

1 Appendices to:

2 Community dynamics and sensitivity to model structure: toward

3 a probabilistic view of process-based model predictions

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10 **1 Model analysis: equilibria existence, stability and bifurcations**

11 The model:

$$\begin{cases} \frac{dx}{dt} = x \left(1 - \frac{x}{K} \right) - f(x)y \\ \frac{dy}{dt} = \varepsilon (f(x) - m) y, \end{cases} \quad (1)$$

has up to three equilibria (with $f^{-1} \circ f = \text{id}$, we assume that f^{-1} exists):

$$E_{(0)} = (0, 0), \quad E_{(1)} = (K, 0), \quad E_{(2)} = \left(x^{(2)} := f^{-1}(m), y^{(2)} := \frac{x^{(2)}}{m} \left(1 - \frac{x^{(2)}}{K} \right) \right)^T.$$

12 The coexistence equilibrium $E_{(2)}$ exists if $\sup_{x \in \mathbb{R}^+} f(x) > m$ and $x^{(2)}, y^{(2)} \in \mathbb{R}^+$. Prey population $x^{(2)}$ is
13 positive as f is a strictly increasing function. Predator population $y^{(2)}$ is positive only if $x^{(2)} < K$. Thus,

14 having $K > f^{-1}(m)$ is a necessary condition for equilibrium $E_{(2)}$ existence.

15 The local stability of equilibria can be studied using the Jacobian matrix:

$$J = \begin{pmatrix} \left(1 - \frac{2x}{K}\right) - f'(x)y & -f(x) \\ \varepsilon f'(x)y & \varepsilon(f(x) - m) \end{pmatrix}, \quad (2)$$

16 where f' is the first-order derivative of f . By evaluating this matrix at equilibria $E_{(0)}$ and $E_{(1)}$, one
 17 can check that the trivial equilibrium $E_{(0)}$ is always a saddle, whereas the prey equilibrium $E_{(1)}$ is a
 18 stable node when $K < f^{-1}(m)$ (i.e. when $E_{(2)}$ does not exist) and is a saddle otherwise. The transition
 19 $K = f^{-1}(m)$ between these two situations involves a non-hyperbolic double equilibrium ($E_{(1,2)} := E_{(1)} =$
 20 $E_{(2)}$) and corresponds to a transcritical bifurcation. This bifurcation implies that $E_{(2)}$ is a stable node
 21 for $K \gtrsim f^{-1}(m)$. So, this bifurcation is a threshold on carrying capacity K above which the predator
 22 population can survive and invade the system.

Jacobian matrix (2) evaluated at coexistence equilibrium $E_{(2)}$:

$$J_{E_{(2)}} = \begin{pmatrix} \left(1 - \frac{2x^{(2)}}{K}\right) - f'(x^{(2)})y^{(2)} & -m \\ \varepsilon f'(x^{(2)})y^{(2)} & 0 \end{pmatrix},$$

23 has eigenvalues with real parts of the same sign ($\lambda_1 \lambda_2 = \det(J_{E_{(2)}}) = m f'(x^{(2)})y^{(2)} > 0$). These real
 24 parts are negative if $\lambda_1 + \lambda_2 = \text{Tr}(J_{E_{(2)}}) < 0$, which gives the following condition for the stability of the
 25 coexistence equilibrium when it exists:

$$f'(x^{(2)}) > \frac{1}{y^{(2)}} \left(1 - \frac{2x^{(2)}}{K}\right). \quad (3)$$

26 At, the limit case $\text{Tr}(J_{E_{(2)}}) = 0$, $J_{E_{(2)}}$ has a pair of conjugate pure imaginary eigenvalues (as $\det(J_{E_{(2)}}) >$
 27 0). This means that the system exhibits a Hopf bifurcation which gives birth to a limit cycle under
 28 genericity conditions.

29 One can prove that the Hopf bifurcation is always supercritical (i.e. it gives birth to a stable limit
 30 cycle) with Holling and Ivlev functional responses, whereas with the hyperbolic tangent the bifurcation

is supercritical if $\sqrt{(1/h^{(t)})^2/3} > m$ and subcritical otherwise (Seo & Wolkowicz, 2018). The point $\sqrt{(1/h^{(t)})^2/3} = m$ on the Hopf bifurcation curve in the plane (m, K) is a Bautin bifurcation where a Limit Point of Cycles bifurcation curve starts and goes toward $m = 0$.

The Limit Point of Cycles bifurcation can only be approximated by numerical techniques. Numerical continuation methods such as Matcont (Dhooge *et al.*, 2006) and AUTO (Doedel *et al.*, 1997) are classic tools to compute bifurcations in a given model of interest. However, they are not the most convenient tools to automatically study complex bifurcations such as Limit Point of Cycles for many parameter sets of the same models. Indeed, these continuation methods rely upon various tuning parameters and tricks (e.g. where to start bifurcation continuation) that become problem specific for bifurcations of limit cycles.

Here, we avoid the above issue by using a *ad hoc* algorithm based on numerical simulations of the model. Each simulation is performed for given m and K values, with very tight error tolerance (absolute and relative errors set to 10^{-12}) and for a simulation of 5000 time units (set based on preliminary tests). At the end of the simulation, we compute the relative system variation $|dx/xdt| + |dy/ydt|$. If this variation is lower than a threshold of 10^{-6} (set based on preliminary tests), we conclude that the system is reaching a stable equilibrium, and otherwise it is reaching a limit cycles (i.e. oscillations). This analysis is performed for parameter values close to the expected location of the Limit Point of Cycles bifurcation. Initial conditions (close to equilibrium $E_{(1)}$ but with small predator population) are such that if the system converges on a limit cycle for those parameter values, that means that we have bistability (K higher than at the Limit Point of Cycles). Conversely, if the system converges on an equilibrium, that means that we have a globally stable coexistence equilibrium (K lower than at the Limit Point of Cycles). Based on that, we search for the K value that delimits the two dynamics, i.e. bifurcation location, for a given m . Doing this for a sample of m values (here 5 equally distributed values) between 0 and $\sqrt{(1/h^{(t)})^2/3}$ (end of the Limit Point of Cycles at the Bautin bifurcation point) gives us a rough estimate of the Limit Point of Cycles bifurcation curve in the plane (m, K) .

The accuracy of our method has been checked by numerical construction of phase portraits around the estimated bifurcation location. As numerical continuation methods are hard to automatise and can also lead to wrong numerical approximations if the algorithm is not properly tuned, our proposed algorithm (which is roughly an automatised algorithm to build phase portraits) is a good trade-off between the desired level of accuracy (we are able to detect changes in bifurcation location with changes in ε value) and a cheap automatic method to analyse the model (we can study 1000 parameter sets within a couple

of hours on a generic laptop). Even if one can always search a better way to perform this analysis, our method is enough to achieve our purpose, i.e. knowing when Rosenzweig-MacArthur model predicts bistability with an hyperbolic tangent as a functional response and when it does not.

2 Markov Chain Monte Carlo algorithm

We estimated the posterior distribution of parameter values for each functional response using a Hamiltonian Monte Carlo algorithm. This algorithm is efficient to describe high-dimensional parameter distributions using a tractable sample size. We used the program Stan (Stan Development Team, 2017) through the *rethinking* package in R language (McElreath, 2015). We optimised a model predicting the amount of prey eaten per unit of predator and unit of time (i.e. functional response) as a Gaussian distribution with a mean depending on prey abundance, as given by a functional response model (Holling, Ivlev or the hyperbolic tangent), and a standard deviation corresponding to a multiplicative noise. For the functional response parameters, we used a uniform prior over $[0, 100]$ to ensure positive values and avoid unrealistic high values. For the standard deviation σ , we used uniform distribution over $[0, 1]$. To compare the efficiency of each functional response to fit data, we used the Widely-Applicable-Information-Criterion (WAIC) that is suited for non-Gaussian posterior distribution. The lower the WAIC is, the better the model would fit new data. Based on WAIC, one can derive Akaike weights (McElreath, 2015, section 6.5) which are the probability that each model gives the best fit to new data, knowing the alternative models considered.

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