Supplementary Material to "RANK: Large-Scale Inference with Graphical Nonlinear Knockoffs"

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This Supplementary Material contains additional technical details for the proofs of Lemmas 3–8. All the notation is the same as in the main body of the paper.

B Additional technical details

B.1 Lemma 3 and its proof

Lemma 3. *Assume that* $\mathbf{X} = (X_{ij}) \in \mathbb{R}^{n\times p}$ *has independent rows with distribution* $N(\mathbf{0}, \mathbf{\Sigma}_0)$, $\Lambda_{\max}(\mathbf{\Sigma}_0) \leq M$, *and* $\boldsymbol{\varepsilon} = (\varepsilon_1, \cdots, \varepsilon_n)^T$ *has i.i.d. components with* $\mathbb{P}\{|\varepsilon_i| > t\} \leq C_1 \exp(-C_1^{-1}t^2)$ *for* $t > 0$ *and some constants* $M, C_1 > 0$. *Then we have*

$$\mathbb{P}\left\{\left\|\frac{1}{n}\mathbf{X}^T\boldsymbol{\varepsilon}\right\|_\infty \leq C\sqrt{(\log p)/n}\right\} \geq 1 - p^{-c}$$

for some constant $c > 0$ *and large enough constant* $C > 0$.

Proof. First observe that $\mathbb{P}(|X_{ij}| > t) \leq 2\exp\{-(2M)^{-1}t^2\}$ for $t > 0$, since $X_{ij} \sim N(0, \mathbf{\Sigma}_{0,jj})$ and $\mathbf{\Sigma}_{0,jj} \leq \Lambda_{\max}(\mathbf{\Sigma}_0) \leq M$, where $\mathbf{\Sigma}_{0,jj}$ denotes the jth diagonal entry of matrix $\mathbf{\Sigma}_0$. By assumption, we also have $\mathbb{P}(|\varepsilon_i| > t) \leq C_1\exp\{-C_1^{-1}t^2\}$. Combining these two inequalities yields

$$\begin{aligned}
\mathbb{P}(|\varepsilon_i X_{ij}| > t) &\leq \mathbb{P}(|\varepsilon_i| > \sqrt{t}) + \mathbb{P}(|X_{ij}| > \sqrt{t}) \\
&\leq C_1\exp\{-C_1^{-1}t\} + 2\exp\{-(2M)^{-1}t\} \\
&\leq C_2\exp\{-C_2^{-1}t\},
\end{aligned}$$

where $C_2 > 0$ is some constant that depends only on constants C_1 and M. Thus by Lemma 6 in [28], there exists some constant $\widetilde{C}_1 > 0$ such that

$$\mathbb{P}(|n^{-1}\sum_{i=1}^n \varepsilon_i X_{ij}| > z) \leq \widetilde{C}_1\exp\{-\widetilde{C}_1 n z^2) \tag{A.1}$$

for all $0 < z < 1$.

Denote by $\mathbf{X}_j$ the jth column of matrix $\mathbf{X}$. Then by (A.1), the union bound leads to

$$\begin{aligned}
1 - \mathbb{P}\left(\left\|n^{-1}\mathbf{X}^T\boldsymbol{\varepsilon}\right\|_\infty \leq z\right) &= \mathbb{P}\left(\left\|n^{-1}\mathbf{X}^T\boldsymbol{\varepsilon}\right\|_\infty > z\right) \\
&= \mathbb{P}\Big(\max_{1\leq j\leq p}|n^{-1}\boldsymbol{\varepsilon}^T\mathbf{X}_j| > z\Big) \\
&\leq \sum_{j=1}^p \mathbb{P}(|n^{-1}\sum_{i=1}^n \varepsilon_i X_{ij}| > z) \\
&\leq p\widetilde{C}_1\exp\{-\widetilde{C}_1 n z^2).
\end{aligned}$$

Letting $z = C\sqrt{(\log p)/n}$ in the above inequality, we obtain

$$\mathbb{P}\Big(\Big\|n^{-1}\mathbf{X}^T\boldsymbol{\varepsilon}\Big\|_\infty \le C\sqrt{(\log p)/n}\Big) \ge 1 - \widetilde{C}_1 p^{-(\widetilde{C}_1 C^2 - 1)}.$$

Taking large enough positive constant C completes the proof of Lemma 3.

B.2 Lemma 4 and its proof

Lemma 4. *Assume that all the conditions of Proposition 2 hold and* $a_n[(L_p+L_p')^{1/2}+K_n^{1/2}] = o(1)$. *Then we have*

$$P\left\{\sup_{\boldsymbol{\Omega}\in\mathcal{A},\,|\mathcal{S}|\le K_n}\Big\|\widetilde{\boldsymbol{\rho}}_{\mathcal{S}} - \widetilde{\mathbf{G}}_{\mathcal{S},\mathcal{S}}\boldsymbol{\beta}_{\mathbb{T},\mathcal{S}}\Big\|_\infty \le C_4\sqrt{(\log p)/n}\right\} = 1 - O(p^{-c_4})$$

for some constants $c_4, C_4 > 0$.

Proof. In this proof, we use c and C to denote generic positive constants and use the same notation as in the proof of Proposition 2 in Section A.6. Since $\boldsymbol{\beta}_{\mathbb{T}} = (\boldsymbol{\beta}_0^T, 0, \ldots, 0)^T$ with $\boldsymbol{\beta}_0$ the true regression coefficient vector, it is easy to check that $\widetilde{\mathbf{X}}_{\mathrm{KO}}\boldsymbol{\beta}_{\mathbb{T}} = \mathbf{X}\boldsymbol{\beta}_0$. In view of $\mathbf{y} = \mathbf{X}\boldsymbol{\beta}_0 + \boldsymbol{\varepsilon}$, it follows from the definitions of $\widetilde{\boldsymbol{\rho}}$ and $\widetilde{\mathbf{G}}$ that

$$\begin{aligned}
\widetilde{\boldsymbol{\rho}}_{\mathcal{S}} - \widetilde{\mathbf{G}}_{\mathcal{S},\mathcal{S}}\boldsymbol{\beta}_{\mathbb{T},\mathcal{S}} &= \frac{1}{n}\widetilde{\mathbf{X}}_{\mathrm{KO},\mathcal{S}}^T\mathbf{X}\boldsymbol{\beta}_0 + \frac{1}{n}\widetilde{\mathbf{X}}_{\mathrm{KO},\mathcal{S}}^T\boldsymbol{\varepsilon} - \frac{1}{n}\widetilde{\mathbf{X}}_{\mathrm{KO},\mathcal{S}}^T\widetilde{\mathbf{X}}_{\mathrm{KO},\mathcal{S}}\boldsymbol{\beta}_{\mathbb{T},\mathcal{S}} \\
&= \frac{1}{n}\mathbf{X}_{\mathrm{KO},\mathcal{S}}^T\boldsymbol{\varepsilon} + \frac{1}{n}(\widetilde{\mathbf{X}}_{\mathrm{KO},\mathcal{S}} - \mathbf{X}_{\mathrm{KO},\mathcal{S}})^T\boldsymbol{\varepsilon}.
\end{aligned}$$

Using the triangle inequality, we deduce

$$\|\widetilde{\boldsymbol{\rho}}_{\mathcal{S}} - \widetilde{\mathbf{G}}_{\mathcal{S},\mathcal{S}}\boldsymbol{\beta}_{\mathbb{T},\mathcal{S}}\|_\infty \le \Big\|\frac{1}{n}\mathbf{X}_{\mathrm{KO},\mathcal{S}}^T\boldsymbol{\varepsilon}\Big\|_\infty + \Big\|\frac{1}{n}(\widetilde{\mathbf{X}}_{\mathrm{KO},\mathcal{S}} - \mathbf{X}_{\mathrm{KO},\mathcal{S}})^T\boldsymbol{\varepsilon}\Big\|_\infty.$$

We will bound both terms on the right hand side of the above inequality.

By Lemma 3, we can show that for the first term,

$$\Big\|\frac{1}{n}\mathbf{X}_{\mathrm{KO},\mathcal{S}}^T\boldsymbol{\varepsilon}\Big\|_\infty \le \Big\|\frac{1}{n}\mathbf{X}_{\mathrm{KO}}^T\boldsymbol{\varepsilon}\Big\|_\infty \le C\sqrt{(\log p)/n}$$

with probability at least $1 - p^{-c}$ for some constants $C, c > 0$. We will prove that with probability at least $1 - o(p^{-c})$,

$$\Big\|\frac{1}{n}(\widetilde{\mathbf{X}}_{\mathrm{KO},\mathcal{S}} - \mathbf{X}_{\mathrm{KO},\mathcal{S}})^T\boldsymbol{\varepsilon}\Big\|_\infty \le Ca_n(L_p + L_p')^{1/2}\sqrt{(\log p)/n} + Ca_n\sqrt{n^{-1}K_n(\log p)}. \qquad \text{(A.2)}$$

Then the desired result in this lemma can be shown by noting that $a_n[(L_p+L_p')^{1/2}+K_n^{1/2}] \to 0$.

It remains to prove (A.2). Recall that matrices $\breve{\mathbf{X}}_{\mathcal{S}}$ and $\breve{\mathbf{X}}_{0,\mathcal{S}}$ can be written as

$$\breve{\mathbf{X}}_{\mathcal{S}} = \mathbf{X}(\mathbf{I} - \boldsymbol{\Omega}\mathrm{diag}\{\mathbf{s}\})_{\mathcal{S}} + \mathbf{Z}\mathbf{B}_{0,\mathcal{S}}(\mathbf{B}_{0,\mathcal{S}}^T\mathbf{B}_{0,\mathcal{S}})^{-1/2}\Big((\mathbf{B}_{\mathcal{S}}^{\boldsymbol{\Omega}})^T\mathbf{B}_{\mathcal{S}}^{\boldsymbol{\Omega}}\Big)^{1/2},$$
$$\breve{\mathbf{X}}_{0,\mathcal{S}} = \mathbf{X}(\mathbf{I} - \boldsymbol{\Omega}_0\mathrm{diag}\{\mathbf{s}\})_{\mathcal{S}} + \mathbf{Z}\mathbf{B}_{0,\mathcal{S}},$$

where the notation is the same as in the proof of Proposition 2 in Section A.6. By the definitions of $\widetilde{\mathbf{X}}_{\mathrm{KO}}$ and $\mathbf{X}_{\mathrm{KO}}$, it holds that

$$\Big\|\frac{1}{n}(\widetilde{\mathbf{X}}_{\mathrm{KO},\mathcal{S}} - \mathbf{X}_{\mathrm{KO},\mathcal{S}})^T\boldsymbol{\varepsilon}\Big\|_\infty = \Big\|\frac{1}{n}(\breve{\mathbf{X}}_{\mathcal{S}} - \breve{\mathbf{X}}_{0,\mathcal{S}})^T\boldsymbol{\varepsilon}\Big\|_\infty, \tag{A.3}$$

where $\breve{\mathbf{X}}_{\mathcal{S}}$ and $\breve{\mathbf{X}}_{0,\mathcal{S}}$ represent the submatrices formed by columns in $\mathcal{S}$. We now turn to analyzing the term $n^{-1}(\breve{\mathbf{X}}_{\mathcal{S}} - \breve{\mathbf{X}}_{0,\mathcal{S}})^T\boldsymbol{\varepsilon}$. Some routine calculations give

$$\begin{aligned}\frac{1}{n}(\breve{\mathbf{X}}_{\mathcal{S}} - \breve{\mathbf{X}}_{0,\mathcal{S}})^T\boldsymbol{\varepsilon} \ &= \ \frac{1}{n}\Big(((\boldsymbol{\Omega}_0 - \boldsymbol{\Omega})\mathrm{diag}\{\mathbf{s}\})_{\mathcal{S}}\Big)^T\mathbf{X}^T\boldsymbol{\varepsilon}\\ &\quad+\frac{1}{n}\Big(((\mathbf{B}_{\mathcal{S}}^{\boldsymbol{\Omega}})^T\mathbf{B}_{\mathcal{S}}^{\boldsymbol{\Omega}})^{1/2}(\mathbf{B}_{0,\mathcal{S}}^T\mathbf{B}_{0,\mathcal{S}})^{-1/2} - \mathbf{I}\Big)\mathbf{B}_{0,\mathcal{S}}^T\mathbf{Z}^T\boldsymbol{\varepsilon}.\end{aligned}$$

Thus it follows from $s_j \leq 2\Lambda_{\max}(\boldsymbol{\Sigma}_0)$ for all $1 \leq j \leq p$ and the triangle inequality that

$$\begin{aligned}\Big\|\frac{1}{n}(\breve{\mathbf{X}}_{\mathcal{S}} - \breve{\mathbf{X}}_{0,\mathcal{S}})^T\boldsymbol{\varepsilon}\Big\|_\infty \ &\leq \ 2\Lambda_{\max}(\boldsymbol{\Sigma}_0)\Big\|\frac{1}{n}(\boldsymbol{\Omega}_{0,\mathcal{S}} - \boldsymbol{\Omega}_{\mathcal{S}})^T\mathbf{X}^T\boldsymbol{\varepsilon}\Big\|_\infty\\ &\quad+ \ \Big\|\frac{1}{n}\Big(((\mathbf{B}_{\mathcal{S}}^{\boldsymbol{\Omega}})^T\mathbf{B}_{\mathcal{S}}^{\boldsymbol{\Omega}})^{1/2}(\mathbf{B}_{0,\mathcal{S}}^T\mathbf{B}_{0,\mathcal{S}})^{-1/2} - \mathbf{I}\Big)\mathbf{B}_{0,\mathcal{S}}^T\mathbf{Z}^T\boldsymbol{\varepsilon}\Big\|_\infty.\end{aligned} \tag{A.4}$$

We first examine the upper bound for $\Big\|\frac{1}{n}(\boldsymbol{\Omega}_{0,\mathcal{S}} - \boldsymbol{\Omega}_{\mathcal{S}})^T\mathbf{X}^T\boldsymbol{\varepsilon}\Big\|_\infty$ in (A.4). Since $\boldsymbol{\Omega} \in \mathcal{A}$ and $\boldsymbol{\Omega}_0$ is L_p-sparse, by Lemma 3 we deduce

$$\begin{aligned}\Big\|\frac{1}{n}(\boldsymbol{\Omega}_{0,\mathcal{S}} - \boldsymbol{\Omega}_{\mathcal{S}})^T\mathbf{X}^T\boldsymbol{\varepsilon}\Big\|_\infty \ &\leq \ \Big\|\frac{1}{n}(\boldsymbol{\Omega}_0 - \boldsymbol{\Omega})\mathbf{X}^T\boldsymbol{\varepsilon}\Big\|_\infty\\ &\leq \ \|\boldsymbol{\Omega}_0 - \boldsymbol{\Omega}\|_1\Big\|\frac{1}{n}\mathbf{X}^T\boldsymbol{\varepsilon}\Big\|_\infty\\ &\leq \ \sqrt{L_p + L_p'}\|\boldsymbol{\Omega} - \boldsymbol{\Omega}_0\|_2 \cdot C\sqrt{(\log p)/n}\\ &\leq \ Ca_n(L_p + L_p')^{1/2}\sqrt{(\log p)/n}.\end{aligned} \tag{A.5}$$

We can also bound the second term on the right hand side of (A.4) as

$$\begin{aligned}&\Big\|\frac{1}{n}\Big(((\mathbf{B}_{\mathcal{S}}^{\boldsymbol{\Omega}})^T\mathbf{B}_{\mathcal{S}}^{\boldsymbol{\Omega}})^{1/2}(\mathbf{B}_{0,\mathcal{S}}^T\mathbf{B}_{0,\mathcal{S}})^{-1/2} - \mathbf{I}\Big)\mathbf{B}_{0,\mathcal{S}}^T\mathbf{Z}^T\boldsymbol{\varepsilon}\Big\|_\infty\\ &\leq \Big\|((\mathbf{B}_{\mathcal{S}}^{\boldsymbol{\Omega}})^T\mathbf{B}_{\mathcal{S}}^{\boldsymbol{\Omega}})^{1/2}(\mathbf{B}_{0,\mathcal{S}}^T\mathbf{B}_{0,\mathcal{S}})^{-1/2} - \mathbf{I}\Big\|_1\Big\|\frac{1}{n}\mathbf{B}_{0,\mathcal{S}}^T\mathbf{Z}^T\boldsymbol{\varepsilon}\Big\|_\infty\\ &\leq \sqrt{2|\mathcal{S}|}\Big\|((\mathbf{B}_{\mathcal{S}}^{\boldsymbol{\Omega}})^T\mathbf{B}_{\mathcal{S}}^{\boldsymbol{\Omega}})^{1/2}(\mathbf{B}_{0,\mathcal{S}}^T\mathbf{B}_{0,\mathcal{S}})^{-1/2} - \mathbf{I}\Big\|_2\Big\|\frac{1}{n}\mathbf{B}_{0,\mathcal{S}}^T\mathbf{Z}^T\boldsymbol{\varepsilon}\Big\|_\infty\\ &\leq \sqrt{2K_n}Ca_n\sqrt{(\log p)/n} = Ca_n\sqrt{n^{-1}K_n(\log p)},\end{aligned}$$

where the second to the last step is entailed by Lemma 2 in Section A.3 and Lemma 5 in Section B.3. Therefore, combining this inequality with (A.3)–(A.5) results in (A.2), which concludes the proof of Lemma 4.

B.3 Lemma 5 and its proof

Lemma 5. *Under the conditions of Proposition 2, it holds that with probability at least* $1 - O(p^{-c})$,

$$\sup_{|\mathcal{S}| \leq K_n} \left\| \frac{1}{n} \mathbf{B}_{0,\mathcal{S}}^T \mathbf{Z}^T \boldsymbol{\varepsilon} \right\|_\infty \geq C\sqrt{(\log p)/n}$$

for some constant $C > 0$.

Proof. Since this is a specific case of Lemma 8 in Section B.6, the proof is omitted.

B.4 Lemma 6 and its proof

Lemma 6. *Under the conditions of Proposition 2 and Lemma 1, there exists some constant* $c \in (2(qs)^{-1}, 1)$ *such that with asymptotic probability one,* $|\widehat{\mathcal{S}}^{\mathbf{\Omega}}| \geq cs$ *holds uniformly over all* $\mathbf{\Omega} \in \mathcal{A}$ *and* $|\mathcal{S}| \leq K_n$, *where* $\widehat{\mathcal{S}}^{\mathbf{\Omega}} = \{j : W_j^{\mathbf{\Omega},\mathcal{S}} \geqslant T\}$.

Proof. Again we use C to denote generic positive constants whose values may change from line to line. By Proposition 2 in Section A.6, we have with probability at least $1 - O(p^{-c_1})$ that uniformly over all $\mathbf{\Omega} \in \mathcal{A}$ and $|\mathcal{S}| \leq K_n$,

$$\max_{1 \leq j \leq p} |\widehat{\beta}_j(\lambda; \mathbf{\Omega}, \mathcal{S}) - \beta_{0,j}| \leq C\sqrt{sn^{-1}(\log p)} \text{ and } \max_{1 \leq j \leq p} |\widehat{\beta}_{j+p}(\lambda; \mathbf{\Omega}, \mathcal{S})| \leq C\sqrt{sn^{-1}(\log p)}$$

for some constants $C, c_1 > 0$. Thus for each $1 \leq j \leq p$, we have

$$\begin{aligned} W_j^{\mathbf{\Omega},\mathcal{S}} &= |\widehat{\beta}_j(\lambda; \mathbf{\Omega}, \mathcal{S})| - |\widehat{\beta}_{j+p}(\lambda; \mathbf{\Omega}, \mathcal{S})| \\ &\geq -|\widehat{\beta}_{j+p}(\lambda; \mathbf{\Omega}, \mathcal{S})| \geq -C\sqrt{sn^{-1}(\log p)}. \end{aligned} \tag{A.6}$$

On the other hand, for each $j \in \mathcal{S}_2 = \{j : \beta_{0,j} \gg \sqrt{sn^{-1}(\log p)}\}$ it holds that

$$\begin{aligned} W_j^{\mathbf{\Omega},\mathcal{S}} &= |\widehat{\beta}_j(\lambda; \mathbf{\Omega}, \mathcal{S})| - |\widehat{\beta}_{j+p}(\lambda; \mathbf{\Omega}, \mathcal{S})| \\ &\geq |\beta_{0,j}| - |\widehat{\beta}_j(\lambda; \mathbf{\Omega}, \mathcal{S}) - \beta_{0,j}| - |\widehat{\beta}_{j+p}(\lambda; \mathbf{\Omega}, \mathcal{S})| \gg C\sqrt{sn^{-1}(\log p)}. \end{aligned} \tag{A.7}$$

Thus in order for any $W_j^{\mathbf{\Omega},\mathcal{S}}$, $1 \leq j \leq p$ to fall below $-T$, we must have $W_j^{\mathbf{\Omega},\mathcal{S}} \geq T$ for all $j \in \mathcal{S}_2$. This entails that

$$\left| \{j : W_j^{\mathbf{\Omega},\mathcal{S}} \geq T\} \right| \geq |\mathcal{S}_2| \geq cs, \tag{A.8}$$

which completes the proof of Lemma 6.

B.5 Lemma 7 and its proof

Lemma 7. *Assume that all the conditions of Proposition 2 hold and* $a_{2n} = a_n + (L'_p + K_n)\{(\log p)/n\}^{1/2} = o(1)$*. Then it holds that*

$$P\left\{\sup_{\boldsymbol{\Omega}\in\mathcal{A},\,|\mathcal{S}|\le K_n}\left\|\widetilde{\mathbf{G}}_{\mathcal{S},\mathcal{S}} - \mathbf{G}_{\mathcal{S},\mathcal{S}}\right\|_{\max} \le C_8 a_{2,n}\right\} = 1 - O(p^{-c_8})$$

for some constants $c_8, C_8 > 0$.

Proof. In this proof, we adopt the same notation as used in the proof of Proposition 2 in Section A.6. In light of (36), we have $\widetilde{\mathbf{G}} = n^{-1}[\mathbf{X}, \breve{\mathbf{X}}^{\boldsymbol{\Omega}}]^T[\mathbf{X}, \breve{\mathbf{X}}^{\boldsymbol{\Omega}}]$. Thus the matrix difference $\widetilde{\mathbf{G}}_{\mathcal{S},\mathcal{S}} - \mathbf{G}_{\mathcal{S},\mathcal{S}}$ can be represented in block form as

$$\begin{aligned}
\widetilde{\mathbf{G}}_{\mathcal{S},\mathcal{S}} - \mathbf{G}_{\mathcal{S},\mathcal{S}} &= \frac{1}{n}\begin{pmatrix} \mathbf{X}_{\mathcal{S}}^T\mathbf{X}_{\mathcal{S}} & (\breve{\mathbf{X}}_{\mathcal{S}}^{\boldsymbol{\Omega}})^T\mathbf{X}_{\mathcal{S}} \\ \mathbf{X}_{\mathcal{S}}^T\breve{\mathbf{X}}_{\mathcal{S}}^{\boldsymbol{\Omega}} & (\breve{\mathbf{X}}_{\mathcal{S}}^{\boldsymbol{\Omega}})^T\breve{\mathbf{X}}_{\mathcal{S}}^{\boldsymbol{\Omega}} \end{pmatrix} - \begin{pmatrix} \boldsymbol{\Sigma}_0 & \boldsymbol{\Sigma}_0 - \mathrm{diag}\{\mathbf{s}\} \\ \boldsymbol{\Sigma}_0 - \mathrm{diag}\{\mathbf{s}\} & \boldsymbol{\Sigma}_0 \end{pmatrix}_{\mathcal{S},\mathcal{S}} \\
&= \begin{pmatrix} n^{-1}\mathbf{X}_{\mathcal{S}}^T\mathbf{X}_{\mathcal{S}} - \boldsymbol{\Sigma}_{0,\mathcal{S},\mathcal{S}} & n^{-1}(\breve{\mathbf{X}}_{\mathcal{S}}^{\boldsymbol{\Omega}})^T\mathbf{X}_{\mathcal{S}} - \big(\boldsymbol{\Sigma}_0 - \mathrm{diag}\{\mathbf{s}\}\big)_{\mathcal{S},\mathcal{S}} \\ n^{-1}\mathbf{X}_{\mathcal{S}}^T\breve{\mathbf{X}}_{\mathcal{S}}^{\boldsymbol{\Omega}} - \big(\boldsymbol{\Sigma}_0 - \mathrm{diag}\{\mathbf{s}\}\big)_{\mathcal{S},\mathcal{S}} & n^{-1}(\breve{\mathbf{X}}_{\mathcal{S}}^{\boldsymbol{\Omega}})^T\breve{\mathbf{X}}_{\mathcal{S}}^{\boldsymbol{\Omega}} - \boldsymbol{\Sigma}_{0,\mathcal{S},\mathcal{S}} \end{pmatrix}.
\end{aligned}$$

Note that the off-diagonal blocks are the transposes of each other. Then we see that $\|\widetilde{\mathbf{G}}_{\mathcal{S},\mathcal{S}} - \mathbf{G}_{\mathcal{S},\mathcal{S}}\|_{\max}$ can be bounded by the maximum of $\|\eta_1\|_{\max}$, $\|\eta_2\|_{\max}$, and $\|\eta_3\|_{\max}$ with

$$\begin{aligned}
\eta_1 &= n^{-1}\mathbf{X}_{\mathcal{S}}^T\mathbf{X}_{\mathcal{S}} - \boldsymbol{\Sigma}_{0,\mathcal{S},\mathcal{S}}, \\
\eta_2 &= n^{-1}\mathbf{X}_{\mathcal{S}}^T\breve{\mathbf{X}}_{\mathcal{S}}^{\boldsymbol{\Omega}} - \big(\boldsymbol{\Sigma}_0 - \mathrm{diag}\{\mathbf{s}\}\big)_{\mathcal{S},\mathcal{S}}, \\
\eta_3 &= n^{-1}(\breve{\mathbf{X}}_{\mathcal{S}}^{\boldsymbol{\Omega}})^T\breve{\mathbf{X}}_{\mathcal{S}}^{\boldsymbol{\Omega}} - \boldsymbol{\Sigma}_{0,\mathcal{S},\mathcal{S}}.
\end{aligned}$$

To bound these three terms, we define three events

$$\begin{aligned}
\mathcal{E}_5 &= \left\{\|n^{-1}\mathbf{X}^T\mathbf{X} - \boldsymbol{\Sigma}_0\|_{\max} \le C\sqrt{(\log p)/n}\right\}, \\
\mathcal{E}_6 &= \left\{\sup_{|\mathcal{S}|\le K_n}\left\|n^{-1}\mathbf{B}_{0,\mathcal{S}}^T\mathbf{Z}^T\mathbf{X}\right\|_\infty \le C\sqrt{(\log p)/n}\right\}, \\
\mathcal{E}_7 &= \left\{\sup_{|\mathcal{S}|\le K_n}\left\|n^{-1}\mathbf{B}_{0,\mathcal{S}}^T\mathbf{Z}^T\mathbf{Z}\mathbf{B}_{0,\mathcal{S}} - \mathbf{B}_{0,\mathcal{S}}^T\mathbf{B}_{0,\mathcal{S}}\right\|_{\max} \le C\sqrt{(\log p)/n}\right\}.
\end{aligned}$$

By Lemma 8 in Section B.6, it holds that $P(\mathcal{E}_6) \ge 1 - O(p^{-c})$ and $P(\mathcal{E}_7) \ge 1 - O(p^{-c})$. Using Lemma A.3 in [6], we also have $P(\mathcal{E}_5) \ge 1 - O(p^{-c})$. Combining these results yields

$$P(\mathcal{E}_5 \cap \mathcal{E}_6 \cap \mathcal{E}_7) \ge 1 - O(p^{-c})$$

with $c > 0$ some constant.

Let us first consider term η_1. Conditional on $\mathcal{E}_5$, it is easy to see that

$$\|\eta_1\|_{\max} \le \|n^{-1}\mathbf{X}^T\mathbf{X} - \boldsymbol{\Sigma}_0\|_{\max} \le C\sqrt{(\log p)/n}. \tag{A.9}$$

We next bound $\|\eta_2\|_{\max}$ conditional on $\mathcal{E}_5 \cap \mathcal{E}_6$. To simplify the notation, denote by $\widetilde{\mathbf{B}}^{\mathcal{S},\boldsymbol{\Omega}} = (\mathbf{B}_{0,\mathcal{S}}^T \mathbf{B}_{0,\mathcal{S}})^{-1/2} \Big((\mathbf{B}_{\mathcal{S}}^{\boldsymbol{\Omega}})^T \mathbf{B}_{\mathcal{S}}^{\boldsymbol{\Omega}} \Big)^{1/2}$. By the definition of $\check{\mathbf{X}}_{\mathcal{S}}$, we deduce

$$
\begin{aligned}
\eta_2 &= n^{-1} \mathbf{X}_{\mathcal{S}}^T \check{\mathbf{X}}_{\mathcal{S}}^{\boldsymbol{\Omega}} - \big(\boldsymbol{\Sigma}_0 - \mathrm{diag}\{\mathbf{s}\}\big)_{\mathcal{S},\mathcal{S}} \\
&= n^{-1} \mathbf{X}_{\mathcal{S}}^T \mathbf{X} (\mathbf{I} - \boldsymbol{\Omega} \mathrm{diag}\{\mathbf{s}\})_{\mathcal{S}} + n^{-1} \mathbf{X}_{\mathcal{S}}^T \mathbf{Z} \mathbf{B}_{0,\mathcal{S}} \widetilde{\mathbf{B}}^{\mathcal{S},\boldsymbol{\Omega}} - \big(\boldsymbol{\Sigma}_0 - \mathrm{diag}\{\mathbf{s}\}\big)_{\mathcal{S},\mathcal{S}} \\
&= \big((n^{-1}\mathbf{X}^T\mathbf{X} - \boldsymbol{\Sigma}_0)(\mathbf{I} - \boldsymbol{\Omega}\mathrm{diag}\{\mathbf{s}\})\big)_{\mathcal{S},\mathcal{S}} + \big(\mathrm{diag}\{\mathbf{s}\} - \boldsymbol{\Sigma}_0 \boldsymbol{\Omega} \mathrm{diag}\{\mathbf{s}\}\big)_{\mathcal{S},\mathcal{S}} + n^{-1} \mathbf{X}_{\mathcal{S}}^T \mathbf{Z} \mathbf{B}_{0,\mathcal{S}} \widetilde{\mathbf{B}}^{\mathcal{S},\boldsymbol{\Omega}} \\
&\equiv \eta_{2,1} + \eta_{2,2} + \eta_{2,3}.
\end{aligned}
$$

We will examine the above three terms separately.

Since $\boldsymbol{\Omega}$ is L_p'-sparse, $\|\mathbf{I} - \boldsymbol{\Omega}_0 \mathrm{diag}(\mathbf{s})\|_2 \le \|\mathbf{I}\|_2 + \|\boldsymbol{\Omega}_0 \mathrm{diag}(\mathbf{s})\|_2 \le C$, and $\|(\boldsymbol{\Omega} - \boldsymbol{\Omega}_0)\mathrm{diag}\{\mathbf{s}\}\|_2 \le C a_n$, we have

$$
\begin{aligned}
\Big\|\mathbf{I} - \boldsymbol{\Omega}\mathrm{diag}\{\mathbf{s}\}\Big\|_1 &\le \sqrt{L_p'} \Big\|\mathbf{I} - \boldsymbol{\Omega}\mathrm{diag}\{\mathbf{s}\}\Big\|_2 \\
&\le \sqrt{L_p'} \Big(\big\|\mathbf{I} - \boldsymbol{\Omega}_0 \mathrm{diag}\{\mathbf{s}\}\big\|_2 + \big\|(\boldsymbol{\Omega} - \boldsymbol{\Omega}_0)\mathrm{diag}\{\mathbf{s}\}\big\|_2 \Big) \\
&\le C\sqrt{L_p'}. \qquad (A.10)
\end{aligned}
$$

Thus it follow from (A.10) that conditional on $\mathcal{E}_5$,

$$
\begin{aligned}
\|\eta_{2,1}\|_{\max} &= \Big\| \big((n^{-1}\mathbf{X}^T\mathbf{X} - \boldsymbol{\Sigma}_0)(\mathbf{I} - \boldsymbol{\Omega}\mathrm{diag}\{\mathbf{s}\})\big)_{\mathcal{S},\mathcal{S}} \Big\|_{\max} \\
&\le \Big\| (n^{-1}\mathbf{X}^T\mathbf{X} - \boldsymbol{\Sigma}_0)(\mathbf{I} - \boldsymbol{\Omega}\mathrm{diag}\{\mathbf{s}\}) \Big\|_{\max} \\
&\le \Big\| n^{-1}\mathbf{X}^T\mathbf{X} - \boldsymbol{\Sigma}_0 \Big\|_{\max} \Big\| \mathbf{I} - \boldsymbol{\Omega}\mathrm{diag}\{\mathbf{s}\} \Big\|_1 \\
&\le C\sqrt{L_p'}\sqrt{(\log p)/n}. \qquad (A.11)
\end{aligned}
$$

For term $\eta_{2,2}$, it holds that

$$
\begin{aligned}
\|\eta_{2,2}\|_{\max} &= \Big\| \big(\mathrm{diag}\{\mathbf{s}\} - \boldsymbol{\Sigma}_0\boldsymbol{\Omega}\mathrm{diag}\{\mathbf{s}\}\big)_{\mathcal{S},\mathcal{S}} \Big\|_{\max} \\
&\le C\|\mathbf{I} - \boldsymbol{\Sigma}_0\boldsymbol{\Omega}\|_{\max} \le C\|\boldsymbol{\Sigma}_0\|_2 \|\boldsymbol{\Omega}_0 - \boldsymbol{\Omega}\|_2 \le C a_n. \qquad (A.12)
\end{aligned}
$$

Note that by Lemma 2 in Section A.3, we have

$$
\|\widetilde{\mathbf{B}}^{\mathcal{S},\boldsymbol{\Omega}}\|_1 \le \sqrt{|\mathcal{S}|}\|\widetilde{\mathbf{B}}^{\mathcal{S},\boldsymbol{\Omega}}\|_2 \le \sqrt{|\mathcal{S}|}(\|\widetilde{\mathbf{B}}^{\mathcal{S},\boldsymbol{\Omega}} - \mathbf{I}\|_2 + 1) \le C\sqrt{|\mathcal{S}|} \le C\sqrt{K_n}
$$

when $|\mathcal{S}| \le K_n$. Then conditional on $\mathcal{E}_6$, it holds that

$$
\begin{aligned}
\|\eta_{2,3}\|_{\max} &= \|n^{-1}\mathbf{X}_{\mathcal{S}}^T\mathbf{Z}\mathbf{B}_{0,\mathcal{S}}\widetilde{\mathbf{B}}^{\mathcal{S},\boldsymbol{\Omega}}\|_{\max} \\
&\le \|n^{-1}\mathbf{X}_{\mathcal{S}}^T\mathbf{Z}\mathbf{B}_{0,\mathcal{S}}\|_{\max} \|\widetilde{\mathbf{B}}^{\mathcal{S},\boldsymbol{\Omega}}\|_1 \\
&\le C\sqrt{n^{-1}K_n(\log p)}. \qquad (A.13)
\end{aligned}
$$

Thus combining (A.11)–(A.13) leads to

$$\|\eta_2\|_{\max} \le C\{a_n + \sqrt{n^{-1}L_p'(\log p)} + \sqrt{n^{-1}K_n(\log p)}\}. \tag{A.14}$$

We finally deal with term η_3. Some routine calculations show that

$$\begin{aligned}
\eta_3 &= n^{-1}(\breve{\mathbf{X}}_{\mathcal{S}}^{\mathbf{\Omega}})^T\breve{\mathbf{X}}_{\mathcal{S}}^{\mathbf{\Omega}} - \mathbf{\Sigma}_{0,\mathcal{S},\mathcal{S}}.\\
&= n^{-1}\big((\mathbf{I}-\mathbf{\Omega}\mathrm{diag}\{\mathbf{s}\})_{\mathcal{S}}^T\mathbf{X}^T + (\widetilde{\mathbf{B}}^{\mathcal{S},\mathbf{\Omega}})^T\mathbf{B}_{0,\mathcal{S}}^T\mathbf{Z}^T\big)\big(\mathbf{X}(\mathbf{I}-\mathbf{\Omega}\mathrm{diag}\{\mathbf{s}\})_{\mathcal{S}} + \mathbf{Z}\mathbf{B}_{0,\mathcal{S}}\widetilde{\mathbf{B}}^{\mathcal{S},\mathbf{\Omega}}\big) - \mathbf{\Sigma}_{0,\mathcal{S},\mathcal{S}}\\
&= \Big(n^{-1}(\mathbf{I}-\mathbf{\Omega}\mathrm{diag}\{\mathbf{s}\})^T\mathbf{X}^T\mathbf{X}(\mathbf{I}-\mathbf{\Omega}\mathrm{diag}\{\mathbf{s}\}) - \mathbf{\Sigma}_0 + \mathbf{B}_0^T\mathbf{B}_0\Big)_{\mathcal{S},\mathcal{S}}\\
&+ n^{-1}(\widetilde{\mathbf{B}}^{\mathcal{S},\mathbf{\Omega}})^T\mathbf{B}_{0,\mathcal{S}}^T\mathbf{Z}^T\mathbf{X}(\mathbf{I}-\mathbf{\Omega}\mathrm{diag}\{\mathbf{s}\})_{\mathcal{S}} + (\mathbf{I}-\mathbf{\Omega}\mathrm{diag}\{\mathbf{s}\})_{\mathcal{S}}^T\mathbf{X}^T\mathbf{Z}\mathbf{B}_{0,\mathcal{S}}\widetilde{\mathbf{B}}^{\mathcal{S},\mathbf{\Omega}}\\
&+ \big((\widetilde{\mathbf{B}}^{\mathcal{S},\mathbf{\Omega}})^T\mathbf{B}_{0,\mathcal{S}}^T\mathbf{Z}^T\mathbf{Z}\mathbf{B}_{0,\mathcal{S}}\widetilde{\mathbf{B}}^{\mathcal{S},\mathbf{\Omega}} - \mathbf{B}_{0,\mathcal{S}}^T\mathbf{B}_{0,\mathcal{S}}\big)\\
&\equiv \eta_{3,1} + \eta_{3,2} + \eta_{3,2}^T + \eta_{3,3}.
\end{aligned}$$

Conditional on event $\mathcal{E}_5$, with some simple matrix algebra we derive

$$\begin{aligned}
\|\eta_{3,1}\| &= \Big\|\Big(n^{-1}(\mathbf{I}-\mathbf{\Omega}\mathrm{diag}\{\mathbf{s}\})^T\mathbf{X}^T\mathbf{X}(\mathbf{I}-\mathbf{\Omega}\mathrm{diag}\{\mathbf{s}\}) - \mathbf{\Sigma}_0 + \mathbf{B}_0^T\mathbf{B}_0\Big)_{\mathcal{S},\mathcal{S}}\Big\|_{\max}\\
&\le \Big\|n^{-1}(\mathbf{I}-\mathbf{\Omega}\mathrm{diag}\{\mathbf{s}\})^T\mathbf{X}^T\mathbf{X}(\mathbf{I}-\mathbf{\Omega}\mathrm{diag}\{\mathbf{s}\}) - \mathbf{\Sigma}_0 + \mathbf{B}_0^T\mathbf{B}_0\Big\|_{\max}\\
&\le \Big\|(\mathbf{I}-\mathbf{\Omega}\mathrm{diag}\{\mathbf{s}\})^T(n^{-1}\mathbf{X}^T\mathbf{X} - \mathbf{\Sigma}_0)(\mathbf{I}-\mathbf{\Omega}\mathrm{diag}\{\mathbf{s}\})\Big\|_{\max}\\
&+ \Big\|(\mathbf{I}-\mathbf{\Omega}\mathrm{diag}\{\mathbf{s}\})^T\mathbf{\Sigma}_0(\mathbf{I}-\mathbf{\Omega}\mathrm{diag}\{\mathbf{s}\}) - \mathbf{\Sigma}_0 + 2\mathrm{diag}\{\mathbf{s}\} - \mathrm{diag}\{\mathbf{s}\}\mathbf{\Omega}_0\mathrm{diag}\{\mathbf{s}\}\Big\|_{\max}\\
&\le \|n^{-1}\mathbf{X}^T\mathbf{X} - \mathbf{\Sigma}_0\|_{\max}\|(\mathbf{I}-\mathbf{\Omega}\mathrm{diag}\{\mathbf{s}\})\|_1^2\\
&+ \|\mathrm{diag}\{\mathbf{s}\}(\mathbf{I}-\mathbf{\Omega}\mathbf{\Sigma}_0)\|_{\max} + \|(\mathbf{I}-\mathbf{\Sigma}_0\mathbf{\Omega})\mathrm{diag}\{\mathbf{s}\}\|_{\max} + \|\mathrm{diag}\{\mathbf{s}\}(\mathbf{\Omega}_0 - \mathbf{\Omega}\mathbf{\Sigma}_0\mathbf{\Omega})\mathrm{diag}\{\mathbf{s}\}\|_{\max}\\
&\le CL_p'\sqrt{(\log p)/n} + Ca_n,
\end{aligned} \tag{A.15}$$

where the last step used (A.10) and calculations similar to (A.12).

It follows from (A.10) and the previously proved result $\|\widetilde{\mathbf{B}}^{\mathcal{S},\mathbf{\Omega}}\|_1 \le C\sqrt{K_n}$ for $|\mathcal{S}| \le K_n$ that conditional on event $\mathcal{E}_6$,

$$\begin{aligned}
\|\eta_{3,2}\| &= \|n^{-1}(\widetilde{\mathbf{B}}^{\mathcal{S},\mathbf{\Omega}})^T\mathbf{B}_{0,\mathcal{S}}^T\mathbf{Z}^T\mathbf{X}(\mathbf{I}-\mathbf{\Omega}\mathrm{diag}\{\mathbf{s}\})_{\mathcal{S}}\|_{\max}\\
&\le \|\widetilde{\mathbf{B}}^{\mathcal{S},\mathbf{\Omega}}\|_1\|n^{-1}\mathbf{B}_{0,\mathcal{S}}^T\mathbf{Z}^T\mathbf{X}\|_{\max}\|(\mathbf{I}-\mathbf{\Omega}\mathrm{diag}\{\mathbf{s}\})_{\mathcal{S}}\|_1\\
&\le C\sqrt{K_n}\sqrt{L_p'n^{-1}(\log p)}\\
&= C\sqrt{n^{-1}K_nL_p'(\log p)}.
\end{aligned} \tag{A.16}$$

Finally, by Lemma 2 it holds that conditioned on $\mathcal{E}_7$,

$$\begin{aligned}
\|\eta_{3,3}\| &= \left\| n^{-1}(\widetilde{\mathbf{B}}^{\mathcal{S},\mathbf{\Omega}})^T \mathbf{B}_{0,\mathcal{S}}^T \mathbf{Z}^T \mathbf{Z} \mathbf{B}_{0,\mathcal{S}} \widetilde{\mathbf{B}}^{\mathcal{S},\mathbf{\Omega}} - \mathbf{B}_{0,\mathcal{S}}^T \mathbf{B}_{0,\mathcal{S}} \right\|_{\max} \\
&\leq \left\| (\widetilde{\mathbf{B}}^{\mathcal{S},\mathbf{\Omega}})^T (n^{-1} \mathbf{B}_{0,\mathcal{S}}^T \mathbf{Z}^T \mathbf{Z} \mathbf{B}_{0,\mathcal{S}} - \mathbf{B}_{0,\mathcal{S}}^T \mathbf{B}_{0,\mathcal{S}}) \widetilde{\mathbf{B}}^{\mathcal{S},\mathbf{\Omega}} \right\|_{\max} \\
&+ \left\| (\widetilde{\mathbf{B}}^{\mathcal{S},\mathbf{\Omega}})^T \mathbf{B}_{0,\mathcal{S}}^T \mathbf{B}_{0,\mathcal{S}} \widetilde{\mathbf{B}}^{\mathcal{S},\mathbf{\Omega}} - \mathbf{B}_{0,\mathcal{S}}^T \mathbf{B}_{0,\mathcal{S}} \right\|_{\max} \\
&\leq \left\| n^{-1} \mathbf{B}_{0,\mathcal{S}}^T \mathbf{Z}^T \mathbf{Z} \mathbf{B}_{0,\mathcal{S}} - \mathbf{B}_{0,\mathcal{S}}^T \mathbf{B}_{0,\mathcal{S}} \right\|_{\max} \|\widetilde{\mathbf{B}}^{\mathcal{S},\mathbf{\Omega}}\|_1^2 + C a_n \\
&\leq C K_n \sqrt{(\log p)/n} + C a_n. \qquad \text{(A.17)}
\end{aligned}$$

Therefore, combining (A.15)–(A.17) results in

$$\begin{aligned}
\|\eta_3\|_{\max} &\leq C a_n + C(L_p' + K_n + \sqrt{K_n L_p'}) \sqrt{(\log p)/n} \\
&\leq C a_n + 2C(L_p' + K_n) \sqrt{(\log p)/n},
\end{aligned}$$

which together with (A.9) and (A.14) concludes the proof of Lemma 7.

B.6 Lemma 8 and its proof

Lemma 8. *Under the conditions of Proposition 2, it holds that with probability at least* $1 - O(p^{-c})$,

$$\begin{aligned}
\sup_{|\mathcal{S}| \leq K_n} \left\| \frac{1}{n} \mathbf{B}_{0,\mathcal{S}}^T \mathbf{Z}^T \mathbf{X} \right\|_{\max} &\geq C \sqrt{(\log p)/n}, \\
\sup_{|\mathcal{S}| \leq K_n} \left\| n^{-1} \mathbf{B}_{0,\mathcal{S}}^T \mathbf{Z}^T \mathbf{Z} \mathbf{B}_{0,\mathcal{S}} - \mathbf{B}_{0,\mathcal{S}}^T \mathbf{B}_{0,\mathcal{S}} \right\|_{\max} &\geq C \sqrt{(\log p)/n}
\end{aligned}$$

for some constants $c, C > 0$.

Proof. We still use c and C to denote generic positive constants. We start with proving the first inequality. Observe that

$$\sup_{|\mathcal{S}| \leq K_n} \left\| \frac{1}{n} \mathbf{B}_{0,\mathcal{S}}^T \mathbf{Z}^T \mathbf{X} \right\|_{\max} \leq \left\| \frac{1}{n} \mathbf{B}_0^T \mathbf{Z}^T \mathbf{X} \right\|_{\max}.$$

Thus it remains to prove

$$P\left(\left\| \frac{1}{n} \mathbf{B}_0^T \mathbf{Z}^T \mathbf{X} \right\|_{\max} \geq C \sqrt{(\log p)/n} \right) \leq o(p^{-c}). \qquad \text{(A.18)}$$

Let $\mathbf{U} = \mathbf{Z}\mathbf{B}_0 \in \mathbb{R}^{n \times p}$ and denote by $\mathbf{U}_j$ the jth column of matrix $\mathbf{U}$. We see that the components of $\mathbf{U}_j$ are i.i.d. Gaussian with mean zero and variance $\mathbf{e}_j^T \mathbf{B}_0^T \mathbf{B}_0 \mathbf{e}_j$, and the vectors $\mathbf{U}_j$ are independent of $\boldsymbol{\varepsilon}$. Let $\widetilde{\mathbf{U}}_j = (\mathbf{e}_j^T \mathbf{B}_0^T \mathbf{B}_0 \mathbf{e}_j)^{-1/2} \mathbf{U}_j$. Then it holds that $\widetilde{\mathbf{U}}_j \sim N(\mathbf{0}, \mathbf{I}_n)$. Since $X_{ij} \sim N(0, \mathbf{\Sigma}_{0,jj})$ and $\mathbf{\Sigma}_{0,jj} \leq \Lambda_{\max}(\mathbf{\Sigma}_0) \leq C$ with $C > 0$ some constant, it follows from Bernstein's inequality that for $t > 0$,

$$\begin{aligned}\mathbb{P}\left(\left\|\frac{1}{n}\mathbf{B}_0^T\mathbf{Z}^T\mathbf{X}\right\|_{\max} \geq t\|\mathbf{B}_0^T\mathbf{B}_0\|_2\right) &\leq \sum_{j=1}^{p}\mathbb{P}\left(\frac{1}{n}\left|(\mathbf{U}_j)^T\mathbf{X}_i\right| \geq t\|\mathbf{B}_0^T\mathbf{B}_0\|_2\right)\\ &\leq \sum_{j=1}^{p}\mathbb{P}\left(\frac{1}{n}\left|(\widetilde{\mathbf{U}}_j)^T\mathbf{X}_i\right| \geq t\right)\\ &\leq Cp\exp(-Cnt^2).\end{aligned}$$

Taking $t = C\sqrt{(\log p)/n}$ with large enough constant $C > 0$ in the above inequality yields

$$\mathbb{P}\left(\left\|\frac{1}{n}\mathbf{B}_0^T\mathbf{Z}^T\mathbf{X}\right\|_{\max} \geq C\sqrt{(\log p)/n}\|\mathbf{B}_0^T\mathbf{B}_0\|_2\right) \leq Cp^{-c}$$

for some constant $c > 0$. Thus with probability at least $1 - O(p^{-c})$, it holds that

$$\begin{aligned}&\left\|\frac{1}{n}\mathbf{B}_0^T\mathbf{Z}^T\mathbf{X}\right\|_{\max} \leq C\sqrt{(\log p)/n}\|\mathbf{B}_0^T\mathbf{B}_0\|_2\\ &= C\sqrt{(\log p)/n}\|\mathrm{diag}(\mathbf{s}) - \mathrm{diag}(\mathbf{s})\boldsymbol{\Omega}_0\mathrm{diag}(\mathbf{s})\|_2\\ &\leq C\sqrt{(\log p)/n},\end{aligned}$$

which establishes (A.18) and thus concludes the proof for the first result.

The second inequality follows from

$$\sup_{|\mathcal{S}|\leq K_n}\left\|n^{-1}\mathbf{B}_{0,\mathcal{S}}^T\mathbf{Z}^T\mathbf{Z}\mathbf{B}_{0,\mathcal{S}} - \mathbf{B}_{0,\mathcal{S}}^T\mathbf{B}_{0,\mathcal{S}}\right\|_{\max} \leq \left\|n^{-1}\mathbf{B}_0^T\mathbf{Z}^T\mathbf{Z}\mathbf{B}_0 - \mathbf{B}_0^T\mathbf{B}_0\right\|_{\max}$$

and Lemma A.3 in [6], which completes the proof of Lemma 8.