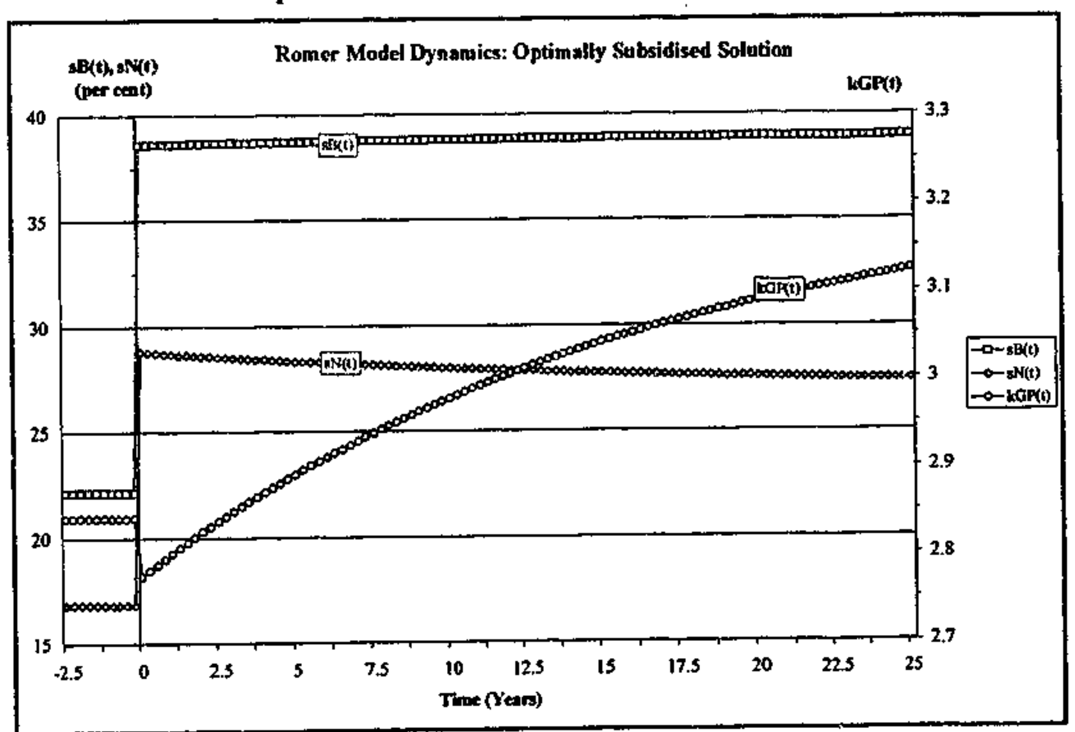
d:\phd\gempack\subrom\sub_sK\sK0sA0\QsK0A0sim.xlw(gA)

Figure 5.11: **Transitional dynamics of moving from the free market to the social optimum ss: effects on r and H_A; unanticipated introduction of the 'optimal subsidies' from time zero, benchmark parameter set.**



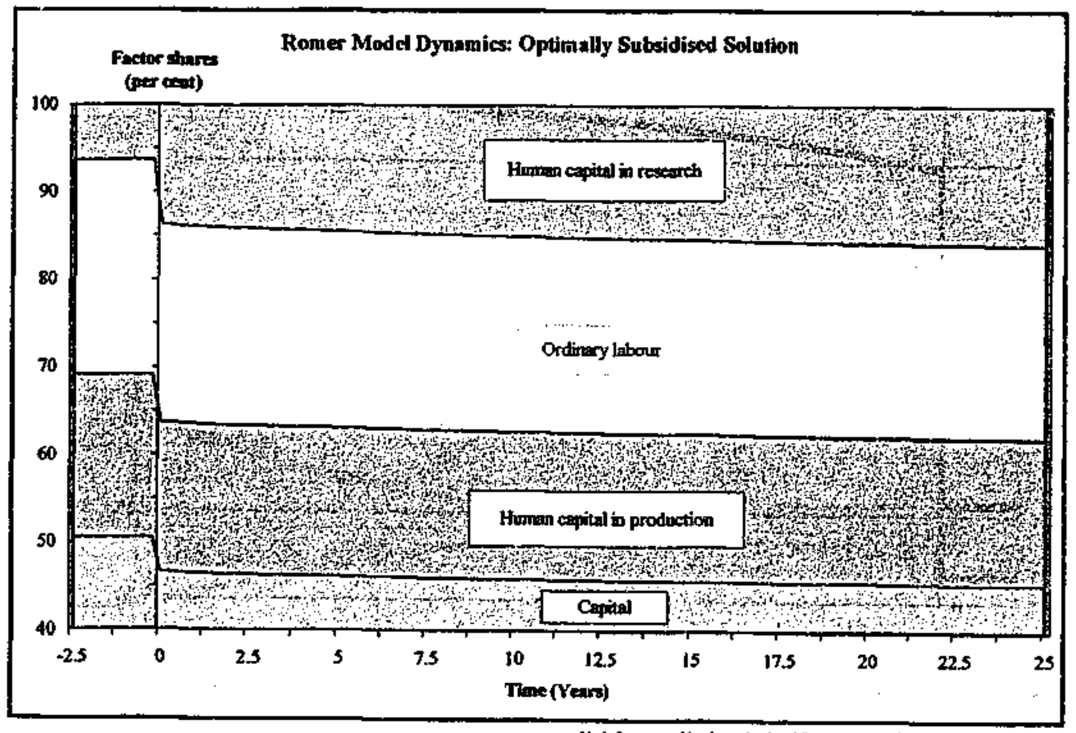
d:\phd\gempack\subrom\sub_sK\sK0sA0\QsK0A0sim.xlw(r,HA)

Figure 5.12: **Transitional dynamics of moving from the free market to the social optimum ss: effects on s_N, s_B, and k_{GP}; unanticipated introduction of the 'optimal subsidies' from time zero, benchmark parameter set.**



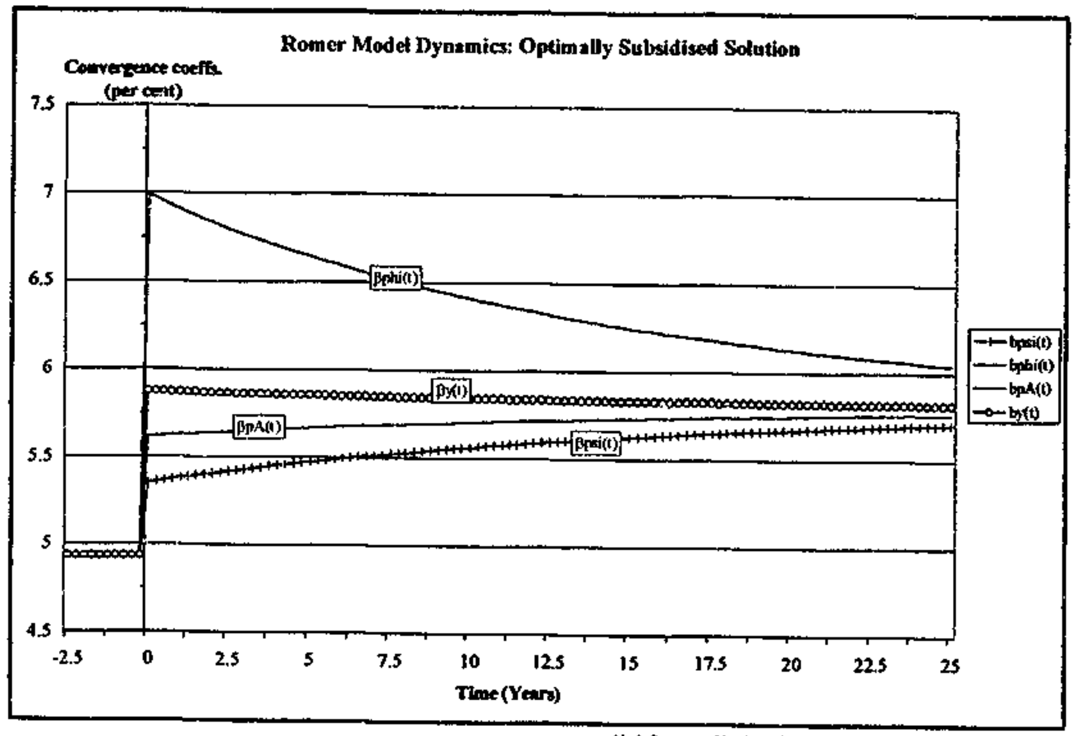
d:\phd\gempack\subrom\sub_sK\sK0sA0\QsK0A0sim.xlw(sB,kGP)

Figure 5.13: Transitional dynamics of moving from the free market to the social optimum ss: effects on the factor shares of gross income; unanticipated introduction of the 'optimal subsidies' from time zero, benchmark parameter set.



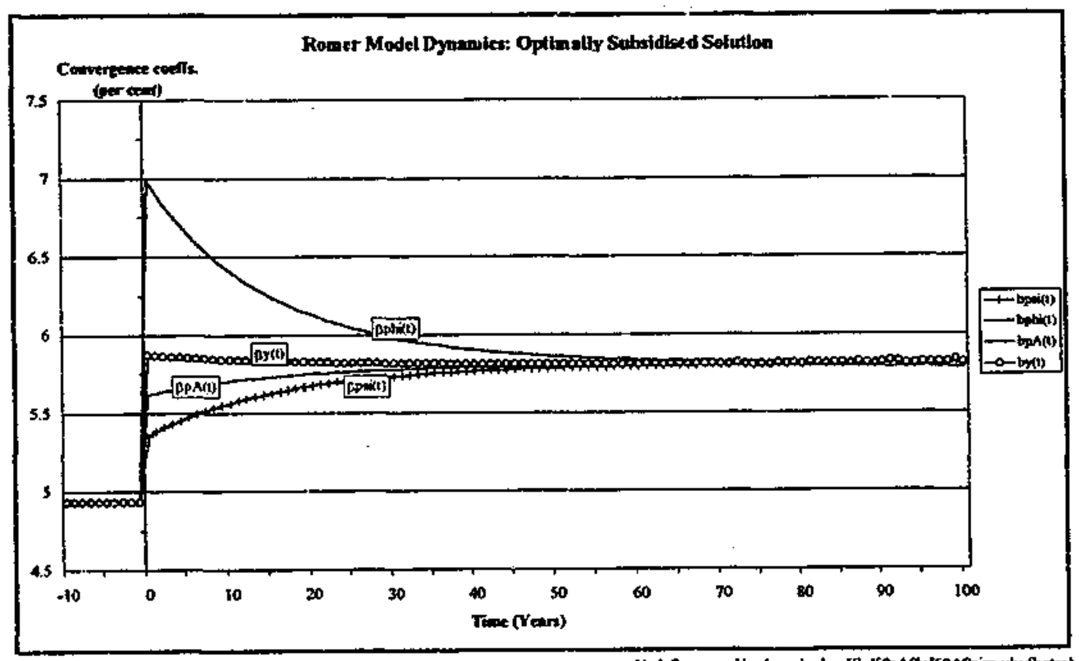
d:\phd\gempack\subrom\sub_sK\sK0sA0\QsK0A0sim.xlw(GP shares)

Figure 5.14: Transitional dynamics of moving from the free market to the social optimum ss: effects on the convergence coefficients (over 25 years); unanticipated introduction of the 'optimal subsidies' from time zero, benchmark parameter set.



d:\phd\gempack\subrom\sub_sK\sK0sA0\QsK0A0sim.xlw(betas)

Figure 5.15: Transitional dynamics of moving from the free market to the social optimum ss: effects on the convergence coefficients (over 100 years); unanticipated introduction of the 'optimal subsidies' from time zero, benchmark parameter set.



d:\phd\gempack\subrom\sub_sK\sK0sA0\sK0A0sim.xlw(betas)

Observation of the asymptotic behaviour of the convergence coefficients for this optimally subsidised benchmark Romer system (Figure 5.14 and Figure 5.15), indicates a 'steady-state' level of around 5.8 per cent. This is significantly less than the asymptotic rate of convergence for the benchmark social optimum system, which was calculated in Appendix 5.1 as β_{ss} = 6.80 per cent (also see Figure 5.6 and Table 5.3). The current result may be verified by linearising the subsidised system and computing the eigenvalues of its coefficients matrix in exactly the same way as was done for the free market and the social optimum systems (Appendices 3.1 and 3.5 and Appendix 5.1). When this is done by log-linearisation the resulting coefficients matrix Ω^{S}_{RL} is:

$$\begin{pmatrix} \dfrac{\gamma\zeta H_{Yss} - \dfrac{(1-\alpha)(1-\gamma)(r_{ss}+\delta)}{subK\gamma^2}}{1-\alpha(1-\gamma)} & -\Phi_{ss} & -\dfrac{\zeta H_{Yss} + \dfrac{\alpha(1-\gamma)(r_{ss}+\delta)}{subK\gamma^2}}{1-\alpha(1-\gamma)} \\ [\dfrac{1}{subK\gamma^2} - \dfrac{1}{\sigma}]\dfrac{(1-\alpha)(1-\gamma)(r_{ss}+\delta)}{1-\alpha(1-\gamma)} & \Phi_{ss} & [\dfrac{1}{subK\gamma^2} - \dfrac{1}{\sigma}]\dfrac{\alpha(1-\gamma)(r_{ss}+\delta)}{1-\alpha(1-\gamma)} \\ -\dfrac{(1-\gamma)(r_{ss}+\delta)[1-\alpha+\dfrac{1}{subA}\dfrac{\Psi_{ss}}{p_{Ass}}]}{1-\alpha(1-\gamma)} & 0 & \dfrac{(1-\gamma)(r_{ss}+\delta)[\dfrac{1}{\gamma subA}\dfrac{\Psi_{ss}}{p_{Ass}} - \alpha]}{1-\alpha(1-\gamma)} \end{pmatrix}$$

It is easy to verify that if subK and subA are both set at unity, the coefficients matrix becomes identical with that for the free market (equation (A3.5.11)). It is also easy to show that if subK and subA are set at their optimum levels as defined by equation (5.41), while the coefficients matrix becomes very similar to that for the social optimum (equation (A5.1.42)), it is **not identical** to it. The only differences lie in the 'Ψ and p_A coefficients of the linearised p_A differential equation' - coefficients Ω_{31} and Ω_{33}. In the optimally subsidised system these coefficients are:

$$\Omega^S_{RL31} = -(1-\alpha)[\frac{(1-\gamma)(r_{ss}+\delta)+\gamma\zeta H_{Yss}/\alpha}{1-\alpha(1-\gamma)}] - \gamma\zeta H$$

and

$$\Omega^S_{RL33} = \frac{\zeta(1-\alpha)H_{Yss}/\alpha - \alpha(1-\gamma)(r_{ss}+\delta)}{1-\alpha(1-\gamma)} + \zeta H$$

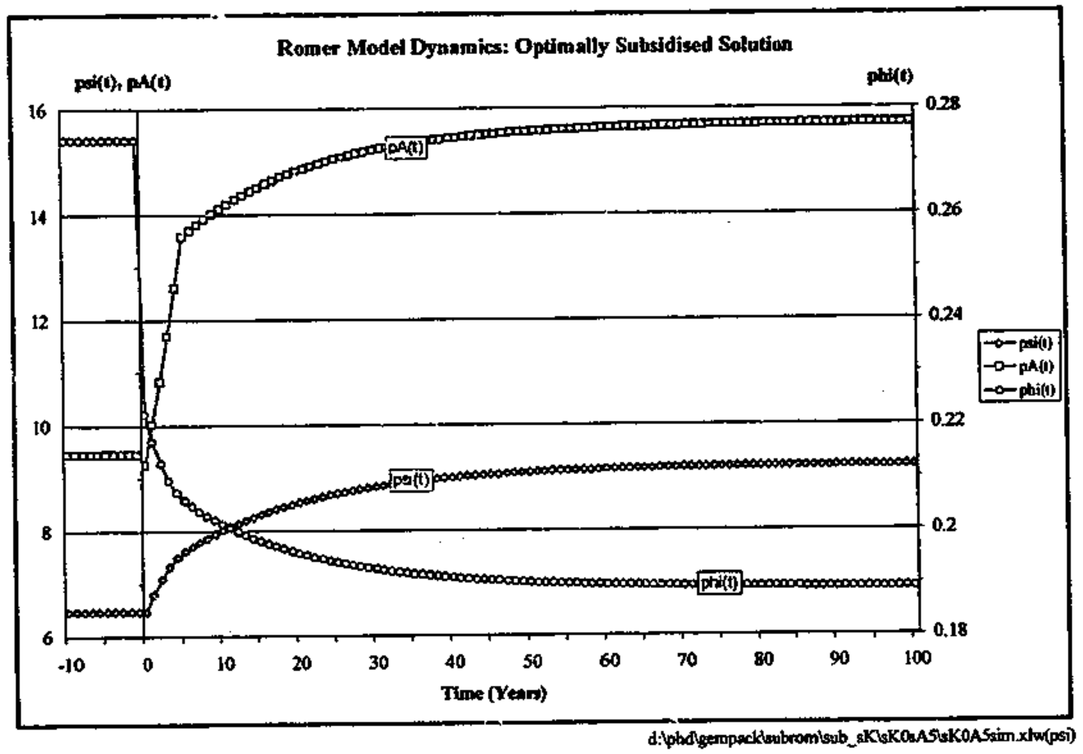
Comparison of these with Ω^O_{RL} in (A5.1.42) shows that the first contains the extra term "$-\gamma\zeta H$"; and the second the extra term "$+\zeta H$". As a result, when the eigenvalues of Ω^S_{RL} above are computed they generate the steady-state speed of convergence β=5.804 per cent, as observed in Figure 5.14 and Figure 5.15. Thus, while implementation of the optimum subsidies shifts the free market system towards a steady-state which seems identical to that of the social optimum, it does so at a somewhat slower asymptotic rate than the dynamic social optimum system itself. Curiously, this means that the steady-states are actually not identical after all!

5.3.2 Unanticipated imposition of s_K; and delayed, anticipated implementation of s_{AK}

Here, as in Section 5.3.1, the subsidy to the rentals received for general-purpose capital s_K, is instantly applied at its full 'optimum' level with no prior announcement. However, unlike the previous simulation it is assumed here that at the time of implementation of s_K, it is announced that a subsidy to the purchase of technology or designs s_{AK}, set at a level corresponding to the 'optimum', will be implemented in exactly five years time. This latter shock is therefore anticipated by the market. The results are shown in the now familiar format of charts (Figure 5.16 to Figure 5.23).

Compared with the previous simulation (in which both subsidies were implemented unannounced), when the implementation of the 'designs subsidy' is announced in advance significant differences arise - mostly only over the (five year) period until the subsidy is actually introduced. It is the growth rates and convergence coefficients that exhibit the most dramatic differences (compare Figure 5.10 with Figure 5.18; and Figure 5.14 with Figure 5.22). Significant differences are also generated between the simulation results over this period for the price of technology p_A (Figure 5.9 and Figure 5.17); and for the allocation of human capital to research, H_A (Figure 5.11 and Figure 5.19). For the capital-gross product ratio, k_{GP}, the differences persist for considerably longer (Figure 5.12 and Figure 5.20).

Figure 5.16: Transitional dynamics of moving from the free market to the social optimum ss: effects on Ψ, Φ, and p_A (over 100 years); unanticipated introduction of s_K from time zero, anticipated introduction of s_{AK} from time t=5 years, benchmark parameter set.



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When the 'designs-subsidy' is anticipated by the market the future prospect of cheaper prices causes some purchases of designs to be postponed. This reduced demand over the period to implementation results in a slower growth of designs (g_A), less human capital (H_A) necessary to generate them, and lower design prices (p_A) than when the subsidy was implemented unannounced (Figure 5.10 & Figure 5.18; Figure 5.9 & Figure 5.17; and Figure 5.11 & Figure 5.19 respectively).

Savings are also switched from research to capital accumulation until the subsidy is introduced. In comparison with the previous simulation the growth of capital g_K is substantially higher (Figure 5.10 and Figure 5.18); an increased share of gross income goes to capital, while the share to research is smaller (Figure 5.13 and Figure 5.21); and the broad measure of savings s_B (which includes savings in the form of research) is less, while the narrow measure s_N (from which research savings are excluded) is higher (Figure 5.12 and Figure 5.20). Higher demand for capital also raises its rental rate and with it the interest rate over this intermediate period (Figure 5.11 and Figure 5.19).

With more resources being directed to capital savings both Ψ and k_{GP} are higher than before (Figure 5.9 & Figure 5.17; and Figure 5.12 & Figure 5.20). Because broad savings remain less than before, there are relatively more resources being devoted to consumption. But since more of the loss of research resources go to capital accumulation than to consumption, the capital-consumption ratio Φ is lower than before (Figure 5.9 and Figure 5.17).

Figure 5.17: **Transitional dynamics of moving from the free market to the social optimum ss: effects on Ψ, Φ, and p_A (over 25 years); unanticipated introduction of s_K from time zero, anticipated introduction of s_{AK} from time t=5 years, benchmark parameter set.**

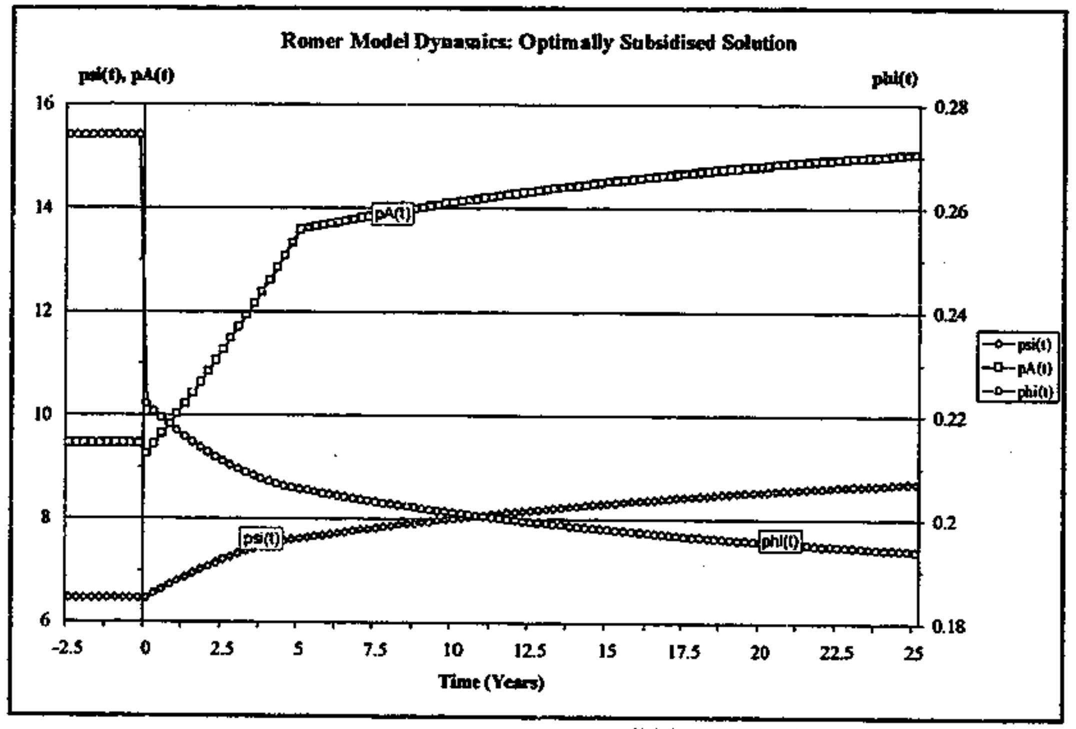

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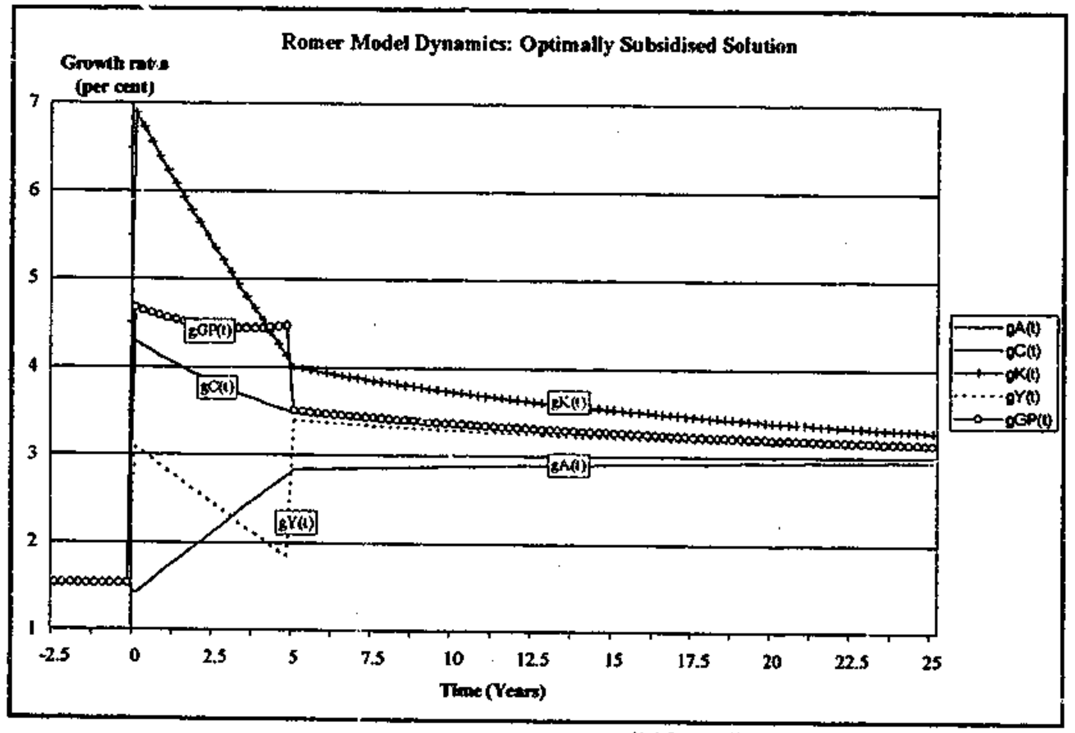
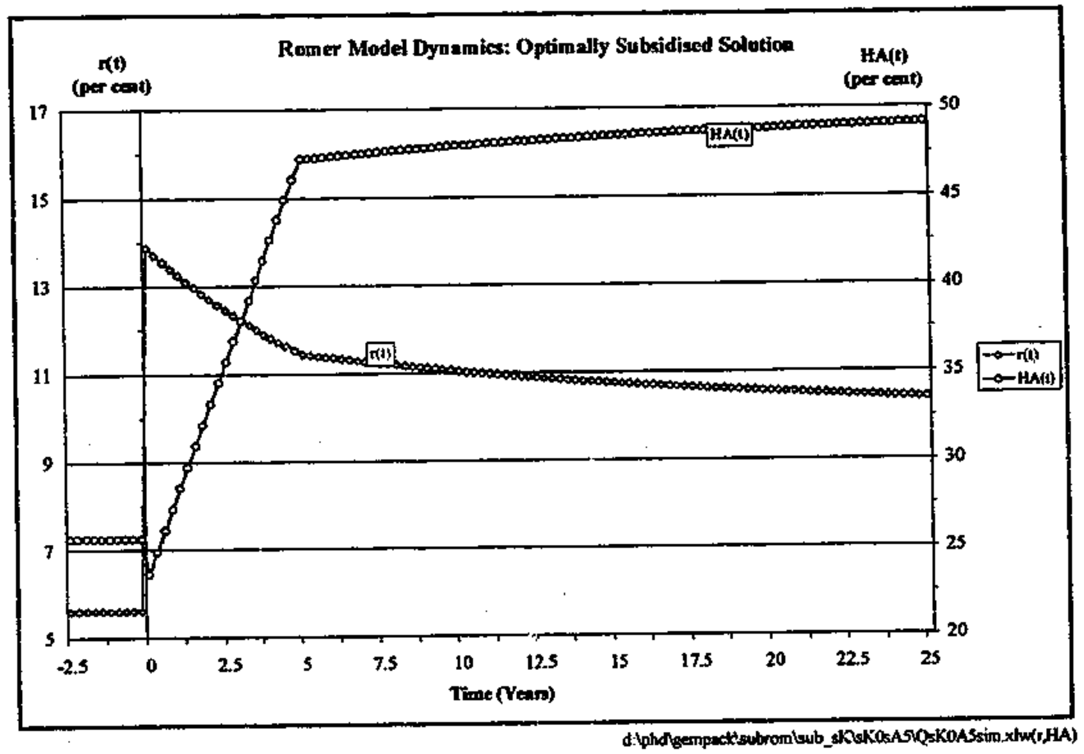
Figure 5.18: **Transitional dynamics of moving from the free market to the social optimum ss: effects on the growth rates; unanticipated introduction of s_K from time zero, anticipated introduction of s_{AK} from time t=5 years, benchmark parameter set.**



d:\phd\gempack\subrom\sub_sK\sK0sA5\QsK0A5sim.xlw(gA)

Figure 5.19: **Transitional dynamics of moving from the free market to the social optimum ss: effects on r and H_A; unanticipated introduction of s_K from time zero, anticipated introduction of s_{AK} from time t=5 years, benchmark parameter set.**



d:\phd\gempack\subrom\sub_sK\sK0sA5\QsK0A5sim.xlw(r,HA)

Figure 5.20: **Transitional dynamics of moving from the free market to the social optimum ss: effects on s_N, s_B, and k_{GP}; unanticipated introduction of s_K from time zero, anticipated introduction of s_{AK} from time t=5 years, benchmark parameter set.**

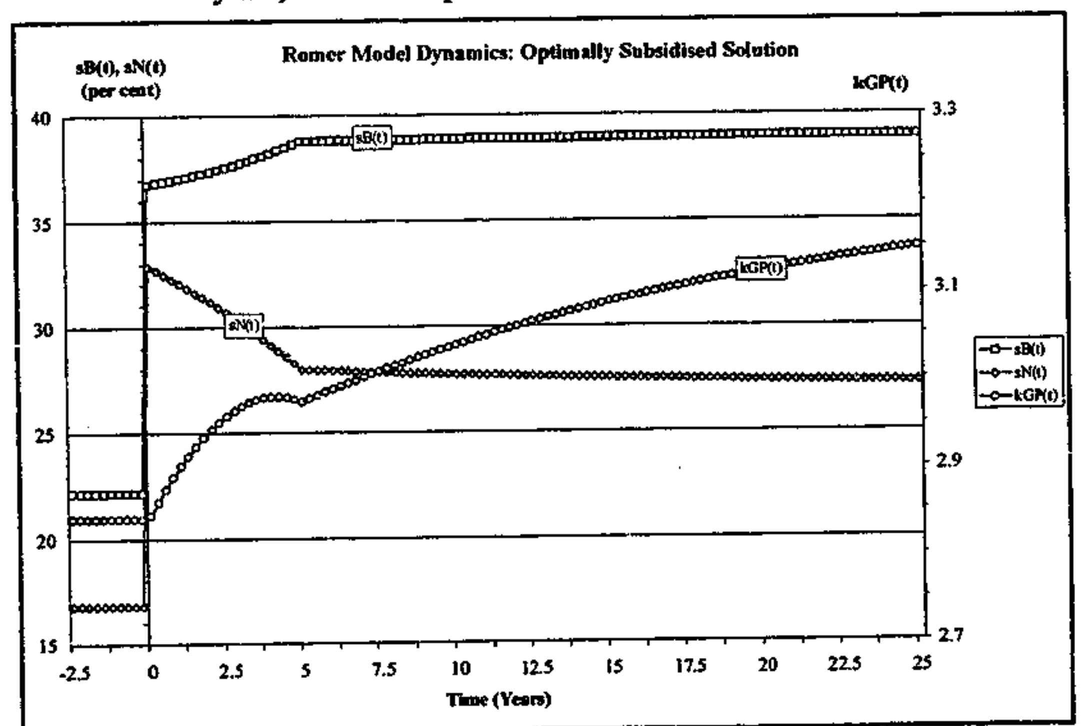


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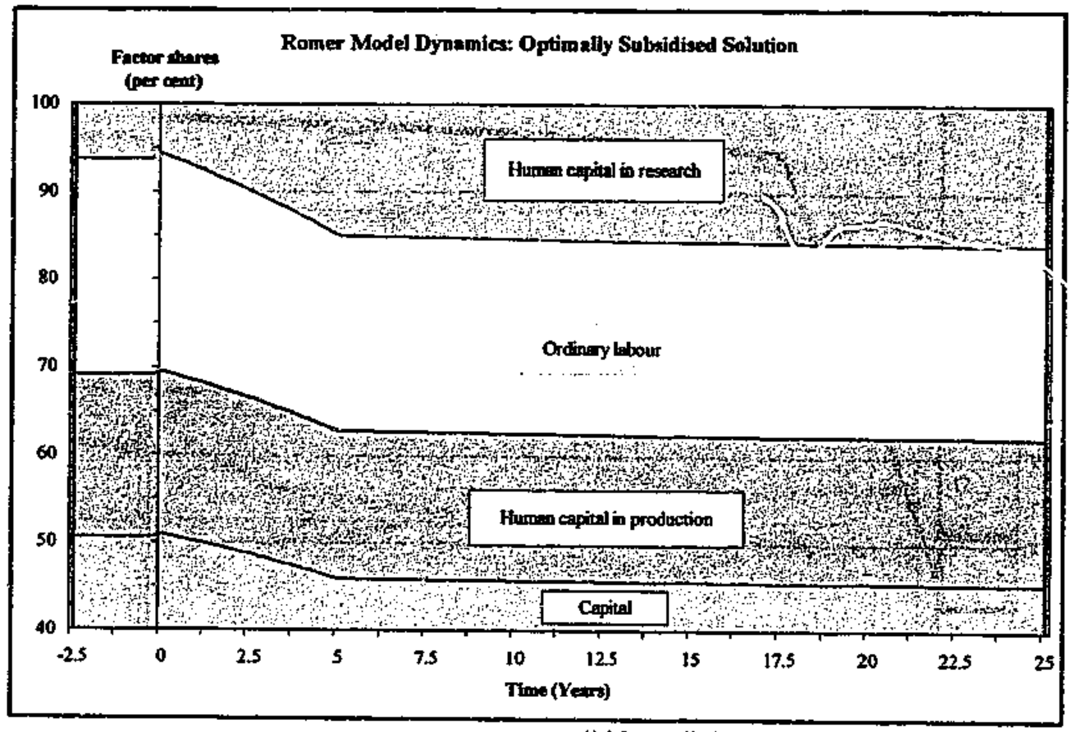
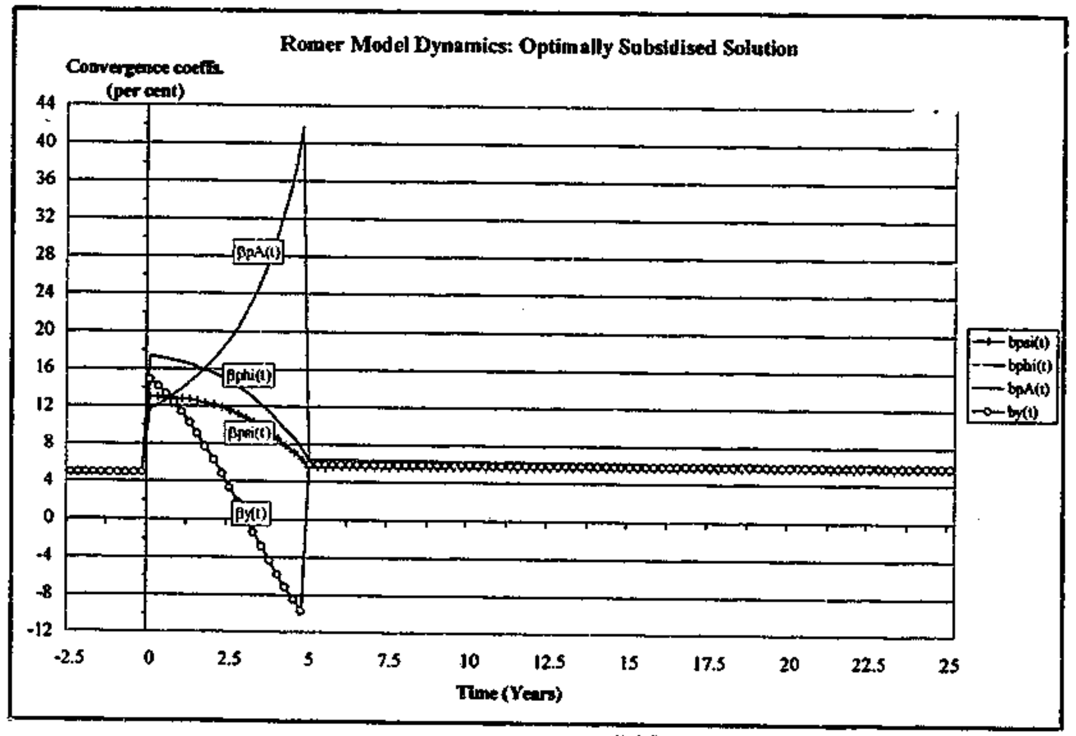
Figure 5.21: **Transitional dynamics of moving from the free market to the social optimum ss: effects on the factor shares of gross income; unanticipated introduction of s_K from time zero, anticipated introduction of s_{AK} from time t=5 years, benchmark parameter set.**



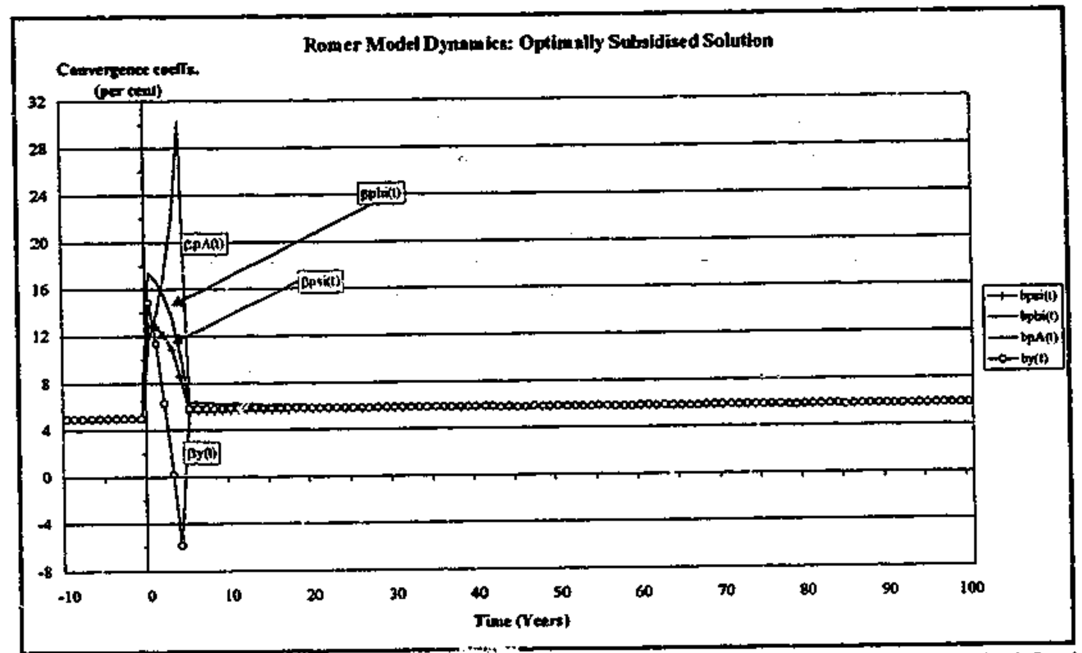
d:\phd\gempack\subrom\sub_sK\sK0sA5\QsK0A5sim.xlw(GPshares)

Figure 5.22: **Transitional dynamics of moving from the free market to the social optimum ss: effects on the convergence coefficients (over 25 years); unanticipated introduction of s_K from time zero, anticipated introduction of s_{AK} from time t=5 years, benchmark parameter set.**



d:\phd\gempack\subrom\sub_sK\sK0sA5\QsK0A5sim.xlw(betas)

Figure 5.23: **Transitional dynamics of moving from the free market to the social optimum ss: effects on the convergence coefficients (over 100 years); unanticipated introduction of s_K from time zero, anticipated introduction of s_{AK} from time t=5 years, benchmark parameter set.[a]**



d:\phd\gempack\subrom\sub_sK\sK0sA5\sK0A5sim.xlw(betas)

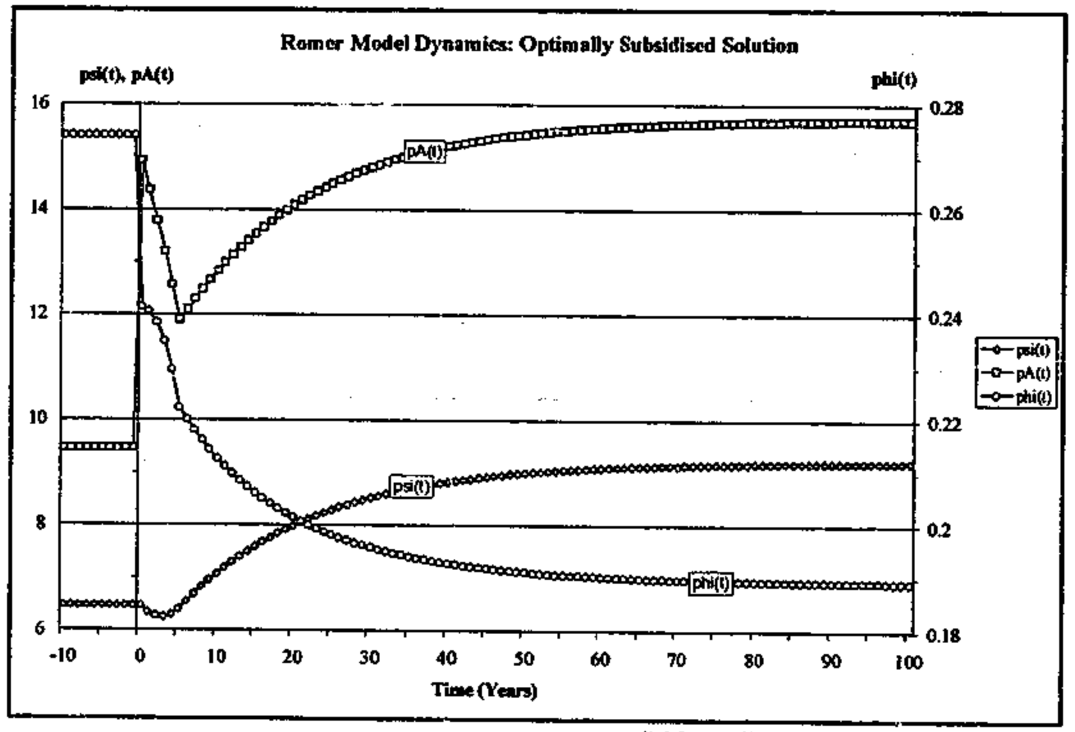
Note: a The difference between the convergence coefficients calculated over annual time intervals and those calculated quarterly is due simply to the fact that they are measured over different time point abscissae. For example, the annual calculation measures β_{pA} at t=4, and t=5 years while the quarterly calculation measures it at t=4.25, 4.5 and 4.75 years as well. And in this case, β_{pA} continues to increase rapidly over the interval t∋[4,5).

5.3.3 Unanticipated imposition of s_{AK}; and delayed, anticipated implementation of s_K

This simulation is simply the mirror image of that of the previous section. Now it is the designs subsidy s_{AK} which is fully implemented unannounced (at time t=0); and the capital rentals subsidy s_K, whose full implementation at time t=5 years is announced in advance - at the same time as the designs subsidy is being introduced. Once again the results are presented in the standard format (Figure 5.24 to Figure 5.31).

Here the divergence of the transitional dynamics from the case when neither subsidy is anticipated is greater than was the case when only the designs-subsidy was expected. This is due to the fact that the capital-subsidy has the greater general impact. The growth rates and convergence coefficients continue to exhibit large (and somewhat eccentric) differences, most significantly over the period to implementation. The allocation of human capital to research H_A and the interest rate r, also exhibit large differences over that initial period, as do the narrow savings measure s_N and the capital-gross product ratio k_{GP}, albeit to a somewhat lesser extent. Unlike the case when it was the designs subsidy that was anticipated (Section 5.3.2), many of the economic variables now show significant differences from the 'unanticipated subsidies simulation' (Section 5.3.1) for much longer periods than the five years to implementation. Prime examples are the price of designs p_A, the consumption-capital ratio Φ, and the capital-technology ratio Ψ (Figure 5.9 and Figure 5.25); the allocation of human capital to research H_A (Figure 5.11 and Figure 5.27); and the capital-gross product ratio k_{GP} (Figure 5.12 and Figure 5.28).

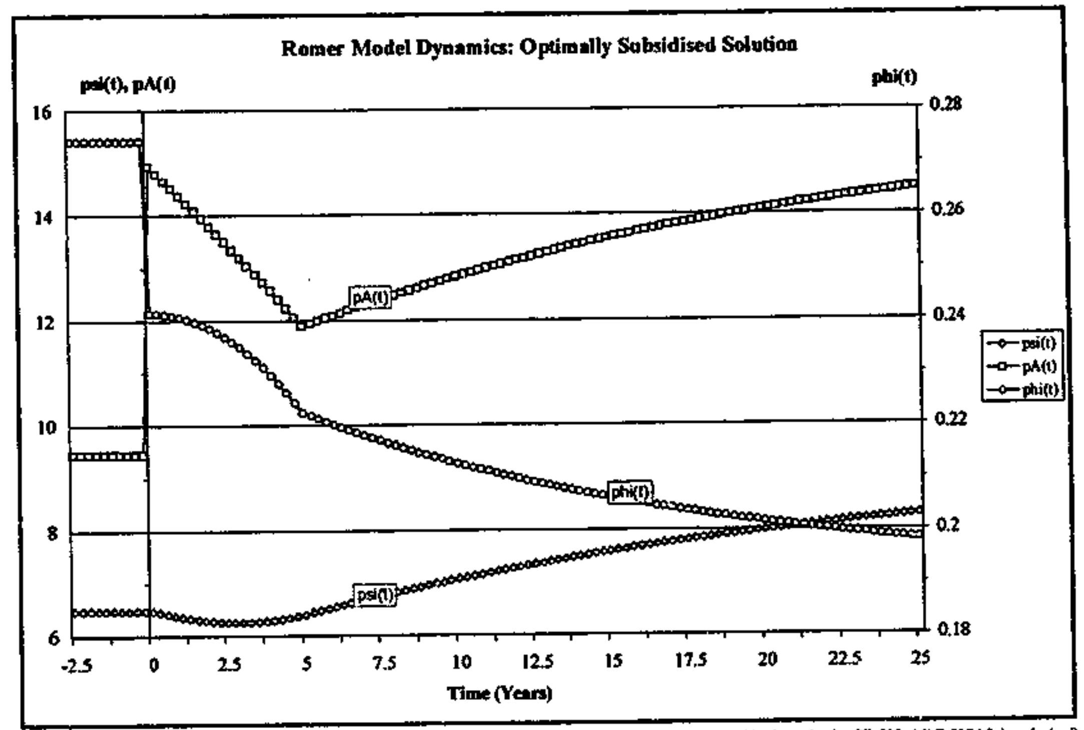
Figure 5.24: **Transitional dynamics of moving from the free market to the social optimum ss: effects on Ψ, Φ, and p_A (over 100 years); unanticipated introduction of s_{AK} from time zero, anticipated introduction of s_K from time t=5 years, benchmark parameter set.**



d:\phd\gempack\subrom\sub_sK\sK5sA0\sK5A0sim.xlw(psi)

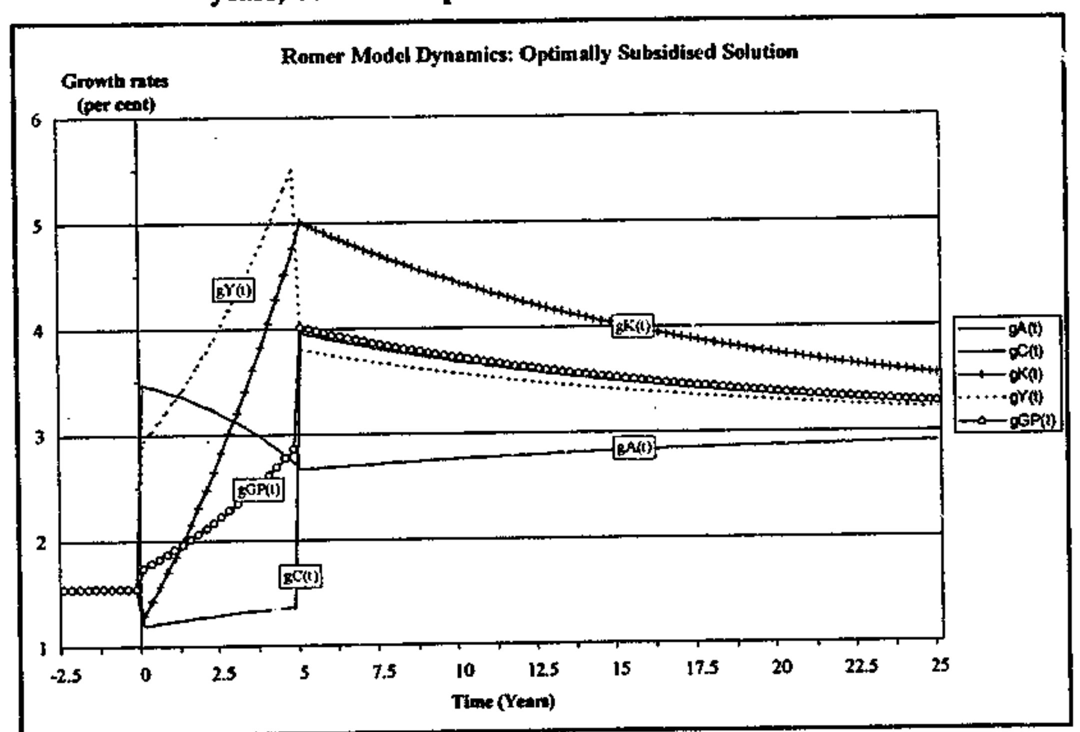
Knowledge that the rentals received for general-purpose capital will soon be subsidised, causes some postponement of capital accumulation: The growth of capital g_K is less than before (Figure 5.10 and Figure 5.26), and so is the investment savings rate s_N. (Figure 5.12 and Figure 5.28). There is also a temporary switch from this form of saving to that of research. The capital share of gross income S_K is much lower, and the research share S_A much higher than when the savings subsidy was not foreseen (Figure 5.13, Figure 5.21 and Figure 5.29). However, extra savings in the form of research do not fully make up for the decline in capital savings over the period before the saving subsidy is implemented. This is reflected by the fact that the broad measure of savings s_B is also lower than before over that period (Figure 5.12 and Figure 5.28). Also, with relatively more resources devoted to consumption and research, Ψ and k_{GP} are lower, and Φ is higher (compare Figure 5.9 with Figure 5.25; and Figure 5.12 with Figure 5.28). Finally, as may readily be seen from equations (5.19) and (5.20), the big rise in rental rates and the interest rate r, is delayed until the subsidy is actually implemented (Figure 5.11 and Figure 5.27).

Figure 5.25: **Transitional dynamics of moving from the free market to the social optimum ss: effects on Ψ, Φ, and p_A (over 25 years); unanticipated introduction of s_{AK} from time zero, anticipated introduction of s_K from time t=5 years, benchmark parameter set.**



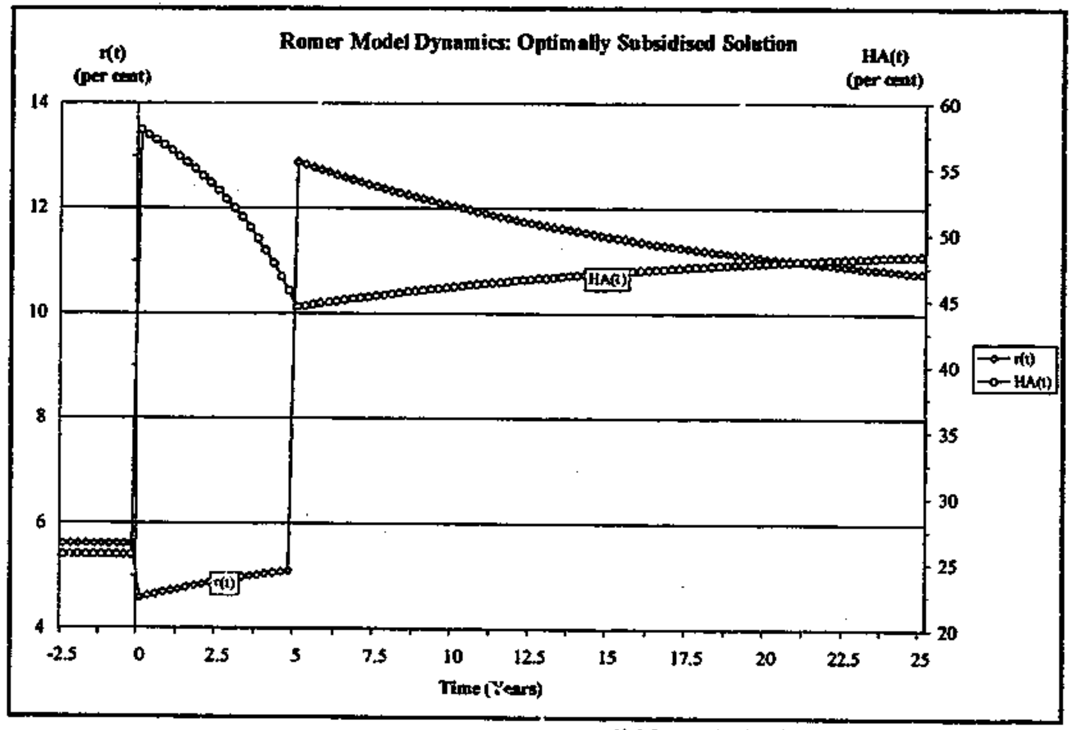
d:\phd\gempack\subrom\sub_sK\sK5sA0\QsK5A0sim.xlw(psi)

Figure 5.26: **Transitional dynamics of moving from the free market to the social optimum ss: effects on the growth rates; unanticipated introduction of s_{AK} from time zero, anticipated introduction of s_K from time t=5 years, benchmark parameter set.**



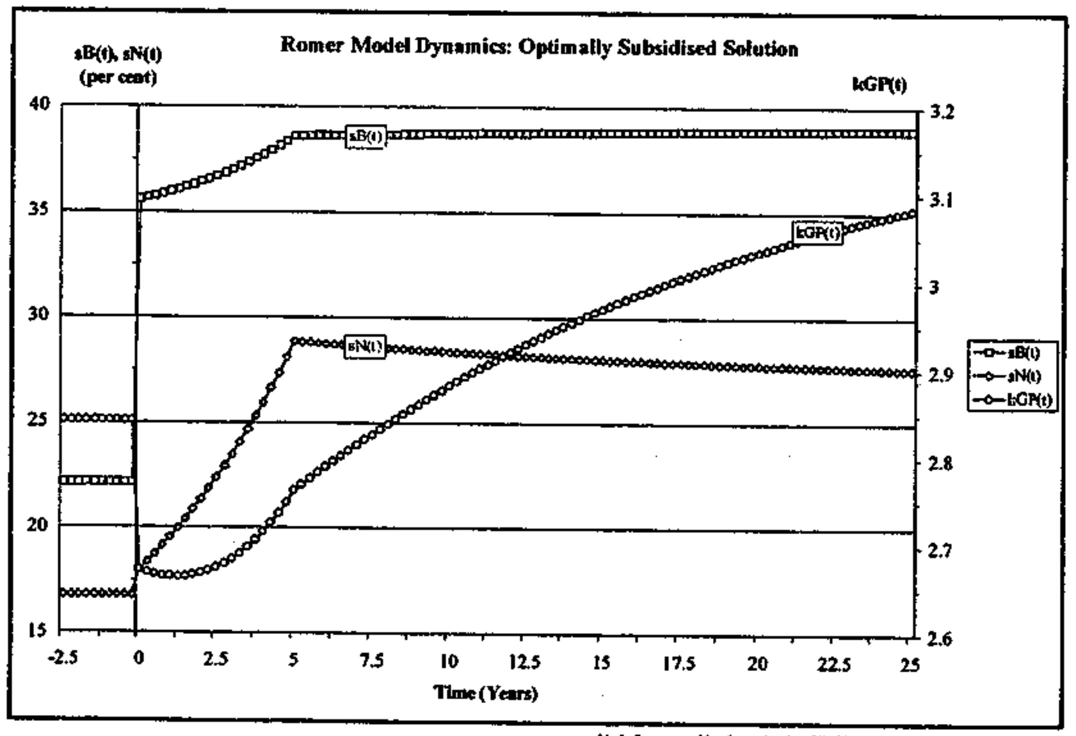
d:\phd\gempack\subrom\sub_sK\sK5sA0\QsK5A0sim.xlw(gA)

Figure 5.27: Transitional dynamics of moving from the free market to the social optimum ss: effects on r and H_A; unanticipated introduction of s_{AK} from time zero, anticipated introduction of s_K from time t=5 years, benchmark parameter set.



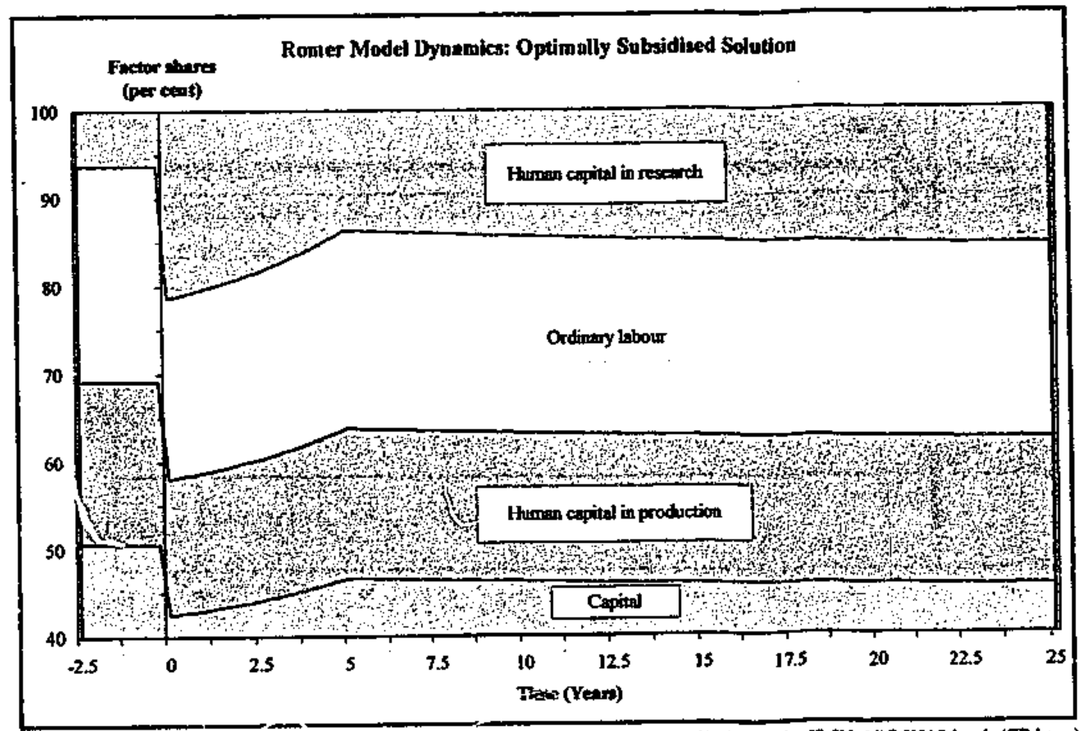
d:\phd\gempack\subrom\sub_sK\sK5sA0\QsK5A0sim.xlw(r,HA)

Figure 5.28: Transitional dynamics of moving from the free market to the social optimum ss: effects on s_N, s_B, and k_{GP}; unanticipated introduction of s_{AK} from time zero, anticipated introduction of s_K from time t=5 years, benchmark parameter set.



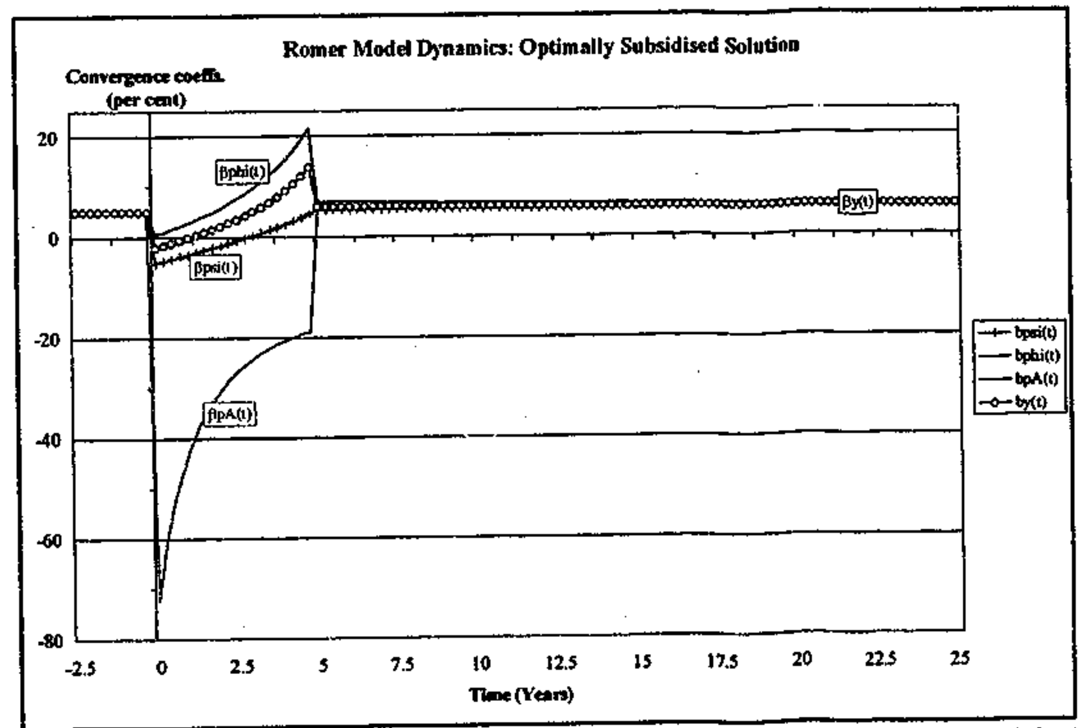
d:\phd\gempack\subrom\sub_sK\sK5sA0\QsK5A0sim.xlw(sB,kGP)

Figure 5.29: Transitional dynamics of moving from the free market to the social optimum ss: effects on the factor shares of gross income; unanticipated introduction of s_{AK} from time zero, anticipated introduction of s_K from time t=5 years, benchmark parameter set.



d:\phd\gempack\subrom\sub_sK\sK5sA0\QsK5A0sim.xlw(GPshares)

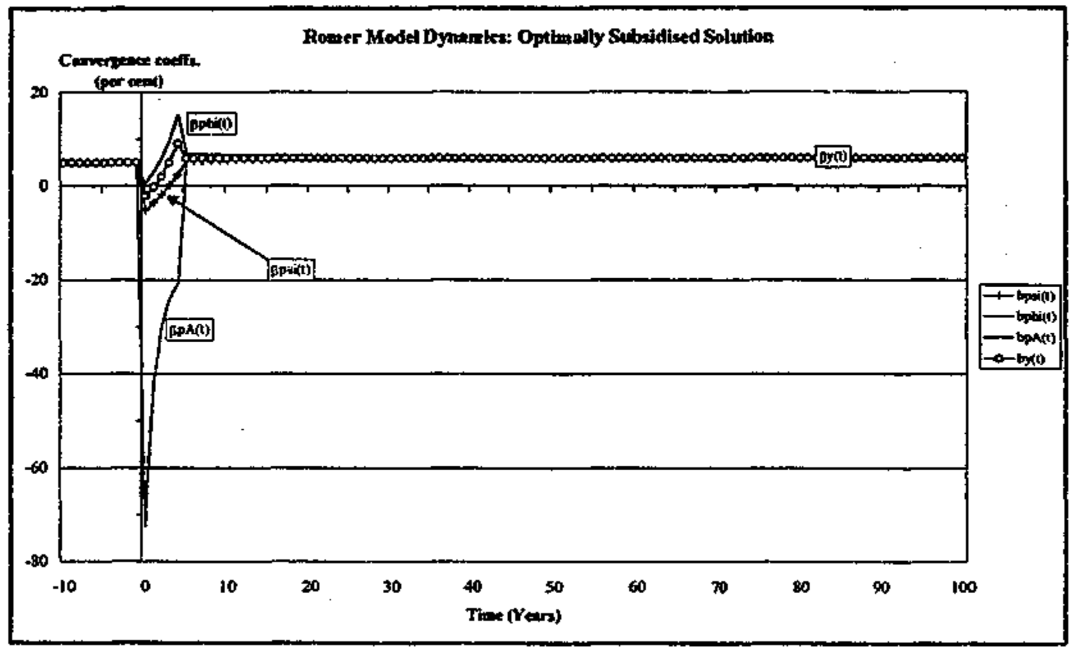
Figure 5.30: Transitional dynamics of moving from the free market to the social optimum ss: effects on the convergence coefficients (over 25 years); unanticipated introduction of s_{AK} from time zero, anticipated introduction of s_K from time t=5 years, benchmark parameter set.[a]



d:\phd\gempack\subrom\sub_sK\sK5sA0\QsK5A0sim.xlw(betas)

Note: a The negative portions of the convergence coefficients reflect temporary movements away from the new steady-states for the relevant variables.

Figure 5.31: **Transitional dynamics of moving from the free market to the social optimum ss: effects on the convergence coefficients (over 100 years); unanticipated introduction of s_{AK} from time zero, anticipated introduction of s_K from time t=5 years, benchmark parameter set.[a]**



d:\phd\gempack\subrom\sub_sK\sK5sA0\sK5A0sim.xlw(betas)

Note: a The negative portions of the convergence coefficients reflect temporary movements away from the new steady-states for the relevant variables.

5.3.4 Dynamic effects of different methods of implementation

The 'method of implementation' is meant here to refer to both the timing of any announcement of the prospective introduction of the optimum subsidies[36], and the timing of the introduction itself - which may be a full implementation at a single point of time, or a phased implementation over a number of different time points. Only three simple methods were considered in the preceding sections. Nevertheless, it is clear from the results of these that alternative methods of implementing the optimum subsidies can produce significant differences in the adjustment paths by which the social welfare maximising equilibrium is approached (compare Figure 5.9, Figure 5.17 and Figure 5.25; Figure 5.10, Figure 5.18 and Figure 5.26 etc.).

Large differences are generally confined to the early period of adjustment, often only until both subsidies are implemented or not long afterwards. However, in some cases and for some variables large differences can persist for considerably longer periods. For example, compare the behaviour of the interest rate r when it is announced in advance that the capital subsidy will be implemented after 5 years, with its behaviour either when it is the designs subsidy that is delayed, or when both subsidies are implemented unannounced. That is, compare Figure 5.27 with Figure 5.11 and Figure 5.19. Naturally, these are issues with which policy makers would need to be aware before selecting the method of implementing the optimum subsidies.

[36] Or, to refer to the timing of any other means by which the subsidies might be anticipated.

Appendix 5.1

A social planning solution to the Romer Model

A5.1.1 Derivation of the dynamic system and its steady-state

The decentralised or market equilibrium for the Romer model, necessarily incorporates certain market imperfections (see Section 5.1). These dictates would not be followed by a social planner, who could allocate resources by administrative fiat. Thus, the social planning problem in the 'Romer economy' is to maximise the discounted sum of all future aggregate utility, subject to the production and research technologies and to the economy's aggregate resource constraint (that consumption and savings together must equal total income). Formally, the problem is:[37]

$$
\begin{aligned}
&\text{Maximise} && \int_0^{\infty} U\{C(t)\}e^{-\rho t}dt \\
&\text{subject to} && \dot{K}(t) = Y(t) - C(t) - \delta K(t) \\
& && \dot{A}(t) = \zeta H_A(t)A(t) \\
& && K(0), A(0) \text{ given; } K(t), A(t) \geq 0 \qquad\qquad (A5.1.1)\\
&\text{where} && U\{C(t)\} = [C(t)^{1-\sigma} - 1]/(1-\sigma) \quad \text{for } \sigma > 0 \\
& && Y(t) = \eta^{-\gamma} H_Y(t)^{\alpha(1-\gamma)} L^{(1-\alpha)(1-\gamma)} K(t)^{\gamma} A(t)^{(1-\gamma)} \\
& && H_A(t) + H_Y(t) = H
\end{aligned}
$$

Then as before in Chapter 2 (Appendix 2.2 in particular), proceeding according to the *Maximum Principle* of Pontryagin et al (1962), the (present valued) Hamiltonian, $\mathcal{H}$, is formed and the first-order conditions for a maximum are applied:

$$\mathcal{H} = [(C(t)^{1-\sigma} - 1)/(1-\sigma)]e^{-\rho t} + \lambda(t)[Y(t) - C(t) - \delta K(t)] + \mu(t)\zeta[H - H_Y(t)]A(t)$$

where $\lambda(t)$ and $\mu(t)$, the dynamic multipliers, are shadow prices of capital $K(t)$, and output $A(t)$ respectively.

The first-order conditions for the problem (A5.1.1) are given in equations (A5.1.2) to (A5.1.6) as follows:

$$\frac{\partial \mathcal{H}}{\partial C(t)} = 0 \Rightarrow \quad \lambda(t) = C(t)^{-\sigma} e^{-\rho t} \qquad (A5.1.2)$$

$$\frac{\partial \mathcal{H}}{\partial H_Y(t)} = 0 \Rightarrow \quad \frac{\lambda(t)}{\mu(t)} = \frac{\zeta A(t) H_Y(t)}{\alpha(1-\gamma)Y(t)} \qquad (A5.1.3)$$

[37] Romer (1990b), and Chiang(1992) in reporting Romer's work, both solve this problem, but only to the point of the (posited) balanced growth equilibrium that results. The addition here is the explicit use of the transversality conditions, and the derivation of the full dynamic system.

$$\dot{\lambda}(t) = -\frac{\partial \mathcal{H}}{\partial K(t)} \Rightarrow \frac{\dot{\lambda}(t)}{\lambda(t)} = \delta - \gamma \frac{Y(t)}{K(t)} \tag{A5.1.4}$$

$$\dot{\mu}(t) = -\frac{\partial \mathcal{H}}{\partial A(t)} \Rightarrow \frac{\dot{\mu}(t)}{\mu(t)} = -(1-\gamma)\frac{\lambda}{\mu}\frac{Y(t)}{A(t)} - \zeta[H - H_Y(t)] \tag{A5.1.5}$$

and the transversality conditions:

$$\begin{aligned} &\lim_{t\to\infty} \mu(t) \geq 0 \ \text{ and } \ \lim_{t\to\infty} \mu(t)A(t) = 0 \\ \text{and}& \\ &\lim_{t\to\infty} \lambda(t) \geq 0 \ \text{ and } \ \lim_{t\to\infty} \lambda(t)K(t) = 0 \end{aligned} \tag{A5.1.6}$$

The dynamic system for the social optimum solution to the Romer model may now be derived from problem statement (A5.1.1) and its first-order conditions (A5.1.2) to (A5.1.6) as follows. First, (A5.1.2) is differentiated with respect to time and re-arranged to produce:

$$\dot{C}(t)/C(t) = [-\dot{\lambda}(t)/\lambda(t) - \rho]/\sigma \tag{A5.1.7}$$

At this point the interest rate variable r(t) is introduced. It is still related to the rental rate on capital $r_K(t)$ by the logic of Section 2.2.5 and by equation (2.12) specifically. That is: $r(t) = r_K(t) - \delta$; but in this *Pareto optimal solution* to the model, where there are no market distortions, the rental rate on general-purpose capital is now equal to the value of its marginal product:

$$r_K(t) = \frac{\partial Y(t)}{\partial K(t)} = \gamma \eta^{-\gamma} L^{(1-\alpha)(1-\gamma)} H_Y(t)^{\alpha(1-\gamma)} [\frac{K(t)}{A(t)}]^{\gamma-1} = \gamma \frac{Y(t)}{K(t)} \tag{A5.1.8}$$

so from equation (A5.1.4) the interest rate is given by:

$$r(t) = -\dot{\lambda}(t)/\lambda(t) \tag{A5.1.9}$$

Equations (A5.1.4) and (A5.1.9) also generate the following alternative expression for output which will prove useful:

$$Y(t) = \frac{r(t) + \delta}{\gamma} K(t) \tag{A5.1.10}$$

A dynamic equation for the growth of technology may be written directly from the conditions of the problem statement (A5.1.1):

$$\dot{A}(t) = \zeta[H - H_Y(t)]A(t) \tag{A5.1.11}$$

Then the capital accumulation equation, also from the problem statement (A5.1.1), is transformed via (A5.1.10) to produce:

$$\dot{K}(t) = \frac{r(t) + \delta}{\gamma} K(t) - C(t) - \delta K(t) \tag{A5.1.12}$$

Next, the consumption growth equation is obtained from (A5.1.7) and (A5.1.10):

$$\dot{C}(t) = \frac{r(t) - \rho}{\sigma} C(t) \tag{A5.1.13}$$

And combining (A5.1.13) and (A5.1.5) generates a dynamic equation for μ:

$$\dot{\mu}(t) = -[\zeta(1/\alpha - 1)H_Y(t) + \zeta H]\mu(t) \tag{A5.1.14}$$

The preceding algebra has now generated a system of four first-order ordinary differential equations, (A5.1.11) to (A5.1.14), in the four variables A(t), K(t), C(t) or $\lambda(t)$, and $\mu(t)$. While there are actually six variables involved in the above ODEs, the two 'extra' variables r(t) and $H_Y(t)$ could readily be eliminated since they are both functions of the other dynamic variables. Substituting for the production function Y(t) from (A5.1.1) into (A5.1.3) and into (A5.1.10) generates expressions for $H_Y(t)$ and r(t) respectively:

$$H_Y(t) = [\frac{\alpha(1-\gamma)}{\delta \eta^{\gamma}} L^{(1-\alpha)(1-\gamma)} [K(t)/A(t)]^{\gamma} [\lambda(t)/\mu(t)]]^{\frac{1}{1-\alpha(1-\gamma)}} \tag{A5.1.15}$$

$$r(t) = \gamma \eta^{-\gamma} H_Y(t)^{\alpha(1-\gamma)} L^{(1-\alpha)(1-\gamma)} [K(t)/A(t)]^{\gamma-1} - \delta \tag{A5.1.16}$$

The four boundary conditions necessary to integrate such a system are provided by the given initial values A(0) and K(0) from the dynamic optimisation problem statement (A5.1.1), and by the transversality conditions (A5.1.6). As usual, the transversality conditions ensure that in order to achieve the social optimum, the correct trajectory (that is, the saddle-path) is the one attained from amongst the infinite number of possible dynamic paths allowed by the initial conditions and differential equations only. Thus, at this stage the complete dynamic system for the social optimum solution to the model comprises equations (A5.1.11) to (A5.1.16) together with the initial and transversality conditions.

Comparison with equations (2.30) to (2.38) from Chapter 2 reveals that this social optimum solution system is extremely similar to that of the market solution to the model. Moreover, the procedures for determining the asymptotic dynamics and the steady-state equilibrium for the social optimum solution follow exactly those used in Section 2.3.3 for the market solution.[38] In particular, it follows from the transversality conditions that the following four growth rate limits must all be constants:

$$\lim_{t\to\infty}[\dot{\lambda}(t)/\lambda(t)],\ \lim_{t\to\infty}[\dot{K}(t)/K(t)],\ \lim_{t\to\infty}[\dot{\mu}(t)/\mu(t)],\text{ and } \lim_{t\to\infty}[\dot{A}(t)/A(t)].$$

Combining this constancy with the necessary conditions for the dynamic optimisation and the consequent dynamic equations above, ensures the system will tend asymptotically to a *balanced growth* equilibrium in which technology, capital, consumption, and output all grow at the same constant rate:

$$\lim_{t\to\infty}[\dot{A}(t)/A(t)] = \lim_{t\to\infty}[\dot{K}(t)/K(t)] = \lim_{t\to\infty}[\dot{C}(t)/C(t)] = \lim_{t\to\infty}[\dot{Y}(t)/Y(t)] = g^O$$

---

[38] The only difference is that the ratio of the two shadow prices μ and λ is considered in place of the price of technology variable p_A.

And where:

$$\lim_{t\to\infty}[\dot{\mu}(t)/\mu(t)] = \lim_{t\to\infty}[\dot{\lambda}(t)/\lambda(t)] = -\lim_{t\to\infty} r(t) = -r^{O}_{ss}$$

$$\lim_{t\to\infty}[K(t)/A(t)] = \Psi^{O}_{ss}; \quad \lim_{t\to\infty}[C(t)/K(t)] = \Phi^{O}_{ss}; \quad \lim_{t\to\infty}[\mu(t)/\lambda(t)] = p^{O}_{Ass}$$

$$\lim_{t\to\infty} H_A(t) = H_{A_{ss}}^{O}; \quad \lim_{t\to\infty} H_Y(t) = H_{Y_{ss}}^{O}$$

for $g^{O}, r^{O}_{ss}, \Psi^{O}_{ss}, \Phi^{O}_{ss}, p^{O}_{Ass}$, and H^{O}_{Yss} all constants of the balanced growth equilibrium.

Continuing to follow the approach of Section 2.3.3, the dynamic system is next transformed so that it yields a stationary equilibrium or steady-state rather than the balanced growth equilibrium described above. For this purpose three new dynamic variables are defined. The first two are the same as those used in the market solution of Section 2.3.3, namely $\Psi(t)=K(t)/A(t)$ and $\Phi(t)=C(t)/K(t)$. The third, necessary to eliminate the shadow price terms $\mu(t)$ and $\lambda(t)$, is the ratio of these terms. Now since μ is the shadow price of technology A, measured in *utils*; and λ is the shadow price of capital K, which is equivalent to that of output Y, and is also measured in *utils*, then the ratio μ/λ is the price of technology measured in units of output.[39] It is therefore equivalent to the variable p_A in the market version of the model. Hence, the third 'not-so-new' dynamic variable to be defined is $p_A(t)=\mu(t)/\lambda(t)$, and with this definition equation (A5.1.3) can be seen to be simply a restatement of equation (2.45).

Taking logs and derivatives, and substituting from equations (A5.1.11) to (A5.1.16) generates the three-variable stationary steady-state dynamic system:

$$\dot{\Psi}(t) = [\frac{r(t)+\delta}{\gamma} - \Phi(t) - \delta - \zeta H + \zeta H_Y(t)]\Psi(t) \qquad \text{(A5.1.17)}$$

$$\dot{\Phi}(t) = [\frac{r(t)-\rho}{\sigma} - \frac{r(t)+\delta}{\gamma} + \Phi(t) + \delta]\Phi(t) \qquad \text{(A5.1.18)}$$

$$\dot{p}_A(t) = [r(t) - \zeta(1/\alpha - 1)H_Y(t) - \zeta H]p_A(t) \qquad \text{(A5.1.19)}$$

where:

$$H_Y(t) = [\frac{\alpha(1-\gamma)}{\zeta\eta^{\gamma}} L^{(1-\alpha)(1-\gamma)} \Psi(t)^{\gamma} p_A(t)^{-1}]^{\frac{1}{1-\alpha(1-\gamma)}} \qquad \text{(A5.1.20)}$$

$$\begin{aligned} r(t) &= \gamma\eta^{-\gamma} H_Y(t)^{\alpha(1-\gamma)} L^{(1-\alpha)(1-\gamma)} \Psi(t)^{\gamma-1} - \delta \\ &= \frac{\zeta\gamma}{\alpha(1-\gamma)} H_Y(t) p_A(t) \Psi(t)^{-1} - \delta \end{aligned} \qquad \text{(A5.1.21)}$$

and for which the boundary conditions easily transform to:

$$\Psi(0) \text{ given;} \qquad \text{(A5.1. 22)}$$

[39] Since it is utility which is being maximised, shadow prices will be expressed in terms of the units in which utility is measured (these are termed *utils*).

$$\lim_{t\to\infty} \Phi(t) = \Phi^{O}_{ss}; \qquad \text{(A5.1.23)}$$

and

$$\lim_{t\to\infty} p_A(t) = p^{O}_{Ass} \qquad \text{(A5.1.24)}$$

Finally, the steady-state equilibrium of the dynamic system (A5.1.17) to (A5.1.21), equivalent to the balanced growth equilibrium of (A5.1.11) to (A5.1.16), is then computed by taking limits and setting each of (A5.1.17) to (A5.1.19) to zero and solving.

$$\frac{r^{O}_{ss}+\delta}{\gamma} - \Phi^{O}_{ss} - \delta - \zeta H + \zeta H^{O}_{Yss} = 0 \qquad \text{(A5.1.25)}$$

$$\frac{r^{O}_{ss}-\rho}{\sigma} - \frac{r^{O}_{ss}+\delta}{\gamma} + \Phi^{O}_{ss} + \delta = 0 \qquad \text{(A5.1.26)}$$

$$r^{O}_{ss} - \zeta(1/\alpha - 1)H^{O}_{Yss} - \zeta H = 0 \qquad \text{(A5.1.27)}$$

Adding equations (A5.1.25) and (A5.1.26) and substituting for r^{O}_{ss} from (A5.1.27) generates the social optimum steady-state allocation of human capital to output production as:

$$H^{O}_{Yss} = \frac{\alpha(\sigma-1)H + \alpha\rho/\zeta}{(1-\alpha)+\alpha\sigma} \qquad \text{(A5.1.28)}$$

And using $H = H^{O}_{Ass} + H^{O}_{Yss}$ and $(\dot{A}/A)^{O}_{ss} = g^{O} = \zeta H^{O}_{Ass}$, the equilibrium levels of human capital in research, and the growth rate are obtained as:

$$H^{O}_{Ass} = \frac{H - \alpha\rho/\zeta}{(1-\alpha)+\alpha\sigma} \quad \text{and} \quad g^{O} = \frac{\zeta H - \alpha\rho}{(1-\alpha)+\alpha\sigma} \qquad \text{(A5.1.29)}$$

Then, substituting the result for H^{O}_{Yss} back into (A5.1.27) generates the steady-state interest rate:

$$r^{O}_{ss} = \frac{\sigma\zeta H + (1-\alpha)\rho}{(1-\alpha)+\alpha\sigma} \qquad \text{(A5.1.30)}$$

Next, the steady-state ratio of consumption to capital is obtained from the capital accumulation relation $(\dot{K}/K)^{O}_{ss} = g^{O} = (r^{O}_{ss}+\delta)/\gamma - \Phi^{O}_{ss} - \delta$; and the steady-state levels of capital intensity of technology and the technology price level from (A5.1.21):

$$\Phi^{O}_{ss} = (r^{O}_{ss}+\delta)/\gamma - g^{O} - \delta \qquad \text{(A5.1.31)}$$

$$\Psi^{O}_{ss} = [\frac{\gamma {H^{O}_{Yss}}^{\alpha(1-\gamma)} L^{(1-\alpha)(1-\gamma)}}{\eta^{\gamma}(r^{O}_{ss}+\delta)}]^{\frac{1}{1-\gamma}} \qquad \text{(A5.1.32)}$$

and

$$p^{O}_{Ass} = \frac{\alpha(1-\gamma)}{\gamma\zeta}\frac{r^{O}_{ss}+\delta}{H^{O}_{Yss}}\Psi^{O}_{ss} \qquad \text{(A5.1.33)}$$

A5.1.2 Calculation of the speed of convergence through linearisation

The sole purpose of this section is to obtain numerical estimates of the asymptotic speed of convergence of the social optimum system towards its steady-state. As discussed in Section 3.2.3, this is given by the so-called convergence coefficient β, which is equal to the absolute value of the negative eigenvalue from the *coefficients matrix* of the linearised system, whether it is ordinarily-linearised or log-linearised. In terms of the Chapter 3 notation:

$$\beta = -\lambda_{R1} = -\lambda_{RL1}$$

Here the system is log-linearised since it is notationally a little simpler. The procedures of Appendix 3.5 are followed exactly: First, the dynamic equations (5.2) to (5.4) are written as:

$$g_\Psi(t) = \frac{1}{\gamma}e^{\ln[r(t)+\delta]} - e^{\ln\Phi(t)} + \zeta e^{\ln H_Y(t)} - (\delta + \zeta H) \qquad \text{(A5.1.34)}$$

$$g_\Phi(t) = \left(\frac{1}{\sigma} - \frac{1}{\gamma}\right)e^{\ln[r(t)+\delta]} + e^{\ln\Phi(t)} + \delta\left(1 - \frac{1}{\sigma}\right) - \frac{\rho}{\sigma} \qquad \text{(A5.1.35)}$$

$$g_{p_A}(t) = e^{\ln[r(t)+\delta]} - \frac{\zeta(1-\alpha)}{\alpha}e^{\ln H_Y(t)} - (\delta + \zeta H) \qquad \text{(A5.1.36)}$$

Logs of equations (5.5) and (5.6) are then taken:

$$\ln H_Y(t) = \frac{\gamma}{1-\alpha(1-\gamma)}\ln\Psi - \frac{1}{1-\alpha(1-\gamma)}\ln p_A(t) + \text{ constant} \qquad \text{(A5.1.37)}$$

$$\ln[r(t)+\delta] = -\frac{(1-\alpha)(1-\gamma)}{1-\alpha(1-\gamma)}\ln\Psi - \frac{\alpha(1-\gamma)}{1-\alpha(1-\gamma)}\ln p_A(t) + \text{ constant} \qquad \text{(A5.1.38)}$$

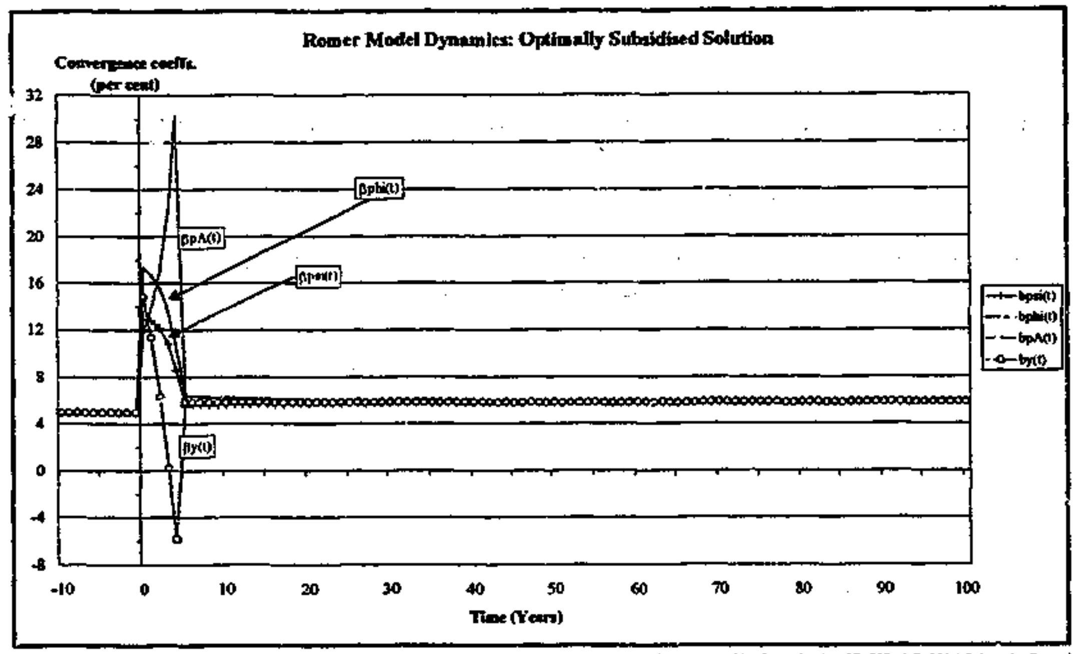
and these are used to linearise (A5.1.34) to (A5.1.36) above by first-order Taylor series expansions about the log of the steady-state. In order to simplify the notation, the time argument is omitted from this point onwards; and the term $[1-\alpha(1-\gamma)]^{-1}$ is written as Λ. Thus, linearising g_Ψ:

$$g_\Psi = \left[\frac{-(1-\alpha)(1-\gamma)\Lambda}{\gamma}e^{\ln(rss+\delta)} + \gamma\zeta\Lambda e^{\ln H_{Yss}}\right]\ln\frac{\Psi}{\Psi_{ss}} - e^{\ln\Phi_{ss}}\ln\frac{\Phi}{\Phi_{ss}}$$
$$-\left[\frac{\alpha(1-\gamma)\Lambda}{\gamma}e^{\ln(rss+\delta)} + \zeta\Lambda e^{\ln H_{Yss}}\right]\ln\frac{p_A}{p_{Ass}}$$

That is:

$$g_\Psi = \Lambda\left[\gamma\zeta H_{Yss} - \frac{(1-\alpha)(1-\gamma)}{\gamma}(r_{ss}+\delta)\right]\ln\frac{\Psi}{\Psi_{ss}} - \Phi_{ss}\ln\frac{\Phi}{\Phi_{ss}}$$
$$-\Lambda\left[\frac{\alpha(1-\gamma)}{\gamma}(r_{ss}+\delta) + \zeta H_{Yss}\right]\ln\frac{p_A}{p_{Ass}} \qquad \text{(A5.1.39)}$$

Figure 5.23: Transitional dynamics of moving from the free market to the social optimum ss: effects on the convergence coefficients (over 100 years); unanticipated introduction of s_K from time zero, anticipated introduction of s_{AK} from time t=5 years, benchmark parameter set.[a]



d:\phd\gempack\subrom\sub_sK\sK0sA5\sK0A5sim.xlw(betas)

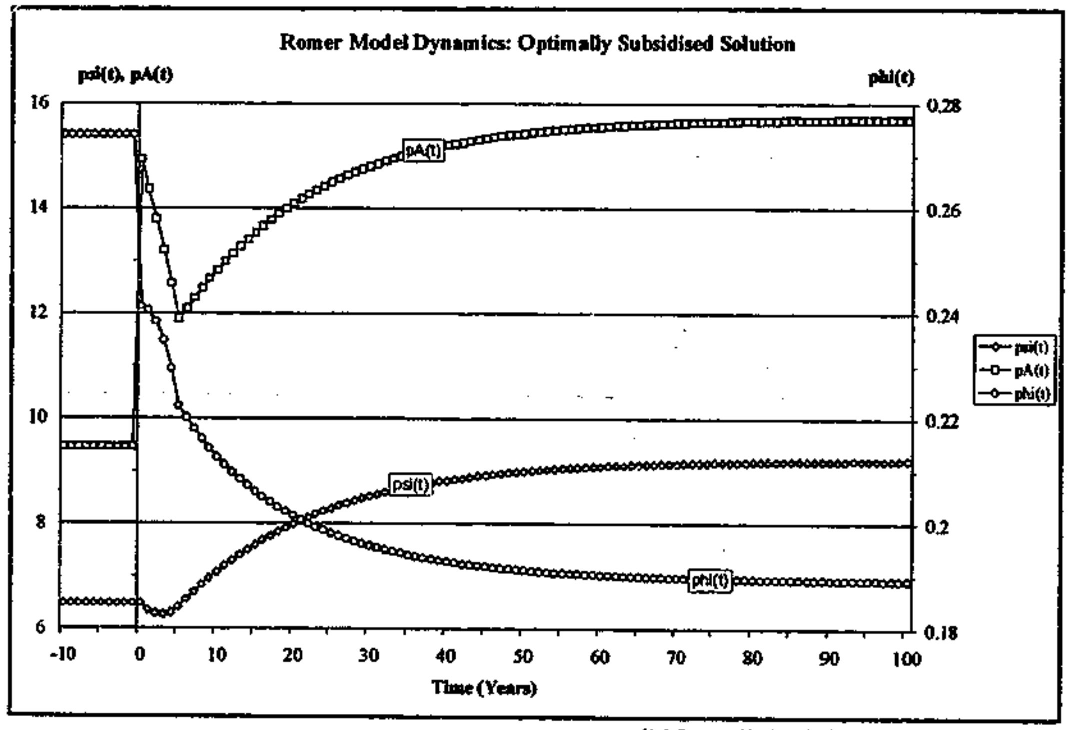
Note: a The difference between the convergence coefficients calculated over annual time intervals and those calculated quarterly is due simply to the fact that they are measured over different time point abscissae. For example, the annual calculation measures β_{pA} at t=4, and t=5 years while the quarterly calculation measures it at t=4.25, 4.5 and 4.75 years as well. And in this case, β_{pA} continues to increase rapidly over the interval t∋[4,5).

5.3.3 Unanticipated imposition of s_{AK}; and delayed, anticipated implementation of s_K

This simulation is simply the mirror image of that of the previous section. Now it is the designs subsidy s_{AK} which is fully implemented unannounced (at time t=0); and the capital rentals subsidy s_K, whose full implementation at time t=5 years is announced in advance - at the same time as the designs subsidy is being introduced. Once again the results are presented in the standard format (Figure 5.24 to Figure 5.31).

Here the divergence of the transitional dynamics from the case when neither subsidy is anticipated is greater than was the case when only the designs-subsidy was expected. This is due to the fact that the capital-subsidy has the greater general impact. The growth rates and convergence coefficients continue to exhibit large (and somewhat eccentric) differences, most significantly over the period to implementation. The allocation of human capital to research H_A and the interest rate r, also exhibit large differences over that initial period, as do the narrow savings measure s_N and the capital-gross product ratio k_{GP}, albeit to a somewhat lesser extent. Unlike the case when it was the designs subsidy that was anticipated (Section 5.3.2), many of the economic variables now show significant differences from the 'unanticipated subsidies simulation' (Section 5.3.1) for much longer periods than the five years to implementation. Prime examples are the price of designs p_A, the consumption-capital ratio Φ, and the capital-technology ratio Ψ (Figure 5.9 and Figure 5.25); the allocation of human capital to research H_A (Figure 5.11 and Figure 5.27); and the capital-gross product ratio k_{GP} (Figure 5.12 and Figure 5.28).

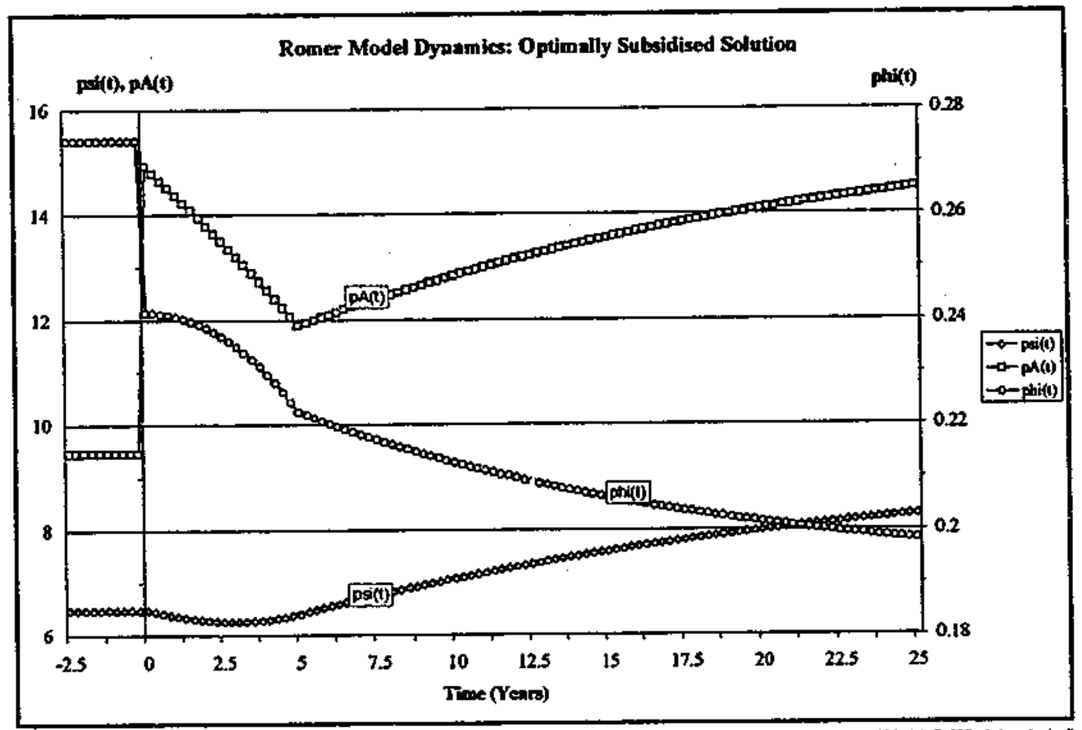
Figure 5.24: Transitional dynamics of moving from the free market to the social optimum ss: effects on Ψ, Φ, and p_A (over 100 years); unanticipated introduction of s_{AK} from time zero, anticipated introduction of s_K from time t=5 years, benchmark parameter set.



d:\phd\gempack\subrom\sub_sK\sK5sA0\sK5A0sim.xlw(psi)

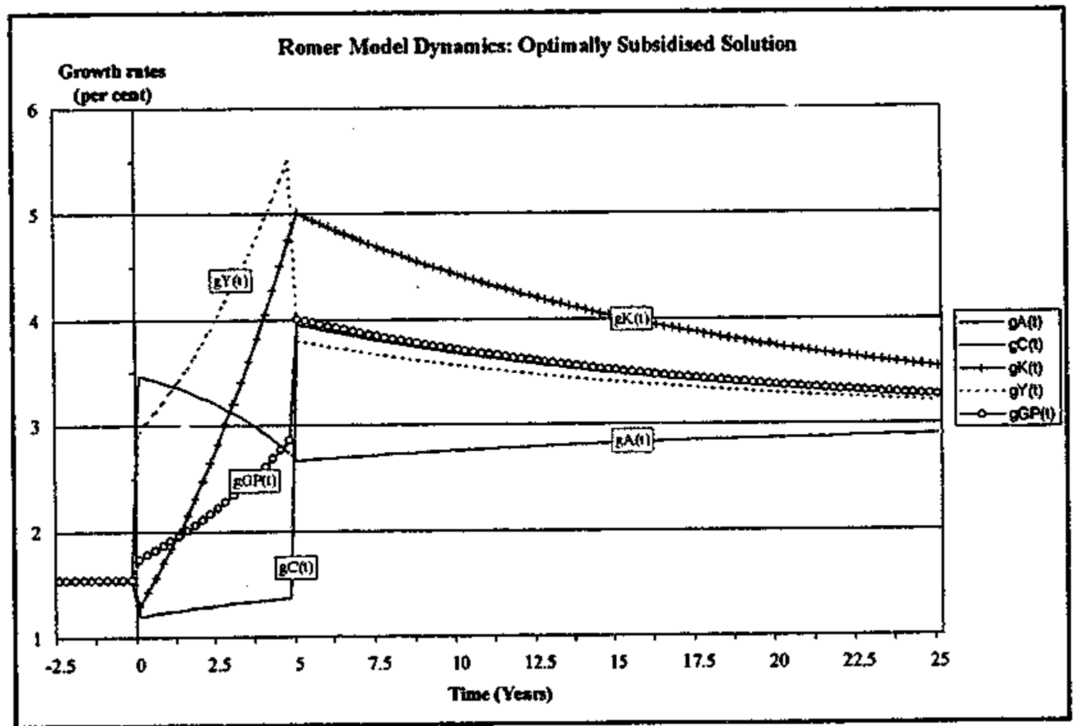
Knowledge that the rentals received for general-purpose capital will soon be subsidised, causes some postponement of capital accumulation: The growth of capital g_K is less than before (Figure 5.10 and Figure 5.26), and so is the investment savings rate s_N. (Figure 5.12 and Figure 5.28). There is also a temporary switch from this form of saving to that of research. The capital share of gross income S_K is much lower, and the research share S_A much higher than when the savings subsidy was not foreseen (Figure 5.13, Figure 5.21 and Figure 5.29). However, extra savings in the form of research do not fully make up for the decline in capital savings over the period before the saving subsidy is implemented. This is reflected by the fact that the broad measure of savings s_B is also lower than before over that period (Figure 5.12 and Figure 5.28). Also, with relatively more resources devoted to consumption and research, Ψ and k_{GP} are lower, and Φ is higher (compare Figure 5.9 with Figure 5.25; and Figure 5.12 with Figure 5.28). Finally, as may readily be seen from equations (5.19) and (5.20), the big rise in rental rates and the interest rate r, is delayed until the subsidy is actually implemented (Figure 5.11 and Figure 5.27).

Figure 5.25: Transitional dynamics of moving from the free market to the social optimum ss: effects on Ψ, Φ, and p_A (over 25 years); unanticipated introduction of s_{AK} from time zero, anticipated introduction of s_K from time t=5 years, benchmark parameter set.



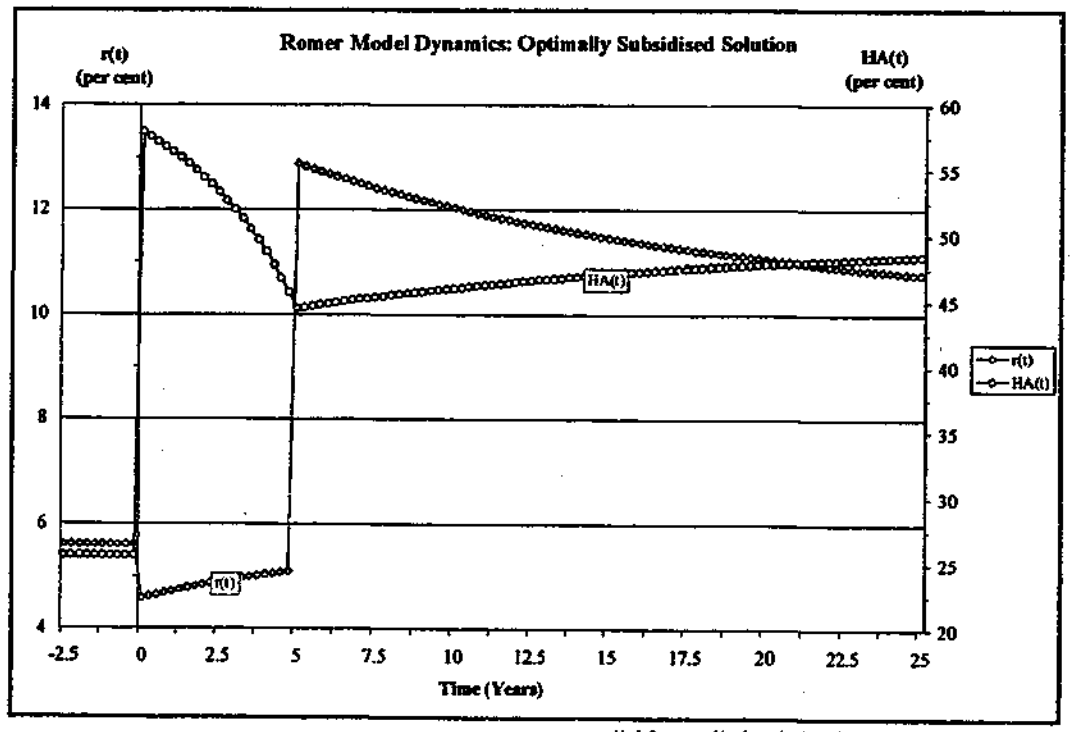
d:\phd\gempack\subrom\sub_sK\sK5sA0\QsK5A0sim.xlw(psi)

Figure 5.26: Transitional dynamics of moving from the free market to the social optimum ss: effects on the growth rates; unanticipated introduction of s_{AK} from time zero, anticipated introduction of s_K from time t=5 years, benchmark parameter set.



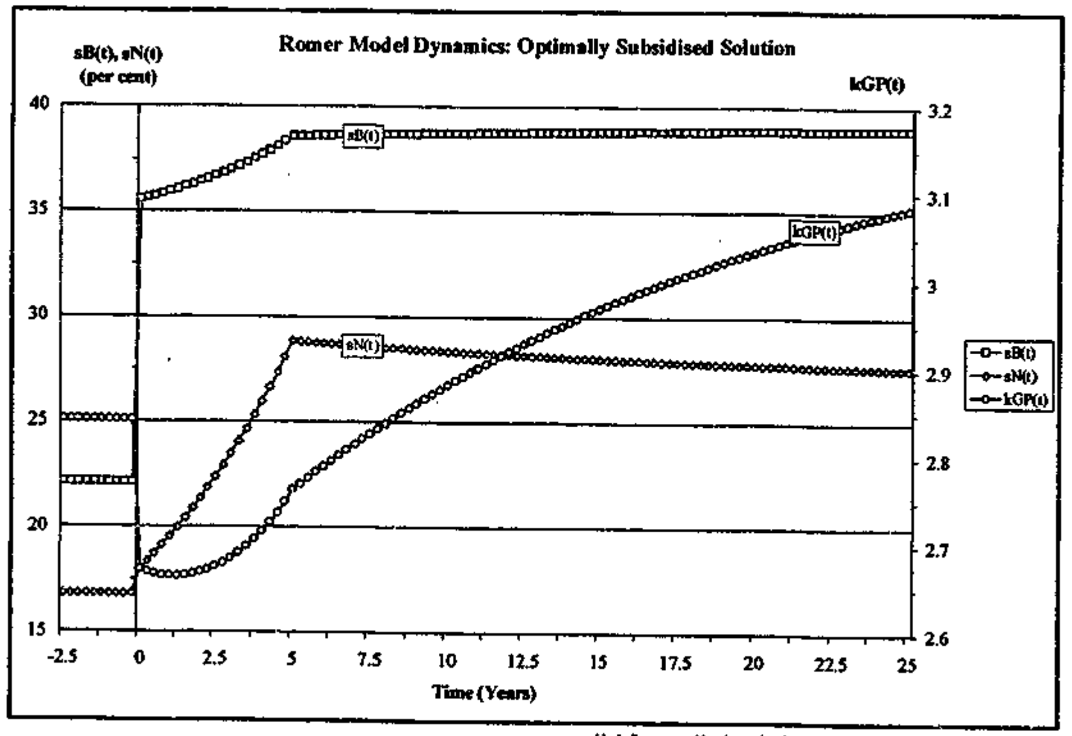
d:\phd\gempack\subrom\sub_sK\sK5sA0\QsK5A0sim.xlw(gA)

Figure 5.27: **Transitional dynamics of moving from the free market to the social optimum ss: effects on r and H_A; unanticipated introduction of s_{AK} from time zero, anticipated introduction of s_K from time t=5 years, benchmark parameter set.**



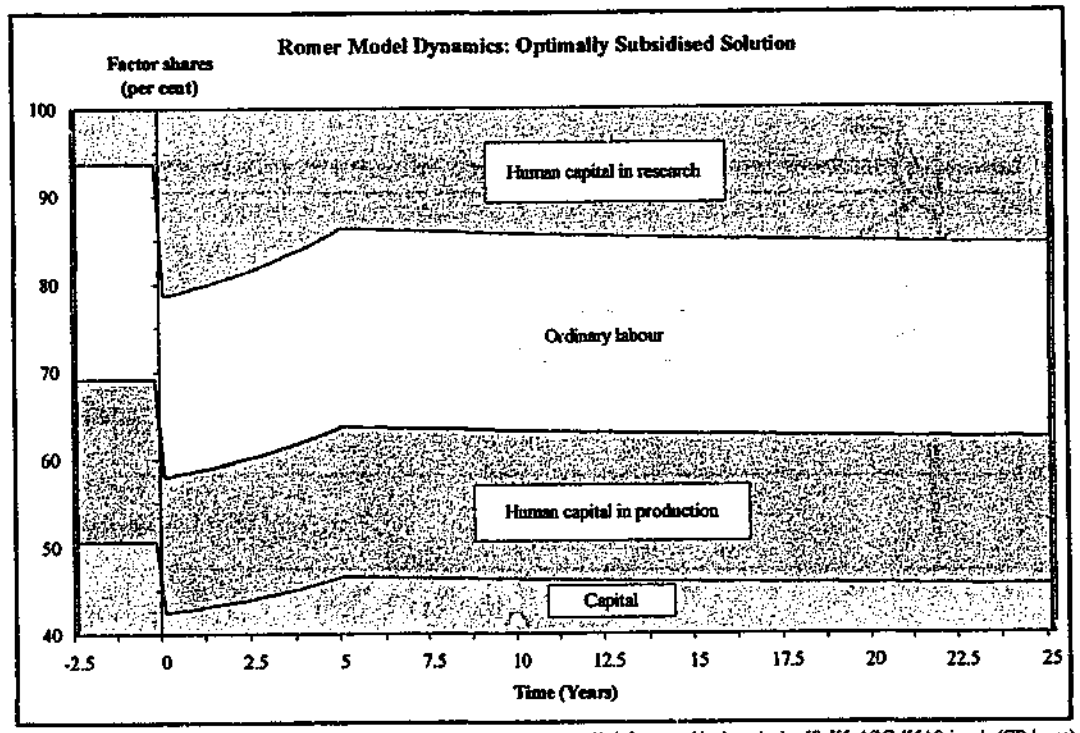
d:\phd\gempack\subrom\sub_sK\sK5sA0\QsK5A0sim.xlw(r,HA)

Figure 5.28: **Transitional dynamics of moving from the free market to the social optimum ss: effects on s_N, s_B, and k_{GP}; unanticipated introduction of s_{AK} from time zero, anticipated introduction of s_K from time t=5 years, benchmark parameter set.**



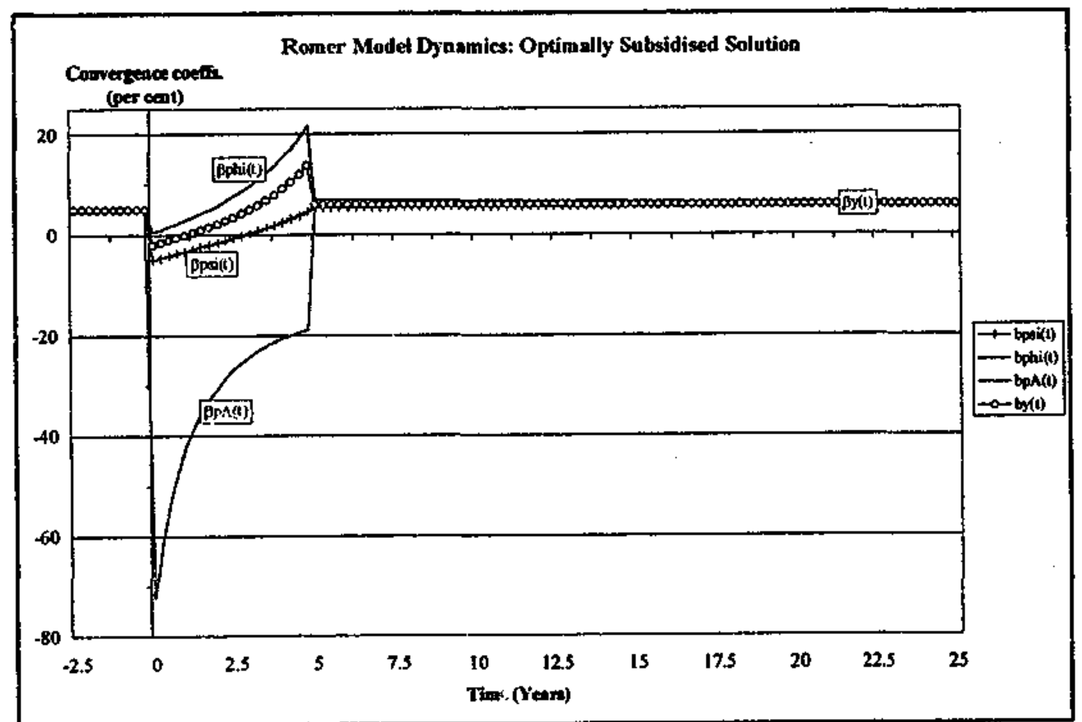
d:\phd\gempack\subrom\sub_sK\sK5sA0\QsK5A0sim.xlw(sB,kGP)

Figure 5.29: **Transitional dynamics of moving from the free market to the social optimum ss: effects on the factor shares of gross income; unanticipated introduction of s_{AK} from time zero, anticipated introduction of s_K from time t=5 years, benchmark parameter set.**



d:\phd\gempack\subrom\sub_sK\sK5sA0\QsK5A0sim.xlw(GPshares)

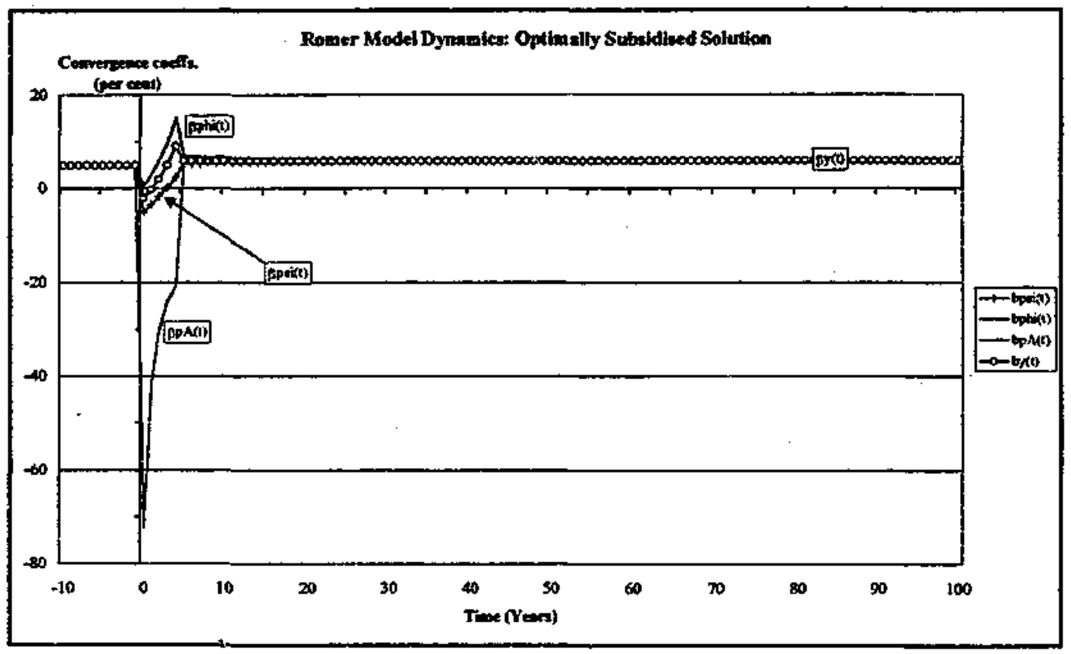
Figure 5.30: **Transitional dynamics of moving from the free market to the social optimum ss: effects on the convergence coefficients (over 25 years); unanticipated introduction of s_{AK} from time zero, anticipated introduction of s_K from time t=5 years, benchmark parameter set.[a]**



d:\phd\gempack\subrom\sub_sK\sK5sA0\QsK5A0sim.xlw(betas)

Note: a The negative portions of the convergence coefficients reflect temporary movements away from the new steady-states for the relevant variables.

Figure 5.31: Transitional dynamics of moving from the free market to the social optimum ss: effects on the convergence coefficients (over 100 years); unanticipated introduction of s_{AK} from time zero, anticipated introduction of s_K from time t=5 years, benchmark parameter set.[a]



d:\phd\gempack\subrom\sub_sK\sK5sA0\sK5A0sim.xlw(betas)

Note: a The negative portions of the convergence coefficients reflect temporary movements away from the new steady-states for the relevant variables.

5.3.4 Dynamic effects of different methods of implementation

The 'method of implementation' is meant here to refer to both the timing of any announcement of the prospective introduction of the optimum subsidies[36], and the timing of the introduction itself - which may be a full implementation at a single point of time, or a phased implementation over a number of different time points. Only three simple methods were considered in the preceding sections. Nevertheless, it is clear from the results of these that alternative methods of implementing the optimum subsidies can produce significant differences in the adjustment paths by which the social welfare maximising equilibrium is approached (compare Figure 5.9, Figure 5.17 and Figure 5.25; Figure 5.10, Figure 5.18 and Figure 5.26 etc.).

Large differences are generally confined to the early period of adjustment, often only until both subsidies are implemented or not long afterwards. However, in some cases and for some variables large differences can persist for considerably longer periods. For example, compare the behaviour of the interest rate r when it is announced in advance that the capital subsidy will be implemented after 5 years, with its behaviour either when it is the designs subsidy that is delayed, or when both subsidies are implemented unannounced. That is, compare Figure 5.27 with Figure 5.11 and Figure 5.19. Naturally, these are issues with which policy makers would need to be aware before selecting the method of implementing the optimum subsidies.

[36] Or, to refer to the timing of any other means by which the subsidies might be anticipated.

Appendix 5.1

A social planning solution to the Romer Model

A5.1.1 Derivation of the dynamic system and its steady-state

The decentralised or market equilibrium for the Romer model, necessarily incorporates certain market imperfections (see Section 5.1). These dictates would not be followed by a social planner, who could allocate resources by administrative fiat. Thus, the social planning problem in the 'Romer economy' is to maximise the discounted sum of all future aggregate utility, subject to the production and research technologies and to the economy's aggregate resource constraint (that consumption and savings together must equal total income). Formally, the problem is:[37]

$$\begin{aligned}
&\text{Maximise} && \int_0^\infty U\{C(t)\}e^{-\rho t}dt \\
&\text{subject to} && \dot{K}(t) = Y(t) - C(t) - \delta K(t) \\
& && \dot{A}(t) = \zeta H_A(t)A(t) \\
& && K(0), A(0) \text{ given; } K(t), A(t) \geq 0 \qquad\qquad (A5.1.1) \\
&\text{where} && U\{C(t)\} = [C(t)^{1-\sigma} - 1]/(1-\sigma) \quad \text{for } \sigma > 0 \\
& && Y(t) = \eta^{-\gamma} H_Y(t)^{\alpha(1-\gamma)} L^{(1-\alpha)(1-\gamma)} K(t)^{\gamma} A(t)^{(1-\gamma)} \\
& && H_A(t) + H_Y(t) = H
\end{aligned}$$

Then as before in Chapter 2 (Appendix 2.2 in particular), proceeding according to the *Maximum Principle* of Pontryagin et al (1962), the (present valued) Hamiltonian, $\mathcal{H}$, is formed and the first-order conditions for a maximum are applied:

$$\mathcal{H} = [(C(t)^{1-\sigma} - 1)/(1-\sigma)]e^{-\rho t} + \lambda(t)[Y(t) - C(t) - \delta K(t)] + \mu(t)\zeta[H - H_Y(t)]A(t)$$

where $\lambda(t)$ and $\mu(t)$, the dynamic multipliers, are shadow prices of capital K(t), and output A(t) respectively.

The first-order conditions for the problem (A5.1.1) are given in equations (A5.1.2) to (A5.1.6) as follows:

$$\frac{\partial \mathcal{H}}{\partial C(t)} = 0 \Rightarrow \quad \lambda(t) = C(t)^{-\sigma} e^{-\rho t} \qquad (A5.1.2)$$

$$\frac{\partial \mathcal{H}}{\partial H_Y(t)} = 0 \Rightarrow \quad \frac{\lambda(t)}{\mu(t)} = \frac{\zeta A(t) H_Y(t)}{\alpha(1-\gamma)Y(t)} \qquad (A5.1.3)$$

[37] Romer (1990b), and Chiang(1992) in reporting Romer's work, both solve this problem, but only to the point of the (posited) balanced growth equilibrium that results. The addition here is the explicit use of the transversality conditions, and the derivation of the full dynamic system.

$$\dot{\lambda}(t) = -\frac{\partial \mathcal{H}}{\partial K(t)} \Rightarrow \frac{\dot{\lambda}(t)}{\lambda(t)} = \delta - \gamma\frac{Y(t)}{K(t)} \qquad \text{(A5.1.4)}$$

$$\dot{\mu}(t) = -\frac{\partial \mathcal{H}}{\partial A(t)} \Rightarrow \frac{\dot{\mu}(t)}{\mu(t)} = -(1-\gamma)\frac{\lambda}{\mu}\frac{Y(t)}{A(t)} - \zeta[H - H_Y(t)] \qquad \text{(A5.1.5)}$$

and the transversality conditions:

$$\lim_{t\to\infty}\mu(t) \ge 0 \text{ and } \lim_{t\to\infty}\mu(t)A(t) = 0$$

and $\qquad\qquad\qquad\qquad$ (A5.1.6)

$$\lim_{t\to\infty}\lambda(t) \ge 0 \text{ and } \lim_{t\to\infty}\lambda(t)K(t) = 0$$

The dynamic system for the social optimum solution to the Romer model may now be derived from problem statement (A5.1.1) and its first-order conditions (A5.1.2) to (A5.1.6) as follows. First, (A5.1.2) is differentiated with respect to time and re-arranged to produce:

$$\dot{C}(t)/C(t) = [-\dot{\lambda}(t)/\lambda(t) - \rho]/\sigma \qquad \text{(A5.1.7)}$$

At this point the interest rate variable r(t) is introduced. It is still related to the rental rate on capital $r_K(t)$ by the logic of Section 2.2.5 and by equation (2.12) specifically. That is: $r(t) = r_K(t)-\delta$; but in this *Pareto optimal solution* to the model, where there are no market distortions, the rental rate on general-purpose capital is now equal to the value of its marginal product:

$$r_K(t) = \frac{\partial Y(t)}{\partial K(t)} = \gamma\eta^{-\gamma}L^{(1-\alpha)(1-\gamma)}H_Y(t)^{\alpha(1-\gamma)}[\frac{K(t)}{A(t)}]^{\gamma-1} = \gamma\frac{Y(t)}{K(t)} \qquad \text{(A5.1.8)}$$

so from equation (A5.1.4) the interest rate is given by:

$$r(t) = -\dot{\lambda}(t)/\lambda(t) \qquad \text{(A5.1.9)}$$

Equations (A5.1.4) and (A5.1.9) also generate the following alternative expression for output which will prove useful:

$$Y(t) = \frac{r(t)+\delta}{\gamma}K(t) \qquad \text{(A5.1.10)}$$

A dynamic equation for the growth of technology may be written directly from the conditions of the problem statement (A5.1.1):

$$\dot{A}(t) = \zeta[H - H_Y(t)]A(t) \qquad \text{(A5.1.11)}$$

Then the capital accumulation equation, also from the problem statement (A5.1.1), is transformed via (A5.1.10) to produce:

$$\dot{K}(t) = \frac{r(t)+\delta}{\gamma}K(t) - C(t) - \delta K(t) \qquad \text{(A5.1.12)}$$

Next, the consumption growth equation is obtained from (A5.1.7) and (A5.1.10):

$$\dot{C}(t) = \frac{r(t)-\rho}{\sigma}C(t) \qquad \text{(A5.1.13)}$$

And combining (A5.1.13) and (A5.1.5) generates a dynamic equation for μ:

$$\dot{\mu}(t) = -[\zeta(1/\alpha - 1)H_Y(t) + \zeta H]\mu(t) \qquad \text{(A5.1.14)}$$

The preceding algebra has now generated a system of four first-order ordinary differential equations, (A5.1.11) to (A5.1.14), in the four variables A(t), K(t), C(t) or $\lambda(t)$, and $\mu(t)$. While there are actually six variables involved in the above ODEs, the two 'extra' variables r(t) and $H_Y(t)$ could readily be eliminated since they are both functions of the other dynamic variables. Substituting for the production function Y(t) from (A5.1.1) into (A5.1.3) and into (A5.1.10) generates expressions for $H_Y(t)$ and r(t) respectively:

$$H_Y(t) = [\frac{\alpha(1-\gamma)}{\delta\eta^{\gamma}}L^{(1-\alpha)(1-\gamma)}[K(t)/A(t)]^{\gamma}[\lambda(t)/\mu(t)]]^{\frac{1}{1-\alpha(1-\gamma)}} \qquad \text{(A5.1.15)}$$

$$r(t) = \gamma\eta^{-\gamma}H_Y(t)^{\alpha(1-\gamma)}L^{(1-\alpha)(1-\gamma)}[K(t)/A(t)]^{\gamma-1} - \delta \qquad \text{(A5.1.16)}$$

The four boundary conditions necessary to integrate such a system are provided by the given initial values A(0) and K(0) from the dynamic optimisation problem statement (A5.1.1), and by the transversality conditions (A5.1.6). As usual, the transversality conditions ensure that in order to achieve the social optimum, the correct trajectory (that is, the saddle-path) is the one attained from amongst the infinite number of possible dynamic paths allowed by the initial conditions and differential equations only. Thus, at this stage the complete dynamic system for the social optimum solution to the model comprises equations (A5.1.11) to (A5.1.16) together with the initial and transversality conditions.

Comparison with equations (2.30) to (2.38) from Chapter 2 reveals that this social optimum solution system is extremely similar to that of the market solution to the model. Moreover, the procedures for determining the asymptotic dynamics and the steady-state equilibrium for the social optimum solution follow exactly those used in Section 2.3.3 for the market solution.[38] In particular, it follows from the transversality conditions that the following four growth rate limits must all be constants:

$$\lim_{t\to\infty}[\dot{\lambda}(t)/\lambda(t)],\ \lim_{t\to\infty}[\dot{K}(t)/K(t)],\ \lim_{t\to\infty}[\dot{\mu}(t)/\mu(t)],\text{ and } \lim_{t\to\infty}[\dot{A}(t)/A(t)].$$

Combining this constancy with the necessary conditions for the dynamic optimisation and the consequent dynamic equations above, ensures the system will tend asymptotically to a *balanced growth* equilibrium in which technology, capital, consumption, and output all grow at the same constant rate:

$$\lim_{t\to\infty}[\dot{A}(t)/A(t)] = \lim_{t\to\infty}[\dot{K}(t)/K(t)] = \lim_{t\to\infty}[\dot{C}(t)/C(t)] = \lim_{t\to\infty}[\dot{Y}(t)/Y(t)] = g^O$$

[38] The only difference is that the ratio of the two shadow prices μ and λ is considered in place of the price of technology variable p_A.

And where:

$$\lim_{t\to\infty}[\dot{\mu}(t)/\mu(t)] = \lim_{t\to\infty}[\dot{\lambda}(t)/\lambda(t)] = -\lim_{t\to\infty} r(t) = -r_{ss}^{O}$$

$$\lim_{t\to\infty}[K(t)/A(t)] = \Psi_{ss}^{O}; \quad \lim_{t\to\infty}[C(t)/K(t)] = \Phi_{ss}^{O}; \quad \lim_{t\to\infty}[\mu(t)/\lambda(t)] = p_{Ass}^{O}$$

$$\lim_{t\to\infty} H_A(t) = H_{A_{ss}}^{O}; \quad \lim_{t\to\infty} H_Y(t) = H_{Y_{ss}}^{O}$$

for $g^O, r_{ss}^O, \Psi_{ss}^O, \Phi_{ss}^O, p_{Ass}^O$, and H_{Yss}^O all constants of the balanced growth equilibrium.

Continuing to follow the approach of Section 2.3.3, the dynamic system is next transformed so that it yields a stationary equilibrium or steady-state rather than the balanced growth equilibrium described above. For this purpose three new dynamic variables are defined. The first two are the same as those used in the market solution of Section 2.3.3, namely $\Psi(t)=K(t)/A(t)$ and $\Phi(t)=C(t)/K(t)$. The third, necessary to eliminate the shadow price terms $\mu(t)$ and $\lambda(t)$, is the ratio of these terms. Now since μ is the shadow price of technology A, measured in *utils*; and λ is the shadow price of capital K, which is equivalent to that of output Y, and is also measured in *utils*, then the ratio μ/λ is the price of technology measured in units of output.[39] It is therefore equivalent to the variable p_A in the market version of the model. Hence, the third 'not-so-new' dynamic variable to be defined is $p_A(t)=\mu(t)/\lambda(t)$, and with this definition equation (A5.1.3) can be seen to be simply a restatement of equation (2.45).

Taking logs and derivatives, and substituting from equations (A5.1.11) to (A5.1.16) generates the three-variable stationary steady-state dynamic system:

$$\dot{\Psi}(t) = [\frac{r(t)+\delta}{\gamma} - \Phi(t) - \delta - \zeta H + \zeta H_Y(t)]\Psi(t) \tag{A5.1.17}$$

$$\dot{\Phi}(t) = [\frac{r(t)-\rho}{\sigma} - \frac{r(t)+\delta}{\gamma} + \Phi(t) + \delta]\Phi(t) \tag{A5.1.18}$$

$$\dot{p}_A(t) = [r(t) - \zeta(1/\alpha - 1)H_Y(t) - \zeta H]p_A(t) \tag{A5.1.19}$$

where:

$$H_Y(t) = [\frac{\alpha(1-\gamma)}{\zeta\eta^{\gamma}} L^{(1-\alpha)(1-\gamma)}\Psi(t)^{\gamma} p_A(t)^{-1}]^{\frac{1}{1-\alpha(1-\gamma)}} \tag{A5.1.20}$$

$$\begin{aligned} r(t) &= \gamma\eta^{-\gamma} H_Y(t)^{\alpha(1-\gamma)} L^{(1-\alpha)(1-\gamma)} \Psi(t)^{\gamma-1} - \delta \\ &= \frac{\zeta\gamma}{\alpha(1-\gamma)} H_Y(t) p_A(t) \Psi(t)^{-1} - \delta \end{aligned} \tag{A5.1.21}$$

and for which the boundary conditions easily transform to:

$$\Psi(0) \text{ given;} \tag{A5.1. 22}$$

[39] Since it is utility which is being maximised, shadow prices will be expressed in terms of the units in which utility is measured (these are termed *utils*).

$$\lim_{t\to\infty}\Phi(t) = \Phi_{ss}^{O}; \tag{A5.1.23}$$

and

$$\lim_{t\to\infty} p_A(t) = p_{Ass}^{O} \tag{A5.1.24}$$

Finally, the steady-state equilibrium of the dynamic system (A5.1.17) to (A5.1.21), equivalent to the balanced growth equilibrium of (A5.1.11) to (A5.1.16), is then computed by taking limits and setting each of (A5.1.17) to (A5.1.19) to zero and solving.

$$\frac{r_{ss}^O+\delta}{\gamma} - \Phi_{ss}^O - \delta - \zeta H + \zeta H_{Yss}^O = 0 \tag{A5.1.25}$$

$$\frac{r_{ss}^O-\rho}{\sigma} - \frac{r_{ss}^O+\delta}{\gamma} + \Phi_{ss}^O + \delta = 0 \tag{A5.1.26}$$

$$r_{ss}^O - \zeta(1/\alpha - 1)H_{Yss}^O - \zeta H = 0 \tag{A5.1.27}$$

Adding equations (A5.1.25) and (A5.1.26) and substituting for r_{ss}^O from (A5.1.27) generates the social optimum steady-state allocation of human capital to output production as:

$$H_{Yss}^O = \frac{\alpha(\sigma-1)H + \alpha\rho/\zeta}{(1-\alpha)+\alpha\sigma} \tag{A5.1.28}$$

And using $H = H_{Ass}^O + H_{Yss}^O$ and $(\dot{A}/A)_{ss}^O = g^O = \zeta H_{Ass}^O$, the equilibrium levels of human capital in research, and the growth rate are obtained as:

$$H_{Ass}^O = \frac{H - \alpha\rho/\zeta}{(1-\alpha)+\alpha\sigma} \quad \text{and} \quad g^O = \frac{\zeta H - \alpha\rho}{(1-\alpha)+\alpha\sigma} \tag{A5.1.29}$$

Then, substituting the result for H_{Yss}^O back into (A5.1.27) generates the steady-state interest rate:

$$r_{ss}^O = \frac{\sigma\zeta H + (1-\alpha)\rho}{(1-\alpha)+\alpha\sigma} \tag{A5.1.30}$$

Next, the steady-state ratio of consumption to capital is obtained from the capital accumulation relation $(\dot{K}/K)_{ss}^O = g^O = (r_{ss}^O + \delta)/\gamma - \Phi_{ss}^O - \delta$; and the steady-state levels of capital intensity of technology and the technology price level from (A5.1.21):

$$\Phi_{ss}^O = (r_{ss}^O + \delta)/\gamma - g^O - \delta \tag{A5.1.31}$$

$$\Psi_{ss}^O = [\frac{\gamma {H_{Yss}^O}^{\alpha(1-\gamma)} L^{(1-\alpha)(1-\gamma)}}{\eta^{\gamma}(r_{ss}^O+\delta)}]^{\frac{1}{1-\gamma}} \tag{A5.1.32}$$

and

$$p_{Ass}^O = \frac{\alpha(1-\gamma)}{\gamma\zeta}\frac{r_{ss}^O+\delta}{H_{Yss}^O}\Psi_{ss}^O \tag{A5.1.33}$$

A5.1.2 Calculation of the speed of convergence through linearisation

The sole purpose of this section is to obtain numerical estimates of the asymptotic speed of convergence of the social optimum system towards its steady-state. As discussed in Section 3.2.3, this is given by the so-called convergence coefficient β, which is equal to the absolute value of the negative eigenvalue from the *coefficients matrix* of the linearised system, whether it is ordinarily-linearised or log-linearised. In terms of the Chapter 3 notation:

$$\beta = -\lambda_{RI} = -\lambda_{RL1}$$

Here the system is log-linearised since it is notationally a little simpler. The procedures of Appendix 3.5 are followed exactly: First, the dynamic equations (5.2) to (5.4) are written as:

$$g_\Psi(t) = \frac{1}{\gamma} e^{\ln[r(t)+\delta]} - e^{\ln\Phi(t)} + \zeta e^{\ln H_Y(t)} - (\delta + \zeta H) \qquad \text{(A5.1.34)}$$

$$g_\Phi(t) = (\frac{1}{\sigma} - \frac{1}{\gamma}) e^{\ln[r(t)+\delta]} + e^{\ln\Phi(t)} + \delta(1 - \frac{1}{\sigma}) - \frac{\rho}{\sigma} \qquad \text{(A5.1.35)}$$

$$g_{p_A}(t) = e^{\ln[r(t)+\delta]} - \frac{\zeta(1-\alpha)}{\alpha} e^{\ln H_Y(t)} - (\delta + \zeta H) \qquad \text{(A5.1.36)}$$

Logs of equations (5.5) and (5.6) are then taken:

$$\ln H_Y(t) = \frac{\gamma}{1-\alpha(1-\gamma)} \ln\Psi - \frac{1}{1-\alpha(1-\gamma)} \ln p_A(t) + \text{ constant} \qquad \text{(A5.1.37)}$$

$$\ln[r(t)+\delta] = -\frac{(1-\alpha)(1-\gamma)}{1-\alpha(1-\gamma)} \ln\Psi - \frac{\alpha(1-\gamma)}{1-\alpha(1-\gamma)} \ln p_A(t) + \text{ constant} \qquad \text{(A5.1.38)}$$

and these are used to linearise (A5.1.34) to (A5.1.36) above by first-order Taylor series expansions about the log of the steady-state. In order to simplify the notation, the time argument is omitted from this point onwards; and the term $[1-\alpha(1-\gamma)]^{-1}$ is written as Λ. Thus, linearising g_Ψ:

$$g_\Psi = [\frac{-(1-\alpha)(1-\gamma)\Lambda}{\gamma} e^{\ln(rss+\delta)} + \gamma\zeta\Lambda e^{\ln H_{Yss}}] \ln\frac{\Psi}{\Psi_{ss}} - e^{\ln\Phi_{ss}} \ln\frac{\Phi}{\Phi_{ss}}$$
$$-[\frac{\alpha(1-\gamma)\Lambda}{\gamma} e^{\ln(rss+\delta)} + \zeta\Lambda e^{\ln H_{Yss}}] \ln\frac{p_A}{p_{Ass}}$$

That is:

$$g_\Psi = \Lambda[\gamma\zeta H_{Yss} - \frac{(1-\alpha)(1-\gamma)}{\gamma}(r_{ss}+\delta)] \ln\frac{\Psi}{\Psi_{ss}} - \Phi_{ss} \ln\frac{\Phi}{\Phi_{ss}}$$
$$-\Lambda[\frac{\alpha(1-\gamma)}{\gamma}(r_{ss}+\delta) + \zeta H_{Yss}] \ln\frac{p_A}{p_{Ass}} \qquad \text{(A5.1.39)}$$

Similarly, linearising g_Φ produces:

$$g_\Phi = (\frac{1}{\gamma} - \frac{1}{\sigma})(1-\alpha)(1-\gamma)\Lambda(r_{ss}+\delta) \ln\frac{\Psi}{\Psi_{ss}} + \Phi_{ss} \ln\frac{\Phi}{\Phi_{ss}}$$
$$+(\frac{1}{\gamma} - \frac{1}{\sigma})\alpha(1-\gamma)\Lambda(r_{ss}+\delta) \ln\frac{p_A}{p_{Ass}} \qquad \text{(A5.1.40)}$$

and finally, linearising g_{pA} generates:

$$g_{p_A} = -\Lambda(1-\alpha)[(1-\gamma)(r_{ss}+\delta) + \gamma\zeta H_{Yss}/\alpha] \ln\frac{\Psi}{\Psi_{ss}}$$
$$+\Lambda[\zeta(1-\alpha)H_{Yss}/\alpha - \alpha(1-\gamma)(r_{ss}+\delta)] \ln\frac{p_A}{p_{Ass}} \qquad \text{(A5.1.41)}$$

Equations (A5.1.39) to (A5.1.41) then express the growth rates of the dynamic variables as linear functions of the logarithms of the ratios of the variables to their steady-state levels:

$$\mathbf{g_R} = \Omega^O_{RL} [\mathbf{\ln(R(t)/R_{ss})}]$$

where $\mathbf{g_R}$; and $\mathbf{\ln(R(t)/R_{ss})}$ are the column vectors: $[g_\Psi, g_\Phi, g_{pA}]^T$; and $[(\ln(\Psi(t)/\Psi_{ss}), \ln(\Phi(t)/\Phi_{ss}), \ln p_A(t)/p_{Ass})]^T$ respectively; and where Ω^O_{RL} is the desired coefficients matrix:

$$\Omega^O_{RL} = \begin{pmatrix} \frac{\gamma\zeta H_{Yss} - (1-\alpha)(1-\gamma)(r_{ss}+\delta)/\gamma}{1-\alpha(1-\gamma)} & -\Phi_{ss} & -\frac{\zeta H_{Yss} + \alpha(1-\gamma)(r_{ss}+\delta)/\gamma}{1-\alpha(1-\gamma)} \\ [\frac{1}{\gamma} - \frac{1}{\sigma}]\frac{(1-\alpha)(1-\gamma)(r_{ss}+\delta)}{1-\alpha(1-\gamma)} & \Phi_{ss} & [\frac{1}{\gamma} - \frac{1}{\sigma}]\frac{\alpha(1-\gamma)(r_{ss}+\delta)}{1-\alpha(1-\gamma)} \\ -(1-\alpha)\frac{(1-\gamma)(r_{ss}+\delta) + \gamma\zeta H_{Yss}/\alpha}{1-\alpha(1-\gamma)} & 0 & \frac{\frac{\zeta(1-\alpha)H_{Yss}}{\alpha} - \alpha(1-\gamma)(r_{ss}+\delta)}{1-\alpha(1-\gamma)} \end{pmatrix} \qquad \text{(A5.1.42)}$$

When this is evaluated for the benchmark parameter set it generates a single negative eigenvalue (as expected) of: $\lambda_{RL1} = -0.0681$. The speed of convergence of the social optimum system in the neighbourhood of its benchmark steady-state is therefore 6.81 per cent a year. When the output elasticity of capital (that is parameter γ) is raised by 10 per cent, as in the simulation conducted on this system, the convergence coefficient falls to 5.60 per cent a year. These are the steady-state values shown in Figure 5.6.

Appendix 5.2

TABLO input files for the 'social optimum' and the 'subsidised market' Romer systems

As already discussed in Chapter 4, TABLO input files are the key to implementing the *finite differences-GEMPACK* method of numerical integration. This appendix records the TABLO files used to conduct simulations on the social optimum and the subsidised market systems via a 4th order Runge-Kutta numerical integration procedure. Both were derived by amending the corresponding file for the free market dynamic system, MKTRK4.TAB, from Table A4.3.4 of Appendix 4.3.

Table A5.2.1 TABLO input file specifying a 4th order Runge-Kutta finite differencing integration method for modelling the dynamics of the social optimum solution to the Romer model (file OPTRK4.TAB).

```
! ROMER MODEL DYNAMICS: SOCIAL OPTIMUM SOLUTION                                  !
! GORDON SCHMIDT, 26 JANUARY 1998.                                               !
! Solution of the "two-point boundary value problem" posed by the social optimum !
! solution of the Romer model of endogenous growth:                              !
!       The base case is a steady-state solution of the model, with the value of !
!       the capital/technology stock ratio equal to the desired initial level.   !
!       The model is implemented in its "levels" form.                           !
!       Finite differencing is by the "4th order Runge-Kutta" method.            !


! Defaults for "levels model"                                                     !
EQUATION(DEFAULT=LEVELS) ;
VARIABLE(DEFAULT=LEVELS) ;
FORMULA(DEFAULT=INITIAL) ;
COEFFICIENT(DEFAULT=PARAMETER) ;


! Number of grid intervals - representing the overall time horizon in years. To be !
!       read from a file of logical name "tperiods"                                 !
FILE (TEXT) tperiods ;
COEFFICIENT (INTEGER) NINTERVAL ;
READ NINTERVAL FROM FILE tperiods ;
```



```
! Sets for describing periods                                                     !
SET (INTERTEMPORAL) alltime # all time periods # MAXIMUM SIZE
2001 ( p[0] - p[NINTERVAL] ) ;
SET (INTERTEMPORAL) fwdtime # domain of fwd diffs # MAXIMUM SIZE
2000 ( p[0] - p[NINTERVAL - 1] ) ;
SET (INTERTEMPORAL) endtime # ending time # SIZE 1 ( p[NINTERVAL] ) ;
SUBSET fwdtime IS SUBSET OF alltime ;
SUBSET endtime IS SUBSET OF alltime ;


! Time intervals: grid points, representing years, to be read from a file        !
!       of logical name "time".                                                   !
FILE (TEXT) time ;
COEFFICIENT (all,t,alltime) year(t) ;
READ year FROM FILE time ;
COEFFICIENT (all,t,fwdtime) dt(t) ;
FORMULA (all,t,fwdtime) dt(t) = year(t+1) - year(t) ;


! Variables                                                                       !
VARIABLE (all,t,alltime) psi(t)    # capital/technology ratio     # ;
VARIABLE (all,t,alltime) phi(t)    # consumption/capital ratio    # ;
VARIABLE (all,t,alltime) pA(t)     # price of technology          # ;
VARIABLE (all,t,alltime) HY(t)     # human capital in mfg         # ;
VARIABLE (all,t,alltime) HA(t)     # human capital in research    # ;
VARIABLE (all,t,alltime) r(t)      # interest rate                # ;
VARIABLE (all,t,alltime) H(t)      # total human capital          # ;
VARIABLE (all,t,alltime) L(t)      # total ordinary labour        # ;


! Intemediate variables: used in the RK4 finite differencing and, since their base !
!       levels are zero, declared such that their linear equivalent                !
!       is a 'change' variable rather than a 'percentage change' one.              !
VARIABLE (CHANGE) (all,t,fwdtime) psiR1(t) ;
VARIABLE (CHANGE) (all,t,fwdtime) phiR1(t) ;
VARIABLE (CHANGE) (all,t,fwdtime) pAR1(t) ;
VARIABLE (CHANGE) (all,t,fwdtime) psiR2(t) ;
VARIABLE (CHANGE) (all,t,fwdtime) phiR2(t) ;
VARIABLE (CHANGE) (all,t,fwdtime) pAR2(t) ;
VARIABLE (CHANGE) (all,t,fwdtime) psiR3(t) ;
VARIABLE (CHANGE) (all,t,fwdtime) phiR3(t) ;
VARIABLE (CHANGE) (all,t,fwdtime) pAR3(t) ;
VARIABLE (CHANGE) (all,t,fwdtime) psiR4(t) ;
VARIABLE (CHANGE) (all,t,fwdtime) phiR4(t) ;
VARIABLE (CHANGE) (all,t,fwdtime) pAR4(t) ;
```

```
! Parameters:  declared as variables, and as functions of time, in order to examine   !
!        the dynamics of changes 'between' equilibria.                                 !
VARIABLE (all,t,alltime) a(t)      # human cap productivity param   # ;
VARIABLE (all,t,alltime) g(t)      # capital productivity param     # ;
VARIABLE (all,t,alltime) z(t)      # research productivity param    # ;
VARIABLE (all,t,alltime) e(t)      # specialised cap cost param     # ;
VARIABLE (all,t,alltime) rho(t)    # consumers' discount factor     # ;
VARIABLE (all,t,alltime) s(t)      # inter-temporal substn param    # ;
VARIABLE (all,t,alltime) d(t)      # depreciation rate on capital   # ;
```

```
! Base case: established by a combination of READ statements from a file of logical  !
!        name "basedata", and dependent FORMULA(INITIAL) statements.                   !
FILE (TEXT)  basedata ;
READ H FROM FILE basedata ;
READ L FROM FILE basedata ;
READ a FROM FILE basedata ;
READ g FROM FILE basedata ;
READ z FROM FILE basedata ;
READ e FROM FILE basedata ;
READ rho FROM FILE basedata ;
READ s FROM FILE basedata ;
READ d FROM FILE basedata ;

FORMULA  (all,t,alltime)
 HY(t) = [a(t)*(s(t)-1)*H(t)+a(t)*rho(t)/z(t)]/[1-a(t)+a(t)*s(t)] ;
FORMULA  (all,t,alltime)
 HA(t) = H(t)-HY(t) ;
FORMULA  (all,t,alltime)
 r(t) = [s(t)*z(t)*H(t)+(1-a(t))*rho(t)]/[1-a(t)+a(t)*s(t)] ;
FORMULA  (all,t,alltime)
 psi(t) = [g(t)/e(t)^g(t)/{r(t)+d(t)}*HY(t)^{a(t)*(1-g(t))}
 *L(t)^{(1-a(t))*(1-g(t))}]^{1/(1-g(t))};
FORMULA  (all,t,alltime)
 phi(t) = {r(t)+d(t)}*/g(t)-z(t)*HA(t)-d(t) ;
FORMULA  (all,t,alltime)
 pA(t) = a(t)*{1/g(t)-1}/z(t)*{r(t)+d(t)}/HY(t)*psi(t) ;

FORMULA  (all,t,fwdtime)
 psiR1(t)=0 ;
FORMULA  (all,t,fwdtime)
 phiR1(t)=0 ;
FORMULA  (all,t,fwdtime)
 pAR1(t)=0 ;
FORMULA  (all,t,fwdtime)
 psiR2(t)=0 ;
```



```
FORMULA  (all,t,fwdtime)
 phiR2(t)=0 ;
FORMULA  (all,t,fwdtime)
 pAR2(t)=0 ;
FORMULA  (all,t,fwdtime)
 psiR3(t)=0 ;
FORMULA  (all,t,fwdtime)
 phiR3(t)=0 ;
FORMULA  (all,t,fwdtime)
 pAR3(t)=0 ;
FORMULA  (all,t,fwdtime)
 psiR4(t)=0 ;
FORMULA  (all,t,fwdtime)
 phiR4(t)=0 ;
FORMULA  (all,t,fwdtime)
 pAR4(t)=0 ;
```

```
! Dynamic equations: relating variables at adjacent time points.                       !
EQUATION psiincr (all,t,fwdtime)
 psi(t+1)=psi(t)+psiR1(t)/6+psiR2(t)/3+psiR3(t)/3+psiR4(t)/6  ;
EQUATION phiincr (all,t,fwdtime)
 phi(t+1)=phi(t)+phiR1(t)/6+phiR2(t)/3+phiR3(t)/3+phiR4(t)/6  ;
EQUATION pAincr (all,t,fwdtime)
 pA(t+1) =pA(t)+pAR1(t)/6+pAR2(t)/3+pAR3(t)/3+pAR4(t)/6  ;

! where the RK4 variables are defined by the time derivative functions of the main     !
! dynamic variables as follows:                                                        !

EQUATION psiRK1 (all,t,fwdtime)
 psiR1(t)=dt(t)*[z(t)*{1+pA(t)/psi(t)/a(t)/(1-g(t))}*{a(t)*(1-g(t))/z(t)/e(t)^g(t)*
        L(t)^{(1-a(t))*(1-g(t))}*psi(t)^g(t)/pA(t)}^[1/(1-a(t)*(1-g(t)))]-
        phi(t)-d(t)-z(t)*H(t)]*psi(t) ;
EQUATION phiRK1 (all,t,fwdtime)
 phiR1(t)=dt(t)*[(g(t)/s(t)-1)*z(t)/a(t)/(1-g(t))*pA(t)/psi(t)*
        {a(t)*(1-g(t))/z(t)/e(t)^g(t)*
        L(t)^{(1-a(t))*(1-g(t))}*psi(t)^g(t)/pA(t)}^[1/(1-a(t)*(1-g(t)))]
        +phi(t)+{d(t)*(s(t)-1)-rho(t)}/s(t)]*phi(t) ;
EQUATION pARK1 (all,t,fwdtime)
 pAR1(t)=dt(t)*[z(t)*g(t)/a(t)/(1-g(t))*[pA(t)/psi(t)-(1-a(t))*(1-g(t))/g(t)]*
        {a(t)*(1-g(t))/z(t)/e(t)^g(t)*L(t)^{(1-a(t))*(1-g(t))}*psi(t)^g(t)/pA(t)}^
        [1/(1-a(t)*(1-g(t)))]-(z(t)*H(t)+d(t))]*pA(t) ;

EQUATION psiRK2 (all,t,fwdtime)
 psiR2(t)=dt(t)*[z(t)*{1+[pA(t)+pAR1(t)/2]/[psi(t)+psiR1(t)/2]/
        a(t)/(1-g(t))}*{a(t)*(1-g(t))/z(t)/e(t)^g(t)*L(t)^{(1-a(t))*(1-g(t))}*
        [psi(t)+psiR1(t)/2]^g(t)/[pA(t)+pAR1(t)/2]}^[1/(1-a(t)*(1-g(t)))]-
        [phi(t)+phiR1(t)/2]-d(t)-z(t)*H(t)]*[psi(t)+psiR1(t)/2] ;
```

```
EQUATION phiRK2 (all,t,fwdtime)
 phiR2(t)=dt(t)*[(g(t)/s(t)-1)*z(t)/a(t)/(1-g(t))*[pA(t)+pAR1(t)/2]/[psi(t)+psiR1(t)/2]*
        {a(t)*(1-g(t))/z(t)/e(t)^g(t)*L(t)^{(1-a(t))*(1-g(t))}*
        [psi(t)+psiR1(t)/2]^g(t)/[pA(t)+pAR1(t)/2]}^[1/(1-a(t)*(1-g(t)))]+
        [phi(t)+phiR1(t)/2]+{d(t)*(s(t)-1)-rho(t)}/s(t)]*[phi(t)+phiR1(t)/2] ;
EQUATION pARK2 (all,t,fwdtime)
 pAR2(t)=dt(t)*[z(t)*g(t)/a(t)/(1-g(t))*[[pA(t)+pAR1(t)/2]/[psi(t)+psiR1(t)/2]-
        (1-a(t))*(1-g(t))/g(t)]*{a(t)*(1-g(t))/z(t)/e(t)^g(t)*L(t)^{(1-a(t))*(1-g(t))}*
        [psi(t)+psiR1(t)/2]^g(t)/[pA(t)+pAR1(t)/2]}^
        [1/(1-a(t)*(1-g(t)))]-(z(t)*H(t)+d(t))]*[pA(t)+pAR1(t)/2] ;

EQUATION psiRK3 (all,t,fwdtime)
 psiR3(t)=dt(t)*[z(t)*{1+[pA(t)+pAR2(t)/2]/[psi(t)+psiR2(t)/2]/a(t)/(1-g(t))}*
        {a(t)*(1-g(t))/z(t)/e(t)^g(t)*L(t)^{(1-a(t))*(1-g(t))}*
        [psi(t)+psiR2(t)/2]^g(t)/[pA(t)+pAR2(t)/2]}^ [1/(1-a(t)*(1-g(t)))]-
        [phi(t)+phiR2(t)/2]-d(t)-z(t)*H(t)]*[psi(t)+psiR2(t)/2] ;
EQUATION phiRK3 (all,t,fwdtime)
 phiR3(t)=dt(t)*[(g(t)/s(t)-1)*z(t)/a(t)/(1-g(t))*[pA(t)+pAR2(t)/2]/[psi(t)+psiR2(t)/2]*
        {a(t)*(1-g(t))/z(t)/e(t)^g(t)*L(t)^{(1-a(t))*(1-g(t))}* [psi(t)+psiR2(t)/2]^g(t)/
        [pA(t)+pAR2(t)/2]}^[1/(1-a(t)*(1-g(t)))]+[phi(t)+phiR2(t)/2]+
        {d(t)*(s(t)-1)-rho(t)}/s(t)]*[phi(t)+phiR2(t)/2] ;
EQUATION pARK3 (all,t,fwdtime)
 pAR3(t)=dt(t)*[z(t)*g(t)/a(t)/(1-g(t))*[[pA(t)+pAR2(t)/2]/[psi(t)+psiR2(t)/2]-
        (1-a(t))*(1-g(t))/g(t)]*{a(t)*(1-g(t))/z(t)/e(t)^g(t)*L(t)^{(1-a(t))*(1-g(t))}*
        [psi(t)+psiR2(t)/2]^g(t)/[pA(t)+pAR2(t)/2]}^
        [1/(1-a(t)*(1-g(t)))]-(z(t)*H(t)+d(t))]*[pA(t)+pAR2(t)/2] ;

EQUATION psiRK4 (all,t,fwdtime)
 psiR4(t)=dt(t)*[z(t)*{1+[pA(t)+pAR3(t)]/[psi(t)+psiR3(t)]/a(t)/(1-g(t))}*
        {a(t)*(1-g(t))/z(t)/e(t)^g(t)*L(t)^{(1-a(t))*(1-g(t))}* [psi(t)+psiR3(t)]^g(t)/
        [pA(t)+pAR3(t)]}^[1/(1-a(t)*(1-g(t)))]-[phi(t)+phiR3(t)]-
        d(t)-z(t)*H(t)]*[psi(t)+psiR3(t)] ;
EQUATION phiRK4 (all,t,fwdtime)
 phiR4(t)=dt(t)*[(g(t)/s(t)-1)*z(t)/a(t)/(1-g(t))*[pA(t)+pAR3(t)]/[psi(t)+psiR3(t)]*
        {a(t)*(1-g(t))/z(t)/e(t)^g(t)*L(t)^{(1-a(t))*(1-g(t))}* [psi(t)+psiR3(t)]^g(t)/
        [pA(t)+pAR3(t)]}^[1/(1-a(t)*(1-g(t)))]+[phi(t)+phiR3(t)]+
        {d(t)*(s(t)-1)-rho(t)}/s(t)]*[phi(t)+phiR3(t)] ;
EQUATION pARK4 (all,t,fwdtime)
 pAR4(t)=dt(t)*[z(t)*g(t)/a(t)/(1-g(t))*[[pA(t)+pAR3(t)]/[psi(t)+psiR3(t)]-
        [[pA(t)+pAR3(t)]/[psi(t)+psiR3(t)]-(1-a(t))*(1-g(t))/g(t)]*
        {a(t)*(1-g(t))/z(t)/e(t)^g(t)*L(t)^{(1-a(t))*(1-g(t))}*[psi(t)+psiR3(t)]^g(t)/
        [pA(t)+pAR3(t)]}^[1/(1-a(t)*(1-g(t)))]-(z(t)*H(t)+d(t))]*[pA(t)+pAR3(t)] ;
```

! Other intraperiod equations: relationships existing over all points of time. !

```
EQUATION humcapY (all,t,alltime)
 HY(t)=[a(t)*(1-g(t))/z(t)/e(t)^g(t)*L(t)^{(1-a(t))*(1-g(t))}*
        psi(t)^g(t)/pA(t)]^{1/(1-a(t)*(1-g(t)))};
```



```
EQUATION humcapA (all,t,alltime)
 HA(t)=H(t)-HY(t) ;
EQUATION irate (all,t,alltime)
 r(t)=[z(t)*g(t)/a(t)/(1-g(t))]*HY(t)*pA(t)/psi(t)-d(t) ;
```

! **Boundary conditions:** !
! 1. Initial capital/technology ratio fixed at its immediate pre-shock level: !
! psi(0)=psi0. Implemented simply by declaring psi(0) exogenous, !
! and setting its shock to zero. !
! 2. 'Final' value of the consumption/capital ratio set at its market solution !
! steady-state level: phi(T)=phiMss. !
! 3. 'Final' value of the price of technology set at its market solution !
! steady- state level: pA(T)=pAMss. !

```
EQUATION phiMss (all,t,endtime)
 phi(t) = {r(t)+d(t)}/g(t)-z(t)*HA(t)-d(t) ;
EQUATION pAMss (all,t,endtime)
 pA(t) = a(t)*{1/g(t)-1}/z(t)*{r(t)+d(t)}/HY(t)*psi(t) ;
```

Table A5.2.2 TABLO input file specifying a 4th order Runge-Kutta finite differencing integration method for modelling the dynamics of the subsidised market solution to the Romer model (file SUBRK4.TAB).

! **ROMER MODEL DYNAMICS: SUBSIDISED MARKET SOLUTION** !

! **GORDON SCHMIDT, 11 NOVEMBER 1998.** !

! Solution of the "two-point boundary value problem" posed by the (optimally) !
! subsidised market solution of the Romer model of endogenous growth: !
! The base case is a steady-state solution of the model, with the value of the !
! capital/technology stock ratio equal to the desired initial level. !
! The model is implemented in its "levels" form. !
! Finite differencing is by the "4th order Runge-Kutta" method. !

! **Defaults for "levels model"** !

```
EQUATION(DEFAULT=LEVELS) ;
VARIABLE(DEFAULT=LEVELS) ;
FORMULA(DEFAULT=INITIAL) ;
COEFFICIENT(DEFAULT=PARAMETER) ;
```

```
! Number of grid intervals - representing the overall time horizon in years. To be    !
!        read from a file of logical name "tperiods"                                    !
FILE (TEXT) tperiods ;
COEFFICIENT (INTEGER) NINTERVAL ;
READ NINTERVAL FROM FILE tperiods ;


! Sets for describing periods                                                           !
SET (INTERTEMPORAL) alltime # all time periods # MAXIMUM SIZE
2001 ( p[0] - p[NINTERVAL] ) ;
SET (INTERTEMPORAL) fwdtime # domain of fwd diffs # MAXIMUM SIZE
2000 ( p[0] - p[NINTERVAL - 1] ) ;
SET (INTERTEMPORAL) endtime # ending time # SIZE 1 ( p[NINTERVAL] ) ;
SUBSET fwdtime IS SUBSET OF alltime ;
SUBSET endtime IS SUBSET OF alltime ;


! Time intervals: grid points, representing years, to be read from a file              !
!        of logical name "time".                                                        !
FILE (TEXT) time ;
COEFFICIENT (all,t,alltime) year(t) ;
READ year FROM FILE time ;
COEFFICIENT (all,t,fwdtime) dt(t) ;
FORMULA (all,t,fwdtime) dt(t) = year(t+1) - year(t) ;


! Variables                                                                             !
VARIABLE (all,t,alltime) psi(t)     # capital/technology ratio      # ;
VARIABLE (all,t,alltime) phi(t)     # consumption/capital ratio     # ;
VARIABLE (all,t,alltime) pA(t)      # price of technology           # ;
VARIABLE (all,t,alltime) H(t)       # total human capital           # ;
VARIABLE (all,t,alltime) L(t)       # total ordinary labour         # ;


! Intemediate variables: used in the RK4 finite differencing and, since their base     !
!        levels are zero, declared such that their linear equivalent                   !
!        is a 'change' variable rather than a 'percentage change' one.                 !
VARIABLE (CHANGE) (all,t,fwdtime) psiR1(t) ;
VARIABLE (CHANGE) (all,t,fwdtime) phiR1(t) ;
VARIABLE (CHANGE) (all,t,fwdtime) pAR1(t) ;
VARIABLE (CHANGE) (all,t,fwdtime) psiR2(t) ;
VARIABLE (CHANGE) (all,t,fwdtime) phiR2(t) ;
VARIABLE (CHANGE) (all,t,fwdtime) pAR2(t) ;
VARIABLE (CHANGE) (all,t,fwdtime) psiR3(t) ;
VARIABLE (CHANGE) (all,t,fwdtime) phiR3(t) ;
```



```
VARIABLE (CHANGE) (all,t,fwdtime) pAR3(t) ;
VARIABLE (CHANGE) (all,t,fwdtime) psiR4(t) ;
VARIABLE (CHANGE) (all,t,fwdtime) phiR4(t) ;
VARIABLE (CHANGE) (all,t,fwdtime) pAR4(t) ;


! Parameters: declared as variables, and as functions of time, in order to examine     !
!        the dynamics of changes 'between' equilibria.                                  !
VARIABLE (all,t,alltime) a(t)       # human cap productivity param   # ;
VARIABLE (all,t,alltime) g(t)       # capital productivity param     # ;
VARIABLE (all,t,alltime) z(t)       # research productivity param    # ;
VARIABLE (all,t,alltime) e(t)       # specialised cap cost param     # ;
VARIABLE (all,t,alltime) rho(t)     # consumers' discount factor     # ;
VARIABLE (all,t,alltime) s(t)       # inter-temporal substn param    # ;
VARIABLE (all,t,alltime) d(t)       # depreciation rate on capital   # ;
VARIABLE (all,t,alltime) subK(t)    # power of capital subsidy       # ;
VARIABLE (all,t,alltime) subA(t)    # power of designs subsidy       # ;


! Base case: established by a combination of READ statements from a file of logical   !
!        name "basedata", and dependent FORMULA(INITIAL) statements.                    !
FILE (TEXT)  basedata ;
READ H FROM FILE basedata ;
READ L FROM FILE basedata ;
READ a FROM FILE basedata ;
READ g FROM FILE basedata ;
READ z FROM FILE basedata ;
READ e FROM FILE basedata ;
READ rho FROM FILE basedata ;
READ s FROM FILE basedata ;
READ d FROM FILE basedata ;
READ subK FROM FILE basedata ;
READ subA FROM FILE basedata ;

FORMULA  (all,t,alltime)
  psi(t) = [g(t)^2/e(t)^g(t)*subK(t)/[{[s(t)*z(t)*H(t)+rho(t)]/
        [1+subA(t)*a(t)*s(t)/g(t)]}+d(t)]*
        {[a(t)*s(t)/g(t)*H(t)+a(t)*rho(t)/g(t)/z(t)]/
        [1/subA(t)+a(t)*s(t)/g(t)]}^[a(t)*(1-g(t))]*
        L(t)^[(1-a(t))*(1-g(t))]]^[1/(1-g(t))];
FORMULA  (all,t,alltime)
  phi(t) = [{[s(t)*z(t)*H(t)+rho(t)]/[1+subA(t)*a(t)*s(t)/g(t)]}+
        d(t)]/subK(t)/g(t)^2-z(t)*[H(t)-{[a(t)*s(t)/g(t)*H(t)+a(t)*
        rho(t)/g(t)/z(t)]/[1/subA(t)+a(t)*s(t)/g(t)]}]-d(t) ;
FORMULA  (all,t,alltime)
  pA(t) = [1/g(t)-1]/subK(t)/subA(t)*[1+d(t)/{[s(t)*z(t)*H(t)+rho(t)]/
        [1+subA(t)*a(t)*s(t)/g(t)]}]*psi(t) ;
```

```
FORMULA (all,t,fwdtime)
 psiR1(t)=0 ;
FORMULA (all,t,fwdtime)
 phiR1(t)=0 ;
FORMULA (all,t,fwdtime)
 pAR1(t)=0 ;
FORMULA (all,t,fwdtime)
 psiR2(t)=0 ;
FORMULA (all,t,fwdtime)
 phiR2(t)=0 ;
FORMULA (all,t,fwdtime)
 pAR2(t)=0 ;
FORMULA (all,t,fwdtime)
 psiR3(t)=0 ;
FORMULA (all,t,fwdtime)
 phiR3(t)=0 ;
FORMULA (all,t,fwdtime)
 pAR3(t)=0 ;
FORMULA (all,t,fwdtime)
 psiR4(t)=0 ;
FORMULA (all,t,fwdtime)
 phiR4(t)=0 ;
FORMULA (all,t,fwdtime)
 pAR4(t)=0 ;
```

```
! Dynamic equations: relating variables at adjacent time points.          !
EQUATION psiincr (all,t,fwdtime)
 psi(t+1)=psi(t)+psiR1(t)/6+psiR2(t)/3+psiR3(t)/3+psiR4(t)/6 ;
EQUATION phiincr (all,t,fwdtime)
 phi(t+1)=phi(t)+phiR1(t)/6+phiR2(t)/3+phiR3(t)/3+phiR4(t)/6 ;
EQUATION pAincr (all,t,fwdtime)
 pA(t+1) =pA(t)+pAR1(t)/6+pAR2(t)/3+pAR3(t)/3+pAR4(t)/6 ;

! where the RK4 variables are defined by the time derivative functions of the main   !
! dynamic variables as follows:                                                      !

EQUATION psiRK1 (all,t,fwdtime)
 psiR1(t)=dt(t)*[z(t)*{1+pA(t)/psi(t)/a(t)/(1-g(t))}*
        {a(t)*(1-g(t))/z(t)/e(t)^g(t)*L(t)^{(1-a(t))*(1-g(t))}*
        psi(t)^g(t)/pA(t)}^[1/(1-a(t)*(1-g(t)))]-
        phi(t)-d(t)-z(t)*H(t)]*psi(t) ;

EQUATION phiRK1 (all,t,fwdtime)
 phiR1(t)=dt(t)*[(g(t)^2/s(t)*subK(t)-1)*z(t)/a(t)/(1-g(t))*
        pA(t)/psi(t)*{a(t)*(1-g(t))/z(t)/e(t)^g(t)*L(t)^{(1-a(t))*(1-g(t))}*
        psi(t)^g(t)/pA(t)}^[1/(1-a(t)*(1-g(t)))]+
        phi(t)+{d(t)*(s(t)-1)-rho(t)}/s(t)]*phi(t) ;
```



```
EQUATION pARK1 (all,t,fwdtime)
 pAR1(t)=dt(t)*[z(t)*g(t)/a(t)*subK(t)*{g(t)/(1-g(t))*pA(t)/psi(t)-1/subK(t)/subA(t)}*
        {a(t)*(1-g(t))/z(t)/e(t)^g(t)*L(t)^{(1-a(t))*(1-g(t))}*
        psi(t)^g(t)/pA(t)}^[1/(1-a(t)*(1-g(t)))]-d(t)]*pA(t) ;

EQUATION psiRK2 (all,t,fwdtime)
 psiR2(t)=dt(t)*[z(t)*{1+[pA(t)+pAR1(t)/2]/[psi(t)+psiR1(t)/2]/a(t)/(1-g(t))}*
        {a(t)*(1-g(t))/z(t)/e(t)^g(t)*L(t)^{(1-a(t))*(1-g(t))}*
        [psi(t)+psiR1(t)/2]^g(t)/[pA(t)+pAR1(t)/2]}^[1/(1-a(t)*(1-g(t)))]-
        [phi(t)+phiR1(t)/2]-d(t)-z(t)*H(t)]*[psi(t)+psiR1(t)/2] ;
EQUATION phiRK2 (all,t,fwdtime)
 phiR2(t)=dt(t)*[(g(t)^2/s(t)*subK(t)-1)*z(t)/a(t)/(1-g(t))*
        [pA(t)+pAR1(t)/2]/[psi(t)+psiR1(t)/2]*
        {a(t)*(1-g(t))/z(t)/e(t)^g(t)*L(t)^{(1-a(t))*(1-g(t))}*
        [psi(t)+psiR1(t)/2]^g(t)/[pA(t)+pAR1(t)/2]}^[1/(1-a(t)*(1-g(t)))]+
        [phi(t)+phiR1(t)/2]+{d(t)*(s(t)-1)-rho(t)}/s(t)]*[phi(t)+phiR1(t)/2] ;
EQUATION pARK2 (all,t,fwdtime)
 pAR2(t)=dt(t)*[z(t)*g(t)/a(t)*subK(t)*{g(t)/(1-g(t))*
        [pA(t)+pAR1(t)/2]/[psi(t)+psiR1(t)/2]-1/subK(t)/subA(t)}*
        {a(t)*(1-g(t))/z(t)/e(t)^g(t)*L(t)^{(1-a(t))*(1-g(t))}*
        [psi(t)+psiR1(t)/2]^g(t)/[pA(t)+pAR1(t)/2]}^
        [1/(1-a(t)*(1-g(t)))]-d(t)]*[pA(t)+pAR1(t)/2] ;

EQUATION psiRK3 (all,t,fwdtime)
 psiR3(t)=dt(t)*[z(t)*{1+[pA(t)+pAR2(t)/2]/[psi(t)+psiR2(t)/2]/a(t)/(1-g(t))}*
        {a(t)*(1-g(t))/z(t)/e(t)^g(t)*L(t)^{(1-a(t))*(1-g(t))}*
        [psi(t)+psiR2(t)/2]^g(t)/[pA(t)+pAR2(t)/2]}^[1/(1-a(t)*(1-g(t)))]-
        [phi(t)+phiR2(t)/2]-d(t)-z(t)*H(t)]*[psi(t)+psiR2(t)/2] ;
EQUATION phiRK3 (all,t,fwdtime)
 phiR3(t)=dt(t)*[(g(t)^2/s(t)*subK(t)-1)*z(t)/a(t)/(1-g(t))*
        [pA(t)+pAR2(t)/2]/[psi(t)+psiR2(t)/2]*
        {a(t)*(1-g(t))/z(t)/e(t)^g(t)*L(t)^{(1-a(t))*(1-g(t))}*[1/(1-a(t)*(1-g(t)))]+
        [psi(t)+psiR2(t)/2]^g(t)/[pA(t)+pAR2(t)/2]}^
        [phi(t)+phiR2(t)/2]+{d(t)*(s(t)-1)-rho(t)}/s(t)]*[phi(t)+phiR2(t)/2] ;
EQUATION pARK3 (all,t,fwdtime)
 pAR3(t)=dt(t)*[z(t)*g(t)/a(t)*subK(t)*{g(t)/(1-g(t))*
        [pA(t)+pAR2(t)/2]/[psi(t)+psiR2(t)/2]-1/subK(t)/subA(t)}*
        {a(t)*(1-g(t))/z(t)/e(t)^g(t)*L(t)^{(1-a(t))*(1-g(t))}*
        [psi(t)+psiR2(t)/2]^g(t)/[pA(t)+pAR2(t)/2]}^[1/(1-a(t)*(1-g(t)))]-
        d(t)]*[pA(t)+pAR2(t)/2] ;

EQUATION psiRK4 (all,t,fwdtime)
 psiR4(t)=dt(t)*[z(t)*{1+[pA(t)+pAR3(t)]/[psi(t)+psiR3(t)]/a(t)/(1-g(t))}*
        {a(t)*(1-g(t))/z(t)/e(t)^g(t)*L(t)^{(1-a(t))*(1-g(t))}*
        [psi(t)+psiR3(t)]^g(t)/[pA(t)+pAR3(t)]}^[1/(1-a(t)*(1-g(t)))]-
        [phi(t)+phiR3(t)]-d(t)-z(t)*H(t)]*[psi(t)+psiR3(t)] ;
EQUATION phiRK4 (all,t,fwdtime)
 phiR4(t)=dt(t)*[(g(t)^2/s(t)*subK(t)-1)*z(t)/a(t)/(1-g(t))*
        [pA(t)+pAR3(t)]/[psi(t)+psiR3(t)]*
        {a(t)*(1-g(t))/z(t)/e(t)^g(t)*L(t)^{(1-a(t))*(1-g(t))}* [1/(1-a(t)*(1-g(t)))]+
```

```
        [psi(t)+psiR3(t)]^g(t)/[pA(t)+pAR3(t)]}^
        [phi(t)+phiR3(t)]+{d(t)*(s(t)-1)-rho(t)}/s(t)]*[phi(t)+phiR3(t)] ;
EQUATION pARK4 (all,t,fwdtime)
 pAR4(t)=dt(t)*[z(t)*g(t)/a(t)*subK(t)*{g(t)/(1-g(t))*
        [pA(t)+pAR3(t)]/[psi(t)+psiR3(t)]-1/subK(t)/subA(t)}*
        {a(t)*(1-g(t))/z(t)/e(t)^g(t)*L(t)^{(1-a(t))*(1-g(t))}*
        [psi(t)+psiR3(t)]^g(t)/[pA(t)+pAR3(t)]}^[1/(1-a(t)*(1-g(t)))]-
        d(t)]*[pA(t)+pAR3(t)] ;
```

```
! Boundary conditions:                                                      !
! 1.  Initial capital/technology ratio fixed at its immediate pre-shock level: !
!     psi(0)=psi0. Implemented simply by declaring psi(0) exogenous,         !
!     and setting its shock to zero.                                          !
! 2.  'Final' value of the consumption/capital ratio set at its market solution !
!     steady-state level: phi(T)=phiMss.                                      !
! 3.  'Final' value of the price of technology set at its market solution    !
!     steady- state level: pA(T)=pAMss.                                       !
EQUATION phiMss (all,t,endtime)
 phi(t) = z(t)*[pA(t)/psi(t)/a(t)/(1-g(t))+1]*
        {a(t)*(1-g(t))/z(t)/e(t)^g(t)*L(t)^{(1-a(t))*(1-g(t))}*
        psi(t)^g(t)/pA(t)}^[1/(1-a(t)*(1-g(t)))]-[z(t)*H(t)+d(t)] ;
EQUATION pAMss (all,t,endtime)
 pA(t) = (1/g(t)-1)/subK(t)/subA(t)*
        {1+d(t)/[z(t)*g(t)^2/a(t)/(1-g(t))*subK(t)*pA(t)/psi(t)*
        {a(t)*(1-g(t))/z(t)/e(t)^g(t)*L(t)^{(1-a(t))*(1-g(t))}*
        psi(t)^g(t)/pA(t)}^[1/(1-a(t)*(1-g(t)))]-d(t)]}*psi(t) ;
```


Appendix 5.3

Limiting levels and shares of human capital

In the market solution to the Romer model the steady-state allocation of human capital to research is given by equation (2.55) of Chapter 2 as:

$$H^M_{Ass} = \frac{H - \alpha\rho/\gamma\zeta}{1+\alpha\sigma/\gamma}$$

Since α, γ, ζ, ρ, σ and H are all positive and H^M_{Ass} must be non-negative, the derivative of H^M_{Ass} with respect to H is less than unity and H^M_{Ass} is zero up to a certain critical minimum level of H:

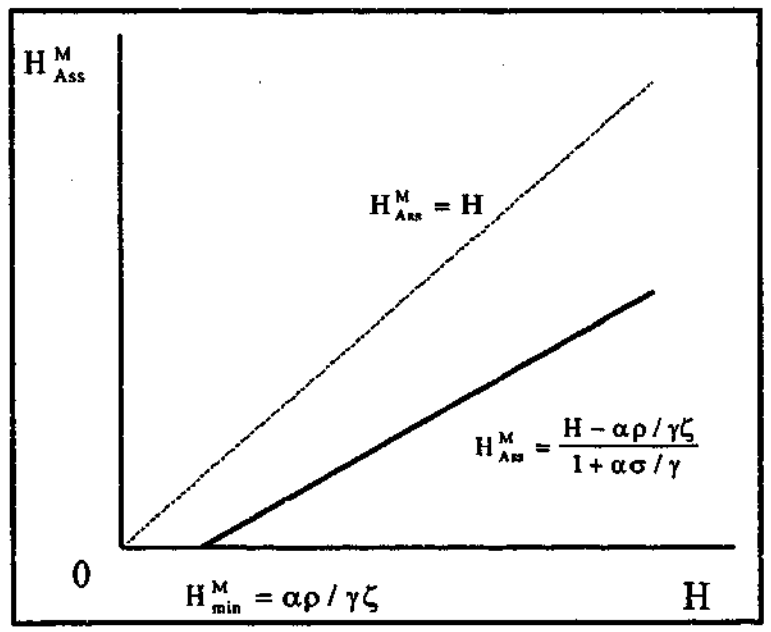
$$\frac{\partial H^M_{Ass}}{\partial H} = \frac{1}{1+\alpha\sigma/\gamma} < 1$$

and (A5.3.1)

$$H^M_{Ass} = 0 \quad \text{for } H \in (0, H^M_{min} = \alpha\rho/\gamma\zeta]$$

The graph of H^M_{Ass} against H is shown in Figure A5.3.1. This is similar to the graph "Figure 2", presented by Romer (1990b). The fact that there is a range of H for which the non-negativity constraint on H^M_{Ass} binds is appealing. As Romer says: It indicates that *"all the feasible growth rates for A are too small relative to the discount rate to justify the sacrifice in current output necessary for growth to take place"*. Even more appealing is the fact that the slope of the $H^M_{Ass} = H^M_{Ass}(H)$ function is less than unity. This ensures that the obvious constraint of the steady-state amount of human capital devoted to research being less than the total human capital available ($H^M_{Ass} < H$), is satisfied for all valid parameter values.

Figure A5.3.1: Steady-state allocation of human capital to research under the market solution of the Romer Model



This property is not necessarily met in the case of the social planning solution. From equation (A5.1.29), the socially optimum allocation of human capital to research is given by:

$$H^{O}_{Ass} = \frac{H - \alpha\rho/\zeta}{(1-\alpha)+\alpha\sigma}$$

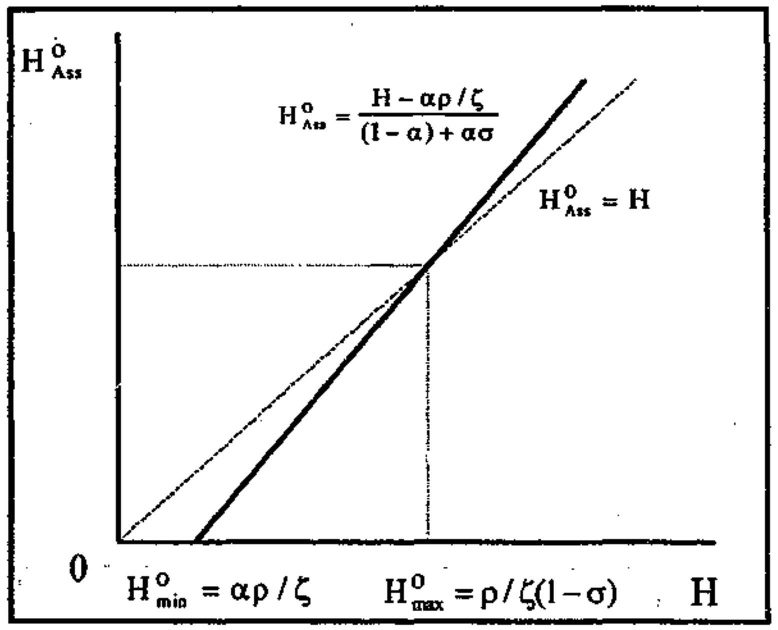
so the sign of the derivative of H^{O}_{Ass} with respect to H depends on the magnitude of the consumer preference parameter σ. Specifically:

$$\frac{\partial H^{O}_{Ass}}{\partial H} = \frac{1}{1+\alpha(\sigma-1)} < 1 \quad \text{for } \sigma > 1 \text{ but} \qquad \text{(A5.3.2)}$$
$$> 1 \quad \text{for } \sigma < 1$$

Also:

$$\exists\, H^{O}_{min} \text{ such that: } H^{O}_{Ass} = 0 \text{ for } H \in (0, H^{O}_{min} = \alpha\rho/\zeta] \qquad \text{(A5.3.3)}$$

Figure A5.3.2: Steady-state allocation of human capital to research under the social optimum solution of the Romer Model, for σ<1.



For $\sigma > 1$ (as is the case for the benchmark parameter set) the position is much the same as for the market solution. However, when $\sigma<1$ the slope of the $H^{O}_{Ass} = H^{O}_{Ass}(H)$ function is greater than unity and the graph looks like that shown in Figure A5.3.2. In this case there appears to be a critical maximum level of overall human capital H^{O}_{max}, for which it is *optimal* for none to be allocated to the production of goods, and beyond which no further output would therefore be possible! By equating H^{O}_{Ass} with H, this level is found to be given by:

$$H^{O}_{max} = \rho/\zeta(1-\sigma) \qquad \text{(A5.3.4)}$$

The constraint imposed by this limiting level of human capital, is equivalent to that of what I shall call a *boundedness condition* associated with the dynamic maximisation of consumer utility (Appendix 5.1). In particular, the maximisation problem in equation (A5.1.1) is only possible and valid if the integrand remains bounded. This imposes the condition that the limit of the rate of growth of current utility, given by $U(t) = [C(t)^{1-\sigma} - 1]/(1-\sigma)$, must be less than the discount rate, ρ. Since the limiting growth rate of current utility is:

$$\lim_{t\to\infty}[\dot{U}(t)/U(t)] = \lim_{t\to\infty}\left[(1-\sigma)\frac{\dot{C}(t)}{C(t)}\frac{C(t)^{1-\sigma}}{C(t)^{1-\sigma}-1}\right] = (1-\sigma)g^{O}$$

the *boundedness condition* is:[40]

$$(1-\sigma)g^{O} < \rho \qquad \text{(A5.3.5)}$$

With $\sigma>1$, as in the benchmark data set, this requirement is automatically satisfied since g^{O} is non-negative and ρ is positive. However, for $0<\sigma<1$, (A5.3.5) imposes a binding restriction on the other parameter values. Substituting for the social optimum steady-state growth rate g^{0} from equation (A5.1.29):

$$(1-\sigma)g^{O} < \rho \;\Rightarrow\; (1-\sigma)\frac{\zeta H - \alpha\rho}{1-\alpha(1-\sigma)} < \rho \qquad \text{(A5.3.6)}$$
$$\Rightarrow\; H < \frac{\rho}{\zeta(1-\sigma)} = H^{O}_{max}$$

Of course the boundedness condition also applies to the consumer utility maximisation problem in the free market (Appendix 2.2). Consequently, for $0<\sigma<1$, it also imposes an upper limit constraint on the level of human capital in the market solution. As above, substitution of the expression for the market steady-state growth rate g^{M} from equation (2.55) into the boundedness condition identifies this constraint as:

$$(1-\sigma)g^{M} < \rho \;\Rightarrow\; (1-\sigma)\frac{\zeta H - \alpha\rho/\gamma}{1+\alpha\sigma/\gamma} < \rho \qquad \text{(A5.3.7)}$$
$$\Rightarrow\; H < \frac{\rho(1+\alpha/\gamma)}{\zeta(1-\sigma)} = H^{M}_{max}$$

Both the market and social planning solutions produce steady-state allocations of human capital to research which take the form $H_{As} = a.H - b$, where b is positive. For this reason, as the total level of human capital rises the overall share allocated to research rises monotonically for both solutions to the model. The relationships are:

$$\frac{H^{M}_{Ass}}{H} = \frac{1}{1+\alpha\sigma/\gamma}\left[1 - \frac{\alpha\rho/\gamma\zeta}{H}\right] \qquad \text{(A5.3.8)}$$

40 That this condition is consistent with the transversality conditions may be demonstrated in the following way. From equations (A5.1.6) and the derived features of the balanced growth equilibrium, satisfaction of the transversality conditions requires that $g^{o} < r^{o}_{\infty}$. Substituting the steady-state relations $r^{o}_{\infty} = \sigma g^{o} + \rho$ then reproduces the boundedness condition.

$$\frac{H^{O}_{Ass}}{H}=\frac{1}{1-\alpha+\alpha\sigma}[1-\frac{\alpha\rho/\zeta}{H}] \qquad (A5.3.9)$$

If the total level of human capital were unconstrained (as would be desirable and is indeed the case when $\sigma>1$), the limiting values of these shares would simply be:

$$\lim_{H\to\infty}\frac{H^{M}_{Ass}}{H}=\frac{1}{1+\alpha\sigma/\gamma} \quad \text{and} \quad \lim_{H\to\infty}\frac{H^{O}_{Ass}}{H}=\frac{1}{1-\alpha+\alpha\sigma} \qquad (A5.3.10)$$

(which evaluate under the benchmark parameter set to 29.5 per cent and 53.8 per cent respectively). However, given that the boundedness condition limits total human capital, when it binds it imposes different limits on the steady-state shares of human capital allocated to research. These are obtained from (A5.3.8) and (A5.3.9) simply by evaluating H at H^{M}_{max} and H^{O}_{max} respectively in the two equations to produce:

$$\frac{H^{M}_{Ass}}{H^{M}_{max}}=\frac{1}{1+\alpha/\gamma} \quad \text{and} \quad \frac{H^{O}_{Ass}}{H^{O}_{max}}=1 \qquad (A5.3.11)$$

Interestingly, these are the same results that would obtain in the unconstrained case with $\sigma=1$ - the value of the elasticity of intertemporal substitution of consumption in the utility function $U(t) = \log C(t)$.[41]

As was seen before, with $\sigma<1$ the human capital share to research in the social optimum solution is effectively unlimited - all the human capital ends up in the research sector and none in manufacturing! The more appealing result is for the market solution. Here the limiting steady-state share of human capital when the boundedness condition binds is less than 100 per cent (for the benchmark parameter set it evaluates to 55.7 per cent). Moreover, though the constraint binds only when $\sigma<1$, and then H^{M}_{max} itself depends on σ; the research sector share of this limiting level of human capital is nevertheless independent of σ.

It would surely be wrong to interpret the boundedness condition as a constraint upon the other parameters with the total level of human capital as given. For one thing, these parameters are determined independently; and for another, a more realistic model would include the endogenous generation and accumulation of human capital, which would thus quickly reach the limiting levels. In that case, for the model to continue 'to work', changes to other parameters, possibly contrary to empirical evidence, would be necessary. Thus, it seems that there is a deficiency in the model, manifested whenever the elasticity of intertemporal substitution of consumption (the reciprocal of σ) is greater than unity. In this sense it is just good luck that $\sigma>1$ is preferred (see Section 2.4.1.1); for then no constraint is imposed by the boundedness condition and there is no deficiency for any of the parameterisations used in this paper. The question of how to overcome this deficiency is left as a topic for future research.

41 As $\sigma\to1$, the current utility function $U\{C(t)\} = [C(t)^{1-\sigma} - 1]/(1-\sigma) \to \log[C(t)]$, (see, for example Barro and Sala-i-Martin, 1995).

Appendix 5.4

Interest rates, capital goods intensities, and technology prices in the market and optimal steady-states

A5.4.1 Interest rates

It was shown in the text that both the allocation of human capital to research and hence the steady-state growth rate, were lower in the market solution to the Romer model than was socially optimal. By utilising this result for the growth rate, the market interest rate was also shown to be 'too low'. However, because it was felt preferable to regard the growth rate result as being a consequence rather than a cause of the interest rate result,[42] a proof of the interest rate being 'too low' which was independent of the previous growth rate result was sought. Such a proof is provided here:

The steady-state market and social optimum interest rate are given by equations (2.56) and (5.12). These results are:

$$r^{M}_{ss}=\frac{\sigma\zeta H+\rho}{1+\alpha\sigma/\gamma} \quad \text{and} \quad r^{O}_{ss}=\frac{\sigma\zeta H+(1-\alpha)\rho}{(1-\alpha)+\alpha\sigma} \qquad (A5.4.1)$$

Two differences in these formulations are apparent: First, is the presence of the term γ in the denominator of the market rate as opposed to a unitary term in the optimal formulation. *Ceteris paribus*, this will tend to make the market rate the lesser of the two. The second difference is the term $(1-\alpha)$ which appears in both the numerator and denominator of the optimal formulation in place of a corresponding term of unity in the market expression. Since $(1-\alpha)<1$, moving from the optimal to the market expression is equivalent to raising $(1-\alpha)$. Thus, this difference in the formulations will also tend to make the market determined interest rate the lower one if $\partial r^{O}_{ss}/\partial(1-\alpha)<0$. This is indeed the case since:

$$\begin{aligned}\partial r^{O}_{ss}/\partial(1-\alpha) &= \frac{\rho}{1-\alpha+\alpha\sigma}-\frac{\sigma\zeta H+(1-\alpha)\rho}{[1-\alpha+\alpha\sigma]^2}\\ &= \frac{\rho}{1-\alpha+\alpha\sigma}[1-\frac{1}{1-\alpha+\alpha\sigma}]-\frac{\sigma\zeta H-\alpha\rho}{[1-\alpha+\alpha\sigma]^2}\\ &= \frac{\alpha\rho\sigma-\alpha\rho}{[1-\alpha+\alpha\sigma]^2}-\frac{\sigma\zeta H-\alpha\rho}{[1-\alpha+\alpha\sigma]^2}\\ &= \frac{\sigma}{[(1-\alpha)+\alpha\sigma]^2}[\alpha\rho-\zeta H]\end{aligned}$$

and, since $H^{O}_{Ass}>0$, equation (5.11) implies $(\alpha\rho-\delta H)<0$. Thus, $\partial r^{O}_{ss}/\partial(1-\alpha)<0$, and it has been proved that $r^{M}_{ss}<r^{O}_{ss}$.

42 It is because consumers face too low a return from savings that the growth rate of capital is sub-optimal.

A5.4.2 Capital goods intensities

As argued in the text, while cursory intuition may lead to the expectation that the capital goods intensity Ψ, or equivalently the steady-state level of specialised capital X, would be lower in the free market than under a social planning optimum, the result actually turns out to be ambiguous. Here the question of when the market level is the lower, and when it is the higher is investigated by evaluating and plotting the ratio $\Psi_{ss}^{O} / \Psi_{ss}^{M}$ for a variety of parameter values.[43]

Figure A5.4.1: Variation of $\Psi_{ss}^{O} / \Psi_{ss}^{M}$ with parameter α, other parameters at benchmark.

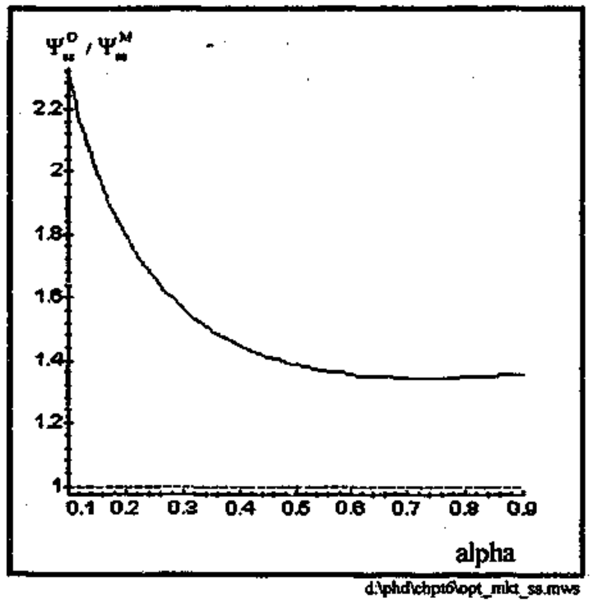


Figure A5.4.2: Variation of $\Psi_{ss}^{O} / \Psi_{ss}^{M}$ with parameter γ, other parameters at benchmark.

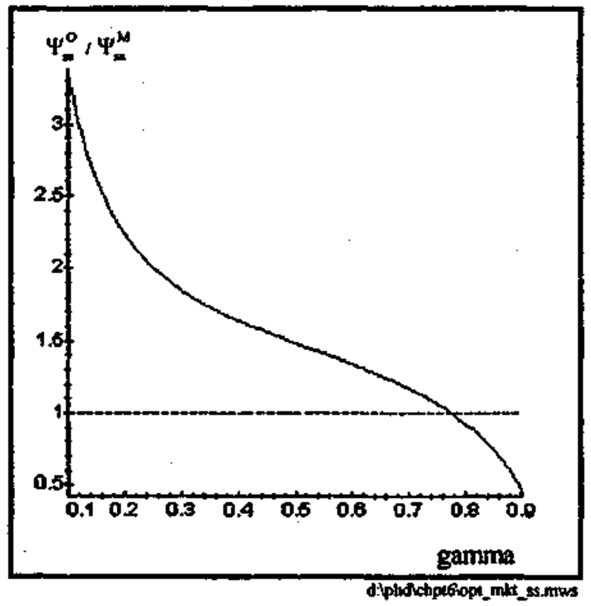


Figure A5.4.3: Variation of $\Psi_{ss}^{O} / \Psi_{ss}^{M}$ with parameter ζ, other parameters at benchmark.

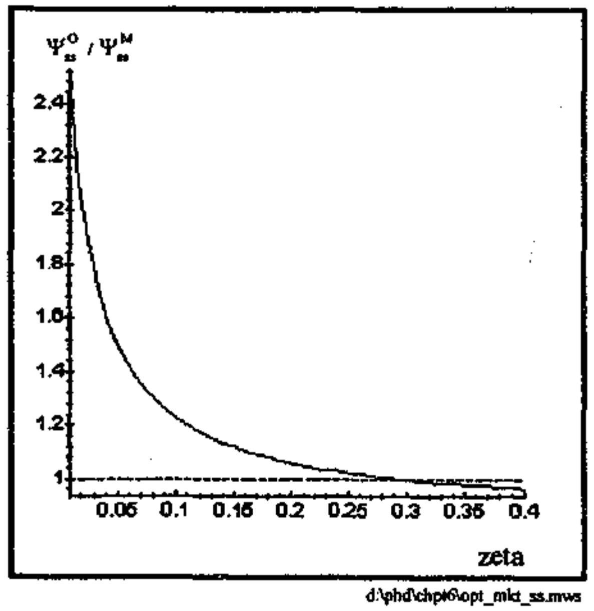
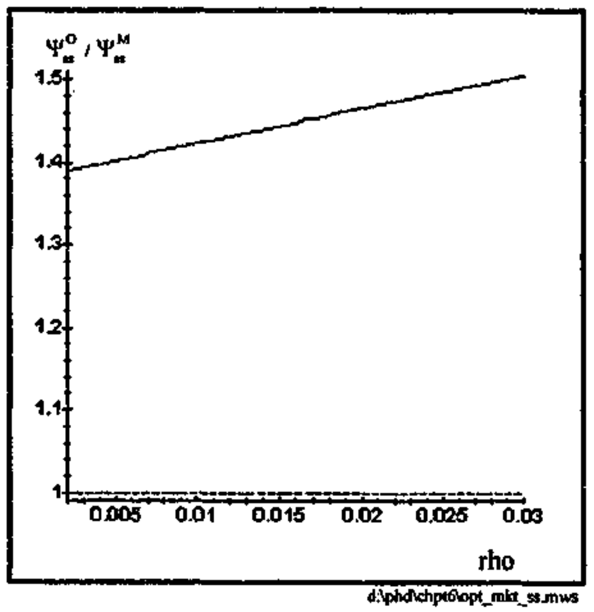


Figure A5.4.4: Variation of $\Psi_{ss}^{O} / \Psi_{ss}^{M}$ with parameter ρ, other parameters at benchmark.



[43] The analysis is confined to the parameters α, γ, ζ, ρ, σ, and δ since neither η nor L affect the ratio $\Psi_{ss}^{O} / \Psi_{ss}^{M}$ and since the effect of H is the same as that of ζ.

Figure A5.4.5: Variation of $\Psi_{ss}^{O} / \Psi_{ss}^{M}$ with parameter σ, other parameters at benchmark.

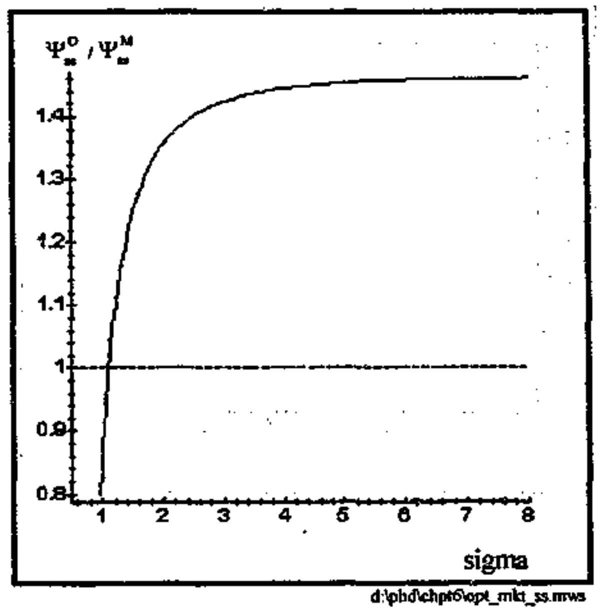


Figure A5.4.6: Variation of $\Psi_{ss}^{O} / \Psi_{ss}^{M}$ with parameter δ, other parameters at benchmark.

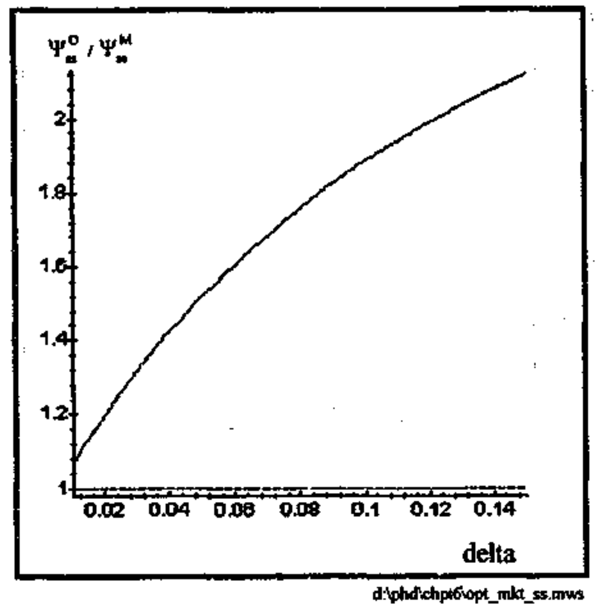


Figure A5.4.7: Variation of $\Psi_{ss}^{O} / \Psi_{ss}^{M}$ with parameter α, other parameters set at: γ=0.72, ζ=0.08, σ=2.0, δ=0.03, rest at benchmark.

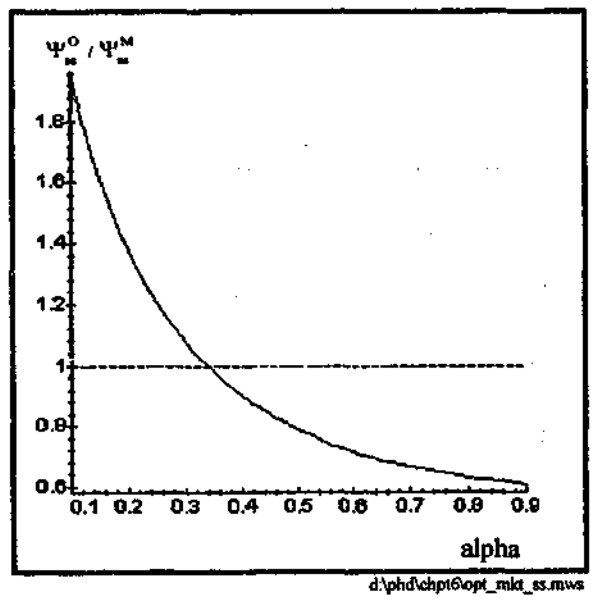


Figure A5.4.8: Variation of $\Psi_{ss}^{O} / \Psi_{ss}^{M}$ with parameter γ, other parameters at: α=0.6, ζ=0.08, σ=2.0, δ=0.03, rest at benchmark.

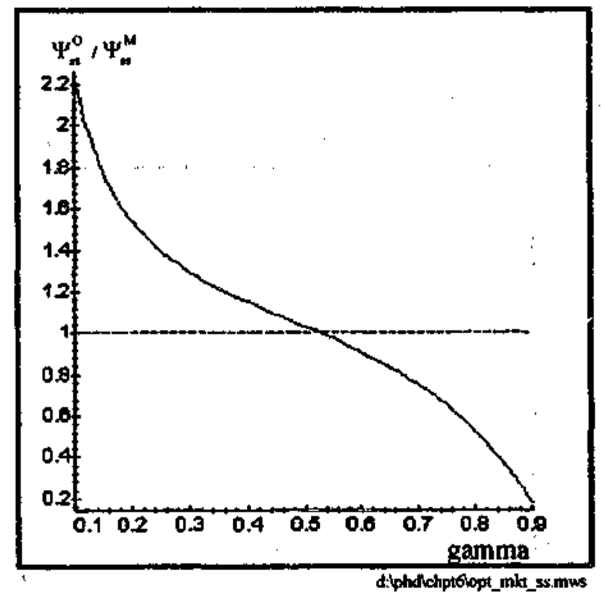


Figure A5.4.9: Variation of $\Psi^O_{ss} / \Psi^M_{ss}$ with parameter ζ, other parameters set at: α=0.6, γ=0.7, σ=2.0, δ=0.03, rest at benchmark.

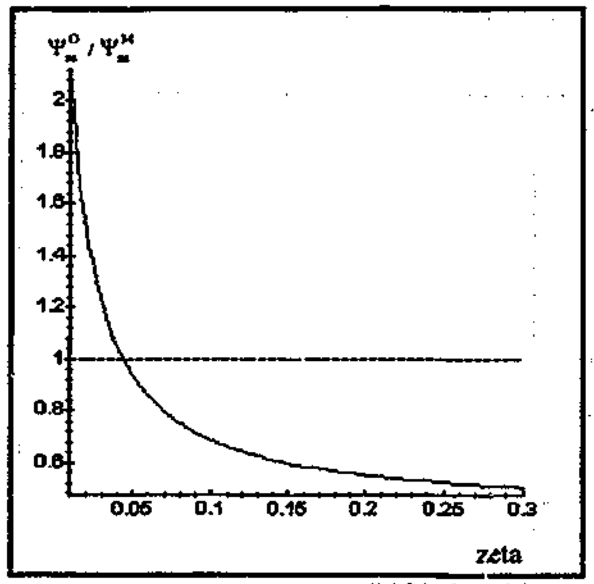


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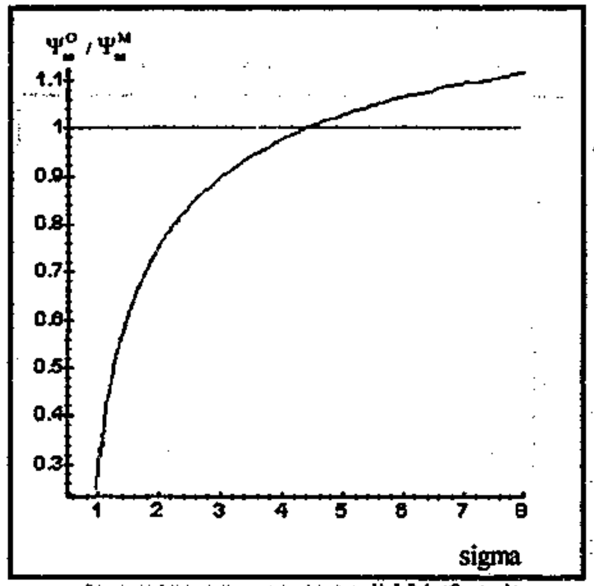
Figure A5.4.10: Variation of $\Psi^O_{ss} / \Psi^M_{ss}$ with parameter σ, other parameters set at: α=0.6, γ=0.7, ζ=0.08, δ=0.03, rest at benchmark.



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A5.4.3 Price of technology

The story of the relative magnitude of the equilibrium price of technology under the free market Romer economy and the social planning optimum, mirrors that of the capital goods intensity: As argued in the text, while the *a priori* intuition may seem to lead to an expectation that the market outcome is 'too low' in a Pareto optimal sense, it turns out that either the market or the optimum level may be the greater, the result depending on parameter values. Like before, this is assessed by evaluating and plotting the ratio p^O_{Ass} / p^M_{Ass} for a variety of parameter values.

Figure A5.4.11: Variation of p^O_{Ass} / p^M_{Ass} with parameter α, other parameters at benchmark.

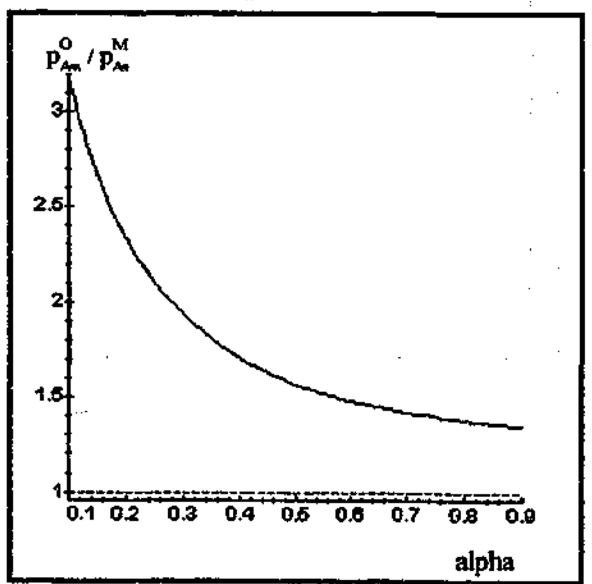

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Figure A5.4.12: Variation of p^O_{Ass} / p^M_{Ass} with parameter γ, other parameters at benchmark.

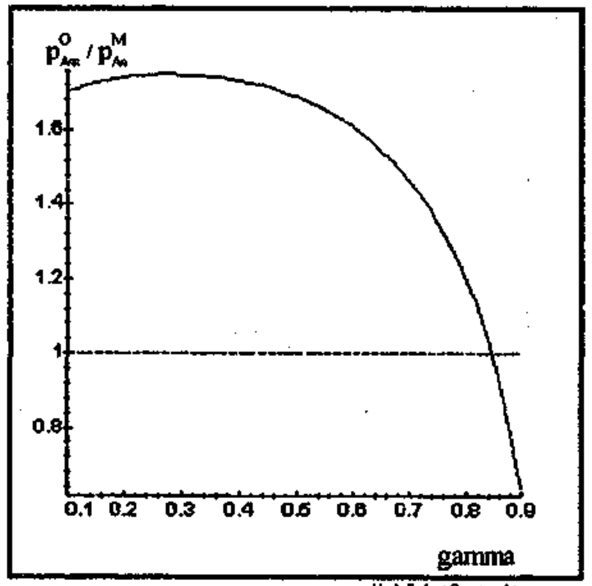

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Figure A5.4.13: Variation of p^O_{Ass} / p^M_{Ass} with parameter ζ, other parameters at benchmark.

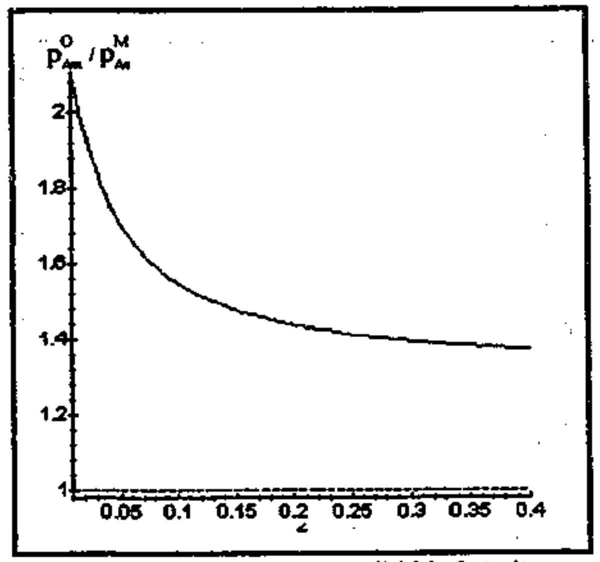

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Figure A5.4.14: Variation of p^O_{Ass} / p^M_{Ass} with parameter ρ, other parameters at benchmark.

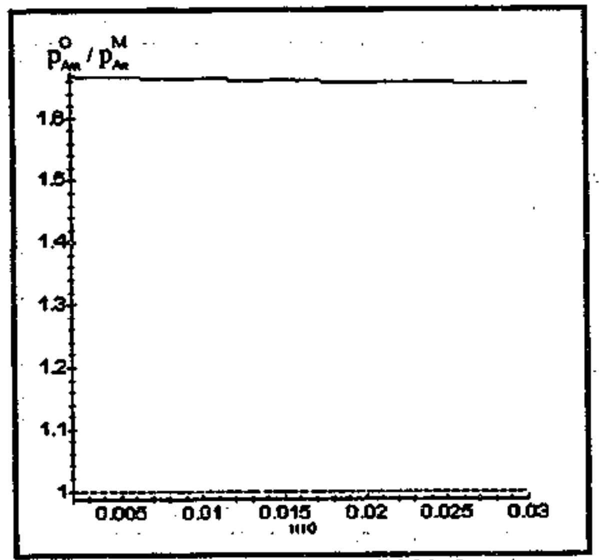

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Figure A5.4.15: Variation of p^O_{Ass} / p^M_{Ass} with parameter σ, other parameters at benchmark.

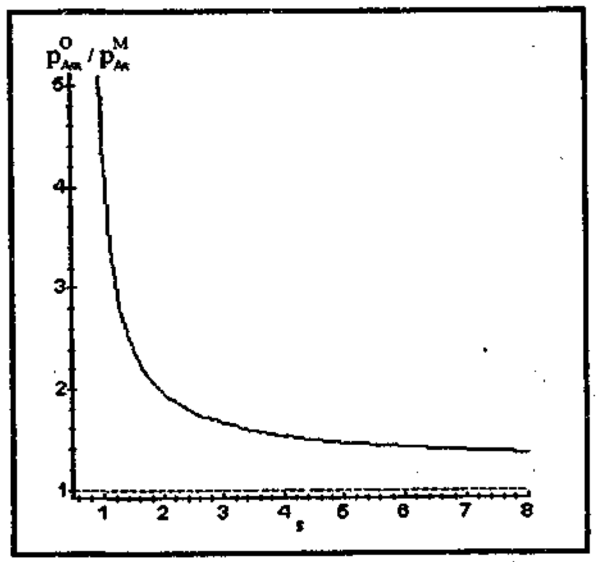

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Figure A5.4.16: Variation of $\Psi^O_{ss} / \Psi^M_{ss}$ with parameter δ, other parameters set at:

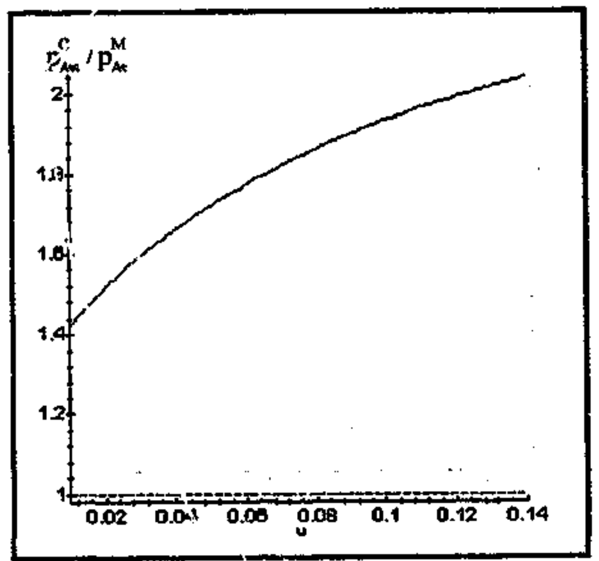

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Figure A5.4.17: Variation of p^O_{Ass}/p^M_{Ass} with parameter α, other parameters set at: γ=0.7, ζ=0.15, σ=5.0, δ=0.03, rest at benchmark.

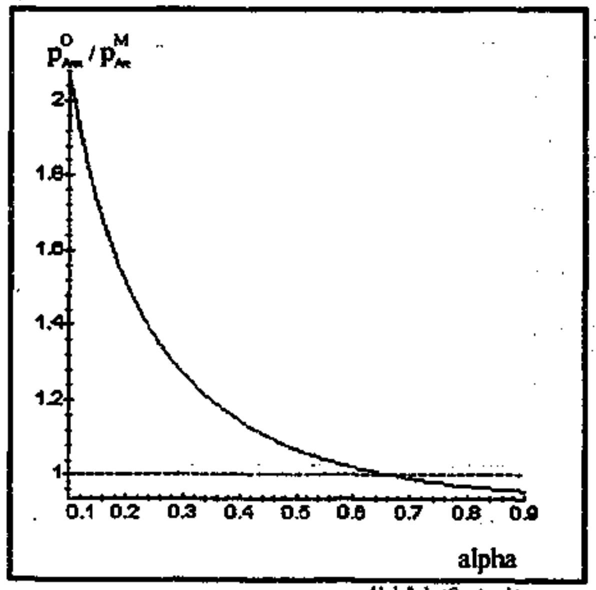

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Figure A5.4.18: Variation of p^O_{Ass}/p^M_{Ass} with parameter γ, other parameters set at: α=0.6, ζ=0.15, σ=5.0, δ=0.03, rest at benchmark.

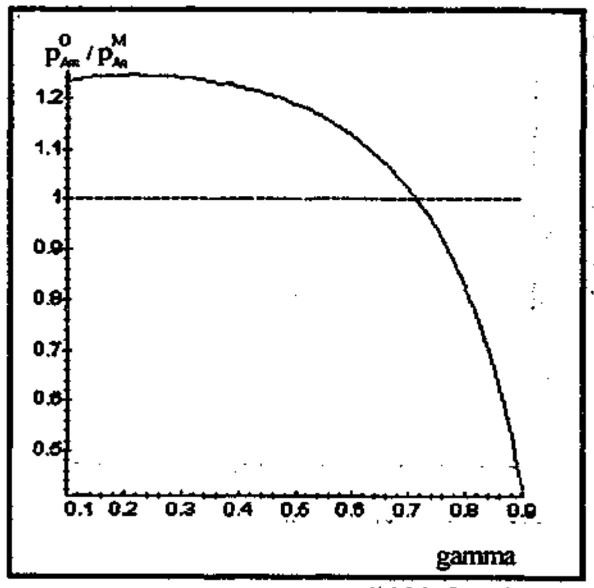

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Figure A5.4.19: Variation of p^O_{Ass}/p^M_{Ass} with parameter ζ, other parameters set at: α=0.65, γ=0.75, σ=5.0, δ=0.03, rest at benchmark.

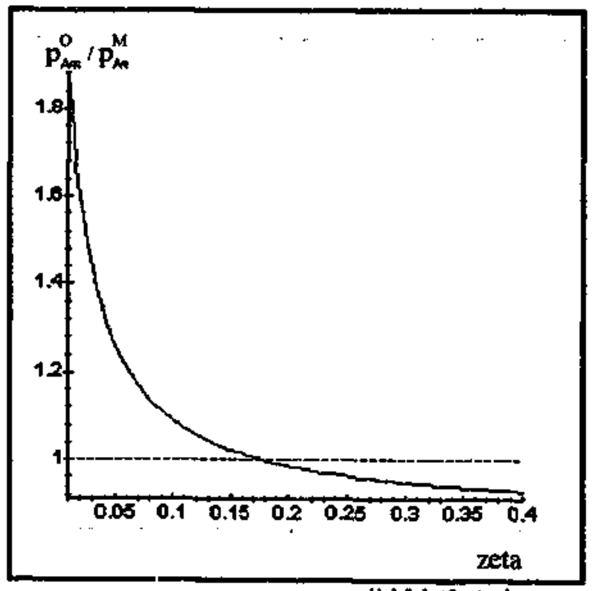

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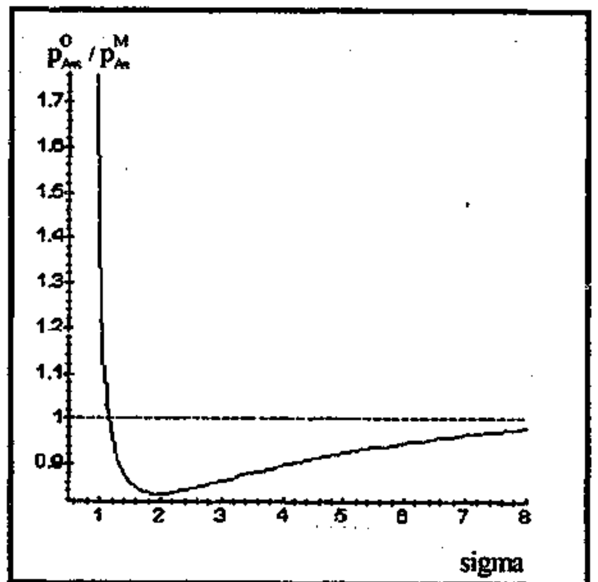
Figure A5.4.20: Variation of p^O_{Ass}/p^M_{Ass} with parameter σ, other parameters set at: α=0.65, γ=0.75, ζ=0.15, δ=0.03, rest at benchmark.



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Appendix 5.5

Alternative subsidisation strategies to attain the social optimum steady-state

It was demonstrated in Section 5.2 that within the Romer model economy the implementation of an appropriate subsidisation policy makes it possible to transform the sub-optimal equilibrium resulting from a decentralised market, into the Pareto optimum equilibrium that would be achieved by a social planner. The subsidisation policy needed to redress several distortions of the free market: Namely, an under-valuation of capital; monopolistic behaviour by capital goods producers; and spillover benefits associated with research output. The particular policy identified was to subsidise the rentals received for general-purpose capital at the rate $100s_K$ per cent, and to subsidise the purchase of designs at a rate of $100s_{AK}$ per cent where, re-writing equation (5.41):

$$s_K = \text{subK} - 1 = 1/\gamma - 1 \text{ ; and}$$

$$s_{AK} = 1 - \text{subA}_K = 1 - \frac{\gamma\zeta}{\alpha}\frac{H^O_{Yss}}{r^O_{Ass}} = 1 - \gamma\frac{\zeta(\sigma-1)H+\rho}{\sigma\zeta H+(1-\alpha)\rho} \qquad \text{(A5.5.1)}$$

Here some alternative subsidisation strategies are considered.

A5.5.1 The production side distortions

Instead of approaching the production side distortions by correcting for the under-valuation of capital, consider an approach designed to correct for the monopoly pricing mark-up on specialised capital. Two methods suggest themselves: direct subsidisation of the rental costs of specialised capital goods faced by final output producers; and somewhat less directly, subsidisation of final output.

A5.5.1.1 Subsidising the rental costs of specialised capital

With a subsidy to the user costs of specialised capital set at the rate $100s_X$ per cent, *purchasers' rental prices* for capital goods are:

$$p^S_X(i,t) = p_X(i,t)(1-s_X) \qquad \text{(A5.5.2)}$$

Final goods producers' maximise profits based upon this subsidised price rather than on the original price p_X and accordingly (see Section 2.2.4) generate the demand function:

$$p_X(i,t) = \gamma H_Y(t)^{\alpha(1-\gamma)} L^{(1-\alpha)(1-\gamma)} X(i,t)^{\gamma-1}/(1-s_X) \qquad \text{(A5.5.3)}$$

And following Section 2.2.5 as before for the 'capital savings' subsidy (s_K), when capital goods producers take this demand function as given and maximise their own profits, results analogous to the pure market conditions (2.13) to (2.15); and to the 'capital savings' subsidy conditions (5.21) to (5.23) are generated. Then, when the purchase of designs is subsidised in exactly the same way as for the (s_K,s_{AK})-subsidy scheme (see

Section 5.2.1.2), the same result as equation (5.27) is obtained.[44] The complete dynamic system for this (s_X,s_{AX})-subsidy scheme may then be derived as before. It turns out that if $(1-s_X)=1/(1+s_K)$ this system is, with only one exception, identical to that for the (s_K,s_{AK})-scheme as given by equations (5.21) to (5.32). The only variation is the differential equation for p_A. In the new scheme equation (5.30) is replaced by:

$$\dot{p}_A(t) = r(t)p_A(t) - \frac{(1-\gamma)}{\gamma}\frac{r(t)+\delta}{(1-s_{AX})}\Psi(t) \tag{A5.5.4}$$

Now the purpose of the s_X subsidy was to correct for the distortion due to the monopoly pricing mark-up on specialised capital over its marginal cost. From Section 2.2.5 (see Figure 2.2 in particular), it is known that the marginal cost of a unit of specialised capital is $[r(t)+\delta]\eta$ while its market price is $[r(t)+\delta]\eta/\gamma$. Thus, $p_X=(1/\gamma)MC_X$ and the monopoly price mark-up is the factor $1/\gamma$. It is then apparent from (A5.5.2) that in order to equate users' prices to the levels which would be faced in a competitive market (or in the social optimum), $subX=(1-s_X)$ must be set to the value γ. The rate of subsidy is then $100s_X=100(1-\gamma)$ per cent. Finally, the optimum level of the other subsidy of this scheme, s_{AX}, may be determined in the same way as before for s_{AK} in Section 5.2.2. The optimum subsidies of this (s_X,s_{AX})-subsidy scheme are thus found to be:

$$\begin{aligned} s_X &= 1 - subX = (1-\gamma) \text{ ;and} \\ s_{AX} &= 1 - subA_X = 1 - \frac{\zeta}{\alpha}\frac{H^O_{Yss}}{r^O_{ss}} = 1 - \frac{\zeta(\sigma-1)H+\rho}{\sigma\zeta H+(1-\alpha)\rho} \end{aligned} \tag{A5.5.5}$$

A5.5.1.2 Subsidising the manufacture of final output

Now consider a *subsidy to final output manufacturers.* With such a subsidy set at s_Y the return to manufacturers per unit of output (the unsubsidised price of which is unity) becomes $subY=(1+s_Y)$. Thus, following the same procedures as in Section 2.2.4, the optimisation problem for final goods producers is:

$$\text{Max.} \int_0^{A(t)} [(1+s_Y)N(H_Y(t),L)^{1-\gamma}X(i,t)^{\gamma} - p_X(i,t)X(i,t)]di$$

yielding the demand function:

$$\begin{aligned} p_X(i,t) &= (1+s_Y)\gamma N(H_Y(t),L)^{1-\gamma}X(i,t)^{\gamma-1} \\ &= (1+s_Y)\gamma H_Y(t)^{\alpha(1-\gamma)}L^{(1-\alpha)(1-\gamma)}X(i,t)^{\gamma-1} \end{aligned} \tag{A5.5.6}$$

The monopoly profits maximisation problem of the capital goods producers may be solved as before to generate results analogous to equations (5.21) to (5.23) *et cetera* of the text. But it is already apparent from comparison of equations (A5.5.6) and (A5.5.3), that the dynamic system applying to a scheme of subsidising final output and the purchase of designs (an (s_Y,s_{AY})-subsidy scheme), will be equivalent to that derived from the (s_X,s_{AX})-subsidy scheme provided the respective rates of production side subsidy are

[44] The only difference is one of notation. Here the subsidy to designs is denoted by s_{AX} to indicate its combination with s_X rather than with s_K.

related by $subY=1/subX$; and that the associated designs subsidies satisfy $subA_Y=subA_X$, or $s_{AY}=s_{AX}$. Thus, the optimum subsidies of this (s_Y,s_{AY})-scheme are given by:

$$\begin{aligned} s_Y &= subY - 1 = 1/\gamma - 1 \text{ ;and} \\ s_{AY} &= 1 - subA_Y = 1 - \frac{\zeta}{\alpha}\frac{H^O_{Yss}}{r^O_{Ass}} = 1 - \frac{\zeta(\sigma-1)H+\rho}{\sigma\zeta H+(1-\alpha)\rho} \end{aligned} \tag{A5.5.7}$$

That all these schemes (A5.5.1), (A5.5.5) and (A5.5.7) do indeed generate the social optimum steady-state is confirmed numerically, for the case of the benchmark parameter set, in Table 5.5 of Section 5.2.2.

A5.5.2 The research externality distortion

A5.5.2.1 Subsidising the wages of researchers

All the designs are produced by human capital in the form of researchers. Because of the externality associated with these designs the researchers are not fully remunerated and so from a socially optimum perspective too few are employed. From this it may seem natural to use subsidisation of research wages to correct for the distortion. Such a subsidy would draw human capital out of the final output producing sector and into research. The amount of human capital involved in final goods production would fall until its marginal product rose to the level of the subsidised wage rate in the research sector, thereby re-equating wages in the two sectors. With the subsidy set at the rate $100s_{HK}$ per cent subsidised wages for human capital are:

$$w^S_{H_A}(t) = (1+s_{HK})w_{H_A}(t) = subH_K.w_{H_A}(t) \tag{A5.5.8}$$

Following Section 2.2.7, unsubsidised wages in each sector are given by the value of the marginal product of the human capital employed there.[45] Thus:

$$w_{H_A}(t) = VMP_{H_A}(t) = \frac{\partial\dot{A}(t)}{\partial H_A(t)}p_A(t) = \zeta A(t)p_A(t) \tag{A5.5.9}$$

$$\begin{aligned} w_{H_Y}(t) &= VMP_{H_Y}(t) = \frac{\partial Y(t)}{\partial H_Y(t)}.1 \\ &= \alpha(1-\gamma)\eta^{-\gamma}H_Y(t)^{\alpha(1-\gamma)-1}L^{(1-\alpha)(1-\gamma)}A\Psi(t)^{\gamma} \end{aligned} \tag{A5.5.10}$$

Then, substituting (A5.5.9) into (A5.5.8) and equating the result with (A5.5.10) determines the allocation of human capital between the sectors. Specifically, $H_Y(t)$ is determined as:

$$H_Y(t) = \left[\frac{\alpha(1-\gamma)}{\zeta\eta^{\gamma}subH_K}L^{(1-\alpha)(1-\gamma)}\Psi(t)^{\gamma}p_A(t)^{-1}\right]^{\frac{1}{1-\alpha(1-\gamma)}} \tag{A5.5.11}$$

[45] There is no proviso to this in the competitive output sector; but in the research sector it must be understood that it holds only when the marginal product of human capital is valued at the market price of designs p_A (see Section 2.2.7).

Accounting for the modifications to the free market system equations that result from introducing the capital savings subsidy s_K, then allows the stationary dynamic system for this (s_K,s_{HK})-subsidy scheme to be derived and for its steady-state to be equated with that of the social optimum as before. This defines the condition:

$$subH_K = \frac{\alpha}{\gamma\zeta}\frac{r^O_{Ass}}{H^O_{Yss}} = \frac{1}{subA_K} \qquad \text{(A5.5.12)}$$

So the optimum subsidies in this (s_K,s_{HK})-scheme are:[46]

$$s_K = subK - 1 = 1/\gamma - 1 \quad ;\text{and}$$
$$s_{HK} = subH_K - 1 = \frac{\alpha}{\gamma\zeta}\frac{r^O_{Ass}}{H^O_{Yss}} = \frac{1}{\gamma}\frac{\sigma\zeta H + (1-\alpha)\rho}{\zeta(\sigma-1)H + \rho} - 1 \qquad \text{(A5.5.13)}$$

However, as mentioned before in Section 5.2.2, this scheme fails to replicate completely the socially optimum steady-state. While it generates the socially optimum levels for the allocation of human capital, the growth and interest rates, the resources devoted to each type of specialised capital, and the consumption-capital ratio, it fails to reproduce the social optimum steady-state price of technology (or any other variables that depend upon it). This remains the case for a 'research wage' subsidy in combination with other production side subsidies, s_X, or s_Y in particular (Table 5.5 demonstrates numerically the case for the benchmark parameter set). It seems that the reason for the failure of this method to correct for the research externality distortion is that although human capital is the only remunerated factor producing designs, there is another input - the existing stock of designs - and omitting to subsidise all factors equally introduces another distortion.

A5.5.2.2 Subsidising the 'accumulation of research'

The precise basis of a subsidy to the *'accumulation of research'* is not immediately obvious. Nor is it clear to whom the subsidy payments would be made. To help clarify matters consider the introduction of another group of economic agents: *research entrepreneurs* who employ researchers and who sell the designs they produce to the manufacturers of specialised capital goods. These agents will then be the recipients of the subsidy, which can be based upon either the prices they receive for designs or the (wage) costs they incur in producing them. But in order to maintain the existing economics of the model these agents would need to be considered as part of the exogenous stock of human capital, and to receive the same wages for their efforts as the researchers they employ.

The research entrepreneurship market is assumed to be competitive so the usual *zero pure profits condition* applies, revenue being equated with costs. In the free market aggregate revenues are $p_A\dot{A}$ and aggregate costs $w_{HA}\,H_A$ so research wages are, as in Section 2.2.7, $w_{HA} = \zeta A p_A$.

[46] It is also easy to calculate the optimum research wage subsidies necessary in combination with other production side subsidies. For example, in either an (s_X,s_{HX})- or a (s_Y,s_{HY})-subsidy scheme the optimum $subH_X = subH_Y = 1/subA_X$.

For a subsidy to the 'accumulation of research' based upon the price of designs at the rate $100s_D$ per cent, the subsidised price received is $p^S_A(t) = (1+s_D)p_A(t)$, aggregate revenues and costs are $(1+s_D)p_A(t)\dot{A}(t)$ and $w_{H_A}(t)H_A(t)$ respectively, and research wages in the subsidised system are:

$$w^S_{H_A}(t) = (1+s_D)\zeta A(t)p_A(t) = (1+s_D)w_{H_A} \qquad \text{(A5.5.14)}$$

Thus, this form of subsidisation is equivalent to subsidising research wages - as in equation (A5.5.8) - and would therefore also fail to achieve the social optimum.

Similarly, for a subsidy based upon research costs at the rate $100s_C$ per cent, subsidised costs are $c^S_A(t) = (1-s_C)c_A(t) = (1-s_C)w_{H_A}(t)H_A(t)$, and from the zero pure profits condition wages in this subsidised system are:

$$w^S_{H_A}(t) = \frac{1}{1-s_C}\zeta A(t)p_A(t) = (1+s_{C'})w_{H_A} \qquad \text{(A5.5.15)}$$

where $s_{C'} = s_C/(1-s_C)$, and once again this form of subsidisation policy is equivalent to subsidising research wages and so fails to achieve the social optimum.

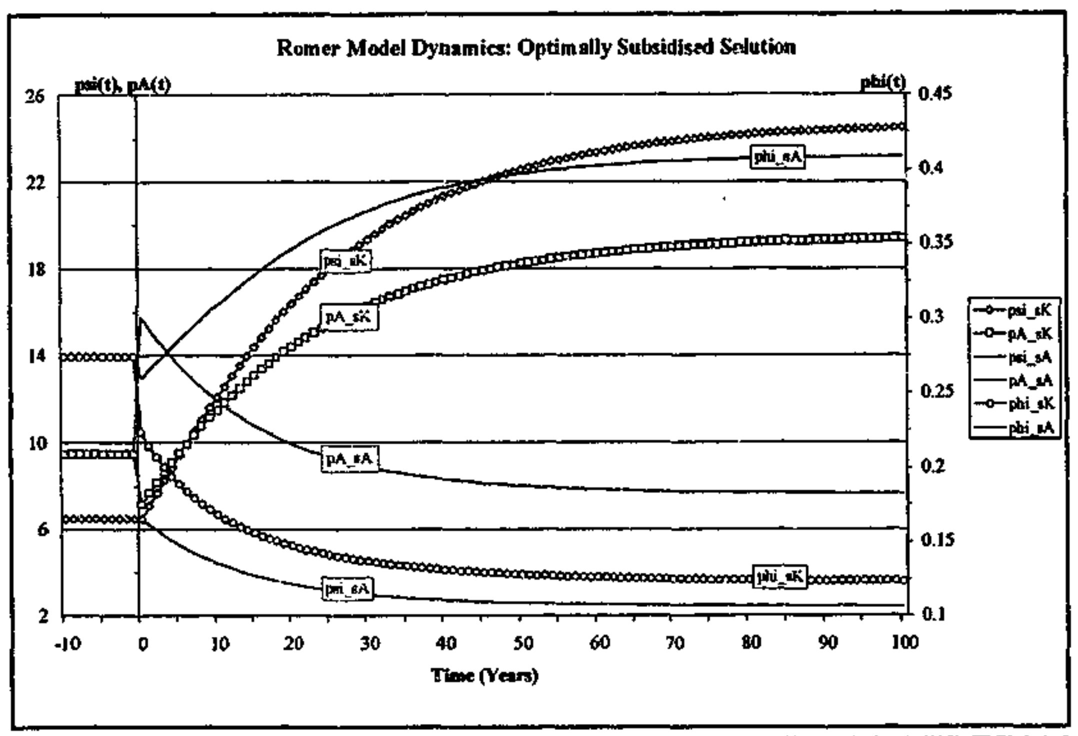
Appendix 5.6

Dynamic impact of the optimum subsidies

A5.6.1 Savings and designs subsidies: (s_K, s_{AK})

Here the fundamental dynamic impacts of the two individual optimum subsidies, $s_K = (1-\gamma)/\gamma$ and $s_{AK} = [1-\gamma\zeta H^{O}_{Yss}/\alpha r^{O}_{ss}]$, are examined by conducting two simulations. Following Section 5.3, introduction of the subsidy to the rental price of capital is simulated by raising the variable subK from its initial value of unity to its benchmark optimum value of 1.852. Similarly, introduction of the subsidy to the purchase price of designs is simulated by lowering the variable subA from unity to 0.378. In both simulations the shocks are applied instantly and are considered to be unanticipated by the market. The results of the simulations are reported in Figure A5.6.1 to Figure A5.6.8 below. As for the simulations reported in Chapter 4, qualitative explanations of the economic mechanisms behind these quantitative results are also offered. In respect of these explanations it should be emphasised again that the economic mechanisms at work are often complex and ambiguous and that while the interplay of the economic forces are accurately captured by the economic model, a full *a priori* explanation is often difficult. Thus, the qualitative comments presented below are (again) something of a verification of the quantitative results.

Figure A5.6.1 Dynamic effects on Ψ, Φ, and p_A of the separate and unanticipated introduction of the 'optimum' subsidies s_K and s_{AK} from time zero, benchmark parameter set.



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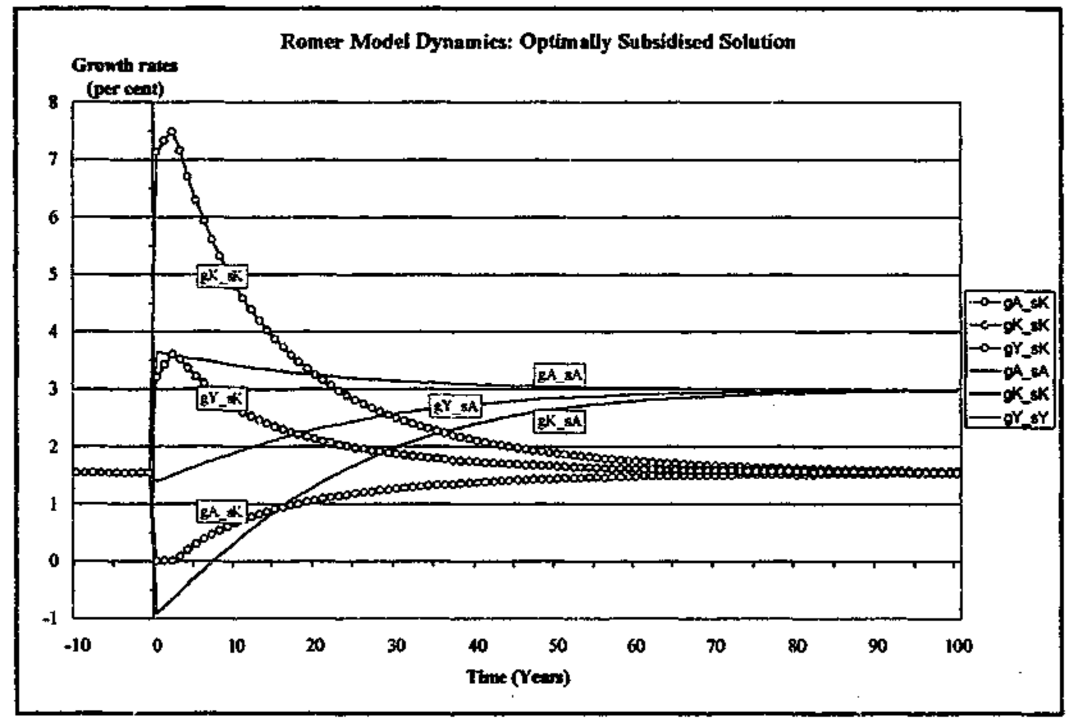
A5.6.1.1 Impact of optimum subsidy s_K

As intended, the subsidy s_K raises savings and investment. Both savings rate measures in the model, s_N and s_B, as well as the rate of growth of capital g_K, exhibit large upward jumps due directly to the sudden imposition of the subsidy (Figure A5.6.6 and Figure A5.6.2). Similarly, the return to savings, measured by either the subsidy inclusive rental rate r_{KS}, or the interest rate r, also rises suddenly (Figure A5.6.4 and Figure A5.6.5). The subsidy has such a large immediate impact on the attractiveness of saving in the form of capital accumulation, that for a short while this replaces research, the other form of saving in the model. Human capital H_A is diverted from the research sector to the output sector for the manufacture of both consumption goods and (general-purpose) capital.

Figure A5.6.2 Dynamic effects on the growth rates g_A, g_K and g_Y of the separate and unanticipated introduction of the 'optimum' subsidies s_K and s_{AK} from time zero, benchmark parameter set.



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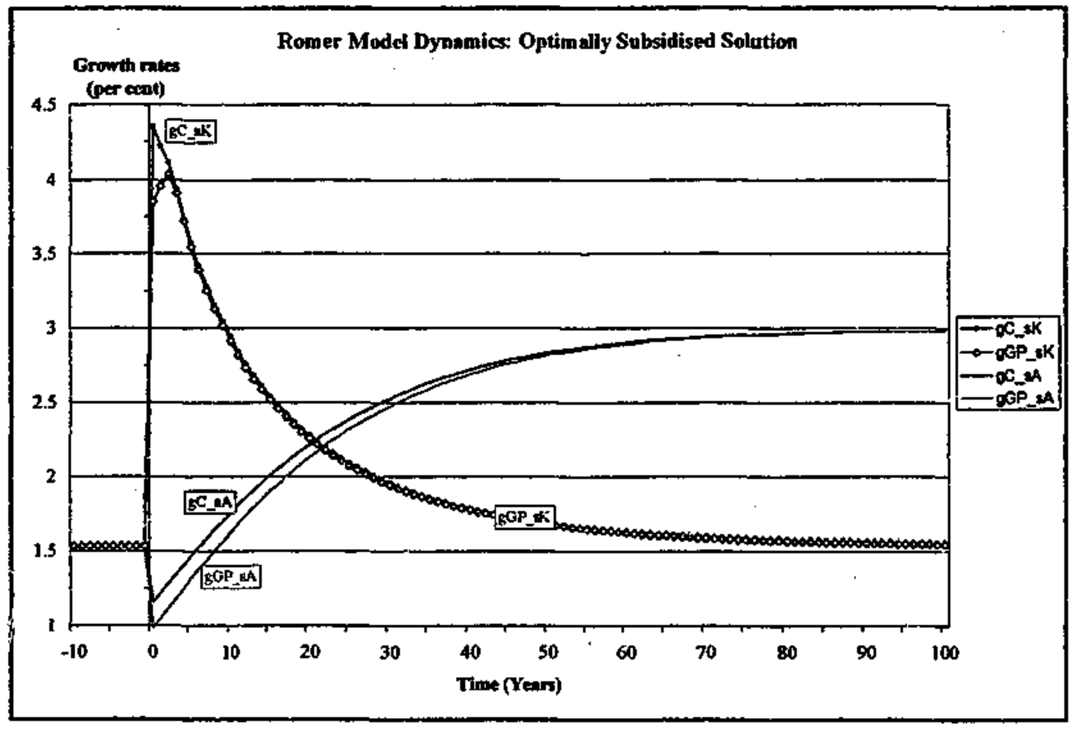
In terms of the 'usual' partial equilibrium demand-supply diagram the subsidy shifts the supply schedule for capital outwards, reducing user prices and stimulating greater usage of capital. However, as a stock variable, increases in capital take time to achieve. As more capital is accumulated and brought into use, its price gradually declines (along with its marginal product) to the new equilibrium. This applies to both the rental price r_K, paid by specialised capital goods producers for general-purpose capital; and consequently, to the subsidised rentals r^S_K received by households as the owners of this general-purpose capital. It also explains the gradual declines of the interest rate and the investment-output ratio, or narrow measure of savings, s_N (Figure A5.6.4, Figure A5.6.5 and Figure A5.6.6).

However, for the broad measure of savings s_B, this influence is counterbalanced by the effect of increased savings in the form of research. As physical capital becomes cheaper[47] producers substitute it for their other variable factor, human capital H_Y, which is thus gradually re-diverted back to the research sector. Human capital in the form of researchers H_A rises gradually, as does the growth of designs g_A; both returning to their pre-shock levels (Figure A5.6.5 and Figure A5.6.2).[48] But since A grows more slowly than K, the ratio Ψ=K/A also rises gradually to its new steady-state equilibrium (Figure A5.6.1). This accords with optimising behaviour since it is productively more efficient to spread K over a greater number of designs. Thus, as K grows the demand for designs increases and with it, their price p_A (Figure A5.6.1).

Figure A5.6.3 Dynamic effects on the growth rates g_C and g_{GP} of the separate and unanticipated introduction of the 'optimum' subsidies, s_K and s_{AK} from time zero, benchmark parameter set.



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The initial downwards jump in the consumption-capital ratio (Figure A5.6.1), is the obverse of the initial sharp rise in the savings rate s_N (Figure A5.6.6). Since the growth of designs, output and consumption are each secondary effects of the growth in capital brought on by the subsidy, they are all lower than it (Figure A5.6.2 and Figure A5.6.3). Though of course, they all approach the same (unchanged) steady-state rate. Nevertheless, as a result of these 'earlier' post-shock differences in the growth rates, Ψ=K/A and k_{GP}=K/GP both rise to their new equilibria, while Φ=C/K declines towards its (Figure A5.6.1 and Figure A5.6.6). The initial fall in the allocation of human capital to research is reflected in its share of total income (gross product), which is temporarily

[47] As the rentals r_K decline and general-purpose capital becomes cheaper, so too does specialised capital. Its rental price p_X can be readily seen - from equations (5.19), (5.20) and (5.22) - to follow r_K.

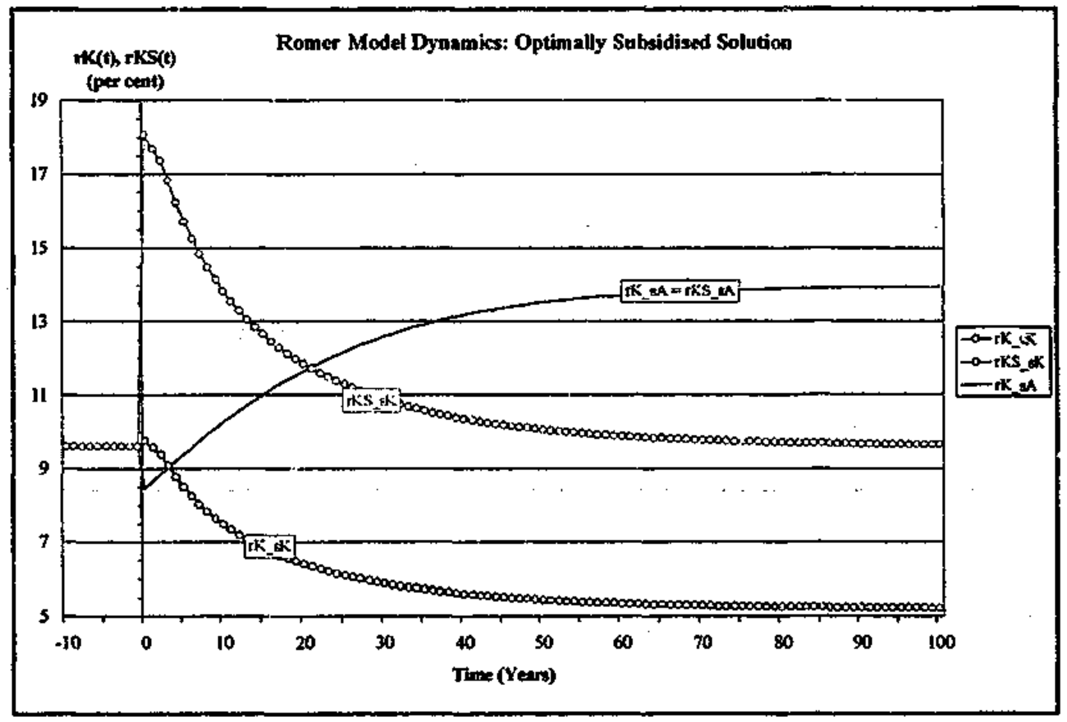
[48] Recall that the equilibrium growth rate g^S and its associated variables H_{Ass}, H_{Yss} and r_{ss} are independent of the subsidy s_K (equations (5.34) to (5.37)).

redistributed to the other factors of production. Eventually however, the shares of all factors revert back to their pre-shock levels (Figure A5.6.7).[49]

A5.6.1.2 Impact of optimum subsidy s_{AK}

The subsidy-induced reduction in the user price of designs raises the demand for them by the capital goods producing sector. This increase in demand is immediate. But because designs are a stock variable they cannot be increased instantaneously. Thus, the immediate impact of the subsidy is to generate an excess demand for them. As a result, the price of designs received by researchers (p_A) jumps upwards. As the supply of designs expands and the excess demand is gradually eliminated, this price declines towards its new equilibrium - where the growth of designs exactly satisfies the growing demand for them (Figure A5.6.1).[50]

Figure A5.6.4 Dynamic effects on capital rentals (r_K and r_{KS}) of the separate and unanticipated introduction of the 'optimum' subsidies s_K and s_{AK} from time zero, benchmark parameter set.



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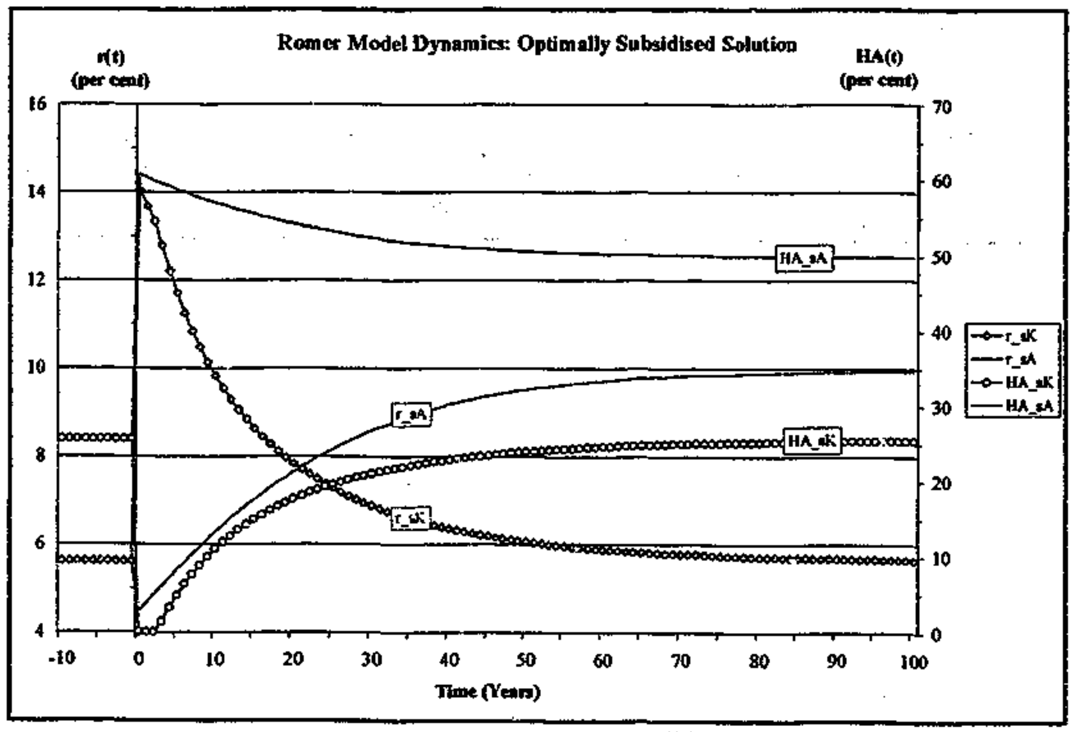
Generation of more designs requires more human capital in the research sector H_A which, (as a flow variable) thereby jumps upwards instantly. Subsequently, as the initial excess demand is gradually eliminated, the need for researchers declines and H_A slowly

[49] As discussed in footnote 30, the subsidy s_K has no influence on steady-state factor shares.

[50] The usual partial equilibrium demand-supply diagram would suggest that while the subsidy lowers the user price of capital, the expansion of demand that it induces will eventually raise the price received for designs. In fact, as the numerical analysis here indicates, the new equilibrium price of designs is lower than the pre-subsidy level. As discussed in Section 5.1.2 the issues are more complex than can be validly handled by a simple partial equilibrium analysis. Once again this emphasises the value of the full general equilibrium dynamic analysis furnished by the model.

falls away towards its new (higher, and in fact socially optimum) equilibrium (Figure A5.6.5). The rate of growth of designs g_A naturally follows these movements (Figure A5.6.2). Increased savings in the form of research causes an immediate rise in the broad measure of savings s_B. However, because less savings by the utility maximising consumers are then required from capital accumulation, the narrow measure of savings s_N falls correspondingly (Figure A5.6.6). This savings measure is equivalent to the investment-output ratio, and its fall is also reflected in the initial fall in the rate of growth of capital g_K (Figure A5.6.2).

Figure A5.6.5 Dynamic effects on r and H_A of the separate and unanticipated introduction of the 'optimum' subsidies s_K and s_{AK} from time zero, benchmark parameter set.



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Since the total supply of human capital resources (H) is fixed in the model, changes in the amount devoted to output production (H_Y) are the exact opposite of the changes in H_A. Thus, H_Y first jumps downwards and then gradually rises. As a result, the growth rates of output and consumption dip momentarily while that of capital drops substantially. Subsequently they all increase steadily towards their new common and higher steady-state rate (Figure A5.6.2 and Figure A5.6.3). These growth rates all reflect the fact that the subsidy to designs induces more resources to be devoted to research and to the production of output for consumption than for capital formation. Their relative magnitudes also mean that Ψ and k_{GP} decline while Φ increases (Figure A5.6.1 and Figure A5.6.6). To support the higher growth of consumption, the return to savings and hence the interest rate r, must also rise (Figure A5.6.5). Finally, the increase in human capital resources devoted to research, and the complementary reduction in those for the production of goods, means that the share of gross income to researchers S_{HA} increases significantly while the shares to the other factors (S_{HY}, S_L, S_K) decline (Figure A5.6.8).

Figure A5.6.6 Dynamic effects on s_B, s_N and k_{GP} of the separate and unanticipated introduction of the 'optimum' subsidies s_K and s_{AK} from time zero, benchmark parameter set.

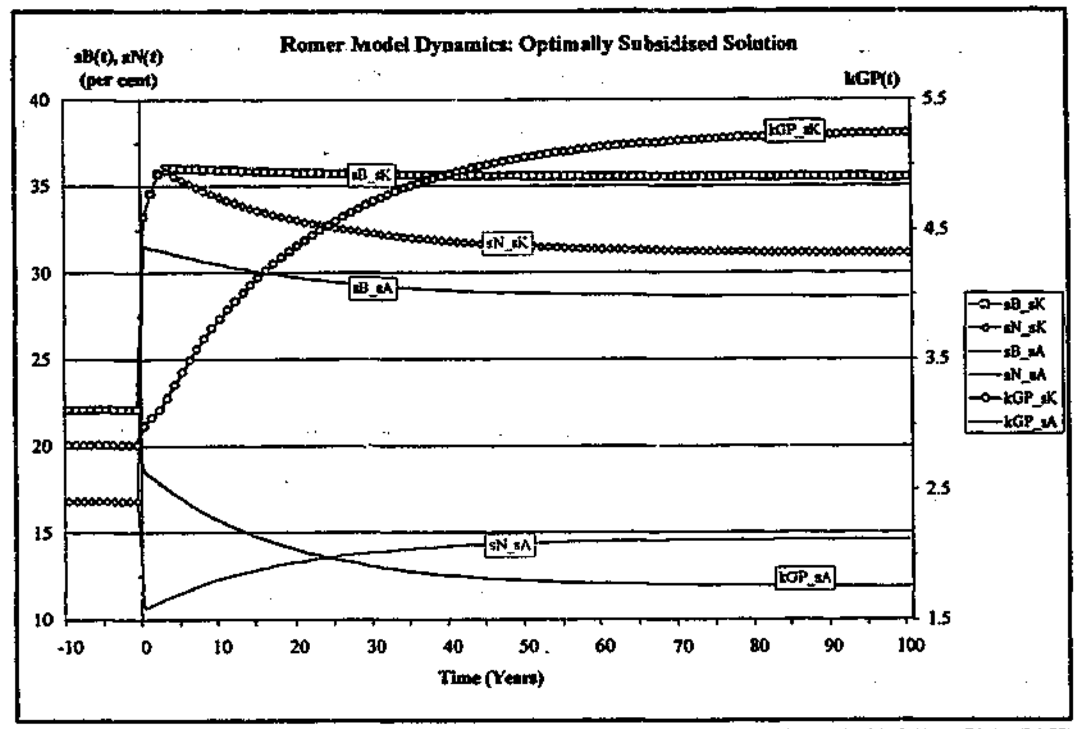

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Figure A5.6.7 Dynamic effects on the factor shares of gross income, of the unanticipated introduction of the 'optimum' subsidy s_K from time zero, benchmark parameter set.

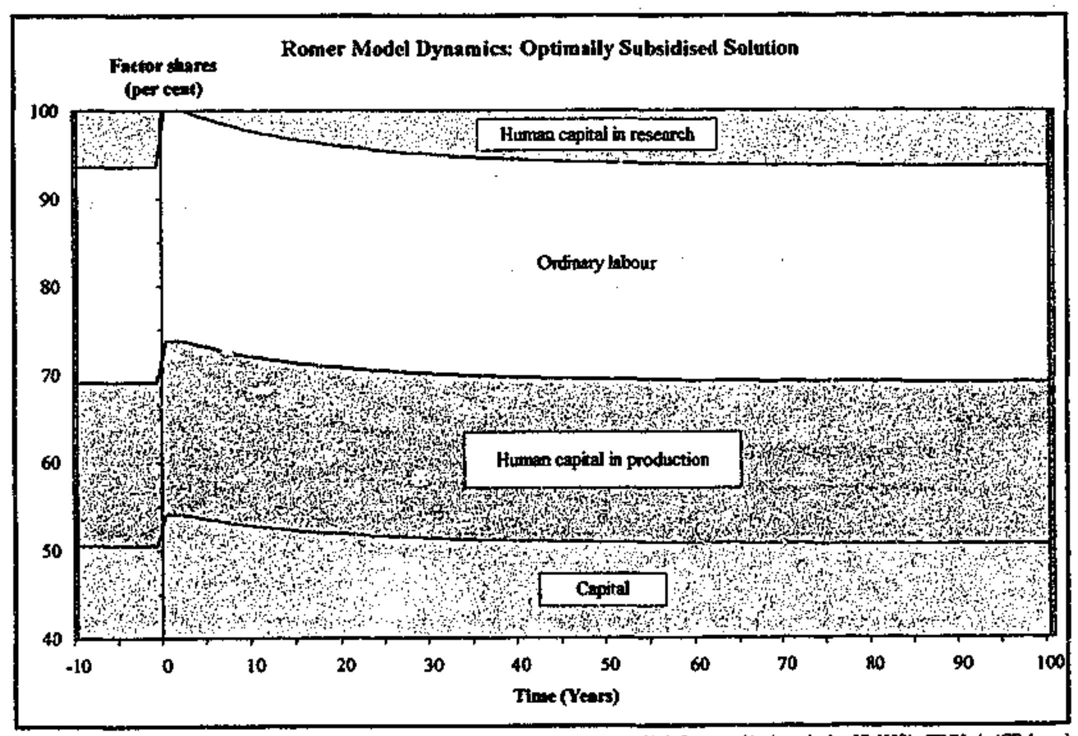

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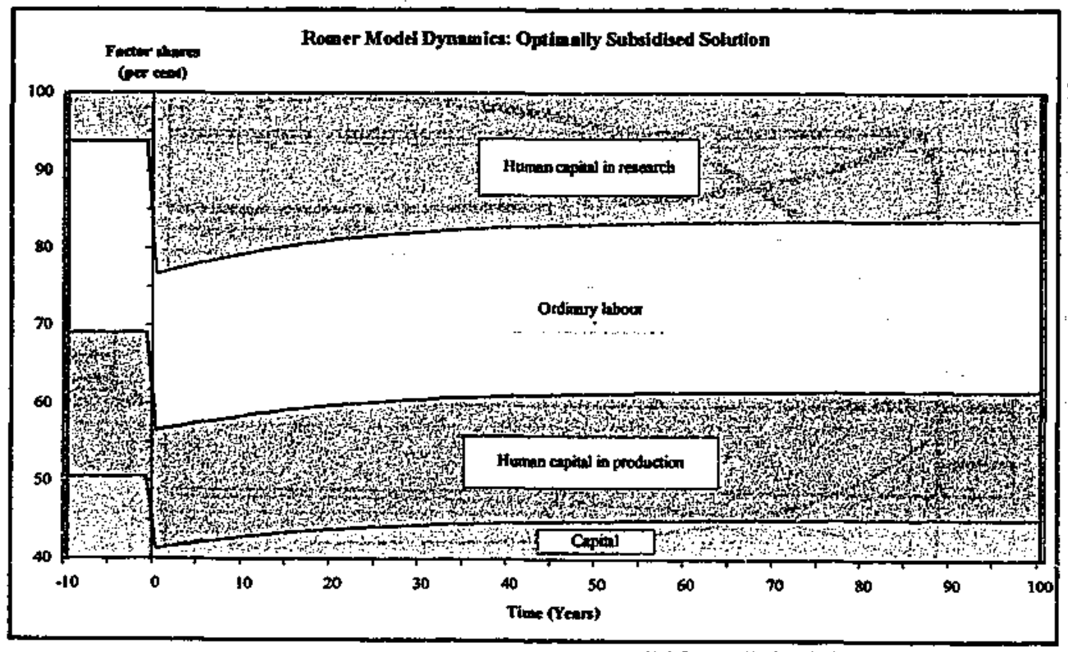
Figure A5.6.8 Dynamic effects on the factor shares of gross income, of the unanticipated introduction of the 'optimum' subsidy s_{AK} from time zero, benchmark parameter set.



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A5.6.2 Monopoly correction and designs subsidies: (s_X,s_{AX}) or (s_Y,s_{AY})

While the final steady-states of these alternative subsidisation systems are the same as that of the (s_K,s_{AK})-scheme, the dynamic paths they take to attain their common steady-state differ. From the appropriate dynamic equation systems it is easily inferred that the dynamics for the (s_X,s_{AX})- and (s_Y,s_{AY})-subsidy schemes are identical to one another, but differ from those of the (s_K,s_{AK})-scheme.

The transitional dynamics of the individual optimum subsidies s_K and s_{AK} were examined in the previous section. Here the corresponding dynamics for the individual optimum subsidies $s_X = (1-\gamma)\, s_X$, and $s_{AX} = 1 - \zeta H^{O}_{Yss} / \alpha r^{O}_{ss}$ are presented. Figure A5.6.9 to Figure A5.6.15 record these dynamics. From them. the two schemes can be seen to differ, but also to be very similar.

Figure A5.6.9 Dynamic effects on Ψ, Φ, and p_A of the separate and unanticipated introduction of the 'optimum' subsidies s_X and s_{AX} from time zero, benchmark parameter set.

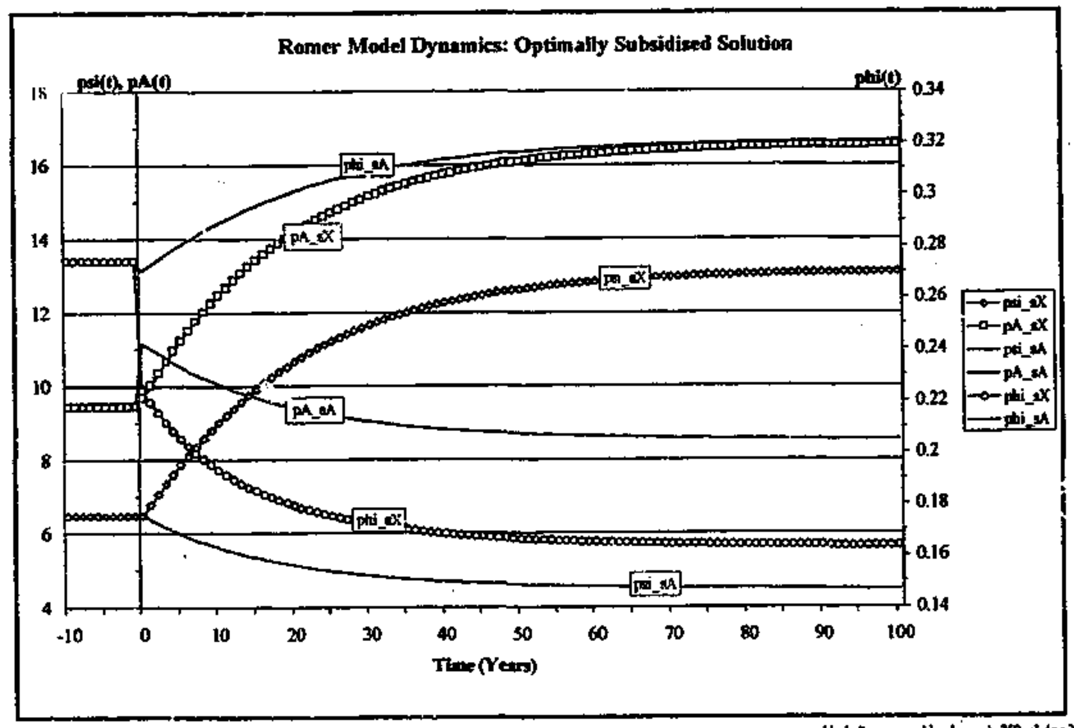

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Figure A5.6.10 Dynamic effects on the growth rates g_A, g_K and g_Y of the separate and unanticipated introduction of the 'optimum' subsidies s_X and s_{AX} from time zero, benchmark parameter set.

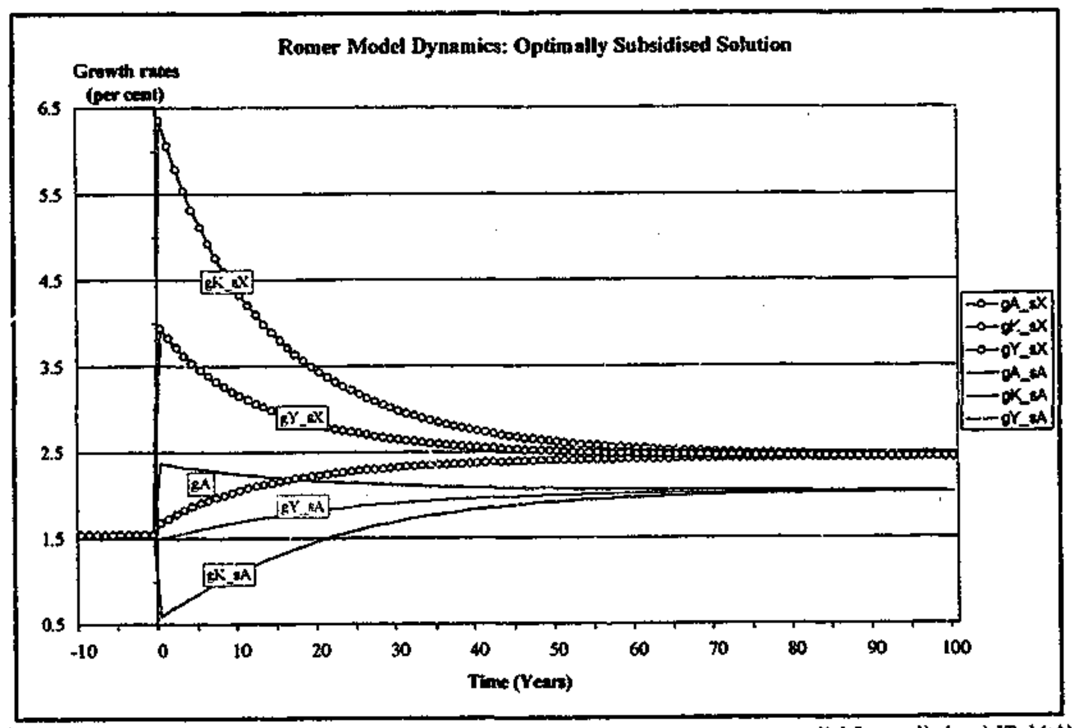

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Figure A5.6.11 **Dynamic effects on the growth rates g_C and g_{GP} of the separate and unanticipated introduction of the 'optimum' subsidies s_X and s_{AX} from time zero, benchmark parameter set.**

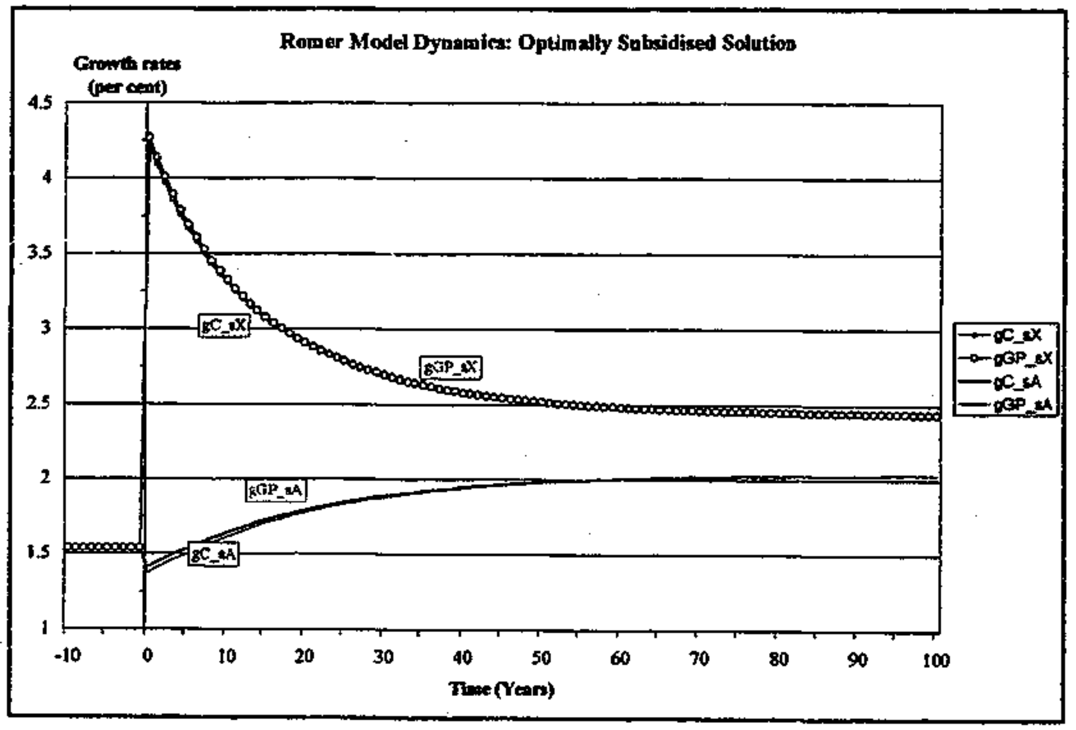

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Figure A5.6.12 **Dynamic effects on r, r_K and H_A of the separate and unanticipated introduction of the 'optimum' subsidies s_X and s_{AX} from time zero, benchmark parameter set.**

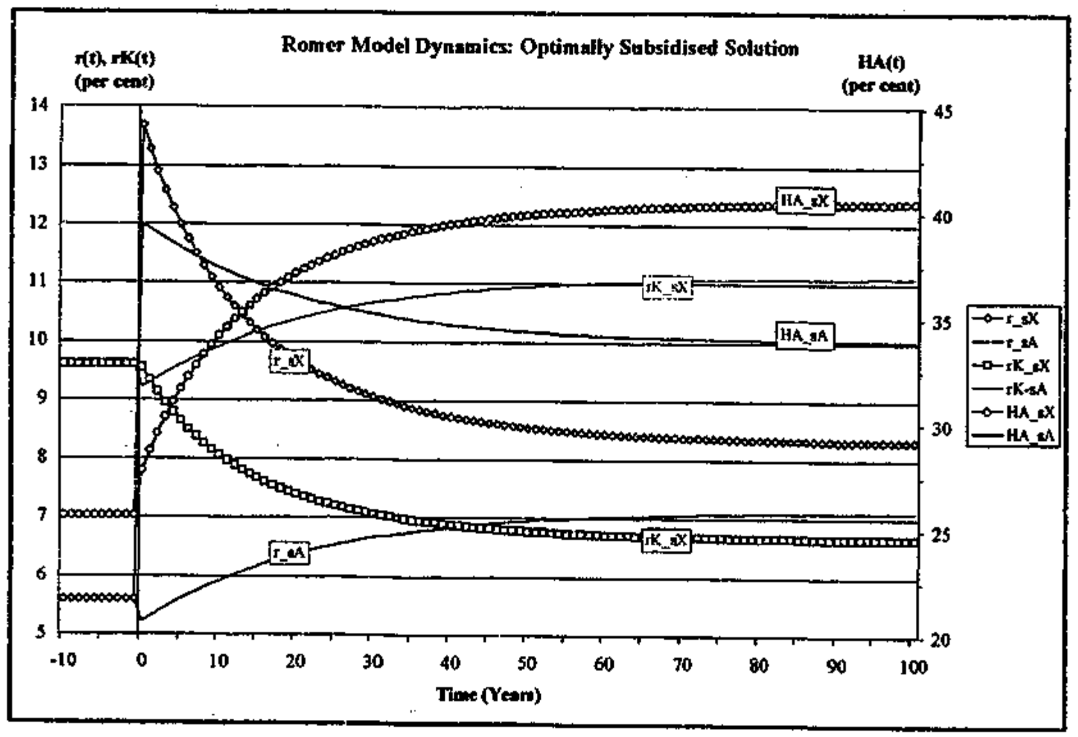

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Figure A5.6.13 **Dynamic effects on s_B, s_N and k_{GP} of the separate and unanticipated introduction of the 'optimum' subsidies s_X and s_{AX} from time zero, benchmark parameter set.**

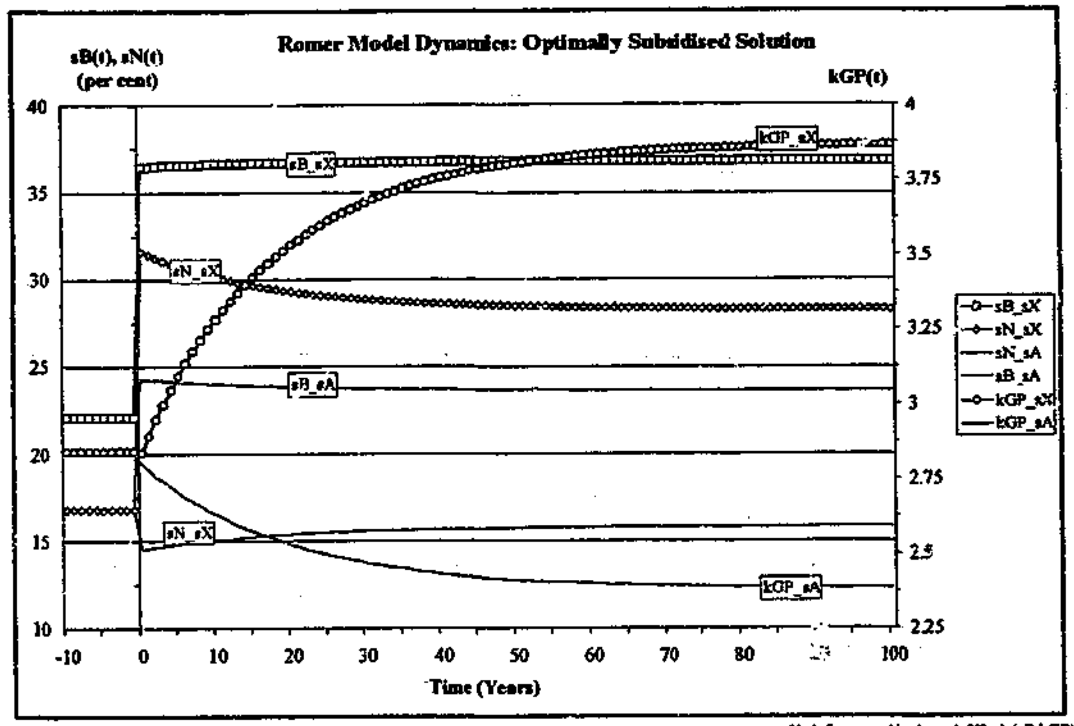

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Figure A5.6.14 **Dynamic effects on the factor shares of gross income of the unanticipated introduction of the 'optimum' subsidy s_X from time zero, benchmark parameter set.**

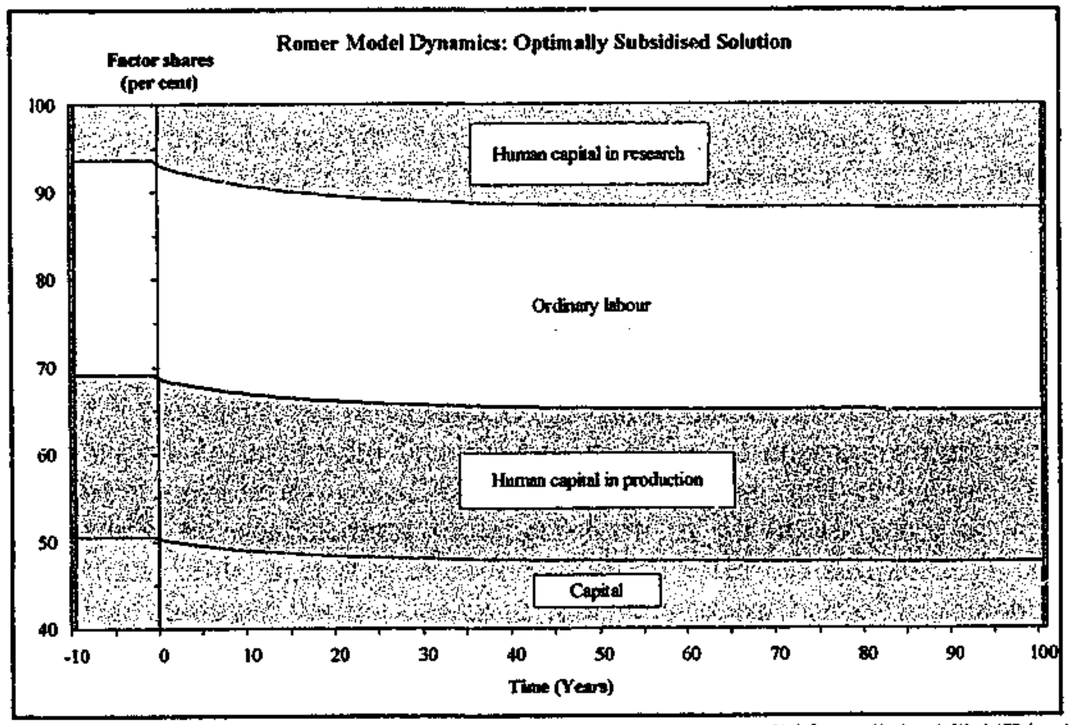

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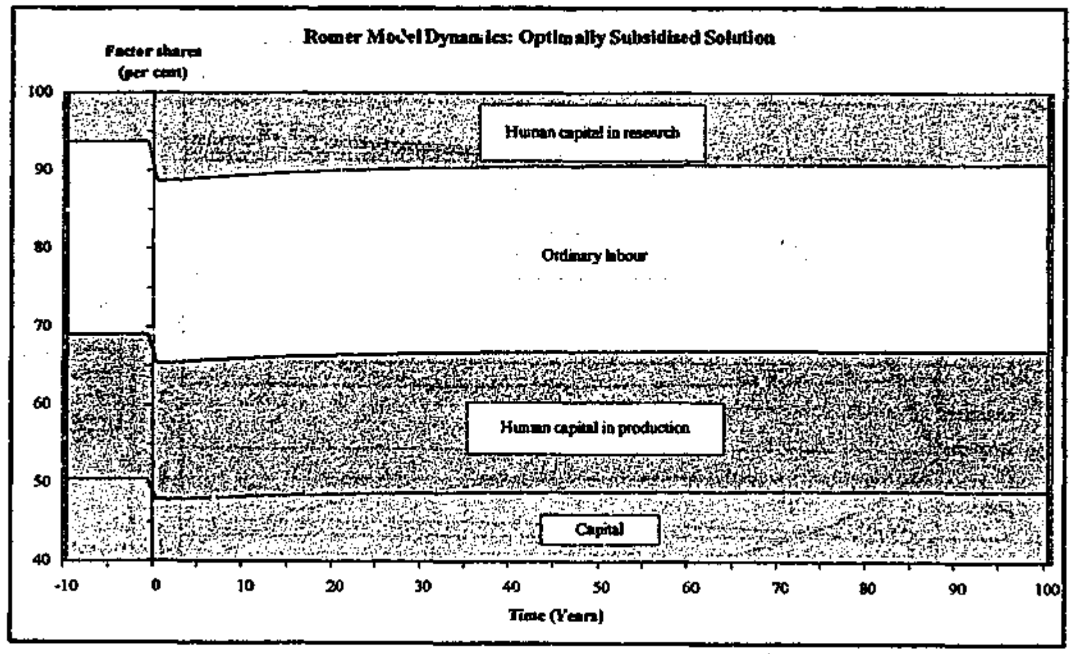
Figure A5.6.15 Dynamic effects on the factor shares of gross income of the unanticipated introduction of the 'optimum' subsidy s_{AX} from time zero, benchmark parameter set.



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Chapter 6

6 Concluding Remarks

This chapter covers three issues. Some conclusions are first drawn from the principal content of the dissertation – the transitional dynamics of endogenous growth. Second, some important practical policy implications to emerge from the work are identified and discussed. And third, some unresolved issues and other matters for further research are examined, including the possibility of incorporating endogenous growth modelling into a large scale general equilibrium model such as "Monash", the original idea for a PhD topic (see Chapter 1).

6.1 Conclusions regarding the dynamics

The dissertation has focussed upon the problem of obtaining actual numerical estimates for the transitional dynamics of endogenous growth (with the Romer, 1990b model as the case study). Issues concerning the technical methods to employ to achieve this were first addressed. Then the question of analysing and evaluating the results of particular applications of the chosen technique in simulating various forms of economic change was confronted. Concluding remarks concerning both these areas are offered.

Technical methods

The first thing that ought to be noted in terms of assessing the performances of the different techniques is that they are **all** approximations of one sort or another, an analytic solution of the complete dynamic system simply not being possible. In particular: While it may be solved analytically, the linearised model is an approximation that is strictly only valid in the neighbourhood of the linearisation point. And while the *Solowian-Romer* model preserves the non-linearity of the full model, by exogenising the consumption-savings choice it too is only an approximation to the full model. Moreover, numerical approximation methods are required for its solution. Finally, direct solution of the full non-linear model of course also relies necessarily on numerical approximations. Nevertheless, it has been shown that varying degrees of confidence may be held for each of these methods.

For a specific, but varied set of simulated economic shocks, both the linearised model and the Solowian-Romer model furnished tolerably good approximations to the non-linear model dynamics computed by the preferred numerical technique. This was particularly true for the medium to longer term of about 10 years or more; but sometimes broke down over shorter periods (though never catastrophically), especially when variables exhibited discontinuities or jumps, or when their early adjustment was rapid. Properties of the simulations and the models themselves that tended to make the 'approximations' good ones were:

- since the steady-states for each of the three models were exactly the same, their asymptotic dynamics were almost identical;

- because the shocks for the *evaluation-simulations* were not 'too big', the linearised dynamic system was not shifted 'too far away' from its (initial steady-state) linearisation point; and the exogenous savings rate in the Solowian-Romer model did not have to be shifted 'too much';
- since the benchmark parameter set, under which the simulations were run, was such that the savings rate in the non-linear model tended to adjust rapidly to new levels, the instantaneous adjustments of the Solowian-Romer model were good approximations[1]; and
- because of the reduced dimension of the model achieved in the Solowian-Romer formulation, it was possible, indeed easy, to solve the latter with a high level of precision by the numerical integration technique of shooting.

While numerical integration of the Solowian-Romer model is undoubtedly simpler (and probably more accurate) than for the full model, the real value of its construction is that it allows the phase-space to be reduced to two dimensions, to a phase-plane which can readily be studied and analysed, greatly facilitating understanding of the model dynamics. Furthermore, with the Solowian-Romer model generating good approximations, some confidence can be placed on the accuracy of this phase-plane analysis in characterising that of the abstruse three-dimensional phase-space of the full model.

Notwithstanding the preceding comments, the main conclusion to draw in respect of the technical methods is how powerful a solution method is the finite differences-GEMPACK technique used to solve the full non-linear Romer model. It delivers adjustment paths over time directly, avoiding the need to solve a second initial value problem (which tends to be unstable) as is the case with the time elimination method.[2] Once some fairly simple model specific algebraic code has been written it is fast and provides enormous flexibility for different simulations. And most tellingly, it easily surmounts the two-point boundary value problems faced by other methods in the simulation of anticipated and temporary shocks, these being just as easy to conduct under the finite differences/GEMPACK technique as for unanticipated shocks.[3]

Dynamic results

The response of growth models to economic change may be analysed in two fundamental ways. Comparative statics considers only the equilibrium or steady-states of the models, comparing final (post-shock) equilibria with initial (pre-shock) ones to determine the changes that would eventually arise if the system were allowed to adjust fully to the shocks being analysed. Dynamic analysis takes this a step further. Not only do such analyses compute and compare pre- and post-shock equilibria, they also

[1] Recall from Appendix 3.7 that there are many parameter sets which generate 'constant transitional savings' in the full Romer model, and for which the Solowian-Romer and the full models are therefore equivalent. The benchmark parameter set is 'close to' one of these *equivalence-generating* sets. Also, the fact that the benchmark set is empirically based provides further assurance in the validity of the the relatively close approximation of the Solowian-Romer model.

[2] The 'eigenvector-backwards integration' also avoids this problem

[3] Both the 'time elimination' and the 'eigenvector-backwards integration' methods faced these problems. While the 'shooting' method avoids them, it is difficult to apply to highly non-linear dynamic systems of more than two differential equations. Also, Judd (1985b and 1987) has devised and applied a technique of converting the differential equations of dynamic systems to algebraic ones via *Laplace transforms*, which also avoids the problems. However Judd's methods appear to require the system of differential equations to be linear, though they have not been closely studied here.

compute the transitional adjustment paths that would be followed from the former to the latter if systems were allowed to adjust fully. From this description one might wonder why analyses are ever confined to comparative statics. The answer is that the problems of computing entire adjustment paths are much the more difficult, and can become intractable for large complex models. Thus, there are trade-off issues between the complexity (and realism) of models and their dynamics. Such distinctions underlay the capital controversy of the late 1960s and early 1970s.[4] However, they are not relevant here and nor are they for most growth models.

It has been amply demonstrated here (in respect of the Romer model), that dynamic analysis is extremely important and revealing. The short and medium run dynamics can reveal movements that are quite unexpected from a comparative statics perspective: They are frequently complex, often 'asymptotically perverse', sometimes counter intuitive, and often of relatively large magnitudes compared to the overall adjustments. Also, adjustment periods are usually lengthy, the 'short and medium terms' persisting for long periods. The policy implications of these issues are examined in the next section. Here they are simply enumerated with reference to examples drawn from the previous chapters.

The first example of how much short term change may be concealed by comparative static analysis was demonstrated in Appendix 2.6, where changes to the rate of depreciation δ, the cost of specialised capital η, and the exogenously given endowment of ordinary labour L, were simulated. Comparison of the pre- and post-shock equilibria for these shocks indicated no change to have occurred for many variables (Table 2.3). But dynamic analysis revealed a great deal of transitory change.[5] For example, although the pre- and post-shock steady states are identical:

- the 25 per cent rise in δ caused the allocation of human capital to research H_A to first rise steeply by 15 per cent, and to still be some 11 per cent above its equilibrium after 20 years (Figure A2.6.3);
- similarly, the 20 per cent rise imposed on L produced an initial rise of about 18 per cent in the interest rate r, which after 15 years had adjusted back to be only about 5 per cent above its equilibrium (Figure A2.6.8); and
- the 10 per cent increase in η induced an immediate and precipitous fall of 28 per cent in the rate of growth of capital g_K, which remained 11 per cent below its equilibrium after 15 years (Figure A2.6.6).

These sorts of movement occurred for every simulation undertaken. In Section 4.5.1, where the output elasticity of capital was shocked by 10 per cent, it was noted that only a single one of the "jumping variables" analysed followed a monotonic path towards the new equilibrium. For all of the other such variables the initial jumps were in the opposite directions to that of the subsequent adjustment paths. For about a half of these the initial jump was *perverse*, but the subsequent adjustment *accorded* with the direction of change from the comparative statics. For the other half, while their initial jumps took them away from their pre-shock levels in the direction of their post-shock

[4] As a result Cambridge UK tended to focus on comparative static analyses of disaggregated capital models, while Cambridge USA tended to concentrate more on dynamic analyses of aggregate capital growth models (see Dixit, 1990).

[5] A similar situation arises for exogenous changes to the elasticity of capital (γ) in the social optimum dynamic system, where that parameter has no permanent but considerable transitory impact on the allocation of human capital, the interest rate, or the growth rates (Section 5.1.1).

steady-states, they *overshot* their new equilibrium levels, resulting in their subsequent adjustment being *perverse* in terms of the comparative statics. From the results of all the simulations conducted it seems that non-monotonic adjustment paths may be more common than monotonic ones. Short run dynamics exhibit greater complexity and an even larger departure from monotonicity in response to shocks that are anticipated and temporary, than for those that are completely unforeseen by the market. Relatively large jumps in one direction, arising at the point of time at which the 'information' about a prospective shock arrives at the market (say at the announcement of some government policy change), are often followed by similarly large jumps in the opposite direction when the shock actually takes place (when the policy is implemented). Examples of this type of dynamic behaviour arose in the simulations of Sections 4.5.2, 4.5.3, 4.5.5 and 4.5.7.

Steady-state analysis, whereby only the initial position and the final position are given, is clearly not capable of identifying non-monotonic adjustment paths. Furthermore, since it is not possible to determine either the extent or direction of the discontinuities from such analysis, neither is it even possible to describe the direction of the smooth, asymptotic adjustment. However, it was shown that by considering the economic mechanisms at work, qualitative reasoning could sometimes enable both the directions of the discontinuities and of the subsequent smooth adjustment to be explained. So, it appears that to some degree qualitative descriptions of the transitional dynamics are sometimes possible. However, in the complete absence of any model results such theorising is extremely difficult. Moreover, it is not always possible. Because there are often opposing influences acting on economic variables, merely qualitative economic reasoning must leave many ambiguities.[6] Empirical results from the dynamic model are then necessary to resolve them. In any case, quantitative measures are important.[7]

In addition to their magnitude and complexity, the importance of transitional dynamics depends on their 'longevity'. In adjusting to economic shocks, the longer it takes economies to closely approach their new growth equilibria, the greater is the importance of the adjustment paths. Conversely, it must be the case to at least some degree that the longer it takes to 'reach' these (asymptotic) steady-state equilibria, the less relevant they become.

Measures of the rate at which the linearised Romer model approaches its steady-state were developed in Section 3.2.3. For the benchmark parameter set the *speed of convergence* (β) was calculated at almost 5 per cent. That is only some 5 per cent of the gap between the current position of the system and its steady-state is closed in a year. This meant that the *half-life* of the system (the time taken to close half of the current gap) was almost 15 years, and the *three quarter-life* almost 30 years. These figures were confirmed for the full non-linear model by the simulations reported in Section 4.5, which indicated that the system could be expected to take almost two decades to complete half of its adjustment, and three to four decades before three-quarters of the change had eventuated. These are long periods of time and suggest that it is unlikely for

[6] Appealing to the quantitative results was precisely the way ambiguous outcomes were resolved in the simulation explanations given here. For example, see 'last paragraph page 211'; 'last paragraph page 218'; 'first paragraph page 231'; and 'first dot-point page 238').

[7] For example, if careful modelling produces the result that a variable will rise by 20 per cent, one may have considerably more confidence that in the real world the variable will indeed rise, than if the computed result had been for an increase of only 2 per cent.

economies ever to be in steady-state equilibrium, some new shock being highly likely to arise before even 10 or 15 per cent of the total potential adjustment to the previous shock is completed.

The generalised conclusion from all this must surely be that dynamic analysis of growth models provides much more useful information and is a far more powerful tool than steady-state analysis. Neither the pattern nor the speed of adjustment of the transitional dynamics can be captured with a static model, particularly in response to shocks that are correctly foreseen by the market. Nor, in general, can they be identified qualitatively, merely by *a priori* theorising about the underlying economics. Also, it is only through the dynamic vehicle that quantitative measures of the adjustment path are available.

6.2 Policy implications

It is the sub-optimality of the free market solution of the Romer model and, no doubt, of other endogenous growth models also, that raise the most significant implications for policy. However, before these matters are examined some policy implications arising from the concluding remarks of the last section are briefly addressed.

Policy implications from the transitional dynamics

Because of the non-monotonic nature of adjustment processes, some of the shorter-term effects of economic change, whether policy induced or otherwise, may have the opposite sign to their longer-term impacts. As a result, it appears that there may be short-term adjustment costs that could be addressed by policy makers. For example, if the overall change in some variable from its initial level to its final steady state is considered beneficial, then an immediate post-shock jump in the opposite direction to this change would presumably be considered deleterious. Or, if the overall change in a variable is thought to represent an acceptable trade-off cost for other perceived benefits of economic change, then an initial jump in this variable that significantly overshoots its new steady-state level will produce even higher trade-off costs. It would seem that such adjustment costs, to be borne in the shorter term, ought to be taken into account in any policy assessment of the overall efficacy of the economic change.

It may be argued however, that since the model is founded upon economic agents who optimise their decision making, the dynamic outcomes represent the best that can be done and so no policy action is warranted. However, in a richer, more disaggregated model with different industrial sectors and different classes of consumers (such as lenders and borrowers), there would be trade-off issues concerning the income distribution effects of economic change.

A second policy issue arising from the conclusions on the importance of the transitional dynamics is that the shorter and medium terms would seem to warrant the greater policy emphasis. This follows directly from the fact that the transitional dynamics over these periods are often characterised by large discontinuities and complex adjustment paths; while adjustment to points which may be considered as approaching equilibria relatively closely, require such long periods of time that they are unlikely ever to be realised.

Aghion and Howitt (1998) argue that 'because innovations may take decades to take effect, they are primarily interested in the long-run' and this justifies their focus on steady-state analysis.[8] This could be taken to suggest that *policy* should also be 'primarily interested' in the longer run and steady-state analysis. However, the opposite view is taken here. While it is important to know where the economy is eventually headed,[9] it seems clearly more important to know, at any point of time, in which direction it is heading, particularly since the former is unlikely ever to be attained, and the latter is highly variable. The policy implication is that the greater focus would seem to be due to transitional issues than to steady-state ones.

Another important implication of the transitional dynamics is perhaps more relevant to economic analysts and advisers than to policy makers themselves, though the indirect link is obvious. Since adjustment takes so long, observed economic data are unlikely to reflect (theoretical) equilibrium levels, so should not be fed into equilibrium models. This raises a general problem for all empirical equilibrium modelling, such as balanced growth analysis, and for econometric work also. The fact that initial jumps for many variables can be in the opposite direction to that implied by differences in their pre- and post-shock equilibria, and can be large in comparison with these differences, exacerbates the problem. It means that observed responses and correlations may not even have the correct long-term signs. Using observed data and correlations to formulate equilibrium theories, and then using more such data to test/confirm them, may easily give spurious results. Alternatively, such transient dynamics could explain why certain observed correlations cannot be made to fit what seem to be otherwise sensible equilibrium models.

Policy implications from the sub-optimality of the model

Many of the implications arising from directing policy prescriptions towards the sources of sub-optimality of the Romer model have already been explicitly addressed in Chapter 5. Two of the distortions present in the model, market power pricing effects by innovators, and spillover benefits from research, are widely recognised as being closely associated with technological, and thereby economic growth. And the third source of divergence from Paretian optimality, the so-called 'specialisation divergence' (which derives from the old *'extent of the market - increasing returns from specialisation - external economy'* ideas of Adam Smith, Alfred Marshall and Allyn Young), is becoming increasingly relevant as specialisation is rediscovered as a source of growth (for example, see Helpman and Krugman, 1985; and Grossman and Helpman, 1991, Chapter3). Thus, the policy implications arising from the model may be taken as conforming with modern views of the processes of growth.

[8] Nevertheless, Aghion and Howitt concede later that "if, as several writers have contended, an economy spends most of its time a long way from a steady-state, then it is important also to analyze nonsteady-state behaviour" (p.109), which they then go on to examine.

[9] One of the reasons for this would be to make judgements about the welfare efficacy of potential policy changes. In this and in most other endogenous growth models, agents are taken to maximise welfare in a dynamic context. But in the real world (and in detailed models with many different categories of agents), there are many market interventions already extant and great scope for inefficient allocations of resources and inequitable distributions of income. As shown in Chapter 5, the market solution of the Romer model exhibits such inefficiencies, insufficient human capital being allocated to research for example. And, to assess the desirability of any policy intervention intended to correct this sub-optimality it is necessary to compare the steady-state equilibria of the pre- and post-policy intervention scenarios.

It was shown that all these divergences could be corrected by subsidisation polices.[10] Various 'subsidy packages', each capable of converting the sub-optimal market steady-state outcomes to the socially optimum ones, were identified. In particular, this was found to be possible by combining a subsidy to the purchase of designs with a (production-side) subsidy to either savings, the purchase of specialised capital, or to final output. Of course such subsidies had to be set at different specific levels, and it was a condition that no new distortions were introduced by the taxes necessary to finance them. For certain other 'potentially optimal' subsidy schemes it was demonstrated that it was not in fact possible to replicate completely the socially optimal steady-state. Specifically, while the socially optimal growth rates could be achieved via the subsidisation of research wages (or equivalently, research incomes or expenditure), other steady-state levels could not be replicated. This was the case irrespective of which production-side subsidy it was combined with. However counter-intuitive it may seem, the obvious policy implication from the model is that direct subsidisation of research (whether based upon income, expenditure or wages) is inappropriate.

Another point that it seems ought to be emphasised in respect of the possible implementation of these subsidies, is that they come as *policy packages*. Implementation of only one of the 'optimum subsidies' of a pair may turn out to increase the divergence of an economy from its social optimum rather than reduce it. While some variables may end up closer to their optimum levels, others will be further away. For example, if the purchase of designs is subsidised at its 'optimum rate' in the scheme in which it is combined with the subsidisation of savings, the so-called (s_K,s_{AK})-scheme, but savings remain unsubsidised, then:

- while the socially optimal steady-state levels for the allocation of human capital, the interest rate, the growth rates, and the factor shares of total income would all be attained precisely;
- the percentage divergences of the capital-technology ratio, the consumption-capital ratio and the capital-gross product ratio would increase from (–30 to –74), (45 to 117) and (–12 to –46) respectively.

And if instead, savings were subsidised at the 'optimum rate' in the (s_K,s_{AK})-scheme, but the purchase of designs remained unsubsidised, then:

- while there would be no change in the allocation of human capital, the interest rate, the growth rates, nor the factor shares of total income;
- the divergences for the investment-output ratio and the broad savings rate would be reduced from (–38 to 15) and from (–43 to –9) respectively.

The transitional dynamics of the conversion from the market steady-state to that of the social optimum also raise significant implications for policy. First, there are short-term adjustment cost and trade-off issues of the kind discussed at the beginning of this section, and which policy makers may wish to address. Second, the manner in which an

[10] Common policy responses aimed at correcting for the market power and research externality distortions face inconsistent outcomes. Anti-monopoly policy is intended to reduce the static inefficiencies of having prices exceed marginal costs. But by removing the incentives from prospective monopoly profits, such policies produce the unintended side effect of discouraging innovative activity and thereby reducing the dynamic gains it generates. And enforceable property rights, which aim to raise dynamic efficiencies by ensuring the incentives are high enough to generate the optimum level of research, unintentionally establish static inefficiencies by encouraging the holders of the property rights to set prices higher than marginal costs.

'optimum subsidy package' is implemented can have profound effects on the transitional dynamics of economies; and there are, of course, a great many different implementation possibilities. For example, either or both of the optimum subsidies may be implemented unannounced and so surprise the market. Or they may be announced in advance, with the time interval to implementation a matter of choice. They may even be phased-in with installments announced or unannounced, and with variable phasing intervals. Only three alternative methods of implementation were examined in the text, but these were enough to demonstrate that the short run dynamics were highly dependent upon the method of implementation. Differences in these dynamics feed back into the issues of adjustment costs and policy responses. Certain methods of implementation may generate short-term adjustment costs perceived as significant enough to warrant some policy intervention, while alternative methods may not. And in richer models the adjustment costs may vary across different economic agents, with further implications for policy. In any event these issues indicate that policy makers may face a great deal of choice.

The third policy issue raised by the transitional dynamics of the conversion from the market steady-state to that of the social optimum involves another choice. Although the alternative 'optimum subsidy schemes' all end up with the economy asymptotically approaching the social optimum steady-state, the dynamics of the transition depend upon which scheme is adopted. The (s_X,s_{AX})- and the (s_Y,s_{AY})-schemes produce exactly the same dynamics, but these differ from those of the (s_K,s_{AK})-scheme. The computations and graphical analysis demonstrating this from the optimum subsidy packages have not been reported here. Nevertheless, an idea of the extent of the differences may be obtained by comparing the dynamics resulting from the simulated imposition of each of the individual 'optimum subsidies' s_K, s_{AK}, s_X, and s_{AX}, as reported in Appendix 5.6.[11]

The theoretical argument in favour of the Government implementing a subsidisation policy package in order to convert the free market equilibrium into the one that would result from a social planner allocating resources by administrative fiat, is simple and compelling: It generates greater welfare for the community as a whole. However, there are considerable practical difficulties in getting such a subsidisation scheme 'right'. To begin with, the Government would have to choose the correct subsidies and avoid introducing any further distortions in financing them, or in addressing perceived short-run adjustment costs. Once established the subsidy scheme would require ongoing management since any shock to the economy would alter the target social optimum steady-state, thereby necessitating adjustments to the optimum subsidies. In the model, the only way for this to arise is through one of the parameters that define the optimum subsidies (and then only a change to γ affects s_K, and only changes to α, ζ, ρ, σ, and H affect s_{AK}). In a richer model, and in the real world, the situation would be far more complex. There would be many other parameters, including tax rates; and probably many more subsidies as well. The 'optimum subsidy scheme' might well comprise more

[11] Simulations of the imposition of the (s_X,s_{AX})-subsidy scheme in exactly the same three ways as reported in Section 5.3 for the (s_K,s_{AK})-scheme were however, conducted. Comparison of these sets of results showed little differences when both optimum subsidies of each scheme were implemented unannounced and at the same time - the method reported in Section 5.3.1. But significant differences emerged when the imposition of corresponding subsidies was announced in advance - as in the methods reported in Sections 5.3.2 and 5.3.3. Predictably, most of the differences arose in the period between announcement and implementation. Also, the largest divergences were for the prices of technology, the growth rates, the allocations of human capital, and the interest rates.

than two general subsidy types.[12] And any such multiplicity could be extended by a need to vary subsidy rates at a disaggregated level - to account for differentials in the external benefits generated by research,[13] or for variable degrees of monopoly power associated with the sale of specialised capital for example. Hopefully, any variability of subsidy rates would be objective, based on carefully measured data. In practice however, data would be limited, subsidy funds would be scarce, judgements would have to be made, and Governments would end up 'picking winners' in the sometimes politically motivated, often arbitrary, and largely unsuccessful manner that has often characterised this sort of activity.[14] And these problems are additional to the fact that taxes are never raised in a completely non-distortionary manner in real economies.

6.3 Directions for further research

Possible directions for further research are discussed here in three broad classes. First, there are some issues that arose directly out of the work on this thesis. Second, there are possibilities for relaxing the specific simplifying assumptions that underlie Romer's (1990a and 1990b) model. And third, there is the grander question of developing the model for use as an empirical policy analytic tool.

Issues from the research of the thesis itself are a small and eclectic group. They were mentioned in the dissertation as they arose and are simply listed below with references back to those points:

- The general point of extending the dynamic analysis to other models of endogenous growth is perhaps the most obvious gap in the research here (Chapter 1: p.3, footnote 5). Other matters are:
- Formally establishing the equivalence between the full Romer model and the abridged Solowian-Romer model when the broad measure of savings, s_B, is used in place of the narrow measure, s_N (Chapter 3: pp.113 and footnote 20).
- Explaining the economic reasoning behind the fact that the allocation of human capital, the interest rate and the growth rate in the steady-state of the social optimum system do not depend on the capital productivity parameter γ (Chapter 5: p.299).
- And, resolving the issue of the "finite limiting value of human capital", giving attention to alternative modelling in which the supply of such human capital is

[12] Recall that the production function used here had the special property of generating exactly offsetting effects from the specialisation divergence and the monopoly pricing distortion (Romer, 1987b). With other production functions individual subsidies may be necessary. Also, for the same production function, Barro and Sala-i-Martin (1995) have shown that an extra subsidy is required once allowance is made for the erosion of monopoly power.
[13] Which could be negative if too many resources (from a social planner's perspective) are devoted to research. This could happen if the advantages from being the first to discover new products and processes induce researchers to race one another.
[14] This sort of selective provision of Government assistance has been widely discredited around the world, both in terms of the 'theoretical' arguments mounted to support it, and in terms of its practical record. For example, see Bureau of Industry Economics (1987 & 1988), Council of Economic Advisers (1984), Gannicott (1979), Krugman (1983), Schmidt (1984), and Schultze (1983).

endogenous and can grow (Chapter 5: p.313, footnote 14; and Appendix 5.3 p.361).[15]

Incomplete excludability, non-rivalry, externalities, market power, specialisation, increasing returns and the accumulation of both knowledge and capital have all long been recognised as important ingredients in the processes of technological change and therefore of economic growth. But up until Romer's work, culminating with his 1990a and 1990b papers, they have never all featured as a complete package in a model that could be analysed quantitatively. Since he only sought to 'highlight the effects of interest' at an 'aggregate level and over long time periods', using the model mostly to just 'sign the effects that various interventions have on the rate of growth', Romer made a number of simplifying assumptions. In particular he assumed:

i. the population and labour supply are both constant, thereby abstracting from questions of fertility, unemployment, and labour participation (and issues of the trade-off between work and leisure in particular);[16]
ii. the total stock of human capital in the population, and the proportion supplied to the market are both fixed;
iii. research output (designs) are infinitely divisible, so there is a constant flow of such output rather than having indivisible lumps arriving at distinct points of time;
iv. the process of generating research output is deterministic (that is, there are no uncertainties);
v. the desired intensive usage of human capital and of the existing stock of knowledge in research are manifested in a research production function in which these are the **only** factors, and for which dependence on each is linear;
vi. the use of technological products, the specialised capital goods, in the production function produce additively separable effects (innovation is purely capital-widening not deepening), which therefore rule out the possibility of obsolescence, irrespective of differences in capital vintages;
vii. there are no adjustment costs associated with the formation, installation or decommissioning of capital - specialised capital can be costlessly produced out of general-purpose capital and, more importantly, it can just as costlessly be reconverted back to it (the *putty-putty* nature of specialised capital);[17]
viii. there is no erosion of the monopoly power of the producers of technological products over time, neither from patent narrowing or expiry, nor from imitation or the development of substitute products by competitors;
ix. research only generates new producer goods, not new consumer goods; and
x. the economy is 'closed', there being no trade in either goods or ideas, and no international diffusion or imitation of technology.

[15] The exogeneity of aggregate human capital is identified as a simplifying assumption of the model that could be relaxed with further research (also see Chapter 2: p.36, footnote 18).
[16] Sala-i-Martin (1990a) and Jones (1995) both raise problems faced by *endogenous growth models* when the population growth rate is positive; namely, because of scale effects in the rewards to innovation, rather than being constant the growth rate of the economy increases over time counterfactually.
[17] This issue was discussed in Appendix 2.1 (pp.62-64). It might seem to be a more serious issue for the analysis of transient dynamics than for balanced growth analysis because on balanced growth paths the number of types of specialised capital is constant ($X(t)=X$). This means there is no disinvestment and hence no need for any reconversion of units back to general-purpose capital. However, it becomes a problem, even for comparative statics, whenever any exogenous shock reduces X. The transient dynamics exist whether they are computed or not.

All of these 'simplifying assumptions' offer prospects for further research. Many of them have been addressed in other models, with their own sets of (different) simplifying assumptions.[18] They represent opportunities to set the model on more realistic foundations. The development of ways of relaxing any of the individual simplifying assumptions above would be significant in this respect. But many of the issues are related, the erosion of monopoly power and obsolescence for example. Thus, well thought-out refinements to the model could replace more than one simplifying assumption at a time. Nevertheless, the next step, that of attempting to put the whole model on a realistic footing and so develop it to a point from which it might be capable of being used empirically and in a policy relevant way, is a far more difficult task.

For it to be achieved, it would have to be possible to implement the model with a fair degree of sectoral disaggregation. This is because the ability to answer questions concerning the resource and income re-allocative impacts of economic changes must surely be considered as central to policy analysis. As mentioned in Chapter 1, some modest work of this sort was the original idea for a PhD topic. In particular, the idea was to investigate the 'new growth theory' with a view to incorporating some of its insights into a large computable general equilibrium model of the Australian economy - the 'MONASH Model'.

MONASH is the latest stage in the evolution of the 'ORANI Model' (see Dixon et al, 1982). Whereas ORANI is basically a comparative statics model[19], MONASH includes some dynamics, principally for the accumulation of certain stock variables like capital and foreign debt. The treatment of technological change in ORANI, where it was mostly simply exogenous, was also an area that was intended to be improved upon in MONASH. Thus, the possibility of introducing some practical modelling of endogenous growth theory (involving accumulation dynamics for the stock of technology or knowledge) to the treatment of technological change in MONASH seemed appealing. However, the task is a very big one:

- The MONASH Model is greatly disaggregated. For example, it has 113 industries, 115 commodities, and some 300 occupations. Implementation of any form of endogenous growth modelling, would require the specification of appropriate

[18] For example see the references below for the corresponding point numbers beginning on page 396:
i. Young (1995a), Barro and Sala-i-Martin (1995, Chapter 9), and Aghion and Howitt (1998, Chapters 3 & 12);
ii. Lucas (1988), Azariadis and Drazen (1990), Laitner (1993), and Aghion and Howitt (1998, Chapter 10);
iv. Aghion and Howitt (1992) and (1998, Chapter 2);
vi. Segerstrom, Anant and Dinopoulos (1990), Grossman and Helpman (1991a), (1991b), and (1991d, Chapter 4), Aghion and Howitt (1992) and (1998), Barro and Sala-i-Martin (1995, Chapter 7);
vii. Abel and Blanchard (1983), King and Rebelo (1993), 'vintage capital models' such as those cited in Appendix 2.1;
viii. Judd (1985a), Barro and Sala-i-Martin (1995, Chapter 6), as well as the references for vi above;
ix. Barro and Sala-i-Martin (1995, Chapter 6); and
x. Yarri (1965), Blanchard (1985), Weil (1989), Grossman and Helpman (1989a), (1989c), (1991c) and (1991d, Chapters 6 &7), Rivera-Batiz and Romer 1991b), Lucas (1993), Barro, Mankiw and Sala-i-Martin (1995), Barro and Sala-i-Martin (1995, Chapters 3 & 8) and Aghion and Howitt (1998, Chapter 11).
[19] There are no explicit dynamics but a broad implicit time dimension exists in that the model results can be interpreted as referring to either the 'short term' or the 'long term' according to its 'closure' (that is its split between endogenous and exogenous variables).

production functions for each industry. While the analytical form of all these might be identical, significant effort would be necessary to establish sensible paramaterisations for different industries (econometric estimation probably being desirable).

- The concept of human capital involved in endogenous growth modelling would have to be reconciled with the occupational disaggregation, both theoretically and empirically.
- While many different forms of endogenous growth model already exist, they have only been specified indicatively - to illustrate broad economic ideas at a macro level. Also, there has been little empirical testing of such models. Thus, much greater care would be necessary in the specification of an operational disaggregated module for MONASH, and at least some empirical validation would be desirable.
- Theoretical solutions for the growth equilibria would also have to be found for any new specifications before they could be implemented.
- There may also be a strong case for having different modelling for different classes of industry. For example, for 'producer goods' and 'consumer goods' sectors.
- Also, in order to include transitional dynamics, the entire MONASH Model would have to be re-specified as dynamic, with each of its several million variables requiring a time dimension.

While it soon became apparent that such a task was not a viable one for a PhD topic, it might at some stage in the not too distant future be feasible for a modelling development team. Obviously a great deal more thought would have to be devoted to the problem than the superficial points made here, but it is at least a potential area for further research.

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