MINIMUM COST TOPOLOGICAL ORDERING

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Abstract. We are given an n vertex directed graph $G = (V, E)$ and also given a cost function $c : V \times [n] \to \mathbb{R}$. Consider a topological ordering of the vertices, $v_1, \ldots, v_n$, the cost of the ordering is $\sum_{i=1}^{n} c(v_i, i)$. We shall prove that finding the minimum cost topological ordering is NP-hard.

1. Proof of hardness

We show a reduction from the Shuffle Problem: given words $w, w_1, \ldots, w_k$ over the alphabet $\{a, b, c\}$, decide whether w can be obtained as an interleaving (aka "shuffle") of $w_1, \ldots, w_n$. This problem is NP-hard as shown by Warmuth & Haussler [3, Theorem 3.1].

Given an instance $w, w_1, \ldots, w_n$ of this problem, and writing $l_i := |w_i|$ for all $1 \leq i \leq n$, we build the DAG G as a union of path graphs $L_1, \ldots, L_n$, where each L_i for $1 \leq i \leq n$ has l_i vertices written $v_1^i, \ldots, v_{l_i}^i$. Now, we define the cost function f as follows: for each $1 \leq i \leq n$ and $1 \leq j \leq l_i$, for each $1 \leq k \leq |w|$, we set $f(v_j^i, k)$ to be 0 if the j-th character of w_i is the same as the k-th character of w, and 1 otherwise.

This reduction is clearly in PTIME, and it is clear that the minimum cost of a topological sort is 0 iff there is an interleaving of the path graphs realizing exactly the word w, showing that the reduction is correct.

2. Conclusion

Garey and Johnson [1] shapes their theory based on previous primal works of Cook, Levin and Karp. Johnson [2] moves on with the guide to this theory. As long as we study a mathematical conjecture, we should encourage ourselves of having enough labouring hours on popular maths books like these. Then, reading some articles on theory of computing like [4] is a good practice. Only after that, could we think of the ultimate final for all mathematics sciences.

References

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