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Mean Curvature Flow With Free Boundary on Smooth Hypersurfaces

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Abstract

In this thesis we study the classical mean curvature flow of hypersurfaces with boundary which satisfy a Neumann boundary condition on an arbitrary, fixed, smooth hypersurface in Euclidean space. In particular, the author focuses on the problem of singularity formation on the free-boundary and the classification of the limiting behaviour thereof. This is achieved by a careful modification of Huisken's monotonicity formula that incorporates a reflection principle of Grüter-Jost, developed in their treatment of the corresponding stationary problem for varifolds, as well as the curvature of the support surface. Using the monotonicity formula thus obtained, the author then classifies the possible limiting behaviour of a natural class of singularities in the case of weakly mean-convex surfaces.

This thesis contains no material which has been accepted for the award of any other degree or diploma in any university or other institution. To the best of my knowledge, this thesis contains no material previously published or written by another person, except where due reference is made in the text.

John A. Buckland

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Contents

Chapter 1

Introduction

1.1 Background

A hypersurface in Euclidean space is said to be evolving by mean curvature flow if, for each point of the surface and at each instant in time, the surface moves in the normal direction at that point with speed equal to the mean curvature at that point. Thus a sphere moving by mean curvature flow will move by homothety, either shrinking to it's centre with increasing speed and becoming singular, or expand indefinitely, depending on the chosen orientation.

In the classical setting, Huisken [18] has shown that compact, initially convex, embedded hypersurfaces contracting under mean curvature flow converge to a single point in finite time and, after suitable rescaling, asymptotically become spherical. For the one dimensional case, referred to as *curve-shortening*, the analogous result - that initially convex planar curves contract to a point - was subsequently obtained by Gage and Hamilton in [13] and [12]. This was later generalized to all closed, embedded planar curves by Grayson [16].

Weak formulations of the flow that allow the continuation past the onset of singularities are also possible. Though the focus in this work is on the classical mean curvature flow, results in this field are due to Brakke [4] and Ilmanen [22] for varifold solutions, and Evans-Spruck [10] for level-sets, amongst others.
The natural question concerning the structure of the singularities formed by non-convex initial data has always been of particular interest, with a classification of the limiting behaviour of surfaces moving under the flow being sought. An important tool in the study of the nature of singularities formed by surfaces under the mean curvature flow has been the monotonicity formula of Huisken [19]. This formula is analogous to the monotonicity formula for minimal surfaces [11] (which are stationary solutions of the mean curvature flow), the monotonicity formula of Giga-Kohn [1[illegible]], that of Struwe for the harmonic map flow [5], as well as Price's result for the Yang-Mills heat flow [23]. Most recently, a localized version of the monotonicity formula for mean curvature flow has also been found by Ecker [6].

Also of particular interest is the behaviour of surfaces evolving by mean curvature flow *with boundary*. Results concerning the Dirichlet problem are due to Huisken [21], who showed that, for boundaries of positive mean curvature, non-parametric

solutions converge to minimal surfaces, and Stone [27], among others. Surfaces possessing boundaries satisfying various contact angle conditions have also been examined. Of particular interest here is the work of Stahl [25], who examined hypersurfaces evolving by mean curvature flow which satisfy a Neumann boundary condition on an arbitrary, fixed, smooth support surface. He proved the existence and uniqueness of solutions for arbitrary smooth support and initial surfaces that exist on a maximal time interval, which either exist eternally or whose curvature becomes unbounded in finite time. He furthermore classified the limiting behaviour of all initially strictly convex surfaces with boundary contained in a sphere which evolve in the sphere's interior (as opposed to its exterior).
This free-boundary problem has also been studied by Grüter-Jost [17] for weak (varifold) stationary solutions of the flow. They established a monotonicity formula perfectly analogous to the standard one of minimal surface theory, which allowed an extension to the regularity results of Allard [2] to be made.

The focus of this work is to obtain a monotonicity formula in this setting that is analogous to Huisken's, and use rescaling techniques and Huisken's classification for boundaryless surfaces [20] to classify a natural class of singularities for mean convex evolving hypersurfaces. The approach undertaken is motivated largely by Huisken's work, but incorporates the key idea of Grüter-Jost's work to deal with the boundary. We remark that, in light of Ecker's recent local monotonicity formula, which can be used locally over hypersurfaces with boundary in regions not containing the boundary, the main concern in the current work is with regions centred directly on the free-boundary.

1.2 Mean Curvature Flow With Free Boundary on Smooth Hypersurfaces

Throughout this work, Σ denotes a hypersurface smoothly embedded in $\mathbb{R}^{n+1}$ which satisfies a *rolling ball* condition with ball of maximal radius $1/\kappa_\Sigma$, and whose second fundamental form A_Σ satisfies

$$\|A_\Sigma\|^2 + \|\nabla A_\Sigma\| \leqslant \kappa_\Sigma^2 < \infty. \tag{1.1}$$

Furthermore, for ease of presentation, we assume that Σ contains the origin,

$$0 \in \Sigma. \tag{1.2}$$

We let M^n denote a smooth, orientable n-dimensional manifold with smooth, compact boundary ∂M^n and set $M_0 := F_0(M^n)$, where $F_0 : M^n \longrightarrow \mathbb{R}^{n+1}$ is a smooth embedding satisfying

$$\begin{aligned} \partial M_0 \equiv F_0(\partial M^n) &= M_0 \cap \Sigma, \\ \langle \nu_0, \nu_\Sigma \circ F_0\rangle(p) &= 0 \qquad \forall p \in \partial M^n, \end{aligned} \tag{1.3}$$

for unit normal fields ν_0 and ν_Σ to M_0 and Σ, respectively. We then have the following formal definition for the flow by mean curvature of M_0 with Neumann free-boundary on the hypersurface Σ:

Definition 1.2.1 (Mean curvature flow with Neumann free-boundary). *Let $I \subset \mathbb{R}$ be an open interval and let $F_t = F(\cdot, t) : M^n \longrightarrow \mathbb{R}^{n+1}$ be a one-parameter family of smooth embeddings for all $t \in I$. The family of hypersurfaces $(M_t)_{t \in I}$, where $M_t = F_t(M^n)$, are said to be evolving by mean curvature flow with Neumann free-boundary condition on Σ if*

$$\begin{aligned} \frac{\partial F}{\partial t}(p,t) &= \vec{H}(p,t) && \forall (p,t) \in M^n \times I, \\ F(\cdot, 0) &= F_0, && \\ F(p,t) &\subset \Sigma && \forall (p,t) \in \partial M^n \times I, \\ \langle \nu, \nu_\Sigma \circ F \rangle (p,t) &= 0 && \forall (p,t) \in \partial M^n \times I. \end{aligned} \tag{1.4}$$

Here $\vec{H}(p,t) = -H(p,t)\,\nu(p,t)$ denotes the mean curvature vector of the immersions M_t at $F(p,t)$, for a choice of unit normal ν for M_t.

Whenever possible we will suppress explicit indication of the embedding map and identify the point $F(p,t)$ simply with its position vector x in $\mathbb{R}^{n+1}$. Thus, the above definition of mean curvature flow with Neumann free-boundary on Σ may be interpreted as saying that, for all $t \in I$, we have

$$\begin{aligned} \frac{\partial x}{\partial t} &= \vec{H}(x) && \forall x \in M_t, \\ \partial M_t &\subset \Sigma && \\ \langle \nu, \nu_\Sigma \rangle (x) &= 0 && \forall x \in \partial M_t. \end{aligned}$$

Moreover, for a given embedding F we adopt the convention that the *exterior* unit normal to Σ *with respect to this embedding* coincides with the unit *inner* co-normal of ∂M_t at all intersection points. Thus, by the Neumann boundary condition, if $\Sigma = \partial G$ for some domain $G \subset \mathbb{R}^{n+1}$ and $M_t \subset \mathbb{R}^{n+1} \backslash G$ (ie. the surface evolves in the *exterior* of the domain), this choice of normal coincides with the standard notion of an exterior normal field to Σ; for the case where $M_t \subset G$, the above convention coincides with an antiparallel vector to the standard exterior normal to Σ.

By choosing special coordinates that account for the curvature of the support surface - so-called *generalized Gaussian coordinates* - one can (locally) transform the quasilinear system of parabolic equations (1.4) into an equivalent initial-boundary-value problem for a scalar function [25], and begin to analyze the problem using the established theory of partial differential equations. Though this approach forfeits much of the insight offered from the geometric approach, standard results from the parabolic theory allow one to proceed as in [25] and establish the short-time existence of a unique solution to this problem. In fact, by proceeding as in the work of Ecker and Huisken in [8], Stahl was able to obtain sharp local gradient estimates that imply the IBV-scalar problem is, in fact, (locally) *uniformly* parabolic. By then appealing to standard results of the linear parabolic theory, the following general existence and regularity result was obtained:

Theorem 1.2.2. [Stahl, [25]] *For any smooth hypersurface Σ and and initial hypersurface M_0 satisfying* (1.3) *there exists a unique solution to* (1.4) *on a maximal* *time interval* $[0, T)$ *which is smooth for* $t > 0$ *and in the class* $C^{2+\alpha, 1+\alpha/2}$ *for* $t \geqslant 0$, *for any* $\alpha \in (0, 1)$. *Moreover, if* $T < \infty$ *then*

$$sup \left\{ |A|^2 (x, t) \ : \ x \in M^n \right\} \longrightarrow \infty \ \ as \ t \longrightarrow T. \tag{1.5}$$

The focus of this work is the case where $T < \infty$, where we wish to study nature of singularities that occur at points within the evolving boundary of M_t, with a further goal being the classification of possible limiting behaviour of the evolving surfaces as the singularity develops.

Chapter 2

Motivation

Here we present a brief synopsis of the method used by Huisken in [19] and [20] to show that hypersurfaces without boundary evolving by the standard mean curvature flow become asymptotically self-similar as the singular time is approached. The key ingredient in this argument is a monotonicity formula, which describes how the area of the evolving surfaces behaves when weighted with a backward heat kernel centred at the singular point in the ambient space. Since the backward heat kernel becomes sharper as the singular time is approached, showing this weighted area monotonically decreases implies that the area does not concentrate at the singular point. More specifically, the monotonicity formula yields that, after appropriate rescaling, the limiting hypersurfaces move by homothety and are thus self-similar. This characterization is captured in an elliptic PDE which, in the case of hypersurfaces with nonnegative mean curvature, has been solved to give a complete classification of the limiting surface.
As this approach directly motivates the subsequent analysis concerning the development of singularities within surfaces possessing Neumann free-boundary on smooth hypersurfaces, the main focus of this section is to simply outline the essential ingredients of Huisken's work and relegate the technical details to the ensuing analysis.

2.1 Singularities For Hypersurfaces Without Boundary

For this section only, we assume that M^n is an n-dimensional manifold without boundary, and that $F_t \equiv F(\cdot,t) : M^n \longrightarrow \mathbb{R}^{n+1}$ is a one-parameter family of smooth hypersurface embeddings with corresponding image surfaces $M_t = F_t(M^n)$ evolving by mean curvature flow. That is, we have

$$\begin{aligned} \frac{\partial F}{\partial t}(p,t) &= \vec{H}(p,t), \qquad \forall\ (p,t) \in M^n \times [0,T), \qquad\qquad \text{(MCF)} \\ F(p,0) &= F_0(p), \qquad \forall\ p \in M^n. \end{aligned}$$

In view of the identity

$$\triangle_{M_t} F(p,t) = \vec{H}(p,t),$$

where Δ_{M_t} denotes the Laplace-Beltrami operator on M_t, one may re-write the governing equation of (MCF) as

$$\frac{\partial F}{\partial t}(p,t) = \Delta_{M_t} F(p,t).$$

Thus it is not surprising that solutions of (MCF) exhibit strong analogies with solutions to the standard linear heat equation and it would seem plausible to expect some insight to come from the linear theory.
Indeed, for any fixed point $(x_0, T) \in \mathbb{R}^{n+1} \times (0, T)$, let $\rho_{x_0,T}$ be the standard backward heat kernel on $\mathbb{R}^{n+1}$, centred at (x_0, T), with time-scaling appropriate to $\mathbb{R}^n$ – that is, let

$$\rho_{x_0,T} \equiv \frac{1}{(4\pi(T-t))^{n/2}} \exp\left(-\frac{|x-x_0|^2}{4(T-t)}\right), \tag{2.1}$$

so that, for all $t < T$, the function $\Psi(x,t) \equiv \frac{1}{\sqrt{4\pi(T-t)}}\rho_{x_0,T}$ solves the linear backward heat equation,

$$\frac{\partial \Psi}{\partial t}(x,t) = -\Delta_{\mathbb{R}^{n+1}} \Psi(p,t).$$

We then have the following theorem, due to Huisken [19].

Theorem 2.1.1 (Monotonicity Formula). *Let M_t be a family of hypersurfaces evolving by the mean curvature flow* (MCF) *for $t \in [0,T)$. Then for all $t < T$ we have*

$$\frac{d}{dt}\int_{M_t} \rho_{x_0,T}\, d\mu_t = -\int_{M_t} \left|\vec{H} - \frac{F^\perp}{2(T-t)}\right|^2 \rho_{x_0,T}\, d\mu_t. \tag{2.2}$$

To deduce the asymptotic behaviour of surfaces near the first first singular time, rescaling techniques are employed. In order to ensure smoothness of the rescaling limit, we make the following distinction between the nature of singularities.

Definition 2.1.2. *A surface M_t develops a singularity of* Type I *if the curvature becomes unbounded as $t \longrightarrow T$ and there exists a constant $C_0 > 0$ such that*

$$\max_{M_t} |A(\cdot,t)|^2 \leqslant \frac{C_0}{2(T-t)}. \tag{2.3}$$

Otherwise, the singularity is called Type II.

For the remainder of this section we assume that M_t develops a singularity of type I at the origin $0 \in \mathbb{R}^{n+1}$. That is, for some $p_0 \in M^n$, we have $F(p_0,t) \longrightarrow 0$ and $|A(p_0,t)|^2 \longrightarrow \infty$ as $t \longrightarrow T$. (For a treatment of singularities of the general type, the reader is referred to the recent work of Ilmanen [22]; here the smoothness of the rescaling limit is not ensured and so one must work in a weak setting to obtain a rescaling limit).

The original rescaling procedure carried out by Huisken to classify type I singularities in [19] involved setting

$$\widetilde{F}(p,s) = \frac{1}{2(T-t)}F(p,t), \qquad \text{where } s(t) = -\frac{1}{2}\log(T-t)$$

for $p \in M^n$, so that the rescaled surfaces $\widetilde{M}_s = \widetilde{F}(\cdot, s)(M^n)$ are defined over $-\frac{1}{2}\log T \leqslant s < \infty$ and satisfy the *normalized* mean curvature flow equation

$$\frac{d}{ds}\widetilde{F}(p,s) = \widetilde{\vec{H}}(p,s) + \widetilde{F}(p,s), \tag{2.4}$$

where $\widetilde{\vec{H}}$ is the mean curvature vector of $\widetilde{M}_s$.
In this rescaled setting we then have

Corollary 2.1.3 (Huisken, [19]). *If the surfaces $\widetilde{M}_s$ satisfy the rescaled evolution equation (2.4), then for all $s \in \left[-\frac{1}{2}\log T, \infty\right)$ we have*

$$\frac{d}{ds}\int_{\widetilde{M}_s} \widetilde{\rho}\, d\widetilde{\mu}_s = -\int_{\widetilde{M}_s} \left|\widetilde{\vec{H}} - \widetilde{F}^{\perp}\right|^2 \widetilde{\rho}\, d\widetilde{\mu}_s, \tag{2.5}$$

where $\widetilde{\rho}(x) = \exp\left(-\frac{|x|^2}{2}\right)$.

For surfaces M_t which develop a singularity of type I, the corresponding rescaled surfaces $\widetilde{M}_s = \widetilde{F}(\cdot, s)(M^n)$ then have uniformly bounded curvature. Moreover, by examining the evolution equation for the norm of the second fundamental form of $\widetilde{M}_s$, one can establish uniform bounds on all higher derivatives of the curvature:

Theorem 2.1.4 (Huisken [19]). *Suppose the Type I hypothesis (2.3) holds. Then for each $k \geqslant 0$ there is a constant $C_k < \infty$ such that the second fundamental form of the rescaled surfaces $\widetilde{M}_s$ satisfies*

$$\max_{\widetilde{M}_s} \left|\nabla^k A\right|^2 \leqslant C_k \tag{2.6}$$

uniformly in s.

Since the type I hypothesis also implies that the term $\widetilde{F}(p_0, s)$ remains bounded (see eg. [19]), for any sequence of times $s_j \longrightarrow \infty$ we obtain the convergence on compact subsets of $\mathbb{R}^{n+1}$ of a subsequence of the surfaces $\widetilde{M}_{s_j}$ to a smooth limiting surface $\widetilde{M}_\infty$.
In view of the uniform regularity estimates (2.6), the behaviour of this limiting hypersurface is then a straightforward consequence of the rescaled monotonicity formula:

Theorem 2.1.5 (Huisken [19]). *Each limiting hypersurface $\widetilde{M}_\infty$ satisfies the equation*

$$H = \langle x, \nu \rangle. \tag{2.7}$$

This is a second-order elliptic equation that implies that the rescaled surfaces become asymptotically self-similar and, therefore, that the surfaces M_t approach a homothetically shrinking solution of (MCF) as $t \longrightarrow T$.
For surfaces of nonnegative mean curvature, the limiting behaviour can then be characterized as follows:

Theorem 2.1.6 (Huisken, [20]). *If $\widetilde{M}_\infty$ is a smooth, embedded limiting hypersurface in $\mathbb{R}^{n+1}$ satisfying* (2.7) *with nonnegative mean curvature, $H \geqslant 0$, then $\widetilde{M}_\infty$ is either S^n or $S^{n-m} \times \mathbb{R}^m$.*

Remark 2.1.7. If the embeddedness hypothesis of Theorem 2.1.6 is removed then, in addition to S^n and $S^{n-m} \times \mathbb{R}^m$, the limiting hypersurface may also be $\Gamma \times \mathbb{R}^{n-1}$, where Γ is one of the immersed, homothetically shrinking curves in $\mathbb{R}^2$ found by Abresch and Langer [1] (see [20] for details).

2.2 Outline of Main Ideas

The essential tool required to mimic the above analysis for solutions to (1.4) is a monotonicity formula for hypersurfaces with free-boundary on some given smooth support surface. Obviously some modifications must be made to the classical formula (2.2), in general, to compensate for the curvature of the support surface, and it is natural to expect that any formula reduce to that for surfaces without boundary in the case of planar support surfaces (since, in this case, the Neumann boundary condition allows the surfaces M_t to be reflected across this hyperplane and be treated as for the surfaces above - cf. proof of Theorem 5.2.5).
The starting point for obtaining such a monotonicity formula is the following general expansion result.

Proposition 2.2.1 (Expansion Formula). *Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a solution of* (1.4) *and U an open subset of $\mathbb{R}^{n+1}$ containing $\mathcal{M}$. For any functions $f, g : U \times [0,T) \longrightarrow \mathbb{R}$, where $f \in C_0^2(U)$, $\frac{\partial f}{\partial t} \in C_0^0(0)$, $g \in C^2(U)$ and $\frac{\partial g}{\partial t} \in C^0(U)$, we have the following general expansion formula:*

$$\begin{aligned}\frac{d}{dt}\int_{M_t} fg\,d\mu_t = \int_{M_t} f\left|\overrightarrow{H} - \frac{D^\perp g}{g}\right|^2 g\,d\mu_t + \int_{M_t} fQ(g)\,d\mu_t \\ + \int_{M_t} g\left(\frac{d}{dt} - \Delta_{M_t}\right) f\,d\mu_t \\ + \int_{\partial M_t} \left(g\langle Df, \nu_\Sigma\rangle - f\langle Dg, \nu_\Sigma\rangle\right) d\sigma_t,\end{aligned} \tag{2.8}$$

where here and henceforth the operator Q is defined by

$$Q(g) := \frac{\partial g}{\partial t} + \mathrm{div}_{M_t} Dg + \frac{\left|D^\perp g\right|^2}{g}. \tag{2.9}$$

Proof. By (A.2) we have

$$\begin{aligned}
\left(\frac{d}{dt}+\Delta_{M_t}\right)g-\left|\vec{H}\right|^2 g &= \frac{\partial g}{\partial t}+\operatorname{div}_{M_t} Dg+2\left\langle \vec{H}, Dg\right\rangle-\left|\vec{H}\right|^2 g \\
&= \frac{\partial g}{\partial t}+\operatorname{div}_{M_t} Dg+2\left\langle \vec{H}, D^{\perp}g\right\rangle-\left|\vec{H}\right|^2 g \\
&= \frac{\partial g}{\partial t}+\operatorname{div}_{M_t} Dg+\frac{\left|D^{\perp}g\right|^2}{g}-\left|\vec{H}-\frac{D^{\perp}g}{g}\right|^2 g \\
&= Q(g)-\left|\vec{H}-\frac{D^{\perp}g}{g}\right|^2 g.
\end{aligned}$$

Therefore, by (A.5) and the Divergence Theorem (A.1),

$$\begin{aligned}
\frac{d}{dt}\int_{M_t} fg\,d\mu_t &= \int_{M_t}\left(f\frac{dg}{dt}+g\frac{df}{dt}-\left|\vec{H}\right|^2 fg\right)d\mu_t \\
&= \int_{M_t}\left(f\left(\left(\frac{d}{dt}+\Delta_{M_t}\right)g-\left|\vec{H}\right|^2 g\right)+g\left(\frac{d}{dt}-\Delta_{M_t}\right)f\right)d\mu_t \\
&\quad -\int_{\partial M_t}\left(f\left\langle Dg,\nu_\Sigma\right\rangle-g\left\langle Df,\nu_\Sigma\right\rangle\right)d\sigma_t,
\end{aligned}$$

and the result follows. □

Remark 2.2.2.
1. Putting $g=\rho$, $f=1$ and noting (by direct computation) that $Q(\rho)\equiv 0$ gives (2.2) in the case where $\partial M_t=\emptyset$.

2. For the case where $\partial M_t\neq\emptyset$, we obtain

$$\frac{d}{dt}\int_{M_t}\rho\,d\mu_t=\int_{M_t}\left|\vec{H}-\frac{D^{\perp}\rho}{\rho}\right|^2\rho\,d\mu_t-\int_{\partial M_t}\left\langle D\rho,\nu_\Sigma\right\rangle d\sigma_t. \tag{2.10}$$

The presence of the boundary integral means that the quantity $\int_{M_t}\rho\,d\mu_t$ is, in general, no longer monotonically decreasing in time. Other than the cases where the boundary integral is non-positive - cf. Lemma 6.3.1 - there are other special cases for which the boundary integral can be sufficiently estimated - cf. Lemma 6.3.3.

In general the boundary term above is difficult to deal with and so the approach undertaken for the general case, motivated by the treatment of the stationary problem for varifolds by Grüter and Jost [17], is to nullify it using reflections. That is, for the function g above we seek a modified version of the backward heat kernel ρ with spatial dependence on some carefully chosen function of x - depending on the distance to, and the curvature of, the support surface Σ - such that the boundary integrands of (2.8) are identically zero.
However, the identity $Q(\rho)\equiv 0$ can no longer be expected to hold for a modified version of the backward heat kernel and thus one challenge then becomes to obtain sufficient control of this term. Secondly, since the distance function will in general

only be well-defined within a tubular neighbourhood of the support surface, it will be necessary to choose f to be an appropriate localization function; again, to ensure the corresponding boundary integrand vanish identically, this function should depend spatially upon reflections. Thus, a second issue is to find such a function for which the term $\left(\frac{d}{dt} - \Delta_{M_t}\right) f$ be sufficiently well-behaved.

The resolution of these two problems forms the basis of chapter 4, the culmination of which is summarized in Proposition 4.1.1 and Lemma 4.2.5. A statement of the general monotonicity formula then follows in Theorem 4.3.1.

The majority of the rest of this work is then concerned with the rescaling analysis, as outlined above, and the extraction and classification of the behaviour of limit surfaces.

Chapter 3

The Distance Function and Reflections

In this section we introduce the Euclidean distance function, which measures the distance to Σ of a point in $\mathbb{R}^{n+1}$, and establish some links between its derivatives and the geometry of Σ, as well as the domain over which it is well-defined. We then introduce a special reflection function, as discussed in the previous chapter, and establish some of its fundamental properties.

3.1 Geometric Properties of Distance Function

Definition 3.1.1 (Signed Distance Function). *For any point $x \in \mathbb{R}^{n+1}$, we denote the minimum distance of x to $\Sigma = \partial G$ (where G is some domain in $\mathbb{R}^{n+1}$) by*

$$d_\Sigma(x) := dist(x, \Sigma),$$

whenever it is well-defined.
We then define the signed distance *function by*

$$d(x) := \begin{cases} -d_\Sigma(x) & \text{if } x \in G \\ d_\Sigma(x) & \text{if } x \in \mathbb{R}^{n+1} \backslash G. \end{cases} \tag{3.1}$$

The following standard result concerning the regularity of the distance function can be found in [15].

Proposition 3.1.2. *Let $G \subset \mathbb{R}^{n+1}$ with $\Sigma = \partial G \in C^k(\mathbb{R}^n)$ for some $k \geqslant 2$. Then there exists an $\varepsilon > 0$ such that $d \in C^k(\Sigma_\varepsilon)$, where Σ_ε is the ε-tubular neighbourhood of Σ given by*

$$\Sigma_\varepsilon := \left\{x \in \mathbb{R}^{n+1} : d_\Sigma(x) < \varepsilon\right\}.$$

As we shall see in the following section, the reflection of points across the surface Σ depends on zero and first order derivatives of the distance function. Hence, to compute the heat- and Q-operator of functions depending on reflected points - as will be required in the expansion formula (2.8) - we will firstly need estimates on derivatives of the distance function up to order three.

Proposition 3.1.3 (Derivatives of Distance Function). *Let $\Sigma \in C^k$ for $k \geqslant 3$ and suppose $x_0 \in \Sigma_\varepsilon$ and $y_0 \in \Sigma$ are such that $d(x_0) = |x_0 - y_0|$. Then in terms of a principal coordinate system at y_0, we have*

1. $Dd(x_0) = \nu_\Sigma^+ (P_\Sigma (x_0))$,
2. $D^2 d (Dd)(x_0) = 0$
3. $D^2 d(x_0) = diag \left[\frac{\kappa_1}{1-\kappa_1 d_\Sigma(x_0)}, \ldots, \frac{\kappa_n}{1-\kappa_n d_\Sigma(x_0)}, 0\right]$,
4. $D_{ijk} d(x_0) = \begin{cases} \frac{D_k h_{ij}^\Sigma(y_0)}{(1-\kappa_i d_\Sigma(x_0))(1-\kappa_j d_\Sigma(x_0))(1-\kappa_k d_\Sigma(x_0))} & \text{if } 1 \leqslant i,j,k \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$

Here ν_Σ^+ is the (conventional) outer unit normal to Σ (ie. points in the direction of increasing signed distance), $P_\Sigma(x_0)$ is the nearest point projection of x_0 onto Σ and $\kappa_i = \kappa_i(y_0) = \kappa_i(P_\Sigma(x_0))$ is the principal curvature of Σ in the direction of the i-th (principal) coordinate.

Remark 3.1.4.
1. If $\Sigma = \partial G$ and M_t evolves in the interior of G, then $Dd = -\nu_\Sigma$ (where ν_Σ is as introduced in section 1.2) whenever it is well-defined. If M_t evolves in $\mathbb{R}^{n+1}\backslash G$ then $Dd = +\nu_\Sigma$.
2. In view of (1.1), the above proposition implies the estimates

$$\|D^2 d\| \leqslant \frac{\kappa_\Sigma}{1 - d\kappa_\Sigma} \tag{3.2}$$

and

$$\|D^3 d\| \leqslant \frac{\kappa_\Sigma^2}{(1 - d\kappa_\Sigma)^3}, \tag{3.3}$$

where here $\|\cdot\|$ refers to the maximum norm.

Proof. 1. To verify the first claim, we will proceed as follows: firstly, we show the inner product of the gradient of the distance function and the unit normal to Σ is one; secondly, we show the distance function is Lipschitz continuous with Lipschitz constant one; then we combine these two facts to obtain the desired conclusion.

First step: For Σ given by $\Phi : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}$, we define the function $\Psi : U \times \mathbb{R} \longrightarrow \mathbb{R}^{n+1}$ by

$$\Psi(q,s) = \Phi(q) + s\nu_\Sigma^+(q).$$

For some ϵ-neighbourhood Σ_ϵ of Σ the distance function $d : \Sigma_\epsilon \longrightarrow \mathbb{R}$ then gives

$$d(\Psi(q,s)) = s$$

and

$$d^{-1}(0) = \Sigma.$$

For any fixed point $q_0 \in U$ let us also define the function $\alpha : \mathbb{R} \longrightarrow \mathbb{R}^{n+1}$ by

$$\begin{aligned}\alpha(s) &= \Psi(q_0, s)\\ &= \phi(q_0) + s\nu_\Sigma^+(q_0).\end{aligned}$$

Then at any point $z_0 = \Psi(q_0, s_0) \in \Sigma_\epsilon$ we have

$$\begin{aligned}Dd(z_0)\cdot\nu_\Sigma^+(q_0) &= Dd(\alpha(s_0))\cdot\alpha'(s_0)\\ &= (d\circ\alpha)'(s_0)\\ &= 1 \qquad (3.4)\end{aligned}$$

Second step: Let $x, y \in \Sigma_\epsilon$ and choose $z \in \Sigma$ such that $|y - z| = d(y)$. Then, by the triangle inequality

$$\begin{aligned}d(x) &\leqslant |x - z|\\ &\leqslant |x-y| + |y-z|\\ &= |x-y| + d(y).\end{aligned}$$

Since x and y are chosen independently, we can interchange their roles above to obtain also

$$d(y) \leqslant |x - y| + d(x).$$

Hence,

$$|d(x) - d(y)| \leqslant |x - y|$$

and so d has Lipschitz constant 1;

$$|Dd(x)| \leqslant 1 \qquad (3.5)$$

for all $x \in \Sigma_\epsilon$.
Third step: Combining (3.4) and (3.5), one obtains

$$\begin{aligned}1 &= Dd(z_0)\cdot\nu_\Sigma^+(q_0)\\ &= |Dd(z_0)|\,cos\theta\end{aligned}$$

where θ is the angle between $Dd(z_0)$ and $\nu_\Sigma^+(q_0)$. Hence θ must be zero and $|Dd(z_0)| = 1$, which implies the result:

$$Dd(z_0) = \nu_\Sigma^+(q_0). \qquad (3.6)$$

For ease of presentation in the rest of this proof, we henceforth drop the superscript $+$ and simply write

$$Dd(z_0) = \nu_\Sigma(q_0).$$

2. Since $|Dd| = 1$, we have

$$\begin{aligned}0 &= \frac{1}{2}D|Dd|^2\\ &= D^2d \circ Dd.\end{aligned}$$

3. We follow the proof of [GT, lemma 14.17] and compute higher derivatives of the distance function, evaluated at a point $x_0 \in \Sigma_\epsilon$, in a principal coordinate system about the point $y_0 \equiv P_\Sigma(x_0) \in \Sigma$.
Since Σ is C^k, $(k \geqslant 3)$, there exists a neighbourhood $N(y_0)$ about the point y_0 such that Σ can be given by the graph

$$x_{n+1} = \phi(\overline{x}),$$

where $\overline{x} = (x_1, \ldots, x_n)$, $\phi \in C^k(T_\Sigma(y_0) \cap N(y_0))$ and $T_\Sigma(y_0)$ is the tangent hyperplane to Σ at y_0. Furthermore, by rotating coordinates we can assume that the $x_1, \ldots, x_n$ axes coincide with the principal curvature directions $\kappa_1^\Sigma, \ldots, \kappa_n^\Sigma$ and that both

$$D\phi(\overline{y_0}) = 0 \tag{3.7}$$

and

$$D^2\phi(\overline{y_0}) = \operatorname{diag}\left[\kappa_1^\Sigma, \ldots, \kappa_n^\Sigma\right]. \tag{3.8}$$

Let us now define the mapping $g : T_\Sigma(y_0) \cap N(y_0) \times \mathbb{R} \longrightarrow \mathbb{R}^{n+1}$ by

$$g(\overline{y}, d) = y + \nu_\Sigma(y)\, d, \quad \text{where } y = (\overline{y}, \phi(\overline{y})).$$

Noting that for any point $y = (\overline{y}, \phi(\overline{y})) \in N(y_0) \cap \Sigma$ we have

$$\nu_\Sigma(y) \equiv \overline{\nu_\Sigma}(\overline{y}) = \frac{1}{\sqrt{1 + |D\phi(\overline{y})|^2}}\left(-D\phi(\overline{y}), 1\right),$$

one computes, for $1 \leqslant i, j \leqslant n$,

$$\begin{aligned} D_i \overline{\nu_\Sigma}_j(\overline{y_0}) &= \frac{-D_i D_j \phi(\overline{y_0})}{\sqrt{1 + |D\phi(\overline{y_0})|^2}} + \frac{D_j\phi(\overline{y_0})\, D_k\phi(\overline{y_0})\, D_i D_k \phi(\overline{y_0})}{\left(1 + |D\phi(\overline{y_0})|^2\right)^{3/2}} && (3.9) \\ &= -\delta_j^i \kappa_i^\Sigma && (3.10) \end{aligned}$$

using (3.7) and (3.8). Hence, the Jacobian matrix of g at $(\overline{y_0}, d(x_0))$ is given by

$$Dg(\overline{y_0}, d(x_0)) = \operatorname{diag}\left[1 - \kappa_1^\Sigma d(x_0), \ldots, 1 - \kappa_n^\Sigma d(x_0), 1\right] \tag{3.11}$$

and it follows that Dg is invertible over the set

$$\Sigma_{1/\kappa} := \{x \in \mathbb{R}^{n+1} : \ d_\Sigma(x) < \frac{1}{\kappa}\}.$$

We are now in a position to compute the Hessian matrix of d at points within $\Sigma_{1/\kappa}$. By part 2 of this proposition, we know that

$$Dd(x^*) = \nu_\Sigma(y_0)$$

for any point x^* of the form $x^* = y_0 + \epsilon\nu_\Sigma(y_0)$, $\epsilon < 1/\kappa$. Therefore it follows that

$$D_{n+1} D_i d(x_0) = 0, \qquad \text{for } 1 \leqslant i \leqslant n+1. \tag{3.12}$$

For the other derivatives, we write for $i, j = 1, \ldots, n$,

$$\begin{aligned} D_i D_j d(x_0) &= D_i \left(\nu_{\Sigma_j}(y_0)\right) \\ &= D_k \overline{\nu}_{\Sigma_j}(\overline{y_0})\, D_i y_k(x_0) \\ &= -\delta_j^k \kappa_k^\Sigma D_i y_k(x_0), \end{aligned}$$

by (3.10). Since $g(\overline{y_0}, d(x_0)) = y_0 + \nu_\Sigma(y_0)\, d(x_0) = x_0$, we have $y(x_0) = g^{-1}(x_0)$ and so in light of (3.11) the Inverse Function Theorem yields

$$\begin{aligned} Dy(x_0) &= Dg^{-1}(x_0) \\ &= \operatorname{diag}\left[\frac{1}{1-\kappa_1 d(x_0)}, \ldots, \frac{1}{1-\kappa_n d(x_0)}, 1\right]. \end{aligned} \tag{3.13}$$

Hence, for $1 \leqslant i, j \leqslant n$ we have

$$\begin{aligned} D_{ij} d(x_0) &= -\frac{\delta_j^k \delta_k^i \kappa_k}{1-\kappa_i d(x_0)} \\ &= \begin{cases} \frac{-\kappa_i}{1-\kappa_i d(x_0)} & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \end{aligned}$$

Combining this result with (3.12) thus gives

$$D^2 d(x_0) = \operatorname{diag}\left[\frac{-\kappa_1}{1-\kappa_1 d(x_0)}, \ldots, \frac{-\kappa_n}{1-\kappa_n d(x_0)}, 0\right].$$

4. For the same reasons that yielded (3.12), we have

$$D_{n+1} D_i D_j d(x_0) = 0 \qquad 1 \leqslant i, j \leqslant n+1.$$

For the other derivatives, where $i, j, k = 1, \ldots, n$, we compute

$$\begin{aligned} D_{ijk} d(x_0) &= D_i D_j \left(D_k d(x_0)\right) \\ &= D_i D_j \left(\nu_{\Sigma_k}(P(x_0))\right) && \text{by (3.6)} \\ &= D_i \left(D_l \nu_{\Sigma_k}(P(x_0))\, D_j P_l(x_0)\right) \\ &= D_m D_l \nu_{\Sigma_k}(P(x_0))\, D_j P_l(x_0)\, D_i P_m(x_0) \\ &\quad + \; D_l \nu_{\Sigma_k}(P(x_0))\, D_i D_j P_l(x_0) \\ &= \frac{D_i D_j \nu_{\Sigma_k}(P(x_0))}{(1-\kappa_j d)(1-\kappa_i d)} - \kappa_k D_i D_j P_k(x_0) && \text{by (3.13).} \end{aligned} \tag{3.14}$$

We now compute explicitly the two derivative terms on the right-hand side of (3.14). Observing (3.7) and that

$$h_{ij}^\Sigma(z) = \frac{D_{ij}\phi(\overline{z})}{\sqrt{1 + |D\phi(\overline{z})|^2}}$$

for any $z \in \Sigma \cap N(y_0)$, we have

$$D_k h_{ij}^\Sigma(P(x_0)) = D_{ijk}\phi\left(\overline{P(x_0)}\right).$$

Thus, noting (3.9) and (3.7) again, we have

$$\begin{aligned} D_i D_j \nu_{\Sigma_k}\left(P\left(x_0\right)\right) &= -D_i D_j D_k \phi\left(\overline{P\left(x_0\right)}\right) \\ &= -D_k h_{ij}^{\Sigma}\left(\overline{P\left(x_0\right)}\right). \end{aligned}$$

We also compute for $1 \leqslant i, j, k \leqslant n$,

$$\begin{aligned} &P_k(x) = x_k - dD_k d, \\ &D_i P_k(x) = \delta_k^i - dD_{ik} d - D_i d D_k d, \\ &D_i D_j P_k(x) = -dD_{ijk} d - \left(D_i d D_{jk} d + D_j d D_{ik} d + D_k d D_{ij} d\right). \end{aligned} \tag{3.15}$$

Since $Dd(x_0) = e_{n+1}$, evaluation of (3.15) at x_0 gives

$$D_i D_j P_k(x_0) = -dD_{ijl} d(x_0)$$

and so from (3.14) we have

$$D_{ijk} d(x_0) = \frac{-D_k h_{ij}^{\Sigma}\left(\overline{P\left(x_0\right)}\right)}{(1-\kappa_j d)(1-\kappa_i d)} + \kappa_k d D_{ijk} d.$$

Transposing to solve for $D_{ijk} d$ yields

$$D_{ijk} d(x_0) = \frac{-D_k h_{ij}^{\Sigma}\left(\overline{P\left(x_0\right)}\right)}{(1-\kappa_i d)(1-\kappa_j d)(1-\kappa_k d)}.$$

□

Convention 3.1.5.
Henceforth, we will suppress all indication of the basepoint x and write simply, for example, $D^2 d$ for the Hessian matrix of d evaluated at x, and denote by $D^2 d(x)$ the linear map $D^2 d$ applied to the position vector x. Also, we adopt the convention that ν_Σ denotes the normal to Σ at the nearest point projection of x onto Σ, wherever it is well-defined.

3.2 Reflections

The following concept concerning the reflection of points across the support surface Σ is inspired by the work of Grüter and Jost [17] on the corresponding free-boundary problem for stationary varifold solutions of (1.4).

Definition 3.2.1 (Tilde-reflection of x across Σ, r). *For any point $x \in \Sigma_{1/\kappa_\Sigma}$ we define*

$$\tilde{x} := x - 2\left(\langle x, \nu_\Sigma\rangle - d_\Sigma\right)\nu_\Sigma$$

to be the tilde-reflection *of x across Σ, where ν_Σ is the exterior unit normal to Σ* with respect to the embedding F *(as per introduction), and set*

$$r := |x|^2 + |\widetilde{x}|^2.$$

Furthermore, for any $x_0 \in \mathbb{R}^{n+1}$ we define the translates

$$r_{x_0} := |x - x_0|^2 + \left|\widetilde{x - x_0}\right|^2.$$

Remark 3.2.2.
1. By construction, we note that $r_{x_0} \equiv 0$ iff $x \equiv x_0$.
2. In view of Proposition 3.1.3 and the definition of the signed distance function, we note that we also have

$$\widetilde{x} = x - 2\left(\langle x, Dd\rangle - d\right) Dd. \tag{3.16}$$

3. Using the fact that $|Dd| = 1$, we have

$$|\widetilde{x}|^2 = |x - 2dDd|^2$$

and so we see that the tilde-reflection has the same length as the standard reflection of x across Σ (and, also, that $|\widetilde{x}|^2 = |x|^2$ for $x \in \Sigma$). However, even though we will predominantly be concerned with the function r, the reason for the introduction of $\widetilde{x}$ is because, on Σ, we have

$$\begin{aligned}
\langle x + \widetilde{x}, \nu_\Sigma\rangle &= 2\left\langle x - \langle x, Dd\rangle Dd, \nu_\Sigma\right\rangle && \text{since } d = 0 \text{ on } \Sigma,\\
&= 2\left(\langle x, \nu_\Sigma\rangle - \langle x, Dd\rangle \langle Dd, \nu_\Sigma\rangle\right)\\
&= 2\left(\langle x, \nu_\Sigma\rangle - \langle x, \nu_\Sigma\rangle\right) && \text{by Remark 3.1.4}\\
&= 0.
\end{aligned}$$

The same result is not true if we replace $\widetilde{x}$ by the standard reflection $\overline{x} = x - 2dDd$. Thus, particularly when we are analyzing boundary integrals that arise from the Divergence Theorem, it will be convenient to know whether the integrand decomposes into an inner product featuring the vector $x + \widetilde{x}$.
4. In the special case of planar support surfaces Σ, we have $\langle x, \nu_\Sigma\rangle = d$ (when $0 \in \Sigma$) and so

$$\widetilde{x} = x \tag{3.17}$$

and

$$r = 2|x|^2. \tag{3.18}$$

The following lemma contains estimates that will be useful in computing the Q- and heat-operator of functions depending on r in the following chapter.

Lemma 3.2.3 (Derivative Estimates for r). *For any $x \in \Sigma_{1/\kappa_\Sigma}$ we have the following estimates:*

1. $|Dr|^2 \leqslant 8r + \frac{32|x|^3\kappa_\Sigma}{1-d\kappa_\Sigma} + \frac{16|x|^4\kappa_\Sigma^2}{(1-d\kappa_\Sigma)^2}$,

2. $|\mathrm{div}_{M_t} Dr - 4n| \leqslant \frac{20n\kappa_\Sigma |x|}{1-d\kappa_\Sigma} + \frac{4n\kappa_\Sigma^2 |x|^2}{(1-d\kappa_\Sigma)^3}$

3. $\langle Dr, \nu_\Sigma \rangle = 0 \qquad$ *for all* $x \in \Sigma$.

Proof. 1. Using $|\widetilde{x}|^2 = |x - 2dDd|^2$, we firstly compute

$$\begin{aligned} Dr &= 2x + D\,|x - 2dDd|^2 \\ &= 2x + \left(Id - 2dD^2d - 2Dd \otimes Dd\right)(2x - 4dDd) \\ &= 2x + 2x - 4dDd - 4dD^2d\,(x) - 4\,\langle x, Dd \rangle\, Dd \\ &= 2x + 2\widetilde{x} - 4dD^2d\,(x)\,. \end{aligned} \tag{3.19}$$

Here we recall the adopted convention that D^2d denotes the Hessian matrix of d evaluated at the point x whereas $D^2d\,(x)$ denotes the linear map D^2d at x applied to the position vector x.
Hence, observing that for any vectorfield Z defined over Σ_{1/κ_Σ} we have

$$D^2d\,(Z, \nabla d) = -\,\langle Dd, \nu \rangle\, D^2d\,(Z, \nu)\,, \tag{3.20}$$

(where $\nabla d = Dd - \langle Dd, \nu \rangle\, \nu$ denotes the component of Dd tangential to M_t at x), which follows directly from part 2 of proposition 3.1.3, we obtain

$$\begin{aligned} |Dr|^2 &= \left|2x + 2\widetilde{x} - 4dD^2d\,(x)\right|^2 \\ &= 4\,|x + \widetilde{x}|^2 - 32dD^2d\,(x, x) + 16d^2\left|D^2d\,(x)\right|^2. \end{aligned} \tag{3.21}$$

Young's inequality gives

$$\begin{aligned} |x + \widetilde{x}|^2 &= r + 2\,\langle x, \widetilde{x} \rangle \\ &\leqslant 2r, \end{aligned} \tag{3.22}$$

and Cauchy's inequality, in conjunction with (3.2), gives

$$\begin{aligned} D^2d\,(x, x) &\leqslant \|D^2d\|\,|x|^2 \\ &\leqslant \frac{\kappa_\Sigma\,|x|^2}{1 - d\kappa_\Sigma} \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} \left|D^2d\,(x)\right|^2 &\leqslant \|D^2d\|^2\,|x|^2 \\ &\leqslant \frac{\kappa_\Sigma^2\,|x|^2}{(1 - d\kappa_\Sigma)^2}. \end{aligned} \tag{3.24}$$

Hence, by (3.21), the estimates (3.22)-(3.24) and the fact that $d \leqslant |x|$ (since $0 \in \Sigma$, by assumption), we have

$$|Dr|^2 \leqslant 8r + \frac{32x^3\kappa_\Sigma}{1 - d\kappa_\Sigma} + \frac{16\,|x|^4\,\kappa_\Sigma^2}{(1 - d\kappa_\Sigma)^2}.$$

2. From (3.19), noting again (3.20), we have

$$\begin{aligned}\mathrm{div}_{M_t} Dr = 2n + 2\mathrm{div}_{M_t}\widetilde{x} - 4d\mathrm{div}_{M_t}\left(D^2 d(x)\right) \\ + 4\langle Dd, \nu\rangle D^2 d(x, \nu).\end{aligned} \tag{3.25}$$

To compute the second term on the right we firstly note that

$$D(\langle x, Dd\rangle - d) = D^2 d(x),$$

and so, from (3.16) and (3.20), a straightforward calculation yields

$$\mathrm{div}_{M_t}\widetilde{x} = n - 2(\langle x, Dd\rangle - d)\,\mathrm{div}_{M_t} Dd + 2\langle Dd, \nu\rangle D^2 d(x, \nu).$$

Hence, from (3.25) we have

$$\begin{aligned}|\mathrm{div}_{M_t} Dr - 4n| \leqslant 4|(\langle x, Dd\rangle - d)\,\mathrm{div}_{M_t} Dd| + 8\left|D^2 d(x, \nu)\right| \\ + 4d\left|\mathrm{div}_{M_t}\left(D^2 d(x)\right)\right|.\end{aligned} \tag{3.26}$$

Introducing an orthonormal basis $\tau_1, \dots \tau_n$ for $T_x M_t$, we have

$$\begin{aligned}\mathrm{div}_{M_t} Dd &= \sum_{i=1}^{n} D^2 d(\tau_i, \tau_j) \\ &\leqslant n\|D^2 d\|,\end{aligned}$$

and so by (3.2) and the estimate $|\langle x, Dd\rangle - d| \leqslant 2|x|$, we obtain

$$|(\langle x, Dd\rangle - d)\,\mathrm{div}_{M_t} Dd| \leqslant \frac{2n\kappa_\Sigma |x|}{1 - d\kappa_\Sigma} \tag{3.27}$$

and also

$$\left|D^2 d(x, \nu)\right| \leqslant \frac{\kappa_\Sigma |x|}{1 - d\kappa_\Sigma}, \tag{3.28}$$

where the omittance of magnitude parenthesis on the denominators is permitted, since we are working over Σ_{1/κ_Σ}.
The remaining term on the right of (3.25) is estimated similarly: using (3.3), we have

$$\begin{aligned}\mathrm{div}_{M_t}\left(D^2 d(x)\right) &= \sum_{i=1}^{n}\left\langle D_{\tau_i}\left(D^2 d(x)\right), \tau_i\right\rangle \\ &= \sum_{i=1}^{n}\left(D^3 d(x, \tau_i, \tau_i) + D^2 d(\tau_i, \tau_i)\right) \\ &\leqslant n\|D^3 d(x)\| + n\|D^2 d\| \\ &\leqslant \frac{n\kappa_\Sigma^2 |x|}{(1 - d\kappa_\Sigma)^3} + \frac{n\kappa_\Sigma}{1 - d\kappa_\Sigma}.\end{aligned} \tag{3.29}$$

Using the estimates (3.27)-(3.29) with $d \leqslant |x|$ in (3.26) then gives the desired estimate.
3. Since $d(x) = 0$ on Σ, we have

$$\widetilde{x} = x - 2\langle x, Dd\rangle Dd$$

and so, by (3.19),

$$Dr = 4x - 4\langle x, Dd\rangle Dd$$

for all $x \in \Sigma$. Hence, for all $x \in \Sigma$,

$$\begin{aligned} \langle Dr, \nu_\Sigma\rangle &= 4\langle x - \langle x, \nu_\Sigma\rangle \nu_\Sigma, \nu_\Sigma\rangle \qquad \text{(see part 3 of Remark 3.2.2)} \\ &= 0. \end{aligned}$$

□

Chapter 4

The Monotonicity Formula

In this section we prove a general monotonicity formula for hypersurfaces evolving by mean curvature flow with free boundary on some given, smooth support surface. As previously stated, in light of the recent discovery of a local monotonicity formula by Ecker [6], which is valid over regions not containing boundary points, the formula obtained is specifically designed for centering on points contained within the support surface Σ.

We begin by introducing and proving crucial estimates for the two functions that, together, form the basis of this result - a localization function and a modified version of the standard backward heat kernel on $\mathbb{R}^{n+1}$.

In light of Lemma 3.2.3, to nullify the boundary integrals that arise from the Divergence Theorem in the expansion formula (2.8), both of these functions depend spatially on the function r of the previous section. The first function is thus required, essentially, to localize the result to a tubular neighbourhood of Σ within which all quantities involving the distance function are well-defined. As well as depending spatially on r, the modified heat kernel also takes into consideration the curvature of the support surface and exhibits similar characteristic behaviour to the standard heat kernel.

4.1 The Cut-off Function

A fundamental property required of the cut-off function is that it be compactly supported over the region

$$\Sigma_{1/\kappa_\Sigma} \equiv \left\{x \in \mathbb{R}^{n+1} : d(x) < 1/\kappa_\Sigma\right\}.$$

Importantly, also, in view of the expansion formula (2.8) and Proposition 3.2.3, is that it should depend spatially on the function r and, ideally, be a sub-solution of the intrinsic heat equation on M_t. In addition to possessing these crucial properties, the following function has compact support over a region which shrinks *slowly* in time as the critical time T is approached. The reason for this latter feature will become apparent in the following section, where we estimate the Q-operator of the modified backward heat kernel over this function's support.

Proposition 4.1.1. *For any $x_0 \in \Sigma$, $\kappa_\Sigma \geqslant 0$ and $\delta \in \left(0, \frac{2}{5}\right]$ let*

$$\eta_{x_0}(x,t) = \left(1 - \left(\frac{2\kappa_\Sigma}{\left(\kappa_\Sigma^2 \tau\right)^\delta}\right)^2 (r_{x_0} - 40n\tau)\right)_+^4, \tag{4.1}$$

where $r_{x_0} = |x - x_0|^2 + \left|\widetilde{x - x_0}\right|^2$ and $\tau = T - t$, and set $\tau_0 := \frac{\left(\frac{3}{160n}\right)^{2/\delta}}{\kappa_\Sigma^2}$. Then for each $t \in [T - \tau_0, T)$ (or, equivalently, $\tau \in (0, \tau_0]$) we have

$$\eta_{x_0} \leqslant 256, \tag{4.2}$$

$$\operatorname{spt} \eta_{x_0} \subset \left\{x \in \mathbb{R}^{n+1} : |x - x_0| \kappa_\Sigma \leqslant \left(\kappa_\Sigma^2 \tau\right)^\delta\right\}, \tag{4.3}$$

$$\frac{1}{1 - d\kappa_\Sigma} \leqslant 2 \quad on \quad \operatorname{spt} \eta_{x_0}, \tag{4.4}$$

$$\left(\frac{d}{dt} - \Delta_{M_t}\right) \eta_{x_0} \leqslant 0, \tag{4.5}$$

$$\operatorname{spt} \eta_{x_0} \longrightarrow \mathbb{R}^{n+1} \quad as \quad \kappa_\Sigma \longrightarrow 0. \tag{4.6}$$

Additionally, for each $t \in [0, T)$ we have

$$\eta \longrightarrow 1 \quad as \quad \kappa_\Sigma \longrightarrow 0. \tag{4.7}$$

Remark 4.1.2. Here the subscript "+" refers to the positive part of the enclosed function and the power 4 is to ensure that η is of class C^2 (any power $p > 2$ would actually suffice).

Proof. Since for all $\tau \leqslant \tau_0$ and $\delta \leqslant \frac{2}{5}$ we have $\left(\kappa_\Sigma^2 \tau\right)^{1-2\delta} \leqslant \frac{3}{160n}$, we estimate

$$\begin{aligned}
\eta_{x_0} &\leqslant \left(1 + 160n \left(\kappa_\Sigma^2 \tau\right)^{1-2\delta}\right)^4 \\
&\leqslant (1+3)^4 \\
&= 256.
\end{aligned}$$

To determine the support of η_{x_0} it is instructive to set $Z = \left(\frac{2\kappa_\Sigma}{\left(\kappa_\Sigma^2 \tau\right)^\delta}\right)^2 (r_{x_0} - 40n\tau)$, so that $\eta_{x_0} = (1 - Z)_+^4$, and then estimate, for each $\tau \in (0, \tau_0]$,

$$\begin{aligned}
\operatorname{spt} \eta_{x_0} &= \left\{x \in \mathbb{R}^{n+1} : Z \leqslant 1\right\} \\
&\subset \left\{x \in \mathbb{R}^{n+1} : |x - x_0|^2 \leqslant \left(\frac{\left(\kappa_\Sigma^2 \tau\right)^\delta}{2\kappa_\Sigma}\right)^2 + 40n\tau\right\} \\
&= \left\{x \in \mathbb{R}^{n+1} : |x - x_0|^2 \leqslant \left(\frac{\left(\kappa_\Sigma^2 \tau\right)^\delta}{\kappa_\Sigma}\right)^2 \left(\frac{1}{4} + 40n \left(\kappa_\Sigma^2 \tau\right)^{1-2\delta}\right)\right\}.
\end{aligned}$$

As above, we note that $\left(\kappa_\Sigma^2\tau\right)^{1-2\delta} \leqslant \left(\frac{3}{160n}\right)$ for all $\delta \leqslant \frac{2}{5}$ and $\tau \leqslant \tau_0$, and thus

$$\operatorname{spt}\eta_{x_0} \subset \left\{x \in \mathbb{R}^{n+1} : |x - x_0|\,\kappa_\Sigma \leqslant \left(\kappa_\Sigma^2\tau\right)^\delta\right\}.$$

Since $x_0 \in \Sigma$ (which implies $d \leqslant |x - x_0|$), the above result implies that for all $\tau \leqslant \tau_0$, over the support of η_{x_0} we have

$$1 - d\kappa_\Sigma \geqslant 1 - |x - x_0|\,\kappa_\Sigma \geqslant 1 - \left(\kappa_\Sigma^2\tau\right)^\delta \geqslant 1 - \left(\frac{3}{160n}\right)^2,$$

from which the looser, more aesthetically pleasing bound (4.4) follows.

To show the intrinsic heat operator acting on η_{x_0} is non-positive, we proceed as follows:

$$\begin{aligned}
\left(\frac{d}{dt} - \Delta_{M_t}\right)\eta_{x_0} &= \eta'_{x_0}\left(\frac{d}{dt} - \Delta_{M_t}\right)Z - \eta''_{x_0}|\nabla Z|^2 && \text{by (A.4)}\\
&= \eta'_{x_0}\left(\frac{\partial}{\partial t} - \operatorname{div}_{M_t} D\right)Z - \eta''_{x_0}|\nabla Z|^2 && \text{by (A.2)}\\
&= -\eta'_{x_0}\left(\frac{\partial}{\partial\tau} + \operatorname{div}_{M_t} D\right)Z && \text{since } \eta''_{x_0} \geqslant 0\\
&= -\eta'_{x_0}\left(\frac{2\kappa_\Sigma}{\left(\kappa_\Sigma^2\tau\right)^\delta}\right)^2\left(40n(2\delta - 1) - \frac{2\delta r_{x_0}}{\tau} + \operatorname{div}_{M_t} Dr_{x_0}\right)\\
&\leqslant -\eta'_{x_0}\left(\frac{2\kappa_\Sigma}{\left(\kappa_\Sigma^2\tau\right)^\delta}\right)^2\left[-4n + 40n\kappa_\Sigma|x - x_0| + 32n\kappa_\Sigma^2|x - x_0|^2\right],
\end{aligned}$$

noting $\eta'_{x_0} \leq 0$, $\delta \leqslant \frac{2}{5}$, (4.4) and Lemma 3.2.3.

Hence, since $|x - x_0|\,\kappa_\Sigma \leqslant \left(\kappa_\Sigma^2\tau\right)^\delta \leqslant \left(\frac{3}{160n}\right)^2$ for all $\tau \leqslant \tau_0$ over the support of η_{x_0}, we have

$$\begin{aligned}
\left(\frac{d}{dt} - \Delta_{M_t}\right)\eta_{x_0} &\leqslant 3n\eta'_{x_0}\left(\frac{2\kappa_\Sigma}{\left(\kappa_\Sigma^2\tau\right)^\delta}\right)^2\\
&\leqslant 0.
\end{aligned}$$

The final claims regarding behaviour as $\kappa_\Sigma \longrightarrow 0$ follow trivially. ☐

Recall from Remark 3.2.2 that $\widetilde{x}$ agrees identically with x when $\kappa_\Sigma = 0$. By working over the support of the above localization function we can extend this result and show that $\widetilde{x}$ actually converges uniformly to x over the support of the above localization function, as κ_Σ tends to zero. This will become important in chapter 5 when we investigate the convergence of rescaled solutions.

Corollary 4.1.3. *Let $x_0 \in \Sigma$. Then for any $x \in \operatorname{spt}\eta_{x_0}$ and each $t \in [0, T)$ we have*

$$\widetilde{x} \longrightarrow x \qquad \text{as} \qquad \kappa_\Sigma \longrightarrow 0. \tag{4.8}$$

Proof. Without loss of generality we take $x_0 = 0$ and write $\eta_0 \equiv \eta$.
By (4.3), for all $t \in \left[T - \frac{\left(\frac{1}{100n}\right)^{2/\delta}}{\kappa_\Sigma^2}, T\right)$ we have $\operatorname{spt} \eta \subset \Sigma_{1/\kappa_\Sigma}$ and so, for any $x \in \operatorname{spt} \eta$ and these times, we can compute

$$D\left(\langle x, Dd\rangle - d\right) = D^2 d\left(x\right). \tag{4.9}$$

Hence, by (3.16) and the mean value theorem, we may estimate

$$\begin{aligned} |x - \widetilde{x}| &= 2\left|\langle x, Dd\rangle - d\right| \\ &\leqslant 2\|D^2 d\|\,|x| \\ &\leqslant 4\kappa_\Sigma\,|x|, \end{aligned} \tag{4.10}$$

using (3.2) and (4.4), where the Hessian matrix of d on the second line is evaluated at some intermediate point.
For sufficiently small κ_Σ this estimate will hold for all $t \in [0, T)$, and for these times we have $\tau^\delta \leqslant (1+T)^\delta =: C_1(T)$. Equation (4.3) then gives

$$\operatorname{spt} \eta \subset \left\{x \in \mathbb{R}^{n+1} : |x| \leqslant C_1 \kappa_\Sigma^{2\delta - 1}\right\}, \tag{4.11}$$

and so, by (4.10), for each $t \in [0, T)$, over the support of η we have

$$|x - \widetilde{x}| \leqslant C_2 \kappa_\Sigma^{2\delta},$$

and the result follows. □

4.2 The Modified Backward Heat Kernel

For the weighting function in our monotonicity formula, we seek a function based on the standard backward heat kernel that is dependent spatially upon the function r and which is perturbed appropriately to take into consideration the curvature of the support surface. Though such functions will always ensure that the corresponding boundary integral in the expansion formula of Proposition 2.2.1 is nullified, the term of (2.8) involving the Q-operator of these functions will, in general, no longer vanish. Any modified version of the standard backward heat kernel should exhibit the same characteristic behaviour as $\rho_{x_0 T}$ and concentrate at the point about which it is centred as the singular time of M_t is approached. Also, for reasons concerning the rescaling analysis (see Chapter 5), it should have time-scaling appropriate to $\mathbb{R}^n$ and become singular at a rate of $(T-t)^{-n/2}$.
Taking these factors into consideration, we introduce the following family of perturbed heat kernels as candidates to play the role of the weighting function in our monotonicity formula.

Definition 4.2.1. *For $x \in \mathbb{R}^{n+1}$, $t < 0$ and $a \geqslant 0$ we define the class of* ***perturbed backward heat kernels*** *$\varrho_a : \mathbb{R}^{n+1} \times (-\infty, 0] \longrightarrow \mathbb{R}$ by*

$$\varrho_a(x, t) := \frac{1}{(-4\pi t)^{n/2}} \exp\left(\frac{r}{8(a+1)t}\right), \tag{4.12}$$

where $r = |x|^2 + |\widetilde{x}|^2$.
Furthermore, for any $x_0 \in \mathbb{R}^{n+1}$ *and* $t < T$, *we define also the translates*

$$\varrho_{a,x_0,T}(x,t) := \frac{1}{(4\pi(T-t))^{n/2}} \exp\left(-\frac{r_{x_0}}{8(a+1)(T-t)}\right). \tag{4.13}$$

Remark 4.2.2.
1. Note that, in view of (3.18), we have $\varrho_0(x,t) \equiv \rho(x,t)$ in the case of a planar support surface.
2. Since ϱ_a and its translates are defined spatially in terms of r, part 3 of Lemma 3.2.3 implies that

$$\int_{\partial M_t} \langle D\varrho_a, \nu_\Sigma \rangle \equiv 0. \tag{4.14}$$

For fixed a, we have $Q(\varrho_a) \sim \frac{\varrho_a}{\tau}$ (see proof of Lemma 4.2.5, below) and so this term becomes infinite as the singular time is approached, with the factor $\frac{1}{\tau}$ not even being time-integrable. If this factor were integrable in time, it would allow, by the introduction of an integrating factor in (2.8), a form of monotonicity formula to be obtained (see following section).
By taking a to be a time-dependent function depending also on the curvature of the support surface - the latter notion naturally suggested by the first remark above - and working over the support of the localization function η, we can obtain a bound for $Q(\varrho)$ in terms of an integrable function of time. This, as we shall see, suffices to yield a monotonicity formula with which we can proceed with our rescaling analysis.

Definition 4.2.3 (Modified Backward Heat Kernel). *For* $\kappa_\Sigma \geqslant 0$ *and any* $\delta > 0$ *we define the* ***modified*** *backward heat kernel* $\rho_{\kappa_\Sigma} : \mathbb{R}^{n+1} \times (-\infty, 0] \longrightarrow \mathbb{R}$ *by* $\rho_{\kappa_\Sigma}(x,t) \equiv \varrho_{16(-t\kappa_\Sigma^2)^\delta}(x,t)$. *That is,*

$$\rho_{\kappa_\Sigma}(x,t) := \frac{1}{(-4\pi t)^{n/2}} \exp\left(\frac{r}{8\left(16(-t\kappa_\Sigma^2)^\delta + 1\right)t}\right), \tag{4.15}$$

where $r = |x|^2 + |\widetilde{x}|^2$.
Furthermore, for any $x_0 \in \mathbb{R}^{n+1}$ *and* $t < T$ *we define the translates*

$$\rho_{\kappa_\Sigma,x_0,T}(x,t) := \frac{1}{(4\pi\tau)^{n/2}} \exp\left(-\frac{r_{x_0}}{8\left(16(\kappa_\Sigma^2\tau)^\delta + 1\right)\tau}\right), \tag{4.16}$$

where $\tau = T - t$.

Remark 4.2.4. Note that, for all $\tau \in (0, \tau_0]$, we have

$$\rho_{\kappa_\Sigma,x_0,T}(x,t) \leqslant \frac{1}{(4\pi\tau)^{n/2}} \exp\left(-\frac{|x-x_0|^2}{C\tau}\right), \tag{4.17}$$

where $C = 8\left(16\left(\frac{3}{160n}\right)^2 + 1\right)$. That is, independent of the parameter κ_Σ, each function ρ_{κ_Σ} is bounded by an integrable function.

We then have the following important estimate.

Lemma 4.2.5. *Let $x_0 \in \Sigma$ and η_{x_0} be as in Proposition 4.1.1, with $\kappa_\Sigma \geqslant 0$ and $\delta \in \left(0, \frac{2}{5}\right]$. Then for all $\tau \in (0, \tau_0]$, over the support of η_{x_0}, we have*

$$Q\left(\rho_{\kappa_\Sigma, x_0, T}\right) \leqslant \widetilde{C} \rho_{\kappa_\Sigma, x_0, T} \kappa_\Sigma^{2\delta} \tau^{\delta - 1} \tag{4.18}$$

where $\widetilde{C} = \widetilde{C}(n)$.

Proof. Without loss of generality, we take $x_0 = 0$.
We show that the modified heat kernel (4.16) arises naturally from the broader class of *perturbed* heat kernels (4.13) as follows: from the definition of $\varrho_{a,0,T} \equiv \varrho$ and the operator Q we compute, for any $a = a(\tau)$,

$$\begin{aligned} Q(\varrho) &= \frac{\partial \varrho}{\partial t} + \operatorname{div}_{M_t} D\varrho + \frac{\left|D^\perp \varrho\right|^2}{\varrho} \\ &= \varrho \left(\frac{n}{2\tau} - \frac{a' r}{8(a+1)^2 \tau} - \frac{r}{8(a+1)\tau^2} - \frac{\operatorname{div}_{M_t} Dr}{8(a+1)\tau} + \frac{|Dr|^2}{(8(a+1)\tau)^2} \right). \end{aligned}$$

Working over the support of η_{x_0} and using results 1 and 2 of Lemma 3.2.3 and (4.4), one may further estimate

$$\begin{aligned} Q(\varrho) \leqslant \varrho \Bigg[& \frac{an}{2(a+1)\tau} - \frac{a' r}{8(a+1)^2 \tau} + \frac{1}{2(a+1)\tau} \left(10 n \kappa_\Sigma |x| + 8 n \kappa_\Sigma^2 |x|^2\right) \\ & + \frac{1}{(8(a+1)\tau)^2} \left(64 |x|^3 \kappa_\Sigma + 64 |x|^4 \kappa_\Sigma^2 - 8ar\right) \Bigg]. \end{aligned}$$

Setting $a(\tau) = c\left(\kappa_\Sigma^2 \tau\right)^\delta$, where $c > 0$ is to be chosen later, and noting $a'(\tau) \geqslant 0$, (4.3) and that $\left(\kappa_\Sigma^2 \tau\right)^\delta < 1$ for all $\tau \leqslant \tau_0$, then gives

$$Q(\varrho) \leqslant \frac{\varrho \left(\kappa_\Sigma^2 \tau\right)^\delta}{(a+1)\tau} \left[\frac{cn}{2} + 9n + \frac{|x|^2}{8(a+1)\tau} (16 - c) \right].$$

Hence, on choosing $c = 16$ we have

$$Q(\varrho) \leqslant \frac{17n \left(\kappa_\Sigma^2 \tau\right)^\delta \varrho}{\tau}$$

or, equivalently,

$$Q\left(\rho_{\kappa_\Sigma, 0, T}\right) \leqslant \frac{17n \left(\kappa_\Sigma^2 \tau\right)^\delta \rho_{\kappa_\Sigma, 0, T}}{\tau}.$$

□

By Remark 4.2.2, we have that $\rho_0 = \rho$. Additionally, we have the following statement, analogous to Corollary 4.1.3.

Corollary 4.2.6.
For any $x_0 \in \Sigma$, let η_{x_0} be as in Proposition 4.1.1 with $\delta \in \left(\frac{1}{3}, \frac{3}{8}\right]$. Then, for any $x \in \operatorname{spt} \eta_{x_0}$ and each $t \in [0, T)$ we have

$$\rho_{\kappa_\Sigma, x_0, T}(x, t) \longrightarrow \rho_{x_0, T}(x, t) \qquad \text{as} \qquad \kappa_\Sigma \longrightarrow 0. \tag{4.19}$$

Proof. Without loss of generality we take $x_0 = 0$. We then have

$$|\rho - \rho_{\kappa_\Sigma}| = \rho \left| 1 - \exp\left(\frac{|x|^2}{4\tau} - \frac{r}{8\left(16\left(\kappa_\Sigma^2 \tau\right)^\delta + 1\right)\tau} \right) \right|,$$

and so, for $\tau > 0$, the result follows provided we can show

$$\left| |x|^2 - \frac{r}{2\left(16\left(\kappa_\Sigma^2 \tau\right)^\delta + 1\right)} \right| \longrightarrow 0 \qquad \text{as} \qquad \kappa_\Sigma \longrightarrow 0. \tag{4.20}$$

To this end, writing $b := 16\left(\kappa_\Sigma^2 \tau\right)^\delta$ and estimating $|\widetilde{x}|^2 = |x - 2dDd|^2 \leqslant 5|x|^2$, we have

$$\begin{aligned} \left| |x|^2 - \frac{r}{2(b+1)} \right| &\leqslant \left| (2b+1)|x|^2 - |\widetilde{x}|^2 \right| \\ &\leqslant (2b+1)\left| |x|^2 - |\widetilde{x}|^2 \right| + 10b|x|^2. \end{aligned}$$

Recalling from (4.11) that $|x| \leqslant C\kappa_\Sigma^{2\delta-1}$ on spt η for each $t \in [0, T)$, and the definition of b, we may further estimate

$$\left| |x|^2 - \frac{r}{2(b+1)} \right| \leqslant \left(32C\kappa_\Sigma^{2\delta} + 1\right)\left| |x|^2 - |\widetilde{x}|^2 \right| + 160C\kappa_\Sigma^{6\delta-2}.$$

Since $\delta > \frac{1}{3}$, equation (4.20) then follows from Corollary 4.1.3. □

4.3 The Monotonicity Formula

Before we state the main monotonicity result, let us recall the following quantities from the previous two sections: for any $x_0 \in \Sigma$ we have

- **localization function** (centred at x_0) -

$$\eta_{x_0}(x, t) = \left(1 - \left(\frac{2\kappa_\Sigma}{\left(\kappa_\Sigma^2 \tau\right)^\delta} \right)^2 \left(r_{x_0} - 40n\tau\right) \right)_+^4$$

- **modified heat kernel** (centred at (x_0, T)) -

$$\rho_{\kappa_\Sigma, x_0, T}(x, t) = \frac{1}{(4\pi\tau)^{n/2}} \exp\left(-\frac{r_{x_0}}{8\left(16\left(\kappa_\Sigma^2 \tau\right)^\delta + 1\right)\tau} \right).$$

Here $r_{x_0} = |x - x_0|^2 + \left|\widetilde{x - x_0}\right|^2$, where the tilde denotes the special reflection introduced in § 3.2, $\tau = T - t$ and δ is a fixed but arbitrary constant, $\frac{1}{3} < \delta \leqslant \frac{2}{5}$.

We recall also that Σ is a smooth hypersurface in $\mathbb{R}^{n+1}$ that satisfies an interior/exterior rolling ball condition and whose curvature satisfies the bound

$$\|A_\Sigma\|^2 + \|\nabla A_\Sigma\| \leqslant \kappa_\Sigma^2 < \infty.$$

Theorem 4.3.1 (General Monotonicity Formula). *Let M_t be a family of hypersurfaces evolving by mean curvature flow with Neumann free-boundary on the hypersurface Σ for all $t \in [0,T)$, as in (1.4), and set $\tau_0 := \frac{\left(\frac{3}{160n}\right)^{2/\delta}}{\kappa_\Sigma^2}$, where $\kappa_\Sigma \geqslant 0$ bounds the curvature of Σ and $\delta \in \left(\frac{1}{3}, \frac{2}{5}\right]$. Then for all $t \in [T - \tau_0, T)$ and any $x_0 \in \Sigma$ we have*

$$\frac{d}{dt}\left(e^{C_8 \kappa_\Sigma^{2\delta} \tau^\delta} \int_{M_t} \eta \rho_{\kappa_\Sigma} \, d\mu_t\right) \leqslant -e^{C_8 \kappa_\Sigma^{2\delta} \tau^\delta} \int_{M_t} \left|\vec{H} - \frac{D^\perp \rho_{\kappa_\Sigma}}{\rho_{\kappa_\Sigma}}\right|^2 \eta \rho_{\kappa_\Sigma} \, d\mu_t, \tag{4.21}$$

where $\rho_{\kappa_\Sigma} \equiv \rho_{\kappa_\Sigma, x_0, T}$, $\eta \equiv \eta_{x_0}$ and C_8 is a positive constant depending only on n.

Proof. Taking $f = \eta$ and $g = \rho_{\kappa_\Sigma}$ in (2.8), noting (4.18), (4.5) and part 3 of Lemma 3.2.3, gives

$$\begin{aligned} \frac{d}{dt}\int_{M_t} \eta \rho_{\kappa_\Sigma} \, d\mu_t \leqslant & -\int_{M_t} \eta \left|\vec{H} - \frac{D^\perp \rho_{\kappa_\Sigma}}{\rho_{\kappa_\Sigma}}\right|^2 \rho_{\kappa_\Sigma} \, d\mu_t \\ & + \frac{17n\left(\kappa_\Sigma^2 (T - t)\right)^\delta}{(T - t)} \int_{M_t} \eta \rho_{\kappa_\Sigma} \, d\mu_t. \end{aligned} \tag{4.22}$$

Equation (4.21) then follows after introducing the integrating factor
$\exp\left(\frac{17n\kappa_\Sigma^{2\delta}(T-t)^\delta}{\delta}\right)$. □

Remark 4.3.2.

1. By Remarks 3.18 and 4.2.2 and (4.7), in the case that $\kappa_\Sigma = 0$ bounds the curvature of Σ – that is, Σ is a hyperplane – the above formula is valid for all $t \in [0, T)$ and is consistent with Huisken's (2.2).
2. Explicitly, the integrand on the right hand side of (4.21) is given by

$$\left|\vec{H} - \frac{D^\perp \rho_{\kappa_\Sigma}}{\rho_{\kappa_\Sigma}}\right|^2 = \left|\vec{H} - \frac{x^\perp + \tilde{x}^\perp - 4dD^2 d(x,\nu)\,\nu}{8\left(16\left(\kappa^2\tau\right)^\delta + 1\right)\tau}\right|^2. \tag{4.23}$$

3. The requirement that $\delta > \frac{1}{3}$, though not strictly necessary here, is included above for convenience in view of Corollary 4.2.6 and the following chapter.

Chapter 5

Classification of Possible Limit Surfaces

In this section we carry out a rescaling analysis of our evolving surfaces and use the monotonicity formula of the previous section to classify the limiting behaviour near singular points.

5.1 Parabolic Rescaling

Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a solution of the mean curvature flow equation (1.4) with Neumann boundary condition on support hypersurface Σ and define, for any $x \in M_t \cup \Sigma$ and any fixed point $x_0 \in \mathbb{R}^{n+1}$, the change of variables

$$(x,t) \mapsto (y,s)$$

by

$$x = \lambda y + x_0, \qquad t = \lambda^2 s + T \tag{5.1}$$

where $\lambda > 0$. This implies the equivalences

$$x \in M_t \Longleftrightarrow y \in \frac{1}{\lambda}\left(M_{\lambda^2 s+T} - x_0\right) \equiv M_s^{(x_0,T),\lambda}$$

and

$$x \in \Sigma \Longleftrightarrow y \in \frac{1}{\lambda}\left(\Sigma - x_0\right) \equiv \Sigma_\lambda^{x_0},$$

and it can be easily seen that, for each $\lambda > 0$ and all $s \in \left[-\frac{T}{\lambda^2}, 0\right)$, the surfaces $M_s^{(x_0,T),\lambda}$ evolve by (MCF) with Neumann boundary condition on support surface $\Sigma_\lambda^{x_0}$. Moreover, if Σ has curvature bounded by κ_Σ then $\Sigma_\lambda^{x_0}$ has curvature bounded by $\lambda\kappa_\Sigma$.
The following two propositions are immediate consequences of the definitions and convergence results of the previous chapter and the parabolic rescaling of the flow.

Proposition 5.1.1. *In the rescaled setting we have*

$$\eta_{x_0} \mapsto \hat{\eta} \qquad \textit{and} \qquad \rho_{\kappa_\Sigma,x_0,T} \mapsto \lambda^{-n}\hat{\rho}_{\lambda\kappa_\Sigma},$$

where

$$\hat{\eta}(y,s)=\left(1-\left(\frac{2(\lambda\kappa_\Sigma)^{1-2\delta}}{(-s)^{\delta}}\right)^2\left(|y|^2+|\widetilde{y}|^2+40ns\right)\right)_+^4 \tag{5.2}$$

and

$$\hat{\rho}_{\lambda\kappa_\Sigma}(y,s)=\frac{1}{(-4\pi s)^{n/2}}\,exp\left(\frac{|y|^2+|\widetilde{y}|^2}{8\left(1+16\left(-(\lambda\kappa_\Sigma)^2 s\right)^{\delta}\right)s}\right). \tag{5.3}$$

Here, $\widetilde{y}=y-2\left(\left\langle y,D\widetilde{d}\right\rangle-\widetilde{d}\right)D\widetilde{d}$, *where* $\widetilde{d}(y):=\text{signed dist}\left(y,\Sigma^{x_0}_{\lambda}\right)$. *Furthermore, the induced measures,* $d\mu_t(x)$ *and* $d\mu_s(y)$*, of the surfaces* M_t *and* $M_s^{(x_0,T),\lambda}$*, respectively, are related by*

$$d\mu_t(x)=\lambda^n\,d\mu_s(y). \tag{5.4}$$

Proposition 5.1.2. *Let* $\kappa_\Sigma\geqslant 0$ *and* $\delta\in\left(\frac{1}{3},\frac{2}{5}\right]$*, and let* $\hat{\eta}$*,* $\widetilde{y}$ *and* $\hat{\rho}_{\lambda\kappa_\Sigma}$ *be as above. Then for each* $s\in\left[\frac{-T_0}{\lambda^2},0\right)$ *and any* $y\in\text{spt}\,\hat{\eta}$ *we have*

$$\widetilde{y}\longrightarrow y, \tag{5.5}$$

$$\hat{\rho}_{\lambda\kappa_\Sigma}(y,s)\longrightarrow\rho(y,s) \tag{5.6}$$

as $\lambda\longrightarrow 0$*. Furthermore, for each fixed* $s<0$ *we have*

$$\text{spt}\,\hat{\eta}\longrightarrow\mathbb{R}^{n+1}\qquad\text{as}\qquad\lambda\longrightarrow 0 \tag{5.7}$$

and

$$\hat{\eta}\longrightarrow\quad 1\qquad\text{as}\qquad\lambda\longrightarrow 0. \tag{5.8}$$

We then have the following re-formulation of Theorem 4.3.1:

Theorem 5.1.3 (Rescaled Monotonicty Formula). *For any* $\lambda>0$ *and* $x_0\in\Sigma$*, let* $M_s^{(x_0,T),\lambda}$ *and* $\Sigma^{x_0}_{\lambda}$ *be as defined above. Then for all* $s\in\left[-\frac{T_0}{\lambda^2},0\right)$ *we have*

$$\frac{d}{ds}\left(e^{C\left(-(\lambda\kappa_\Sigma)^2 s\right)^{\delta}}\int_{M_s^{(x_0,T),\lambda}}\hat{\eta}\hat{\rho}_{\lambda\kappa_\Sigma}\,d\mu_s\right)\leqslant$$
$$-\,e^{C\left(-(\lambda\kappa_\Sigma)^2 s\right)^{\delta}}\int_{M_s^{(x_0,T),\lambda}}\left|\vec{H}-\frac{D^{\perp}\hat{\rho}_{\lambda\kappa_\Sigma}}{\hat{\rho}_{\lambda\kappa_\Sigma}}\right|^2\hat{\eta}\hat{\rho}_{\lambda\kappa_\Sigma}\,d\mu_s, \tag{5.9}$$

where $\vec{H}$ *is the mean curvature vector of the surfaces* $M_s^{(x_0,T),\lambda}$ *and* $C=C(n)$*.*

Proof. We check the scaling behaviour of each of the three terms in (4.22); from (5.1) we have

$$\frac{d}{dt} = \frac{1}{\lambda^2}\frac{d}{ds}$$

and so, by Proposition 5.1.1, we have

$$\frac{d}{dt}\int_{M_t} \eta\rho_{\kappa_\Sigma}\, d\mu_t = \frac{1}{\lambda^2}\frac{d}{ds}\int_{M_s^{(x_0,T),\lambda}} \hat{\eta}\hat{\rho}_{\lambda\kappa_\Sigma}\, d\mu_s.$$

For the second term, we firstly note

$$\vec{H} = \frac{\vec{\hat{H}}}{\lambda}.$$

and so, by Proposition 5.1.1,

$$\left|\vec{H} - \frac{D^\perp\rho_{\kappa_\Sigma}}{\rho_{\kappa_\Sigma}}\right|^2 = \frac{1}{\lambda^2}\left|\vec{\hat{H}} - \frac{D^\perp\hat{\rho}_{\lambda\kappa_\Sigma}}{\hat{\rho}_{\lambda\kappa_\Sigma}}\right|^2$$

and

$$\int_{M_t}\left|\vec{H} - \frac{D^\perp\rho_{\kappa_\Sigma}}{\rho_{\kappa_\Sigma}}\right|^2 \eta\rho_{\kappa_\Sigma}\, d\mu_t = \frac{1}{\lambda^2}\int_{M_s^{(x_0,T),\lambda}}\left|\vec{\hat{H}} - \frac{D^\perp\hat{\rho}_{\kappa_\Sigma}}{\hat{\rho}_{\kappa_\Sigma}}\right|^2 \hat{\eta}\hat{\rho}_{\kappa_\Sigma}\, d\mu_s.$$

Finally, for the third term we have

$$\frac{17n\left(\kappa_\Sigma^2\,(T-t)\right)^\delta}{(T-t)}\int_{M_t}\eta\rho_{\kappa_\Sigma}\, d\mu_t = -\frac{17n\left(-\left(\lambda\kappa_\Sigma\right)^2 s\right)^\delta}{\lambda^2 s}\int_{M_s^{(x_0,T),\lambda}}\hat{\eta}\hat{\rho}_{\kappa_\Sigma}\, d\mu_s.$$

Equation (5.9) then follows after cancellation of λ^2 throughout each of the rescaled quantities above and introducing the appropriate integrating factor. □

5.2 Limit Surfaces

We now use the above rescaled monotonicity formula to characterize the limiting behaviour of the evolving hypersurfaces as the singular time is approached.
To this end, we consider the sequence of rescaled solutions $\left(M_s^{(x_0,T),\lambda_j}\right)$, for some sequence $\lambda_j \searrow 0$. In general, such solutions will not converge smoothly to a solution of (MCF) as $j \longrightarrow \infty$ (corresponding to $t \nearrow T$) but, rather, measure-theoretically to a generalized solution in the sense of Brakke's weak formulation of the flow (see [22]). However, as in § 2.1, we can ensure the existence of a smooth, nontrivial limiting solution of (1.4) by imposing the *Type 1* curvature assumption

$$|A\,(\cdot,t)|^2 \leqslant \frac{C_0}{T-t} \tag{5.10}$$

on the surfaces M_t, for all $t \in [0,T)$ and some constant $C_0 > 0$.

The question then becomes one of which point to rescale about, as, in general, parabolically rescaling the evolving surfaces about an arbitrary point $x_0 \in \Sigma$ will cause the resulting surface $M_s^{(x_0,T),\lambda}$ to drift off to infinity as $\lambda_j \searrow 0$. This prompts the following definition.

Definition 5.2.1 (Limit point). *For any point $p \in M^n$, we define the* limit point *function* $\Upsilon : M^n \longrightarrow \mathbb{R}^{n+1}$ *by*

$$\Upsilon(p) = \lim_{t \to T} F(p,t).$$

The existence of this limit exists follows directly from the type I assumption and (1.4).
We are now ready to state the first result of this rescaling analysis.

Theorem 5.2.2 (Existence of Smooth Limiting Surface). *Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a smooth, embedded solution of (1.4) satisfying the type I curvature assumption (5.10), and let $x_0 = \Upsilon(p)$, for some $p \in M^n$. Then for every sequence $\lambda_j \searrow 0$, corresponding to $t \nearrow T$, there is a subsequence $\{\lambda_{j_k}\}$ such that the rescaled surfaces $M_s^{(x_0,T),\lambda_{j_k}}$ converge smoothly on compact subsets of $\mathbb{R}^{n+1} \times (-\infty, 0)$ to a non-empty, embedded limit-surface, $\mathcal{M}' = (M'_s)_{s<0}$ such that*

1. *(M'_s) evolves by mean curvature flow for $s < 0$;*
2. *If $p \notin \partial M^n$ then M'_s has no boundary;*
3. *If $p \in \partial M^n$ then M'_s has boundary $\partial M'_s \subset \Sigma'_{x_0}$, where Σ'_{x_0} is a hyperplane through the origin $y = 0$, and $\langle \widehat{\nu}, \widehat{\nu_{\Sigma'}} \rangle = 0$ on $\partial M'_s$.*

Proof. We proceed as in [7], Chapter 3. Since x_0 is a limit point of the flow, by the type 1 hypothesis we have

$$\begin{aligned}
|F(p,t) - x_0| &= \left| \int_t^T \frac{dF}{d\gamma}(p,\gamma)\, d\gamma \right| \\
&\leqslant \int_t^T |H(p,\gamma)|\, d\gamma \\
&\leqslant \int_t^T \sqrt{\frac{nC_0}{T-\gamma}}\, d\gamma \\
&= 2\sqrt{nC_0(T-t)},
\end{aligned}$$

and so

$$\operatorname{dist}(x_0, M_t) \leqslant 2\sqrt{nC_0(T-t)}. \tag{5.11}$$

Furthermore, since for any fixed $R > 0$ we have

$$|A(x,t)|^2 \leqslant \frac{C_0}{T-t}$$

for all $x \in M_t \cap B_R(x_0)$ and $t \in [0,T)$, for any sequence $(\lambda_j) \searrow 0$ the corresponding sequence of rescaled solutions $\left(M_s^{(x_0,T),\lambda_j}\right)$ satisfy

$$\left|\widehat{A}(y,s)\right|^2 \leqslant \frac{C_0}{-s} \tag{5.12}$$

for all $y \in M_s^{(x_0,T),\lambda_j} \cap B_{\frac{R}{\lambda_j}}(0)$ and $s \in \left[-\frac{T}{\lambda_j^2}, 0\right)$. Hence, for any fixed $\vartheta \in \left(0, \frac{1}{2}\right)$ the inequality

$$\left|\widehat{A}(y,s)\right|^2 \leqslant \frac{C_0}{\vartheta^2}$$

holds for $y \in M_s^{(x_0,T),\lambda_j} \cap B_{\frac{R}{\lambda_j}}(0)$ and $s \in \left[-\frac{T}{\lambda_j^2}, \vartheta^2\right]$ and therefore, in particular, for $y \in M_s^{(x_0,T),\lambda_j} \cap B_{\frac{R}{\vartheta}}(0)$ and $s \in \left[-\frac{T}{\vartheta^2}, \vartheta^2\right]$ if j is large enough to ensure $\lambda_j < \vartheta$. The interior estimates of Stahl [25] then imply that

$$\left|\nabla^k \widehat{A}(y,s)\right|^2 \leqslant \frac{C_1}{\vartheta^{2(k+1)}}$$

for $y \in M_s^{(x_0,T),\lambda_j} \cap B_{\frac{R}{2\vartheta(1+C)}}(0)$ and $s \in \left[-\frac{T}{4\vartheta^2}, \vartheta^2\right]$ for sufficiently large j, for each $k \geqslant 0$, where C is a bounded constant depending on the curvature of the rescaled surfaces.
Moreover, by (5.11) we have

$$\begin{aligned}
\operatorname{dist}\left(0, M_s^{(x_0,T),\lambda_j}\right) &= \frac{1}{\lambda_j}\operatorname{dist}\left(x_0, M_{\lambda_j^2 s+T}\right) \\
&\leqslant \frac{1}{\lambda_j} 2\sqrt{nC_0\left(T - \left(\lambda_j^2 s + T\right)\right)} \\
&= 2\sqrt{-nC_0 s}.
\end{aligned}$$

By the Arzela-Ascoli theorem combined with a diagonal sequence argument when letting $\vartheta \searrow 0$ [and hence $\lambda_j \searrow 0$] for local graph representations of $M_s^{(x_0,T),\lambda_j}$, we can therefore find a subsequence of the rescaled solutions which converges smoothly on compact subsets of $\mathbb{R}^{n+1} \times (-\infty, 0)$ to a smooth solution $(M_s')_{s<0}$ of (MCF).
The subsequent claims regarding boundaries are then a direct consequence of the rescaling procedure - if $p \notin \partial M^n$, (which, be embeddedness, implies $x_0 \notin \partial M_t$) all points within the boundary will be translated by $-x_0$ and homothetically sent to infinity, whereas if $p \in \partial M^n$, (which implies $x_0 \in \partial M_t \subset \Sigma$) the boundary remains anchored to the origin $y = 0$ and the rescaled support surface will be straightened out to a hyperplane under homothetic expansion as $\lambda \searrow 0$. □

The monotonicity formula then allows the behaviour of this limit flow to be characterized.

Theorem 5.2.3 (Characterization of Limit Surface). *The limiting hypersurfaces M_s' as obtained in Theorem 5.2.2 satisfy the equation*

$$\vec{H} = \frac{y^\perp}{2s} \tag{5.13}$$

for all $y \in M_s'$ and $s < 0$.

Proof. If the limit surface has been obtained by rescaling about a boundary limit point, then we firstly note that, as an immediate consequence of the non-positivity of the right hand side of the monotonicity formula of Theorem 4.3.1, the quantity

$$\Theta(\mathcal{M}, x_0, T) \equiv \lim_{t \nearrow T} \left(e^{C\kappa_\Sigma^{2\delta} \tau^\delta} \int_{M_t} \eta \rho_{\kappa_\Sigma} \, d\mu_t \right) \tag{5.14}$$

exists. By virtue of the rescaling procedure, for each fixed $s < 0$, we have

$$\lim_{t \nearrow T} \left(e^{C\kappa_\Sigma^{2\delta} \tau^\delta} \int_{M_t} \eta \rho_{\kappa_\Sigma} \, d\mu_t \right) = \lim_{\lambda \searrow 0} \left(e^{C\left(-(\kappa_\Sigma \lambda)^2 s\right)^\delta} \int_{M_s^{(x_0,T),\lambda}} \hat{\eta} \hat{\rho}_{\kappa_\Sigma} \, d\mu_s \right)$$

and so, in view of Proposition 5.1.2 and Theorem 5.2.2, the Dominated Convergence Theorem [9] implies

$$\Theta(\mathcal{M}, x_0, T) = \int_{M_s'} \hat{\rho} \, d\mu_s', \tag{5.15}$$

where $\hat{\rho}(y, s) = \frac{1}{(-4\pi s)^{n/2}} \exp\left(\frac{|y|^2}{4s}\right)$. Since M_s' satsfies a Neumann boundary condition on a supporting hyperplane, one can use standard reflection across the plane to obtain a complete, boundaryless, symmetric limit surface, M_s'', which evolves by standard mean curvature flow. For such surfaces, Huisken's monotonicity formula (2.2) applies, implying

$$\frac{d}{ds} \int_{M_s''} \hat{\rho} \, d\mu_s = - \int_{M_s''} \left| \vec{H} - \frac{y^\perp}{2s} \right|^2 \hat{\rho} \, d\mu_s. \tag{5.16}$$

However, since $\Theta(\mathcal{M}, x_0, T)$ is independent of s, equation (5.15) implies that the integrand on the right of (5.16) must vanish identically, giving the result.
For the case where we have rescaled about an interior limit point, the resulting limit surface is smooth, possesses no boundary and evolves by standard mean curvature flow. Thus, Huisken's monotonicity formula immediately applies and we can proceed similarly to above to conclude the desired result - cf. Theorem 2.1.5.

□

Corollary 5.2.4. *The rescaling limit $\mathcal{M}' = (M_s')_{s<0}$ satisfies*

$$M_s' = \sqrt{-s} M_{-1}' \tag{5.17}$$

for all $s < 0$.

Proof. Using (5.13) and the definition of (MCF), one readily checks that the embeddings for (M_s') satisfy

$$\frac{d}{ds} \left(\frac{\widehat{F}(\phi(p,s), s)}{\sqrt{-s}} \right) = 0, \tag{5.18}$$

where $\phi(\cdot, s)\ M^n \longrightarrow M^n$ is a family of diffeomorphisms satisfying

$$D\widehat{F}(\phi(p,s), s) \left(\frac{\partial \phi}{\partial s}(p, s) \right) = - \left(\frac{\partial \widehat{F}}{\partial s}(\phi(p,s), s) \right)^T.$$

Here T denotes the projection onto the tangent space of $\widehat{F}(\cdot, s)(M^n)$.
The result follows upon integration of (5.18). □

In the case of surfaces without boundary, there is a extensive variety of self-similar, contracting hypersurfaces evolving by (MCF) that satisfy condition (5.13), (see, for example, [1] and [3]) and a complete classification of all such possible limit hypersurfaces resulting from the above rescaling procedure has not been obtained. However, a complete classification in the class of embedded limit hypersurfaces having non-negative mean curvature has been obtained, which we can carry over to the current free-boundary setting.

Theorem 5.2.5 (Classification of Limit Surfaces with Non-negative Mean Curvature). *If M_s' is a smooth, embedded limiting hypersurface in $\mathbb{R}^{n+1}$, as obtained by the above rescaling procedure, atisfying* (5.13) *with nonnegative mean curvature $H \geqslant 0$, then M_s' is one of the following:*

1. $\mathbb{S}^n$;
2. $\mathbb{S}^{n-m} \times \mathbb{R}^m$,
3. $\mathbb{S}^n \cap \Pi$;
4. $\mathbb{S}^{n-m} \times \mathbb{R}^m \cap \Pi$

for some $m \leqslant n$, where Π *is an $n+1$-dimensional half-space through the origin.*

Proof. As above, the Neumann boundary condition allows limiting solutions possessing boundary to be reflected across their supporting hyperplane and be treated as complete (symmetric) hypersurfaces without boundary. Thus all limit surfaces can be treated as boundaryless hypersurfaces evolving by standard mean curvature flow. Hence, for surfaces of weak mean-convexity, Huisken's classification (Theorem 2.1.6) yields the result. □

Remark 5.2.6. For compact solutions of (1.4), mean convexity of the limit surface is guaranteed for initially weakly convex surfaces M_0 by [26, Theorem 3.1], which follows from the maximum principle in [25].

Chapter 6

Area/Boundary Length Estimates and Special Cases

In this chapter we prove a local area and boundary length estimate and also obtain a monotonicity formula for solutions of (1.4) in two special cases, using more elementary means than those used for the general case.

6.1 A Local Area Estimate

The focus of this section is to obtain a local area bound for solutions of (1.4), analogous to that of Brakke [4, Chapter 3] and Ecker [7] for the standard mean curvature flow, but in balls that are allowed to contain the boundary.
The starting point for obtaining such estimates is the following proposition, which follows directly from Proposition 2.2.1 on taking $g = 1$ and re-labelling f to ϕ.

Proposition 6.1.1. *Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a smooth, embedded solution of* (1.4) *and U an open subset of $\mathbb{R}^{n+1}$ containing $\mathcal{M}$. Then, for any function $\phi : U \times [0,T) \longrightarrow \mathbb{R}$ which satisfies $\phi \in C_0^2(U)$ and $\frac{\partial\phi}{\partial t} \in C_0^0$ we have*

$$\frac{d}{dt}\int_{M_t}\phi\, d\mu_t = \int_{M_t}\left(\left(\frac{d}{dt} - \Delta_{M_t}\right)\phi - |H|^2\phi\right) d\mu_t + \int_{\partial M_t}\langle D\phi, \nu_\Sigma\rangle\, d\sigma_t. \tag{6.1}$$

Furthermore, if ϕ satisfies $\left(\frac{d}{dt} - \Delta_{M_t}\right)\phi \leqslant 0$ then

$$\frac{d}{dt}\int_{M_t}\phi\, d\mu_t \leqslant -\int_{M_t}|H|^2\phi\, d\mu_t + \int_{\partial M_t}\langle D\phi, \nu_\Sigma\rangle\, d\sigma_t. \tag{6.2}$$

Inspired by the work of Brakke, for any $R > 0$ and $(x,t) \in \mathbb{R}^{n+1} \times [0,T)$ we consider the class of testfunctions given by

$$\phi_R(z) = \left(1 - \frac{z}{R^2}\right)_+^4, \qquad \text{where } z = z(x,t)\,. \tag{6.3}$$

We then have

$$\begin{aligned}\left(\frac{d}{dt}-\Delta_{M_t}\right)\phi_R &= \phi_R'\left(\frac{d}{dt}-\Delta_{M_t}\right)z-\phi_R''|\nabla z|^2 && \text{by (A.4)}\\ &= \phi_R'\left(\frac{\partial}{\partial t}-\mathrm{div}_{M_t}D\right)z-\phi_R''|\nabla z|^2 && \text{by (A.2)}\\ &\leqslant \phi_R'\left(\frac{\partial}{\partial t}-\mathrm{div}_{M_t}D\right)z, && (6.4)\end{aligned}$$

since $\phi_R''(z)=\frac{12}{R^4}\left(1-\frac{z}{R^2}\right)_+^2\geqslant 0$. Thus, in light of the identity

$$\left(\frac{\partial}{\partial t}-\mathrm{div}_{M_t}D\right)\left(|x|^2+2nt\right)=0$$

we have established that the *spherically shrinking* testfunction

$$\varphi_R(x,t)=\left(1-\frac{|x|^2+2nt}{R^2}\right)_+^4$$

satisfies

$$\frac{d}{dt}\int_{M_t}\varphi_R\,d\mu_t\leqslant-\int_{M_t}|H|^2\varphi_R\,d\mu_t+\int_{\partial M_t}\langle D\varphi_R,\nu_\Sigma\rangle\,d\sigma_t. \tag{6.5}$$

For surfaces M_t evolving by mean curvature flow without boundary, or possessing boundary but with $\partial M_t\cap B_R(0)=\emptyset$ (so that the boundary integral vanishes, since spt $\phi_R\subset B_R(0)$), one can proceed as in [7] and use equation (6.5) to establish the following local area estimate.

Proposition 6.1.2. (Brakke [4], Ecker [7]) *Let $(M_t)_{t\in\left[0,\frac{R^2}{8n}\right)}$ be a smooth, embedded solution of the mean curvature flow in $B_R(x_0)$ with $\partial M_t\cap B_R(x_0)=\emptyset$. Then for all $t\in\left[0,\frac{R^2}{8n}\right)$ we have*

$$\mathcal{H}^n\left(M_t\cap B_{\frac{R}{2}}\right)+\int_0^t\int_{M_s\cap B_{\frac{R}{2}}(x_0)}|H|^2\,d\mu_s\,ds\leqslant 8\mathcal{H}^n\left(M_0\cap B_R\right). \tag{6.6}$$

We would like to mimic this result and obtain a local area estimate for balls containing part of the boundary of surfaces evolving by mean curvature flow with Neumann free-boundary on a given support surface Σ. To do this we must account for the fact that any ball intersecting the boundary need not be centred on it, and so the idea is to reflect M_t across the support surface and consider the sum of the area of M_t and the area of the reflected surface, in this ball.

Proposition 6.1.3. *For any $x_0\in\Sigma$, let (M_t) be a smooth, embedded solution of* (1.4) *in $B_R(x_0)$ for all $t\in\left[0,\frac{R^2}{128n}\right)$ and any $R\leqslant\frac{1}{2\kappa_\Sigma}$. Then for all $t\in\left[0,\frac{R^2}{128n}\right)$ we have*

$$\mathcal{H}^n\left(M_t\cap\left(B_{\frac{R}{\sqrt{8}}}\cup\widetilde{B}_{\frac{R}{\sqrt{8}}}\right)\right)+\int_0^t\int_{M_s\cap\left(B_{\frac{R}{\sqrt{8}}}\cup\widetilde{B}_{\frac{R}{\sqrt{8}}}\right)}|H|^2\,d\mu_s\,ds$$
$$\leqslant 16\mathcal{H}^n\left(M_0\cap\left(B_R\cup\widetilde{B_R}\right)\right), \tag{6.7}$$

where

$$B_R = B_R(x_0) := \{x \in \mathbb{R}^{n+1} : |x - x_0| \leqslant R\}$$

and

$$\widetilde{B}_R = \widetilde{B}_R(x_0) := \{x \in \mathbb{R}^{n+1} : \left|\widetilde{x - x_0}\right| \leqslant R\}.$$

Proof. Setting $z = r_{x_0} + ct$ in the testfunction $\phi_R(z)$ of (6.3), where $r_{x_0} = |x - x_0|^2 + \left|\widetilde{x - x_0}\right|^2$ and c is a constant to be chosen later, and noting (6.1), (6.4) and Lemma 3.2.3, we have

$$\frac{d}{dt}\int_{M_t} \phi_R \, d\mu_t \leqslant \int_{M_t} \left(\phi_R' \left(\frac{\partial z}{\partial t} - \operatorname{div}_{M_t} Dz\right) - \phi_R |H|^2\right) d\mu_t. \tag{6.8}$$

Since

$$\phi_R'(z) = -\frac{4}{R^2}\left(1 - \frac{z}{R^2}\right)_+^3 \qquad \text{and} \qquad \frac{\partial z}{\partial t} = c,$$

we have

$$\begin{aligned}
\phi_R' \left(\frac{\partial z}{\partial t} - \operatorname{div}_{M_t} Dz\right) &= \frac{4}{R^2}\left(1 - \frac{z}{R^2}\right)_+^3 \left(\operatorname{div}_{M_t} Dz - c\right) \\
&\leqslant \frac{4}{R^2}\left(1 - \frac{z}{R^2}\right)_+^3 \left(4n + \frac{20n\kappa_\Sigma |x - x_0|}{1 - d\kappa_\Sigma}\right. \\
&\qquad \left. + \frac{4n\kappa_\Sigma^2 |x - x_0|^2}{(1 - d\kappa_\Sigma)^3} - c\right),
\end{aligned}$$

using the estimate for $\operatorname{div}_{M_t} Dr$ of Lemma 3.2.3.
Moreover, since

$$\begin{aligned}
\operatorname{spt}\left(1 - \frac{z}{R^2}\right)_+^3 &= \{(x,t) \in \mathbb{R}^{n+1} \times \mathbb{R} : r_{x_0} + ct \leqslant R^2\} \\
&\subset \{x \in \mathbb{R}^{n+1} : |x - x_0| \leqslant R\},
\end{aligned}$$

for all $R \leqslant \frac{1}{2\kappa_\Sigma}$ and $x_0 \in \Sigma$ (for which $d(x) \leqslant |x - x_0|$) we can therefore estimate

$$\begin{aligned}
\phi_R' \left(\frac{\partial z}{\partial t} - \operatorname{div}_{M_t} Dz\right) &\leqslant \frac{4}{R^2}\left(1 - \frac{z}{R^2}\right)_+^3 (4n + 20n + 8n - c) \\
&= \frac{4}{R^2}\left(1 - \frac{z}{R^2}\right)_+^3 (32n - c).
\end{aligned}$$

On setting $c = 32n$, equation (6.8) therefore gives

$$\frac{d}{dt}\int_{M_t} \phi_R \, d\mu_t \leqslant -\int_{M_t} \phi_R |H|^2 \, d\mu_t, \tag{6.9}$$

where

$$\phi_R(x,t) = \left(1 - \frac{r_{x_0} + 32nt}{R^2}\right)_+^4.$$

Integrating (6.9) in time yields

$$\int_{M_t} \phi_R \, d\mu_t + \int_0^t \int_{M_s} \phi_R |H|^2 \, d\mu_s \, ds \leqslant \int_{M_0} \phi_R \, d\mu_0, \tag{6.10}$$

where, since $\phi_R \leqslant 1$ and

$$\begin{aligned} \operatorname{spt} \phi_R(\cdot, 0) &= \left\{x \in \mathbb{R}^{n+1} : r_{x_0} \leqslant R^2\right\} \\ &\subset \left(B_R(x_0) \cup \widetilde{B}_R(x_0)\right), \end{aligned}$$

we have

$$\int_{M_0} \phi_R \, d\mu_0 \leqslant \mathcal{H}^n \left(M_0 \cap \left(B_R(x_0) \cup B_R(x_0)\right)\right). \tag{6.11}$$

For $r_{x_0} \leqslant \frac{R^2}{4}$ and $32nt \leqslant \frac{R^2}{4}$ we have $\phi_R \geqslant \frac{1}{16}$, and so, since

$$\begin{aligned} \left\{x : r_{x_0} \leqslant \frac{R^2}{4}\right\} &= \left\{x : |x - x_0|^2 + \left|\widetilde{x - x_0}\right|^2 \leqslant \frac{R^2}{4}\right\} \\ &\supset \left(\left\{x : |x - x_0|^2 \leqslant \frac{R^2}{8}\right\} \cup \left\{x : \left|\widetilde{x - x_0}\right|^2 \leqslant \frac{R^2}{8}\right\}\right) \\ &= \left(B_{\frac{R}{\sqrt{8}}}(x_0) \cup \widetilde{B}_{\frac{R}{\sqrt{8}}}(x_0)\right), \end{aligned}$$

we have

$$\int_{M_t} \phi_R \, d\mu_t \geqslant \frac{1}{16} \mathcal{H}^n \left(M_t \cap \left(B_{\frac{R}{\sqrt{8}}}(x_0) \cup \widetilde{B}_{\frac{R}{\sqrt{8}}}(x_0)\right)\right) \tag{6.12}$$

for all $t \leqslant \frac{R^2}{128n}$.

Using the estimates (6.11) and (6.12) in (6.10) and multiplying through by 16 then yields the result.

□

6.2 Boundary Length Estimate

For minimal surfaces which intersect a given support surface orthogonally - and are thus degenerate (stationary) solutions of the mean curvature flow problem (1.4) - the Divergence Theorem (A.1) provides a natural starting point for obtaining estimates on the length of the boundary.

Indeed, taking $\mathbf{X} = (1 - 2d\kappa_\Sigma)_+^4 \, Dd$, for any minimal surface M and support surface Σ the Divergence Theorem yields

$$\begin{aligned} \mathcal{H}^{n-1}(\partial M) &= \int_M \operatorname{div}_M \left((1 - 2d\kappa_\Sigma)_+^4 \, Dd\right) d\mu \\ &= \int_M \left((1 - 2d\kappa_\Sigma)_+^4 \operatorname{div}_M (Dd) - 8\kappa_\Sigma (1 - 2d\kappa_\Sigma)_+^3 |\nabla d|^2\right) d\mu \\ &\leqslant 2n\kappa_\Sigma \mathcal{H}^n \left(M \cap \Sigma_{1/2\kappa_\Sigma}\right), \end{aligned}$$

using (3.2), where $\Sigma_{1/2\kappa_\Sigma}$ is the $\frac{1}{2\kappa_\Sigma}$-tubular neighbourhood of Σ.

The following lemma establishes a bound, analogous to that above, for general solutions to (1.4).

Lemma 6.2.1 (Time-integrated Boundary Length Estimate). *Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a smooth, embedded solution to (1.4), where $\kappa_\Sigma \geqslant 0$ bounds the curvature of the support surface Σ. Then for any $t \in [0,T)$ we have*

$$\kappa_\Sigma \int_0^t \mathcal{H}^{n-1}\left(\partial M_s\right) ds \leqslant 2n\kappa_\Sigma^2 \int_0^t \mathcal{H}^n\left(M_s \cap \Sigma_{1/2\kappa_\Sigma}\right) ds + \frac{1}{10}\mathcal{H}^n\left(M_0 \cap \Sigma_{1/2\kappa_\Sigma}\right). \tag{6.13}$$

Proof. In the general setting, for the vectorfield $\mathbf{X} = (1-2d\kappa_\Sigma)_+^4\, Dd$ the Divergence Theorem yields

$$\begin{aligned}\mathcal{H}^{n-1}\left(\partial M_t\right) &= \int_{M_t} (1-2d\kappa_\Sigma)_+^4 \left(\mathrm{div}_{M_t} Dd + \left\langle Dd, \vec{H}\right\rangle\right) d\mu_t \\ &\quad - \int_{M_t} 8\kappa_\Sigma (1-2d\kappa_\Sigma)_+^3 \left|\nabla d\right|^2 d\mu_t, \\ &\leqslant \int_{M_t} (1-2d\kappa_\Sigma)_+^4 \left(2n\kappa_\Sigma + \left\langle Dd, \vec{H}\right\rangle\right) d\mu_t, \end{aligned}\tag{6.14}$$

using (3.2) and noting $d\kappa_\Sigma \leqslant \frac{1}{2}$ over the support of $(1-2d\kappa_\Sigma)_+^4$. Noting then (A.5), one can also compute

$$\begin{aligned}\frac{d}{dt}\int_{M_t} (1-2d\kappa_\Sigma)_+^5\, d\mu_t &= -10\kappa_\Sigma \int_{M_t} (1-2d\kappa_\Sigma)_+^4 \left\langle Dd, \vec{H}\right\rangle d\mu_t \\ &\quad - \int_{M_t} (1-2d\kappa_\Sigma)_+^5 \left|H\right|^2 d\mu_t,\end{aligned}$$

and so

$$\kappa_\Sigma \int_{M_t} (1-2d\kappa_\Sigma)_+^4 \left\langle Dd, \vec{H}\right\rangle d\mu_t \leqslant -\frac{1}{10}\frac{d}{dt}\int_{M_t} (1-2d\kappa_\Sigma)_+^5\, d\mu_t.$$

Hence, by (6.14), we have

$$\kappa_\Sigma \mathcal{H}^{n-1}\left(\partial M_t\right) \leqslant 2n\kappa_\Sigma^2 \mathcal{H}^n\left(M_t \cap \Sigma_{1/2\kappa_\Sigma}\right) - \frac{1}{10}\frac{d}{dt}\int_{M_t} (1-2d\kappa_\Sigma)_+^5\, d\mu_t.$$

Integration from 0 to t then gives the desired result. □

Remark 6.2.2. Since κ_Σ is merely an upper bound for the curvature of the support surface Σ, the above formula should be viewed accordingly. That is, in the case that Σ is a plane, the above result is valid also for any $\kappa_\Sigma > 0$.

6.3 Monotonicity Formula for Special Cases

In this section we establish a monotonicity formula for solutions of (1.4) by directly estimating the boundary integral

$$-\int_{\partial M_t} \langle D\rho, \nu_\Sigma \rangle \, d\sigma_t = \frac{1}{2\tau} \int_{\partial M_t} \rho \, \langle x, \nu_\Sigma \rangle \, d\sigma_t \tag{6.15}$$

of (2.10) in two special cases.

6.3.1 Concave\Convex Support Surfaces

In the special case where the support surface Σ is concave\convex and M_t meets Σ from the interior\exterior, we have

$$\langle x, \nu_\Sigma \rangle \leqslant 0 \qquad \text{for all } x \in \Sigma,$$

which implies, since $\partial M_t \subset \Sigma$, that

$$\int_{\partial M_t} \langle x, \nu_\Sigma \rangle \, d\sigma_t \leqslant 0.$$

Though this statement relies on the assumption $0 \in \Sigma$, more generally, for any point $x_0 \in \Sigma$ we always have

$$\langle x - x_0, \nu_\Sigma \rangle \leqslant 0 \qquad \text{for all } x \in \Sigma.$$

Hence, the boundary integral (6.15) has the right sign.

Lemma 6.3.1 (Monotonicity Formula for Concave\Convex Support Surfaces). *Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a smooth solution of* (1.4). *Then, if Σ is concave\convex and M_t intersects Σ from the interior\exterior, for all $t \in [0,T)$ we have*

$$\frac{d}{dt} \int_{M_t} \rho \, d\mu_t \leqslant -\int_{M_t} \left| \vec{H} - \frac{x^\perp}{2\tau} \right|^2 \rho \, d\mu_t. \tag{6.16}$$

6.3.2 Slow Boundary Growth

The second special case for which we obtain a monotonicity formula is that where the evolving boundary ∂M_t satisfies a certain, specified growth-rate. In this instance we obtain a uniform bound on the the quantity

$$\int_0^T \int_{\partial M_t} \langle D\rho, \nu_\Sigma \rangle \, d\sigma_t \, dt = \int_0^T \frac{1}{2\tau} \int_{\partial M_t} \rho \, \langle x, \nu_\Sigma \rangle \, d\sigma_t \, dt,$$

which arises when we integrate equation (2.10) in time. Though such an estimate actually suffices for the business of extracting limit surfaces from sequences of rescalings, we merely present the critical bound and refer the reader to the work of Stone [27] for the subsequent analysis.

Definition 6.3.2 (Slow Boundary Growth). *Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a smooth solution of* (1.4). *Then M_t has* slow boundary growth *if, for each $x_0 \in \Sigma$, there exists positive constants C and R_0 such that, for all $R < R_0$ and some $\alpha \in (0,1)$, we have*

$$\mathcal{H}^{n-1}\left(\partial M_t \cap B_R(x_0)\right) \leqslant \frac{CR^{n-1}}{(T-t)^{\alpha/2}} \tag{6.17}$$

for all $t < T$.

Note that compact boundaries satisfying the slow growth rate (6.17) must also be integrable in time. That is, we have

$$\int_0^T \mathcal{H}^{n-1}\left(\partial M_t\right) dt \leqslant C. \tag{6.18}$$

The method employed to estimate the boundary integral (6.15) is based largely on the work of Stone [27] for the Dirichlet problem. It involves splitting the boundary ∂M_t into two sets - one, a ball of a specified time-dependent, shrinking radius, and the other its complement. The more troublesome boundary integral, that over the former set, is then estimated by utilizing the bound (6.17).

Lemma 6.3.3 (Boundary Estimate for Slow Growth Boundaries). *Let $\mathcal{M} = (M_t)_{t\in[0,T)}$ be a smooth solution of* (1.4) *with boundary $\partial M_t \subset \Sigma$ satisfying the slow-growth rate* (6.17) *for all $t \in [0,T)$, where Σ is compact. Then*

$$\int_0^T \int_{\partial M_t} \left|\left\langle \frac{x-x_0}{2\tau}, \nu_\Sigma \right\rangle\right| \rho_{x_0}\, d\mu_t\, dt \leqslant C. \tag{6.19}$$

Proof. Let $R = R(t)$ be a given funciton, to be chosen explicitly later, satisfying

$$R(t) \searrow 0 \quad \text{as} \quad t \longrightarrow T$$

and define

$$\begin{aligned} I(t_1) &:= \int_{t_1}^T \int_{\partial M_t} \rho_{x_0} \left\langle \frac{x-x_0}{2\tau}, Dd \right\rangle d\sigma_t\, dt \\ &= \int_{t_1}^T \int_{\partial M_t \cap B_R(x_0)} \rho_{x_0} \left\langle \frac{x-x_0}{2\tau}, Dd \right\rangle d\sigma_t\, dt \\ &\quad + \int_{t_1}^T \int_{\partial M_t \setminus B_R(x_0)} \rho_{x_0} \left\langle \frac{x-x_0}{2\tau}, Dd \right\rangle d\sigma_t\, dt \\ &:= I_1(t_1) + I_2(t_1). \end{aligned} \tag{6.20}$$

The estimate (6.19) follows by suitably estimating each of the integrals $I_1(t)$ and $I_2(t)$. Regarding the latter of these, using the compactness of Σ and (6.18), we have

$$\begin{aligned} I_2(t_1) &= \int_{t_1}^T \int_{\partial M_t \setminus B_R(x_0)} \rho_{x_0} \left\langle \frac{x-x_0}{2\tau}, \nu_\Sigma \right\rangle d\sigma_t\, dt \\ &\leqslant \frac{\operatorname{diam}(\Sigma)}{2} (4\pi)^{-n/2} \int_{t_1}^T \int_{\partial M_t \setminus B_R(x_0)} \tau^{-n/2-1} e^{-R^2/4\tau}\, d\sigma_t\, dt \\ &\leqslant C \int_{t_1}^T \tau^{-n/2-1} e^{-R^2/4\tau}\, dt. \end{aligned} \tag{6.21}$$

This estimate will be complete after choosing a suitable function $R = R(t)$, which we shall do shortly.

Turning our attention to $I_1(t)$ now, we firstly note that, as in [27], since Σ is smooth, there exists constants R_1 and C (depending on κ_Σ) such that

$$\langle x - x_0, \nu_\Sigma \rangle \leqslant C_6 |x - x_0|^2 \qquad \forall x \in \Sigma \cap B_{R_1}(x_0). \tag{6.22}$$

Thus, for all t_1 sufficiently close to T to ensure both $R(t_1) \leqslant R_1$ and $R(t_1) \leqslant R_0$ – $t_1 \geqslant t_0$, say – where R_0 is as in Definition 6.3.2, we have

$$\begin{aligned} I_1(t_1) &= \int_{t_1}^{T} \int_{\partial M_t \cap B_R(x_0)} \rho_{x_0} \left\langle \frac{x - x_0}{2\tau}, \nu_\Sigma \right\rangle d\sigma_t \, dt \\ &\leqslant C \int_{t_1}^{T} \int_{\partial M_t \cap B_R(x_0)} \tau^{-n/2-1} R^2 \, d\sigma_t \, dt \\ &\leqslant C \int_{t_1}^{T} \tau^{-n/2-1-\alpha/2} R^{n+1} \, dt, \end{aligned} \tag{6.23}$$

using (6.17).

We now make a choice for $R(t)$. In view of (6.21), it is constructive to choose $R(t)$ to satisfy

$$-\frac{R^2}{4\tau} = \log\left(\tau^{n/2+1}\right);$$

that is,

$$R(t) = \sqrt{-4\tau\,(n/2+1)\log\tau}. \tag{6.24}$$

This then ensures for all

$$T - 1 \leqslant t_0 \leqslant t \leqslant T$$

that

$$I_2(t) \leqslant C. \tag{6.25}$$

Turning our attention back to our estimate for $I_1(t)$, we firstly observe that, since α is strictly less than 1, there exists constants $\epsilon, \gamma > 0$, sufficiently small, such that $1/2 + \alpha/2 + \epsilon =: 1 - \gamma < 1$. Noting also that, for any $\delta < 1$ and $y \geqslant 0$, the function $f(y) = -y^\delta \log y$ is bounded above by $(e\delta)^{-1}$, we use the above choice of $R(t)$ to estimate for any $t \geqslant t_0$

$$\begin{aligned} I_1(t) &\leqslant (2n+4)^{\frac{n+1}{2}} C \int_t^T \tau^{-1/2-\alpha/2-\epsilon} \tau^\epsilon (-\log\tau)^{\frac{n+1}{2}} \, dt, \\ &= C \int_t^T \tau^{\gamma-1} \left(-\tau^{\frac{2\epsilon}{n+1}} \log(\tau)\right)^{\frac{n+1}{2}} dt, \\ &\leqslant C \left(\frac{2e\epsilon}{n+1}\right)^{-\left(\frac{n+1}{2}\right)} \int_t^T \tau^{\gamma-1} \, dt \\ &= C \frac{(T-t)^\gamma}{\gamma}. \end{aligned} \tag{6.26}$$

Hence, in view of (6.20), (6.25) and (6.26) the boundary estimate (6.19) follows. □

Appendix A

Appendices

A.1 Miscellaneous Formulae

Theorem A.1.1 (Divergence Theorem). *Let M be a smooth, orientable hypersurface with boundary, embedded in $\mathbb{R}^{n+1}$. Then for any C^1-vectorfield $X : M \longrightarrow \mathbb{R}^{n+1}$ with compact support we have*

$$\int_M \mathrm{div}_M X \, d\mu = -\int_M \left\langle X, \vec{H} \right\rangle d\mu + \int_{\partial M} \langle X, \nu_{\partial M} \rangle \, d\sigma, \tag{A.1}$$

where $\vec{H} = -H\nu = -(\mathrm{div}_M \nu)\,\nu$ denotes the mean curvature vector of M for a choice of unit normal ν to M, and $\nu_{\partial M}$ is the unit inner co-normal to ∂M.

Proof. See eg. [24]. □

Lemma A.1.2 (Heat Operator). *Let $M_t \equiv F(M^n, t)$ be a family of hypersurfaces evolving by mean curvature flow and let $f = f(x,t)$ for $x = F(p,t)$, $p \in M^n$. Then*

$$\left(\frac{d}{dt} - \Delta_{M_t}\right) f = \left(\frac{\partial}{\partial t} - \mathrm{div}_{M_t} D\right) f. \tag{A.2}$$

Proof. One computes,

$$\begin{aligned} \frac{d}{dt} f(x,t) &= \frac{d}{dt} f(F(p,t), t) \\ &= \frac{d}{dt} f(x,t) + \left\langle Df(x,t), \frac{\partial F}{\partial t}(p.t) \right\rangle, \end{aligned}$$

and so, by (MCF), we have

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \left\langle Df, \vec{H} \right\rangle.$$

Also,

$$\begin{aligned} \Delta_{M_t} f &= \mathrm{div}_{M_t} (\nabla f) \\ &= \mathrm{div}_{M_t} (Df - \langle Df, \nu \rangle \nu) \\ &= \mathrm{div}_{M_t} Df - \langle Df, \nu \rangle H - \langle \nabla \langle Df, \nu \rangle, \nu \rangle \\ &= \mathrm{div}_{M_t} Df + \left\langle Df, \vec{H} \right\rangle. \end{aligned}$$

Thus,

$$\frac{df}{dt} - \Delta_{M_t} f = \frac{\partial f}{\partial t} - \operatorname{div}_{M_t} Df.$$

□

Lemma A.1.3 (Product/Chain-rule for Heat Operator). *For any twice differentiable functions f, g defined on M_t, we have*

$$\left(\frac{d}{dt} - \Delta_{M_t}\right)(fg) = f\left(\frac{d}{dt} - \Delta_{M_t}\right)g + g\left(\frac{d}{dt} - \Delta_{M_t}\right)f - 2\langle\nabla f, \nabla g\rangle. \tag{A.3}$$

Furthermore, if $\eta : \mathbb{R} \longrightarrow \mathbb{R}$ is twice-differentiable then the composite function $\eta(f)$ satisfies

$$\left(\frac{d}{dt} - \Delta_{M_t}\right)\eta(f) = \eta'(f)\left(\frac{d}{dt} - \Delta_{M_t}\right)f - \eta''|\nabla f|^2. \tag{A.4}$$

Proof. One computes

$$\begin{aligned}
\Delta_{M_t}(fg) &= \operatorname{div}_{M_t}(\nabla(fg)) \\
&= \operatorname{div}_{M_t}(f\nabla g + g\nabla f) \\
&= f\Delta_{M_t} g + g\Delta_{M_t} f + 2\langle\nabla f, \nabla g\rangle,
\end{aligned}$$

from which the first result follows. For the second result, one simply observes

$$\frac{d}{dt}\eta(f) = \eta'(f)\frac{df}{dt}$$

and

$$\begin{aligned}
\Delta_{M_t}\eta(f) &= \operatorname{div}_{M_t}(\nabla\eta(f)) \\
&= \operatorname{div}_{M_t}(\eta'(f)\nabla f) \\
&= \eta'(f)\Delta_{M_t} f + \eta''|\nabla f|^2.
\end{aligned}$$

□

Lemma A.1.4 (Evolution of Area Element). *The area element of a solution $(M_t)_{t\in I}$ of* (MCF) *satisfies the evolution equation*

$$\frac{d}{dt}d\mu_t = -|H|^2\, d\mu_t \tag{A.5}$$

for all $t \in I$.

Proof. The area elements of the hypersurfaces M_t are given by

$$d\mu_t(p) = \sqrt{\det g_{ij}(p,t)}\, d\mu_{M^n}(p),$$

where $g_{ij} = \left\langle \frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j} \right\rangle (p, t)$ denotes the evolving metric and $d\mu_{M^n}$ the volume measure on the fixed parameter manifold M^n.
One computes

$$\begin{aligned} \frac{d}{dt} d\mu_t &= \frac{d}{dt} \sqrt{\det g_{ij}}\, d\mu_{M^n} \\ &= \frac{1}{2} \sqrt{\det g_{ij}}\, g^{ij} \frac{d}{dt} g_{ij}\, d\mu_{M^n}, \end{aligned}$$

where $g^{ij} := g_{ij}^{-1}$. Moreover, by (MCF) we have

$$\begin{aligned} \frac{d}{dt} g_{ij} &= \frac{d}{dt} \left\langle \frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j} \right\rangle \\ &= \left\langle \frac{\partial}{\partial x_i} (-H\nu), \frac{\partial F}{\partial x_j} \right\rangle + \left\langle \frac{\partial F}{\partial x_i}, \frac{\partial}{\partial x_j} (-H\nu) \right\rangle \\ &= -H \left\langle \frac{\partial \nu}{\partial x_i}, \frac{\partial F}{\partial x_j} \right\rangle - H \left\langle \frac{\partial F}{\partial x_i}, \frac{\partial \nu}{\partial x_j} \right\rangle. \\ &= -2H h_{ij} \end{aligned}$$

Hence,

$$\begin{aligned} \frac{d}{dt} d\mu_t &= -H g^{ij} h_{ij} \sqrt{\det g_{ij}}\, d\mu_{M^n} \\ &= -H^2\, d\mu_t. \end{aligned}$$

□

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