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PRESENTED BY: Yue Pu

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Robert Pego — MAJOR PROFESSOR — August 2018 — DATE

Thomas Bohman — DEPARTMENT HEAD — August 2018 — DATE

APPROVED BY THE COLLEGE COUNCIL

Rebecca W. Doerge — DEAN — August 2018 — DATE

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Yue Pu

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Abstract

This thesis studies a regularization of the classical Saint-Venant (shallow-water) system, namely the regularized shallow-water (Airy or Saint-Venant) system, recently introduced by D. Clamond and D. Dutykh. This regularization is non-dispersive and formally conserves mass, momentum and energy.

We show that for every classical shock wave, this system admits a corresponding non-oscillatory traveling wave solution which is continuous and piecewise smooth, having a weak singularity at a single point where the energy is dissipated as it is for the classical shock. This system also admits cusped solitary waves of both elevation and depression.

The H^s $(s \geqslant 2)$ large time existence with respect to the scaling of initial data and uniqueness are established using an iteration scheme. The solution exists so long as the first derivatives are bounded in L^∞. When the energy is small, the height of water admits a positive lower bound dictated by the smallness of the energy.

Lastly, we show that there exists smooth initial data with which the L^∞ norm of the first derivatives go unbounded in finite amount of time. This is proved by a Riccati-type analysis with the help from Landau-Kolmogorov inequality that addresses the nonlocal part.

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Contents

Chapter 1

Introduction

This thesis studies a regularization of the classical Saint-Venant (shallow-water) system, or the regularized Saint-Venant system (rSV), with a focus on its weakly singular shock profiles, large-time well-posedness and existence of blow-up phenomena. The first part of the work is contained in the paper [45].

1.1 Background of the classical shallow-water equations

The classical shallow-water equations in one dimension with flat bottom take the form

$$h_t + (hu)_x = 0\,, \tag{1.1.1}$$

$$(hu)_t + (hu^2 + \frac{1}{2}gh^2)_x = 0. \tag{1.1.2}$$

where $u = u(x,t)$ is the depth-averaged horizontal velocity and $h = h(x,t)$ is the water depth and g is the gravitational acceleration. These equations describe nonlinear non-dispersive long surface gravity waves propagating in shallow water, especially when modeling large scale phenomena without resolving the details at small scales. Common applications include flood routing along rivers, dam breaking analysis, storm pulses in an open channel, and storm runoff in overland flow.

The classical shallow water system is a hyperbolic system and therefore can be tackled analytically and numerically by many powerful methods (e.g. characteristics, finite volume, discontinuous Galerkin). See Chapter 2 for details in analytical results. More can be found in [21,32,48].

Due to the complexity of Euler equations, many approximate models have been derived in various wave regimes. In shallow water, the main restriction comes from the shallowness parameter

$$\beta = \frac{h_0}{l} \ll 1,$$

where h_0 is the water depth and l is the characteristic wavelength. Restrictions on the free surface elevation are characterized by the dimensionless parameter $\alpha = a/h_0$ where a is the amplitude of the free surface. Many approximate equations have been derived for waves in shallow water, such as the classical shallow-water system (1.1.1)–(1.1.2) for bidirectional non-dispersive waves, the Green-Naghdi system (GN) (3.2.3), (3.2.7) for bidirectional dispersive waves, the Korteweg-De Vries (KdV) equation for unidirectional waves, and many variants of the Boussinesq system for dispersive waves propagating in both directions. In addition to the shallowness, KdV and Boussinesq systems assume small amplitudes.

Among these approximations, the Green-Naghdi system

$$h_t + (hu)_x = 0\,, \tag{1.1.3}$$

$$(hu)_t + (hu^2 + \frac{1}{2}gh^2 + \frac{1}{3}h^2\gamma)_x = 0\,, \tag{1.1.4}$$

$$\text{with } \gamma \stackrel{\text{def}}{=} h(u_x^2 - u_{xt} - uu_{xx}). \tag{1.1.5}$$

is a prototypical second-order improvement of the classical shallow-water system. Numerical computations [39] have been conducted to compare the solutions of the GN system, the KdV equation, and the Boussinesq system with those of the water wave problem and show that the solutions of the GN system approximate the water wave problem better within the region of large amplitude than the other two weakly nonlinear models.

1.2 The regularized Saint-Venant system

The GN system consists of Euler-Lagrange equations of the Lagrangian density

$$\mathcal{L} = \frac{1}{2}hu^2 + \frac{1}{6}h^3u_x^2 - \frac{1}{2}gh^2. \tag{1.2.1}$$

Clamond and Dutykh modified in [10] the second order term in $\mathcal{L}$ using the first order approximation (i.e., the shallow water equations) and derive a family of modified Serre-Green-Naghdi (mSGN) systems equivalent in accuracy. As second order approximations of the water wave equations, mSGN has, if the free parameter is chosen properly, better dispersion relation than the Green-Naghdi system. Then in [9], Clamond and Dutykh show that the coefficients in the second order terms can be chosen so that the resulting system is *dispersionless*:

$$h_t + (hu)_x = 0\,, \tag{1.2.2}$$

$$(hu)_t + (hu^2 + \frac{1}{2}gh^2 + \varepsilon\mathcal{R}h^2)_x = 0, \tag{1.2.3}$$

$$\mathcal{R} \stackrel{\text{def}}{=} h(u_x^2 - u_{xt} - uu_{xx}) - g\left(hh_{xx} + \frac{1}{2}h_x^2\right)\,. \tag{1.2.4}$$

This is the main system to be studied in the thesis, namely *the regularized Saint-Venant (rSV) system.* Obviously when ε is 0, the system (1.2.2)–(1.2.3) is reduced to (1.1.1)–(1.1.2).

The particular coefficients appearing here, however, do not yield improved accuracy for modeling exact water-wave dispersion at long wavelengths. Instead, they are designed to *eliminate* linear dispersion, resulting in a regularization that faithfully reproduces the original shallow-water dispersion relation. The balance of terms in $\mathcal{R}$ ensures that the rSV equations are *non-dispersive*—linearized about a constant state (h_0, u_0), solutions proportional to $e^{ikx-i\omega t}$ necessarily have

$$(\omega - u_0k)^2 = gh_0k^2\,,$$

implying that phase velocity is independent of frequency.

This system formally conserves mass, momentum, and energy. Smooth solutions of these equations also satisfy a conservation law for energy, in the form

$$\mathcal{E}_t^\varepsilon + \mathcal{Q}_x^\varepsilon = 0\,, \tag{1.2.5}$$

where

$$\mathcal{E}^\varepsilon \stackrel{\text{def}}{=} \frac{1}{2}hu^2 + \frac{1}{2}gh^2 + \varepsilon\left(\frac{1}{2}h^3u_x^2 + \frac{1}{2}gh^2h_x^2\right)\,, \tag{1.2.6}$$

$$\mathcal{Q}^\varepsilon \stackrel{\text{def}}{=} \frac{1}{2}hu^3 + gh^2u + \varepsilon\left(\left(\frac{1}{2}h^2u_x^2 + \frac{1}{2}ghh_x^2 + h\mathcal{R}\right)hu + gh^3h_xu_x\right)\,. \tag{1.2.7}$$

The rSV equations (1.2.2)–(1.2.3) above were derived in [9] as the Euler-Lagrange equations corresponding to a least action principle for a Lagrangian of the form (see Chapter 3 or [9, Eq. (3.2)])

$$\mathcal{L} \stackrel{\text{def}}{=} \frac{1}{2}hu^2 - \frac{1}{2}gh^2 + \varepsilon\left(\frac{1}{2}h^3u_x^2 - \frac{1}{2}gh^2h_x^2\right) + (h_t + (hu)_x)\phi\,. \tag{1.2.8}$$

Here ϕ is a Lagrange multiplier field that enforces mass conservation. The terms proportional to ε in (1.2.3) have a form similar to terms that appear in improved Green-Naghdi or Serre equations that approximate shallow-water dynamics for waves of small slopes, see [10].

The rSV equations also admit a non-canonical Hamiltonian structure like one known for the Green-Naghdi equations. Namely, with

$$\mathcal{H} = \int \frac{1}{2}hu^2 + \frac{1}{2}g(h-h_\infty)^2 + \varepsilon\left(\frac{1}{2}h^3u_x^2 + \frac{1}{2}gh^2h_x^2\right)\,dx\,, \tag{1.2.9}$$

and $m = hu - \varepsilon(h^3u_x)_x$, the rSV system is formally equivalent to

$$\partial_t \begin{pmatrix} m \\ h \end{pmatrix} = -\begin{pmatrix} \partial_x\, m + m\,\partial_x & h\partial_x \\ \partial_x h & 0 \end{pmatrix}\begin{pmatrix} \delta\mathcal{H}/\delta m \\ \delta\mathcal{H}/\delta h \end{pmatrix}. \tag{1.2.10}$$

This is a simple variant of the Hamiltonian structure well-known for the Green-Naghdi equations [12, 22, 28, 37], obtained by replacing the Green-Naghdi Hamiltonian with a Hamiltonian derived from (1.2.6).

The presence of squared derivatives in the energy $\mathcal{E}^\varepsilon$ indicates that the rSV equations will not admit classical shock wave solutions with discontinuities in h and u. Numerical experiments reported in [9] suggest, in fact, that with smooth initial data that produce hydraulic jumps (shock wave solutions) for the shallow-water equations, one obtains front-like solutions of the rSV equations that remain smooth and non-oscillatory, yet propagate at the correct speed determined by classical jump conditions corresponding to limiting states on the left and right. These solutions were computed numerically by a pseudo-spectral scheme that is highly accurate for smooth solutions and fairly uncomplicated. This is a hint that a similar approach could perhaps be taken to approximate shallow water dynamics by non-dispersive regularization in multidimensional geometries with more complicated topography and other physics.

At this point, a paradox arises. The energy of smooth solutions of the rSV equations satisfies the conservation law (1.2.5), whereas in the case of shallow water equations, energy is dissipated at a shock-wave discontinuity, satisfying a distributional identity of the form

$$\mathcal{E}^0_t + \mathcal{Q}^0_x = \mu \tag{1.2.11}$$

where μ is a non-positive measure supported along the shock curve. How can it be that front-like solutions of the rSV equations approximate classical shallow-water shocks well while conserving an energy similar to the one dissipated for shallow-water shocks?

1.3 Weakly singular shock waves

We describe in Chapter 4 a novel wave-propagation mechanism that may explain this paradox. We show that the regularized Saint-Venant equations (1.2.2)–(1.2.3) admit regularized shock-wave solutions with profiles that are *continuous but only piecewise smooth*, with derivatives having a weak singularity at a single point, see figure 1.1 and equation (4.1.21). Such a wave exists corresponding to every classical shallow-water shock. These waves are traveling-wave weak solutions of the rSV equations that conserve

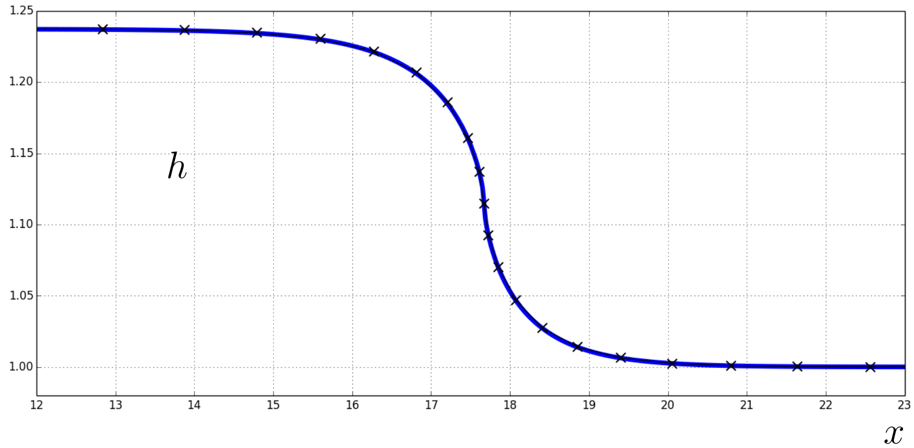


Figure 1.1: From [45]: a weakly singular shock profile for the regularized Saint-Venant equations, where the derivative blows up at the reflection point.

mass and momentum. They *dissipate energy at the singular point*, however, at the precise rate that the corresponding classical shock does, see (4.4.1).

We also find that the rSV equations admit weak solutions in the form of *cusped solitary waves*. These waves loosely resemble the famous 'peakon' solutions of the Camassa-Holm equation in the non-dispersive case [7]. One difference is that the wave slope of our cusped solitary waves becomes infinite approaching the crest, while that of a peakon remains finite. The rSV equations also loosely resemble various 2-component generalizations of the Camassa-Holm equation which have appeared in the literature—for a sample see [8, 23, 26, 27, 31]. One of the most well-studied of these is the integrable 2-component Camassa-Holm system appearing in [8, 27, 31],

$$h_t + (hu)_x = 0, \tag{1.3.1}$$

$$u_t + 3uu_x - u_{txx} - 2u_x u_{xx} - uu_{xxx} + ghh_x = 0, \tag{1.3.2}$$

which has been derived in the context of shallow-water theory by Constantin and Ivanov [13] (also see [25]). This system admits peakon-type solutions, and as noted in [16], it admits some degenerate front-type traveling wave solutions, which however necessarily have $h \to 0$ as either $x \to +\infty$ or $-\infty$.

The existence of weakly singular weak solutions of the rSV equations raises many interesting analytical and numerical issues. For example, do smooth solutions develop weak singularities in finite time? The answer is yes and we shall address it step by step.

1.4 Large time existence, uniqueness and blow-up criterion

First of all, we shall prove the well-posedness of an initial value problem for this system. Let $\alpha > 0$ and write

$$h = h_0 + \alpha\eta, \quad u \text{ replaced with } \alpha u,$$

we have the following result.

Theorem 1.4.1. *Let $s \geqslant 2$ be an integer and $\alpha > 0, h_0 > 0$. Assume the initial value $W^0 = (\eta^0, u^0)^T \in H^s(\mathbb{R})$ satisfies*

$$\exists\, h_{\min} > 0 \text{ such that } \inf_{x\in\mathbb{R}} h = \inf_{x\in\mathbb{R}}(h_0 + \alpha\eta^0) \geqslant h_{\min}. \tag{1.4.1}$$

Then there exist $T_{\max} = T_{\max}(s, W^0) > 0$ bounded below uniformly with respect to ε and α such that the regularized shallow-water system (1.2.2)–(1.2.3) admit a unique solution $W = (\eta, u)^T \in C([0, T_{\max}/\alpha]; H^s(\mathbb{R}))$ with the initial condition W^0 and preserving the nonzero depth condition (1.4.1).

In particular if $T_{\max} < \infty$ we have

$$\|DW(t,\cdot)\|_{L^\infty} \to \infty \quad \text{as } t \to \frac{T_{\max}}{\alpha}, \tag{1.4.2}$$

or

$$\inf_{\mathbb{R}} h(t,\cdot) = \inf_{\mathbb{R}}(h_0 + \alpha\eta(t,\cdot)) \to 0 \quad \text{as } t \to \frac{T_{\max}}{\alpha}. \tag{1.4.3}$$

Moreover, the following conservation of energy property holds

$$\partial_t \int \tilde{\mathcal{E}}\, dx = \partial_t \int \left(\frac{1}{2}hu^2 + \frac{1}{2}\eta^2 + \varepsilon\Big(\frac{1}{2}h^3u_x^2 + \frac{1}{2}h^2\eta_x^2\Big)\right) dx = 0. \tag{1.4.4}$$

In this theorem, α is a scaling constant that indicates the smallness of the initial data. As in the case of Burger's equation, the total existence time $T_{\max}/\alpha$ is proportional to $1/\alpha$, which is why we call this a "large time" existence result.

The idea of the proof starts with the observation that there are two terms having time derivatives in (1.2.3). It is natural to combine these two terms by inverting the corresponding spatial operator. We formally write

$$\mathcal{I}_h(w) \stackrel{\text{def}}{=} hw - \varepsilon(h^3 w_x)_x, \quad \text{or} \quad \mathcal{I}_h = h - \varepsilon\partial_x \circ h^3 \circ \partial_x. \tag{1.4.5}$$

Acting $\mathcal{I}_h^{-1}$ on both sides of (1.2.3), we obtain

$$u_t + gh_x + uu_x + \varepsilon\mathcal{I}_h^{-1}\partial_x\Big(2h^3u_x^2 - \frac{1}{2}gh^2h_x^2\Big) = 0 \tag{1.4.6}$$

When h is bounded below, $\mathcal{I}_h$ is a strictly elliptic operator, so the last term in the above equation is a "zero-th order term" which has two derivatives inside while gaining two from $\mathcal{I}_h^{-1}$ (see lemma 5.1.3, lemma 6.2.2 for precise results). Therefore a standard energy estimate applies, controlling the high norm of the solution. The proof of existence goes by using an iteration scheme in which the controlled high norm helps argue that the sequence converges. See Theorem 5.0.1 and its proof in Chapter 5. This standard treatment of symmetrizable hyperbolic systems can be found in [40, Chapter 2]

Another important feature of the system (1.2.2)–(1.2.3) is that when energy $\mathcal{E}^\varepsilon$ in (1.2.6) is small and the smooth solution conserves it, the height stays bounded from below by a positive constant that is dictated by the initial energy. See Theorem 6.1.1 for more details.

1.5 Blow-up phenomena

For the existence of blow-up phenomena, we have the following theorem:

Theorem 1.5.1. *Let the domain $D = \mathbb{T}$ or $\mathbb{R}$. Then there exists certain smooth initial data with which the system (1.2.2)–(1.2.3) would blow up in finite time. The precise* *meaning of blowing up is that there exists* $0 < T < \infty$ *s.t. the solution exists and stays smooth on* $\mathbb{T} \times [0, T)$ *with*

$$\|(h_x, u_x)(\cdot, t)\|_{L^\infty} \to +\infty \quad \textit{as } t \uparrow T. \tag{1.5.1}$$

The proof goes as the following: as in the classical shallow-water system case, we write Riemann invariants ($R_\pm$) and the two characteristic speeds ($\lambda_\pm$) as

$$R_+ = u + 2\sqrt{gh}, \qquad \lambda_+ = u + \sqrt{gh},$$
$$R_- = u - 2\sqrt{gh}, \qquad \lambda_- = u - \sqrt{gh},$$

Write also the Riccati-type quantities as

$$P_+ = (R_+)_x, \quad P_- = (R_-)_x$$

Then rewriting the system (1.2.2), (1.4.6) into its characteristic form and differentiating, we have

$$\frac{d^+}{dt} P_+ = -\frac{1}{4}(3P_+ + P_-)P_+ - 2\varepsilon \partial_x \mathcal{I}_h^{-1} \partial_x \Big(h^3 (\lambda_+)_x (\lambda_-)_x \Big), \tag{1.5.2}$$

$$\frac{d^-}{dt} P_- = -\frac{1}{4}(P_+ + 3P_-)P_- - 2\varepsilon \partial_x \mathcal{I}_h^{-1} \partial_x \Big(h^3 (\lambda_+)_x (\lambda_-)_x \Big). \tag{1.5.3}$$

One key observation here is that the operator $\varepsilon \partial_x \mathcal{I}_h^{-1} \partial_x \circ h^3 - \mathrm{Id}$ takes a neat form which enables us to bound the resulting term in L^∞ by the initial energy. So what is left is just two coupled Riccati-type equations.

We use a smoothed-out weakly singular shock profile as our initial data. From the exact form of the weakly singular shock profile, asymptotic behaviors of various functions near the singular point are presented in Section 4.2, from which we know that near singularity: P_- stays bounded and P_+ blows up. Moreover, the integral of P_+ along the "-" characteristic curves should be bounded! These intuitions provide a guideline of the argument and eventually leads to a proof in Section 6.4.

1.6 Thesis overview

The remainder of the thesis is organized as follows.

- In section 2 we introduce the classical shallow-water (Airy or Saint-Venant) system as well as some important features of the system that are useful to later sections.
- In section 3 we derive the regularized shallow-water system (rSV) through a variational principle argument.
- In section 4 we prove the existence of weakly singular shock profiles for traveling wave solutions to the rSV equations. Such solutions conserves mass, momentum and dissipates energy at the same rate as that of the classical shallow-water system.
- In section 5 we establish the finite-time (large time) existence, uniqueness and blow-up criterion of the rSV system.
- In section 6 we prove that for certain smooth initial data the singularity happens in finite amount of time.
- In section 7 we discuss the main contributions of the thesis and some possible future directions.

Chapter 2

The 1-D Classical Shallow-water System

In fluid mechanics, many phenomena can be described by hyperbolic partial differential equations, among which the classical shallow-water (Airy or Saint-Venant) system plays an important role and is commonly used to model transient open-channel flow and surface runoff. Particularly, 1-D classical shallow-water system are used extensively in various computer models because they are significantly easier to solve than 2-D shallow-water systems. Common applications of the 1-D system include flood routing along rivers, dam breaking analysis, storm pulses in an open channel, as well as storm runoff in overland flow.

All the following content is limited to the 1-D system, so we will omit '1-D' from now on.

Chapter 1.1 introduces three different derivations of the classical shallow-water system. In Chapter 1.2 we show that there are infinitely many conservation laws for the classical shallow-water system including formal conservation of mass, momentum and energy. Chapter 1.3 introduces the jump conditions for discontinuous solutions which typically happen when characteristics cross each other. Chapter 1.4 computes the energy dissipation at shocks and Chapter 1.5 we use three different approaches to illustrate how mass and momentum are conserved by Rankine-Hugoniot condition while energy is necessarily dissipated.

2.1 Derivation of the classical shallow-water system

Consider an inviscid incompressible fluid (water) in a constant gravitational field. The space coordinates are denoted by (x, y) and the corresponding component of the velocity vector $\overline{\mathbf{u}}$ by $(\overline{u}, \overline{v})$. The gravitational acceleration g is in the negative y direction. We also assume that the density ρ remains constant and that there is an external force $F = -\rho g\mathbf{j}$, where $\mathbf{j}$ is the unit vector in the y direction. The equations are

$$\nabla \cdot \overline{\mathbf{u}} = 0, \tag{2.1.1}$$

$$\frac{D\overline{\mathbf{u}}}{Dt} = \frac{\partial \overline{\mathbf{u}}}{\partial t} + (\overline{\mathbf{u}} \cdot \nabla)\overline{\mathbf{u}} = -\frac{1}{\rho}\nabla p - g\mathbf{j}. \tag{2.1.2}$$

Assuming the irrotationality of the fluids, we have velocity potential $\overline{\varphi}$ such that

$$\overline{\mathbf{u}} = (\overline{u}, \overline{v}) = \nabla\overline{\varphi}.$$

Assuming further that the bottom is flat, together with the kinematic conditions, we can expand (2.1.1)–(2.1.2) as

$$\overline{\varphi}_{xx} + \overline{\varphi}_{yy} = 0 \qquad \text{if } 0 < y < h, \tag{2.1.3}$$

$$\overline{\varphi}_t + \frac{1}{2}|\nabla\overline{\varphi}|^2 + gy = 0 \qquad \text{if } y = h, \tag{2.1.4}$$

$$h_t + \overline{\varphi}_x h_x - \overline{\varphi}_y = 0 \qquad \text{if } y = h, \tag{2.1.5}$$

$$\overline{\varphi}_y = 0 \qquad \text{if } y = 0. \tag{2.1.6}$$

In shallow-water regime, the depth h_0 is small compared to the wavelength l, i.e.

$$\beta = \frac{h_0}{l} \ll 1.$$

Note that all variables have their own physical units. However, in mathematical analysis, one prefers unitless forms. A proper set of changes of variables is

$$x \to lx, \quad y \to h_0 y, \quad t \to \frac{lt}{c_0}, \quad h \to h_0 + a\eta, \quad \overline{\varphi} \to c_0 l\alpha\varphi, \tag{2.1.7}$$

where h_0 is the average depth of the water, a is the amplitude, and $c_0 = \sqrt{gh_0}$. It is convenient to write the dimensionless constant

$$\alpha = \frac{a}{h_0}.$$

Adopting the new notations and applying the changes of variables, one obtains

$$\beta^2 \varphi_{xx} + \varphi_{yy} = 0 \qquad \text{if } 0 < y < 1 + \alpha\eta, \tag{2.1.8}$$

$$\eta + \varphi_t + \frac{1}{2}\alpha\varphi_x^2 + \frac{1}{2}\frac{\alpha}{\beta^2}\varphi_y^2 = 0 \qquad \text{if } y = 1 + \alpha\eta, \tag{2.1.9}$$

$$\eta_t + \alpha\varphi_x\eta_x - \frac{1}{\beta^2}\varphi_y = 0 \qquad \text{if } y = 1 + \alpha\eta, \tag{2.1.10}$$

$$\varphi_y = 0 \qquad \text{if } y = 0. \tag{2.1.11}$$

In this set of equations the unknowns η, φ and the constants α, β are all dimensionless. Set dimensionless velocity potential potential at the bottom

$$f := \varphi|_{y=0}.$$

Kinematic condition (2.1.11) says $f_y = 0$. Expanding φ at the bottom and replacing y derivative by x derivative using the harmonicity (2.1.8), one has

$$\begin{aligned} \varphi &= \sum_{k=0}^{\infty} \frac{\partial^k \varphi}{\partial y^k}\bigg|_{y=0} \frac{y^k}{k!} = \sum_{m=0}^{\infty} (-\beta^2)^m \frac{\partial^{2m} f}{\partial x^{2m}} \frac{y^{2m}}{(2m)!} \\ &= f - \frac{1}{2}\beta^2 f_{xx} y^2 + O(\beta^4). \end{aligned} \tag{2.1.12}$$

2.1.1 An asymptotic procedure.

This is an approach that directly uses Taylor expansion (2.1.12), plugs in, and considers terms that aren't 'too small'. It is used a lot in various derivation of asymptotic system to water-wave equations, e.g. Korteweg-deVries and Boussinesq equations in Chapter 13.11 in [48] and more in [29].

Substitute φ in (2.1.9)–(2.1.10), and set $y = 1 + \alpha\eta$,

$$\eta_t + \alpha f_x \eta_x + f_{xx}(1 + \alpha\eta) = O(\beta^2), \tag{2.1.13}$$

$$\eta + f_t + \frac{1}{2}\alpha f_x^2 = O(\beta^2). \tag{2.1.14}$$

Let the dimensionless horizontal velocity at the bottom be $w = f_x$ and differentiate the second equation with respect to x, we have

$$\eta_t + ((1+\alpha\eta)w)_x = O(\beta^2), \tag{2.1.15}$$

$$w_t + \eta_x + \alpha w w_x = O(\beta^2). \tag{2.1.16}$$

Let

$$u = c_0 \alpha w$$

be the physical horizontal velocity at the bottom. Changing back to the physical variables according to (2.1.7), one has the shallow-water system:

$$h_t + (hu)_x = O(\beta^3)c_0, \tag{2.1.17}$$

$$u_t + gh_x + uu_x = O(\beta^3)g, \tag{2.1.18}$$

There are physical constants following big O terms here indicating that these are equations with physical variables and the exact units are given by the units of the constants.

2.1.2 A "physics" approach.

Chapter 13.10 in [48] shows a more physically meaningful way to derive the system. From Taylor expansion (2.1.12) one can derive that, in the dimensionless variables,

$$\varphi_x = f_x - \frac{1}{2}\beta^2 f_{xxx} y^2 + O(\beta^4), \tag{2.1.19}$$

$$\varphi_y = -\beta^2 f_{xx} y + O(\beta^4). \tag{2.1.20}$$

Changing back to physical variables using (2.1.7), the physical velocities admit approximations

$$\overline{u} = u + O(\beta^2)c_0 = u - \frac{1}{2}u_{xx}y^2 + O(\beta^4)c_0, \tag{2.1.21}$$

$$\overline{v} = O(\beta)c_0 = -u_x y + O(\beta^3)c_0. \tag{2.1.22}$$

Approximations of various order are presented here and will be useful in derivation of various approximations to the water wave equations.

Following the same procedure, one can check the orders of β for all dependent variables and differential operators in shallow-water regime:

$$O(\beta^{-1}) : \overline{\varphi}, \quad O(\beta^0) : \partial_y, \overline{u}, h, \quad O(\beta^1) : \partial_x, \partial_t, \overline{v}. \tag{2.1.23}$$

Note that approximation of $\overline{v}$ (2.1.22) implies that

$$\overline{v}_t = O(\beta^2)g, \quad \overline{v}_x = O(\beta^2)\frac{c_0}{h_0}, \quad \overline{v}_y = O(\beta)\frac{c_0}{h_0},$$

the vertical component of Euler equation (2.1.2) gives that

$$\frac{1}{\rho}\frac{\partial p}{\partial y} + g = O(\beta^2)g. \tag{2.1.24}$$

Integrating from a generic position (x, y) up to the water surface

$$\frac{p - p_0}{\rho} = g(h - h_0 - y) + O(\beta^2)c_0^2. \tag{2.1.25}$$

Substituting the approximations of $\overline{u}, \overline{v}$ into the horizontal component of Euler equation (2.1.2), one obtains

$$u_t + uu_x = -\frac{1}{\rho}p_x + O(\beta^2)g = -gh_x + O(\beta^2)g. \tag{2.1.26}$$

This is the horizontal velocity conservation equation (2.1.18).

The other equation is conservation of mass, so it follows directly from incompressibility (2.1.1). Integrating (2.1.1) with respect to y, one obtains

$$\begin{aligned} 0 &= \int_0^h \overline{u}_x + \overline{v}_y \, dy = \partial_x\left(\int_0^h \overline{u}\, dy\right) - \overline{u}(x,h)h_x + \overline{v}(x,h) - \overline{v}(x,0) \\ &= \partial_x(uh) + h_t + O(\beta^2)c_0, \end{aligned} \tag{2.1.27}$$

where the last equality follows from approximation (2.1.21) and the kinematic condition (2.1.10).

2.1.3 A Lagrangian approach.

It is interesting that, the kinetic and potential energies per water volume, respectively $\mathcal{K}_2$ and $\mathcal{V}$, can be derived using approximations (2.1.21)–(2.1.22)

$$\frac{\mathcal{K}_2}{\rho} = \int_0^h \frac{\overline{u}^2 + \overline{v}^2}{2} = \frac{1}{2}u^2 h + O(\beta^2)c_0^2 h_0, \quad \frac{\mathcal{V}}{\rho} = \int_0^h gy = \frac{1}{2}gh^2. \tag{2.1.28}$$

where the subscript 2 in $\mathcal{K}_2$ indicates all terms of order $O(\beta^2)$ or higher are dropped. A Lagrangian density $\mathcal{L}_2$ can then be introduced as the kinetic energy minus the potential energy plus a constraint for mass conservation

$$\frac{\mathcal{L}_2}{\rho} = \frac{1}{2}u^2 h - \frac{1}{2}gh^2 + (h_t + (hu)_x)\phi + O(\beta^2)c_0^2 h_0 \tag{2.1.29}$$

where ϕ is a Lagrange multiplier. The Euler-Lagrange equations are computed as

$$\begin{aligned} \delta\phi: &\quad 0 = h_t + (hu)_x, \\ \delta u: &\quad 0 = hu - h\phi_x + O(\beta^2), \\ \delta h: &\quad 0 = \frac{1}{2}u^2 - gh - \phi_t - u\phi_x. \end{aligned}$$

Hence eliminating ϕ, one gets the shallow-water system (2.1.17)–(2.1.18).

Lastly, omitted the $O(\beta^2)$ terms, the classical shallow-water system (2.1.17)–(2.1.18) is written as

$$h_t + (hu)_x = 0, \tag{2.1.30}$$

$$u_t + gh_x + uu_x = 0. \tag{2.1.31}$$

From the physics approach one sees that if one chose u to be the average horizontal velocity (i.e. $\int_0^h \overline{u}\,dy$) instead of the horizontal velocity at the bottom, equation (2.1.30) would be exact. Both choices of horizontal velocities derives (2.1.30)–(2.1.31).

For this reason, we will from now on change the definition of u and view u as *average horizontal velocity.* Approximations (2.1.21)–(2.1.22)become

$$\overline{u} \;=\; u + O(\beta^2)c_0 = u - \frac{1}{3}u_{xx}y^2 + O(\beta^4)c_0, \tag{2.1.32}$$

(2.1.33) $$\overline{v} \;=\; O(\beta)c_0 = -u_x y + O(\beta^3)c_0.$$

Note that the only modification is the coefficient in front of $u_{xx}y^2$ in (2.1.32). When one changes the height u sits on, this coefficient typically changes accordingly. This phenomena is very useful when one wants to choose a specific coefficient to kill certain terms, e.g. in the derivation of Camassa-Holm equation in [29].

2.2 Further Conservation Equations

In the scalar conservation law case, one has infinitely many entropy-flux pairs that formally satisfy the equation. Analogously, it is interesting that the classical shallow-water system (2.1.30)–(2.1.31) also admits an infinite number of conservation equations of the general form

$$\frac{\partial}{\partial t}P(h,u) + \frac{\partial}{\partial x}Q(h,u) = 0.$$

The following derivation comes from Chapter 13.10 in [48].

The previous equation formally expands to

(2.2.1) $$Q_h h_x + Q_u u_x = -(P_h h_t + P_u u_t) = P_h(uh_x + hu_x) + P_u(gh_x + uu_x),$$

comparing the coefficients for h_x and u_x yields

(2.2.2) $$Q_u = uP_u + hP_h, \quad Q_h = gP_u + uP_h.$$

Thus any solution of

(2.2.3) $$gP_{uu} = hP_{hh}$$

will lead to a conservation equation. Among all possibilities we are particularly interested in polynomials in u and h. Take

$$P_n(h,u) = \sum_{m=0}^{n} p_m(u)h^m$$

from which it follows that

$$p_0'' = 0, p_n'' = 0, \text{ and } gp_m'' = m(m+1)p_{m+1} \text{ for } m = 1, \ldots, n-1. \tag{2.2.4}$$

The first four are:

$$\begin{array}{ll} P_0 = u & Q_0 = \frac{1}{2}u^2 + gh \\ P_1 = h & Q_1 = hu \\ P_2 = hu & Q_2 = hu^2 + \frac{1}{2}gh^2 \\ P_3 = \frac{1}{2}hu^2 + \frac{1}{2}gh^2 & Q_3 = \frac{1}{2}hu^3 + gh^2u \end{array}$$

in which the pair (P_1, Q_1) gives the conservation of mass,

$$\partial_t P_2 + \partial_x Q_2 = 0, \quad \text{ or } (hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = 0 \tag{2.2.5}$$

gives the conservation of momentum, and

$$\partial_t P_3 + \partial_x Q_3 = 0, \quad \text{ or } \left(\frac{1}{2}hu^2 + \frac{1}{2}gh^2\right)_t + \left(\frac{1}{2}hu^3 + gh^2u\right)_x = 0 \tag{2.2.6}$$

gives the conservation of energy for smooth solutions to the system.

One significance of having infinitely many conservation equations is that, since each provides a constant integral

$$\int_{-\infty}^{\infty} (P(u,h) - P(u_0, h_0))\, dx = \text{constant}$$

in any problem such that $u \to 0, h \to h_0$ at $\pm\infty$, an infinite number of integrals of the solution are known. Therefore, it is in principle feasible to find the solution analytically, see [44].

2.3 Jump conditions.

All solutions to the classical shallow-water system carry an increasing number of jumps as time goes. A piecewise smooth solution that jumps along a curve $x = X(t)$ is a weak solution if and only if the Rankine-Hugoniot conditions hold at each point of the curve:

$$-s[h] + [hu] = 0\,, \tag{2.3.1}$$

$$-s[hu]+\left[hu^2+\frac{1}{2}gh^2\right]=0\,. \tag{2.3.2}$$

Here $s=\dot{X}(t)$ is the jump speed and $[h]\overset{\text{def}}{=}h_+-h_-$ is the difference of right and left limits at the shock location, with similar definitions for the other brackets, e.g., $[hu]\overset{\text{def}}{=}h_+u_+-h_-u_-$.

This system has two Riemann invariants $u\pm 2\sqrt{gh}$, and two characteristic speeds

$$\lambda_1=u-\sqrt{gh},\qquad \lambda_2=u+\sqrt{gh}.$$

After eliminating s from the Rankine-Hugoniot conditions one finds

$$\left[\frac{1}{2}gh^2\right][h]=\frac{g(h_++h_-)}{2}[h]^2=[hu]^2-[hu^2][h]=h_+h_-[u]^2\,,$$

so that the states $(h_\pm,u_\pm)$ lie on the Hugoniot curves given by

$$u_+-u_-=\pm\gamma(h_+-h_-),\qquad \gamma\overset{\text{def}}{=}\sqrt{\frac{g(h_++h_-)}{2h_+h_-}}\,.$$

Correspondingly the jump speed is determined by

$$s=\frac{[hu]}{[h]}=u_+\pm\gamma h_-=u_-\pm\gamma h_+\,. \tag{2.3.3}$$

In these relations, the $-$ sign corresponds to 1-waves and the $+$ sign corresponds to 2-waves. Physically meaningful shock waves satisfy the Lax shock conditions:

$$\begin{aligned} &u_--\sqrt{gh_-}>s>u_+-\sqrt{gh_+} &&\text{for 1-shocks,}\\ &u_-+\sqrt{gh_-}>s>u_++\sqrt{gh_+} &&\text{for 2-shocks.}\end{aligned} \tag{2.3.4}$$

From (2.3.3) one finds that the Lax conditions hold if and only if

$$\begin{aligned} &h_-<h_+ &&\text{for 1-shocks,}\\ &h_->h_+ &&\text{for 2-shocks.}\end{aligned} \tag{2.3.5}$$

The two wave families are related via the natural spatial reflection symmetry of the shallow water equations:

$$(x,t)\to(-x,t),\qquad (h,u)\to(h,-u)$$

Under this symmetry, 1-shocks are mapped to 2-shocks and vice versa.

2.4 Energy dissipation.

The energy dissipation identity for a piecewise-smooth solution with shock curve $\Gamma = \{(x,t) : x = X(t)\}$ takes the form

$$\mathcal{E}^0_t + \mathcal{Q}^0_x = \mu \tag{2.4.1}$$

where

$$\mathcal{E}^0 = \frac{1}{2}hu^2 + \frac{1}{2}gh^2, \quad \mathcal{Q}^0 = \frac{1}{2}hu^3 + gh^2u$$

and the measure μ is absolutely continuous with respect to 1-dimensional Hausdorff measure (arc length measure) restricted to the shock curve Γ. Denoting this Hausdorff measure by σ, in terms of the parametrization $x = X(t)$ we can write informally that $d\sigma = \sqrt{1+s^2}\,dt$ and

$$\mathcal{D} \stackrel{\text{def}}{=} \frac{d\mu}{dt} = -s[\mathcal{E}^0] + [\mathcal{Q}^0] = \left[\frac{1}{2}h(u-s)^3 + gh^2(u-s)\right]. \tag{2.4.2}$$

One verifies this identity by expanding $(u-s)^3$ and using that

$$s^3[h] = s^2[hu] = s\left[hu^2 + \frac{1}{2}gh^2\right]$$

from the Rankine-Hugoniot conditions. The precise meaning of (2.4.2) and (2.4.1) is that for any smooth test function φ with support in a small neighborhood of the shock curve Γ and contained in the half-plane where $x \in \mathbb{R}$ and $t > 0$, we have

$$\int_0^\infty \int_{-\infty}^\infty (-\mathcal{E}^0\partial_t\varphi - \mathcal{Q}^0\partial_x\varphi)\,dx\,dt = \int_\Gamma \varphi\,d\mu = \int_0^\infty \varphi(X(t),t)\,\mathcal{D}(t)\,dt\,.$$

The identity (2.4.2) is related to the Galilean invariance of shallow-water equations after changing to a frame moving with constant speed s frozen at some instant of time. To conveniently compute further we introduce $v = u - s$ and write

$$v_- = u_- - s, \qquad v_+ = u_+ - s, \tag{2.4.3}$$

and note that by the Rankine-Hugoniot conditions,

$$M \stackrel{\text{def}}{=} h_+v_+ = h_-v_-\,, \tag{2.4.4}$$

$$N \stackrel{\text{def}}{=} h_+ v_+^2 + \frac{1}{2} g h_+^2 = h_- v_-^2 + \frac{1}{2} g h_-^2 \,. \tag{2.4.5}$$

With the same choice of sign as in (2.3.3) we find

$$M = \mp \gamma h_+ h_- \,, \tag{2.4.6}$$

$$N = \frac{M^2}{h_\pm} + \frac{1}{2} g h_\pm^2 = \frac{1}{2} g (h_+^2 + h_+ h_- + h_-^2) \,. \tag{2.4.7}$$

Then using (2.4.6) and (2.3.5), we compute

$$\mathcal{D} = \frac{M^3}{2} \left[\frac{1}{h^2} \right] + g M [h] = \pm \frac{1}{4} g \gamma [h]^3 < 0 \,, \tag{2.4.8}$$

for both 1-shocks and 2-shocks. Note that the dissipation is of the order of the amplitude cubed for small shocks.

2.5 Interpretations of mass, momentum and energy.

The energy conservation equation (2.2.6) would provide a third potential jump condition (and there are potentially infinitely many), but only two conditions can be used with the classical shallow-water system. Rayleigh proposed that the energy is not conserved across a jump, attributing this to the observed turbulence, so that the jump condition corresponding to energy conservation (2.2.6) should not be used.

What follows will show that mass and momentum are conserved across a jump and that, whereas the system implies a formal energy conservation equation (2.2.6), Rankine-Hugoniot conditions imply energy dissipation equation (2.4.2) and hence energy is not conserved.

2.5.1 A direct computation

Consider a piecewise smooth solution at time t with the only discontinuity at $X(t)$, the position of the shock at time t. Two particles at $X_1(t), X_2(t)$ are such that $X_1(t) < X(t) < X_2(t)$. Let $A = A(h, u), B = B(h, u)$ be a generic entropy-flux pair.

Then we compute that

$$
\begin{aligned}
\frac{d}{dt}\int_{X_1}^{X_2} A\,dx &= \frac{d}{dt}\int_{X_1}^{X} A\,dx + \frac{d}{dt}\int_{X}^{X_2} A\,dx \\
&= -s[A] + (Au)|_{X_1}^{X_2} - \int_{X_1}^{X} B_x\,dx - \int_{X}^{X_2} B_x\,dx \\
&= -s[A] + [B] + (Au - B)|_{X_1}^{X_2} \qquad (2.5.1)
\end{aligned}
$$

For $(A, B) = (h, hu)$, we have

$$\frac{d}{dt}\int_{X_1}^{X_2} h\,dx = -s[h] + [hu] = 0 \qquad (2.5.2)$$

which is mass conservation.

For $(A, B) = (hu, hu^2 + \frac{1}{2}gh^2)$, we have

$$\frac{d}{dt}\int_{X_1}^{X_2} hu\,dx = -s[hu] + [hu^2 + \frac{1}{2}gh^2] - (\frac{1}{2}gh^2)|_{X_1}^{X_2} = (\frac{1}{2}gh^2)|_{X_2}^{X_1} \qquad (2.5.3)$$

which is momentum conservation.

For $(A, B) = (\mathcal{E}^0, \mathcal{Q}^0)$, we have

$$\frac{d}{dt}\int_{X_1}^{X_2} \mathcal{E}^0\,dx = -s[\mathcal{E}^0] + [\mathcal{Q}^0] - (\frac{1}{2}gh^2u)|_{X_1}^{X_2} = \mathcal{D} + (\frac{1}{2}gh^2u)|_{X_2}^{X_1} \qquad (2.5.4)$$

which shows energy dissipation (or energy conservation after dissipation being encoded in the equation).

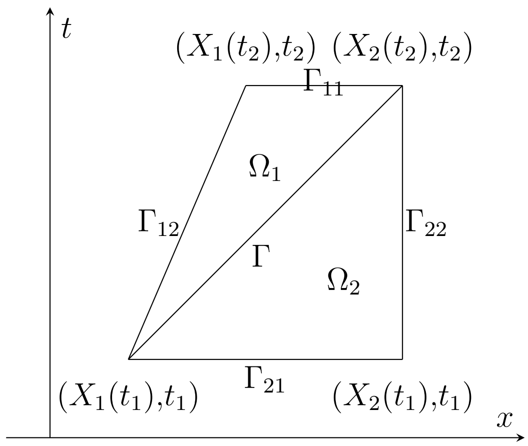
2.5.2 A space-time domain approach

There is a more elementary mathematical justification. Consider two particles at $X_1(t), X_2(t)$ within the time interval $[t_1, t_2]$. Assume the position of the shock satisfies

$$X(t) = \begin{cases} X_1(t) & \text{if } t = t_1 \\ X_2(t) & \text{if } t = t_2. \end{cases} \qquad (2.5.5)$$

For simplicity, also assume that

$$(h, u)(x, t) = \begin{cases} (h_+, u_+) & \text{if } x < X(t) \\ (h_-, u_-) & \text{if } x > X(t) \end{cases} \qquad (2.5.6)$$



Let A, B be a generic entropy-flux pair. Divergence theorem gives that

$$\begin{aligned} 0 = \int_{\Omega_1} A_t + B_x &= \int_{X_1(t_2)}^{X_2(t_2)} A_- \, dx + \int_{\Gamma_{12}} -\frac{1}{\sqrt{1+u_-^2}} B_- + \frac{u_-}{\sqrt{1+u_-^2}} A_- \, dS \\ &\quad + \int_{\Gamma} \frac{1}{\sqrt{1+s^2}} B_- - \frac{s}{\sqrt{1+s^2}} A_- \, dS \end{aligned} \tag{2.5.7}$$

and

$$\begin{aligned} 0 = \int_{\Omega_2} A_t + B_x &= \int_{X_1(t_1)}^{X_2(t_1)} -A_+ \, dx + \int_{\Gamma_{22}} \frac{1}{\sqrt{1+u_+^2}} B_+ - \frac{u_+}{\sqrt{1+u_+^2}} A_+ \, dS \\ &\quad + \int_{\Gamma} -\frac{1}{\sqrt{1+s^2}} B_+ + \frac{s}{\sqrt{1+s^2}} A_+ \, dS \end{aligned} \tag{2.5.8}$$

Adding the above two, we obtain

$$\begin{aligned} 0 &= \int_{X_1(t_2)}^{X_2(t_2)} A_- \, dx - \int_{X_1(t_1)}^{X_2(t_1)} A_+ \, dx + \int_{\Gamma_{12}} \frac{u_- A_- - B_-}{\sqrt{1+u_-^2}} \, dS \\ &\quad + \int_{\Gamma_{22}} \frac{B_+ - u_+ A_+}{\sqrt{1+u_+^2}} \, dS + \int_{\Gamma} \frac{-[B] + s[A]}{\sqrt{1+s^2}} \, dS \\ &= A_-(X_2(t_2) - X_1(t_2)) - A_+(X_2(t_1) - X_1(t_1)) + (t_2 - t_1)[(s-u)A] \end{aligned} \tag{2.5.9}$$

which would lead to mass and momentum conservation for proper entropy-flux pairs. For energy, one extra dissipation term would pop out as expected.

2.5.3 A physics approach

We make the same assumptions as we did above including (2.5.5) and (2.5.6).

$$t_2 - t_1 = \frac{X_2(t_1) - X_1(t_1)}{s - u_+} \tag{2.5.10}$$

implies that

$$\begin{aligned} X_2(t_2) - X_1(t_2) &= X_2(t_1) - X_1(t_1) - (u_- - u_+)(t_2 - t_1) \\ &= X_2(t_1) - X_1(t_1) - (u_- - u_+)\frac{X_2(t_1) - X_1(t_1)}{s - u_+} \\ &= \frac{s - u_-}{s - u_+}(X_2(t_1) - X_1(t_1)) \qquad (2.5.11) \end{aligned}$$

Rankine-Hugoniot (2.3.1) says

$$\frac{s - u_-}{s - u_+} = \frac{h_+}{h_-} \tag{2.5.12}$$

hence

$$h_-(X_2(t_2) - X_1(t_2)) = h_+(X_2(t_1) - X_1(t_1)) \tag{2.5.13}$$

which is mass conservation.

Momentum involves one more object: the forces acting on two sides of the fluids.

$$\begin{aligned} &\text{Accumulated Force} && F = \frac{1}{2}\Big(gh_-^2 - gh_+^2\Big)\frac{X_2(t_1) - X_1(t_1)}{s - u_+} \\ &\text{Momentum before:} && M_b = (X_2(t_1) - X_1(t_1))h_+u_+ \\ &\text{Momentum after:} && M_a = (X_2(t_1) - X_1(t_1))\frac{h_+}{h_-}h_-u_-. \end{aligned}$$

The momentum density dissipated through the shock is

$$\frac{F + M_b - M_a}{X_2(t_1) - X_1(t_1)} = \frac{gh_-^2 - gh_+^2}{2(s - u_+)} + h_+u_+ - h_+u_- = 0. \tag{2.5.14}$$

To compute energy,

$$\begin{aligned} &\text{Accumulated Work:} && W = \frac{1}{2}gh_-^2(X_2(t_1) - X_1(t_1))\frac{u_-}{s - u_+} \\ &\text{Energy before:} && E_b = (X_2(t_1) - X_1(t_1))\Big(\frac{1}{2}h_+u_+^2 + \frac{1}{2}gh_+^2\Big) \end{aligned}$$

$$\text{Energy after:} \qquad E_a = (X_2(t_1) - X_1(t_1))\frac{h_+}{h_-}\left(\frac{1}{2}h_- u_-^2 + \frac{1}{2}gh_-^2\right).$$

Hence the energy density dissipated through the shock is

$$\begin{aligned}\frac{W + E_b - E_a}{X_2(t_1) - X_1(t_1)} &= \frac{1}{2}h_+ u_+^2 + \frac{1}{2}gh_+^2 + \frac{1}{2}gh_-^2\frac{u_-}{s - u_+} - \frac{h_+}{h_-}\left(\frac{1}{2}h_- u_-^2 + \frac{1}{2}gh_-^2\right) \\ &= (t_2 - t_1)(s[\mathcal{E}^0] - [\mathcal{Q}^0]) > 0. \qquad (2.5.15)\end{aligned}$$

Chapter 3

Derivation of the Regularized Shallow-water System

A major inconvenience when tackling the classical shallow-water system (2.1.30) and (2.1.31) is that the weak solutions (with entropy condition to guarantee the uniqueness) corresponding to shock waves lack regularity, which can be problematic, in particular for computations using spectral methods. Several methods have been introduced to regularize such equations and, in particular, to avoid the formation of sharp discontinuous shocks.

An excellent example of such a regularization was proposed by J. Leray [35] in the context of incompressible Navier-Stokes equations. His theoretical program consisted in showing the existence of solutions in regularized equations, subsequently taking the limit $\varepsilon \to 0$ (ε a small regularizing parameter) to obtain weak solutions of Navier Stokes.

A method of regularization consists in first adding an artificial viscosity into the equations, and in taking the limit of vanishing viscosity in a second time. This method was introduced by von Neumann and Richtmyer [46]. It allows to generalize the classical concept of a solution and to prove eventually the uniqueness, existence and stability results for viscous regularized solutions [4, 14]. Due to the added viscosity, the energy is no longer conserved, which can be a serious drawback for some applications, for instance for long time simulations when the shocks represent (unresolved) small scale phenomena that are not dissipative.

Another regularization consists in adding weak dispersive effects to the equations [30]. As shown by Lax and Levermore [33], the dispersive regularization is not always sufficient to obtain a reasonable limit to weak entropy solutions as the dispersion vanishes. Consequently, a more successful approach to study non-classical shock waves is to consider the combined dispersive-diffusive approximations [20]. Also, the added dispersion can generate high-frequency oscillations that must be resolved by the numerical scheme, resulting in a significant increase in the computational time. Nonlinear diffusive-dispersive regularizations for the scalar case were considered in [1]. The main goal was to obtain a regularized model which admits the existence of classical solutions globally in time.

Yet another, less known, regularization inspired by Leray's method [35], filters the velocity field such that the resulting equations are non-dissipative and non-dispersive. Such regularizations have been proposed for the Burgers [3], for isentropic Euler [2,42,43] and other [6] equations. In the literature, this regularization method appears with various denominations, such as *Leray-type regularization, α-regularization* and *Helmholtz regularization*. A drawback of this method is that the regularized (then smooth) shocks propagate at a speed different than that of the original equations.

Lastly, Clamond and Dutykh proposed in [9] a new type of regularization to the classical shallow-water system. They call it the regularized shallow-water (Airy or Saint-Venant) system. We will discuss the derivation of several variants of the Serre-Green-Naghdi system, and from there derive the regularized shallow-water system. Most derivations and ideas in this subsection are from [10].

3.1 Approximations to the water wave equation in shallow water regime

Due to the complexity of Euler equations, many approximate models have been derived in various wave regimes including the classical shallow-water system (2.1.30)–(2.1.31). Once a new mathematical model is proposed, the limits of its applicability have to be determined. In shallow water, the main restriction comes from the shallowness parameter

$$\beta = \frac{h_0}{l} \ll 1,$$

where h_0 is the water depth and l is the characteristic wavelength. Restrictions on the free surface elevation are characterized by the dimensionless parameter $\alpha = a/h_0$ where a is the amplitude of the free surface. Many approximate equations have been derived for waves in the shallow water regime ($\beta \ll 1$). Depending on the size of α, one can identify the following important sub-regimes for which a rich variety of asymptotic models can be derived:

- Large amplitude models. If no assumption is made on the nonlinearity parameter α (i.e. $0 \leqslant \alpha \leqslant 1$), one obtains at first order (with respect to β^2) the shallow-water (or Saint-Venant) system, and at second order the Green-Naghdi (or Serre, or fully nonlinear Boussinesq) system.

- Medium wave models. If $\beta^2 \ll \alpha \ll \beta$, one obtains at the second order (with respect to β^2) the Camassa-Holm equation for horizontal velocity component evaluated at a specific depth.

- Long wave models. If in addition it is assumed that $\alpha \leqslant \beta^2$, then the Green-Naghdi system can be simplified into the Boussinesq system. If one further assumes $\alpha = \beta^2$ and focuses on unidirectional waves, the KdV equation can be derived from Boussinesq system.

We will utilize the structure of the Green-Naghdi system to seek a regularization of the classical shallow-water system and create another system in the shallow-water regime.

3.2 The Classical Serre-Green-Naghdi system

We have seen the Lagrangian (2.1.29) to the classical shallow water system, in which 0-th order approximation of u and v from (2.1.32)–(2.1.33) is considered. In order to encode more nonlinearity, all terms no less than β would be considered here and hence the kinetic energy density (using notation without tilde) is written as

$$\frac{\mathcal{K}_4}{\rho} = \int_0^h \frac{1}{2}(u^2 + u_x^2 y^2) + O(\beta^4)c_0^2 h_0 = \frac{1}{2}hu^2 + \frac{1}{6}h^3 u_x^2 + O(\beta^4)c_0^2 h_0. \tag{3.2.1}$$

where the subscript 4 in $\mathcal{K}_4$ indicates that all terms of order $O(\beta^4)$ or higher are dropped. The corresponding Lagrangian density takes the form

$$\begin{aligned} \frac{\mathcal{L}_4}{\rho} &= \frac{\mathcal{K}_4 - \mathcal{V}}{\rho} + (h_t + (hu)_x)\phi \\ &= \frac{1}{2}hu^2 + \frac{1}{6}h^3 u_x^2 - \frac{1}{2}gh^2 + (h_t + (hu)_x)\phi + O(\beta^4)c_0^2 h_0, \end{aligned} \tag{3.2.2}$$

where ϕ is a Lagrange multiplier to enforce the mass conservation equation. The Euler-Lagrange equations of $\mathcal{L}_4$ give

$$\begin{aligned} \delta\phi : &\quad 0 = h_t + (hu)_x, \\ \delta u : &\quad 0 = hu - \left(\frac{1}{3}h^3 u_x\right)_x - h\phi_x + O(\beta^4)h_0 c_0, \\ \delta h : &\quad 0 = \frac{1}{2}u^2 + \frac{1}{2}h^2 u_x^2 - gh - \phi_t - u\phi_x + O(\beta^4)c_0^2, \end{aligned} \tag{3.2.3}$$

hence we have

$$\phi_x = u - \frac{1}{3}h^{-1}(h^3 u_x)_x + O(\beta^4)c_0, \tag{3.2.4}$$

$$\phi_t = -\frac{1}{2}u^2 + \frac{1}{2}h^2 u_x^2 - gh + \frac{1}{3}h^{-1}u(h^3 u_x)_x + O(\beta^4)c_0^2. \tag{3.2.5}$$

Differentiating the first equation with respect to t and the second equation with respect to x and eliminating ϕ, one obtains the surface tangential velocity equation

$$\left(u - \frac{1}{3}h^{-1}(h^3 u_x)_x\right)_t + \left(\frac{1}{2}u^2 + gh - \frac{1}{2}h^2 u_x^2 - \frac{1}{3}h^{-1}u(h^3 u_x)_x\right)_x = O(\beta^5)g \tag{3.2.6}$$

which can be simplified to

$$u_t + uu_x + gh_x + \frac{1}{3}h^{-1}(h^2\gamma)_x = O(\beta^5)g. \tag{3.2.7}$$

Here γ is the material derivative of the vertical velocity at the free surface

$$\gamma \stackrel{\text{def}}{=} \frac{D\overline{v}}{Dt}\Big|_{y=h} = (\overline{v}_t + \nabla\overline{\varphi}\cdot\nabla\overline{v})|_{y=h} = h(u_x^2 - u_{xt} - uu_{xx}) + O(\beta^4)g. \tag{3.2.8}$$

where we used (2.1.12) and (2.1.32)–(2.1.33) for approximation.

Equations (3.2.3), (3.2.7) and (3.2.8) comprise the Serre equations. We refer to these equations as the Green-Naghdi system, or the *classical Serre-Green-Naghdi system* (cSGN). There are of course other ways of deriving this system, e.g. see [38] for deriving Hamiltonian structure of the cSGN system from the water wave problem using the Taylor expansion of Dirichlet-Neumann operator.

Momentum conservation equation can also be derived as

$$(hu)_t + (hu^2 + \frac{1}{2}gh^2 + \frac{1}{3}h^2\gamma)_x = O(\beta^5)h_0 g \tag{3.2.9}$$

which also takes the form

$$\left(hu - \frac{1}{3}(h^3u_x)_x\right)_t + \left(hu^2 + \frac{1}{2}gh^2 - \frac{2}{3}h^3u_x^2 - \frac{1}{3}h^3uu_{xx} - h^2h_xuu_x\right)_x = O(\beta^5)h_0g. \tag{3.2.10}$$

Conservation of energy is written as

$$\left(\frac{1}{2}hu^2 + \frac{1}{6}h^3u_x^2 + \frac{1}{2}gh^2\right)_t + \left(hu\left(\frac{1}{2}u^2 + gh + \frac{1}{6}h^2u_x^2 + \frac{1}{3}h\gamma\right)\right)_x = O(\beta^5)h_0c_0g. \tag{3.2.11}$$

Lastly, to see that (3.2.6) is the conservation equation for surface tangential velocity, one sets

$$\tilde{\varphi}(x,t) = \overline{\varphi}(x, h(x,t), t) \tag{3.2.12}$$

then the kinematic condition and Bernoulli equation at the free surface imply that

$$\tilde{\varphi}_x = \overline{u} + \overline{v}h_x$$

$$\begin{aligned} \tilde{\varphi}_t &= \overline{\varphi}_t + \overline{v}h_t = \overline{\varphi}_t + \overline{v}(\overline{v} - \overline{u}h_x) = \overline{\varphi}_t + \overline{v}^2 - \overline{u}(\overline{v}h_x) \\ &= -\frac{1}{2}\overline{u}^2 + \frac{1}{2}\overline{v}^2 - gh - \overline{u}(\tilde{\varphi}_x - \overline{u}). \end{aligned} \tag{3.2.13}$$

Differentiating with respect to x gives

$$(\tilde{\varphi}_x)_t + (-\frac{1}{2}\overline{u}^2 - \frac{1}{2}\overline{v}^2 + gh + \overline{u}\tilde{\varphi}_x)_x = 0. \tag{3.2.14}$$

Note that from (2.1.32) and (2.1.33), one has that on the free surface

$$\overline{u} = u - \frac{1}{3}u_{xx}h^2 + O(\beta^4)c_0, \quad \overline{v} = -u_x h + O(\beta^3)c_0, \tag{3.2.15}$$

and

$$\tilde{\varphi}_x = \overline{u} + \overline{v}h_x = u - \frac{1}{3}u_{xx}h^2 - u_x h h_x + O(\beta^4)c_0 = u - \frac{1}{3}h^{-1}(h^3 u_x)_x + O(\beta^4)c_0. \tag{3.2.16}$$

So one recovers (3.2.6) when approximated up to $O(\beta^5)$ terms.

3.3 The Modified Serre-Green-Naghdi system

In the velocity equation (3.2.7), one can check that the term $\frac{1}{3}h^{-1}(h^2\gamma)_x$ is $O(\beta^3)g$. Hence (2.1.31) and (3.2.7) are consistent. Either one implies that

$$(u_t + uu_x + gh_x)_x = O(\beta^4)\frac{g}{h_0}. \tag{3.3.1}$$

Note that in the computation (3.2.8), $O(\beta^4)$ terms are neglected, so we can add the above expression onto γ,

$$\gamma = h(u_x^2 - u_{xt} - uu_{xx}) + \theta h(u_t + uu_x + gh_x)_x + O(\beta^4)g. \tag{3.3.2}$$

Thus, (3.2.3), (3.2.7) and (3.3.2) comprise the *modified Serre-Green-Naghdi system*. This system admits a secondary relation

$$\begin{aligned} &\Big(hu + \frac{1}{3}(\theta-1)(h^3u_x)_x\Big)_t + \Big(hu^2 + \frac{1}{2}gh^2 + \frac{1}{3}\theta g h^3 h_{xx} \\ &\quad + \frac{2}{3}(2\theta-1)h^3u_x^2 + \frac{1}{3}(\theta-1)h^3uu_{xx} + (\theta-1)h^2h_x uu_x\Big)_x = O(\beta^5)h_0 g \end{aligned} \tag{3.3.3}$$

and yet has no conservation equations analogous to (3.2.6) and (3.2.11). Particularly, this means that the energy is not conserved for mSGN system and that a variational principle cannot be obtained if $\beta \neq 0$.

3.4 The Improved and generalized Serre-Green-Naghdi system

Here we derive another modification of the Serre's equations that have a free parameter and conserves both the momentum and the energy. Instead of modifying γ or the equations, we modify the Lagrangian density (3.2.2) with observation (3.3.1)

$$\frac{\tilde{\mathcal{L}}^i}{\rho} \stackrel{\text{def}}{=} \frac{\mathcal{L}_{1/2}}{\rho} + \frac{1}{12}\theta h^3(u_t + uu_x + gh_x)_x + O(\beta^4)c_0^2 h_0.$$

For the added term, integrating by parts and using (3.2.3) to remove h_t, one gets

$$h^3(u_t + uu_x + gh_x)_x = (h^3 u_x)_t + (h^3 uu_x)_x + (gh^3 h_x)_x + 3h^3 u_x^2 - 3gh^2 h_x^2. \tag{3.4.1}$$

Hence omitting full-derivative terms $(\cdot)_t, (\cdot)_x$ we obtain that

$$\begin{aligned}\frac{\mathcal{L}^i}{\rho} &\stackrel{\text{def}}{=} \frac{\mathcal{L}_{1/2}}{\rho} + \frac{1}{4}\theta(h^3 u_x^2 - gh^2 h_x^2) + O(\beta^4)c_0^2 h_0 \\ &= \frac{1}{2}hu^2 + \left(\frac{1}{6} + \frac{1}{4}\theta\right)h^3 u_x^2 - \frac{1}{2}gh^2\left(1 + \frac{1}{2}\theta h_x^2\right) + (h_t + (hu)_x)\phi + O(\beta^4)c_0^2 h_0\end{aligned}$$

where θ is a free parameter. $\mathcal{L}^i$ and $\mathcal{L}_{1/2}$ have the same order of approximation and $\mathcal{L}_{1/2}$ is a special case when $\theta = 0$. The Euler-Lagrange equations of $\mathcal{L}^i$ give the *improved Serre-Green-Naghdi system* as discovered in [10].

We consider a natural two-parameter extension of $\mathcal{L}_i$ of the form

$$\mathcal{L}^g \stackrel{\text{def}}{=} \frac{1}{2}hu^2 + \left(\frac{1}{6} + \frac{1}{4}\theta_1\right)h^3 u_x^2 - \frac{1}{2}gh^2\left(1 + \frac{1}{2}\theta_2 h_x^2\right) + (h_t + (hu)_x)\phi. \tag{3.4.2}$$

Here the density ρ is omitted for simplicity and θ_1, θ_2 are free parameters at our disposal. This Lagrange density obviously generalizes the above ones. The corresponding Euler-Lagrange equations are

$$\begin{aligned}\delta\phi :&\quad 0 = h_t + (hu)_x, \\ \delta u :&\quad 0 = hu - h\phi_x - \left(\frac{1}{3} + \frac{1}{2}\theta_1\right)(h^3 u_x)_x, \\ \delta h :&\quad 0 = \frac{1}{2}u^2 - gh - \phi_t - u\phi_x\end{aligned} \tag{3.4.3}$$

$$+\Big(\frac{1}{2}+\frac{3}{4}\theta_1\Big)h^2u_x^2-\frac{1}{2}\theta_2ghh_x^2+\frac{1}{2}\theta_2g(h^2h_x)_x.$$

So

$$\phi_x = u-h^{-1}\Big(\frac{1}{3}+\frac{1}{2}\theta_1\Big)(h^3u_x)_x, \tag{3.4.4}$$

$$\phi_t = \frac{1}{2}u^2-gh-u\phi_x+\Big(\frac{1}{2}+\frac{3}{4}\theta_1\Big)h^2u_x^2+\frac{1}{2}\theta_2gh(h_x^2+hh_{xx}). \tag{3.4.5}$$

With the above formulas, another interpretation of conservation of surface tangential velocity is formally written

$$(\phi_x)_t+(-\phi_t)_x=0. \tag{3.4.6}$$

Eliminating ϕ_{xt}, one obtains the non-conservative momentum equation

$$u_t+uu_x+gh_x+\frac{1}{3}h^{-1}(h^2\Gamma)_x=0, \tag{3.4.7}$$

where

$$\Gamma \stackrel{\text{def}}{=} \Big(1+\frac{3}{2}\theta_1\Big)h(u_x^2-u_{xt}-uu_{xx})-\frac{3}{2}\theta_2g(hh_{xx}+\frac{1}{2}h_x^2). \tag{3.4.8}$$

Equations (3.4.3) and (3.4.7) comprise the *generalized Serre-Green-Naghdi system.* The quantity Γ is a generalization of γ, or a relaxed version of material derivative at the free surface. One can derive the momentum conservation equation

$$0=(hu)_t+(hu^2+\frac{1}{2}gh^2+\frac{1}{3}h^2\Gamma)_x. \tag{3.4.9}$$

Energy conservation equation is written as

$$0=\mathcal{E}_t^g+\mathcal{Q}_x^g=0 \tag{3.4.10}$$

with

$$\mathcal{E}^g = \frac{1}{2}hu^2+\Big(\frac{1}{6}+\frac{1}{4}\theta_1\Big)h^3u_x^2+\frac{1}{2}gh^2\Big(1+\frac{1}{2}\theta_2h_x^2\Big),$$

$$\mathcal{Q}^g = hu\Big(\frac{1}{2}u^2+\Big(\frac{1}{6}+\frac{1}{4}\theta_1\Big)h^2u_x^2+gh\Big(1+\frac{1}{4}\theta_2h_x^2\Big)+\frac{1}{3}h\Gamma\Big)+\frac{1}{2}\theta_2gh^3h_xu_x.$$

Thus, for any choice of parameter θ_j, gSGN formally conserves mass, momentum and energy.

3.5 Linear Approximation

The generalized SGN equations (3.4.3)–(3.4.7) are linearized about a constant state (h_0, u_0) as

$$h_t + u_0 h_x + h_0 u_x = 0, \tag{3.5.1}$$

$$u_t + u_0 u_x + g h_x - h_0^2 \Big((\frac{1}{3} + \frac{1}{2}\theta_1)(u_{xxt} + u_0 u_{xxx}) + \frac{1}{2}\theta_2 g h_{xxx} \Big) = 0. \tag{3.5.2}$$

A nontrivial solution proportional to $e^{ikx - i\omega t}$ necessarily have

$$\begin{aligned} \frac{(w - u_0 k)^2}{g h_0 k^2} &= \frac{1 + \frac{1}{2}\theta_2 h_0^2 k^2}{1 + (\frac{1}{3} + \frac{1}{2}\theta_1) h_0^2 k^2} \\ &= 1 - \Big(\frac{1}{3} + \frac{1}{2}(\theta_1 - \theta_2)\Big)(h_0 k)^2 + \Big(\frac{1}{3} + \frac{1}{2}\theta_1\Big)\Big(\frac{1}{3} + \frac{1}{2}(\theta_1 - \theta_2)\Big)(h_0 k)^4 \\ &\quad -\Big(\frac{1}{3} + \frac{1}{2}\theta_1\Big)^2 \Big(\frac{1}{3} + \frac{1}{2}(\theta_1 - \theta_2)\Big)(h_0 k)^6 + \cdots \end{aligned} \tag{3.5.3}$$

This should be compared with the dispersion relation of the water water equation

$$\frac{(w - u_0 k)^2}{g h_0 k^2} = \frac{\tanh(h_0 k)}{h_0 k} = 1 - \frac{1}{3}(h_0 k)^2 + \frac{2}{15}(h_0 k)^4 - \frac{17}{315}(h_0 k)^6 + \cdots \tag{3.5.4}$$

One could choose $\theta_1 = \theta_2 = 2/15$ (which reduces to the case of the improved SGN) so that the linear dispersion relations coincide up to the highest order of $(h_0 k)^4$.

3.6 The regularized shallow-water system

We further consider the non-dispersive version of the generalized SGN system, which will leads us to a regularized modification of classical shallow-water (Airy or Saint-Venant) system.

From dispersion relation (3.5.3), a non-dispersive model is obtained by taking $\frac{1}{3} + \frac{1}{2}\theta_1 = \frac{1}{2}\theta_2$. Choosing

$$\theta_1 \stackrel{\text{def}}{=} 2\varepsilon - \frac{2}{3}, \quad \theta_2 = 2\varepsilon, \quad \mathcal{R} \stackrel{\text{def}}{=} \frac{1}{3\varepsilon}\Gamma$$

where ε being a free parameter, one obtains the Lagrangian density

$$\mathcal{L}^\varepsilon \stackrel{\text{def}}{=} \frac{1}{2} h u^2 + \frac{1}{2}\varepsilon h^3 u_x^2 - \frac{1}{2} g h^2 (1 + \varepsilon h_x^2) + (h_t + (hu)_x)\phi \tag{3.6.1}$$

and the resulting equations

$$h_t + (hu)_x = 0\,, \tag{3.6.2}$$

$$(hu)_t + (hu^2 + \frac{1}{2}gh^2 + \varepsilon h^2\mathcal{R})_x = 0, \tag{3.6.3}$$

$$\mathcal{R} \stackrel{\text{def}}{=} h(u_x^2 - u_{xt} - uu_{xx}) - g\left(hh_{xx} + \frac{1}{2}h_x^2\right)\,. \tag{3.6.4}$$

which we called regularized shallow-water (Airy or Saint-Venant) system. Conservation of surface tangential velocity takes the form

$$(\phi_x)_t + \left(u\phi_x - \frac{1}{2}u^2 + gh - \frac{3}{2}\varepsilon h^2u_x^2 - \varepsilon gh(hh_{xx} + h_x^2)\right)_x = 0 \tag{3.6.5}$$

with $\phi_x = u - \varepsilon h^{-1}(h^3u_x)_x$. Conservation of momentum (3.6.3) can also be written as

$$(h\phi_x)_t + \left(hu\phi_x + \frac{1}{2}gh^2 - \varepsilon h^2(2hu_x^2 + ghh_{xx} + \frac{1}{2}gh_x^2)\right)_x = 0. \tag{3.6.6}$$

Smooth solutions of these equations also satisfy a conservation law for energy, in the form

$$\mathcal{E}_t^\varepsilon + \mathcal{Q}_x^\varepsilon = 0\,, \tag{3.6.7}$$

where

$$\mathcal{E}^\varepsilon \stackrel{\text{def}}{=} \frac{1}{2}hu^2 + \frac{1}{2}gh^2 + \varepsilon\left(\frac{1}{2}h^3u_x^2 + \frac{1}{2}gh^2h_x^2\right)\,, \tag{3.6.8}$$

$$\mathcal{Q}^\varepsilon \stackrel{\text{def}}{=} \frac{1}{2}hu^3 + gh^2u + \varepsilon\left(\left(\frac{1}{2}h^2u_x^2 + \frac{1}{2}ghh_x^2 + h\mathcal{R}\right)hu + gh^3h_xu_x\right)\,. \tag{3.6.9}$$

It should be noted that this regularized system has the same order of approximation as Serre equations and that the classical shallow-water system is recovered if $\varepsilon = 0$.

The definition of $\mathcal{R}$ involves a time derivative, so the momentum equation (3.6.3) has two terms differentiated in time. It is natural to combine them and invert the spacial operator

$$\mathcal{I}_h = h - \varepsilon\partial_x \circ h^3 \circ \partial_x,\ \text{i.e.}\ \mathcal{I}_h(u) = hu - \varepsilon(h^3u_x)_x, \tag{3.6.10}$$

on both side of (3.6.3). Therefore, the system (3.6.2)–(3.6.3) can also be written as a system of evolution equations

$$h_t + (hu)_x = 0\,, \tag{3.6.11}$$

$$u_t + gh_x + uu_x + \varepsilon \mathcal{I}_h^{-1} \partial_x\Big(2h^3 u_x^2 - \frac{1}{2} g h^2 h_x^2\Big) = 0. \tag{3.6.12}$$

This form is particularly useful when analyzing the well-posedness of the system, see Chapter 5. Moreover, the above system recovers the classical shallow-water system except the ε term. So one can write the system in terms of the Riemann invariants and characteristic speeds of the classical shallow-water system. This method is introduced in Chapter 6 to prove the existence of blow-up phenomena.

Chapter 4

Weakly Singular Shock Profiles for the Regularized Shallow-water System

Our purpose here is to describe a novel wave-propagation mechanism. We shall show that the regularized shallow-water system (3.6.2)–(3.6.3) admit regularized shock-wave solutions with profiles that are *continuous but only piecewise smooth*, with derivatives having a weak singularity at a single point. Such a wave exists corresponding to every classical shallow-water shock. These waves are traveling-wave weak solutions of the rSV equations that conserve mass and momentum. They *dissipate energy at the singular point*, however, at the precise rate that the corresponding classical shock does.

We also find that the rSV equations admit weak solutions in the form of *cusped solitary waves*. These waves loosely resemble the famous 'peakon' solutions of the Camassa-Holm equation in the non-dispersive case [7]. One difference is that the wave slope of our cusped solitary waves becomes infinite approaching the crest, while that of a peakon remains finite. The rSV equations also loosely resemble various 2-component generalizations of the Camassa-Holm equation which have appeared in the literature—for a sample see [8, 23, 26, 27, 31]. One of the most well-studied of these is the integrable

2-component Camassa-Holm system appearing in [8, 27, 31],

$$h_t + (hu)_x = 0, \tag{4.0.1}$$

$$u_t + 3uu_x - u_{txx} - 2u_x u_{xx} - uu_{xxx} + ghh_x = 0, \tag{4.0.2}$$

which has been derived in the context of shallow-water theory by Constantin and Ivanov [13] (also see [25]). This system admits peakon-type solutions, and as noted in [16], it admits some degenerate front-type traveling wave solutions, which however necessarily have $h \to 0$ as either $x \to +\infty$ or $-\infty$.

Now consider any simple piecewise-constant shock-wave solution of the shallow water equations, in the form

$$(h, u) = \begin{cases} (h_-, u_-) & x < st \\ (h_+, u_+) & x > st \end{cases} \tag{4.0.3}$$

where s, $h_\pm$, and $u_\pm$ are constants with $h_\pm > 0$. Our goal in this section is to show that the regularized Saint-Venant equations (3.6.2)–(3.6.3) admit a corresponding traveling-wave solution having shock profile that is continuous and piecewise smooth.

4.1 Construction of shock profiles

Because both the rSV and shallow water equations are invariant under spatial reflection, we may assume the shock is a 2-shock without loss of generality. Moreover, the rSV equations and shallow water equations are invariant under the Galilean transformation taking

$$u \to u + s, \quad \partial_t \to -s\partial_x + \partial_t. \tag{4.1.1}$$

Thus it is natural to work in the frame of reference moving with the shock at speed s and seek a steady wave profile that is smooth except at the origin $x = 0$. Adopting the notation in (2.4.3) and writing $v = u - s$ for convenience, therefore we seek time-independent functions $h : \mathbb{R} \to (0, \infty)$ and $v : \mathbb{R} \to \mathbb{R}$ such that h and v are continuous, smooth except at $x = 0$, take the limiting values

$$(h, v) \to \begin{cases} (h_-, v_-) & x \to -\infty, \\ (h_+, v_+) & x \to +\infty, \end{cases} \tag{4.1.2}$$

and provide a weak solution of the steady rSV equations

$$(hv)_x = 0, \qquad \left(hv^2 + \frac{1}{2}gh^2 + \varepsilon \mathcal{R} h^2\right)_x = 0, \tag{4.1.3}$$

$$\mathcal{R} = hv_x^2 - hvv_{xx} - g(hh_{xx} + \frac{1}{2}h_x^2). \tag{4.1.4}$$

As is natural, we will find solutions whose derivatives approach zero as $x \to \pm\infty$. Thus upon integration we find that

$$hv = M\,, \tag{4.1.5}$$

$$hv^2 + \frac{1}{2}gh^2 + \varepsilon \mathcal{R} h^2 = N\,, \tag{4.1.6}$$

where M and N are the Rankine-Hugoniot constants defined in (2.4.4) and (2.4.5) and are given by (2.4.6) and (2.4.7), respectively.

Let us first work on the right half-line where $x > 0$. In terms of the dimensionless variables given by

$$H = \frac{h}{h_+}, \quad V = \frac{v}{v_+}, \quad z = \frac{x}{h_+},$$

and the squared Froude number on the right,

$$\mathcal{F}_+ = \frac{v_+^2}{gh_+}\,,$$

the equations take the form

$$HV = 1\,, \tag{4.1.7}$$

$$\mathcal{F}HV^2 + \frac{1}{2}H^2 + \varepsilon \mathcal{F} H^3 (V_z^2 - VV_{zz}) - \varepsilon(H^3 H_{zz} + \frac{1}{2}H^2 H_z^2) = \mathcal{F} + \frac{1}{2}. \tag{4.1.8}$$

(For simplicity we temporarily drop the subscript on $\mathcal{F}_+$ here.) Eliminating V we obtain a single equation for the dimensionless wave height H,

$$\frac{\mathcal{F}}{H} + \frac{1}{2}H^2 + \frac{\varepsilon \mathcal{F}}{H}(HH_{zz} - H_z^2) - \varepsilon(H^3 H_{zz} + \frac{1}{2}H^2 H_z^2) = \mathcal{F} + \frac{1}{2}. \tag{4.1.9}$$

Dividing this equation by H^2 we can rewrite it as

$$\frac{\mathcal{F}}{H^3}+\frac{1}{2}+\varepsilon\mathcal{F}(H^{-1}H_z)_zH^{-1}-\varepsilon(H^{\frac{1}{2}}H_z)_zH^{\frac{1}{2}}=\frac{\mathcal{F}+\frac{1}{2}}{H^2}\,. \tag{4.1.10}$$

Further multiplying by H_z one can integrate this equation to obtain

$$\varepsilon H_z^2=G(\mathcal{F},H)\stackrel{\text{def}}{=}\frac{(H-\mathcal{F})(H-1)^2}{H^3-\mathcal{F}}\,. \tag{4.1.11}$$

Here the integration constant is determined by requiring $H\to 1$ as $z\to\infty$.

In terms of the original dimensional variables this equation takes the form

$$\varepsilon h_x^2=G\left(\mathcal{F}_+,\frac{h}{h_+}\right)=\frac{(h-h_+\mathcal{F}_+)(h-h_+)^2}{h^3-h_+^3\mathcal{F}_+}\,. \tag{4.1.12}$$

On the left half-line where $x<0$, a similar integration procedure yields

$$\varepsilon h_x^2=G\left(\mathcal{F}_-,\frac{h}{h_-}\right)=\frac{(h-h_-\mathcal{F}_-)(h-h_-)^2}{h^3-h_-^3\mathcal{F}_-}\,, \tag{4.1.13}$$

with $\mathcal{F}_-=v_-^2/gh_-$. We note that these equations correspond to equation (29) of [9] with the appropriate choice of integration constants.

Recalling that we are dealing with a 2-shock for which $h_+<h_-$, we note

$$h_+^3\mathcal{F}_+=h_-^3\mathcal{F}_-=\frac{M^2}{g}=\frac{1}{2}(h_++h_-)h_+h_-\quad\in(h_+^3,h_-^3)\,. \tag{4.1.14}$$

Therefore

$$\mathcal{F}_-<1<\mathcal{F}_+\,, \tag{4.1.15}$$

and furthermore, the denominators in (4.1.12) and (4.1.13) vanish at the *same critical height* h_c satisfying $h_+<h_c<h_-$, where

$$h_c^3=\frac{1}{2}(h_++h_-)h_+h_-=\frac{M^2}{g}\,. \tag{4.1.16}$$

On the right half line where $x>0$, note the denominator in (4.1.12) changes sign from negative to positive as h increases from h_+ past the critical height h_c, while the numerator is negative for $h_+<h<h_+\mathcal{F}_+$. Because $h_+\mathcal{F}_+=h_c^3/h_+^2>h_c$, this means that the right-hand side of (4.1.12) changes sign as h increases past h_c: for h near h_c we have

$$G\left(\mathcal{F}_+, \frac{h}{h_+}\right) > 0 \quad \text{for } h < h_c, \qquad G\left(\mathcal{F}_+, \frac{h}{h_+}\right) < 0 \quad \text{for } h > h_c.$$

Thus a solution of (4.1.12) taking values between h_+ and h_- can exist only as long as $h < h_c$. Because we require $h \to h_+$ as $x \to +\infty$, such a solution must be monotone decreasing and satisfy

$$\sqrt{\varepsilon} h_x = -\sqrt{G(\mathcal{F}_+, h/h_+)}\,. \tag{4.1.17}$$

Actually, we have $h(x) = \eta_+(x/\sqrt{\varepsilon})$ for a unique continuous function $\eta_+ \colon [0,\infty) \to (0,\infty)$ which is a smooth decreasing solution of (4.1.17) with $\varepsilon = 1$ for $x > 0$ and satisfies

$$\eta_+(0) = h_c, \qquad \eta_+(x) \to h_+ \quad \text{as } x \to +\infty.$$

To see that this is true, one can separate variables in (4.1.17) and determine the solution implicitly according to the relation

$$\int_h^{h_c} \frac{dk}{\sqrt{G(\mathcal{F}_+, k/h_+)}} = \frac{x}{\sqrt{\varepsilon}}\,, \quad x \ge 0, \quad h \in (h_+, h_c], \tag{4.1.18}$$

since the integral converges on any interval $[h, h_c] \subset (h_+, h_c]$.

On the left half line where $x < 0$, the reasoning is similar. The numerator in (4.1.13) is positive for $h_- > h > h_-\mathcal{F}_-$ while the denominator changes sign from positive to negative as h decreases past the critical height h_c. The solution we seek takes values between h_- and h_c, satisfying

$$\sqrt{\varepsilon} h_x = -\sqrt{G(\mathcal{F}_-, h/h_-)}\,. \tag{4.1.19}$$

Again, we have $h(x) = \eta_-(x/\sqrt{\varepsilon})$ for a unique continuous function $\eta_- \colon (-\infty, 0] \to (0,\infty)$ which is a smooth decreasing solution of (4.1.19) with $\varepsilon = 1$ for $x < 0$ and satisfies

$$\eta_-(0) = h_c, \qquad \eta_- \to h_- \quad \text{as } x \to -\infty.$$

The solution is determined implicitly in this case according to the relation

$$\int_h^{h_c} \frac{dk}{\sqrt{G(\mathcal{F}_-, k/h_-)}} = \frac{x}{\sqrt{\varepsilon}}\,, \quad x < 0, \quad h \in (h_c, h_-). \tag{4.1.20}$$

Summary. Let us summarize: Given the 2-shock solution (4.0.3) of the shallow water equations, our corresponding weakly singular traveling wave solution of the rSV equations satisfies (4.1.2) and takes the form

$$h(x,t) = \begin{cases} \eta_+\left(\dfrac{x-st}{\sqrt{\varepsilon}}\right) & x \geq st, \\ \eta_-\left(\dfrac{x-st}{\sqrt{\varepsilon}}\right) & x < st, \end{cases} \qquad u(x,t) = s + \frac{M}{h}\,, \tag{4.1.21}$$

where $\eta_\pm$ are determined by h_+ and h_- implicitly from (4.1.18) and (4.1.20) respectively with $\varepsilon = 1$, using (4.1.14) to determine $\mathcal{F}_\pm$, and h_c is given by (4.1.16).

4.2 Behavior near the singular point and infinity

The nature of the singularity at $x = st$ for the solution above may be described as follows. For the function G in (4.1.12), because $h_+\mathcal{F}_+ = h_c^3/h_+^2$ we have

$$\frac{1}{G(\mathcal{F}_+, h/h_+)} = \frac{(h^3 - h_c^3)h_+^2}{(h_+^2 h - h_c^3)(h - h_+)^2} \sim K_+^2(h_c - h) \tag{4.2.1}$$

as $h \to h_c$, where

$$K_+^2 = \frac{3h_c h_+^2}{(h_c^2 - h_+^2)(h_c - h_+)^2}\,.$$

From this asymptotic description we infer from (4.1.18) that for small $x > 0$,

$$h_c - h \sim c_+ x^{2/3}, \qquad h_x \sim -\frac{2}{3}c_+ x^{-1/3}, \qquad h_{xx} \sim \frac{2}{9}c_+ x^{-4/3}, \tag{4.2.2}$$

where $c_+ = (2K_+\sqrt{\varepsilon}/3)^{-2/3}$.

A similar description holds on the other side of the singularity: From (4.1.13) we have

$$\frac{1}{G(\mathcal{F}_-, h/h_-)} = \frac{(h^3 - h_c^3)h_-^2}{(h_-^2 h - h_c^3)(h - h_-)^2} \sim K_-^2(h - h_c) \tag{4.2.3}$$

as $h \to h_c$, where

$$K_-^2 = \frac{3h_c h_-^2}{(h_-^2 - h_c^2)(h_c - h_-)^2}\,.$$

So for small $x < 0$,

$$h - h_c \sim c_-|x|^{2/3}, \qquad h_x \sim -\frac{2}{3}c_-|x|^{-1/3}, \qquad h_{xx} \sim \frac{2}{9}c_-|x|^{-4/3}, \tag{4.2.4}$$

where $c_- = (2K_-\sqrt{\varepsilon}/3)^{-2/3}$.

The behavior of v follows by differentiation from (4.1.5). Thus we see that h_x and v_x are square integrable in any neighborhood of $x = 0$ (and belong to L^p for $p < 3$), while h_{xx} and v_{xx} are not integrable functions. The singularities due to second derivatives in (4.1.6) cancel however (see below), to produce the constant value N. This yields a valid distributional solution of the steady rSV equations (4.1.3) written in conservation form.

As $x \to \pm\infty$, it is straightforward to check that the limits in (4.1.2) are achieved at an exponential rate.

4.3 Distributional derivatives

Because of the blow-up of h_x at the origin, the distributional derivative of h_x is no longer a classical function. Rather, it is a generalized function or a distribution which can be computed as follows.

We write h_{xx} to denote the distributional derivative of h_x and write $\overline{h_{xx}}$ for the classical derivative of h_x that is not defined at 0. Let $\varphi \in C_c^\infty(\mathbb{R})$ be a test function with support $\operatorname{supp}\varphi \subset (-L, L)$. Let τ be a subtracting operator acting on functions from $\mathbb{R}$ to $\mathbb{R}$ such that

$$\tau\varphi(x) = \varphi(x) - \varphi(0) \tag{4.3.1}$$

Then the distributional pairing of φ with the distribution h_{xx} is

$$\langle h_{xx}, \varphi\rangle \;=\; -\int_{\mathbb{R}} h_x\varphi_x\,dx = -\int_{\mathbb{R}} h_x(\tau\varphi)_x\,dx$$

$$
\begin{aligned}
&= -\lim_{\varepsilon\to 0^+}\left(\int_{-L}^{-\varepsilon} h_x(\tau\varphi)_x\,dx + \int_{\varepsilon}^{L} h_x(\tau\varphi)_x\,dx\right)\\
&= -h_x(-L)\varphi(0) + h_x(L)\varphi(0) + \int_{-L}^{L} \overline{h_{xx}}(\tau\varphi)\,dx
\end{aligned}
\tag{4.3.2}
$$

where in the last step we use the fact that $(\tau\varphi)(x) \sim x\varphi_x(0)$ when x is small and the fact that $\overline{h_{xx}}\tau\varphi$ is integrable near 0. Furthermore, the above equality is true for all L large enough, so sending L to infinity we have that

$$
\langle h_{xx}, \varphi\rangle = \int_{\mathbb{R}} \overline{h_{xx}}(\tau\varphi)\,dx. \tag{4.3.3}
$$

Due to this result, the distribution $h_{xx}(h-h_c)$ satisfies

$$
\begin{aligned}
\langle h_{xx}(h-h_c), \varphi\rangle &= \langle h_{xx}, (h-h_c)\varphi\rangle\\
&= \int_{\mathbb{R}} \overline{h_{xx}}\tau((h-h_c)\varphi)\,dx = \int_{\mathbb{R}} \overline{h_{xx}}(h-h_c)\varphi\,dx\\
&= \left\langle \overline{h_{xx}}(h-h_c), \varphi\right\rangle
\end{aligned}
\tag{4.3.4}
$$

where the first line is justified by the fact that h_{xx} is a continuous linear functional on $W^{1,p}(\mathbb{R})$ for any $p \in (1,\infty)$. This implies that in the sense of distributions,

$$
h_{xx}(h-h_c) = \overline{h_{xx}}(h-h_c), \tag{4.3.5}
$$

where the right-hand side is a locally integrable function.

From this we can find a locally integrable representation of the quantity $h^2\mathcal{R}$ from (4.1.4). Differentiating (4.1.5) twice and multiplying by $h^2 v$, we find $h^3 v_x^2 = M^2 h_x^2/h$ and

$$
-h^3 v v_{xx} = M^2\left(h_{xx} - \frac{2h_x^2}{h}\right).
$$

Because $M^2 = g h_c^3$, using (4.3.5) it follows

$$
h^2\mathcal{R} = g(h_c^3 - h^3)\overline{h_{xx}} - \frac{g}{2h}(2h_c^3 + h^3)h_x^2. \tag{4.3.6}
$$

So we conclude that the singularities appearing in h_{xx} and v_{xx} do cancel each other in a way that makes the stationary momentum flux locally integrable with distributional derivative 0.

Another way to see this cancellation is that the singular terms in $h^2\mathcal{R}$ sum up to give

$$\begin{aligned} h^3vv_{xx} + gh^3h_{xx} &= \Big(h^3vv_x + gh^3h_x\Big)_x - (h^3v)_xv_x - g(h^3)_xh_x \\ &= \Big(h^3v\Big(-\frac{M}{h^2}h_x\Big) + gh^3h_x\Big)_x - (h^3v)_xv_x - g(h^3)_xh_x \\ &= g\Big((h^3 - h_c^3)h_x\Big)_x - (h^3v)_xv_x - g(h^3)_xh_x\,, \end{aligned} \tag{4.3.7}$$

in which every term is a locally integrable function.

4.4 Energy dissipation of weakly singular waves.

Here our aim is to show that the regularized shock-wave solutions of the rSV equations that correspond to the simple shallow-water shock (4.0.3) satisfy the distributional identity

$$\mathcal{E}_t^\varepsilon + \mathcal{Q}_x^\varepsilon = \mu \tag{4.4.1}$$

where the dissipation measure μ is a constant multiple of 1-dimensional Hausdorff measure restricted to the simple shock curve $\{(x,t) : x = st\}$, satisfying

$$\mathcal{D} = \frac{d\mu}{dt} = \pm\frac{1}{4}g\gamma(h_+ - h_-)^3 < 0, \tag{4.4.2}$$

exactly the same as the simple shallow-water shock in (4.0.3).

Indeed, the steady solution constructed above is a smooth solution of the rSV equations (3.6.2)–(3.6.3) on both the right and left half-lines, hence satisfies the conservation law (3.6.7) except at $x = 0$. In this time-independent situation this means

$$\mathcal{Q}_x^\varepsilon = 0, \qquad x \in \mathbb{R} \setminus \{0\}.$$

Now, integration of this equation separately on the right and left half lines yields

$$\mathcal{Q}^\varepsilon = \begin{cases} \mathcal{Q}_-\,, & x < 0, \\ \mathcal{Q}_+\,, & x > 0, \end{cases}$$

where the constants $\mathcal{Q}_\pm$ can be evaluated by taking $x \to \pm\infty$ in the expression for $\mathcal{Q}^\varepsilon$ in (3.6.7) and invoking the limits in (4.1.2). The result is that the constants $\mathcal{Q}_\pm$ take the same values as appear in (2.4.2) for the simple shallow-water shock. Namely,

$$\mathcal{Q}_\pm = \frac{1}{2} h_\pm v_\pm^3 + g h_\pm^2 v_\pm \,.$$

Therefore, by the same calculation that leads to (2.4.8), the weak derivative of $\mathcal{Q}^\varepsilon$ on all of $\mathbb{R}$ is a multiple of the Dirac delta measure δ_0 at $x = 0$, satisfying

$$\mathcal{Q}_x^\varepsilon = (\mathcal{Q}_+ - \mathcal{Q}_-)\delta_0 = \mathcal{D}\,\delta_0, \tag{4.4.3}$$

where $\mathcal{D}$ is the same as in (2.4.8). By undoing the Galilean transformation to the frame moving with the simple shock speed, we obtain (4.4.1) with dissipation measure μ exactly as claimed above.

4.5 Cusped solitary waves for the regularized system

The construction of weakly singular shock profiles in the previous section also enables us to describe cusped solitary waves for the rSV equations. These are weak traveling-wave solutions whose limits as $x \to -\infty$ are the same as those as $x \to +\infty$.

The point is that weak solutions of the steady rSV equations (4.1.3)–(4.1.4) can be constructed *by reflection* from either piece $\eta_\pm$ of the 2-shock profile in the previous section. For each of these pieces, the quantities on the left-hand sides in (4.1.5) and (4.1.6) are locally integrable (in total, though not term-wise) and indeed constant on $\mathbb{R} \setminus \{0\}$. Thus the construction above yields two valid distributional solutions of the steady rSV equations with height profiles

$$h(x,t) = \eta_\pm\left(\frac{|x - st|}{\sqrt{\varepsilon}}\right), \tag{4.5.1}$$

respectively satisfying $h(x,t) \to h_\pm$ as $|x| \to \infty$. The energy of these solitary wave solutions satisfies the conservation law (3.6.7) without alteration.

4.5.1 Solitary waves of elevation

We note that for the solution using η_+, the value of h_- has no direct interpretation in terms of the wave shape. However, from (4.2.1) we see that the solitary-wave height profile with the + sign can be determined from any independently chosen values of $h_\infty \stackrel{\text{def}}{=} h_+$ and h_c with

$$0 < h_\infty < h_c.$$

Here h_c is the maximum height of the wave and h_∞ is the limiting value at ∞. The wave everywhere is a wave of elevation, with $h_\infty < h(x,t) \le h_c$, determined implicitly as in (4.1.18) and (4.2.1) by

$$\int_h^{h_c} \left(\frac{h_c^3 - k^3}{h_c^3 - h_\infty^2 k} \right)^{\frac{1}{2}} \frac{h_\infty}{k - h_\infty}\, dk = \frac{|x - st|}{\sqrt{\varepsilon}}\,, \qquad x \in \mathbb{R}, \quad h \in (h_\infty, h_c]. \tag{4.5.2}$$

It is natural for solitary waves to consider $u_+ = 0$ to be the limiting velocity as $|x| \to \infty$ in the original frame. Then by (2.3.3), $v_+ = -s = -\gamma h_-$, whence we find using (4.1.16) that $\gamma = \sqrt{gh_c}\, h_c/(h_+ h_-)$ and

$$s = \sqrt{gh_c}\, \frac{h_c}{h_\infty}\,. \tag{4.5.3}$$

This determines the velocity profile according to

$$u(x,t) = s + \frac{M}{h} = s\left(1 - \frac{h_\infty}{h}\right)\,. \tag{4.5.4}$$

This velocity is everywhere positive, as a consequence of the fact that we started with a 2-shock profile. We note that these solitary waves travel to the right, with speed s that exceeds the characteristic speed $\sqrt{gh_\infty}$ at the constant state $(h_\infty, 0)$ in this case. The spatial reflection symmetry yields solitary waves that travel to the left instead. This symmetry also recovers the solitary waves that can be constructed from 1-shock profiles.

4.5.2 Solitary waves of depression

We obtain solitary waves of depression by using η_- in (4.5.1) instead of η_+, choosing $h_\infty \stackrel{\text{def}}{=} h_-$ (the wave height at ∞) and h_c (the minimum wave height) arbitrary subject to the requirement that

$$0 < h_c < h_\infty.$$

Similarly to (4.5.2), the wave height $h(x,t) \in [h_c, h_\infty)$ is determined implicitly by

$$\int_{h_c}^{h} \left(\frac{k^3 - h_c^3}{h_\infty^2 k - h_c^3} \right)^{\frac{1}{2}} \frac{h_\infty}{h_\infty - k} \, dk = \frac{|x - st|}{\sqrt{\varepsilon}}, \qquad x \in \mathbb{R}, \quad h \in [h_c, h_\infty). \tag{4.5.5}$$

Considering $u_- = 0$ to be the limiting velocity as $|x| \to \infty$, we find $v_- = -s = -\gamma h_+$ from (2.3.3), whence the solitary wave speed is again given by equation (4.5.3), and again the corresponding velocity profile is given by (4.5.4). This time, the velocity is everywhere negative (when starting with the 2-shock profile), while the solitary wave travels to the right ($s > 0$) but with speed s *less than* the characteristic speed $\sqrt{gh_\infty}$ of the state at infinity. Again, spatial reflection yields waves of depression traveling to the left.

4.6 Non-existence of 'stumpons'

Solitary waves of elevation and depression are valid solitary wave solutions. In the above equation, there is one more degree of freedom: the critical height h_c is at our disposal. Inspired by Lenells [34], one might ask whether it is possible to add between two halves of the cusp a flat plateau conjuncting them and inquire whether such "stumpon" is a legitimate weak solution.

Actually, we shall check that there is no 'stumpon' solitary wave solution. In other words, if we inserted a flat part in the middle of the 'cuspon' solution, we will show that it is impossible to have conserved mass and momentum simultaneously.

By contradiction, suppose we do have such a solution whose unknowns are h_c and v_c on the flat part. Then they must satisfies (4.5.3) and

$$-h_\infty s = h_c v_c \tag{4.6.1}$$

$$h_\infty s^2 + \frac{1}{2} g h_\infty^2 = h_c v_c^2 + \frac{1}{2} g h_c^2 \tag{4.6.2}$$

By (4.5.3) we get $v_c = -\sqrt{gh_c}$, so setting $x = h_c/h_\infty$, we reduce to the cubic equation

$$0 = x^3 - \frac{3}{2}x^2 + \frac{1}{2} = (x-1)^2\Big(x + \frac{1}{2}\Big) \tag{4.6.3}$$

which doesn't have any nontrivial positive solution. Hence there is no 'stumpon' solitary wave solution.

4.7 Parametric formulae for shock profiles and cusped waves

Here we describe how weakly singular shock profiles and cusped waves can be determined in a parametric form,

$$h = h(\xi), \quad x = x(\xi), \qquad \xi \in \mathbb{R},$$

by a quadrature procedure that eliminates having to deal with the singularities present in the ODEs (4.1.12), (4.1.13) and in the integrands of the implicit relations (4.1.18), (4.1.20), (4.5.2). Inspired by the fact that classical solitary wave profiles of the form $f(\xi) = \kappa \operatorname{sech}^2(\frac{1}{2}\xi)$ (and their translates) satisfy an equation with cubic polynomial as right-hand side,

$$f_\xi^2 = f^2\left(1 - \frac{f}{\kappa}\right), \tag{4.7.1}$$

we modify the dimensionless ODE (4.1.11) by replacing H^3 in the denominator by its asymptotic value 1. Thus we seek the solution of (4.1.11) in parametric form $H = H(\xi)$, $z = z(\xi)$ by solving

$$H_\xi^2 = \frac{(H-\mathcal{F})(H-1)^2}{1-\mathcal{F}} = (H-1)^2\left(1 - \frac{H-1}{\mathcal{F}-1}\right), \tag{4.7.2}$$

$$z_\xi^2 = \varepsilon \frac{H^3 - \mathcal{F}}{1-\mathcal{F}}. \tag{4.7.3}$$

We require $H^3 = \mathcal{F}$ when $z = 0$. It is convenient to require $z(0) = 0$. Comparing the form of (4.7.2) with (4.7.1) we find the appropriate solution of (4.7.2) on either half-line $\xi \geq 0$ or $\xi \leq 0$ can be written in the form

$$H(\xi) = 1 + (\mathcal{F} - 1)\,\mathrm{sech}^2\left(\frac{1}{2}|\xi| + \alpha\right), \tag{4.7.4}$$

where $H(0) = \mathcal{F}^{1/3}$ provided

$$\cosh^2 \alpha = \frac{\mathcal{F} - 1}{\mathcal{F}^{1/3} - 1}. \tag{4.7.5}$$

A unique $\alpha > 0$ solving this equation exists in either case $\mathcal{F} > 1$ or $0 < \mathcal{F} < 1$, namely

$$\alpha = \ln(\sqrt{\gamma} + \sqrt{\gamma - 1}), \qquad \gamma = \frac{\mathcal{F} - 1}{\mathcal{F}^{1/3} - 1}, \tag{4.7.6}$$

because $\gamma > 1$. Now $z(\xi)$ is recovered by quadrature from (4.7.3) as

$$z(\xi) = \sqrt{\varepsilon} \int_0^\xi \left(\frac{H(\zeta)^3 - \mathcal{F}}{1 - \mathcal{F}}\right)^{1/2} d\zeta \tag{4.7.7}$$

To express this result in dimensional terms for $h = h_\pm H$ in each case as appropriate, we recall $\mathcal{F}_\pm = h_c^3/h_\pm^3$ where h_c may be determined from h_+, h_- by (4.1.16). We obtain

$$h(\xi) = h_\pm + \frac{(h_c - h_\pm)\cosh^2 \alpha_\pm}{\cosh^2(\frac{1}{2}|\xi| + \alpha_\pm)}, \tag{4.7.8}$$

$$x(\xi) = \sqrt{\varepsilon} h_\pm \int_0^\xi \left(\frac{h(\zeta)^3 - h_c^3}{h_\pm^3 - h_c^3}\right)^{1/2} d\zeta, \tag{4.7.9}$$

where $\alpha_\pm$ is determined from (4.7.6) using $\mathcal{F} = \mathcal{F}_\pm$.

Cusped solitary waves profiles are expressed parametrically by the same formulae after replacing $h_\pm$ with h_∞.

An explicit expression for $x(\xi)$ remains to be obtained. Even if this expression could be obtained in closed form, it likely would involve special functions that may not be easily computed. In any case, it is straightforward to compute $x(\xi)$ directly from the integral by an efficient quadrature method. We note, however, that Taylor expansion of $\mathrm{sech}^2(\frac{1}{2}|\xi| + \alpha_\pm)$ implies that for small $|\xi|$,

$$\frac{h(\xi) - h_c}{h_\pm - h_c} = |\xi| \tanh \alpha_\pm + O(|\xi|^2).$$

Consequently the integrand of (4.7.9) has a weak singularity at 0, with

$$\left(\frac{h(\zeta)^3 - h_c^3}{h_\pm^3 - h_c^3}\right)^{1/2} = K|\zeta|^{1/2} + O(|\zeta|), \qquad K = \left(\frac{3h_c^2 \tanh \alpha_\pm}{h_\pm^2 + h_\pm h_c + h_c^2}\right)^{1/2}.$$

This singularity can be eliminated by a change of variable $\zeta = \pm y^2$—then simple quadratures will yield accurate numerical approximations.

4.8 Numerical simulations

In this section we examine how the theory of weakly singular shock profiles developed in this paper fits the smoothed shocks observed in the computations carried out in [9]. Computations and figures are taken from [45].

4.8.1 A dynamically generated wave front.

In Fig. 4.1 we compare a shock profile computed by the theory developed in this paper with a solution to the rSV system computed as in [9] for "dam-break" initial data. The solid line is from the numerically computed solution to the rSV system at time $t = 15$ with $\varepsilon = 0.5$ and smoothed step function ("dam break") initial data

$$h_0(x) = h_- + \frac{1}{2}(h_+ - h_-)(1 + \tanh(\delta x))$$

for $h_- = 1.5$, $h_+ = 1$, $g = 1$, $\delta = 1$, as indicated in [9]. The numerical computation was performed with a Fourier pseudo-spectral method as described in [15], using an Erfc-Log filter for anti-aliasing [5] and with $N = 8192$ modes on a periodic domain of length $4L$ with $L = 25$.

The crosses mark the shock profile solution computed parametrically using formulae (4.7.8)-(4.7.9) of the previous section with $h_- = 1.2374$ and $h_+ = 1$, with x shifted by 17.67. The bottom part of the figure is a zoom-in on the indicated region of the upper part. We remark that the computed rSV solution in Fig. 4.1 corresponds directly

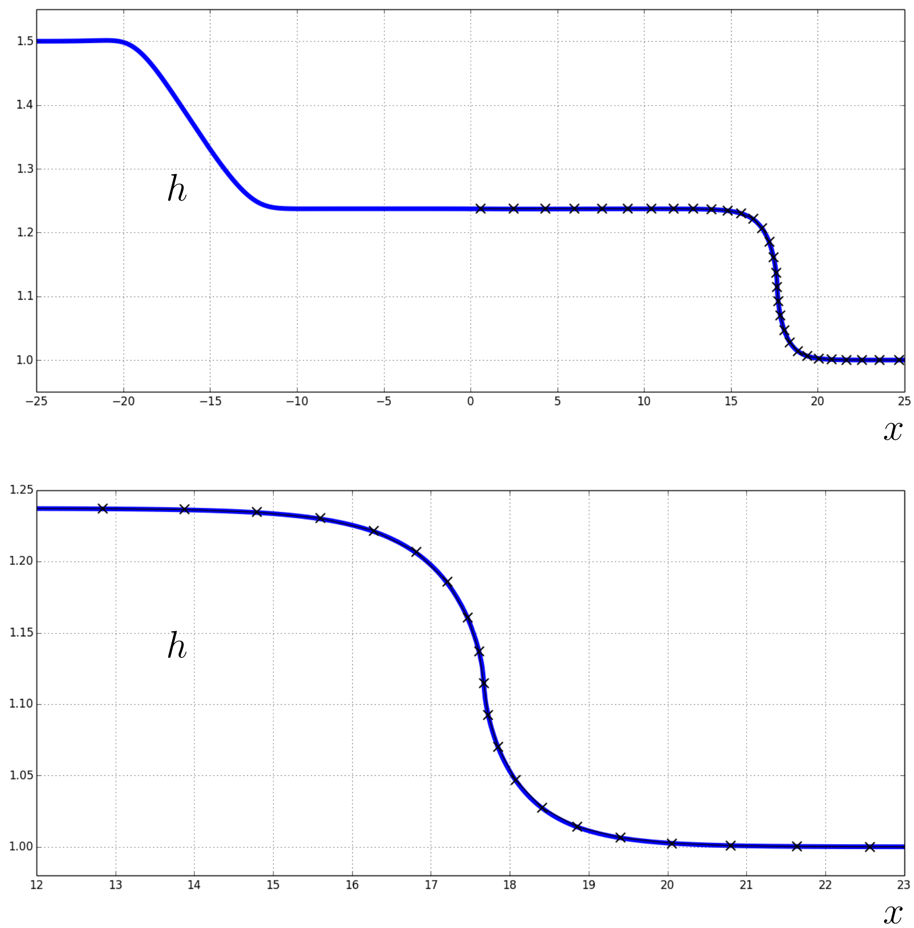


Figure 4.1: Comparison of shock profile with dam-break computation of [9]. The solid line is the rSV solution with $\varepsilon = 0.5$ computed by a pseudo-spectral method. Crosses mark the shock profile computed as in (4.7.8)–(4.7.9), shifted by 17.67.

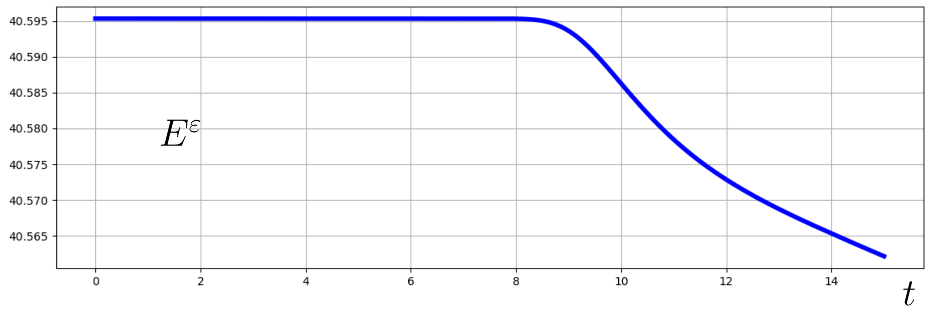


Figure 4.2: Total energy E^{ε} vs. t in the smoothed dam break problem as in Fig. 4.1 with $\varepsilon = 0.5$.

to Fig. 3(c) of [9]—due to a late change of notation the values of ε reported for the computations in [9] correspond to 2ε in the present notation.

4.8.2 Energy dissipation.

In Fig. 4.2 we plot the total energy from (3.6.7),

$$E^{\varepsilon}(t) = \int_{-L}^{L} \mathcal{E}^{\varepsilon}\, dx, \tag{4.8.1}$$

as a function of time, for a solution computed as in Fig. 4.1 but with anti-aliasing performed using the filter employed by Hou and Li in [24], namely

$$\rho(2k/N) = \exp(-36|2k/N|^{36}), \quad k = -N/2, \ldots, N/2-1,$$

applied on each time step. From this data, we estimate the average energy decay rate $dE^{\varepsilon}/dt \approx -0.00326$ over the range $t \in [14, 15]$. Corresponding to $h_- = 1.2374$, $h_+ = 1$, the dissipation formula (2.4.8) predicts $dE^{\varepsilon}/dt = -0.00318$, giving a relative error of less than 2.6 percent.

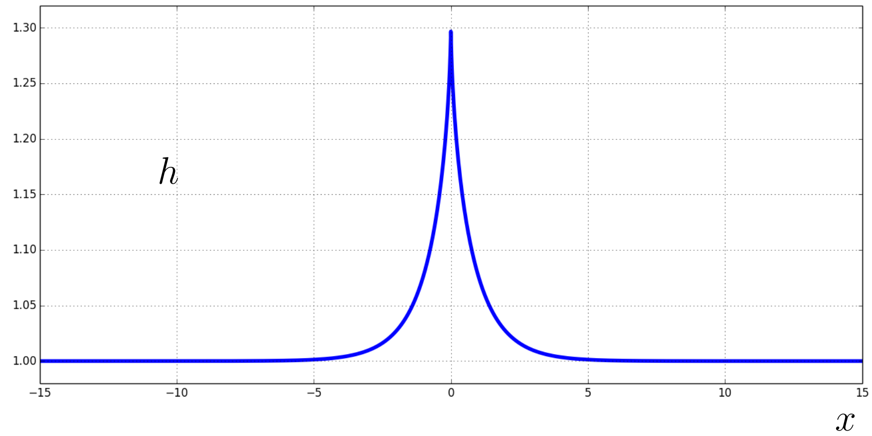


Figure 4.3: Cusped solitary wave profile for $h_\infty = 1$, $h_c = 1.3$

4.8.3 Cusped waves

The profile of a cusped solitary wave of elevation is plotted in Fig. 4.3 for $h_\infty = h_+ = 1$ and maximum height $h_c = 1.3$. We were not able to compute a clean isolated traveling cusped wave by taking the numerically computed wave profile for (h, u) as initial data on a regular grid. Indeed, there is no particular reason our pseudo-spectral code should work well for such a singular solution, and anyway it may not be numerically stable. However, when taking the h-profile in Fig. 4.3 as initial data with *zero* initial velocity, the numerical solution develops two peaked waves traveling in opposite direction as indicated Fig. 4.4. While hardly conclusive, this evidence suggests that cusped solutions may be relevant in the dynamics of the rSV system.

The two peaks here are slightly skewed compared to the profile of a cusped solitary wave. Our limited exploration uncovered no convincing evidence that cusped waves collide “cleanly” enough to justify calling them ‘cuspons’ or suggest that the rSV system is formally integrable—It may be difficult to tell, though, as perturbed cusped waves do not leave behind a dispersive “tail” in this non-dispersive system.

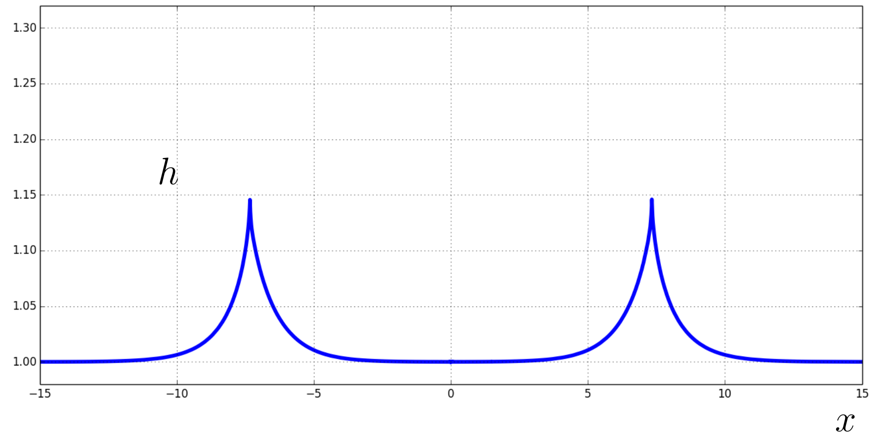


Figure 4.4: Numerical solution at $t = 6$ with initial height from Fig. 4.3, initial velocity zero.

4.9 Discussion and outlook

Our analysis of traveling wave profiles for the rSV system proves that, as the authors of [9] stated, the regularized system admits 'smoothed shocks' that propagate at exactly the same speed as corresponding classical discontinuous shocks for the shallow water equations. The new waves are indeed piecewise smooth and continuous, but have weak singularities which correctly generate the same energy dissipation as the classical shocks.

This ability of the rSV system to correctly model shock wave propagation non-dispersively without oscillations while conserving energy for smooth solutions is an interesting feature which deserves further investigation. As demonstrated in [9], it means that a rather straightforward pseudo-spectral method (albeit one which involves careful dealiasing, and iteration to eliminate the time derivative term in $\mathcal{R}$) computes shock speeds accurately over a wide range of values of ε, with 2ε ranging from 0.001 to 5 in the examples treated in [9].

The comparisons made in the previous section above make it plausible that the pseudo-spectral method used to produce Figs. 4.1 and 4.2 is computing an accurate approximation to a solution of the rSV system which ceases to conserve energy (hence loses smoothness) around $t = 7$ or 8, and develops afterward a traveling wave whose shape closely matches a weakly singular shock profile. We speculate that an important source of energy dissipation in this pseudo-spectral computation may be the damping of high frequency components induced for dealiasing purposes.

How this actually happens and what it may mean with regard to the design and accuracy of numerical approximations remains to be investigated in detail. Often, energy conservation, or preservation of some variational (Lagrangian) or symplectic (Hamiltonian) structure, is a desirable feature of a numerical scheme designed for long-time computations in an energy-conserving system. (See [11, 19, 36, 41] for discussion of variational and symplectic integrators.) But for the rSV system considered here, exact conservation of energy appears to be *inappropriate* for approximating solutions containing weakly singular shock profiles, which dissipate energy as we have shown.

At present, the issue of preservation of symplectic structure may be moot anyways, since we are not aware of a canonical Hamiltonian structure for the rSV system. It seems worth mentioning, however, that the rSV system admits the following non-canonical Hamiltonian structure. Namely, with

$$\mathcal{H} = \int \frac{1}{2}hu^2 + \frac{1}{2}g(h-h_\infty)^2 + \varepsilon\left(\frac{1}{2}h^3u_x^2 + \frac{1}{2}gh^2h_x^2\right)\,dx\,, \tag{4.9.1}$$

and $m = hu - \varepsilon(h^3u_x)_x$, the rSV system is formally equivalent to

$$\partial_t \begin{pmatrix} m \\ h \end{pmatrix} = -\begin{pmatrix} \partial_x\, m + m\,\partial_x & h\partial_x \\ \partial_x h & 0 \end{pmatrix}\begin{pmatrix} \delta\mathcal{H}/\delta m \\ \delta\mathcal{H}/\delta h \end{pmatrix}. \tag{4.9.2}$$

This is a simple variant of the Hamiltonian structure well-known for the Green-Naghdi equations [12, 22, 28, 37], obtained by replacing the Green-Naghdi Hamiltonian with a Hamiltonian derived from (3.6.8).

Chapter 5

Large Time Existence, Uniqueness and Blow-up Criterion

In this section, we will establish existence, uniqueness and blow-up criterion of the rSV system.

We are interested in how the existence time varies according to the value of $\alpha = a/h_0$, the nonlinearity parameter. For example, in the inviscid Burger's equation, a Ricatti-type calculation shows that the existence time for smooth solutions is proportional to $1/\alpha$.

For this reason, we make the following change of variables

$$h = h_0 + \alpha\eta, \quad u \text{ replaced by } \alpha u, \tag{5.0.1}$$

and obtain

$$\eta_t + (hu)_x = 0\,, \tag{5.0.2}$$

$$h(u_t + g\eta_x + \alpha u u_x) + \varepsilon(h^2\tilde{\mathcal{R}})_x = 0, \tag{5.0.3}$$

$$\tilde{\mathcal{R}} \stackrel{\text{def}}{=} h(\alpha u_x^2 - u_{xt} - \alpha u u_{xx}) - g\left(h\eta_{xx} + \frac{1}{2}\alpha\eta_x^2\right)\,, \tag{5.0.4}$$

The energy equation is written as

$$\tilde{\mathcal{E}}_t + \tilde{\mathcal{Q}}_x = 0\,, \tag{5.0.5}$$

where

$$\tilde{\mathcal{E}}^\varepsilon \overset{\text{def}}{=} \frac{1}{2}\alpha^2 hu^2 + \frac{1}{2}gh^2 + \varepsilon\alpha^2\left(\frac{1}{2}h^3u_x^2 + \frac{1}{2}gh^2\eta_x^2\right), \tag{5.0.6}$$

$$\tilde{\mathcal{Q}}^\varepsilon \overset{\text{def}}{=} \frac{1}{2}\alpha^3 hu^3 + g\alpha h^2 u + \varepsilon\alpha^2\left(\left(\frac{1}{2}\alpha h^2u_x^2 + \frac{1}{2}g\alpha h\eta_x^2 + h\tilde{\mathcal{R}}\right)hu + gh^3\eta_x u_x\right), \tag{5.0.7}$$

where $\tilde{\mathcal{R}}$ is consistent with the one in the momentum equation (5.0.3).

With $\mathcal{I}_h = h - \varepsilon\partial_x \circ h^3 \circ \partial_x$, the momentum equation (5.0.3) also takes the form

$$u_t + g\eta_x + \alpha uu_x + \varepsilon\alpha\mathcal{I}_h^{-1}\partial_x\Big(2h^3u_x^2 - \frac{1}{2}gh^2\eta_x^2\Big) = 0. \tag{5.0.8}$$

Equations (5.0.2), (5.0.8) form a hyperbolic system, for which we shall use an iteration scheme to prove the main theorem of this section:

Theorem 5.0.1. *Let $s \geqslant 2$ be an integer. Assume the initial value $W^0 = (\eta^0, u^0)^T \in H^s(\mathbb{R})$ satisfies*

$$\exists\, h_{\min} > 0, \quad \inf_{x\in\mathbb{R}}(h_0 + \alpha\eta^0) \geqslant h_{\min}. \tag{5.0.9}$$

Then there exist $T_{\max} = T_{\max}(s, W^0) > 0$ bounded below uniformly with respect to ε and α such that the regularized shallow-water system (5.0.2)–(5.0.3) admit a unique solution $W = (\eta, u)^T \in C([0, T_{\max}/\alpha]; H^s(\mathbb{R}))$ with the initial condition W^0 and preserving the nonzero depth condition (5.0.9).

In particular if $T_{\max} < \infty$ we have

$$\|DW(t,\cdot)\|_{L^\infty} \to \infty \quad \text{as } t \to \frac{T_{\max}}{\alpha}, \tag{5.0.10}$$

or

$$\inf_{\mathbb{R}} h(t,\cdot) = \inf_{\mathbb{R}}(h_0 + \alpha\eta(t,\cdot)) \to 0 \quad \text{as } t \to \frac{T_{\max}}{\alpha}. \tag{5.0.11}$$

Moreover, the following conservation of energy property holds

$$\partial_t \int \tilde{\mathcal{E}}^\varepsilon\, dx = \partial_t \int \left(\frac{1}{2}\alpha^2 hu^2 + \frac{1}{2}gh^2 + \varepsilon\alpha^2\left(\frac{1}{2}h^3u_x^2 + \frac{1}{2}gh^2\eta_x^2\right)\right) dx = 0. \tag{5.0.12}$$

Remark 5.0.2. a) The theorem also works on 1D periodic domains.

b) Although continuous dependence on the initial data is not mentioned in the theorem, we do have it in the following sense: for all $\tilde{W}^0$ with $\left\|\tilde{W}^0\right\|_{H^s} \leqslant 2\|W^0\|_{H^s}$,

$$\left\|\tilde{W} - W\right\|_{L^\infty([0,t];H^{s-1})} \leqslant C(s, h_{\min}, \|W\|_{L^\infty([0,t];H^s)})\left\|\tilde{W}^0 - W^0\right\|_{H^s}. \tag{5.0.13}$$

The proof follows directly from the energy estimate (5.3.13).

c) The dependence on $h_{\min}$ can be dropped if the initial energy is so small that Theorem 6.1.1 can be applied.

The proof of Theorem 5.0.1 is structured as below:

Section 5.1 contains the technical preparation for the operator $\mathcal{I}_h$. Section 5.2 discusses the iteration step in the iteration scheme and introduces the energy estimate. The main proof of Theorem 5.0.1 is presented in Section 5.3.

5.1 Preliminary results

The elliptic operator $\mathcal{I}_h$ plays an important role in the energy estimate and well-posedness of the regularized shallow-water system. In this subsection, we shall introduce the main technical tool to handle $\mathcal{I}_h$ and the nonlocal term in (5.0.8).

Before getting into the details of the estimate, we cite two well-known harmonic analysis results here without proofs.

The first one is a classical Moser-Tame product estimate, and is used to prove energy estimate. The second one is a generalized Kato-Ponce commutator estimate that is shaper in the integer exponent case, which is used to get blow-up criterion.

Let Λ^s be the operator associated with Fourier symbol $(1+\xi^2)^{s/2}$:

$$\widehat{\Lambda^s u} \overset{\text{def}}{=} (1+\xi^2)^{s/2}\hat{u} \quad \text{for all tempered distribution } u. \tag{5.1.1}$$

Lemma 5.1.1. *Let $t_0 > \frac{1}{2}$ and $s \geqslant 0$.*

1) For all $f, g \in H^s(\mathbb{R})$,

$$\|fg\|_{H^s} \leqslant \begin{cases} C(s)(\|f\|_{H^{t_0}}\|g\|_{H^s} + \|f\|_{H^s}\|g\|_{H^{t_0}}) & \textit{if } s > t_0, \\ C(s)\|f\|_{H^{t_0}}\|g\|_{H^s} & \textit{if } 0 \leqslant s \leqslant t_0. \end{cases} \tag{5.1.2}$$

2) For all $f \in H^s(\mathbb{R})$ and $u \in H^{s-1}$,

(5.1.3)
$$\|[\Lambda^s, f]u\|_{L^2} \leqslant \begin{cases} C(s)(\|\nabla f\|_{H^{t_0}}\|u\|_{H^{s-1}} + \|\nabla f\|_{H^{s-1}}\|u\|_{H^{t_0}}) & \text{if } s > t_0+1, \\ C(s)\|\nabla f\|_{H^{t_0}}\|u\|_{H^{s-1}} & \text{if } 0 \leqslant s \leqslant t_0+1. \end{cases}$$

When s is an integer, we have slightly stronger results which are crucial to the blow up criteria in the main theorem.

Lemma 5.1.2. *Let $s \geqslant 0$ be an integer.*

1) For all $f, g \in H^s(\mathbb{R}) \cap L^\infty(\mathbb{R})$,

$$\|D^s(fg)\|_{L^2} \leqslant C(s)(\|f\|_{L^\infty}\|D^s g\|_{L^2} + \|D^s f\|_{L^2}\|g\|_{L^\infty}), \tag{5.1.4}$$

where the differential operator $D \overset{\text{def}}{=} \frac{d}{dx}$.

2) For all $f \in H^s(\mathbb{R})$, $Df \in L^\infty(\mathbb{R})$ and $g \in H^{s-1}(\mathbb{R}) \cap L^\infty(\mathbb{R})$,

$$\|[D^s, f]g\|_{L^2} \leqslant C(s)\Big(\|Df\|_{L^\infty}\Big\|D^{s-1}g\Big\|_{L^2} + \|D^s f\|_{L^2}\|g\|_{L^\infty}\Big). \tag{5.1.5}$$

The following lemma gives the invertibility of $\mathcal{I}_h$ and boundedness of $\mathcal{I}_h^{-1}\partial_x$.

Lemma 5.1.3. *Let $h \in W^{1,\infty}(\mathbb{R})$ be such that (5.0.9) is satisfied.*

1) The operator

$$\mathcal{I}_h : H^2(\mathbb{R}) \to L^2(\mathbb{R}) \tag{5.1.6}$$

is well-defined, one-to-one and onto.

2) Let $t_0 > 1/2$, $s \geqslant 0$, $\tilde{s} = \max\{s, t_0+1\}$ and $h - h_0 \in H^{\tilde{s}}(\mathbb{R})$, then

$$\Big\|\mathcal{I}_h^{-1}\Big\|_{H^s(\mathbb{R})\to H^s(\mathbb{R})} + \sqrt{\varepsilon}\Big\|\mathcal{I}_h^{-1}\partial_x\Big\|_{H^s(\mathbb{R})\to H^s(\mathbb{R})} \leqslant C \tag{5.1.7}$$

$$\sqrt{\varepsilon}\Big\|\mathcal{I}_h^{-1}\Big\|_{H^s(\mathbb{R})\to H^{s+1}(\mathbb{R})} + \varepsilon\Big\|\mathcal{I}_h^{-1}\partial_x\Big\|_{H^s(\mathbb{R})\to H^{s+1}(\mathbb{R})} \leqslant C \tag{5.1.8}$$

where $C = C(s, h_0, h_{\min}, h)$ is a constant depending on $s, h_0, h_{\min}, \|h-h_0\|_{H^{\tilde{s}}}$.

Remark 5.1.4. After applying a spatial change of variable

$$\sqrt{\varepsilon}\frac{\partial}{\partial x} = \frac{\partial}{\partial z} \quad \text{or} \quad x = \sqrt{\varepsilon} z \tag{5.1.9}$$

$\mathcal{I}_h$ will be an operator independent of ε. Thus a dimension check on ε shows that in (5.1.8) the exponents of ε are sharp.

Proof. 1) The idea is that $\mathcal{I}_h$ is in essence a very well-behaved elliptic operator such that the basic Lax-Milgram approach works on it.

We define the bilinear mapping $a : H^1(\mathbb{R}) \times H^1(\mathbb{R}) \to \mathbb{R}$ such that

$$a(u, v) = (hu, v)_{L^2} + \varepsilon(h^3 u_x, v_x)_{L^2} \quad \forall\, u, v \in H^1(\mathbb{R}). \tag{5.1.10}$$

Next, we will show that a is not only bounded but also coercive. We have

$$\begin{aligned} |a(u, v)| &\leqslant \|h\|_{L^\infty}\|u\|_{L^2}\|v\|_{L^2} + \varepsilon\|h\|^3_{L^\infty}\|u\|_{H^1}\|v\|_{H^1} \\ &\leqslant C(h)\|u\|_{H^1}\|v\|_{H^1}. \end{aligned}$$

and by (5.0.9)

$$|a(u, u)| \geqslant h_{\min}\|u\|^2_{L^2} + \varepsilon(h_{\min})^3\|u_x\|^2_{L^2} \geqslant \varepsilon C(h_{\min})\|u\|^2_{H^1}.$$

So by Lax-Milgram, there is a bounded bijective linear operator $I : H^1(\mathbb{R}) \to H^{-1}(\mathbb{R})$ such that

$$a(u, v) = (Iu, v)_{H^{-1}\times H^1} \quad \forall\, u, v \in H^1(\mathbb{R}). \tag{5.1.11}$$

Therefore, given $f \in L^2(\mathbb{R}) \hookrightarrow H^{-1}(\mathbb{R})$, $u := I^{-1} f$ satisfies

$$(f, v)_{L^2} = \langle f, v\rangle_{H^{-1}\times H^1} = a(u, v) = (\mathcal{I}_h u, v)_{L^2} \quad \forall\, v \in H^1(\mathbb{R}), \tag{5.1.12}$$

hence $u = \mathcal{I}_h^{-1} f$ is well-defined, and by elliptic regularity

$$u_{xx} = -\frac{f - hu + 3h^2 h_x u_x}{h^3} \quad \text{exists a.e. and } \in L^2(\mathbb{R}). \tag{5.1.13}$$

2) Let $\|\cdot\|_{H^1_*}$ be the norm on $H^1(\mathbb{R})$ (equivalent to $\|\|_{H^1}$ but not uniformly w.r.t. ε) determined by

$$\|f\|_{H^1_*}^2 := \|f\|_{L^2}^2 + \varepsilon\|f_x\|_{L^2}^2. \tag{5.1.14}$$

Consider $f, g \in C_c^\infty(\mathbb{R})$ such that

$$\mathcal{I}_h u = f + \sqrt{\varepsilon} g_x \tag{5.1.15}$$

then invoke the coercivity estimate from above

$$\begin{aligned} C(h_{\min})\|u\|_{H^1_*}^2 &\leqslant a(u,u) = (\mathcal{I}_h u, u)_{L^2} = (f,u)_{L^2} - (g, \sqrt{\varepsilon} u_x)_{L^2} \\ &\leqslant (\|f\|_{L^2} + \|g\|_{L^2})\|u\|_{H^1_*} \end{aligned} \tag{5.1.16}$$

hence

$$\|u\|_{H^1_*} \leqslant C(h_{\min})(\|f\|_{L^2} + \|g\|_{L^2}). \tag{5.1.17}$$

This proves the case $s = 0$.

Now for a generic $s \geqslant 1$, assume $\mathcal{I}_h u = f \in C_c^\infty$, we compute that

$$\begin{aligned} \mathcal{I}_h(\Lambda^s u) &= \Lambda^s \mathcal{I}_h u + [\mathcal{I}_h, \Lambda^s]u \\ &= \Lambda^s f + [h - \varepsilon\partial_x \circ h^3 \circ \partial_x, \Lambda^s]u \\ &= \Lambda^s f + [h, \Lambda^s]u - \varepsilon\partial_x\Big([h^3, \Lambda^s]u_x\Big), \end{aligned} \tag{5.1.18}$$

so consider applying (5.1.17) here while taking

$$\tilde{f} = \Lambda^s f + [h, \Lambda^s]u, \quad \tilde{g} = \sqrt{\varepsilon}[h^3, \Lambda^s]u_x \tag{5.1.19}$$

we would have

$$\begin{aligned} \|\Lambda^s u\|_{H^1_*} &\leqslant C(h_{\min})\Big(\Lambda^s f + \|[h, \Lambda^s]u\|_{L^2} + \Big\|\sqrt{\varepsilon}[h^3, \Lambda^s]u_x\Big\|_{L^2}\Big) \\ &\leqslant C(s, h_{\min})\bigg(\|f\|_{H^s} + \|h - h_0\|_{H^{\tilde{s}}}\|u\|_{H^{s-1}} \\ &\quad + \sqrt{\varepsilon} C(h_0)(1 + \|h - h_0\|_{H^{\tilde{s}}}^3)\|u\|_{H^s}\bigg) \end{aligned}$$

$$\leqslant \quad C(s, h_0, h_{\min}, \|h-h_0\|_{H^{\bar{s}}})\Big(\|f\|_{H^s} + \left\|\Lambda^{s-1}u\right\|_{H^1_*}\Big) \tag{5.1.20}$$

where in the last inequality we already use (5.1.2) and (5.1.3).

Inducting on integer-valued s and interpolating for fractional-valued s will yield the result

$$\|\Lambda^s u\|_{H^1_*} \leqslant C(s, h_0, h_{\min}, \|h-h_0\|_{H^{\bar{s}}})\|f\|_{H^s} \quad \forall\, s \geqslant 0. \tag{5.1.21}$$

Similarly, assume $\mathcal{I}_h u = \sqrt{\varepsilon} g_x \in C_c^\infty$, we compute that

$$\begin{aligned}
\mathcal{I}_h(\Lambda^s u) &= \Lambda^s \mathcal{I}_h u + [\mathcal{I}_h, \Lambda^s]u \\
&= \sqrt{\varepsilon}\Lambda^s g_x + [h - \varepsilon\partial_x \circ h^3 \circ \partial_x, \Lambda^s]u \\
&= [h, \Lambda^s]u + \partial_x\Big(\sqrt{\varepsilon}\Lambda^s g - \varepsilon[h^3, \Lambda^s]u_x\Big), \qquad (5.1.22)
\end{aligned}$$

so using (5.1.17) with

$$\tilde{f} = [h, \Lambda^s]u, \quad \tilde{g} = \Lambda^s g - \sqrt{\varepsilon}[h^3, \Lambda^s]u_x \tag{5.1.23}$$

we would have

$$\begin{aligned}
\|\Lambda^s u\|_{H^1_*} &\leqslant C(h_{\min})\Big(\|[h, \Lambda^s]u\|_{L^2} + \left\|\Lambda^s g - \sqrt{\varepsilon}[h^3, \Lambda^s]u_x\right\|_{L^2}\Big) \\
&\leqslant C(s, h_{\min})\Big(\|h-h_0\|_{H^{\bar{s}}}\|u\|_{H^{s-1}} + \|g\|_{H^s} \\
&\quad + \sqrt{\varepsilon}C(h_0)(1 + \|h-h_0\|^3_{H^{\bar{s}}})\|u\|_{H^s}\Big) \\
&\leqslant C(s, h_0, h_{\min}, \|h-h_0\|_{H^{\bar{s}}})\Big(\|g\|_{H^s} + \left\|\Lambda^{s-1}u\right\|_{H^1_*}\Big), \qquad (5.1.24)
\end{aligned}$$

which yields

$$\|\Lambda^s u\|_{H^1_*} \leqslant C(s, h_0, h_{\min}, \|h-h_0\|_{H^{\bar{s}}})\|g\|_{H^s} \quad \forall\, s \geqslant 0. \tag{5.1.25}$$

□

5.2 Linear analysis

We can reformulate the hyperbolic system (5.0.2), (5.0.8) as

$$W_t + B(W)W_x + F(W) = 0, \tag{5.2.1}$$

with $W = (\eta, u)^T$ and where

$$B(W) = \begin{pmatrix} \alpha u & h \\ g & \alpha u \end{pmatrix}, \quad F(W) = \begin{pmatrix} 0 \\ f(W) \end{pmatrix} \tag{5.2.2}$$

with $f(W) = \varepsilon\alpha\mathcal{I}_h^{-1}\partial_x\Big(2h^3u_x^2 - \frac{1}{2}gh^2\eta_x^2\Big)$.

This subsection is devoted to the proof of energy estimates for the following initial value problem around some reference state $\underline{W} = (\underline{h}, \underline{u})^T$:

$$\begin{cases} W_t + B(\underline{W})W_x + F(\underline{W}) = 0; \\ W|_{t=0} = W_0 \end{cases} \tag{5.2.3}$$

A symmetrizer for $B(\underline{W})$ is given by

$$A(\underline{W}) = \begin{pmatrix} g & 0 \\ 0 & \underline{h} \end{pmatrix}. \tag{5.2.4}$$

We also introduce a natural energy for the IVP (5.2.3)

$$E^s(W)^2 = (\Lambda^s W, \underline{A}\Lambda^s W) = g\|\eta\|_{H^s} + (\Lambda^s u, \underline{h}\Lambda^s u). \tag{5.2.5}$$

which is equivalent to $\|W\|_{H^s}$ provided that $0 < h_{\min} \leqslant |h| \leqslant \|\underline{h}\|_{L^\infty} < \infty$.

The following theorem justifies the iteration scheme and provides an energy estimate that controls the norms of all the solutions in the scheme.

Theorem 5.2.1 (energy estimate)**.** *Let $t_0 > 1/2, s \geqslant t_0 + 1$, $\alpha \ll 1$, and $R > 0$. $T = T(s, h_{\min}, R) > 0$ is chosen to be independent of ε and α. Let also $\underline{W} = (\underline{\eta}, \underline{u})^T \in C([0, T/\alpha]; H^s)$ satisfy nonzero depth condition (5.0.9) and $E^s(\underline{W}) < R$ on $[0, T/\alpha]$. Then for all $W_0 = (h_0, u_0) \in H^s$ satisfying*

$$h_0 > 2h_{\min} \quad \textit{and} \quad E^s(W_0) < \frac{R}{3} \tag{5.2.6}$$

there exists a unique solution $W = (h,u)^T \in C([0,T/\alpha];H^s)$ *such that* (5.2.3) *and* (5.0.9) *are still satisfied,* $E^s(W) < R$ *on* $[0,T/\alpha]$*, and*

$$E^s(W) \leqslant e^{C\alpha t}E^s(W_0) + e^{C\alpha t} - 1. \tag{5.2.7}$$

for some $C = C(s,h_{\min},R) > 0$*.*

Proof. Since all coefficients of this initial value problem are independent of unknowns, by a standard Friedrich mollification approach we have the well-posedness of the symmetrizable hyperbolic system. We will focus on the proof of the energy estimate.

For simplicity, we use underlines to denote the dependence on $\underline{W}$:

$$\underline{A} := A(\underline{W}), \quad \underline{B} := B(\underline{W}), \quad \underline{F} := F(\underline{W}), \quad \underline{f} := f(\underline{W}).$$

We compute that

$$\begin{aligned}\partial_t\Big(E^s(W)^2\Big) &= \partial_t(\Lambda^s W, \underline{A}\Lambda^s W)\\ &= (\underline{h_t}\Lambda^s u, \Lambda^s u) + 2(\Lambda^s W_t, \underline{A}\Lambda^s W)\end{aligned} \tag{5.2.8}$$

Using the equation (5.2.3) and integrating by parts, we obtain

$$\begin{aligned}\partial_t\Big(E^s(W)^2\Big) &= (\underline{h_t}\Lambda^s u, \Lambda^s u) - 2(\underline{AB}\Lambda^s W_x, \Lambda^s W)\\ &\quad +2([\underline{B}, \Lambda^s]W_x, \underline{A}\Lambda^s W) - 2(\Lambda^s \underline{F}, \underline{A}\Lambda^s W).\end{aligned} \tag{5.2.9}$$

Now we turn to bound the four terms in the RHS of (5.2.9) respectively.

1) The fact that $H^{t_0}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ and (5.1.2) indicate that

$$\begin{aligned}\Big|\underline{h_t}\Big| &= \alpha\Big|\underline{(hu)_x}\Big| \leqslant \alpha C(s)\|\underline{hu}\|_{H^s} = \alpha C(s)\Big\|\underline{u} + \alpha\underline{\eta u}\Big\|_{H^s}\\ &\leqslant \alpha C(s)\Big(\|\underline{u}\|_{H^s} + \alpha\Big\|\underline{\eta}\Big\|_{H^s}\|\underline{u}\|_{H^s}\Big)\\ &\leqslant \alpha C(s,h_{\min},R).\end{aligned} \tag{5.2.10}$$

Note that the constant C might change from line to line without changing the notation. So

$$\Big|(\underline{h_t}\Lambda^s u, \Lambda^s u)\Big| \leqslant \Big\|\underline{h_t}\Big\|_{L^\infty}\|u\|_{H^s}^2 \leqslant \alpha C(s,h_{\min},R)E^s(W)^2 \tag{5.2.11}$$

2) For the second term, note that

$$\underline{AB} = \begin{pmatrix} \alpha g\underline{u} & g\underline{h} \\ g\underline{h} & \alpha \underline{hu} \end{pmatrix} \tag{5.2.12}$$

is symmetric, so we can take advantage of this symmetry and move the derivative from W terms to $\underline{AB}$ term. We use

$$|\underline{h}_x| \leqslant \alpha C(s)\left\|\underline{\eta}\right\|_{H^s} \leqslant \alpha C(s)E^s(\underline{W}) \leqslant \alpha C(s,R) \tag{5.2.13}$$

together with (5.1.2) and obtain

$$\begin{aligned} 2|(\underline{AB}\Lambda^s W_x, \Lambda^s W)| &= \left|(\underline{(AB)_x}\Lambda^s W, \Lambda^s W)\right| \\ \leqslant \ & |(\alpha g\underline{u}_x\Lambda^s\eta, \Lambda^s\eta)| + |2(g\underline{h}_x\Lambda^s\eta, \Lambda^s u)| + |(\alpha(\underline{hu})_x\Lambda^s u, \Lambda^s u)| \\ \leqslant \ & \alpha C(s, h_{\min}, R)E^s(W)^2. \end{aligned} \tag{5.2.14}$$

3) For the third term, it is crucial to use Kato-Ponce commutator estimate (5.1.3)

$$\begin{aligned} & |([\underline{B}, \Lambda^s]W_x, \underline{A}\Lambda^s W)| \\ = \ & \left|\Big(\alpha[\underline{u}, \Lambda^s]\eta_x + \alpha[\underline{\eta}, \Lambda^s]u_x, \Lambda^s\eta\Big) + (\alpha[\underline{u}, \Lambda^s]u_x, \underline{h}\Lambda^s u)\right| \\ \leqslant \ & \alpha C(s)\Big(\|\underline{u}\|_{H^s}\|\eta_x\|_{H^{s-1}} + \left\|\underline{\eta}\right\|_{H^s}\|u_x\|_{H^{s-1}}\Big)\|\eta\|_{H^s} \\ & +\alpha C(s)\|\underline{u}\|_{H^s}\|u_x\|_{H^{s-1}}\|\underline{h}\|_{L^\infty}\|u\|_{H^s} \\ \leqslant \ & \alpha C(s, h_{\min}, R)\Big(\|\eta\|_{H^s}^2 + \|\eta\|_{H^s}\|u\|_{H^s}\Big) + \alpha C(s, h_{\min}, R)\|u\|_{H^s}^2 \\ \leqslant \ & \alpha C(s, h_{\min}, R)E^s(W)^2. \end{aligned} \tag{5.2.15}$$

4) For the fourth term, we exploit (5.1.8)

$$\begin{aligned} \left\|\underline{f}\right\|_{H^s} &= \left\|\varepsilon\alpha\mathcal{I}_h^{-1}\partial_x(2\underline{h}^3\underline{u_x}^2 - \frac{1}{2}g\underline{h}^2\underline{\eta_x}^2)\right\|_{H^s} \\ &\leqslant \alpha\left\|2\underline{h}^3\underline{u_x}^2 - \frac{1}{2}g\underline{h}^2\underline{\eta_x}^2\right\|_{H^{s-1}} \leqslant \alpha C(s, h_{\min}, R) \end{aligned} \tag{5.2.16}$$

where the last inequality is obtained by expanding $\underline{h} = h_0 + \alpha\underline{\eta}$ and applying (5.1.2) multiple times.

Then we have

$$|(\Lambda^s \underline{F}, \underline{A}\Lambda^s W)| = |\Lambda^s f, \underline{h}\Lambda^s u| \leqslant C(s, h_{\min}, R)\left\|\underline{f}\right\|_{H^s}\|u\|_{H^s}$$
$$\leqslant \alpha C(s, h_{\min}, R) E^s(W). \tag{5.2.17}$$

Now, substituting all estimates back into (5.2.9), we have that

$$\partial_t\Big(E^s(W)^2\Big) \leqslant \alpha C(s, h_{\min}, R)\Big(E^s(W)^2 + E^s(W)\Big) \tag{5.2.18}$$

which reduces to a Gronwall type differential inequality

$$\partial_t E^s(W) \leqslant \alpha C(s, h_{\min}, R)(E^s(W) + 1), \tag{5.2.19}$$

which in turn gives the inequality

$$E^s(W) \leqslant e^{C\alpha t} E^s(W_0) + e^{C\alpha t} - 1. \tag{5.2.20}$$

In view of this, we set $T = T(s, h_{\min}, R)$ small enough so that

$$e^{CT} < 3/2 \tag{5.2.21}$$

then $E^s(W) < R$ for all $t \in [0, T/\alpha]$.

To check that h is bounded below by $h_{\min}$, we note

$$h(t,x) = h(0,x) + \int_0^t h_t(s,x)\,ds \geqslant \operatorname{ess\,inf}_{\mathbb{R}} h_0 - t\|h_t\|_{L^\infty(\mathbb{R}\times[0,T/\alpha])} \tag{5.2.22}$$

so we obtain from (5.2.10) that

$$\operatorname{ess\,inf}_{\mathbb{R}\times[0,T]} h \geqslant \operatorname{ess\,inf}_{\mathbb{R}} h_0 - \frac{T}{\alpha}\alpha C(s, h_{\min}, R) \geqslant 2h_{\min} - TC(s, h_{\min}, R) \tag{5.2.23}$$

Further choosing T small enough would ensure that nonzero depth condition holds for our solution h. $\square$

5.3 Proof of the main theorem

In this section we present the proof of theorem 5.0.1. The first two steps are standard for symmetrizable hyperbolic system.

Proof of Theorem 5.0.1. We use an iteration scheme to show the existence.

Step 1

We inductively construct a sequence of approximate solutions $(W_{n+1} = (h_{n+1}, u_{n+1}))_{n\geqslant 0}$ such that

$$W_0 = W^0, \quad \text{and } \forall n \in \mathbb{N}, \begin{cases} \partial_t W_{n+1} + B(W_n)\partial_x W_{n+1} + F(W_n) = 0 \\ W_{n+1}|_{t=0} = W^0 \end{cases}. \tag{5.3.1}$$

We write

$$E_0 := (\Lambda^s W_0, A_0 \Lambda^s W_0), \quad E_{n+1} := (\Lambda^s W_{n+1}, A_n \Lambda^s W_{n+1}) \text{ for all } n \geqslant 0. \tag{5.3.2}$$

where

$$A_n = \begin{pmatrix} 1 & 0 \\ 0 & h_n \end{pmatrix} \text{ for all } n \geqslant 0. \tag{5.3.3}$$

Similar notations like B_n, F_n are also adopted. Let $R > 0, h_{\min} > 0$ such that

$$E_0 < \frac{R}{3} \quad \text{and} \quad \min_{\mathbb{R}} h^0 \geqslant 2h_{\min}. \tag{5.3.4}$$

Theorem 5.2.1 exactly provides the induction step for the scheme, i.e. assume that W_n satisfies (5.0.9) and $E_n < R$ on $[0, T/\alpha]$, then there exists an unique solution W_{n+1} to (5.3.1), and the solution satisfies (5.0.9) and $E_{n+1} < R$ on $[0, T/\alpha]$.

To obtain the solution, let $t_0 \leqslant r \leqslant s-1$. Set

$$E^r(W_{n+1} - W_n) := (\Lambda^r(W_{n+1} - W_n), A_n \Lambda^r(W_{n+1} - W_n)) \tag{5.3.5}$$

and subtract the equations at the n-th step from that at the $(n+1)$-th step and do inner product with $W_{n+1} - W_n$ to obtain

$$\partial_t E^r(W_{n+1} - W_n)^2$$

$$
\begin{aligned}
&= (\partial_t h_n \Lambda^r(\eta_{n+1}-\eta_n), \Lambda^r(\eta_{n+1}-\eta_n)) \\
&\quad -2(A_nB_n\partial_x\Lambda^r(W_{n+1}-W_n), \Lambda^r(W_{n+1}-W_n)) \\
&\quad +2([B_n,\Lambda^r]\partial_x(W_{n+1}-W_n), A_n\Lambda^r(W_{n+1}-W_n)) \\
&\quad -2(\Lambda^r[(B_n-B_{n-1})\partial_x W_n + F_n - F_{n-1}], A_n\Lambda^r(W_{n+1}-W_n)) \\
&=: I + II + III + IV. \qquad (5.3.6)
\end{aligned}
$$

For the former three of the four terms here, we do exactly the same estimate as that in Theorem 5.2.1 and have

$$
|I| + |II| + |III| \leqslant \alpha C(r, h_{\min}, R) E^r(W_{n+1}-W_n)^2 \tag{5.3.7}
$$

For the fourth term, note that

$$
\|F_n - F_{n-1}\|_{H^r} \leqslant \alpha C(r, h_{\min}, R) E^r(W_{n+1}-W_n) \tag{5.3.8}
$$

and

$$
\begin{aligned}
&\|(B_n-B_{n-1})\partial_x W_n\|_{H^r} \\
\leqslant\ & C(r)\|B_n-B_{n-1}\|_{H_r}\|W_n\|_{H^{r+1}} \\
\leqslant\ & \alpha C(r, h_{\min}, R) E^r(W_n - W_{n-1}). \qquad (5.3.9)
\end{aligned}
$$

Hence

$$
\begin{aligned}
&|IV| \\
\leqslant\ & C(r, h_{\min}, R)\|(B_n-B_{n-1})\partial_x W_n + F_n - F_{n-1}\|_{H^r}\|W_{n+1}-W_n\|_{H^r} \\
\leqslant\ & \alpha C(r, h_{\min}, R)\Big(E^r(W_{n+1}-W_n)^2 + E^r(W_{n+1}-W_n)E^r(W_n-W_{n-1})\Big). \qquad (5.3.10)
\end{aligned}
$$

Here we use the assumption that $r \leqslant s-1$ so that $\|\partial_x W_n\|_{H^r} \leqslant C(r, h_{\min}, R)$. Gluing things together, we obtain

$$
\begin{aligned}
&\partial_t E^r(W_{n+1}-W_n)^2 \\
\leqslant\ & \alpha C(r, h_{\min}, R)\Big(E^r(W_{n+1}-W_n)^2 + E^r(W_{n+1}-W_n)E^r(W_n-W_{n-1})\Big), \qquad (5.3.11)
\end{aligned}
$$

which reduces to

$$\partial_t E^r(W_{n+1} - W_n) \leqslant \alpha C(r, h_{\min}, R)(E^r(W_{n+1} - W_n) + E^r(W_n - W_{n-1})) \tag{5.3.12}$$

which in turn implies that

$$E^r(W_{n+1} - W_n) \leqslant (e^{\alpha C(r,h_{\min},R)t} - 1)E^r(W_n - W_{n-1}). \tag{5.3.13}$$

because by the choice of initial value $E^r(W_{n+1} - W_n)|_{t=0} = 0$.

We can ensure that T is so small that

$$e^{C(r,h_{\min},R)T} < \frac{3}{2} \tag{5.3.14}$$

(which is the second time we adjust T after we did this in Theorem 5.2.1). Therefore, for all $t \in [0, T/\alpha]$,

$$E^r(W_{n+1} - W_n) \leqslant \frac{1}{2}E^r(W_n - W_{n-1}). \tag{5.3.15}$$

So there is $W \in C([0, T/\alpha]; H^r)$ so that

$$W_n \to W \quad \text{in } C([0, T/\alpha]; H^r). \tag{5.3.16}$$

Taking $r = s - 1$, by the equations, this implies that

$$\partial_t W_n \to \partial_t W \quad \text{in } C([0, T/\alpha]; H^{s-2}). \tag{5.3.17}$$

Henceforth, it follows that W is a weak solution to the system (5.2.1) and the system (5.0.2)–(5.0.3).

Step 2

Furthermore, since we have $\|W_n\|_{H^s}$ uniformly bounded for all t and n, a standard interpolation argument would give that

$$\lim_{n\to\infty} \sup_{t\in[0,T/\alpha]} \|W_n - W\|_{H^{s'}} = 0 \quad \text{for all } 0 \leqslant s' < s. \tag{5.3.18}$$

So we actually have $W_n \to W$ in $C([0, T/\alpha]; H^{s'})$ for all $s' < s$.

To move further, notice that $H^{-s'}$ is dense in H^{-s} and that the dual pairing between $H^{-s'}, H^{s'}$ is just L^2-inner product. So for fixed $\phi \in H^{-s}$, choose a proper $\tilde{\phi} \in H^{-s'}$ so that

$$\begin{aligned} |(\phi, W_n - W)| &\leqslant \left\|\phi - \tilde{\phi}\right\|_{H^{-s}} \|W_n - W\|_{H^s} + \left|(\tilde{\phi}, W_n - W)\right| \\ &\leqslant C(h_{\min}, R)\left\|\phi - \tilde{\phi}\right\|_{H^{-s}} + \left|(\tilde{\phi}, W_n - W)\right| \\ &\to C(h_{\min}, R)\left\|\phi - \tilde{\phi}\right\|_{H^{-s}} \quad \text{as } n \to \infty. \end{aligned} \tag{5.3.19}$$

This proves that W_n converges uniformly to W in the weak topology of H^s regardless of the time t, and hence $W \in C_w([0, T/\alpha]; H^s)$.

We can actually improve this weak continuity to strong continuity, i.e. $W \in C([0, T/\alpha]; H^s)$. To show the strong continuity, it suffices to show the right continuity because the equation (and the approach we will use) is time-reversible. H^s is a Hilbert space and we have weak continuity already, so all we need is to prove that

$$(\Lambda^s W, A_0 \Lambda^s W)(t_0) \geqslant \limsup_{t \downarrow t_0} (\Lambda^s W, A_0 \Lambda^s W)(t) \quad \forall t_0 \in [0, \frac{T}{\alpha}). \tag{5.3.20}$$

To see this, we do the a priori estimate as what we did in Theorem 5.2.1 and obtain an analogy of (5.2.19)

$$\partial_t (\Lambda^s W, A_0 \Lambda^s W) \leqslant \alpha C(s, h_{\min}, R)((\Lambda^s W, A_0 \Lambda^s W) + 1). \tag{5.3.21}$$

Solving it will grant us

$$(\Lambda^s W, A_0 \Lambda^s W)(t) \leqslant e^{C\alpha(t - t_0)} (\Lambda^s W, A_0 \Lambda^s W)(t_0) + e^{C\alpha(t - t_0)} - 1. \tag{5.3.22}$$

Sending t down to t_0 will give us the desired inequality.

Step 3

To see the blow up criteria, we take advantage of the fact that s is an integer and modify our energy estimate (5.2.19).

Recall that the time derivative of the $E^s(W)$ is written as

$$\begin{aligned} \partial_t\Big(E^s(W)^2\Big) &= (h_t \Lambda^s u, \Lambda^s u) - 2(AB\Lambda^s W_x, \Lambda^s W) \\ &\quad + 2([B, \Lambda^s] W_x, A\Lambda^s W) - 2(\Lambda^s F, A\Lambda^s W). \end{aligned} \tag{5.3.23}$$

1) For the first term,

$$|h_t| = \alpha|(hu)_x| \leqslant \alpha\|W\|_{W^{1,\infty}} \tag{5.3.24}$$

so

$$|(h_t\Lambda^s u, \Lambda^s u)| \leqslant \alpha C\|W\|_{W^{1,\infty}}\|W\|_{H^s}^2. \tag{5.3.25}$$

2) For the second term,

$$\begin{aligned}
&|2(AB\Lambda^s W_x, \Lambda^s W)| \\
\leqslant\ & |(\alpha g u_x \Lambda^s\eta, \Lambda^s\eta)| + |2(g h_x \Lambda^s\eta, \Lambda^s u)| + |(\alpha(hu)_x\Lambda^s u, \Lambda^s u)| \\
\leqslant\ & \alpha C\|W\|_{W^{1,\infty}}\|W\|_{H^s}^2. \qquad (5.3.26)
\end{aligned}$$

3) For the third term, we use the sharper results (5.1.4) and (5.1.5),

$$\begin{aligned}
&|([B, \Lambda^s]W_x, A\Lambda^s W)| \\
\leqslant\ & |(\alpha[u, \Lambda^s]\eta_x, \Lambda^s\eta)| + |(\alpha[\eta, \Lambda^s]u_x, \Lambda^s\eta)| + |(\alpha[u, \Lambda^s]u_x, h\Lambda^s u)| \\
\leqslant\ & \alpha C(s)(1 + \|h\|_{L^\infty})\|W\|_{W^{1,\infty}}\|W\|_{H^s}^2 \qquad (5.3.27)
\end{aligned}$$

4) For the fourth term,

$$\begin{aligned}
\|f\|_{H^s} &= \left\|\varepsilon\alpha\mathcal{I}_h^{-1}\partial_x(2h^3u_x^2 - \frac{1}{2}gh^2\eta_x^2)\right\|_{H^s} \\
&\leqslant \alpha\left\|2h^3u_x^2 - \frac{1}{2}gh^2\eta_x^2\right\|_{H^{s-1}} \\
&\leqslant \alpha C(s)(\|W\|_{W^{1,\infty}}^4 + \|W\|_{W^{1,\infty}}^3)\|W\|_{H^s} \qquad (5.3.28)
\end{aligned}$$

by applying (5.1.4) several times. Hence

(5.3.29)

$$|(\Lambda^s F, A\Lambda^s W)| = |\Lambda^s f, h\Lambda^s u| \leqslant \alpha C(s)\|h\|_{L^\infty}(\|W\|_{W^{1,\infty}}^4 + \|W\|_{W^{1,\infty}}^3)\|W\|_{H^s}^2$$

Collecting everything yields

$$\partial_t\Big(E^s(W)^2\Big) \leqslant \alpha C(s)(1 + \|W\|_{W^{1,\infty}}^5)\|W\|_{H^s}^2$$

$$\leqslant \quad \frac{\alpha C(s)}{h_{\min}}(1+\|W\|_{W^{1,\infty}}^5)E^s(W)^2 \tag{5.3.30}$$

So long as $h \geqslant h_{\min}$ bounded below and $\|W\|_{W^{1,\infty}}$ stays bounded, $E^s(W)$ is bounded and the standard continuation argument applies. Actually, since $W \in C([0, T_{\max}/\alpha); H^2)$, the energy $\int_{\mathbb{R}} \mathcal{E}$ is conserved, and hence

$$\|W(t,\cdot)\|_{L^\infty} \leqslant C\|W(t,\cdot)\|_{H^1} \leqslant C(h_{\min},\varepsilon)\left\|W^0(t,\cdot)\right\|_{H^1}, \tag{5.3.31}$$

so $\|W(t,\cdot)\|_{L^\infty}$ can be dropped from the blow up condition. $\square$

Chapter 6

Existence of Blow-up Phenomena

In dimensional variables, with the operator $\mathcal{I}_h = h - \partial_x \circ \partial_x \circ h^3$, the regularized Saint-Venant equations (rSV) are written as

$$h_t + (hu)_x = 0\,, \tag{6.0.1}$$

$$u_t + gh_x + uu_x + \varepsilon \mathcal{I}_h^{-1} \partial_x \Big(2h^3 u_x^2 - \frac{1}{2} g h^2 h_x^2 \Big) = 0. \tag{6.0.2}$$

Here $h : D \to [0, \infty)$ represents the depth of the fluid and $u : D \to \mathbb{R}$ represents the horizontal velocity of the fluid. The one-dimensional domain D is either the real line $\mathbb{R}$ or the 1D torus $\mathbb{T} = \mathbb{R}/l\mathbb{Z}$, where $l > 0$ is the perimeter of the torus. All the results in this section does not depend on l.

We have shown in Chapter 4 that for every classical shock wave, the above equation admits a non-oscillatory traveling wave solution which is continuous and piecewise smooth, having a weak singularity at a single point where the energy is dissipated as it is for the classical shock.

It is tempting to explore whether a smooth initial data governed by this system would generate singularities in finite time. This section addresses this question and gives a positive answer.

In Section 6.1, we present how the smallness of the energy controls the lower bound of the water depth. In Section 6.2, we prove a analysis result that handles the nonlocal part. In Section 6.3, the Riccati-type quantities and equations are introduced along with the ideas and intuitions that guide the proof. In Section 6.4, the proof of the existence of the blow-up phenomena is presented in the order that the construction of the initial data comes first, followed by a detailed analysis that consolidates the intuitions from Section 6.3.

6.1 Non-zero depth condition

It is desirable to understand when the non-zero depth condition holds:

$$\exists\, h_{\min} > 0, \quad \inf_{x\in D} h \geqslant h_{\min}. \tag{6.1.1}$$

However, it is usually not easy to ensure this: in general, even if the energy is small, there could be a narrow downwards cusp at which height goes to zero The two-component Camassa-Holm equation admits some degenerate front-type traveling wave solutions (see [16]) which however necessarily have $h \to 0$ as either $x \to +\infty$ or $-\infty$.

In the regularized shallow-water system, since the energy assembles the structure of $H^1(\mathbb{R})$ norm of (h, u), we do have control over the depth of the water when initial energy is small.

Theorem 6.1.1. *Assume $h, u : \mathbb{T}\times[0,T] \to \mathbb{R}$ are measurable functions with $h(t), u(t) \in H^1(\mathbb{T})$ for all $t \in [0,T]$. Also assume that the system conserves mass and momentum while dissipating energy, i.e. for all $t \in [0,T]$*

$$\frac{d}{dt}\int_{\mathbb{T}} h = 0, \quad \frac{d}{dt}\int_{\mathbb{T}} hu = 0, \quad \frac{d}{dt}\int_{\mathbb{T}} \frac{1}{2}hu^2 + \frac{1}{2}gh^2 + \frac{1}{2}\varepsilon\Big(h^3u_x^2 + gh^2h_x^2\Big) \leqslant 0.$$

Let $h_0 > 0, u_0 \in \mathbb{R}$. If $\frac{1}{|\mathbb{T}|}\int_{\mathbb{T}} h\Big|_{t=0} \geqslant h_0$ and the modified initial energy

$$E_0 := \int_{\mathbb{T}} \frac{1}{2}h(u-u_0)^2 + \frac{1}{2}g(h-h_0)^2 + \frac{1}{2}\varepsilon\Big(h^3u_x^2 + gh^2h_x^2\Big)\Big|_{t=0} \leqslant \frac{1}{3}g\sqrt{\varepsilon}h_0^3$$

for some $u_0 \in \mathbb{R}$, we will have

$$\inf_{\mathbb{T}\times[0,T]} h \geqslant h_{\min}$$

where $h_{\min} \in [0, h_0]$ is the unique root in $[0, h_0]$ of the polynomial equation (w.r.t. w)

$$2w^3 - 3h_0 w^2 + h_0^3 - \frac{3E_0}{g\sqrt{\varepsilon}} = 0.$$

Proof. First of all, note that by conservation of mass and momentum and the energy inequality, the modified energy is also a dissipating quantity, i.e.

$$\frac{d}{dt}\int_{\mathbb{T}} \frac{1}{2}h(u-u_0)^2 + \frac{1}{2}g(h-h_0)^2 + \frac{1}{2}\varepsilon\Big(h^3 u_x^2 + gh^2 h_x^2\Big) \leqslant 0.$$

Fix time $t \in [0, T]$. Since we will only work on time t, $h(\cdot, t), u(\cdot, t)$ are simply written as h, u in this proof and the time variable will be omitted.

Suppose $h(0) = w \in [0, h_0), h(a) = h_0$ for some $a > 0$. Also assume that $h([0, a]) \subset [w, h_0]$ (which is always possible by finding the smallest such interval).

Then the modified energy on the interval $[0, a]$ is

$$\begin{aligned}
& \int_0^a \left(\frac{1}{2}h(u-u_0)^2 + \frac{1}{2}g(h-h_0)^2 + \frac{1}{2}\varepsilon\Big(h^3 u_x^2 + gh^2 h_x^2\Big)\right) dx \\
\geqslant\ & \frac{1}{2}g\int_0^a \Big((h-h_0)^2 + \varepsilon h^2 h_x^2\Big)\, dx \\
\overset{x=az}{=}\ & \frac{1}{2}g\int_0^1 \Big(a(\tilde{h}-h_0)^2 + \frac{\varepsilon}{a}\tilde{h}^2\tilde{h}_z^2\Big)\, dz
\end{aligned}$$

where $\tilde{h} : [0, 1] \to [w, h_0]$ is such that $\tilde{h}(z) = h(az)$. Then by AM-GM the right hand side of the above is greater than or equals to

$$g\sqrt{\varepsilon}\int_0^1 (h_0 - \tilde{h})\tilde{h}\tilde{h}_z\, dz = g\sqrt{\varepsilon}\int_{\tilde{h}(0)}^{\tilde{h}(1)} \left(\frac{1}{2}h_0\tilde{h}^2 - \frac{1}{3}\tilde{h}^3\right)_z dz = \frac{g\sqrt{\varepsilon}}{6}(2w^3 - 3h_0 w^2 + h_0^3),$$

which is an decreasing function of w on $[0, h_0]$.

Since domain $\mathbb{T}$ is periodic, there must be another interval on which h goes from h_0 down to w. So the total energy on $\mathbb{T}$ at time t is at least

$$\frac{g\sqrt{\varepsilon}}{3}(2w^3 - 3h_0 w^2 + h_0^3) \leqslant E_0$$

by the assumption that energy is dissipating. This gives the lower bound on $w = h(0)$. $\square$

Remark 6.1.2.

a) The lower bound estimate is sharp, i.e. we have the equality that

$$\inf_{\substack{\{(a,h):h\in H^1([0,a]\\ h(0)=w,h(a)=h_0\}}} \int_0^a \left((h-h_0)^2+\varepsilon h^2 h_x^2\right) dx = \frac{\sqrt{\varepsilon}}{3}(2w^3-3h_0w^2+h_0^3),$$

but the infimum is not attainable on a finite domain.

b) The above proof doesn't essentially rely on the finite length of the domain, so the argument can be extended to infinite domain as long as $h(+\infty)=h(-\infty)=h_0, u(+\infty)=u(-\infty)=u_0$ and the modified energy is finite.

c) If for some initial modified energy E_0 the theorem implies $h \geqslant h_{\min} \geqslant 0$, it also implies that $h \leqslant 2h_0 - h_{\min}$ by the same proof.

d) Intuitively, with a small amount of water particles a narrow downwards cusp could be formed within a tiny horizontal length scale while the total energy of it remains small. However, this is not the case for rSGN.

6.2 A preliminary result

To handle the nonlocal term in (6.0.2), we need a L^∞ bound on the solution of the elliptic operator $\mathcal{I}_h$. The following Landau-Kolmogorov interpolation inequality is crucial

Lemma 6.2.1. *Let $f \in C^2(\mathbb{R};\mathbb{R})$ be such that*

$$\|f\|_{L^\infty} < \infty, \quad \|f''\|_{L^\infty} < \infty.$$

Then

$$\|f'\|_{L^\infty}^2 \leqslant 2\|f\|_{L^\infty}\|f''\|_{L^\infty}. \tag{6.2.1}$$

Proof. Assume $\|f''\|_{L^\infty}=1$ and $f'(0)=a>0$. Then

$$f(a) = f(0)+\int_0^a f'(x)\,dx = f(0)+\int_0^a \left(f'(0)+\int_0^x f''(y)\,dy\right) dx$$

$$\geqslant \; f(0) + \int_0^a \left(a + \int_0^x (-1)\, dy \right) dx = f(0) + \frac{1}{2}a^2 \tag{6.2.2}$$

Similarly, $f(-a) \leqslant f(0) - \frac{1}{2}a^2$. So

$$2\|f\|_{L^\infty} \geqslant f(a) - f(-a) \geqslant a^2 = |f'(0)|^2. \tag{6.2.3}$$

□

The nonlocal term can be bounded using the following important lemma.

Lemma 6.2.2. *Let $h \in C^1(\mathbb{R})$ and $0 < h_{\min} \leqslant h \leqslant h_{\max} < \infty$. Assume $f, v : \mathbb{R} \to \mathbb{R}$ satisfy $\mathcal{I}_h v = f$ in the classical sense. Then*

$$\|v\|_{L^\infty} \leqslant \frac{1}{h_{\min}} \|f\|_{L^\infty} \quad \textit{and} \quad \|v_x\|_{L^\infty} \leqslant \frac{2}{\sqrt{\varepsilon}} \frac{h_{\max}^2}{h_{\min}^4} \|f\|_{L^\infty}. \tag{6.2.4}$$

Proof. Suppose a new variable z on $\mathbb{R}$ is such that

$$\frac{d}{dz} = h^3 \frac{d}{dx}, \tag{6.2.5}$$

then under the new variable

$$hv - \varepsilon h^{-3} v_{zz} = f. \tag{6.2.6}$$

By the weak maximum principle and the fact that h is bounded below, one has $\|v\|_{L^\infty} \leqslant \frac{1}{h_{\min}} \|f\|_{L^\infty}$. Therefore

$$\|v_{zz}\|_{L^\infty} = \frac{1}{\varepsilon} \left\| h^4 v - h^3 f \right\|_{L^\infty} \leqslant \frac{1}{\varepsilon} \left(\frac{h_{\max}^4}{h_{\min}} + h_{\max}^3 \right) \|f\|_{L^\infty} \leqslant \frac{2}{\varepsilon} \frac{h_{\max}^4}{h_{\min}} \|f\|_{L^\infty}, \tag{6.2.7}$$

hence the Landau-Kolmogorov interpolation inequality (6.2.1) says

$$\begin{aligned} &(h_{\min})^3 \|v_x\|_{L^\infty} \leqslant \|v_z\|_{L^\infty} \leqslant \left(2\|v\|_{L^\infty} \|v_{zz}\|_{L^\infty} \right)^{1/2} \\ \leqslant \;& \frac{2}{\sqrt{\varepsilon}} \frac{h_{\max}^2}{h_{\min}} \|f\|_{L^\infty}. \end{aligned} \tag{6.2.8}$$

□

6.3 Riccati-type analysis

In this section, the main goal is to present the fundamental ideas that convince us the solutions to (6.0.1)–(6.0.2) with certain initial condition do blow up.

Write Riemann invariants ($R_\pm$) of the classical shallow water system and the two corresponding characteristic speeds ($\lambda_\pm$) as

$$\begin{aligned} R_+ &= u + 2\sqrt{gh}, \qquad \lambda_+ = u + \sqrt{gh}, \\ R_- &= u - 2\sqrt{gh}, \qquad \lambda_- = u - \sqrt{gh}, \end{aligned} \tag{6.3.1}$$

which satisfy

$$\lambda_+ = \frac{1}{4}(3R_+ + R_-), \quad \lambda_- = \frac{1}{4}(R_+ + 3R_-). \tag{6.3.2}$$

Noting that the function inside the nonlocal term in (6.0.2) takes the form

$$2h^3u_x^2 - \frac{1}{2}gh^2h_x^2 = 2h^3(u_x^2 - \frac{1}{4}gh^{-1}h_x^2) = 2h^3(\lambda_+)_x(\lambda_-)_x, \tag{6.3.3}$$

one can derive the corresponding equations along characteristic curves

$$\frac{d^+}{dt}R_+ := (R_+)_t + \lambda_+(R_+)_x = -2\varepsilon\mathcal{I}_h^{-1}\partial_x\Big(h^3(\lambda_+)_x(\lambda_-)_x\Big), \tag{6.3.4}$$

$$\frac{d^-}{dt}R_- := (R_-)_t + \lambda_-(R_-)_x = -2\varepsilon\mathcal{I}_h^{-1}\partial_x\Big(h^3(\lambda_+)_x(\lambda_-)_x\Big). \tag{6.3.5}$$

Here $\frac{d^+}{dt}, \frac{d^-}{dt}$ indicate the derivatives along "+" and "−" characteristic curves, respectively. Now we have rewritten the entire equations in terms of the classical shallow water Riemann invariants.

The flow maps $X_+, X_- : \mathbb{T}\times[0,\infty) \to \mathbb{T}$ along the "+" and "−" characteristic curves are defined through

$$\begin{cases} \frac{\partial X_+}{\partial t}(\xi,t) = \lambda_+(X_+(\xi,t),t) \\ X_+(\xi,0) = \xi \end{cases}, \quad \begin{cases} \frac{\partial X_-}{\partial t}(\zeta,t) = \lambda_-(X_-(\zeta,t),t) \\ X_-(\zeta,0) = \zeta \end{cases}, \tag{6.3.6}$$

where ξ, ζ are Lagrangian variables. Differentiating the first set of equations w.r.t. ξ, one obtains

$$\left(\frac{\partial X_+}{\partial \xi}\right)_t = (\lambda_+)_x\frac{\partial X_+}{\partial \xi}, \quad \frac{\partial X_+}{\partial \xi}(\xi,0) = 1. \tag{6.3.7}$$

So along a certain "+" characteristic curve, $(\lambda_+)_x \leqslant 0$ everywhere implies that $\frac{\partial X_+}{\partial \xi} \in (0, 1]$ everywhere on the curve. Such characteristics curves are concentrating towards each other and become denser as time goes. The same holds true for the "-" characteristic curves.

Write the Riccati-type quantities

$$P_+ = (R_+)_x, \quad P_- = (R_-)_x \tag{6.3.8}$$

which clearly have

$$(\lambda_+)_x = \frac{1}{4}(3P_+ + P_-), \quad (\lambda_-)_x = \frac{1}{4}(P_+ + 3P_-) \tag{6.3.9}$$

From asymptotic analysis in Section 4.2, we know that near singularity: P_- stays bounded and P_+ blows up to $-\infty$. So both $(\lambda_+)_x, (\lambda_-)_x$ blows up to $-\infty$, which certainly implies that characteristics curves are concentrating. Actually, since P_+ blows up along "+" characteristic curves in finite time, "+" characteristic curves are absorbed by the singularity point in the steady wave profile.

Differentiating (6.3.4)–(6.3.5), one obtains that P_+ and P_- satisfy Riccati-type equations with the presence of a nonlocal term

$$\begin{aligned}
\frac{d^+}{dt}P_+ &= -\frac{1}{4}(3P_+ + P_-)P_+ - 2\varepsilon\partial_x\mathcal{I}_h^{-1}\partial_x\Big(h^3(\lambda_+)_x(\lambda_-)_x\Big), \\
\frac{d^-}{dt}P_- &= -\frac{1}{4}(P_+ + 3P_-)P_- - 2\varepsilon\partial_x\mathcal{I}_h^{-1}\partial_x\Big(h^3(\lambda_+)_x(\lambda_-)_x\Big).
\end{aligned}$$

The next step is to extract the local part of the nonlocal term. Let G denote a primitive function of $(\lambda_+)_x(\lambda_-)_x$. Recall that $\mathcal{I}_h = h - \varepsilon\partial_x \circ h^3 \circ \partial_x$, the nonlocal term can be rearranged as

$$\begin{aligned}
& 2\varepsilon\partial_x\mathcal{I}_h^{-1}\partial_x\Big(h^3(\lambda_+)_x(\lambda_-)_x\Big) \\
= \; & 2\partial_x\mathcal{I}_h^{-1} \circ \varepsilon\partial_x \circ h^3 \circ \partial_x(G) = 2\partial_x\mathcal{I}_h^{-1} \circ (-\mathcal{I}_h + h)(G) \\
= \; & -2\partial_x G + 2\partial_x\mathcal{I}_h^{-1}(hG) = -2(\lambda_+)_x(\lambda_-)_x + 2\partial_x\mathcal{I}_h^{-1}(hG) \qquad (6.3.10)
\end{aligned}$$

Within this identity, $-2(\lambda_+)_x(\lambda_-)_x$ is a local term, the other term can be bounded by the H^1 norms of h, u by Lemma 6.2.2. Whence one can rewrite the Riccati-type equations as

$$\frac{d^+}{dt}P_+ = -\frac{1}{2}(\lambda_+)_x(P_+ - 3P_-) - 2\partial_x\mathcal{I}_h^{-1}(hG),$$

$$= -\frac{3}{8}P_+^2 + P_+P_- + \frac{3}{8}P_-^2 - 2\partial_x\mathcal{I}_h^{-1}(hG), \tag{6.3.11}$$

$$\frac{d^-}{dt}P_- = \frac{3}{8}P_+^2 + P_+P_- - \frac{3}{8}P_-^2 - 2\partial_x\mathcal{I}_h^{-1}(hG). \tag{6.3.12}$$

These two equations are of central importance because the nonlocal term that appears here is essentially a constant, and after that (6.3.11)–(6.3.12) is a system whose behaviors are governed by the quadratic terms in P_+ and P_-.

We shall show that P_+ blows up and P_- stays bounded. The equation (6.3.11) perfectly agrees with this goal while (6.3.12) implies that if the goal were true, the integral of P_+^2 along the "-" characteristics had to be bounded. This gives rise to the following two observations:

a) $\frac{\partial X_+}{\partial \xi}P_+^2$ is a constant along the "+" characteristic curves, i.e. the concentrating effect of the "+" characteristic curves and the blow-up effect of P_+^2 offset and exactly balance each other.

b) The integrals of P_+^2 along the "-" characteristic curves are bounded.

We shall use a) to derive b), and the exact meaning of a) is the following identity:

$$\frac{d^+}{dt}\left(\frac{\partial X_+}{\partial \xi}P_+^2\right) = \frac{d^+}{dt}\left(\frac{\partial X_+}{\partial \xi}\right)P_+^2 + \frac{\partial X_+}{\partial \xi}2P_+\frac{d^+}{dt}P_+$$

$$= (\lambda_+)_x\frac{\partial X_+}{\partial \xi}P_+^2 + \frac{\partial X_+}{\partial \xi}2P_+\Big(-\frac{1}{2}(\lambda_+)_x(P_+ - 3P_-) - 2\partial_x\mathcal{I}_h^{-1}(hG)\Big)$$

$$= \frac{\partial X_+}{\partial \xi}P_+\Big(3(\lambda_+)_xP_- - 4\partial_x\mathcal{I}_h^{-1}(hG)\Big) \tag{6.3.13}$$

Here the upshot is that the highest order terms (cubic in P_+) match exactly and go away.

6.4 Existence of blow-up phenomena

Now we are ready to state and prove the main theorem.

Theorem 6.4.1. *Let the domain $D = \mathbb{T}$. Then there exists certain smooth initial data with which the system* (3.6.2)–(3.6.3) *will blow up in finite time. The precise meaning of blowing up is that there exists $0 < T < \infty$ such that the solution exists and stays smooth on $\mathbb{T} \times [0, T)$ with*

$$\|(h_x, u_x)(\cdot, t)\|_{L^\infty} \to +\infty \quad \textit{as } t \uparrow T. \tag{6.4.1}$$

Remark 6.4.2. a) We can prove that

$$P_+ = u_x + \sqrt{\frac{g}{h}} h_x \to -\infty \quad \text{as } t \uparrow T. \tag{6.4.2}$$

is the true blow-up behavior instead of (6.4.1) with one more assumption on the initial data.

The idea is that, although the main proof goes by a contradiction argument, once we have (6.4.2) the solution exists and stays smooth before the blow-up time $T_{\max}$, so all the arguments in the proof works until time $T_{\max}$. If we had one more assumption that asserts the flatness of the solution elsewhere, we could argue that P_+ does blow up as described.

b) The proof works in both a bounded domain $\mathbb{T}$ and an unbounded domain $\mathbb{R}$. The estimate of the nonlocal term does not depend on the length of domain and we only need L^2 norms of P_+, P_- to control the nonlocal term.

c) One should NOT expect finite time blow-up for all smooth initial data, because in the numerical simulation in Figure 4.1 we see that the smoothed-out dam breaking initial data admits a rarefaction wave traveling to the left which stays smooth for all time.

Proof. **Step Zero**. First of all, the constants in this proof are chosen as the following: the initial average height $h_0 > 0$ is arbitrary. Define positive constants $C_1(\varepsilon), C_2(\varepsilon, h_0), C_3(\varepsilon, h_0)$ explicitly by

$$C_1(\varepsilon) \overset{\text{def}}{=} \frac{6g}{\sqrt{\varepsilon}}, \quad C_2(\varepsilon, h_0) \overset{\text{def}}{=} \frac{216 C_1}{\sqrt{\varepsilon} h_0}, \quad C_3(\varepsilon, h_0) \overset{\text{def}}{=} \frac{8 C_1}{\sqrt{2 g h_0}}. \tag{6.4.3}$$

Choose real numbers $M>0, N>0, \delta>0$ such that

$$
\begin{aligned}
&(6.4.4) && M &\geqslant\ & 4C_2+6C_3+12C_3^2,\\
& && \delta^{-1} &\geqslant\ & \Big(M+8(1+C_3^2)+\frac{2C_2}{C_3}\Big)\frac{1}{\sqrt{gh_0}}+\frac{\sqrt{3}(C_2^2+C_3^2)}{C_1}\\
&(6.4.5) && & & +\frac{500(h_0+\varepsilon h_0^3)M^2}{g\sqrt{\varepsilon}h_0^3},\\
&(6.4.6) && N &\geqslant\ & \frac{5}{3}C_3+\frac{8\sqrt{6gh_0}}{\delta}.
\end{aligned}
$$

Fix $h_0>0$. The initial data $\bar h, \bar u : \mathbb{T}\to\mathbb{R}$ are chosen to be smooth and satisfy the following:

$$
\frac{1}{|\mathbb{T}|}\int_{\mathbb{T}}\bar h \geqslant h_0, \tag{6.4.7}
$$

$$
E_0=\frac{1}{2}\int_{\mathbb{T}}\bar h\bar u^2+g(\bar h-h_0)^2+\varepsilon(\bar h^3\bar u_x^2+g\bar h^2\bar h_x^2)\leqslant \frac{1}{6}g\sqrt{\varepsilon}h_0^3, \tag{6.4.8}
$$

$$
P_-|_{t=0}=\bar u_x-\sqrt{\frac{g}{\bar h}}\bar h_x\equiv 0, \tag{6.4.9}
$$

$$
P_+(x,0)=\bigg(\bar u_x+\sqrt{\frac{g}{\bar h}}\bar h_x\bigg)(x,0)\leqslant -M \quad \forall\, x\in I\subset\mathbb{T}, \tag{6.4.10}
$$

$$
P_+(x_I,0)=-N \text{ with } x_I \text{ the midpoint of } I. \tag{6.4.11}
$$

where I is a non-degenerate interval whose length $|I|=\delta$.

It is not straightforward to see the existence of such functions $(\bar h,\bar u)$. The idea is that, if only the first three conditions (6.4.7)–(6.4.9) were present, one would take $\bar h\equiv h_0, \bar u\equiv 0$. Starting from there, one can adjust the value in a small interval $2I$ (the symmetric dilation of the interval I w.r.t. its midpoint) to ensure that $P_+\leqslant -M$ on I while keeping (6.4.7)–(6.4.9) valid. The same strategy works for the one point condition (6.4.11).

Actually, here is a step-by-step construction that guarantees (6.4.7)–(6.4.10). Unfold the torus $\mathbb{T}$ to form a bounded interval $\mathbb{T}_0\subset\mathbb{R}$. The goal is to construct some functions on $\mathbb{R}$ which are only non-constant on a small interval within $\mathbb{T}_0$. Let $x_I\in\mathbb{T}_0$, φ be a smooth cutoff function and $\tilde I=(x_I-\frac{1}{2}, x_I+\frac{1}{2})$ such that

$$
\varphi|_{\tilde I}=1,\quad \varphi|_{2\tilde I\setminus\tilde I}\in[0,1],\quad \varphi|_{(2\tilde I)^c}=0,\quad \|\varphi\|_{L^2}^2\leqslant 2. \tag{6.4.12}
$$

Define $\tilde{h}, \tilde{u} : \mathbb{R} \to \mathbb{R}$ to be two bump functions by the following: for all $x \in \mathbb{R}$

$$\begin{aligned}
\overline{h}(x) &\stackrel{\text{def}}{=} \left(\sqrt{h_0} + \frac{M}{2\sqrt{g}} \int_{-\infty}^{x} (\varphi_\delta(y - 2\delta) - \varphi_\delta(y)\, dy)\, dy\right)^2, \\
\overline{u}(x) &\stackrel{\text{def}}{=} \frac{M}{2} \int_{-\infty}^{x} (\varphi_\delta(y - 2\delta) - \varphi_\delta(y)\, dy)\, dy,
\end{aligned}$$

where φ_δ is such that $\varphi_\delta(x) = \varphi(\frac{x - x_I}{\delta} + x_I)$ with its support on $[x_I - \delta, x_I + \delta]$. One can check that $\overline{h}(\pm\infty) = h_0, \overline{u}(\pm\infty) = 0$, and

$$\overline{u}_x(x) = \sqrt{\frac{g}{\overline{h}(x)}}\overline{h}_x(x) = -\frac{1}{2}M\varphi_\delta(x) + \frac{1}{2}M\varphi_\delta(x - 2\delta), \tag{6.4.13}$$

so $\overline{h}_x, \overline{u}_x$ are only non-zero on the supports of φ_δ and form bumps respectively. This pair of functions agrees with (6.4.9) and (6.4.10). Moreover, since $\overline{h}$ is an upward bump function, (6.4.7) is fulfilled.

To ensure (6.4.8), one needs to choose a proper δ. Noting that the supports of φ_δ have lengths 2δ, one computes that

$$\begin{gathered}
\left\|\overline{h}\right\|_{L^\infty} \leqslant \left(\sqrt{h_0} + \tfrac{M}{\sqrt{g}}\delta\right)^2, \quad \left\|\overline{h} - h_0\right\|_{L^\infty} \leqslant 2\sqrt{\tfrac{h_0}{g}}M\delta + \tfrac{M^2\delta^2}{g} \quad , \|\overline{u}\|_{L^\infty} \leqslant M\delta, \\
\left\|\overline{h}_x\right\|_{L^2}^2 \leqslant M^2 \frac{\|\overline{h}\|_{L^\infty}}{g} \|\varphi_\delta\|_{L^2}^2, \quad \text{and } \|\overline{u}_x\|_{L^2}^2 \leqslant M^2\|\varphi_\delta\|_{L^2}^2.
\end{gathered} \tag{6.4.14}$$

Since $\|\varphi_\delta\|_{L^2}^2 = \delta\|\varphi\|_{L^2}^2$, by the choice (6.4.5) of δ, initial energy condition (6.4.8) is fulfilled. The interval $I \stackrel{\text{def}}{=} \varphi_\delta^{-1}(1)$ whose length $|I| = \delta$. One can see that the initial conditions (6.4.7)–(6.4.10) are fulfilled.

Lastly, by virtue of the above adjustment, one can enforce (6.4.11) by adjusting values in a tinier interval within I (one point adjustment is much easier than that on an interval, hence skipped).

Step One. Assume for the sake of contradiction that, under such initial condition, the system doesn't blow up in the sense of (6.4.1). By the short-time wellposedness theorem and the blow-up criterion, one must have that for all $t \in (0, \infty)$

$$\sup_{[0,t]} \|(h_x, u_x)\|_{L^\infty(\mathbb{T})} < \infty. \tag{6.4.15}$$

Therefore, by well-posedness theorem, there is a unique solution defined on $\mathbb{T}\times[0,\infty)$ that stays smooth on the whole domain. With enough regularity, (6.0.1)–(6.0.2) hold in the classical sense, hence the system conserves mass, momentum, and energy. By non-zero depth theorem(6.1.1) and initial conditions (6.4.7)–(6.4.8), one obtains

$$\frac{1}{2}h_0 \leqslant h \leqslant \frac{3}{2}h_0. \tag{6.4.16}$$

Recall that $P_\pm = u_x \pm \sqrt{\frac{g}{h}}h_x$,

$$P_\pm^2 \leqslant 2(u_x^2 + \frac{g}{h}h_x^2) \leqslant 2(u_x^2 + \frac{2g}{h_0}h_x^2), \tag{6.4.17}$$

then by (6.4.8) and energy conservation, for all $t \geqslant 0$

$$\|P_\pm(\cdot,t)\|_{L^2}^2 \leqslant 2\int_{\mathbb{T}} u_x^2\,dx + \frac{4g}{h_0}\int_{\mathbb{T}} h_x^2\,dx \leqslant \frac{6g}{\sqrt{\varepsilon}} = C_1. \tag{6.4.18}$$

Define $G : \mathbb{T}\times[0,\infty) \to \mathbb{R}$ such that for all $y \in \mathbb{T}$

$$G(y,t) = \int_{x_I}^{y} (\lambda_+)_x(\lambda_-)_x\,dx. \tag{6.4.19}$$

Here the specific starting point x_I is only to specify a certain primitive function of $(\lambda_+)_x(\lambda_-)_x$. Since $(\lambda_+)_x, (\lambda_-)_x$ are just convex combinations of P_+, P_-,

$$\|G\|_{L^\infty(\mathbb{T}\times[0,\infty))} \leqslant \sup_{t\in[0,\infty)} \|(\lambda_+)_x(\lambda_-)_x\|_{L^1(\mathbb{T})} \leqslant C_1. \tag{6.4.20}$$

Therefore, by lemma (6.2.2),

$$\left\|2\partial_x \mathcal{I}_h^{-1}(hG)\right\|_{L^\infty(\mathbb{T})} \leqslant \frac{144}{\sqrt{\varepsilon}h_0^2}\|hG\|_{L^\infty(\mathbb{T})} \leqslant \frac{216}{\sqrt{\varepsilon}h_0}C_1 = C_2$$

for all time $t \in [0,\infty)$.

Moreover, since h, u are smooth, so are λ_+, λ_-. By the flow map definitions (6.3.6), $X_+(\xi), X_-(\zeta)$ solve first order ODEs that are locally Lipschitz in the X variable, so "+" characteristic curves don't cross with themselves. The same holds for the "−" characteristic curves. In other words, $X_+(t), X_-(t)$ are diffeomorphisms on $\mathbb{T}$ for all time $t \in [0,\infty)$.

Now we can start to get estimates on P_+, P_-.

First of all, we clarify the region $\mathcal{R}_I$ on which we do most of the work. Denote the endpoints of the interval I in initial condition (6.4.10) by x_0, x_1 so that $I = [x_0, x_1]$. Note that by construction of initial data (6.4.5) $|I| = \delta$ is rather small compared to quantities defined through M, ε, h_0. Consider two characteristic curves

$$t \mapsto X_+(x_0, t), \quad \text{and } t \mapsto X_-(x_1, t). \tag{6.4.21}$$

Since the relative speed between two characteristic curves

$$\lambda_+ - \lambda_- = 2\sqrt{gh} \in [\sqrt{2gh_0}, \sqrt{6gh_0}] \tag{6.4.22}$$

is bounded above and below, they intersect at time

$$T_I \in \left(\delta/\sqrt{6gh_0}, \delta/\sqrt{2gh_0}\right). \tag{6.4.23}$$

Denote the "triangular" region enclosed by these two characteristic curves and x-axis by $\mathcal{R}_I$. That is,

$$\mathcal{R}_I \stackrel{\text{def}}{=} \{(x,t) \in \mathbb{T} \times [0, \infty) : 0 \leqslant t \leqslant T_I, X_+(x_0, t) \leqslant x \leqslant X_-(x_1, t)\} \tag{6.4.24}$$

Note that by definition $\mathcal{R}_I \cap (\mathbb{T} \times \{0\}) = I \times \{0\}$, so the initial condition in this region is such that

$$P_+(x, 0) \leqslant -M, \quad P_-(x, 0) = 0 \quad \text{for all } x \in I. \tag{6.4.25}$$

We shall prove that

$$P_+ \leqslant -M \quad \text{and} \quad 0 \leqslant P_- \leqslant C_3 \quad \text{on } R_I. \tag{6.4.26}$$

The justification of this assertion (6.4.26) is the major part of the remaining proof. Actually, with (6.4.26), one can easily compute the blow-up time as shown in the last part of step 3.

The bounded spatial derivatives (6.4.15) implies that

$$F(t) \stackrel{\text{def}}{=} \sup_{\{(x,s) \in \mathcal{R}_I : 0 \leqslant s \leqslant t\}} |P_-(x, s)| < +\infty \tag{6.4.27}$$

is well-defined, continuous, increasing, and finite everywhere. By initial condition (6.4.25), $F(0)=0$.

Let $T\in(0,T_I]$ be such that $F(T)\leqslant C_3=\frac{100C_1}{\sqrt{2gh_0}}$, the constant defined in step zero. Such T exists since F starts from 0. Then by equation (6.3.11) and largeness of M (6.4.4),

$$\begin{aligned}\frac{d^+}{dt}P_+ &\leqslant -\frac{3}{8}P_+^2+P_+P_-+\frac{3}{8}P_-^2+C_2\leqslant-\frac{1}{4}P_+^2+3P_-^2+C_2\\ &\leqslant -\frac{1}{4}M^2+3C_3^2+C_2\leqslant 0.\end{aligned}\tag{6.4.28}$$

Therefore for $t\in[0,T],(x,t)\in\mathcal{R}_I$,

$$P_-(x,t)\leqslant C_3,\quad P_+(x,t)\leqslant -M,\quad (\lambda_+)_x(x,t)\leqslant 0.\tag{6.4.29}$$

Step Two.

The goal of this step is to prove the following:

Claim: Fix any point $\zeta_0\in I$ and let $t\in(0,T]\subset(0,T_I]$ such that $(X^-(\zeta_0,t),t)\in\mathcal{R}_I$. Given that $|P_-|\leqslant F(T)\leqslant C_3$ in $(\mathbb{T}\times[0,T])\cap\mathcal{R}_I$, one can bound the integral of P_+^2 on the "$-$" characteristic curves by

$$\int_0^t P_+^2(X_-(\zeta_0,s),s)\,ds\leqslant\frac{1}{4}C_3(\varepsilon,h_0).$$

Now we shall present the proof of the claim.

For every $\xi\in\mathbb{T}$, $\xi<\zeta$, let $s(\xi)$ be the time that the "+" characteristic curve starting from ξ and the "$-$" characteristic curve starting from ζ_0 intersect, i.e.

$$X_+(\xi,s(\xi))=X_-(\zeta_0,s(\xi)).\tag{6.4.30}$$

Differentiating w.r.t. ξ, one has

$$\frac{\partial X_+}{\partial\xi}+\lambda_+\frac{ds}{d\xi}=\lambda_-\frac{ds}{d\xi}\qquad\Longrightarrow\qquad\frac{ds}{d\xi}=\frac{1}{\lambda_--\lambda_+}\frac{\partial X_+}{\partial\xi}=-\frac{1}{2\sqrt{gh}}\frac{\partial X_+}{\partial\xi}.\tag{6.4.31}$$

Let $\xi_0=X_+(t)^{-1}(X_-(\zeta_0,t))\in I$. In view of the above change of variable, one can pull P_+^2 back to $\mathbb{T}\times\{0\}$ (and everything still stays inside $\mathcal{R}_I$)

$$\int_0^t P_+^2(X_-(\zeta_0,s),s)\,ds$$

$$
\begin{aligned}
&= \int_{\xi_0}^{\zeta_0} \left(P_+^2 \frac{1}{2\sqrt{gh}}\right)(X_+(\xi, s(\xi)), s(\xi)) \frac{\partial X_+}{\partial \xi}(\xi, s(\xi))\, d\xi \\
&\leqslant \frac{1}{\sqrt{2gh_0}} \int_{\xi_0}^{\zeta_0} P_+^2(X_+(\xi, s(\xi)), s(\xi)) \frac{\partial X_+}{\partial \xi}(\xi, s(\xi))\, d\xi \\
&= \frac{1}{\sqrt{2gh_0}} \int_{\xi_0}^{\zeta_0} \left(P_+^2(\xi, 0) + \int_0^{s(\xi)} \frac{d^+}{dt}\left(P_+^2 \frac{\partial X_+}{\partial \xi}\right)(X_+(\tau, s(\tau)), s(\tau))\, d\tau\right) d\xi \\
&\leqslant \frac{1}{\sqrt{2gh_0}} \|P_+(\cdot, 0)\|_{L^2(\mathbb{T})}^2 + \frac{1}{\sqrt{2gh_0}} \int_{\xi_0}^{\zeta_0} \int_0^{s(\xi)} \frac{d^+}{dt}\left(P_+^2 \frac{\partial X_+}{\partial \xi}\right) d\tau\, d\xi.
\end{aligned}
$$

In the last line, we bound the initial spatial integral of P_+^2 by its L^2 norm, which is only true when ξ_0, ζ_0 stays in the same period of $\mathbb{T}$, which is valid since $[\xi_0, \zeta_0] \subset I \subset \mathbb{T}$.

Let

$$
\mathcal{A} := \int_{\xi_0}^{\zeta_0} \int_0^{s(\xi)} \left| \frac{d^+}{dt}\left(P_+^2 \frac{\partial X_+}{\partial \xi}\right)\right| d\tau\, d\xi. \tag{6.4.32}
$$

By the crucial derivative computation (6.3.13)

$$
\mathcal{A} \;=\; \int_{\xi_0}^{\zeta_0} \int_0^{s(\xi)} \left| \frac{\partial X_+}{\partial \xi} P_+\Big(3(\lambda_+)_x P_- - 4\partial_x \mathcal{I}_h^{-1}(hG)\Big)\right| d\tau\, d\xi. \tag{6.4.33}
$$

Since $4(\lambda_+)_x = 3P_+ + P_-$ and $P_- \leqslant F(t) \leqslant C_3$, the integrand above is a quadratic polynomial in P_+ times $\frac{\partial X_+}{\partial \xi}$, in which the linear term can be bounded by the quadratic term and the constant term:

$$
\begin{aligned}
&\left| \frac{\partial X_+}{\partial \xi} P_+\Big(3(\lambda_+)_x P_- - 4\partial_x \mathcal{I}_h^{-1}(hG)\Big)\right| \\
=\; & \frac{\partial X_+}{\partial \xi}\Big(\frac{3}{4}|P_+P_-|\,|3P_+ + P_-| + 2|P_+|C_2\Big) \\
\leqslant\; & \frac{\partial X_+}{\partial \xi}\Big(3(1+C_3^2)P_+^2 + C_3^2 + C_2^2\Big)
\end{aligned}
$$

Here $(\lambda_+)_x \leqslant 0$ so by (6.3.7) $\frac{\partial X_+}{\partial \xi}$ is a nonnegative decreasing function on $[0, T]$ with value $\leqslant 1$. Note that

$$
\zeta_0 - \xi_0 \leqslant \|\lambda_+ - \lambda_-\|_{L^\infty} t \leqslant \sqrt{6gh_0}\, t, \tag{6.4.34}
$$

so

$$
\int_{\xi_0}^{\zeta_0} \int_0^{s(\xi)} \frac{\partial X_+}{\partial \xi}\, d\tau\, d\xi \leqslant (\zeta_0 - \xi_0) t \leqslant \sqrt{6gh_0}\, t^2. \tag{6.4.35}
$$

For the P_+^2 term, one use the inequality

$$\frac{1}{s}\int_0^s f(\tau)\,d\tau \leqslant f(0) + \int_0^s |f'(\tau)|\,d\tau \quad \forall\, f \in C^1(\mathbb{R}) \tag{6.4.36}$$

to obtain

$$\begin{aligned} &\int_{\xi_0}^{\zeta_0}\int_0^{s(\xi)} \frac{\partial X_+}{\partial \xi} P_+^2\,d\tau\,d\xi \\ \leqslant\;& t\int_{\xi_0}^{\zeta_0}\left(P_+^2(\xi,0) + \int_0^{s(\xi)}\left|\frac{d^+}{dt}\left(\frac{\partial X_+}{\partial \xi}P_+^2\right)\right|d\tau\right)d\xi \\ \leqslant\;& t(\|P_+(\cdot,0)\|_{L^2(\mathbb{T})}^2 + \mathcal{A}). \end{aligned} \tag{6.4.37}$$

Putting everything together, one obtains

$$\begin{aligned} & \mathcal{A} \leqslant 3(1+C_3^2)\Big(\|P_+(\cdot,0)\|_{L^2(\mathbb{T})}^2 + \mathcal{A}\Big)t + (C_3^2 + C_2^2)\sqrt{6gh_0}\,t^2 \\ \text{whence}\quad & (1-3(1+C_3^2)t)\mathcal{A} \leqslant 3(1+C_3^2)C_1 t + C_1 t \\ \Longrightarrow\quad & \mathcal{A} \leqslant 8(1+C_3^2)C_1 t. \end{aligned} \tag{6.4.38}$$

Here one uses the smallness of t twice: by the bounds (6.4.23) on T_I and the defining inequality (6.4.5) of δ,

$$t \leqslant T_I \leqslant \frac{\delta}{\sqrt{2gh_0}} \tag{6.4.39}$$

implies that

$$(C_3^2 + C_2^2)\sqrt{6gh_0}\,t \leqslant C_1 \quad \text{and} \quad 3(1+C_3^3)t \leqslant \frac{1}{2}. \tag{6.4.40}$$

So we can conclude that if $F(T) \leqslant C_3$, for all $\zeta_0 \in I, t \in [0,T]$ with $X^-(\zeta_0,t) \in \mathcal{R}_I$,

$$\int_0^t P_+^2(X_-(\zeta_0,s),s)\,ds \leqslant \frac{C_1 + 8(1+C_3^2)C_1 t}{\sqrt{2gh_0}} \leqslant \frac{2C_1}{\sqrt{2gh_0}} = \frac{1}{4}C_3 \tag{6.4.41}$$

where (6.4.23) and (6.4.5) are used again.

Step Three.

By the Riccati-type equation (6.3.12) for P_-, one has for $t \in [0,T], (\zeta,t) \in \mathcal{R}_I$

$$\frac{d^-}{dt}P_-(X_-(\zeta,t),t) \leqslant \frac{3}{8}P_+^2 + P_+P_- - \frac{3}{8}P_-^2 + C_2 \leqslant 2P_+^2 + C_2$$

$$(6.4.42) \qquad \Longrightarrow \quad P_-(X_-(\zeta,t),t) \leqslant 2\int_0^t P_+^2(X_-(\zeta,s),s)\,ds + C_2 t \leqslant \frac{1}{2}C_3 + C_2 t,$$

hence $F(t) \leqslant \frac{1}{2}C_3 + C_2 t$. On the other hand, for $(x,t) \in \mathcal{R}_I, t \in [0,T_I]$, since

$$\begin{aligned} \frac{d^-}{dt}P_- &\geqslant \frac{3}{8}P_+^2 + P_+P_- - \frac{3}{8}P_-^2 - C_2 \geqslant \frac{1}{4}P_+^2 - 3P_-^2 - C_2 \\ &\geqslant \frac{1}{4}M^2 - 3C_3^2 - C_2 \geqslant 0, \end{aligned} \tag{6.4.43}$$

one has that

$$P_-(x,t) \geqslant 0.$$

This is essentially saying that for $(x,t) \in \mathcal{R}_I, t \in [0,T_I]$

$$(6.4.44) \qquad F(t) \leqslant C_3 \quad \Longrightarrow \quad 0 \leqslant P_-(x,t) \leqslant F(t) \leqslant \frac{1}{2}C_3 + C_2 t.$$

One sees from (6.4.23) and (6.4.5) that $C_2 T_I \leqslant \frac{1}{2}C_3$. Therefore, one must have

$$(6.4.45) \qquad |P_-(x,t)| \leqslant F(t) \leqslant \frac{1}{2}C_3 + C_2 t \leqslant C_3 \quad \text{for all } (x,t) \in \mathcal{R}_I, t \in [0,T_I].$$

Therefore, we have assertion (6.4.26).

Finally, by the Riccati-type equation (6.3.11) for P_+, for points (x,t) lie on the "+" characteristic curve that starts from x_I with $(x,t) \in \mathcal{R}_I$

$$\frac{d^+}{dt}P_+ \leqslant -\frac{3}{8}(P_+ - \frac{1}{3}C_3)(P_+ + 3C_3) + C_2.$$

Note that we already maximize the right hand side over all possible P_- and replace P_- by C_3. All terms in this ordinary differential inequality have definite signs. Formally solving it, one gets

$$\ln\left(\frac{P_+ - \frac{1}{3}C_3}{P_+ + 3C_3}\right)\Bigg|_0^t \leqslant \frac{10}{3}C_3\Big(-\frac{3}{8} + \frac{2C_2}{M^2}\Big)t \leqslant -C_3 t.$$

Using the initial condition $P_+(x_I, 0) = -N$ one has

$$P_+(x,t) \leqslant -3C_3 - \frac{10}{3}C_3\bigg(\frac{N + \frac{1}{3}C_3}{N - 3C_3}e^{-C_3 t} - 1\bigg)^{-1}.$$

So P_+ blows up to $-\infty$ before

$$t_0 = \frac{1}{C_3}\Big(\ln(N + \frac{1}{3}C_3) - \ln(N - 3C_3)\Big) \leqslant \frac{4}{N}$$

which is valid since $\frac{C_3}{N} \leqslant \frac{3}{5}$ by defining inequality (6.4.6) of N. Again by (6.4.6),

$$\frac{\delta/2}{\lambda_+ - \lambda_-} \geqslant \frac{\delta}{2\sqrt{6gh_0}} \geqslant \frac{4}{N},$$

so the formal blow-up time is less than the minimum time needed to escape $\mathcal{R}_I$, hence the "+" characteristic curve starts from x_I stays in $\mathcal{R}_I$ before its blow-up time and this formal calculation is valid. This argument contradicts our assumption that the solution doesn't blow up in the sense of (6.4.1). □

6.5 Asymptotic blow-up profile

We want to derive the asymptotic behavior near the singularity when the blow-up just happens.

6.5.1 Blow-up profile for the rSV equations

Suppose $P_0 < 0$ is the initial value of P_+ that loosely resembles that of the weakly singular shock profile (as we constructed in the proof of theorem 6.4.1). P_0 admits a global minimum at 0, from which the singularity is produced. Since P_- is bounded before P_+ blows up, we assume

$$|P_-| \ll |P_+| \tag{6.5.1}$$

in the space time domain. Therefore we can rewrite (6.3.7) and (6.3.11) as

$$\left(\frac{\partial X_+}{\partial \xi}\right)_t = \frac{3}{4}P_+\frac{\partial X_+}{\partial \xi}, \tag{6.5.2}$$

$$\frac{d^+}{dt}P_+ = \frac{\partial}{\partial t}P_+(X_+(\xi,t),t) = -\frac{3}{8}P_+^2. \tag{6.5.3}$$

With the initial data P_0 we can solve (6.5.3) along the "+" characteristic curves:

$$P_+(X_+(\xi,t),t) = \frac{P_0(\xi)}{1+\frac{3}{8}tP_0(\xi)}. \tag{6.5.4}$$

Following the "+" characteristic curve emitting from the global minimum point 0 of P_0 the blow-up happens at

$$T = -\frac{8}{3P_0(0)} \ll 1. \tag{6.5.5}$$

From (6.5.2) and (6.5.3) one can compute that

$$\frac{d^+}{dt}\left(\frac{\partial X_+}{\partial \xi}P_+^p\right) = (\frac{3}{4}-\frac{3}{8}p)\frac{\partial X_+}{\partial \xi}P_+^{p+1}, \tag{6.5.6}$$

which is 0 if and only if $p=2$. This balancing effect agrees with the rigorous computation (6.3.13). Actually, the right hand side of equation (6.3.13) has a leading term $(\frac{9}{4}P_-)\frac{\partial X_+}{\partial \xi}P_+^2$ where $P_- \geqslant 0$ is bounded. So

$$0 \leqslant \frac{d^+}{dt}\left(\frac{\partial X_+}{\partial \xi}P_+^2\right) \leqslant C\frac{\partial X_+}{\partial \xi}P_+^2 \tag{6.5.7}$$

which leads to the inequality that

$$P_0^2 \leqslant \frac{\partial X_+}{\partial \xi}P_+^2 \leqslant P_0^2 + Ct. \tag{6.5.8}$$

Since $t \leqslant T \ll 1$, we have when $p=2$

$$\left(\frac{\partial X_+}{\partial \xi}P_+^p\right)(\xi,\cdot) \approx P_0^p(\xi) \quad \text{for all } \xi. \tag{6.5.9}$$

Now we can use (6.5.9) to compute that

$$\frac{\partial X_+}{\partial \xi}(\xi,t) = \frac{P_0^p(\xi)}{P_+^p(X_+(\xi,t),t)} = \left(1+\frac{3}{8}tP_0(\xi)\right)^p. \tag{6.5.10}$$

At the vicinity of $0 \in D$, assume the initial data satisfies

$$P_0 \sim P_0(0) + C\xi^n \quad \text{with } n \geqslant 2. \tag{6.5.11}$$

Then near $(X_+(0,T),T)$,

$$\frac{\partial X_+}{\partial \xi}(\xi,T) \sim C\xi^{np} \implies X_+(\xi,T) \sim C\xi^{np+1}. \tag{6.5.12}$$

Consequently, for ξ close to 0

$$(6.5.13)\qquad \begin{aligned} P_+(X_+(\xi,T),T) &\sim C\frac{P_0(0)}{\xi^n} = C\frac{P_0(0)}{X_+(\xi,T)^{\frac{n}{np+1}}} \\ \Longrightarrow\quad P_+(x,T) &\sim Cx^{-\frac{n}{np+1}}, \end{aligned}$$

which is in $L^p_{\mathrm{loc}}(\mathbb{R})$.

In the rSV case, $p = 2$. Then $n = 2$ would lead to an asymptotic behavior

$$(6.5.14)\qquad P_+(x,T) \sim Cx^{-\frac{2}{5}}.$$

6.5.2 Blow-up profile for the inviscid Burger's equation

As a comparison, asymptotic blow-up profile for the inviscid Burger's equation

$$(6.5.15)\qquad u_t + uu_x = 0$$

is derived. In Burger's equation $v \stackrel{\text{def}}{=} u_x$ satisfies

$$(6.5.16)\qquad v_t + uv_x = -v^2$$

which implies that along characteristic curves $X(\xi,\cdot)$ with $\frac{\partial X}{\partial t}(\xi,t) = u(X(\xi,t),t)$ we have analogues of (6.5.2) and (6.5.3)

$$(6.5.17)\qquad \left(\frac{\partial X}{\partial \xi}(\xi,t)\right)_t = v\frac{\partial X}{\partial \xi}(\xi,t)$$

$$(6.5.18)\qquad \frac{\partial}{\partial t}\Big(v(X(\xi,t),t)\Big) = -v^2$$

so similar to (6.5.6), we obtain that

$$(6.5.19)\qquad \frac{\partial}{\partial t}\left(\frac{\partial X}{\partial \xi}v^p\right) = 0 \quad \text{if and only if} \quad p = 1.$$

Hence Burger's equation also admits asymptotic behavior near singularity at the blow-up moment as described in (6.5.13). When $n = 2$, the asymptotic behavior of the blow-up profile is that

$$(6.5.20)\qquad v \sim x^{-2/3} \quad \text{and} \quad u \sim x^{1/3}.$$

Different powers p in two systems lead to different asymptotic behaviors. The reason for the rSV equations having a different power p is that in the rSV equations, the local part of the nonlocal term (see the decomposition in (6.3.10)) contributes to the ODEs (6.3.11),(6.5.3) of P_+ while not altering the characteristic equations (6.3.7),(6.5.2). In other words, the nonlocal term changes how P_+ evolves along the "+" characteristic curves while leaving the "+" characteristic curves themselves unchanged.

Chapter 7

Future Directions

There are many interesting analytical and numerical issues that we cannot address.

a) Rarefaction wave profile.

Consider the rSV equations with a dam-breaking initial data. Analogous to the classical shallow-water system, solutions to such a Riemann problem should consist of two parts: one is the shock wave traveling to the right which has a weakly singular shock profile described in Chapter 4; the other one is a rarefaction wave traveling to the left that gets flatter as time goes. See numerical simulations in Figure 4.1.

The question is whether there is a way to describe the rarefaction wave part in the rSV case. Although in numerical simulation it behaves very closely as that in the classical case, the profile is not a function of a single variable xt^{α} ($\alpha \in \mathbb{R}$) as in the classical case due to the presence of higher order terms. The construction of such profile, if possible, gives an explicit example of smooth initial data that don't blow up in finite time.

b) Further exploration on the cusped solitary waves.

In the Camassa-Holm case, a complete list of traveling wave solutions is derived in [34]. Similarly, we want to understand all traveling wave solutions to the rSV equations. Solving the steady rSV system (4.1.3)–(4.1.4), we find essentially two possibilities: the weakly singular shock profile and the (upward or downward) cusped solitary wave.

The cusped solitary wave is a valid distributional weak solution, i.e. the resulting equations are locally integrable and have distributional derivative zero. The profile of a cusped solitary wave of elevation is plotted in Fig. 4.3.

However, we were not able to compute a clean isolated traveling cusped wave by taking the numerically computed wave profile for (h, u) as initial data on a regular grid.

Indeed, there is no particular reason our pseudo-spectral code should work well for such a singular solution, and anyway it may not be numerically stable. However, when taking the h-profile in Fig. 4.3 as initial data with *zero* initial velocity, the numerical solution develops two peaked waves traveling in opposite direction as indicated Fig. 4.4. While hardly conclusive, this evidence suggests that cusped solutions may be relevant in the dynamics of the rSV system.

The two peaks here are slightly skewed compared to the profile of a cusped solitary wave. Our limited exploration uncovered no convincing evidence that cusped waves collide "cleanly" enough to justify calling them "cuspons" or suggest that the rSV system is formally integrable—It may be difficult to tell, though, as perturbed cusped waves do not leave behind a dispersive "tail" in this non-dispersive system.

c) Generalization of the blow-up proof.

The regularized Saint-Venant system (rSV) (1.2.2)–(1.2.3) and the Green-Naghdi system can be unified as Euler-Lagrange equations of this Lagrangian density as in the derivations in Chapter 3:

$$\mathcal{L} = \frac{1}{2}hu^2 - \frac{1}{2}gh^2 + \frac{1}{2}\gamma_1 h^3 u_x^2 - \frac{1}{2}\gamma_2 g h^2 h_x^2. \tag{7.0.1}$$

Here if $\gamma_1 = \gamma_2 = \varepsilon$, the resulting system is rSV while $\gamma_1 = \frac{1}{3}, \gamma_2 = 0$ corresponds to the Green-Naghdi system (see Chapter 3 for more details). It is still open whether the Green-Naghdi system admits any blow-up phenomena with smooth initial data.

In our case, the proof cannot be extended directly to the Green-Naghdi case due to the following: under the Lagrangian density (7.0.1) the counterpart of the velocity equation (1.4.6) is

$$u_t + uu_x + gh_x + \mathcal{I}_h^{-1}\partial_x(2\gamma_1 h^3 u_x^2 - \frac{1}{2}\gamma_2 g h^2 h_x^2 + (\gamma_1 - \gamma_2) g h^3 h_{xx}) = 0. \tag{7.0.2}$$

The nonlocal term appearing here becomes a "first order term" due to the presence of $h^3 h_{xx}$, for which we don't have a bound that only involves initial energy.

However, this proof can be generalized to all system that can be formulated as (6.3.11)–(6.3.12) with an asymptotic behavior prescribed by (6.5.2)–(6.5.3).

d) Global existence in $H^1(\mathbb{R})$ for the Cauchy problem.

We have seen that in Section 4.2 that the weakly singular shock profile has derivatives that behave like $x^{-1/3}$ which stays in $L^2(\mathbb{R})$. In Section 6.5, the profile at the blow up time has derivatives that are asymptotically $x^{-\frac{n}{2n+1}}$ which is in $L^2_{\text{loc}}(\mathbb{R})$ and the exponent is sharp (i.e. the derivatives are not $L^p_{\text{loc}}(\mathbb{R})$ for $p > 2$). It would be interesting to know whether the rSV system admits a global existence of solutions in $H^1(\mathbb{R})$ as that in the scalar Camassa-Holm equation (see [18,47,49]).

The system is derived through a variational principle and formally conserves energy which controls the H^1 norm provided that the height is bounded below. It also admits a non-canonical Hamiltonian structure (1.2.9)–(1.2.10) like the one known for the the Green-Naghdi equations. A natural way to attack the problem is to add some dispersion term with a free parameter so that the modified systems has global smooth solutions with H^1 norm controlled independent of the free parameter. However, the regularization term with the desired properties remains elusive.

Another inspiring example is [17] for the modified Camassa-Holm equation, in which “peakon” solutions are considered in their notion of weak solutions. For the rSV system, it is also desirable to include the weakly singular shock profiles and the cusped solitary waves as possible weak solutions.

e) $\varepsilon \to 0$.

The rSV system is reduced to the classical shallow-water system when taking $\varepsilon = 0$, so we want to understand how close these two system are, measured by the difference of their solutions under certain norms (e.g. $L^\infty([0,T]; L^p(\mathbb{R}))$ for $1 \leqslant p \leqslant 2$) and how the spatial norms (e.g. $L^p(\mathbb{R})$ for $1 \leqslant p \leqslant 2$) vary over time.

A first attempt to this question can be derived by the virtue of the energy estimate (5.3.13): let W^ε be the solution to the rSV system with parameter ε; let the smooth initial data $\overline{W}$ be fixed and independent of ε, then

$$\partial_t E^r(W^\varepsilon - W^0)^2 \leqslant \alpha C(r, h_{\min}, E^{r+1}(W^0))(E^r(W^\varepsilon - W^0)^2 + E^r(W^\varepsilon)^2), \tag{7.0.3}$$

which is a crude estimate. Since $\mathcal{I}_h$ operator depends on ε and the bounds (5.1.8) on $\mathcal{I}_h^{-1}\partial_x$ cannot be improved in terms of the exponent of ε, a better estimate requires a comparison between $\mathcal{I}_h$ operators with different ε parameters.

f) Regularizations of other models.

The ability of the rSV system to correctly model shock wave propagation non-dispersively without oscillations while conserving energy for smooth solutions is an interesting feature which deserves further investigation. As demonstrated in [9], it means that a rather straightforward pseudo-spectral method (albeit one which involves careful dealiasing, and iteration to eliminate the time derivative term in $\mathcal{R}$) computes shock speeds accurately over a wide range of values of ε, with 2ε ranging from 0.001 to 5 in the examples treated in [9].

It would be exciting if this regularization technique can be applied to other models.

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