Punjab University
Journal of Mathematics
Vol. xxx (1997) pp.109 - 123

A REPRESENTATION OF THE PROLONGATIONS OF A G-STRUCTURE

Ebrahim Esrafilian, Mehdi Nadjafikhah
Department of Mathematics, Iran University of Science and Technology,
Narmak, 16-Tehran, Iran.

[September 2, 1995]

ABSTRACT: In this paper, we describe the general group of order two GP_n^2. We prove an arbitrary prolongation of a Lie subgroup of $GL(n,\Re)$ is a direct sum of additive Lie group of the form $\Re^{\tilde{n}}$ and a Lie sub-group of $GL(n,\Re)$. Then we show that an arbitrary prologation of a Lie subalgebra of $Mat(n\times n)$ is a direct sum of an additive Lie subalgebra of the form $\Re^{\tilde{n}}$ and a Lie subalgebra of $Mat(n\times n)$. In conclusion structure group of every k'th order Geometric structure on a given n dimmentinal manifold is isomorphic to an additive standard group $\Re^{\tilde{n}}$, with $0\le\tilde{n}\le k\times\dfrac{n^2(3n-1)}{2}$, and a Lie subgroup of $GL(n,\Re)$.

Key Words: G-structure, Matrix Lie group, Prolongation, Vector bundle. 1991 MSC: 53 C 15.

1. INTRODUCTION

In this paper, all manifolds are finite dimentional paracompact and, all mappings and functions are smooth.

Let M and M' be two manifolds and $\phi:M\to M'$ be an immersion, and also assume that $m\in Dom(\phi)$, $\phi(m)=m'$, (x,U) is a chart contains m, and

$((x',U')$ is a chart around of m'. The k'th order jet $j^k_m\phi$ of ϕ at m is denoted by the following coordinates:

$$\left(x^i, x^{lj}, x^{lj}_{i_1}, \ldots, x^{lj}_{i_1,\ldots,i_k}\right)$$

$$x^{lj}_{i_1,\ldots,i_t} := \frac{\partial^t\left(x^l o\phi o x^{-1}\right)}{\partial x_{i_1},\ldots,\partial x_{i_t}}\Big|_{x(m)},$$

where $i, i_1, i_2,\ldots, i_k$ vary in the set $\{1,2,\ldots, dimM'\}$. The $x^j_{i_1,\ldots,ik}$ will not change by any permutations in the lower indices.

Let G be a Lie subgroup of $GL(n,\Re)$ (the general linear group) and G a Lie subalgebra of $Mat(n\times n)$ (Lie algebra of $n\times n$ square matrices with real entries). We denote the $k'th$ prolongation of G and G, by $G^{(k)}$ and $G^{(k)}$ respectively.

The group of all invertible *k-jets* with source and target in 0 (the zero of $\Re^n$), is denoted by GP^k_n This is a Lie group which is proved that $GP^k_n \cong [GL(n,\Re)]^k$

By Reinhart's notation, an element of GP^k_n can be represented by an n-tuple $(f_1, f_2,\ldots, f_n)$, where f_i, for $i = 1,2,\ldots,n$, is a polynomial of variables of the form

$$f_i(0) = 0,\ \det\left[\frac{\partial f_i}{\partial x_j}\right] \neq 0$$

In this notation, the operation in GP^k_n is

$$(f_1,\ldots f_n) \bigstar (g_1, \ldots, g_n) = (f_1(g_1,\ldots,g_n),\ldots f_n(g_1,\ldots,g_n)).$$

2. STRUCTURE OF GP_n^2

Proposition 1

Let n be a natural number. Then there exist Lie group isomorphism

$$GP_n^2 \cong \Re^{\frac{n^2(3n-1)}{2}} \oplus GL(n,\Re),$$

where $\Re^{\frac{n^2(3n-1)}{2}}$ *is standard additive Lie group of* $\Re^{\frac{n^2(3n-1)}{2}}$

Proof

Let $M: = \left\{ j_0^2\phi \in GP_n^2 \mid j_0^1\phi = [\delta_{ij}] \right\},$

$N: = \left\{ j_0^2\phi \in GP_n^2 \mid j_0^2\phi = j_0^1\phi \right\}.$

We prove this proposition in steps (a) to (g).

a) M is Lie subgroup of GP_n^2.

For, let $j_0^2\phi$ and $j_0^2\psi$ are in M. Then $(j_0^2\phi)*(j_0^2\psi) = j_0^2(\phi o \psi)$ and we have

$$j_0^1(\phi o \psi) = (j_0^1\phi)*(j_0^1\psi) = [S_{ij}]$$

Moreover, since $(j_0^2\phi)^{-1} = j_0^2(\phi^{-1})$ we obtain

$$j_0^1(\phi^{-1}) = (j_0^1\phi)^{-1} = [S_{ij}]$$

Therefore M is a subgroup of GP_n^2, and furthermore with charts

$$M \ni j_0^2\phi \to \left(\frac{\partial^2\phi^j}{\partial x_{i_1}\partial x_{x_2}}\right) \in \Re^{\frac{n^2(3n-1)}{2}},$$

is a Lie subgroup of GP_n^2.

b) M is a normal subgroup in GP_n^2

For, let $j_0^2\phi$ belongs to M and $j_0^2\psi$ be a member of GP_n^2, then

$$(j_0^2\psi^{-1})\star(j_0^2\phi)\star(j_0^2\psi) = j_0^2(\psi^{-1} \circ \phi \circ \psi).$$

On the other hand,

$$\begin{aligned} j_0^1(\psi^{-1}\phi\psi) &= (j_0^1\psi)^{-1})\star(j_0^1\phi)\star(j_0^1\psi) \\ &= (j_0^1\psi^{-1})\star[\delta_{ij}]\star(j_0^1\psi) \\ &= [\delta_{ij}] \end{aligned}$$

therefore $(j_0^2\psi)^{-1}\star(j_0^2\phi)\star(j_0^2\psi)$ belongs to M.

c) M is isomorphic with the additive Lie group $\Re^{\frac{n^2(3n-1)}{2}}$.

For, we define the function $\eta : M \to \Re^{\frac{n^2(3n-1)}{2}}$ as follow:

$$M \ni \left(\delta_{ij}, \frac{\partial^2\phi^j}{\partial x_{i_1}\partial x_{i_2}}|0\right) \to \left(\frac{\partial^2\phi^j}{\partial x_{i_1}\partial x_{i_2}}|0\right) \in \Re^{\frac{n^2(3n-1)}{2}}$$

The smoothness of function is easily proved. Then it is just enough to prove that it is a "group isomorphism".

For this, suppose that

$$\left(\ldots,x_k + a^k_{i_1i_2}x_{i_1}x_{i_2},\ldots\right), \left(\ldots,x_1 + b^l_{j_1j_2}x_{j_1}x_{j_2},\ldots\right),$$

two elements of M. Then

$$\eta\left((\ldots, x_k + a^k_{i_1 i_2} x_{i_1} x_{i_2}, \ldots) \star (\ldots, x_1 + b^l_{j_1 j_2} x_{j_1} x_{j_2}, \ldots)\right),$$

$$= \eta\ (\ldots, \left(x_k + b^l_{j_1^k j_2^k} x_{j_1^k} x_{j_2^k}\right) + a^k_{i_1 i_2}\left(x_{i_1} + b^{i_1}_{j_1^{i_1} j_2^{i_1}}\ x_{j_1^{i_1}}\ x_{j_2^{i_1}}\right)$$

$$\times \left(x_{i_2} + b^{i_2}_{j_1^{i_2} j_2^{i_2}}\ x_{j_1^{i_2}}\ x_{j_2^{i_2}}\right), \ldots.$$

$$= \eta\left(\ldots, x_k + b^k_{j_1^{i_1} j_2^{i_2}} + a^k_{i_1 i_2}\ x_{i_1}\ x_{i_2}, \ldots\right)$$

$$= \eta\left(\ldots, x_k + \left(a^k_{i_1 i_2} + b^k_{i_1 i_2}\right) x_{i_1} x_{i_2}, \ldots\right)$$

$$= \left(\ldots, a^k_{i_1 i_2} + b^k_{i_1 i_2}, \ldots\right)$$

$$= \eta\left(\ldots, x_k + a^k_{i_1 i_2} x_{i_1} x_{i_2}, \ldots\right) + \eta\left(\ldots, x_k + b^k_{i_1 i_2} x_{i_1} x_{i_2}, \ldots\right)$$

d) N is a normal Lie subgroup of GP_n^2

For, let $j_0^2\phi$ and $j_0^2\psi$ are in N. Then $(j_0^2\phi)\star(j_0^2\psi) = j_0^2(\phi o \psi)$ and

$$j_0^2(\phi o \psi) = (j_0^2\phi)\star(j_0^2\psi)$$

$$= (j_0^1\phi)\star(j_0^1\psi)$$

$$= j_0^1(\phi o \psi)$$

also $(j_0^2\phi)^{-1} = j_0^2(\phi^{-1})$, and

$$j_0^2(\phi^{-1}) = (j_0^2\phi)^{-1} = (j_0^1\phi)^{-1} = j_0^1(\phi^{-1})$$

therefore N is a subgroup of GP_n^2

Let $j_0^2\phi$ belongs to N and $j_0^2\phi$ belongs to GP_n^2. Then

$$(j_0^2\psi)^{-1}\star(j_0^2\phi)\star(j_0^2\psi) = j_0^2(\psi^{-1} \circ \phi \circ \psi).$$

Since $j_0^1\phi = j_0^2\phi$, then $(j_0^1\phi)\star(j_0^2\psi) = (j_0^2\phi)\star(j_0^2\psi)$; but $(j_0^1\phi)\star(j_0^2\psi) = (j_0^1\phi)\star(j_0^1\psi)$, therefore $(j_0^1\phi)\star(j_0^1\psi) = (j_0^2\phi)\star(j_0^2\psi)$. Hence we have

$$(j_0^1\phi)\star(j_0^1\psi) = (j_0^2\psi)\star [(j_0^2\psi^{-1})\star(j_0^2\phi)\star(j_0^2\psi)]$$

$$= (j_0^1\psi)\star [(j_0^2\psi^{-1})\star(j_0^2\phi)\star(j_0^2\psi)].$$

Hence $j_0^1(\psi^{-1} o \phi o \psi) = j_0^2(\psi^{-1} o \phi o \psi)$, and N is normal in GP_n^2. On the other hand the function

$$\eta : N \ni j_0^2\phi \rightarrow j_0^1\phi \in GL(n,\Re) \subseteq \Re^{n^2}$$

induced a Lie subgroup structure on N.

e) N is isomorphic with $GL(n,\Re)$.

For, Let η be a function which is defined in (d) step. We have $\eta\left(\left(\ldots, a_i^k x_i, \ldots\right)\star\left(\ldots, b_i^k x_i, \ldots\right)\right) = \eta\left(\ldots, \sum_j a_j^k b_i^j x_i, \ldots\right)$

$$= \left[\sum_j a_j^k b_i^j\right]$$

$$= \left[a_j^k\right]\left[b_j^k\right]$$

$$= \eta\left(\ldots, a_i^k x_i, \ldots\right)\eta\left(\ldots, b_i^k x_i, \ldots\right)$$

therefore η is a Lie group isomorphism from N onto $GL(n,\Re)$.

f) $M \cap N = \{j_0^2\, id\}$, where id is identity function on $\Re^n$. For, let $j_0^2\phi \in M \cup N$. Then $j_0^2\phi \in M$ and $j_0^1\phi = j_0^1\, id$. On the other hand $j_0^2\phi \in N$ and $j_0^2\phi = j_0^1\phi$. Therefore $j_0^2\phi = j_0^2 id$.

g) GP_n^2 as a Lie group, is isomorphic to $M \oplus N$.

For, let $j_0^2\phi = \left(\ldots,\ a_i^k x_i + a^2_{i_1 i_2} x_{i_1} x_{i_2}\ ,\ldots\right)$ belongs to GP_n^2, $j_0^2\zeta = (\ldots,\ A_i^k\ x_i,\ \ldots)$ belongs to N and $j_0^2\psi = \left(\ldots,\ x_k + A^k_{i_1 i_2} x_{i_1} x_{i_2},\ \ldots\right)$ belongs to M such that

$$j_0^2\phi = \left(j_0^2\psi\right) * \left(j_0^2\zeta\right),$$

then we have

$$A_i^k = a_i^k,\ \sum_{i_1, i_2} A^k_{i_1 i_2} A_j^{i_1} A_l^{i_2} = a^k_{jl},$$

thus, for all k, l and j

$$\sum_{i_1, i_2} A^k_{i_1 i_2} a_j^{i_1} a_l^{i_2} = a^k_{jl}.$$

Let l be fixed, then

$$\left[\sum_{i_1} A^k_{i_1 i_2} a_j^{i_1}\right] \left[a_l^{i_2}\right] = \left[a^k_{jl}\right],$$

and

$$\sum_{i_1} A^k_{i_1, i_2} a_l^{i_1} = \sum_s a^k_{js} (a^{-1})^{i_2}_s,$$

now if i_2 is fixed, then

$$\left[A^k_{i_1 i_2}\right]\left[a^{i_1}_j\right]=\left[\sum_s a^k_{js}(a^{-1})^{i_2}_s\right],$$

therefore $$A^k_{i_1 i_2}=\sum_t\sum_s a^k_{ts}(a^{-1})^{i_2}_s(a^{-1})^{i_1}_t.$$

Where $\left[a^i_j\right]^{-1}=\left[(a^{-1})^j_i\right]$. Hence GP_n^2, as an abstract group, is a direct sum of M and N.

Finally by corollary at page 96 of [3], we access which we required.□

Corollary 1

Let G be a Lie subgroup of $GL(n,\Re)$. Then there exists a Lie subgroup $\tilde{G}$ of $GL(n,\Re)$ and an integer $\tilde{n}$ such that $0\le\tilde{n}\le\dfrac{n^2(3n-1)}{2}$, and the Lie group $G^{(1)}\cong\Re^{\tilde{n}}\oplus\tilde{G}$, where $\Re^{\tilde{n}}$ is the standard additive Lie group of $\tilde{\Re}^{\tilde{n}}$

3. STRUCTURE OF GP_n^k

Lemma 1

Let G and H be two Lie subalgebras of $Mat(n\times n)$, then

$$(H\oplus G)^{(1)}\cong H^{(1)}\oplus G^{(1)}$$

Proof

We note that (refer to [1])

$$G^{(1)} \cong Hom(\mathfrak{R}^n, G) \cap (\mathfrak{R}^n \otimes S^2(\mathfrak{R}^{n*})) \qquad (1)$$

therefore

$$\begin{aligned}(G \oplus H)^{(1)} &\cong Hom(\mathfrak{R}^n, G \oplus H) \cap (\mathfrak{R}^n \otimes S^2(\mathfrak{R}^{n*})) \\ &\cong (Hom(\mathfrak{R}^n, G) \oplus Hom(\mathfrak{R}^n, H)) \\ &\quad \cap [(\mathfrak{R}^n \otimes S^2(\mathfrak{R}^{n*})) \oplus (\mathfrak{R}^n \otimes S^2(\mathfrak{R}^{n*}))] \\ &= [Hom(\mathfrak{R}^n, G) \cap (\mathfrak{R}^n \otimes S^2(\mathfrak{R}^{n*}))] \\ &\quad \oplus [Hom(\mathfrak{R}^n, H) \cap (\mathfrak{R}^n \otimes S^2(\mathfrak{R}^{n*}))] \\ &= G^{(1)} \otimes H^{(1)}. \square\end{aligned}$$

Example 1

We have proved that $<\mathfrak{R}^n, +>^{(1)} \cong <\mathfrak{R}^n, +>$.

a) It is proved that $L(<\mathfrak{R}^n, +>) \cong (\mathfrak{R}^n, +)$.

b) As a Lie algebra $<\mathfrak{R}^n, +>$ is isomorphic to $\Delta Mat(n \times n)$, where $\Delta Mat(n \times n)$ is the Lie subgroup of all $n \times n$ diagonal matrices of $Mat(n \times n)$.

For we define

$$\Psi : <\mathfrak{R}^{n+}, \times> \to \Delta Mat(n \times n)$$

$$(x_1, \ldots, x_n) \to [\delta_{ij} x_i].$$

c) We prove that "as a vector space $<\mathfrak{R}^n, +>^{(1)}$ (prolongation of Lie algebra $<\mathfrak{R}^n, +>$) is isomorphic to $<\mathfrak{R}^n, +>$".

For this let T belongs to $(\mathfrak{R}^n)^{(1)}$. Therefore T is a linear mapping of $\mathfrak{R}^n \times \mathfrak{R}^n$ into $\mathfrak{R}^n$. Let $T(e_i.e_j)\ \sum_{ij} T_{ij}^k\ e_k$, where $\{e_1, \ldots, e_2\}$ is standard basis for $\mathfrak{R}^n$, and by definition, T_{ij}^k =

T_{ji}^{k} and $T_{ij}^{k} \in \Delta\, Mat(n\times n)$ for all i,j,k. Thus $T_{ij}^{k} = \delta_{jk}$ and $T_{ij}^{k} \neq 0 \Leftrightarrow i = j = k$. To see the result we define the mapping

$$\Gamma : <\Re^n,+>^{(1)} \ni [T_{ij}^{k}] \rightarrow (T_{ii}^{i}) \in <\Re^n,+>$$

e) By [4], if G is a Lie subgroup of $GL(n,\Re)$, then Lie group $G^{(1)}$ is isomorphic with group of all linear mappings of $\Re^n + G$ of the form a_T (for $T \in G^{(1)}$) where

$$a_T(A,v) = (v, A+T(v,.)),\quad A \in G,\ v \in \Re^n$$

Therefore prolongation of Lie group $(\Re^n,+>$ consist of all linear mappings of $\Re^n + L(\Re^n) \cong \Re^n$ of the form a_T (for $T = [T_{ij}^{k}] \in (\Re^n)^{(1)} \cong \Re^n$) where

$$\begin{aligned} a_T(A,v) &= (v, A + [T_{ij}^{k}](v,.)) \\ &= \left(v,\ A + tras \sum \left(T_{ii}^{i} v_i + a_i\right) \vec{e}_i(.)\right) \\ &= \left(\sum v_i \vec{e}_i,\ trans\ \Sigma\left(t_{ii}^{i} v_i +\right) \vec{e}_i(.)\right) \end{aligned}$$

Here the $trans_{\vec{\omega}}$ (.) is translation by $\vec{\omega}$ in $\Re^n$. It completes the proof. □

Lemma 2

Let G and H be two matrix Lie subgroups. Then

$$[H \oplus G]^{(1)} \cong H^{(1)} \oplus G^{(1)}.$$

Proof

Suppose that H be a Lie subgroup of $GL(n, V)$ with $L(H) = \mathcal{H}$ and G be a Lie subgroup of $GL(n, W)$ with $L(G) = G$, and let $(v_1 \oplus \omega_1) \otimes (s_1 \oplus t_1) = (v_1 \otimes s_1) \oplus (\omega_1 \otimes t_1)$ be an elements of $(V \oplus W) \otimes S^{k+1}((V \oplus W)^*)$ and $(v_2 \oplus w_2) \otimes (s_2 \oplus t_2) = (v_2 \otimes s_2) \oplus (w_2 \otimes t_2)$ be an elements of $(V \oplus W) \otimes S^{k+1}((V \oplus W)^*)$.

Now, we define the bracket of these two elements (denoted by symbol"[,]") as follows:

$$(v_2 \oplus w_2) \otimes D_{(v1 \oplus w1)} (s_2 \oplus t_2) \circ (s_1 \oplus t_1)$$

$$- (v_1 \oplus w_1) \otimes D_{(v1 \oplus w1)} (s_1 \oplus t_1) \circ (s_2 \oplus t_2)$$

$$= [v_2 \otimes D_{v1}\ s_2 \circ s_1 - v_1 \otimes D_{v2}\ s_1 \circ s_2]$$

$$\oplus [w_2 \otimes D_{w1}\ (t_2 \circ t_1) - w_1 \otimes D_{w2}\ (t_1 \circ t_2)].$$

Note that this lies in $(V \oplus W) \otimes S^{k+l+1}((V \oplus W)^*)$.

"[,]" extends a bilinear mapping of $((V \oplus W) \otimes S^{k+1}((V \oplus W)^*) \times (V \oplus W) \otimes S^{l+1}((V \oplus W)^*))$ into $(V \oplus W) \otimes S^{k+l+1}((V \oplus W)^*)$. Recalling (1), this induces a bilinear mapping of $(\mathcal{H} \oplus G)^{(k)} \otimes (\mathcal{H} \oplus G)^{(l)} = (\mathcal{H}^{(k)} \otimes \mathcal{H}^{(l)} \oplus G^{(k)} \otimes G^{(l)})$ into $(V \oplus W) \otimes S^{k+l+1}((V \oplus W)^*)$, which is in fact a bilinear mapping into $(\mathcal{H} \oplus G)^{(k+l)} = \mathcal{H}^{(k+l+1)} \oplus G^{(k+l+1)}$. Moreover "[,]" makes the vector space

$$(V \oplus W) + (\mathcal{H} \oplus \mathrm{G}) + (\mathcal{H} \oplus \mathrm{G})^{(1)} + \ldots$$

$$= [V + \mathcal{H} + \mathcal{H}^{(1)} + \ldots] \oplus [W + \mathrm{G} + \mathrm{G}^{(1)} + \ldots].$$

into a Lie algebra. But, the bracket operation on the Lie algebra $(G \oplus H)^{(1)}$ coincides with the bracket operation already defined on

$$(\mathcal{H} \oplus \mathrm{G}) + (\mathcal{H} \oplus \mathrm{G})^{(1)} + \ldots + (\mathcal{H} \oplus \mathrm{G}) + (\mathcal{H} \oplus \mathrm{G})^{(k)} + (\mathcal{H} \oplus \mathrm{G})^{(k+1)} + \ldots.$$

truncated at degree k (refer to [1]). Thus as a Lie algebra

$$L((\mathcal{H} \oplus G)^{(1)}) \cong L(H)^{(1)} \oplus L(G)^{(1)}.$$

This proves the lemma. □

Lemma 3

Let G be a Lie subalgebra of $Mat(n \times n)$. There exist a Lie subalgebra $\tilde{G}$ of $Mat(n \times n)$ and an integer $\tilde{n}$ such that $0 \le \tilde{n} \le \frac{n^2(3n-1)}{2}$ and the Lie algebra $G^{(1)} \cong \Re^{\tilde{n}} \oplus \tilde{G}$, where $\Re^{\tilde{n}}$ is the standard additive Lie algebra of $\Re^{\tilde{n}}$.

Proof

Let G be a Lie subgroup of $GL(n,\Re)$ where it's Lie algebra is G (in this case we write $L(G)$ = G). Now we have, by corollary 1:

$$G^{(1)} \cong \Re^{n'} \oplus \tilde{G},\ \tilde{G} \le GL(n,\Re),\ 0 \le n' \le \frac{n^2(3n-1)}{2}$$

On the other hand (refer to [1]) we know that $L(G^{(1)}) \cong \mathrm{G} \oplus \mathrm{G}^{(1)}$; therefore

$$zG \oplus G^{(1)} \cong L(\Re^{n'}) \oplus L(\tilde{G})$$

Hence, there exists a Lie subalgebra G of $L(\tilde{G})$ (and Lie subalgebra of $Mat(n \times n)$ and an integer n' with $0 \le n' \le n$ such that $G^{(1)} \cong \Re^{n'} \oplus (\tilde{G})$.□

Proposition 2

Let n and m are two natural numbers. Then there exists Lie group isomorphism

$$GP_n^m \cong \Re^{m \times \frac{n^2(3n-1)}{2}} \oplus GL(n,\Re),$$

where $\Re^{m \times \frac{n^2(3n-1)}{2}}$ has standard Lie group structure.

Proof

Let m be an integer greater than 2. Assume that result is proved for $m-1$. Then

$$GP_n^m \cong \left(GP_n^{m-1}\right)^{(1)}$$

$$\cong \left[\Re^{\frac{n^2(3n-1)}{2}} \oplus GL(N,\Re)\right]^{(1)}$$

$$\cong \left[\Re^{\frac{n^2(3n-1)}{2}}\right]^{(1)} \oplus [GL(n,\Re)]^{(1)}$$

$$\cong \Re^{\frac{n^2(3n-1)}{2}} \oplus GP_n^{m-1} \qquad \text{(by example 1)}$$

$$\cong \Re^{m-1 \times \frac{n^2(3n-1)}{2}} \oplus GL(n,\Re) \qquad \text{(by assumption)}$$

Then by induction Proposition is proved.□

Corollary 2

Let G be a Lie subgroup of $GL(n,\Re)$, and k be an integer. Then there exists a Lie subgroup $\tilde{G}$ of $GL(n\Re)$ and an integer $\tilde{n}$ such that $0 \le \tilde{n} \le \frac{n^2(3n-1)}{2}$, and the Lie group $G^{(1)} \cong \Re^{\tilde{n}} \bigoplus \tilde{G}$, where $\Re^{n}$ is the standard additive Lie group of $\Re^{n}$.□

Example 2

We study the $GP_1^3 = \{ax^3+bx^2+cx \mid a,b,c \in \Re,\ c\neq 0\}$ where proved that

$$(ax^3+bx^2+cx) * (Ax^3+Bx^2+Bx^2+Cx)$$

$$= (aC^3+2bBC+cA)x^3 + (bC^2+cB)x^2 + cCx.$$

Let $M = \{ax^3+bx^2+x \mid a,b \in \Re\}$ and $N = \{ax \mid a \in \Re-\{0\}\}$. Then N and M are normal subgroup of GP_1^3 and $GP_1^3 \cong M \oplus N$. On the other hand we have proved that N is isomorphic with multiplicative group $\Re-\{0\}$. For M, assume $T = \{ax^3+x \mid a \in \Re\}$ and $S = \{a^2x^3+ax^2+x \mid a \in \Re\}$. We know that T and S are normal subgroups of M and $M \cong S \oplus T$. But with respect to above operation we have

$$(a^2x^3+ax^2+x)*(A^2x^3+Ax^2+x) = (A+a)^2x^3+(A+a)x^2+x,$$

$$(ax^3+x)*(Ax^3+x) = (A+a)x^2+x$$

Therefore S and T are isomorphic with additive group $\Re$. In conclusion

$$GP_1^3 \cong (\Re^2,+) \oplus (\Re-\{0\}, x)$$

Corollary 3

Structure group of every k'th order geometric structure on a given n dimmentinal manifold is isomorphic with an additive standard group $\Re^m$,

where $0 \leq m \leq k \times \dfrac{n^2(3n-1)}{2}$, and Lie subgroup of $GL(n,\Re)$.□

Proposition 3

Let G be a Lie subalgebra of Mat(n×n), and k be a natural number.

Then there exists a Lie subalgebra $\tilde{G}$ of mat(n×n) and an integer $\tilde{n}$

such that $0 \le \tilde{n} \le k \times \frac{n^2(3n-1)}{2}$ and the Lie algebra where $G^{(1)} \cong \Re^{\tilde{n}} \oplus \tilde{G}$

$\Re^{\tilde{n}}$ has the standard Lie algebra structure. □

REFERENCES

1. Guilemin, V.; The integrability problem for G-structures; *Tran. Am. Math. Soc.,* Vol. 116, 1965, pp. 544-560.

2. Molino, P.; Teorie des G-structures: Le problem d'equivalence; L.M.N., Springer Verlag, 1979.

3. Postnikov, M.; Lie groups and Lie algebras; Lectures in geometry, semester V, Mir Pub., Moscow, 1986.

4. Strenberg, S.; Lectures on Differential Geometry; Chelsea Pub. Co., New York, 1983.

5. Yang, K.; Exterior Differential Systems and Equivalence problems; Kluwer Academic Publishers, Netherland, 1992.