1 Online Supplemental Material

Proof of Proposition 1 From equation (2) we can write the process for the vector z_t as:

$$z_t = \sqrt{\widetilde{T}_t}\varepsilon_t + \rho\sqrt{\widetilde{T}_t}\sqrt{\widetilde{T}_{t-1}}\varepsilon_{t-1} + \rho^2\sqrt{\widetilde{T}_t}\sqrt{\widetilde{T}_{t-1}}\sqrt{\widetilde{T}_{t-2}}\varepsilon_{t-2} + \rho^3\sqrt{\widetilde{T}_t}\sqrt{\widetilde{T}_{t-1}}\sqrt{\widetilde{T}_{t-2}}\sqrt{\widetilde{T}_{t-3}}\varepsilon_{t-3} + ...$$

which implies that conditional on $\widetilde{T}$, z_t is the sum of independent normals. Hence, $z_t|\widetilde{T}$ is also a normal, with mean 0 and variance-covariance $\theta^2 v_{c,t} I_n$, where I_n is the identity matrix and $v_{c,t}$ is the scalar defined in Proposition 1. This implies that $k_t = z_t' z_t$ conditional on $\widetilde{T}$ is a $G(n/2, 2\theta^2 v_{c,t})$, and therefore $(k_t/v_{c,t})|\widetilde{T}$ is a $G(n/2, 2\theta^2)$ (i.e. independent of $\widetilde{T}$). Note that $(\varepsilon_t'\varepsilon_t)$ is also distributed as a $G(n/2, 2\theta^2)$, and therefore we can write $E((k_t/v_{c,t})^s) = E((\varepsilon_t'\varepsilon_t)^s)$. By the law of iterated expectations we can calculate the moments of k_t as $E(k_t^s) = E(E(k_t^s|\widetilde{T})) = E(v_{c,t}^s E((k_t/v_{c,t})^s|\widetilde{T})) = E(v_{c,t}^s)E((\varepsilon_t'\varepsilon_t)^s)$. Because $(\varepsilon_t'\varepsilon_t)$ is distributed as a $G(n/2, 2\theta^2)$, its moments are given by (e.g. Johnson et al. (1994 p. 339)):

$$E((\varepsilon_t'\varepsilon_t)^s) = \left(\theta^2\right)^s \prod_{i=0}^{s-1}(n+2i)$$

To calculate $E(v_{c,t}^s)$ note that we can write $v_{c,t}$ as $v_{c,t} = \widetilde{T}_t + \rho^2\widetilde{T}_t v_{c,(t-1)}$. so that $E(v_{c,t}^s) = E((\widetilde{T}_t + \rho^2\widetilde{T}_t v_{c,(t-1)})^s)$. Using the binomial theorem we can write:

$$E((\widetilde{T}_t + \rho^2\widetilde{T}_t v_{c,(t-1)})^s) = E(\widetilde{T}_t^s)\sum_{i=0}^{s}\binom{s}{i}\rho^{2i}E(v_{c,(t-1)}^i) \tag{O.1}$$

Because $E(v_{c,t}^s) = E(v_{c,(t-1)}^s)$, (O.1) implies property (9) and the other unconditional moments stated in Proposition 1. To obtain the conditional moments, note that equation (3) can be written as:

$$k_t = \frac{\widetilde{T}_t}{E(\widetilde{T}_t)}(\widetilde{\rho}^2 k_{t-1} + \widetilde{\varepsilon}_t'\widetilde{\varepsilon}_t + 2\widetilde{\rho}\widetilde{\varepsilon}_t' z_{t-1}) \tag{O.2}$$

Because $\widetilde{\varepsilon}_t$ is independent of z_{t-1} and $E(\widetilde{\varepsilon}_t) = 0$ we obtain that $E(\widetilde{\varepsilon}_t' z_{t-1}) = 0$. Taking into account that $E(\widetilde{\varepsilon}_t'\widetilde{\varepsilon}_t) = n\widetilde{\theta}^2$ we can take conditional expectations on both sides of (O.2) to get equations (4) and (6).

Let us calculate $cov(k_t, k_{t-h})$ as $cov(k_t, k_{t-h}) = E(k_t k_{t-h}) - [E(k_t)]^2$. To derive $E(k_t k_{t-h})$ let us use iterative expectations to rewrite equation (4) as:

$$E(k_t|k_{t-h}) = \widetilde{\rho}^{2h}k_{t-h} + \sum_{i=0}^{h-1}\widetilde{\rho}^{2i}(1-\widetilde{\rho}^2)E(k_t) \tag{O.3}$$

Multiplying both sides of (O.3) by k_{t-h} and then taking expectations with respect to k_{t-h} we obtain:

$$E(k_t k_{t-h}) = \widetilde{\rho}^{2h} E(k_{t-h}^2) + \sum_{i=0}^{h-1} \widetilde{\rho}^{2i}(1-\widetilde{\rho}^2)\left[E(k_t)\right]^2 = \widetilde{\rho}^{2h} E(k_{t-h}^2) + (1-\widetilde{\rho}^{2h})\left[E(k_t)\right]^2$$

where we have used the formula for the sum of a geometric series. Thus $cov(k_t, k_{t-h}) = E(k_t k_{t-h}) - \left[E(k_t)\right]^2 = \widetilde{\rho}^{2h}(E(k_{t-h}^2) - \left[E(k_t)\right]^2) = \widetilde{\rho}^{2h} var(k_t)$. Thus, the correlation between k_t and k_{t-h} is $\widetilde{\rho}^{2h}$.

Because the stationary distribution of $\sigma_t^2 = 1/k_t$ is that of the product of $(v_{c,t})^{-1}$ and $(\varepsilon_t'\varepsilon_t)^{-1}$, with $(v_{c,t})^{-1}$ being independent of $(\varepsilon_t'\varepsilon_t)^{-1}$, the expectation $E(\sigma_t^{2s})$ is finite if and only if both $E((v_{c,t})^{-s})$ and $E((\varepsilon_t'\varepsilon_t)^{-s})$ are finite. Because $(\varepsilon_t'\varepsilon_t)^{-1}$ is an inverted gamma with n degrees of freedom, $E((\varepsilon_t'\varepsilon_t)^{-s})$ is finite only if $2s < n$. In addition, from $v_{c,t} = \widetilde{T}_t(1+\rho^2 v_{c,(t-1)})$ it follows that:

$$\frac{1}{v_{c,t}} = \frac{1}{\widetilde{T}_t}\frac{1}{1+\rho^2 v_{c,(t-1)}}$$

Because $(1+\rho^2 v_{c,(t-1)})^{-s} < 1$, it follows that $E((1+\rho^2 v_{c,(t-1)})^{-s})$ is finite because the density function of $v_{c,(t-1)}$ integrates up to 1. Because $\widetilde{T}_t$ follows a $B(\underline{\alpha}, \underline{\beta})$, $E(\widetilde{T}_t^{-s})$ is finite if and only if $\underline{\alpha} > s$. Putting both conditions together, $E(\sigma_t^{2s})$ is finite when $\underline{\alpha} > s$ and $n > 2s$.

For the ARG model (i.e. $\widetilde{T}_t = 1$ for all t), the expressions for the expected value and variance of σ_t^2 are derived from the properties of the inverted gamma distribution (e.g. Bauwens et al. (1999. p.292)). To calculate the correlations between σ_t^2 and σ_{t-s}^2 in the ARG model, let us first proof the following property:

$$E(\sigma_t^2|\sigma_{t-s}^2) = \int \left(\prod_{i=2}^{s}(u_i)^{n/2}\right)\frac{1}{\theta^2(n-2)}\exp\left(-\frac{(1-u_s)}{2\theta^2}\rho^2\frac{1}{\sigma_{t-s}^2}\right)p(u_1)du_1 \tag{O.4}$$

where $u_1 \sim B((n-2)/2, 1)$, $p(u_1)$ is the density function of u_1 and $u_s = 1/(1+\rho^2(1-u_{s-1}))$ for $s \geq 2$. To proof this note that the Poisson representation in (10) implies that $k_t|(k_{t-1}, h_t)$ is a Gamma which in turn implies that $\sigma_t^2|(\sigma_{t-1}^2, h_t)$ is an $IG_2(\theta^{-2}, n+2h_t)$, such that $E(\sigma_t^2|(\sigma_{t-1}^2, h_t)) = \theta^{-2}/(n+2h_t-2)$. We can therefore integrate out h_t to obtain $E(\sigma_t^2|\sigma_{t-1}^2)$.as:

$$E(\sigma_t^2|\sigma_{t-1}^2) = \frac{1}{\theta^2\exp(\lambda_t)}\sum_{i=0}^{\infty}\frac{\lambda_t^i}{i!}\left(\frac{1}{n+2i-2}\right), \text{ where } \lambda_t = \frac{\rho^2}{2\sigma_{t-1}^2\theta^2} \tag{O.5}$$

Note that $1/(n+2i-2) = (n-2)^{-1}[n/2-1]_i/[n/2]_i = (n-2)^{-1}E((u_1)^i)$, where $[n/2]_i$ is the rising factorial. Therefore (O.5) can be written as:

$$
\begin{aligned}
E(\sigma_t^2|\sigma_{t-1}^2) &= \int \frac{1}{\theta^2 \exp(\lambda_t)} \frac{1}{(n-2)} \sum_{i=0}^{\infty} \frac{\lambda_t^i}{i!} (u_1)^i \, p(u_1) du_1 \\
&= \int \frac{1}{\theta^2 \exp(\lambda_t)} \frac{1}{(n-2)} \exp(\lambda_t u_1) p(u_1) du_1 \\
&= \int \frac{1}{\theta^2} \frac{1}{(n-2)} \exp\left(-\frac{(1-u_1)}{2\theta^2}\rho^2 \frac{1}{\sigma_{t-1}^2}\right) p(u_1) du_1
\end{aligned} \tag{O.6}
$$

which is the same as (O.4) for the case $s = 1$. To proof (O.4) for $s = 2$ we need to integrate $E(\sigma_t^2|\sigma_{t-1}^2)$ with respect to $p(\sigma_{t-1}^2|\sigma_{t-2}^2)$ using expression (O.6). This can be done by first integrating with respect to $p(\sigma_{t-1}^2|h_{t-1}, \sigma_{t-2}^2)$ (which is a $IG_2(\theta^{-2}, n + 2h_{t-1})$) and then integrating out h_{t-1} (using a $P(\lambda_{t-1})$) as follows:

$$
\begin{aligned}
E(\sigma_t^2|\sigma_{t-2}^2) &= \int E(\sigma_t^2|\sigma_{t-1}^2) p(\sigma_{t-1}^2|\sigma_{t-2}^2) d\sigma_{t-1}^2 \\
&= \int E(\sigma_t^2|\sigma_{t-1}^2) \sum_{h_{t-1}=0}^{\infty} p(\sigma_{t-1}^2|h_{t-1}, \sigma_{t-2}^2) p(h_{t-1}) d\sigma_{t-1}^2 \\
&= \sum_{h_{t-1}=0}^{\infty} \int E(\sigma_t^2|\sigma_{t-1}^2) p(\sigma_{t-1}^2|h_{t-1}, \sigma_{t-2}^2) p(h_{t-1}) d\sigma_{t-1}^2
\end{aligned} \tag{O.7}
$$

Using the properties of the inverse Gamma distribution, we can obtain that:

$$
\int E(\sigma_t^2|\sigma_{t-1}^2) p(\sigma_{t-1}^2|h_{t-1}, \sigma_{t-2}^2) d\sigma_{t-1}^2 = \int (u_2)^{\frac{n+2h_{t-1}}{2}} \frac{1}{\theta^2} \frac{1}{(n-2)} p(u_1) du_1
$$

Therefore, (O.7) can be written as:

$$
E(\sigma_t^2|\sigma_{t-2}^2) = \sum_{h_{t-1}=0}^{\infty} \int (u_2)^{\frac{n+2h_{t-1}}{2}} \frac{1}{\theta^2} \frac{1}{(n-2)} p(u_1) du_1 p(h_{t-1})
$$

Using the properties of the Poisson distribution, we can obtain that:

$$
\sum_{h_{t-1}=0}^{\infty} \int (u_2)^{\frac{n+2h_{t-1}}{2}} \frac{1}{\theta^2} \frac{1}{(n-2)} p(u_1) du_1 p(h_{t-1}) = \int (u_2)^{\frac{n}{2}} \frac{1}{\theta^2 (n-2)} \exp\left(-\frac{(1-u_2)}{2\theta^2}\rho^2 \frac{1}{\sigma_{t-2}^2}\right) p(u_1) du_1
$$

The proof for $s > 2$ can be obtained by repeating the same process, that is, integrate out with respect to $p(\sigma_{t-s+1}^2|h_{t-s+1}, \sigma_{t-s}^2)$ and then integrate out h_{t-s+1} (using a $P(\lambda_{t-s+1})$).

$E(\sigma_t^2 \sigma_{t-s}^2)$ can be obtained by using expression (O.4) to calculate $E(\sigma_t^2 \sigma_{t-s}^2|\sigma_{t-s}^2)$ and then integrate out σ_{t-s}^2 using the stationary distribution $IG_2((1-\rho^2)/\theta^2, n)$. This gives:

$$
E(\sigma_t^2 \sigma_{t-s}^2) = E(\sigma_{t-s}^2 E(\sigma_t^2|\sigma_{t-s}^2)) = \sigma_{t-s}^2 E(\sigma_t^2|\sigma_{t-s}^2) p(\sigma_{t-s}^2) d\sigma_{t-s}^2
$$

Using the properties of the gamma function we have that $\Gamma(n/2-1)/\Gamma(n/2) = (n/2-1)^{-1}$ and therefore $E(\sigma_t^2\sigma_{t-s}^2)$ can be written as:

$$E(\sigma_t^2\sigma_{t-s}^2) = \frac{(1-\rho^2)^{n/2}}{(2\theta^2)^2\,(n/2-1)^2}\int\left(\prod_{i=2}^{s}(u_i)^{n/2}\right)\left(\frac{1}{1-\rho^2 u_s}\right)^{n/2-1} p(u_1)du_1 \tag{O.8}$$

By using the definition of u_s it is possible to verify that:

$$\left(\prod_{i=2}^{s}u_i\right)\frac{1}{1-\rho^2 u_s} = \left(\prod_{i=2}^{s-1}u_i\right)\frac{u_s}{1-\rho^2 u_s} = \left(\prod_{i=2}^{s-1}u_i\right)\frac{1}{1-\rho^2 u_{s-1}} = \frac{1}{1-\rho^2 u_1}$$

and:

$$\prod_{i=2}^{s}u_i = u_s u_{s-1}\prod_{i=2}^{s-2}u_i = \frac{1}{1+(\rho^2+\rho^4)(1-u_{s-2})}\prod_{i=2}^{s-2}u_i = \frac{1}{1+\rho_s^2(1-u_1)}$$

where $\rho_s^2 = \sum_{g=1}^{s-1}\rho^{2g}$. Hence, the integral in expression (O.8) can be written as:

$$\begin{aligned} E\left[\left(\prod_{i=2}^{s}(u_i)^{n/2}\right)\left(\frac{1}{1-\rho^2 u_s}\right)^{n/2-1}\right] &= E\left[\frac{(1-\rho^2 u_1)^{-(n/2-1)}}{1+\rho_s^2(1-u_1)}\right] \\ &= E\left[\frac{(1+\rho_s^2)^{-1}}{1-\widehat{\rho}_s^2 u_1}\left(\frac{1}{1-\rho^2 u_1}\right)^{n/2-1}\right] \end{aligned} \tag{O.9}$$

where the expectation is calculated with respect to u_1 and $\widehat{\rho}_s^2 = \rho_s^2/(1+\rho_s^2)$. By expanding $(1/(1-\rho^2 u_1))^{n/2-1}$ as a hypergeometric series (e.g. Muirhead (1985, p. 259)) and using basic properties of the beta distribution, it is possible to show that:

$$E\left[(u_1^h)\left(\frac{1}{1-\rho^2 u_1}\right)^{n/2-1}\right] = \left(\frac{[n/2-1]_h}{[n/2]_h}\right)\left({}_2F_1(\frac{n}{2}-1,\frac{n}{2}-1+h;\frac{n}{2}+h;\rho^2)\right)$$

and therefore the expectation in (O.9) can be written as:

$$\begin{aligned} &\frac{1}{1+\rho_s^2}\sum_{h=0}^{\infty}\left[\left(\widehat{\rho}_s^2\right)^h\left(\frac{[n/2-1]_h}{[n/2]_h}\right)\left({}_2F_1(\frac{n}{2}-1,\frac{n}{2}-1+h;\frac{n}{2}+h;\rho^2)\right)\right] \\ =\ &\frac{1}{1+\rho_s^2}\sum_{h=0}^{\infty}\sum_{i=0}^{\infty}\left[\frac{\left(\widehat{\rho}_s^2\right)^h\left(\rho^2\right)^i}{h!i!}[1]_h\frac{[n/2-1]_{h+i}}{[n/2]_{h+i}}[n/2-1]_i\right] = \frac{1}{1+\rho_s^2}F_1\left[\frac{n}{2}-1;1,\frac{n}{2}-1;\frac{n}{2};\widehat{\rho}_s^2,\rho^2\right] \end{aligned}$$

where $F_1[.]$ is an Appell series of the first type (e.g. Slater (1966, p. 210)), which in our case can be reduced to a ${}_2F_1(.)$ series (Slater (1966, p. 219)):

$$\begin{aligned} F_1\left[\frac{n}{2}-1;1,\frac{n}{2}-1;\frac{n}{2};\widehat{\rho}_s^2,\rho^2\right] &= \left(\frac{1}{1-\rho^2}\right)^{n/2-1}\left[{}_2F_1\left(\frac{n}{2}-1,1;\frac{n}{2};\frac{\widehat{\rho}_s^2-\rho^2}{1-\rho^2}\right)\right] = \\ &\left(\frac{1}{1-\rho^2}\right)^{n/2-1}\left[{}_2F_1\left(\frac{n}{2}-1,1;\frac{n}{2};\frac{-\rho^{2s}}{1-\rho^{2s}}\right)\right] \end{aligned}$$

Using the Euler relationships (e.g. Muirhead (1982, p. 265)), the ${}_2F_1(.)$ series can be written as:

$$ {}_2F_1\left(\frac{n}{2}-1, 1; \frac{n}{2}; \frac{-\rho^{2s}}{1-\rho^{2s}}\right) = (1-\rho^{2s})\left[{}_2F_1\left(1,1;\frac{n}{2};\rho^{2s}\right)\right] $$

Putting all this together the expectation in (O.9) can be written as:

$$ \begin{aligned} E\left[\left(\prod_{i=2}^{s}(u_i)^{n/2}\right)\left(\frac{1}{1-\rho^2 u_s}\right)^{n/2-1}\right] &= \frac{1-\rho^{2s}}{1+\rho_s^2}\left(\frac{1}{1-\rho^2}\right)^{n/2-1}\left[{}_2F_1\left(1,1;\frac{n}{2};\rho^{2s}\right)\right] = \\ & (1-\rho^2)\left(\frac{1}{1-\rho^2}\right)^{n/2-1}\left[{}_2F_1\left(1,1;\frac{n}{2};\rho^{2s}\right)\right] \end{aligned} $$

where we have used that a geometric series can be written as $1+\rho_s^2 = (1-\rho^{2s})/(1-\rho^2)$. This proves that (O.8) is equal to:

$$ E(\sigma_t^2\sigma_{t-s}^2) = \frac{(1-\rho^2)^2}{(2\theta^2)^2\,(n/2-1)^2}\left[{}_2F_1\left(1,1;\frac{n}{2};\rho^{2s}\right)\right] $$

The correlation $corr(\sigma_t^2, \sigma_{t-s}^2)$ can then be calculated as $\left(E(\sigma_t^2\sigma_{t-s}^2) - E(\sigma_t^2)\right)/var(\sigma_t^2)$, where $E(\sigma_t^2)$ and $var(\sigma_t^2)$ are obtained from the properties of the inverted gamma distribution (e.g. Bauwens et al. (1999. p.292)).