Spatial patterns in a diffusive modified Holling-Tanner predator-prey model

Queensland University of Technology
Faculty of Science and Engineering

Claudio Arancibia-Ibarra
claudio.arancibia@hdr.qut.edu.au

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Abstract

In this work, we consider the dynamics of a spatio-temporal Holling--Tanner predator-prey models with an alternative food for the predator. From our result of the temporal model, we present the analytical conditions for the bifurcation diagram. Additionally, we identify regions in parameter space in which Turing instability in the spatio-temporal model are expected. We use simulations to illustrate the behaviour of both the temporal and spatio-temporal model.

Introduction

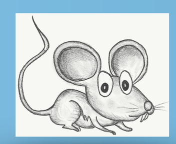
The Holling-Tanner model has been used extensively to model many real-world predator-prey interactions [1,2,3]. For instance, Hanski *et al.* [4] used this model to investigate the predator-prey interaction between the least weasel (*Mustela nivalis*) and the field vole (*Microtus agrestis*). This study was based under the hypothesis that generalist predators predicts correctly the geographic gradient in rodent oscillations in Fennoscandia. Additionally, the authors showed that the amplitude and cycle period decreasing from north to south.

The following pair of equations is a typical representation of a purely Holling-Tanner predator-prey models. In this model we consider that the predator and prey population contains a logistic growth function and the predator environmental carrying capacity is a prey dependant. Moreover, the functional response is hyperbolic in form, and is referred to as a Holling Type II function [1,5]. Here X and Y indicate the prey and predator population sizes respectively, r and s are the intrinsic growth rate for the prey and predator respectively, K is the prey environmental carrying capacity, n is a measure of the quality of the prey as food for the predator, nX can be a prey dependent carrying capacity for the predator, q is the maximum predation rate per capita and a is half of the saturated level.

$$
(1)\quad \frac{dX}{dt} = rX\left(1-\frac{X}{K}\right) - \frac{qXY}{X+a}, \qquad \frac{dY}{dt} = sY\left(1-\frac{Y}{nX}\right)
$$

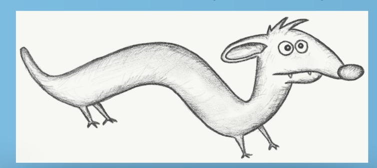
Logistic growth of the prey

X: Prey (Field Voles)



Functional response (Holling Type II)

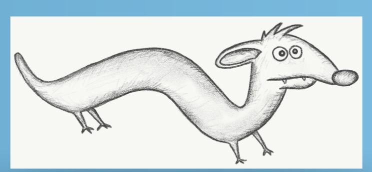
Y: Predator (Weasels)



Logistic growth of the predator

Weasels diet

System (1) does not consider that some predators can survive under different environments and utilise a large range of food resources [6,7,8]. For instance, weasels can switch to another available food such us small rodents (%68 of percentage of occurrence), lagomorphs (%25 of percentage of occurrence) and birds and birds' eggs (%5 of percentage of occurrence) [9], although its population growth may still be limited by the fact that its preferred food, are not available abundantly [1,4,10].



Alternative food "c" (Lagomorphs, Birds and birds' eggs)

$$
(2)\quad \frac{dX}{dt} = rX\left(1-\frac{X}{K}\right) - \frac{qXY}{X+a}, \qquad \frac{dY}{dt} = sY\left(1-\frac{Y}{nX+c}\right)
$$

$$
(3)\quad \frac{\partial X}{\partial t} = rX\left(1-\frac{X}{K}\right) - \frac{qXY}{X+a} + D_1\nabla^2 X, \qquad \frac{\partial Y}{\partial t} = sY\left(1-\frac{Y}{nX+c}\right) + D_2\nabla^2 Y
$$

Purely diffusive Holling-Tanner predator-prey model

Small rodent (67%)

Nondimensionalisation

In order to simplify the analysis we introduce the dimensionless variable by setting: $u = X/K$, $v = Y/nK$, $S = s/r$, $C = c/(nK)$, $A = a/K$, $Q = qn/r$, $\tau = rt$, $y = \sqrt{r/D_1}x$ and $d = D_1/D_2$ into (3) we obtain (4).

$$
(3)\quad \frac{\partial X}{\partial t} = rX\left(1-\frac{X}{K}\right) - \frac{qXY}{X+a} + D_1\nabla^2 X, \qquad \frac{\partial Y}{\partial t} = sY\left(1-\frac{Y}{nX+c}\right) + D_2\nabla^2 Y
$$

$$
(4)\quad \frac{\partial u}{\partial \tau} = u(1-u) - \frac{uvQ}{u+A} + \nabla^2 u, \qquad \frac{\partial v}{\partial \tau} = Sv\left(1-\frac{v}{u+C}\right) + d\nabla^2 v.
$$

References

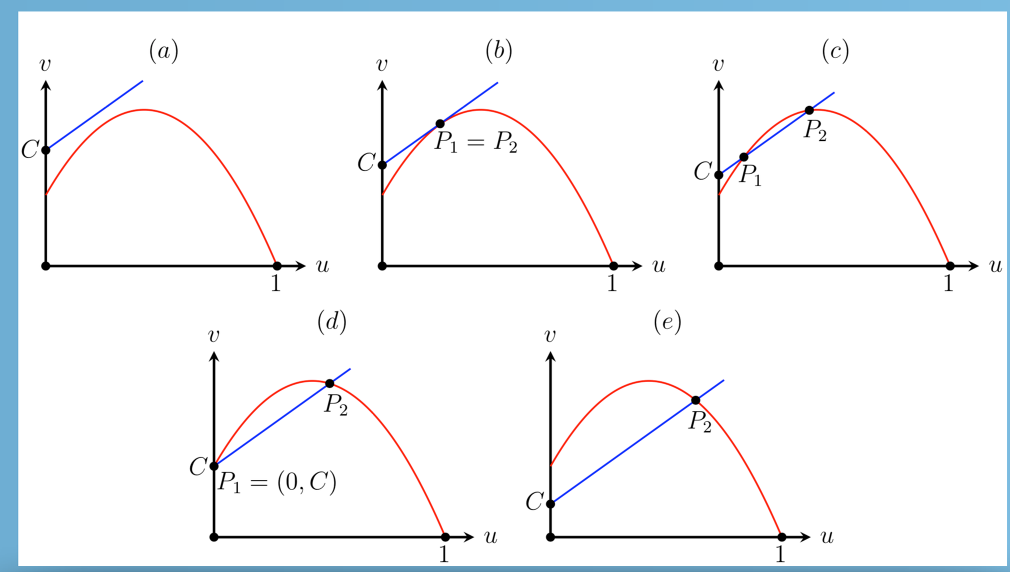
[1] P. Turchin, Complex population dynamics: a theoretical/empirical synthesis, Vol. 35 of Monographs in population biology, 2003.
[2] I. Hanski, L. Hansson and H. Henttonen, Specialist predators, generalist predators, and the microtine rodent cycle, The Journal of Animal Ecology (1991) 353-367.
[3] D. Wollkind, J. Collings and J. Logan, Metastability in a temperature-dependent model system for predator-prey mite outbreak interactions on fruit trees, Bulletin of Mathematical Biology (1988) 379-409.
[4] I. Hanski, H. Henttonen, E. Korpimaki, L. Oksanen and P. Turchin, Small-rodent dynamics and predation, Ecology (2001) 1505-1520.
[5] R. May, Stability and complexity in model ecosystems, Vol. 6 of Monographs in population biology, 1974.
[6] M. Andersson and S. Erlinge, Influence of predation on rodent populations, Oikos (1977) 591-597.
[7] S. Erlinge, Predation and noncyclicity in a microtine population in southern Sweden, Oikos (1987) 347-352.
[8] L. Hansson, Competition between rodents in successional stages of taiga forests: Microtus agrestis vs. Clethrionomys glareolus, Oikos (1983) 258-266.
[9] R. McDonald, C. Webbon and S. Harris, The diet of stoats (Mustela erminea) and weasels (Mustela nivalis) in great britain, Journal of Zoology (2000) 363-371.
[10] A. Korobeinikov, A Lyapunov function for Leslie-Gower predator-prey models, Applied Mathematics Letters (2001) 697-699.
[11] C. Arancibia-Ibarra and E. González-Olivares, The Holling-Tanner model considering an alternative food for predator, Proceedings of CMMSE 2015 (2015) 130-141.
[12] A. Turing, The chemical basis of morphogenesis, Bulletin of mathematical biology (1990) 153–197.

Temporal Model

The equilibrium points of system (4) without diffusion are (0,0), (1,0), (0,C) and the interior equilibrium points:

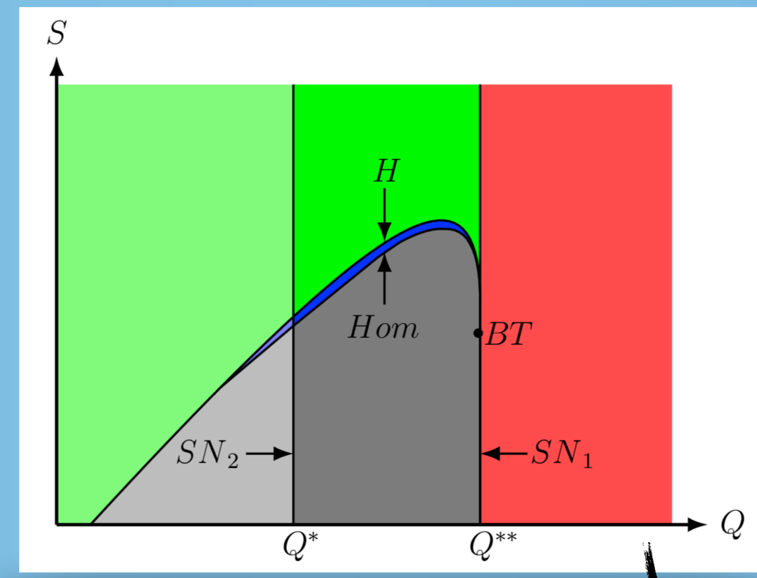
$$
P_{1,2} = (u_{1,2}, u_{1,2}+C) \text{ where } u_{1,2} = \frac{1}{2}\left(1-A-Q \pm \sqrt{(1-A-Q)^2+4(A-CQ)}\right)
$$

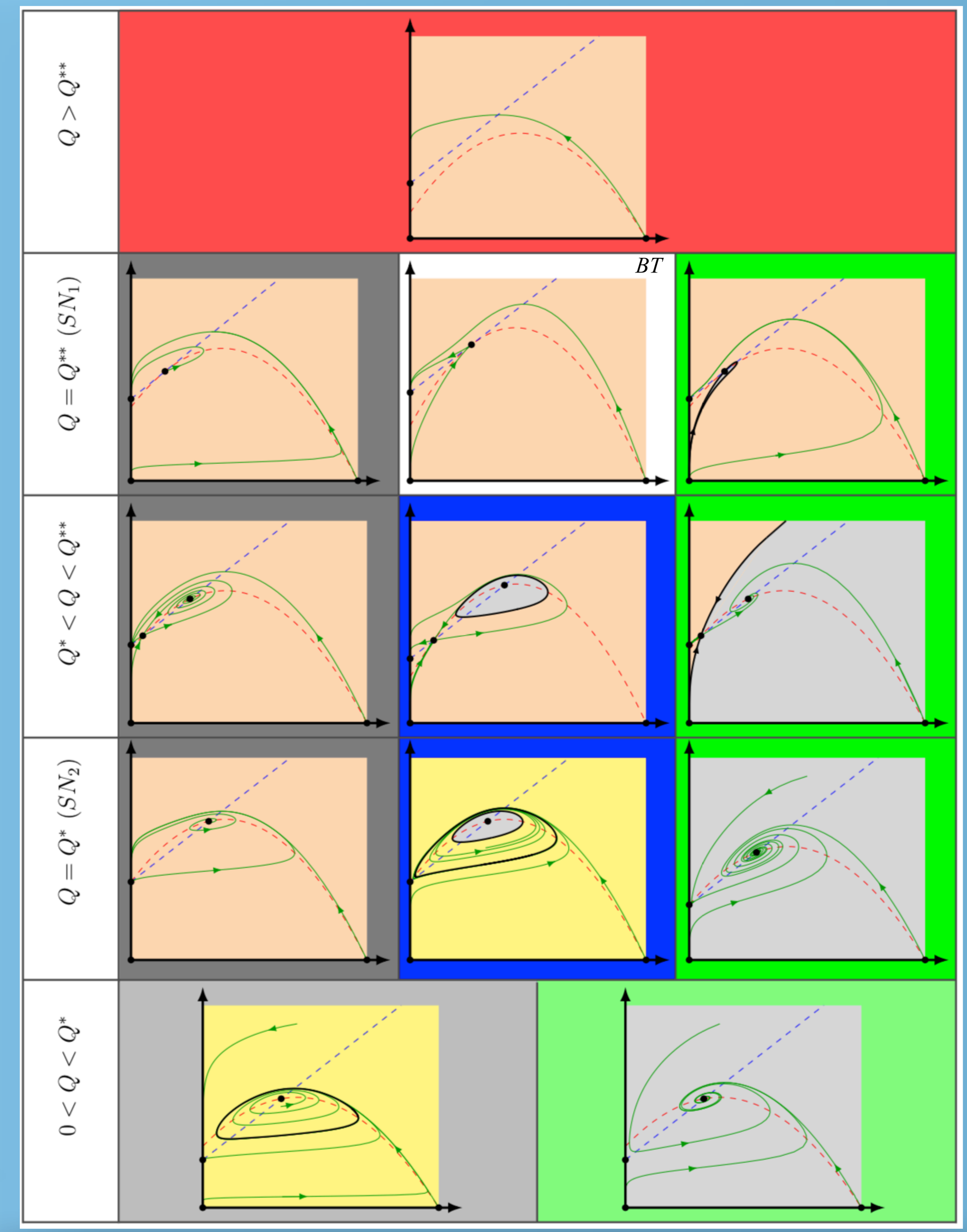
The equilibrium point *(0,0)* is always unstable, *(1,0)* which is always a saddle point. Moreover, *(0,C)* can be attractor if $CQ-A>0$ (see (a)-(c)), saddle node if $CQ-A=0$ (see (d)) and a saddle point if $CQ-A<0$ (see (e)). While, there are at most two positive equilibrium points P_1 and P_2. The positive equilibrium point P_1 is always a saddle point. when it is located in the firs quadrant. While, P_2 can be stable or unstable [11].



Temporal Simulation

The bifurcation diagram of system (4) without diffusion for *(A,C)=(0.15,0.28)* fixed and created with the numerical bifurcation package MATCONT. The curve H represents the Hopf curve where P_2 changes stability, Hom represents the homoclinic curve of P_1 and $SN_{1,2}$ represents the saddle-node curve where $\Delta=0$ and $u_1=0$ respectively. Moreover, BT represents the Bogdanov-Takens bifurcation.





Spatio-temporal Model

We first recall that the mathematical criteria for the Turing pattern formation for spatio-temporal predator-prey model where individuals are assumed to be distributed over one-dimensional and two-dimensional bounded domain. Moreover, if we linearise the full system that includes the diffusive terms [12], about the equilibrium point P_2 we arrive at

$$
w_t = J(u, u+C)w + D\nabla^2 w \text{ where } D = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}
$$

Therefore, the dispersion relation $\lambda(k)$ is determined by the roots of the characteristic polynomial:

$$
|\lambda I - AP_2 + Dk^2| = \left| \begin{pmatrix} F_u(u_2) & F_v(u_2) \\ G_u(u_2) & G_v(u_2) \end{pmatrix} - \begin{pmatrix} k^2 & 0 \\ 0 & dk^2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right|
$$

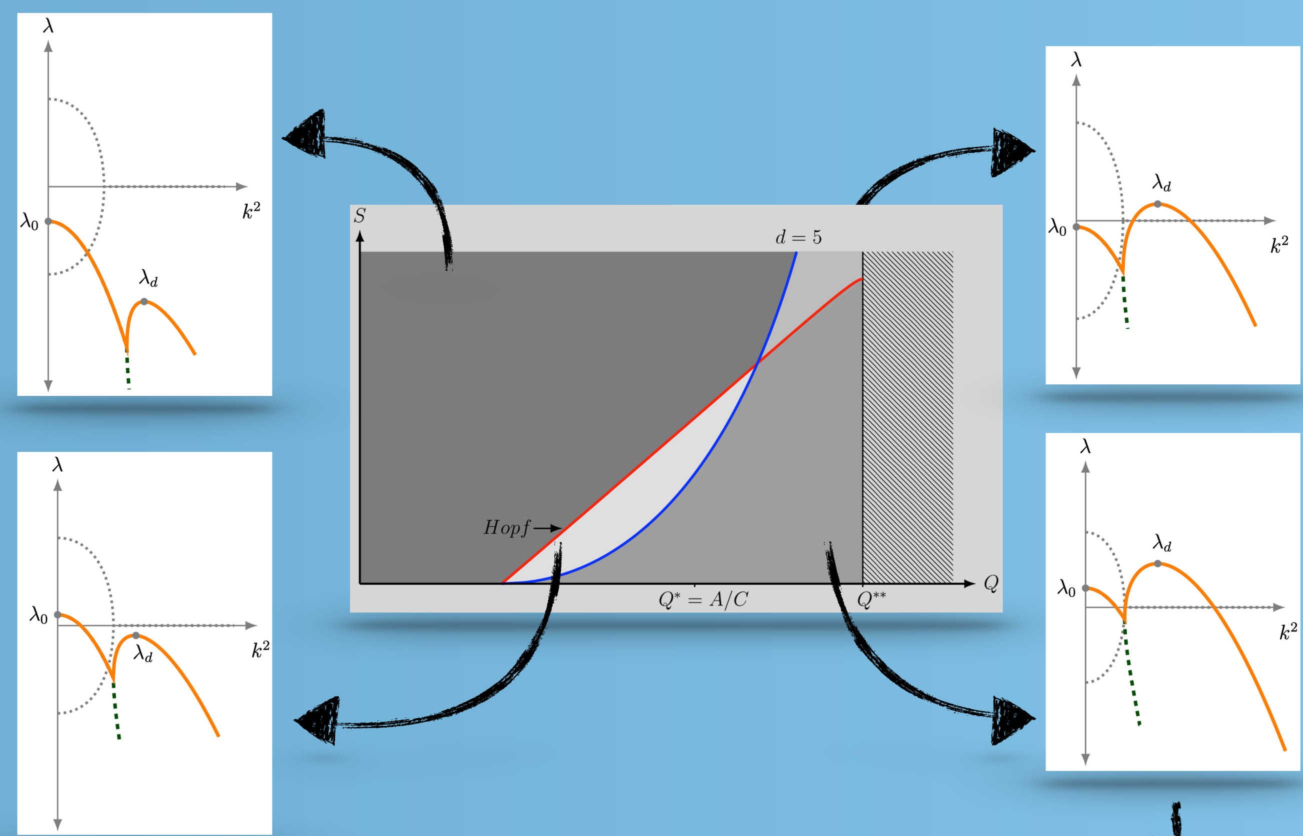
With all of the conditions met at the equilibrium point P_2, that is for the diffusion-free system:

$$
(i)\ \frac{u_2 S}{u_2+A}(-1+A+Q+2u_2) > 0 \text{ and } (ii)\ \frac{u_2 S}{u_2+A}(1-A-2u_2) - S < 0
$$

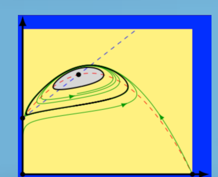
and the conditions for diffusive instability:

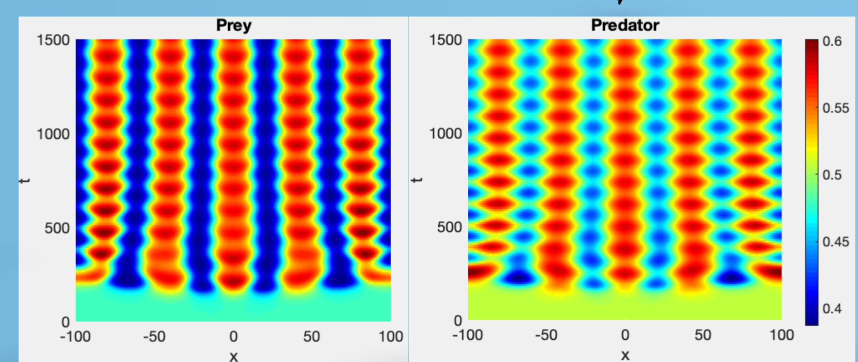
$$
(iii)\ \frac{du_2}{u_2+A}(1-A-2u_2) - S > 0 \text{ and } (iv)\ \left(\frac{du_2}{u_2+A}(1-A-2u_2) - S\right)^2 - \frac{4du_2S}{u_2+A}(-1+A+Q+2u_2) > 0
$$

The Turing parameter space *(Q,S)* of the equilibrium point P_2 for the system parameters *(A,C,d)=(0.15,0.28,5)* fixed. Note that if $Q>Q^{**}$ then there are no positive equilibrium points in system (4) without diffusion and *(0,C)* is stable.



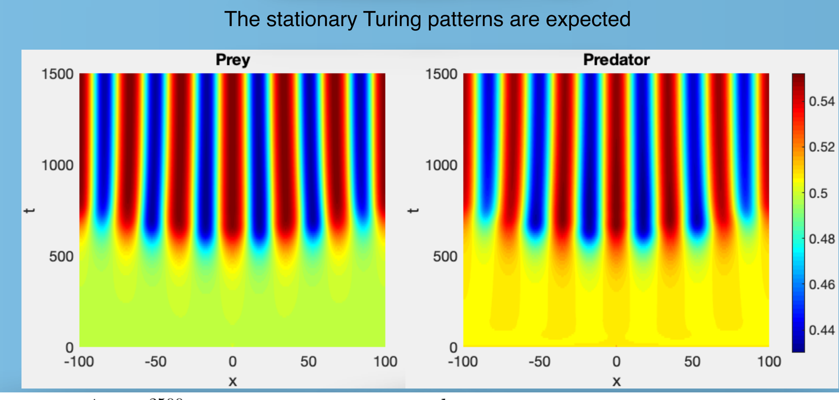
Simulation in 1D

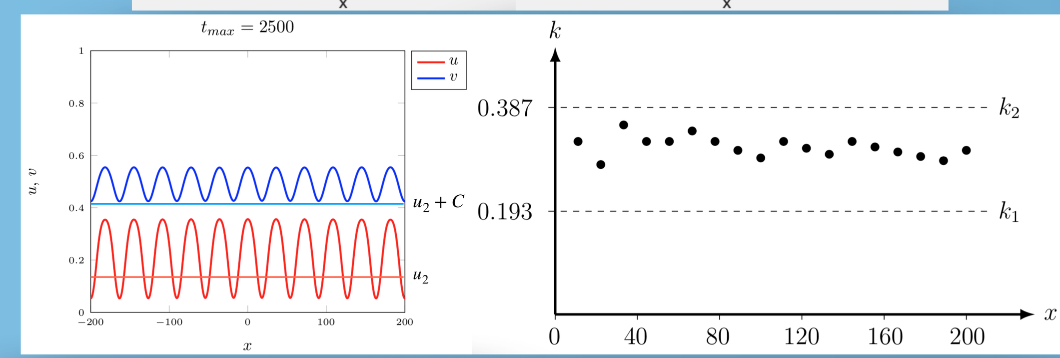




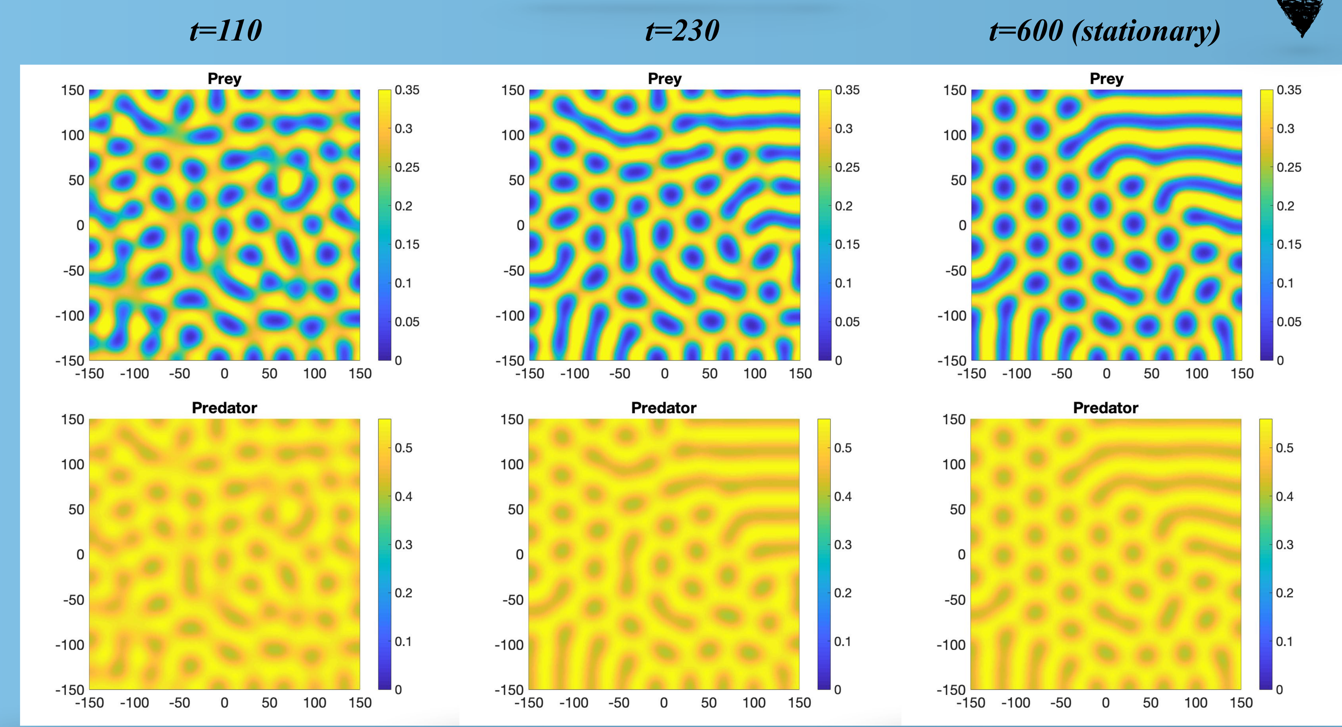
Simulation in 1D

The stationary Turing patterns are expected





Simulation in 2D



Variation of the diffusion coefficient