Supplementary Material for Inverse-square law between time and amplitude for crossing tipping thresholds in Proceedings of the Royal Society A

Paul Ritchie, Özkan Karabacak, Jan Sieber

January 18, 2019

Contents Section S1 repeats the notation and assumptions introduced in the main paper for reference. Sections S2 and S3 justify the reduction to a scalar stochastic differential equation (SDE) under the assumption of the main paper and Section S1. Section S2 considers the case where the noise is large relative to the parameter drift speed, while Section S3 considers the case of noise variance small or comparable with the parameter drift speed. Section S4 gives details on approximations for tipping probabilities in non-transversal crossings of tipping thresholds (corresponding to fold bifurcations) discussed in the main article, Section 4. Section S5 contains a table of fitting coefficients that approximate a-priori computable functions over argument ranges of interest.

S1 Reference to notation used in main article

We consider an n-dimensional system of ordinary differential equations (ODEs) with a scalar output y_o. The general procedure to simplify a nonlinear system near a bifurcation requires a center manifold reduction and normal form transformations [1]. For the fold bifurcation (or saddle-node) this procedure simplifies to a simple rescaling of all variables. We also include an additive Gaussian noise term from the beginning.

$$\begin{aligned} \mathrm{d}\mathbf{y}(t) &= f(\mathbf{y}(t), q(t))\mathrm{d}t + \Sigma \mathrm{d}W_t, &\quad \mathbf{y}(t) \in \mathbb{R}^n, q(t) \in \mathbb{R}, \Sigma \in \mathbb{R}^{n\times\ell} \\ y_o(t) &= \mathbf{w}^T\mathbf{y}(t), & y_o(t) \in \mathbb{R}, \mathbf{w} \in \mathbb{R}^n, \end{aligned} \tag{S1}$$

where $\mathrm{d}W_t$ are the increments of ℓ independent Wiener processes ($W_{j,0} = 0$, $W_{j,t} - W_{j,s} \sim \mathcal{N}(0, t-s)$, $W_{j,t_1+s_1} - W_{j,t_1}$ is independent of $W_{j,t_2+s_2} - W_{j,t_2}$ if $t_1 + s_1 \leq t_2$, s_1, $s_2 > 0$, and $W_{j,t}$ is almost surely continuous in t for all $j \in \{1, \ldots, \ell\}$).

Deterministic part We assume that the deterministic part has a fold for fixed q at $q = q^b$ and $\mathbf{y} = \mathbf{y}^b$, such that

(A1) $f(\mathbf{y}^b, q^b) = 0$,

(A2) $A_1 := \partial_1 f(\mathbf{y}^b, q^b)$ is singular with single right nullvector $\mathbf{v}_0$ and single left nullvector $\mathbf{w}_0$ ($A_1\mathbf{v}_0 = 0$, $\mathbf{w}_0^T A_1 = 0$), scaled such that $\mathbf{w}_0^T\mathbf{v}_0 = 1$;

(A3) all other eigenvalues of A_1 have negative real part;

(A4) $a_0 := \mathbf{w}_0^T\partial_2 f(\mathbf{y}^b, q^b) > 0$;

(A5) $\kappa := \frac{1}{2a_0}\mathbf{w}_0^T\partial_1^2 f(\mathbf{y}^b, q^b)\mathbf{v}_0^2 > 0$;

(A6) $\mathbf{w}^T\mathbf{v}_0 \neq 0$ (such that we may scale $\mathbf{v}_0$ to achieve $\mathbf{w}^T\mathbf{v}_0 = 1$).

The *center* projection $\mathbf{v}_0\mathbf{w}_0^T$ is the spectral projection onto the nullspace of A_1, while the *stable* projection $I - \mathbf{v}_0\mathbf{w}_0^T$ projects along the nullspace onto the stable subspace of A_1 (spanned by all eigenvectors for eigenvalues with negative real part). Thus, the stable projection can be written in the form $I - \mathbf{v}_0\mathbf{w}_0^T = \mathbf{V}_s\mathbf{W}_s^T$, where $\mathbf{V}_s$, $\mathbf{W}_s \in \mathbb{R}^{n\times n-1}$ have full rank and satisfy $\mathbf{W}_s^T\mathbf{V}_s = I_{n-1}$ (the identity in $\mathbb{R}^{n-1}$), $\mathbf{W}_s^T\mathbf{v}_0 = 0$, $\mathbf{w}_0^T\mathbf{V}_s = 0$, and all eigenvalues of $A_s := \mathbf{W}_s^T A_1\mathbf{V}_s \in \mathbb{R}^{(n-1)\times(n-1)}$ have negative real part.

Time dependence of parameter Furthermore, we assume that $q(t)$ changes slowly with t and has a regular maximum at $t = 0$, such that we may introduce a small parameter ϵ and expand

$$q(t) = q^{\max} + q_h(\epsilon t) = q^b + \epsilon R_0 + q_h(\epsilon t), \tag{S2}$$

where

(P1) $q_h(0) = q_h'(0) = 0$,

(P2) $R_2 := -\frac{1}{2}q_h''(0) > 0$.

The parameter R_0 in the expansion is of order at most 1, because for $R_0 \gg 1$ the deterministic system (with noise amplitude $\Sigma = 0$) will escape from the neighborhood of $\mathbf{y}^b$ for all sufficiently small ϵ, and for $q^{\max} - q^b < 0$ of order 1 we are in the regime noise-induced escape over a potential barrier [2].

Noise amplitude We assume that variance $\Delta = \Sigma\Sigma^T$ of the noise increments is small. We introduce another small parameter σ^2 that scales the noise variance,

$$D_0 = \frac{1}{2}\sigma^{-2}\mathbf{w}_0^T\Delta\mathbf{w}_0 \in [0, \infty),$$
$$\Sigma_s = \sigma^{-1}\mathbf{W}_s^T\Sigma \in \mathbb{R}^{(n-1)\times\ell},$$

where we assume that for the case with non-zero noise, D_0 is of order 1 and the matrix norm of $\|\Sigma_s\|$ is at most of order 1, excluding the case that $D_0 = 0$ and $\Sigma_s \neq 0$ for simplicity.

Section S2 shows that for $0 < \epsilon^{3/2} \ll \sigma^2 \ll 1$ tipping (leaving the neighborhood of $\mathbf{y}^b$ before time 0) has probability close to 1 for $\sigma \to 0$. Section S3 shows that for $\sigma^2 \ll \epsilon^{3/2} \ll 1$ tipping is determined by the deterministic part, while for $\sigma^2 \sim \epsilon^{3/2}$ the tipping probability is the same as the probability of $x \to +\infty$ from $x(t_0) \leq 0$, $t_0 \ll -1$ in finite time, when x satisfies the SDE

$$\mathrm{d}x = a_0[R_0 - R_2t^2 + \kappa x^2]\mathrm{d}t + \sqrt{2D}\mathrm{d}W_t, \tag{S3}$$

where $D = \sigma^2\epsilon^{-3/2}D_0$, as claimed in the main paper. The basic arguments are following those in the textbook by Berglund and Gentz [3]. However, we use slightly less precise estimates, as we do not establish that certain small probabilities are exponentially small for $\epsilon \to 0$, but simply that they go to zero for $\epsilon \to 0$, $\sigma \to 0$.

S2 Escape from $(\mathbf{y}^b, q^b)$ for $\epsilon^{3/4} \ll \sigma$ — Large-noise or slow drift regime

Let as assume that $0 < \epsilon^{3/4} \ll \sigma \ll 1$ (or $0 < \epsilon \ll \sigma^{4/3} \ll 1$), and that the initial condition $\mathbf{y}(t_0)$ is in a neighborhood of order $\sigma^{2/3}$ of $\mathbf{y}^b$ at time $t_0 \sim -\epsilon^{-1/2} \ll -\sigma^{-2/3}$. First we aim to confirm that the probability of $\mathbf{y}$ staying in a neighborhood of $\mathbf{y}^b$ of size $O(\sigma^{2/3})$ for all times up to $O(\epsilon^{-1/2})$ goes to zero as $\sigma \to 0$ (and, hence, $\epsilon \to 0$). In order to do so, we assume that $\mathbf{y} - \mathbf{y}^b \sim \sigma^{2/3}$ for all t from initial time $t_0 \sim -\epsilon^{-1/2}$ up to some exit time t_E (which is at most of order $O(\epsilon^{-1/2})$). For times in $[t_0, t_E]$ the difference of the forcing $q(t)$ from q^b is of order $O(\epsilon)$ or less such that for the large-noise regime we can estimate the forcing as $q(t) = q^b + O(\epsilon) = q^b + o(\sigma^{4/3})$. Hence, the precise form of the forcing does not play a role in the following expansions of Section S2. This is unsurprising as we aim to show that $\mathbf{y}(t)$ escapes independent of the forcing parameters.

The trajectory $\mathbf{y}(t)$ can be described using the following rescaled and projected quantities by zooming into the neighborhood of $\mathbf{y}^b$ using the small parameter σ:

$$y_c := \sigma^{-2/3}\mathbf{w}_0^T(\mathbf{y} - \mathbf{y}^b) \in \mathbb{R}, \quad \mathbf{y}_s := \sigma^{-2/3}\mathbf{W}_s^T(\mathbf{y} - \mathbf{y}^b) \in \mathbb{R}^{n-1}. \tag{S4}$$

The new quantities $y_c(t)$, $\mathbf{y}_s(t)$ have magnitude at most $O(1)$ for times in $[t_0, t_E]$ (in fact, we may define the exit time for a given large threshold $c \gg 1$ as the first time when $|y_c| \geq c$ or $|\mathbf{y}_s| \geq c$). The stable component $\mathbf{y}_s$ satisfies the stochastic differential equation (obtained by multiplying the SDE in (S1) by $\mathbf{W}_s^T$)

$$\mathrm{d}\mathbf{y}_s = [A_s\mathbf{y}_s + o(1)]\mathrm{d}t + \sigma^{1/3}\Sigma_s\mathrm{d}W_t, \tag{S5}$$

where $o(1)$ is a bounded nonlinear term depending on y_c, $\mathbf{y}_s$ and t. We use the notation that an expression η is $o(1)$, if the probability $P(|\eta| > c) \to 0$ for $\sigma \to 0$ for all c. Consequently, for times $t > t_1 := t_0 + c_2|\log\sigma|$ (where c_2 is a constant that depends on the eigenvalues of A_s) the stable component $\mathbf{y}_s$ will be $o(1)$ in this sense. The new time t_1 is still of order $-\epsilon^{-1/2} \ll -\sigma^{-2/3}$.

For the center component y_c we multiply the SDE in (S1) by $\mathbf{w}_0^T$ and rescale time

$$t_{\text{new}} := \sigma^{2/3}t_{\text{old}} \tag{S6}$$

such that time in the new scale is still ranging over the large time interval $(-\epsilon^{-1/2}\sigma^{2/3}, \epsilon^{-1/2}\sigma^{2/3})$. We obtain the SDE

$$\begin{aligned}\mathrm{d}y_c = &\Big[O(\sigma^{-4/3}\epsilon) + a_0\kappa y_c^2 + A_{11}^c[\mathbf{v}_0 y_c][\mathbf{V}_s\mathbf{y}_s] + A_{11}^c[\mathbf{V}_s\mathbf{y}_s]^2 + o(1)\Big]\,\mathrm{d}t \\ &+ \sqrt{2D_0}\mathrm{d}W_t, \text{ where } \qquad A_{11}^c := \mathbf{w}_0^T\partial_1^2 f(\mathbf{y}^b, q^b).\end{aligned} \tag{S7}$$

The first term $O(\epsilon/\sigma^{4/3})$ arises from the expansion of the forcing $q(t)$. After time t_1 the terms involving $\mathbf{y}_s$ are all small with probability approaching 1 for $\sigma \to 0$.

Similarly, the terms η collected in the remainder $o(1)$ satisfy $P(|\eta| > c) \to 0$ for $\sigma \to 0$ and all $c > 0$ and all times $t \in [t_0, t_E]$. Since $O(\epsilon/\sigma^{4/3}) = o(1)$, y_c satisfies for all $t \in [t_1, t_E]$

$$\mathrm{d}y_c = \left[a_0 \kappa y_c^2 + o(1)\right] \mathrm{d}t + \sqrt{2D_0}\mathrm{d}W_t$$

with positive a_0, κ and D_0 in the rescaled time. Since $t_1 \sim -\sigma^{2/3}/\epsilon^{1/2} \ll -1$ for small σ and the $o(1)$ term is bounded and satisfies $P(|\eta| > c) \to 0$ for $\sigma \to 0$ and all $c > 0$, y_c reaches every arbitrary $C > 0$ with probability approaching 1 for $\sigma \to 0$ before time $t_E = 0$ (since $t_1 \to -\infty$ for $\sigma^{2/3} \gg \epsilon^{1/2}$ and $\sigma \to 0$, one could choose any other time t_E as long as t_E stays bounded for $\sigma \to 0$). Consequently, the output

$$y_o = \mathbf{w}^T \mathbf{y} = \mathbf{w}^T \mathbf{y}^b + \sigma^{2/3} y_c + \sigma^{2/3} \mathbf{w}^T \mathbf{V}_s \mathbf{y}_s = y_o^b + \sigma^{2/3} y_c + \sigma^{2/3} o(1)$$

leaves the $\sigma^{2/3}$-neighborhood of its critical value y_o^b also at its upper end, regardless of the forcing parameters.

S3 Escape from $(\mathbf{y}^b, q^b)$ for σ of order $\epsilon^{3/4}$ or smaller — the balanced or small-noise regime

If the noise amplitude σ is less or equal to $\epsilon^{3/4}$ (any constant factor can be absorbed into D_0 and Σ_s), then we may rescale the components of $\mathbf{y}(t)$ and time depending on ϵ, knowing that the noise amplitude is simultaneously bounded. We now assume that $\mathbf{y} - \mathbf{y}^b \sim \epsilon^{1/2}$ for all t from initial time $t_0 \sim -\epsilon^{-1/2}$ up to some exit time t_E (which is at most of order $O(\epsilon^{-1/2})$). Slightly stronger, we assume that $\mathbf{y}(t_0)$ is close to $\mathbf{y}^s(q(t_0))$, the stable equilibrium of the deterministic part for fixed $q < q^b$ (for example, we may assume $\mathbf{w}_0^T(\mathbf{y}(t_0) - \mathbf{y}^b) < 0$). Whether the exit occurs before time reaches order $\epsilon^{-1/2}$ or not, depends now on the forcing parameters R_0 and R_2. We again rescale the trajectory $\mathbf{y}(t)$, but now using the small parameter ϵ:

$$y_c := \epsilon^{-1/2} \mathbf{w}_0^T (\mathbf{y} - \mathbf{y}^b) \in \mathbb{R}, \quad \mathbf{y}_s := \epsilon^{-1/2} \mathbf{W}_s^T (\mathbf{y} - \mathbf{y}^b) \in \mathbb{R}^{n-1}. \tag{S8}$$

The new quantities $y_c(t_0)$, $\mathbf{y}_s(t_0)$ have magnitude at most $O(1)$ before exit time t_E. The stable component $\mathbf{y}_s$ satisfies the stochastic differential equation (obtained by multiplying the SDE in (S1) by $\mathbf{W}_s^T$)

$$\mathrm{d}\mathbf{y}_s = [A_s \mathbf{y}_s + o(1)]\mathrm{d}t + \sigma\epsilon^{-1/2} \Sigma_s \mathrm{d}W_t, \tag{S9}$$

where $\sigma\epsilon^{-1/2} \ll 1$, and $o(1)$ is bounded satisfying $P(|\eta| > c) \to 0$ for $\epsilon \to 0$ and all $c > 0$. Consequently, if ϵ is small, for times $t > t_1 := t_0 + c_2|\log \epsilon|$ the stable component $\mathbf{y}_s$ will be of order $o(1)$ with probability close to 1. The new time t_1 is still of order $-\epsilon^{-1/2}$. For the center component y_c we multiply the SDE in (S1) by $\mathbf{w}_0^T$ and rescale time

$$t_{\text{new}} := \epsilon^{1/2} t_{\text{old}} \tag{S10}$$

such that time in the new scale ranges over a finite time interval $[-t_{0,\text{new}}, t_{E,\text{new}}]$, where we now look for exit times $t_{E,\text{new}}$ within a bounded time interval. We drop the subscript from t_{new} below. By our assumption on the starting point of the trajectory $y_c(t_0)$ is less than 0. It satisfies the SDE

$$\mathrm{d}y_c = \left[a_0\left(R_0 - R_2 t^2 + \kappa y_c^2\right)\right]\mathrm{d}t + \frac{\sigma}{\epsilon^{3/4}}\sqrt{2D_0}\mathrm{d}W_t \tag{S11}$$

$$+\left[A_{11}^c[\mathbf{v}_0 y_c][\mathbf{V}_s\mathbf{y}_s] + A_{11}^c[\mathbf{V}_s\mathbf{y}_s]^2 + o(1)\right]\mathrm{d}t, \tag{S12}$$

where $A_{11}^c = \mathbf{w}_0^T\partial_1^2 f(\mathbf{y}^b, q^b)$, and κ, a_0, R_0 and R_2 are the expansion coefficients defined in A4, A5, (S2) and P2. Furthermore, the noise amplitude is bounded, since $\sigma \leq \epsilon^{3/4}$. In the new time scaling the term $\mathbf{y}_s(t)$ decays rapidly (within time of order $\epsilon^{1/2}|\log\epsilon|$) to $o(1)$, with probability close to 1 as $\epsilon \to 0$. Thus, for any positive threshold $c > 0$, the probability that $y_c(t_1) > c > 0$ goes to zero as $\epsilon \to 0$ (the time t_1 is the time when $\mathbf{y}_s$ reaches small values with probability close to 1). Furthermore, from t_1 onward all terms in the line (S12) are small with probability close to 1 as $\epsilon \to 0$. Furthermore, the rescaled output $x = \epsilon^{-1/2}\mathbf{w}^T(\mathbf{y} - \mathbf{y}^b)$ satisfies $x = y_c + \mathbf{w}^T V_s\mathbf{y}_s = y_c + o(1)$ for $\epsilon \to 0$. Thus, up to terms of order $o(1)$ it satisfies the same SDE (S11), starting from the same initial condition.

For $\sigma \ll \epsilon^{3/4}$ (S11) is a small-noise perturbation of the deterministic differential equation

$$\dot{x} = a_0\left(R_0 - R_2 t^2 + \kappa x^2\right). \tag{S13}$$

If $\sigma \sim \epsilon^{3/4}$, we may define $D = \sigma^2/\epsilon^{3/2}D_0$ such that equation (S11) becomes the scalar stochastic differential equation for x (up to terms that are $o(1)$ for $\epsilon \to 0$)

$$\mathrm{d}x = a_0[R_0 - R_2 t^2 + \kappa x^2]\mathrm{d}t + \sqrt{2D}\mathrm{d}W_t \tag{S14}$$

where the rescaled noise variance $2D$ is of order 1 and a_0, $\kappa > 0$. The initial condition is at $x(t_0) < c$, where $c > 0$ is a small positive threshold, and $t_0 < 0$. By further rescaling x and time and introducing correspondingly rescaled versions of the parameters R_0 and R_2,

$$\begin{aligned} x_{\mathrm{new}} &= \frac{(a_0\kappa)^{1/3}}{D^{1/3}}x_{\mathrm{old}}, & t_{\mathrm{new}} &= D^{1/3}(a_0\kappa)^{2/3}t_{\mathrm{old}}, \\ p_0 &= \frac{a_0^{2/3}R_0}{D^{2/3}\kappa^{1/3}}, & p_2 &= \frac{R_2}{D^{4/3}\kappa^{5/3}a_0^{2/3}}, \end{aligned} \tag{S15}$$

we may simplify (S14) to the SDE

$$\mathrm{d}x = [p_0 - p_2 t^2 + x^2]\mathrm{d}t + \sqrt{2}\mathrm{d}W_t \tag{S16}$$

Remark for the case of $\sigma = 0$ For the case without noise ($\sigma = 0$), the dynamics of the stable direction $\mathbf{y}_s$ can be decoupled into an exponentially decaying term (the stable fibres or isochrones) and the center manifold, which is a graph of the form $\epsilon^{1/2}\mathbf{y}_s(y_c, t)$. Thus, all terms involving $\mathbf{y}_s$ in the projection by $\mathbf{w}_0^T$, (S12) are of order $\epsilon^{1/2}$ and can be neglected in the deterministic case, leading to the sharper error term $O(\epsilon^{1/2})$ in equation (2.4) of the main paper.

S4 Approximations of tipping probability P_{esc}

The tipping probability P_{esc} in (S16) is defined as the probability of leaving the domain $I = [-x_{\mathrm{bd}}, x_{\mathrm{bd}}]$ before time T_0, when starting from some $x_0 \le 0$ at time T_0. The probability density $u(x,t)$ determining P_{esc} is governed by the Fokker-Planck equation (FPE)

$$\partial_t u(x,t) = \partial_x^2 u(x,t) - \partial_x[(p_0 - p_2 t^2 + x^2)u(x,t)] \tag{S17}$$

with Dirichlet boundary conditions ($u(-x_{\mathrm{bd}},t) = u(x_{\mathrm{bd}},t) = 0$) starting from u concentrated near $x = x_0 < 0$ at $t = -T_0 = -\sqrt{(x_0^2 + p_0)/p_2}$ and $x_0 \ll -1$ with $\int_I u(x,t)\mathrm{d}x = 1$. Then $P_{\mathrm{esc}} = 1 - \int_I u(x,T_0)\mathrm{d}x$. Since escape through $-x_{\mathrm{bd}}$ is extremely unlikely, this difference corresponds almost exclusively to escape through the upper boundary x_{bd}.

For large T_0 this probability is nearly independent from the values x_0, T_0, x_{bd}. The main article's Figure 5 shows numerical results in the order-1 region of the $(q_1, q_2) = (\sqrt{p_2}, p_0 - \sqrt{p_2})$ plane for $x_{\mathrm{bd}} = 8$, $T_0 = \sqrt{(x_0^2 + p_0)/p_2}$ and $x_0 = -4$.

In the region of the main articles Figure 5 the double exponential of $1 - P_{\mathrm{esc}}$ satisfies a cubic fit accurately over the region shown in Figure 5 of the main article:

$$1 - P_{\mathrm{esc}} \approx \exp\left[-\exp \sum_{k=0, j\le k}^{3} c_{kj} q_1^j q_2^{k-j}\right], \tag{S18}$$

where the coefficients c_{kj} are given in Table S1. The absolute error over the region of the main article's Figure 5 is 0.024 and the cut-off relative error

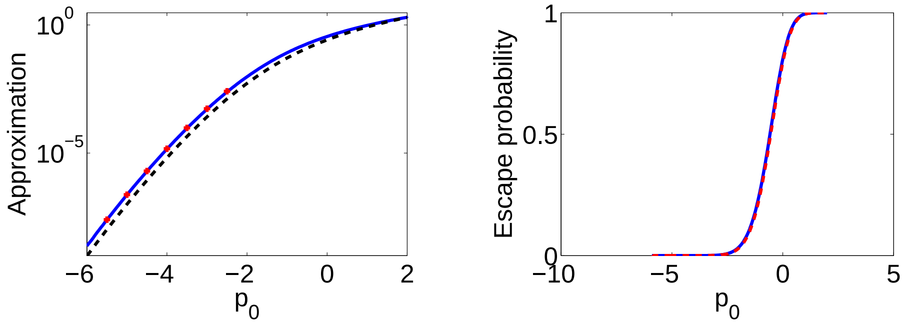
$$\mathrm{err}_{\mathrm{rel}} = |P_{\mathrm{esc}} - P_{\mathrm{esc}}^{\mathrm{approx}}| / \max(0.1, P_{\mathrm{esc}}) \tag{S19}$$

is less than 10%. Since the two-parameter fit does not give insight in the origin of its terms, we provide approximations in two limiting regimes.

Slow drift approximation For small parameters p_2 (or q_1) the integration of the FPE (S17) would require long time intervals (only for times of order $1/\sqrt{p_2}$ are well separated stable and unstable slow manifolds present in the deterministic part $\dot{x} = p_0 - p_2 t^2 + x^2$). However, in this regime the time-dependence of (S16) is weak: the time derivative of the right-hand side $p_0 - p_2 t^2 + x^2$ is of order $\sqrt{p_2}$ for $|t|\sqrt{p_2}$ of order 1 or less. Hence, we may approximate the rate of escape at each time t using the escape rate for the static potential well corresponding to the right-hand side $p_0 - p_2 t^2 + x^2$. This escape rate is given by the dominant eigenvalue λ_0 of the linear operator on the right-hand side of the Fokker-Planck equation (S17) [3]. Solving the parameter-dependent eigenvalue problem

$$-\lambda(p)u(x,p) = \partial_x^2 u(x,p) - \partial_x[(p + x^2)u(x,p)] \tag{S20}$$

for its first eigenvalue λ_0 (specifically, with Dirichlet boundary conditions on the interval $[-8, 8]$ using `chebfun`[4]), provides the escape rate. The eigenvalue $\lambda_0(p)$ is real and positive (due to the minus sign on the left-hand side in (S20)), and exponentially small for $p \ll -1$, where the approximation with Kramers' escape rate ($\lambda_0 \approx \sqrt{-p}/\pi \exp(4p\sqrt{-p}/3)$ for the drift term in (S20) and $D = 1$)



(a) (b)

Figure S1: Slow drift approximation: (a) shows the leading eigenvalue $\lambda_0(p)$ (blue solid) as numerically computed using (S20), the approximation given by Kramers' escape rate (red markers) and μ_0 defined below (S21) (black dashed). (b) compares the slow drift approximation (S21) (blue solid) to the numerical value (from Figure 5, main article) (red dashed) at $p_2 = 0.1$. The difference always below 0.02.

is valid. Figure S1a shows λ_0 and the Kramers' escape rate approximation. The probability of *not* escaping is then the product of all probabilities of not escaping near all times t, such that overall:

$$\begin{aligned} P_{\mathrm{esc}} &\approx 1 - \exp\left(-2\int_0^\infty \lambda_0(p_0 - p_2 t^2)\mathrm{d}t\right) \\ &\approx 1 - \exp\left(-2p_2^{-1/2}\mu_0(p_0)\right) \end{aligned} \tag{S21}$$

where $\mu_0(p_0) = \int_0^\infty \lambda_0(p_0 - s^2)\mathrm{d}s$ is also shown in Figure S1a. Since the eigenvalue $\lambda_0(p)$ (and its integral $\mu_0(p)$) are exponentials, approximation (S21) explains the double exponential nature of the probability P_{esc}. The logarithms of λ_0 and μ_0 fit accurately to cubic polynomials over the range shown in Figure S1 ($\mu_0(p) \approx \exp(1.35(p-1))$ fits up to 0.02 in absolute value for $p < 0.3$; see Table S1. Figure S1b compares the slow-drift approximation (S21) to the numerical result at $p_2 = 0.1$ from Figure 5 of the main article ($q_1 = 0.316$ in Figure 5). The absolute error is always below 0.02 and the cut-off relative error $\mathrm{err}_{\mathrm{rel}}$ in (S19) is less than 10%. The slow drift approximation becomes more accurate for values of p_2 smaller than 0.1, such that

$$P_{\mathrm{esc}}(p_0, p_2) \approx 1 - \exp\left(-2p_2^{-1/2}\exp(1.35(p_0 - 1))\right) \tag{S22}$$

is a good approximation for the probability of escape of $x \to \infty$ satisfying $\mathrm{d}x = p_0 - p_2 t^2 + x + \sqrt{2}\mathrm{d}W_t$, starting from $x(t_0) \leq 0$ for $t_0 \ll -1$, for all $p_2 \leq 0.1$.

Mode approximation in moving coordinates An approach explored by Ritchie and Sieber [5] extends the slow drift approximation to a region of the parameter plane where p_2 is not small. We consider the unique solution $\bar{x}(t; p_0)$

of the deterministic ODE (dropping $\sqrt{2}\mathrm{d}W_t$ from (S16))

$$\mathrm{d}x(t) = [p_0 - p_2 t^2 + x(t)^2]\mathrm{d}t \tag{S23}$$

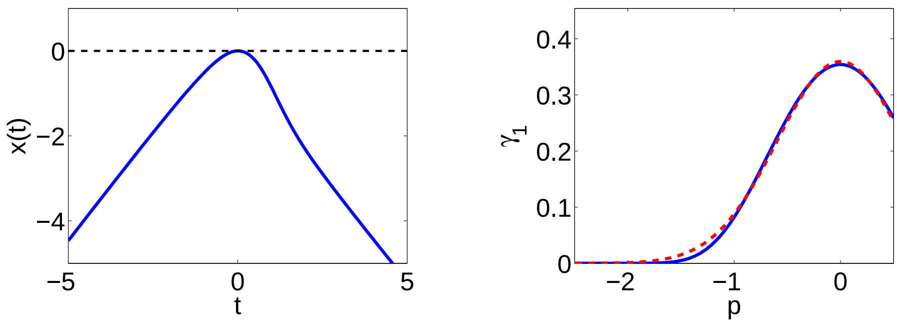
satisfying $\bar{x}(t; p_0) + \sqrt{p_2}|t| \to 0$ for $t \to \infty$ (see Figure S2a). Then we make a time-dependent coordinate shift $z(t) = x(t) - \bar{x}(t; p_0, p_2)$ and consider escape of a realization of z from the vicinity of the origin when adding stochastic disturbances to this shifted system:

$$\mathrm{d}z = [z^2 + 2\bar{x}(t; p_0, p_2)z]\mathrm{d}t + \sqrt{2}\mathrm{d}W_t. \tag{S24}$$

Now we apply the slow-drift approximation in the coordinate system for z (after the time-dependent shift). The eigenvalue problem for the operator on the right-hand side of the Fokker-Planck equation for (S24) is now (with Dirichlet boundary conditions)

$$-\gamma(p)u(z; p) = \partial_z^2 u(z; p) - \partial_z[(z^2 + 2pz)u(z; p)], \tag{S25}$$

where the parameter p is equal to $\bar{x}(t; p_0, p_2)$. Ritchie and Sieber [5] give a way to approximate the escape rate $\gamma_1(p)$. Its numerically computed value is shown in Figure S2b (computation performed with `chebfun`[4] on the interval $[-8, 8]$). As one can see, the escape rate $\gamma_1(p)$ has a maximum at $p = 0$. This points to a



(a) (b)

Figure S2: (a) Trajectory of deterministic system (S23) with $p_0 = 0.59$ and $p_2 = 1$. (b) Slow drift approximation (blue solid) and numerical value (red) of escape rate $\gamma_1(p)$ for escape problem (S25).

limitation of the validity for the mode approximation. When the deterministic trajectory $\bar{x}(t)$ enters the region $x > 0$, it becomes locally repelling, such that the potential $-z^3/3 - pz^2$ corresponding to (S24) has a hill top at 0, but a well at $-2\bar{x}$. The region of validity for the mode approximation is thus limited to the region where $p = \bar{x} \le 0$. This implies that the deterministic reference trajectory $\bar{x}(t; p_0, p_2)$ has to lie in $\{x \le 0\}$ for all t. This is the case when $p_0 \le 0.59\sqrt{p_2}$ (corresponding to the area below the red line in Figure 5 of the main article).

Figure S2b also shows a fitted curve of the form $\gamma_{1,2}(\bar{x}) = \exp(-c_0 - c_2\bar{x}^2)$ with $c_0 = 1.01$ and $c_2 = 1.41$. A 4th-order fit $\gamma_{1,4}(\bar{x}) = \exp\left(-\sum_{j=0}^4 c_j \bar{x}^{4-j}\right)$ with $c = (0.33, 0.04, 1.17, -0.01, 1.04)$ has an absolute error less than 10^{-3} and a cut-off relative error $(|\gamma_1(p) - \gamma_{1,4}(p)|/\max(0.1, \gamma_1))$ less than 10^{-2}.

Again, the probability of *not* escaping is the product of all probabilities of not escaping near all times t, such that overall

$$P_{\mathrm{esc}} \approx 1 - \exp\left(-\int_{-\infty}^{\infty} \gamma_1(\bar{x}(t;p_0,p_2))\mathrm{d}t\right), \tag{S26}$$

which equals the approximation (4.8) in Section 4 of the main article. In contrast to the slow drift approximation (S21) the integrand γ_1 depends on the deterministic trajectory $\bar{x}(t;p_0,p_2)$. This trajectory is typically non-symmetric about $t=0$ (see Figure S2a; in contrast to the simple parabolic path $p_0 - p_2t^2$) such that the escape rate has to be integrated over all times. As there is no good approximation formula for the trajectory $\bar{x}(t;p_0,p_2)$ (a quadratic approximation at its maximum is typically poor), the integral has to be evaluated numerically. This evaluation can be performed in parallel to the computation of the trajectory $\bar{x}(t;p_0,p_2)$ itself, in particular also covering the case $0 \leq p_2 \ll 1$ for which the slow drift approximation (S21) is valid. For the normal form this would be an extension of the form (assuming that the integration interval is $[-T_0, T_0]$)

$$\dot{x} = p_0 - p_2t^2 + x^2, \qquad x(-T_0) = -T_0 \tag{S27}$$

$$\dot{\gamma}_{\mathrm{acc}} = \gamma_{1,4}(x), \qquad \gamma_{\mathrm{acc}}(-T_0) = 0. \tag{S28}$$

Then $P_{\mathrm{esc}} \approx 1 - \exp(-\gamma_{\mathrm{acc}}(T_0))$. More generally, the parameter path does not have to be parabolic: $p_0 - p_2t^2$ may be replaced with an arbitrary function $p(t)$ satisfying $p(t) \ll -1$ for $|t| \gg 1$ (after rescaling). Moreover, x in (S28) can be the rescaled scalar output of the simulated large system (after applying scalings (S8) and (S15): $x(t) = \epsilon^{-1/2}D^{1/3}(a_0\kappa)^{-1/3}w^T(y(t) - y^b)$). The initial condition for γ_{acc} should be set to 0, as (S28) evaluates the integral in (S26).

S5 Fitting coefficients

The values of the fitting coefficients for the approximations of tipping probability P_{esc} are listed in Table S1.

Table S1: Table of fitting coefficients

Expression	coefficient	Value
Eq. (S18)	c_0	0.98
	c_1	$(1.41, -0.97)$
	c_2	$(-0.22, -0.28, 0.33)$
	c_3	$(0.01, 0.03, 0.04, -0.04)$
$\log \lambda_0(s) = \sum_{k=0}^{3} c_k s^k$ in Eq. (S21)	c	$(-1.3433, 1.3659, -0.2347, 0.0277)$
$\log \gamma_{1,4}(\bar{x}) = \sum_{k=0}^{4} c_k \bar{x}^k$ in Eq. (S28)	c	$(-1.0388, 0.0058, -1.1687, -0.0409, -0.3326)$

References

[1] Y. A Kuznetsov. *Elements of Applied Bifurcation Theory*, volume 112 of *Applied Mathematical Sciences*. Springer-Verlag, New York, third edition, 2004. ISBN 0-387-21906-4.

[2] Peter Hänggi, Peter Talkner, and Michal Borkovec. Reaction-rate theory: fifty years after Kramers. *Reviews of modern physics*, 62(2):251, 1990.

[3] Nils Berglund and Barbara Gentz. *Noise-induced phenomena in slow-fast dynamical systems: a sample-paths approach*. Springer Science & Business Media, 2006.

[4] Tobin A Driscoll, Nicholas Hale, and Lloyd N Trefethen. *Chebfun guide*. Pafnuty Publications, Oxford, `http://www.chebfun.org/`, 2014.

[5] Paul Ritchie and Jan Sieber. Probability of noise- and rate-induced tipping. *Phys. Rev. E*, 95:052209, 2017.