Supplementary information for Liu and Shi, 2019

Table 1. Summarizes of the geometrical, optical, and atmospheric thermal and radiational profiles quantities and symbols involved in this paper.[i]

Symbol	Unit	Definition
$\theta,\ \mu,\ \gamma$	Radian (1)	$\mu = \cos\theta,\ \gamma = \sec\theta$, where θ is the local viewing zenith angle.[ii]
$\theta',\ \mu'$		$\mu' = \cos\theta'$, where θ' is the local incident zenith angle.
$\theta'_0,\ \mu'_0$		$\mu'_0 = \cos\theta'_0$, where θ'_0 is the local solar incident zenith angle.
$\varphi,\ \varphi',\ \varphi'_0$		Observing, incident, and solar incident azimuthal angles, respectively.
$\vec{\Omega}$	1	Local emission, observing, or reflection direction.
$\vec{\Omega}',\ \vec{\Omega}'_0$		Local incident direction, local solar incident direction.
$f_{r,\lambda}(\cdot,\cdot)$	1	Local surface Bidirectional Reflectance Distribution Function (BRDF).
$\alpha_{r,\lambda}(\cdot,\cdot)$		Local surface BRDF anisotropic factor.
ρ_λ		local surface reflectance (also inferred to as albedo or hemispherical reflectance in visible and near-IR range).
$\bar{\varepsilon}_\lambda$		Mean local surface emissivity at wavelength λ.
$\varepsilon_\lambda(\vec{\Omega})$		Local surface emissivity in direction $\vec{\Omega}$.
T_s	K	Surface temperature.
T_{a0}		Near surface air temperature.
$T(z)$		Temperature profile along altitude.
$T(\tau_\lambda(z,\mu))$		Temperature profile along upward transmittance $\tau_\lambda(z,\mu)$.
$t_\lambda(\mu)$	1	Atmospheric transmittance from the ground to the TOA in direction μ.
$\tau_\lambda(z,\mu)$		Atmospheric transmittance from the ground to an altitude z in direction μ.
$t'_\lambda(\mu')$		Atmospheric transmittance from the TOA to the ground in direction μ'.
$\tau'_\lambda(z,\mu')$		Atmospheric transmittance from the TOA to an altitude z in direction μ'.
$t_{\lambda,2}(\mu'_0,\mu)$		$t_{\lambda,2}(\mu'_0,\mu) = t'_\lambda(\mu'_0)t_\lambda(\mu)$.
$\tau'_d(\tau'_\lambda,\mu')$		Downward transmittance in direction μ' from the altitude with a solar-beam transmittance $\tau'_\lambda(z,\mu'_0)$ to the ground.
$\tau_\lambda(\tau'_\lambda,\mu)$		Upward transmittance defined from the altitude that corresponding to $\tau'_\lambda(z,\mu'_0)$ to the TOA along a viewing direction μ.
$P(\mu,\mu'_0)$		Scattering phase function of an intercepting particle that distributed in the wave traveling path.
$\bar{\omega}$		Single scattering albedo of a particle that distributed in the wave traveling path.
$\rho_c(\tau'_\lambda)$		Molar density of a bulk atmosphere at an altitude with transmittance $\tau'_\lambda(z,\mu'_0)$.
$L_{\lambda,TOA}(\mu)$, $L_\lambda(\mu)$	$\frac{W}{m^2}Sr^{-1}\mu m^{-1}$	TOA outgoing radiance at the entrance slit of a radiometer in direction μ.
$L_i(\mu)$, L_i	$\frac{W}{m^2}Sr^{-1}$	Band effective radiance collected by a radiometer channel.
$\phi(\lambda)$	1	Spectral response function for a specific channel of a radiometer to calibrate the observed signal to the radiative transfer equation.
$R_{\lambda,SL0}(\mu)$	$\frac{W}{m^2}Sr^{-1}\mu m^{-1}$	Surface leaving radiance at ground level in direction μ.
$R_{\lambda,SL1}(\mu)$		Attenuated Surface leaving radiance at TOA in direction μ.
$L_{\lambda,BS\downarrow}(\mu'_0,\mu')$		Downward solar scattering radiance in direction μ' when illuminated in μ'_0, sky radiance in direction μ' when illuminated in μ'_0.
$B_\lambda(T_s)$		Planck's function.
$L_{\lambda,AE\downarrow}(T(\tau'),\mu')$		Downward atmospheric emitting radiance in direction μ' when the atmosphere with transmittance profile $\tau'_\lambda(z,\mu')$ and temperature profile $T(\tau')$, also called atmospheric radiance for short.
$R_{\lambda,BS\uparrow}(\mu'_0,\mu)$		Upward solar scattering radiance in direction μ when illuminated in μ'_0,
$R_{\lambda,AE\uparrow}(T(\tau),\mu)$		Upward atmospheric emitting radiance in direction μ when the atmosphere with transmittance profile $\tau_\lambda(z,\mu)$ and temperature profile $T(\tau)$, also called atmospheric upward radiance for short.
$J_\lambda(\tau_\lambda)$		Atmospheric source radiance.
$E_{\lambda,0}$	$\frac{W}{m^2}\mu m^{-1}$	Solar irradiance at the TOA.
$E_{\lambda,AE}$		Atmospheric downward irradiance.
$E_{\lambda,SS}$		Sky or downward solar diffuse irradiance.

Table 2. Summarize of the quantities involved in the Planck's function and its derivational equations involved in the IR radiometry.

Symbol	Physical quantity	Description	Value/Unit
$B_\lambda(T)$[1]	Spectral radiance	The energy (J) emitted per second per unit wavelength (μm) per steradian (sr) from one square meter of a perfect blackbody-surface in thermodynamic equilibrium at temperature T (K)	$\left(\frac{W}{m^2}\right) sr^{-1}\mu m^{-1}$, Where $W = \frac{J}{s}$ and $1 \cdot \mu m = 10^{-6} m$
$B(T)$	radiance	The power (W) per steradian (sr) from one square meter of a perfect blackbody-surface in thermodynamic equilibrium at temperature T (K)	$\left(\frac{W}{m^2}\right) sr^{-1}$
$M_\lambda(T), E_\lambda(T)$	Spectral exitance	defined as spectral hemispherical radiance with M for outgoing and E for incoming exitance	$\int_{2\pi} B_\lambda(T) d\Omega$, $\left(\frac{W}{m^2}\right)\mu m^{-1}$
$M(T), E(T)$	radiant flux density, radiant exitance or Irradiance,	defined as hemispherical radiance with M for outgoing and E for incoming radiation	$\int_{2\pi}\left(\int_0^\infty B_\lambda(T) d\lambda\right) d\Omega$, $\frac{W}{m^2}$
I	Radiant intensity	defined as Radiant flux emitted, reflected, transmitted or received, per unit solid angle[iii].	$\frac{W}{sr}$
ϕ	Radiant flux,	defined as Radiant energy, denoted by Q, emitted, reflected, transmitted or received per unit time for some giving surface area and sometimes also called "radiant power".	$W = \frac{J}{s}$
T	Physical temperature of the Earth surface systems. For Earth surface, it is denoted by T_s, and for the atmosphere at altitude ξ, it is referred to as $T(\xi)$. For the brightness temperature of the Earth surface systems in channel i of a radiometer, it is designated by T_i		K
h	Planck's constant (or the altitude of the satellite with unit m)		$6.626070040(81) \times 10^{-34}\ Js$
λ	Wavelength involved in this paper		μm
k	Boltzmann's constant		$1.380658 \times 10^{-23}\ J \cdot K^{-1}$
c	speed of light in a medium, whether material or vacuum		$\sim 2.99792458 \times 10^{8}\ m \cdot s^{-1}$
σ	the Stefan–Boltzmann constant or irradiance coefficient, mathematical short hand for $\frac{2\pi^5 k^4}{15c^2h^3}$		$5.670373 \times 10^{-8}\ w/(m^2K^4)$

[1] In $B_\lambda(T)$, B is for Blackbody or Black-surface which can be absorbed all the incident energy and emitted out, if it is in the state of thermodynamic equilibrium, all the absorbed energy to keep thermodynamic equilibrium. For a real-body surface, their emitted spectral radiance is denoted by $R_\lambda(T)$ with R for Radiance. A real surface is of course less emissive than a black surface and their emissive ability is described by a factor called emissivity through comparing with the black surface at the same condition. i.e., $R_\lambda(T) = \varepsilon_\lambda(\mu)B_\lambda(T)$. In honor of the brilliant contributions of Johann Heinrich Lambert (1728–1777) to the absorbance of a material sample and the reflectance of an ideal surface, a radiometer collected radiance is commonly denoted by L_i with i for the corresponding channel number. Furthermore, for convenience and mathematical shorthand, L_i is converted to and recorded by its corresponding blackbody's physical temperature called brightness temperature and denoted by T_i. i.e. T_i is the solution of $L_i = \int_{\lambda_1}^{\lambda_2} \phi_i(\lambda) B_\lambda(T_i) d\lambda$, where $\phi_i(\lambda)$ is the SRF of the channel i, λ_1 and λ_2 is the lower and upper boundaries of the channel spectral range.

Table 3. Summarize of the quantities and symbols involved in the relationship between T_s and L_i.

Physical and effective quantities	symbols	definitions
Water vapor concentration at altitude ξ		$e(\xi)$
Pressure of atmosphere at altitude ξ		$P(\xi)$
Spectral absorption coefficient of water vapor.		α_λ
Spectral absorption coefficient of water vapor at altitude ξ.		$\alpha_\lambda(\xi)$
The tuning function for Spectral absorption coefficient of water vapor at altitude ξ	$F\big(P(\xi), T(\xi)\big)$	$F\big(P(\xi), T(\xi)\big) = \frac{\alpha_\lambda(\xi)}{\alpha_\lambda}$
Effective absorption of atmosphere	W	$\mathrm{W} = \int_0^h F\big(P(\xi), T(\xi)\big) e(\xi) d\xi$
Effective radiative temperature of atmosphere	T_a	$T_a = \frac{\int_0^h T(\xi) F\big(P(\xi), T(\xi)\big) e(\xi) d\xi}{W}$
effective absorptive factor in band i.	A_i	$A_i(f_i, T_i) = \frac{\int_0^\infty f_i(\lambda) \left.\frac{dR_\lambda}{dT}\right\|_{T_i} \alpha_\lambda d\lambda}{\int_0^\infty f_i(\lambda) \left.\frac{dR_\lambda}{dT}\right\|_{T_i} d\lambda}$
A mathematical shorthand to describe the linearization of the Planck's function	$\tilde{L}_i$	$\tilde{L}_i = \frac{R_\lambda(T_i)}{\frac{d}{dT} R_\lambda(T_i)}$

Results

The process of surface emitted radiance over the spectral region 8~14 μm (which is generally called SW region and specially used to observe the Earth surface emitted radiance, see path ④ of the figure) transferred to a space borne radiometer in a distance h under local thermal equilibrium (LTE) in clear sky could be modeled by

$$I_\lambda(\mu,h)=t_\lambda(\mu)*\underbrace{\left(\varepsilon_\lambda(\mu)B_\lambda(T_s)+\int_0^{2\pi}\int_0^1 f_{r,\lambda}(\mu,\mu')R'_{\lambda,0}(\mu')\mu' d\mu' d\varphi'\right)}_{\text{Surface leaving radiance}}+R_{\lambda,AE\uparrow}(\mu), \qquad [1]$$

The LTE condition is required because the atmospheric correction is simulated by the Planck's function and Kirchhoff's law and the condition clear sky is the guarantee of the surface emitted radiance to reach the satellite. In order to parameterize the relationship between T_s and T_i in one channel, Prabhakara et al (1974), McMillin (1975), Deschamps and Phulpin (1980), and Becker (1987) proposed four approximations as

1. The Earth surface is assumed to be a Lambertian reflector (which is generally not a good approximation (Becker et al 1985));
2. The Planck's function is linearized in the vicinity of T_i, where T_i is the BT in channel i;
3. The atmospheric absorption (WV gives the dominant effect in these wavelengths) is small enough to approximate the transmission $\tau_\lambda(\mu,h)$ by

$$\tau_\lambda(\mu,h)=1-\mu^{-1}\int_0^h \alpha(\lambda,z)e(z)dz, \qquad [2]$$

   where $e(z)$ is the WV-concentration at altitude z; and
4. The dependence of WV-absorption coefficient $\alpha(\lambda,z)$ on λ and z may be factored by

$$\alpha(\lambda,z)=\alpha(\lambda)F\big(P(z),T(z)\big). \qquad [3]$$

The relationship derived by Becker is

$$T_s=\left(T_i+\frac{1-\varepsilon_i}{\varepsilon_i}\tilde{L}_i\right)-\frac{A_iW}{\mu\varepsilon_i-A_iW\varepsilon_i}(T_a-T_i)-2A_iW\frac{1-\varepsilon_i}{\varepsilon_i}(T_a-T_i-\tilde{L}_i), \qquad [4]$$

which was wrong, and the correct one we proposed is

$$T_s=\left(T_i+\frac{1-\varepsilon_i+A_i\gamma W}{\varepsilon_i-A_i\gamma W}\tilde{L}_i\right)-\frac{A_iW}{\mu\varepsilon_i-A_iW}\left(T_a-T_i+\tilde{L}_i\right)-2A_iW(1-\varepsilon_i)\frac{1-A_i\gamma W}{\varepsilon_i-A_i\gamma W}\left(T_a-T_i+\tilde{L}_i\right). \qquad [5]$$

The physical meaning and description of the symbols involved in the above equations (i.e., Eq. [1]~[5]) are depicted in Liu et al (2019) and summarized in the above tables 1-3. The derivation of Eq. [5] is specified in the following materials and methods section. For comparison purpose, the symbols are tried to kept consistent with Becker (1987) as closely as possible.

Materials and methods

The detailed mathematical derivation of the correct equation is given as follows. Inheriting the symbols or notations in Becker (1987), expanding the factors in Eq.[1], we have

$$\begin{aligned} I_\lambda(\theta,h) \quad = \quad & \left[\varepsilon_\lambda R_\lambda(T_s)+\int_0^{2\pi}\int_0^{\frac{\pi}{2}} f_{r,\lambda}(\theta,\theta',\varphi')\int_0^h R_\lambda\big(T(\xi)\big)\frac{\partial\tau'_\lambda(\theta',\xi)}{\partial\xi}d\xi\,\mu' d\mu' d\varphi'\right]\tau_\lambda(\theta,0) \\ & +\int_0^h R_\lambda(T(\xi))\frac{\partial\tau_\lambda(\theta,\xi)}{\partial\xi}d\xi \end{aligned} \qquad [6]$$

where $I_\lambda(\theta,h)$ is the radiance at the entrance silt of a radiometer that observing the surface with local zenith angle θ at wavelength λ from a distance of h. the terms in the square brackets is the surface leaving radiance.

Lambertian reflection for the downward radiance

For a Lambertian reflector, $f_{r,\lambda}(\theta,\theta',\varphi') = \frac{1-\varepsilon_\lambda}{\pi}$. Thus, the reflected atmospheric downward radiance at the surface level in Eq. [6] is

$$\int_0^{2\pi}\int_0^{\frac{\pi}{2}} f_{r,\lambda}(\theta,\theta',\varphi')\int_0^h R_\lambda(T(\xi))\frac{\partial\tau'_\lambda(\theta',\xi)}{\partial\xi}d\xi\,\mu' d\mu' d\varphi' \quad = \quad 2(1-\varepsilon_\lambda)\int_0^{\frac{\pi}{2}}\int_0^h R_\lambda(T(\xi))\frac{\partial\tau'_\lambda(\theta',\xi)d\xi}{\partial\xi}\mu' d\mu' \qquad [7]$$

Linearization of Planck's function

Planck's function is linearized in the vicinity of T_i rather than T_s is generally accurate because the main contribution to the upward and downward atmospheric thermal radiances are principally emitted from the lower atmosphere, where the temperatures, $T(z)$, are close to each other and generally closer to T_i than to T_s. This yield

$$R_\lambda(T) = R_\lambda(T_i) + \left.\frac{dR_\lambda}{dT}\right|_{T_i}(T-T_i)$$
$$= \frac{d}{dT}R_\lambda(T_i) * \left(\frac{R_\lambda(T_i)}{\frac{d}{dT}R_\lambda(T_i)} + T - T_i\right).$$

For mathematical shorthand, define

$$\tilde{L}_i = \frac{R_\lambda(T_i)}{\frac{d}{dT}R_\lambda(T_i)}, \qquad [8]$$

the linearization of Planck's function is

$$R_\lambda(T) = \left.\frac{dR_\lambda}{dT}\right|_{T_i}\left(\tilde{L}_i + T - T_i\right) \qquad [9]$$

Substitute Eq. [7] and [9] into Eq. [6], after rearrangement, we have

$$I_\lambda(\theta,h) = \left.\frac{dR_\lambda}{dT}\right|_{T_i}\left(\left[\varepsilon_\lambda(T_s - T_i + \tilde{L}_i) + 2(1-\varepsilon_\lambda)\int_0^{\frac{\pi}{2}}\left(\int_h^0 (T(\xi) - T_i + \tilde{L}_i)\frac{\partial}{\partial\xi}\tau'_\lambda(\theta',\xi)d\xi\right)cos\theta' sin\theta' d\theta'\right]\tau_\lambda(\theta,0) + \int_0^h (T(\xi) - T_i + \tilde{L}_i)\frac{\partial}{\partial\xi}\tau_\lambda(\theta,\xi)d\xi\right) \quad [10]$$

Simplification of the transmittance and factorization of the absorption coefficient

Suppose the absorption of the atmosphere is small enough to approximate the transmission $\tau_\lambda(\theta,\xi)$ by the first order approximation of Taylor's expansion, we have,

$$\tau_\lambda(\theta,\xi) = e^{-sec\theta\int_\xi^h \alpha_\lambda(\xi)e(\xi)d\xi} = 1 - sec\theta\int_\xi^h \alpha_\lambda(\xi)e(\xi)d\xi.$$

Recall the dependence of absorption coefficient $\alpha_\lambda(z)$ on λ and z (Eq.[3]), we have,

$$\tau_\lambda(\theta,\xi) = 1 - sec\theta\int_\xi^h \alpha_\lambda(\xi)e(\xi)d\xi = 1 - sec\theta\cdot\alpha_\lambda\int_\xi^h F(P(\xi),T(\xi))e(\xi)d\xi$$

and thus

$$\frac{\partial}{\partial\xi}\tau_\lambda(\theta,\xi) = \frac{\partial}{\partial\xi}\left(1 - sec\theta\alpha_\lambda\int_\xi^h F(P(\xi),T(\xi))e(\xi)d\xi\right) = sec\theta\alpha_\lambda\frac{\partial}{\partial\xi}\int_\xi^h F(P(\xi),T(\xi))e(\xi)d\xi = sec\theta\alpha_\lambda F(P(\xi),T(\xi))e(\xi)d\xi$$

Similarly,

$$\frac{\partial}{\partial\xi}\tau'_\lambda(\theta',\xi) = -sec\theta'\alpha_\lambda F(P(\xi),T(\xi))e(\xi)d\xi$$

therefore, Eq. [10] reduces to

$$\begin{aligned} I_\lambda(\theta,h) \quad = \quad & \left.\frac{dR_\lambda}{dT}\right|_{T_i}\left(\left[\varepsilon_\lambda(T_s - T_i + \tilde{L}_i) - 2(1-\varepsilon_\lambda)\int_0^{\frac{\pi}{2}}\left(\int_h^0 (T(\xi) - T_i + \tilde{L}_i)\sec\theta'\,\alpha_\lambda F(P(\xi),T(\xi))e(\xi)d\xi\right)\cos\theta'\sin\theta'\,d\theta'\right]\right. \\ & * \left(1 - \sec\theta\,\alpha_\lambda\int_0^h F(P(\xi),T(\xi))e(\xi)d\xi\right) \\ & \left. + \int_0^h (T(\xi) - T_i + \tilde{L}_i)\sec\theta\,\alpha_\lambda F(P(\xi),T(\xi))e(\xi)d\xi\right) \end{aligned}$$

Since

$$\int_0^{\frac{\pi}{2}}\sec\theta'\cos\theta'\sin\theta'\,d\theta' = 1,$$

and use the shorthand γ for $\sec\theta$, we have

$$\begin{aligned} I_\lambda(\theta,h) &= \left.\frac{dR_\lambda}{dT}\right|_{T_i}\left(\left[\varepsilon_\lambda(T_s-T_i+\tilde{L}_i)+2(1-\varepsilon_\lambda)\alpha_\lambda\int_0^h(T(\xi)-T_i+\tilde{L}_i)F(P(\xi),T(\xi))e(\xi)d\xi\right]\right.\\ &\qquad \left. *\left(1-\gamma\alpha_\lambda\int_0^h F(P(\xi),T(\xi))e(\xi)d\xi\right)\right)\\ &\quad +\left.\frac{dR_\lambda}{dT}\right|_{T_i}\left(\gamma\alpha_\lambda\int_0^h(T(\xi)-T_i+\tilde{L}_i)F(P(\xi),T(\xi))e(\xi)d\xi\right) \end{aligned} \qquad [11]$$

and since

$$\int_0^\infty f_i(\lambda)I_\lambda(\theta,h)d\lambda=\int_0^\infty f_i(\lambda)R_\lambda(T_i(\theta))d\lambda \overset{\text{Applying Eq.[9]}}{=\!=\!=} \int_0^\infty f_i(\lambda)\tilde{L}_i\left.\frac{dR_\lambda}{dT}\right|_{T_i}d\lambda$$

multiplying both sides of Eq. [11] by the SRF $f_i(\lambda)$ and integrating over the channel region, we have

$$\begin{aligned} \int_0^\infty \boldsymbol{f_i(\lambda)}\tilde{L}_i\left.\frac{dR_\lambda}{dT}\right|_{T_i}\boldsymbol{d\lambda} &= \int_0^\infty \boldsymbol{f_i}(\lambda)\left.\frac{dR_\lambda}{dT}\right|_{T_i}\left(\left[\varepsilon_\lambda(T_s-T_i+\tilde{L}_i)+2(1-\varepsilon_\lambda)\alpha_\lambda\int_0^h(T(\xi)-T_i+\tilde{L}_i)F(P(\xi),T(\xi))e(\xi)d\xi\right]\right.\\ &\qquad *\left(1-\gamma\alpha_\lambda\int_0^h F(P(\xi),T(\xi))e(\xi)d\xi\right)\\ &\qquad \left.+\gamma\alpha_\lambda\int_0^h(T(\xi)-T_i+\tilde{L}_i)F(P(\xi),T(\xi))e(\xi)d\xi\right)d\lambda \end{aligned}$$

Define

$$\mathrm{W}=\int_0^h F(P(\xi),T(\xi))e(\xi)d\xi$$

and

$$T_a=\frac{\int_0^h T(\xi)F(P(\xi),T(\xi))e(\xi)d\xi}{W}$$

(Sobrino et al 1991) and respectively called "absorption weighted WV column" and "atmospheric effective radiative temperature" to describe the atmospheric profiles, we have

$$\begin{aligned} \int_0^h(T(\xi)-T_i+\tilde{L}_i)F(P(\xi),T(\xi))e(\xi)d\xi &= \int_0^h T(\xi)F(P(\xi),T(\xi))e(\xi)d\xi+\int_0^h(\tilde{L}_i-T_i)F(P(\xi),T(\xi))e(\xi)d\xi\\ &= WT_a+(\tilde{L}_i-T_i)W=W(T_a+\tilde{L}_i-T_i) \end{aligned}$$

we have,

$$\begin{aligned} &\int_0^\infty f_i(\lambda)\tilde{L}_i\left.\frac{dR_\lambda}{dT}\right|_{T_i}d\lambda\\ &=\int_0^\infty f_i(\lambda)\left.\frac{dR_\lambda}{dT}\right|_{T_i}\left(\left[\varepsilon_\lambda(T_s-T_i+\tilde{L}_i)+2(1-\varepsilon_\lambda)\alpha_\lambda(T_a+\tilde{L}_i-T_i)W\right](1-\alpha_\lambda\gamma W)+\alpha_\lambda(T_a+\tilde{L}_i-T_i)\gamma W\right)d\lambda\\ &=\int_0^\infty f_i(\lambda)\left.\frac{dR_\lambda}{dT}\right|_{T_i}\varepsilon_\lambda(T_s-T_i+\tilde{L}_i)(1-\alpha_\lambda\gamma W)d\lambda\\ &+2W\int_0^\infty f_i(\lambda)\left.\frac{dR_\lambda}{dT}\right|_{T_i}(1-\varepsilon_\lambda)\alpha_\lambda(T_a+\tilde{L}_i-T_i)(1-\alpha_\lambda\gamma W)d\lambda+\gamma W\int_0^\infty f_i(\lambda)\left.\frac{dR_\lambda}{dT}\right|_{T_i}\alpha_\lambda(T_a+\tilde{L}_i-T_i)d\lambda \end{aligned} \qquad [12]$$

Suppose $\tilde{L}_i$ is independent of wavelength within a SW band

With the assumption of $\tilde{L}_i$ is independent of wavelength within a SW band, Eq. [12] could be reduced to

$$\begin{aligned} &\tilde{L}_i\int_0^\infty f_i(\lambda)\left.\frac{dR_\lambda}{dT}\right|_{T_i}d\lambda\\ &=(T_s-T_i+\tilde{L}_i)\int_0^\infty f_i(\lambda)\left.\frac{dR_\lambda}{dT}\right|_{T_i}\varepsilon_\lambda(1-\alpha_\lambda\gamma W)d\lambda\\ &\quad +2W(T_a+\tilde{L}_i-T_i)\int_0^\infty f_i(\lambda)\left.\frac{dR_\lambda}{dT}\right|_{T_i}(1-\varepsilon_\lambda)\alpha_\lambda(1-\alpha_\lambda\gamma W)d\lambda\\ &\quad +\gamma W(T_a+\tilde{L}_i-T_i)\int_0^\infty f_i(\lambda)\left.\frac{dR_\lambda}{dT}\right|_{T_i}\alpha_\lambda d\lambda \end{aligned}$$

Thus,

$$
\begin{aligned}
\tilde{L}_i \;=\;& (T_s - T_i + \tilde{L}_i)\frac{\int_0^\infty f_i(\lambda)\frac{dR_\lambda}{dT}\Big|_{T_i}\varepsilon_\lambda(1-\alpha_\lambda\gamma W)d\lambda}{\int_0^\infty f_i(\lambda)\frac{dR_\lambda}{dT}\Big|_{T_i}d\lambda} \\
&+2W(T_a + \tilde{L}_i - T_i)\frac{\int_0^\infty f_i(\lambda)\frac{dR_\lambda}{dT}\Big|_{T_i}(1-\varepsilon_\lambda)\alpha_\lambda(1-\alpha_\lambda\gamma W)d\lambda}{\int_0^\infty f_i(\lambda)\frac{dR_\lambda}{dT}\Big|_{T_i}d\lambda} \\
&+\gamma W(T_a + \tilde{L}_i - T_i)\frac{\int_0^\infty f_i(\lambda)\frac{dR_\lambda}{dT}\Big|_{T_i}\alpha_\lambda d\lambda}{\int_0^\infty f_i(\lambda)\frac{dR_\lambda}{dT}\Big|_{T_i}d\lambda}
\end{aligned} \qquad [13]
$$

Suppose the integral of products equals to the product of integrals within a SW band

Define the band effective absorptive factor and emissivity (channel i for this case) by

$$
A_i(f_i, T_i) = \frac{\int_0^\infty f_i(\lambda)\frac{dR_\lambda}{dT}\Big|_{T_i}\alpha_\lambda d\lambda}{\int_0^\infty f_i(\lambda)\frac{dR_\lambda}{dT}\Big|_{T_i}d\lambda}
$$

and

$$
\varepsilon_i = \frac{\int_0^\infty f_i(\lambda)\frac{dR_\lambda}{dT}\Big|_{T_i}\varepsilon_\lambda d\lambda}{\int_0^\infty f_i(\lambda)\frac{dR_\lambda}{dT}\Big|_{T_i}d\lambda}
$$

respectively and applying the integral approximation descripted in Liu et al (2019), the factors in the first two terms of the RHS of [13] reduces to

$$
\frac{\int_0^\infty f_i(\lambda)\frac{dR_\lambda}{dT}\Big|_{T_i}\varepsilon_\lambda(1-\alpha_\lambda\gamma W)d\lambda}{\int_0^\infty f_i(\lambda)\frac{dR_\lambda}{dT}\Big|_{T_i}d\lambda} = \frac{\int_0^\infty f_i(\lambda)\frac{dR_\lambda}{dT}\Big|_{T_i}\varepsilon_\lambda d\lambda}{\int_0^\infty f_i(\lambda)\frac{dR_\lambda}{dT}\Big|_{T_i}d\lambda} - \frac{\int_0^\infty f_i(\lambda)\frac{dR_\lambda}{dT}\Big|_{T_i}\alpha_\lambda d\lambda}{\int_0^\infty f_i(\lambda)\frac{dR_\lambda}{dT}\Big|_{T_i}d\lambda}\gamma W = \varepsilon_i - \gamma A_i W \qquad [14]
$$

and

$$
\frac{\int_0^\infty f_i(\lambda)\frac{dR_\lambda}{dT}\Big|_{T_i}(1-\varepsilon_\lambda)\alpha_\lambda(1-\gamma\alpha_\lambda W)d\lambda}{\int_0^\infty f_i(\lambda)\frac{dR_\lambda}{dT}\Big|_{T_i}d\lambda} = (1-\varepsilon_i)A_i(1-\gamma A_i W)
$$

Thus, [13] reduces to

$$
\tilde{L}_i = [\ \varepsilon_i - A_i\gamma W \qquad \{2(1-\varepsilon_i)(1-A_i\gamma W)+\gamma\}A_i W\ ]\cdot\begin{bmatrix} T_s + \tilde{L}_i - T_i \\ \\ T_a + \tilde{L}_i - T_i \end{bmatrix} \qquad [15]
$$

or explicitly written T_s as Eq. [5], which completes the derivation.

Eq. [5] or [15] is the formula that we proposed to relate T_s and T_i in channel i over the SW region. It is different from the one developed by Becker (1987).

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[i] The superscript "$'$ "and subscript "$_0$"is incident and solar related quantities respectively.

[ii] μ and γ are introduced for mathematical shorthand.

[iii] In concurrent radiometry, radiance is referred to as L_λ in honor of Lambert. We use I_λ in this paper is intended only to keep pace with Becker 1987.