Third Online Supplement:

Part (a) Recursions and Proof of Proposition 4.1

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- For the part (a) recursions (of Proposition 4.1):

  – <u>m- step</u>: Write $RS^{\{0\}} := RS$, ..., $r^{r+1}q^{\{0\}} := r^{r+1}q$ and define for $m = 1, \dots, d^*$ and $r = 1, 2, \dots, k-1$, that

$$RS(d)_{t,l}^{\{m\}} := \begin{cases} RS(d)_{t,l}^{\{m-1\}} + RS(d)_{t,m}^{\{m-1\}}\ RS(d)_{t-m,l-m}, & \text{if } l = 1+m, \dots, d^*+m-1, \\ RS(d)_{t,m}^{\{m-1\}} RS(d)_{t-m,l-m}, & \text{if } l = d^*+m \end{cases} \tag{1}$$

and

$$R^{r+1}S(d)_{t,l}^{\{m\}} := \begin{cases} R^{r+1}S(d)_{t,l}^{\{m-1\}} + R^{r+1}S(d)_{t,m}^{\{m-1\}}\ R^{r+1}S(d)_{t-m,l-m}, \\ \qquad \text{if } l = 1+m, \dots, d^*+m-1, \\ \Big[L_{1:k+1-r,1:k-r}^{\left((k+1)^{q-2-d}\right)} \left(R^{r+1}S(d)_{t,m}^{\{m-1\}}\ R^{r+1}S(d)_{t-m,l-m}\right) \mid OR^rS(d)_{t,l}^{\{m-1\}} + \\ L_{1:k+1-r,k+1-r}^{\left((k+1)^{q-2-d}\right)} \left(R^{r+1}S(d)_{t,m}^{\{m-1\}}\ R^{r+1}S(d)_{t-m,l-m}\right)\Big], \text{ if } l = d^*+m \end{cases} \tag{2}$$

and

$$OR^rS(d)_{t,l}^{\{m\}} := \begin{cases} OR^rS(d)_{t,l}^{\{m-1\}} + R^{r+1}S(d)_{t,m}^{\{m-1\}}\ OR^rS(d)_{t-m,l-m}, \\ \qquad \text{if } l = d^*+m+1, \dots, (r+1)d^*+m-1, \\ R^{r+1}S(d)_{t,m}^{\{m-1\}}\ OR^rS(d)_{t-m,l-m}, \\ \qquad \text{if } l = (r+1)d^*+m \end{cases} \tag{3}$$

and

$$r^{r+1}s(d)_{t,l}^{\{m\}} := \begin{cases} r^{r+1}s(d)_{t,l}^{\{m-1\}}, \text{ if } l = 1, \dots, m, \\ r^{r+1}s(d)_{t,l}^{\{m-1\}} + R^{r+1}S(d)_{t,m}^{\{m-1\}}\ r^{r+1}s(d)_{t-m,l-m}, \\ \qquad \text{if } l = 1+m, \dots, (r+1)d^*+m-1, \\ R^{r+1}S(d)_{t,m}^{\{m-1\}}\ r^{r+1}s(d)_{t-m,l-m}, \\ \qquad \text{if } l = (r+1)d^*+m \end{cases}, \tag{4}$$

where this last one is set for $r = 0$ as well; similarly,

$$RQ(d)_{t,l}^{\{m\}} := \begin{cases} RQ(d)_{t,l}^{\{m-1\}} + RQ(d)_{t,m}^{\{m-1\}}\ RS(d)_{t-m,l-m}, & \text{if } l = 1+m, \dots, d^*+m-1, \\ RQ(d)_{t,m}^{\{m-1\}}\ RS(d)_{t-m,l-m}, & \text{if } l = d^*+m \end{cases} \tag{5}$$

and

$$R^{r+1}Q(d)_{t,l}^{\{m\}} := \begin{cases} R^{r+1}Q(d)_{t,l}^{\{m-1\}} + R^{r+1}Q(d)_{t,m}^{\{m-1\}}\ R^{r+1}S(d)_{t-m,l-m}, \\ \qquad \text{if } l = 1+m, \dots, d^*+m-1, \\ \Big[L_{1:k-r}^{\left((k+1)^{q-2-d}\right)} \left(R^{r+1}Q(d)_{t,m}^{\{m-1\}}\ R^{r+1}S(d)_{t-m,l-m}\right) \mid OR^rQ(d)_{t,l}^{\{m-1\}} + \\ L_{k-r+1}^{\left((k+1)^{q-2-d}\right)} \left(R^{r+1}Q(d)_{t,m}^{\{m-1\}}\ R^{r+1}S(d)_{t-m,l-m}\right)\Big], \text{ if } l = d^*+m \end{cases} \tag{6}$$

and

$$OR^rQ(d)^{\{m\}}_{t,l} := \begin{cases} OR^rQ(d)^{\{m-1\}}_{t,l} + R^{r+1}Q(d)^{\{m-1\}}_{t,m}\ OR^rS(d)_{t-m,l-m}, \\ \qquad \text{if } l = d^* + m + 1, \dots, (r+1)d^* + m - 1, \\ R^{r+1}Q(d)^{\{m-1\}}_{t,m}\ OR^rS(d)_{t-m,l-m}, \\ \qquad \text{if } l = (r+1)d^* + m \end{cases} \tag{7}$$

and

$$r^{r+1}q(d)^{\{m\}}_{t,l} := \begin{cases} r^{r+1}q(d)^{\{m-1\}}_{t,l}, \text{ if } l = 1, \dots, m, \\ r^{r+1}q(d)^{\{m-1\}}_{t,l} + R^{r+1}Q(d)^{\{m-1\}}_{t,m}\ r^{r+1}s(d)_{t-m,l-m}, \\ \qquad \text{if } l = 1+m, \dots, (r+1)d^* + m - 1, \\ R^{r+1}Q(d)^{\{m-1\}}_{t,m} r^{r+1}s(d)_{t-m,l-m}, \\ \qquad \text{if } l = (r+1)d^* + m \end{cases}, \tag{8}$$

where this last one is also set for $r = 0$.

– <u>r- step</u>: For $r = 1, \dots, k-1$, define $R^{r+1}S(d)_{t,l}$, $l = 1, \dots, d^*$ to be square $(k+1)^{q-2-d}(k+1-r)$ matrices, such that

$$L^{\left((k+1)^{q-2-d}\right)}_{i,j}\left(R^{r+1}S(d)_{t,l}\right) := B^{l(k-r+1)-j}HB^{(d^*-l)(k-r+1)+j-1}R^rS(d)^{(i+r),\{d^*\}}_{t,d^*+l} \tag{9}$$

for $i, j = 1, \dots, k+1-r$.

Define $OR^rS(d)_{t,l}$, $l = 1 + d^*, \dots, (r+1)d^*$ to be $[(k+1)^{q-2-d}(k+1-r)] \times (k+1)^{q-2-d}$ matrices, such that

$$L^{\left((k+1)^{q-2-d}\right)}_{i,1}(OR^rS(d)_{t,l}) := \begin{cases} B^{(k-r+1)d^*}R^rS(d)^{(i+r,\ k+1),\{d^*\}}_{t,l}, & \text{if } l = 1 + d^*, \dots, 2d^*, \\ B^{(k-r+1)d^*}OR^{r-1}S(d)^{(i+r),\{d^*\}}_{t,l}, & \text{if } l = 2d^* + 1, \dots, (r+1)d^* \end{cases} \tag{10}$$

for $i = 1, \dots, k+1-r$.

Define $r^{r+1}s(d)_{t,l}$, $l = 1, \dots, (r+1)d^*$ to be the $(k+1)^{q-2-d}(k+1-r)$ long column vectors, such that

$$L^{\left((k+1)^{q-2-d}\right)}_{i}\left(r^{r+1}s(d)_{t,l}\right) := \begin{cases} L^{\left((k+1)^{q-2-d}\right)}_{i+1}\left(r^rs(d)^{\{d^*\}}_{t,l}\right), & \text{if } l = 1 \\ b^{(k-r+1)(l-1)}r^rs(d)^{(i+r),(\{d^*\},\{d^*-1\})}_{t,l}, & \text{if } l = 2, \dots, d^*, \\ b^{(k-r+1)d^*}r^rs(d)^{(i+r),\{d^*\}}_{t,l}, \\ \text{if } l = 1 + d^*, \dots, (r+1)d^* \end{cases} \tag{11}$$

for $i = 1, \dots, k+1-r$.

Similarly, define $R^{r+1}Q(d)_{t,l}$, $l = 1, \dots, d^*$ to be $(k+1)^{q-2-d}(k+1-r)$ long row vectors, such that

$$L^{\left((k+1)^{q-2-d}\right)}_{j}(R^{r+1}Q(d)_{t,l}) := B^{l(k-r+1)-j}HB^{(d^*-l)(k-r+1)+j-1}R^rQ(d)^{\{d^*\}}_{t,d^*+l} \tag{12}$$

for $j = 1, \dots, k+1-r$.

Define $OR^rQ(d)_{t,l}$, $l = 1+d^*, \dots, (r+1)d^*$ to be the $(k+1)^{q-2-d}$ long row vectors, such that

$$OR^rQ(d)_{t,l} := \begin{cases} B^{(k-r+1)d^*} R^rQ(d)^{(k+1),\{d^*\}}_{t,l}, & \text{if } l = 1+d^*, \dots, 2d^*, \\ B^{(k-r+1)d^*} OR^{r-1}Q(d)^{\{d^*\}}_{t,l}, & \text{if } l = 2d^*+1, \dots, (r+1)d^* \end{cases} . \tag{13}$$

Define $r^{r+1}q(d)_{t,l}$, $l = 1, \dots, (r+1)d^*$ to be scalars, such that

$$r^{r+1}q(d)_{t,l} := \begin{cases} r^rq(d)^{\{d^*\}}_{t,l}, & \text{if } l = 1, \\ b^{(k-r+1)(l-1)} r^rq(d)^{(\{d^*\},\{d^*-1\})}_{t,l}, & \text{if } l = 2, \dots, d^*, \\ b^{(k-r+1)d^*} r^rq(d)^{\{d^*\}}_{t,l}, & \text{if } l = 1+d^*, \dots, (r+1)d^* \end{cases} . \tag{14}$$

– <u>Basic step</u>: The fixed d is often implied within this step; for $m = 1, \dots, d^*$ (fixed $r = 1, \dots, k$), define

$$HR^rS^{(i),\{m\}}_{t,u} := \{L^{\left((k+1)^{q-2-d}\right)}_{i-r+1,1}(R^rS^{\{m\}}_{t,u})\}\{L^{\left((k+1)^{q-2-d}\right)}_{2,1}(R^rS_{t-m,u-m})\}^{-1}, \tag{15}$$

$$HR^rQ^{\{m\}}_{t,u} := \{L^{\left((k+1)^{q-2-d}\right)}_{1}(R^rQ^{\{m\}}_{t,u})\}\{L^{\left((k+1)^{q-2-d}\right)}_{2,1}(R^rS_{t-m,u-m})\}^{-1}, \tag{16}$$

followed by

$$BR^rS^{(i,j),\{m\}}_{t,l} := L^{\left((k+1)^{q-2-d}\right)}_{i-r+1,j-r+1}(R^rS^{\{m\}}_{t,l}) - HR^rS^{(i),\{m\}}_{t,d^*+m} \cdot L^{\left((k+1)^{q-2-d}\right)}_{2,j-r+1}(R^rS_{t-m,l-m}), \tag{17}$$

$$BR^rQ^{(j),\{m\}}_{t,l} := L^{\left((k+1)^{q-2-d}\right)}_{j-r+1}(R^rQ^{\{m\}}_{t,l}) - HR^rQ^{\{m\}}_{t,d^*+m} \cdot L^{\left((k+1)^{q-2-d}\right)}_{2,j-r+1}(R^rS_{t-m,l-m}), \tag{18}$$

$$br^rs^{(i),\{m\}}_{t,l} := L^{\left((k+1)^{q-2-d}\right)}_{i-r+1}(r^rs^{\{m\}}_{t,l}) - HR^rS^{(i),\{m\}}_{t,d^*+m} \cdot L^{\left((k+1)^{q-2-d}\right)}_{2}(r^rs_{t-m,l-m}), \tag{19}$$

$$br^rq^{\{m\}}_{t,l} := r^rq^{\{m\}}_{t,l} - HR^rQ^{\{m\}}_{t,d^*+m} \cdot L^{\left((k+1)^{q-2-d}\right)}_{2}(r^rs_{t-m,l-m}); \tag{20}$$

for $r = 2, \dots, k$ only, also define

$$\begin{aligned} BOR^{r-1}S^{(i),\{m\}}_{t,l} &:= L^{\left((k+1)^{q-2-d}\right)}_{i-r+1,1}(OR^{r-1}S^{\{m\}}_{t,l}) - HR^rS^{(i),\{m\}}_{t,d^*+m} \cdot \\ &\quad L^{\left((k+1)^{q-2-d}\right)}_{2,1}(OR^{r-1}S_{t-m,l-m}), \\ BOR^{r-1}Q^{\{m\}}_{t,l} &:= OR^{r-1}Q^{\{m\}}_{t,l} - HR^rQ^{\{m\}}_{t,d^*+m} \cdot L^{\left((k+1)^{q-2-d}\right)}_{2,1}(OR^{r-1}S_{t-m,l-m}). \end{aligned}$$

Next, for $i_* = r+1, \dots, k$, define

$$HB^{i_*-r}R^rS^{(i),\{m\}}_{t,u} := \{B^{i_*-r}R^rS^{(i,i_*),\{m\}}_{t,u}\}\{C^{i_*-r}R^rS^{(i_*+1,i_*)}_{t-m,u-m}\}^{-1}, \tag{21}$$

$$HB^{i_*-r}R^rQ^{\{m\}}_{t,u} := \{B^{i_*-r}R^rQ^{(i_*),\{m\}}_{t,u}\}\{C^{i_*-r}R^rS^{(i_*+1,i_*)}_{t-m,u-m}\}^{-1} \tag{22}$$

followed by

$$\begin{aligned}
BHB^{i_*-r-1}R^rS_{t,l}^{(i),\{m\}} &:= HB^{i_*-r-1}R^rS_{t,l}^{(i),\{m\}} - HB^{i_*-r}R^rS_{t,d^*+m}^{(i),\{m\}} \cdot \\
&\quad IC^{i_*-r-1}R^rS_{t-m,l-m}^{(i_*+1)}, \qquad (23)\\
BHB^{i_*-r-1}R^rQ_{t,l}^{\{m\}} &:= HB^{i_*-r-1}R^rQ_{t,l}^{\{m\}} - HB^{i_*-r}R^rQ_{t,d^*+m}^{\{m\}} \cdot \\
&\quad IC^{i_*-r-1}R^rS_{t-m,l-m}^{(i_*+1)}, \qquad (24)\\
&\vdots\\
B^{i_*-r}HR^rS_{t,l}^{(i),\{m\}} &:= B^{i_*-r-1}HR^rS_{t,l}^{(i),\{m\}} - HB^{i_*-r}R^rS_{t,d^*+m}^{(i),\{m\}} \cdot \\
&\quad C^{i_*-r-1}IR^rS_{t-m,l-m}^{(i_*+1)},\\
B^{i_*-r}HR^rQ_{t,l}^{\{m\}} &:= B^{i_*-r-1}HR^rQ_{t,l}^{\{m\}} - HB^{i_*-r}R^rQ_{t,d^*+m}^{\{m\}} \cdot \\
&\quad C^{i_*-r-1}IR^rS_{t-m,l-m}^{(i_*+1)},
\end{aligned}$$

$$\begin{aligned}
B^{i_*-r+1}R^rS_{t,l}^{(i,j),\{m\}} &:= B^{i_*-r}R^rS_{t,l}^{(i,j),\{m\}} - HB^{i_*-r}R^rS_{t,d^*+m}^{(i),\{m\}} \cdot \\
&\quad C^{i_*-r}R^rS_{t-m,l-m}^{(i_*+1,j)}, \qquad (25)\\
B^{i_*-r+1}R^rQ_{t,l}^{(j),\{m\}} &:= B^{i_*-r}R^rQ_{t,l}^{(j),\{m\}} - HB^{i_*-r}R^rQ_{t,d^*+m}^{\{m\}} \cdot \\
&\quad C^{i_*-r}R^rS_{t-m,l-m}^{(i_*+1,j)}, \qquad (26)\\
b^{i_*-r+1}r^rs_{t,l}^{(i),\{m\}} &:= b^{i_*-r}r^rs_{t,l}^{(i),\{m\}} - HB^{i_*-r}R^rS_{t,d^*+m}^{(i),\{m\}} \cdot \\
&\quad c^{i_*-r}r^rs_{t-m,l-m}^{(i_*+1)}, \qquad (27)\\
b^{i_*-r+1}r^rq_{t,l}^{\{m\}} &:= b^{i_*-r}r^rq_{t,l}^{\{m\}} - HB^{i_*-r}R^rQ_{t,d^*+m}^{\{m\}} \cdot \\
&\quad c^{i_*-r}r^rs_{t-m,l-m}^{(i_*+1)}; \qquad (28)
\end{aligned}$$

for $r = 2, \dots, k$ only, also define

$$\begin{aligned}
B^{i_*-r+1}OR^{r-1}S_{t,l}^{(i),\{m\}} &:= B^{i_*-r}OR^{r-1}S_{t,l}^{(i),\{m\}} - HB^{i_*-r}R^rS_{t,d^*+m}^{(i),\{m\}} \cdot \\
&\quad C^{i_*-r}OR^{r-1}S_{t-m,l-m}^{(i_*+1)},\\
B^{i_*-r+1}OR^{r-1}Q_{t,l}^{\{m\}} &:= B^{i_*-r}OR^{r-1}Q_{t,l}^{\{m\}} - HB^{i_*-r}R^rQ_{t,d^*+m}^{\{m\}} \cdot \\
&\quad C^{i_*-r}OR^{r-1}S_{t-m,l-m}^{(i_*+1)}.
\end{aligned}$$

In the equations above, it is defined that

$$IR^rS_{t,u}^{(i)} := \left\{L_{i-r+1,1}^{\left((k+1)^{q-2-d}\right)}(R^rS_{t,u})\right\}\left\{L_{2,1}^{\left((k+1)^{q-2-d}\right)}(R^rS_{t,u})\right\}^{-1}, \qquad (29)$$

followed by

$$\begin{aligned}
CR^rS_{t,l}^{(i,j)} &:= L_{i-r+1,j-r+1}^{\left((k+1)^{q-2-d}\right)}(R^rS_{t,l}) - IR^rS_{t,d^*}^{(i)} \cdot L_{2,j-r+1}^{\left((k+1)^{q-2-d}\right)}(R^rS_{t,l}), \qquad (30)\\
cr^rs_{t,l}^{(i)} &:= L_{i-r+1}^{\left((k+1)^{q-2-d}\right)}(r^rs_{t,l}) - IR^rS_{t,d^*}^{(i)} \cdot L_{2}^{\left((k+1)^{q-2-d}\right)}(r^rs_{t,l}); \qquad (31)
\end{aligned}$$

for $r = 2, \ldots, k$ only, also define

$$COR^{r-1}S^{(i)}_{t,l} \;\; := \;\; L^{\left((k+1)^{q-2-d}\right)}_{i-r+1,1}(OR^{r-1}S_{t,l}) - IR^rS^{(i)}_{t,d^*} \cdot L^{\left((k+1)^{q-2-d}\right)}_{2,1}(OR^{r-1}S_{t,l}).$$

It is also defined for $i_* = r+1, \ldots, k$, that

$$IC^{i_*-r}R^rS^{(i)}_{t,u} := \{C^{i_*-r}R^rS^{(i,i_*)}_{t,u}\}\{C^{i_*-r}R^rS^{(i_*+1,i_*)}_{t,u}\}^{-1}, \tag{32}$$

followed by

$$CIC^{i_*-r-1}R^rS^{(i)}_{t,l} \;\; := \;\; IC^{i_*-r-1}R^rS^{(i)}_{t,l} - IC^{i_*-r}R^rS^{(i)}_{t,d^*} \cdot IC^{i_*-r-1}R^rS^{(i_*+1)}_{t,l}, \tag{33}$$

$$\vdots$$

$$C^{i_*-r}IR^rS^{(i)}_{t,l} \;\; := \;\; C^{i_*-r-1}IR^rS^{(i)}_{t,l} - IC^{i_*-r}R^rS^{(i)}_{t,d^*} \cdot C^{i_*-r-1}IR^rS^{(i_*+1)}_{t,l},$$

$$C^{i_*-r+1}R^rS^{(i,j)}_{t,l} \;\; := \;\; C^{i_*-r}R^rS^{(i,j)}_{t,l} - IC^{i_*-r}R^rS^{(i)}_{t,d^*} \cdot C^{i_*-r}R^rS^{(i_*+1,j)}_{t,l}, \tag{34}$$

$$c^{i_*-r+1}r^rs^{(i)}_{t,l} \;\; := \;\; c^{i_*-r}r^rs^{(i)}_{t,l} - IC^{i_*-r}R^rS^{(i)}_{t,d^*} \cdot c^{i_*-r}r^rs^{(i_*+1)}_{t,l}; \tag{35}$$

for $r = 2, \ldots, k$ only, also define

$$C^{i_*-r+1}OR^{r-1}S^{(i)}_{t,l} \;\; := \;\; C^{i_*-r}OR^{r-1}S^{(i)}_{t,l} - IC^{i_*-r}R^rS^{(i)}_{t,d^*} \cdot C^{i_*-r}OR^{r-1}S^{(i_*+1)}_{t,l}.$$

For $n = 1, 2, \ldots, d^*-2$ or $n = d^*-1$ and $m = n+1, n+2, \ldots, d^*$ (fixed $r = 1, \ldots, k$), define

$$HB^{n(k-r+1)}R^rS^{(i),\{m\}}_{t,u} \;\; := \;\; \{B^{n(k-r+1)}R^rS^{(i,r),\{m\}}_{t,u}\}\{B^{n(k-r+1)}R^rS^{(r+1,r),\{n\}}_{t-m+n,u-m+n}\}^{-1}, \tag{36}$$

$$HB^{n(k-r+1)}R^rQ^{\{m\}}_{t,u} \;\; := \;\; \{B^{n(k-r+1)}R^rQ^{(r),\{m\}}_{t,u}\}\{B^{n(k-r+1)}R^rS^{(r+1,r),\{n\}}_{t-m+n,u-m+n}\}^{-1}, \tag{37}$$

followed by

$$
\begin{aligned}
BHB^{n(k-r+1)-1}R^rS_{t,l}^{(i),\{m\}} &:= HB^{n(k-r+1)-1}R^rS_{t,l}^{(i),\{m\}} - HB^{n(k-r+1)}R^rS_{t,d^*+m-n}^{(i),\{m\}} \cdot \\
&\quad HB^{n(k-r+1)-1}R^rS_{t-m+n,l-m+n}^{(r+1),\{n\}}, \qquad (38) \\
BHB^{n(k-r+1)-1}R^rQ_{t,l}^{\{m\}} &:= HB^{n(k-r+1)-1}R^rQ_{t,l}^{\{m\}} - HB^{n(k-r+1)}R^rQ_{t,d^*+m-n}^{\{m\}} \cdot \\
&\quad HB^{n(k-r+1)-1}R^rS_{t-m+n,l-m+n}^{(r+1),\{n\}}, \qquad (39) \\
B^2HB^{n(k-r+1)-2}R^rS_{t,l}^{(i),\{m\}} &:= BHB^{n(k-r+1)-2}R^rS_{t,l}^{(i),\{m\}} - HB^{n(k-r+1)}R^rS_{t,d^*+m-n}^{(i),\{m\}} \cdot \\
&\quad BHB^{n(k-r+1)-2}R^rS_{t-m+n,l-m+n}^{(r+1),\{n\}}, \qquad (40) \\
B^2HB^{n(k-r+1)-2}R^rQ_{t,l}^{\{m\}} &:= BHB^{n(k-r+1)-2}R^rQ_{t,l}^{\{m\}} - HB^{n(k-r+1)}R^rQ_{t,d^*+m-n}^{\{m\}} \cdot \\
&\quad BHB^{n(k-r+1)-2}R^rS_{t-m+n,l-m+n}^{(r+1),\{n\}}, \qquad (41) \\
&\vdots \\
B^{n(k-r+1)}HR^rS_{t,l}^{(i),\{m\}} &:= B^{n(k-r+1)-1}HR^rS_{t,l}^{(i),\{m\}} - HB^{n(k-r+1)}R^rS_{t,d^*+m-n}^{(i),\{m\}} \cdot \\
&\quad B^{n(k-r+1)-1}HR^rS_{t-m+n,l-m+n}^{(r+1),\{n\}}, \qquad (42) \\
B^{n(k-r+1)}HR^rQ_{t,l}^{\{m\}} &:= B^{n(k-r+1)-1}HR^rQ_{t,l}^{\{m\}} - HB^{n(k-r+1)}R^rQ_{t,d^*+m-n}^{\{m\}} \cdot \\
&\quad B^{n(k-r+1)-1}HR^rS_{t-m+n,l-m+n}^{(r+1),\{n\}}, \qquad (43) \\
B^{n(k-r+1)+1}R^rS_{t,l}^{(i.j),\{m\}} &:= B^{n(k-r+1)}R^rS_{t,l}^{(i,j),\{m\}} - HB^{n(k-r+1)}R^rS_{t,d^*+m-n}^{(i),\{m\}} \cdot \\
&\quad B^{n(k-r+1)}R^rS_{t-m+n,l-m+n}^{(r+1,j),\{n\}}, \qquad (44) \\
B^{n(k-r+1)+1}R^rQ_{t,l}^{(j),\{m\}} &:= B^{n(k-r+1)}R^rQ_{t,l}^{(j),\{m\}} - HB^{n(k-r+1)}R^rQ_{t,d^*+m-n}^{\{m\}} \cdot \\
&\quad B^{n(k-r+1)}R^rS_{t-m+n,l-m+n}^{(r+1,j),\{n\}}, \qquad (45) \\
b^{n(k-r+1)+1}r^rs_{t,l}^{(i),\{m\}} &:= b^{n(k-r+1)}r^rs_{t,l}^{(i),\{m\}} - HB^{n(k-r+1)}R^rS_{t,d^*+m-n}^{(i),\{m\}} \cdot \\
&\quad b^{n(k-r+1)}r^rs_{t-m+n,l-m+n}^{(r+1),\{n\}}, \qquad (46) \\
b^{n(k-r+1)+1}r^rq_{t,l}^{\{m\}} &:= b^{n(k-r+1)}r^rq_{t,l}^{\{m\}} - HB^{n(k-r+1)}R^rQ_{t,d^*+m-n}^{\{m\}} \cdot \\
&\quad b^{n(k-r+1)}r^rs_{t-m+n,l-m+n}^{(r+1),\{n\}}, \qquad (47) \\
b^{(n-1)(k-r+1)+1}r^rs_{t,l}^{(i),(\{m\},\{n\})} &:= b^{(n-1)(k-r+1)}r^rs_{t,l}^{(i),(\{m\},\{n-1\})} - HB^{n(k-r+1)}R^rS_{t,d^*+m-n}^{(i),\{m\}} \cdot \\
&\quad b^{(n-1)(k-r+1)}r^rs_{t-m+n,l-m+n}^{(r+1),(\{n\},\{n-1\})}, \text{(if } n=1, \text{ omit)} \\
b^{(n-1)(k-r+1)+1}r^rq_{t,l}^{(\{m\},\{n\})} &:= b^{(n-1)(k-r+1)}r^rq_{t,l}^{(\{m\},\{n-1\})} - HB^{n(k-r+1)}R^rQ_{t,d^*+m-n}^{\{m\}} \cdot \\
&\quad b^{(n-1)(k-r+1)}r^rs_{t-m+n,l-m+n}^{(r+1),(\{n\},\{n-1\})}, \text{(if } n=1, \text{ omit)} \\
&\vdots
\end{aligned}
$$

$$
\begin{aligned}
b^{(k-r+1)+1}r^r s_{t,l}^{(i),(\{m\},\{n\})} &:= b^{(k-r+1)}r^r s_{t,l}^{(i),(\{m\},\{n-1\})} - HB^{n(k-r+1)}R^r S_{t,d^*+m-n}^{(i),\{m\}} \cdot \\
&\quad b^{(k-r+1)}r^r s_{t-m+n,l-m+n}^{(r+1),(\{n\},\{n-1\})}, (\text{if } n=1,\ \text{omit}) \\
b^{(k-r+1)+1}r^r q_{t,l}^{(\{m\},\{n\})} &:= b^{(k-r+1)}r^r q_{t,l}^{(\{m\},\{n-1\})} - HB^{n(k-r+1)}R^r Q_{t,d^*+m-n}^{\{m\}} \cdot \\
&\quad b^{(k-r+1)}r^r s_{t-m+n,l-m+n}^{(r+1),(\{n\},\{n-1\})}, (\text{if } n=1,\ \text{omit}) \\
br^r s_{t,l}^{(i),(\{m\},\{n\})} &:= L_{i-r+1}^{\left((k+1)^{q-2-d}\right)}(r^r s_{t,l}^{\{m\}}) - HB^{n(k-r+1)}R^r S_{t,d^*+m-n}^{(i),\{m\}} \cdot \\
&\quad L_2^{\left((k+1)^{q-2-d}\right)}(r^r s_{t-m+n,l-m+n}^{\{n\}}), \qquad (48) \\
br^r q_{t,l}^{(\{m\},\{n\})} &:= r^r q_{t,l}^{\{m\}} - HB^{n(k-r+1)}R^r Q_{t,d^*+m-n}^{\{m\}} \cdot \\
&\quad L_2^{\left((k+1)^{q-2-d}\right)}(r^r s_{t-m+n,l-m+n}^{\{n\}}); \qquad (49)
\end{aligned}
$$

for $r = 2, \ldots, k$ only, also define

$$
\begin{aligned}
B^{n(k-r+1)+1}OR^{r-1}S_{t,l}^{(i),\{m\}} &:= B^{n(k-r+1)}OR^{r-1}S_{t,l}^{(i),\{m\}} - HB^{n(k-r+1)}R^r S_{t,d^*+m-n}^{(i),\{m\}} \cdot \\
&\quad B^{n(k-r+1)}OR^{r-1}S_{t-m+n,l-m+n}^{(r+1),\{n\}}, \\
B^{n(k-r+1)+1}OR^{r-1}Q_{t,l}^{\{m\}} &:= B^{n(k-r+1)}OR^{r-1}Q_{t,l}^{\{m\}} - HB^{n(k-r+1)}R^r Q_{t,d^*+m-n}^{\{m\}} \cdot \\
&\quad B^{n(k-r+1)}OR^{r-1}S_{t-m+n,l-m+n}^{(r+1),\{n\}}.
\end{aligned}
$$

Next, for $i_* = r+1, \ldots, k$, define

$$
\begin{aligned}
HB^{i_*-r+n(k-r+1)}R^r S_{t,u}^{(i),\{m\}} &:= \{B^{i_*-r+n(k-r+1)}R^r S_{t,u}^{(i,i_*),\{m\}}\} \\
&\quad \{C^{i_*-r}B^{n(k-r+1)}R^r S_{t-m+n,u-m+n}^{(i_*+1,i_*)}\}^{-1}, \qquad (50) \\
HB^{i_*-r+n(k-r+1)}R^r Q_{t,u}^{\{m\}} &:= \{B^{i_*-r+n(k-r+1)}R^r Q_{t,u}^{(i_*),\{m\}}\} \\
&\quad \{C^{i_*-r}B^{n(k-r+1)}R^r S_{t-m+n,u-m+n}^{(i_*+1,i_*)}\}^{-1}, \qquad (51)
\end{aligned}
$$

followed by

$$BHB^{i_*-r-1+n(k-r+1)}R^rS_{t,l}^{(i),\{m\}} := HB^{i_*-r-1+n(k-r+1)}R^rS_{t,l}^{(i),\{m\}} -$$
$$HB^{i_*-r+n(k-r+1)}R^rS_{t,d^*+m-n}^{(i),\{m\}} \cdot IC^{i_*-r-1}B^{n(k-r+1)}R^rS_{t-m+n,l-m+n}^{(i_*+1)}, \tag{52}$$
$$BHB^{i_*-r-1+n(k-r+1)}R^rQ_{t,l}^{\{m\}} := HB^{i_*-r-1+n(k-r+1)}R^rQ_{t,l}^{\{m\}} -$$
$$HB^{i_*-r+n(k-r+1)}R^rQ_{t,d^*+m-n}^{\{m\}} \cdot IC^{i_*-r-1}B^{n(k-r+1)}R^rS_{t-m+n,l-m+n}^{(i_*+1)}, \tag{53}$$
$$\vdots$$
$$B^{i_*-r-1}HB^{n(k-r+1)+1}R^rS_{t,l}^{(i),\{m\}} := B^{i_*-r-2}HB^{n(k-r+1)+1}R^rS_{t,l}^{(i),\{m\}} -$$
$$HB^{i_*-r+n(k-r+1)}R^rS_{t,d^*+m-n}^{(i),\{m\}} \cdot C^{i_*-r-2}ICB^{n(k-r+1)}R^rS_{t-m+n,l-m+n}^{(i_*+1)},$$
$$B^{i_*-r-1}HB^{n(k-r+1)+1}R^rQ_{t,l}^{\{m\}} := B^{i_*-r-2}HB^{n(k-r+1)+1}R^rQ_{t,l}^{\{m\}} -$$
$$HB^{i_*-r+n(k-r+1)}R^rQ_{t,d^*+m-n}^{\{m\}} \cdot C^{i_*-r-2}ICB^{n(k-r+1)}R^rS_{t-m+n,l-m+n}^{(i_*+1)},$$
$$B^{i_*-r}HB^{n(k-r+1)}R^rS_{t,l}^{(i),\{m\}} := B^{i_*-r-1}HB^{n(k-r+1)}R^rS_{t,l}^{(i),\{m\}} -$$
$$HB^{i_*-r+n(k-r+1)}R^rS_{t,d^*+m-n}^{(i),\{m\}} \cdot C^{i_*-r-1}IB^{n(k-r+1)}R^rS_{t-m+n,l-m+n}^{(i_*+1)}, \tag{54}$$
$$B^{i_*-r}HB^{n(k-r+1)}R^rQ_{t,l}^{\{m\}} := B^{i_*-r-1}HB^{n(k-r+1)}R^rQ_{t,l}^{\{m\}} -$$
$$HB^{i_*-r+n(k-r+1)}R^rQ_{t,d^*+m-n}^{\{m\}} \cdot C^{i_*-r-1}IB^{n(k-r+1)}R^rS_{t-m+n,l-m+n}^{(i_*+1)}, \tag{55}$$
$$B^{i_*-r+1}HB^{n(k-r+1)-1}R^rS_{t,l}^{(i),\{m\}} := B^{i_*-r}HB^{n(k-r+1)-1}R^rS_{t,l}^{(i),\{m\}} -$$
$$HB^{i_*-r+n(k-r+1)}R^rS_{t,d^*+m-n}^{(i),\{m\}} \cdot C^{i_*-r}HB^{n(k-r+1)-1}R^rS_{t-m+n,l-m+n}^{(i_*+1)}, \tag{56}$$
$$B^{i_*-r+1}HB^{n(k-r+1)-1}R^rQ_{t,l}^{\{m\}} := B^{i_*-r}HB^{n(k-r+1)-1}R^rQ_{t,l}^{\{m\}} -$$
$$HB^{i_*-r+n(k-r+1)}R^rQ_{t,d^*+m-n}^{\{m\}} \cdot C^{i_*-r}HB^{n(k-r+1)-1}R^rS_{t-m+n,l-m+n}^{(i_*+1)}, \tag{57}$$
$$\vdots$$
$$B^{i_*-r+1+n(k-r+1)-1}HR^rS_{t,l}^{(i),\{m\}} := B^{i_*-r+n(k-r+1)-1}HR^rS_{t,l}^{(i),\{m\}} -$$
$$HB^{i_*-r+n(k-r+1)}R^rS_{t,d^*+m-n}^{(i),\{m\}} \cdot C^{i_*-r}B^{n(k-r+1)-1}HR^rS_{t-m+n,l-m+n}^{(i_*+1)}, \tag{58}$$
$$B^{i_*-r+1+n(k-r+1)-1}HR^rQ_{t,l}^{\{m\}} := B^{i_*-r+n(k-r+1)-1}HR^rQ_{t,l}^{\{m\}} -$$
$$HB^{i_*-r+n(k-r+1)}R^rQ_{t,d^*+m-n}^{\{m\}} \cdot C^{i_*-r}B^{n(k-r+1)-1}HR^rS_{t-m+n,l-m+n}^{(i_*+1)}, \tag{59}$$
$$B^{i_*-r+1+n(k-r+1)}R^rS_{t,l}^{(i,j),\{m\}} := B^{i_*-r+n(k-r+1)}R^rS_{t,l}^{(i,j),\{m\}} -$$
$$HB^{i_*-r+n(k-r+1)}R^rS_{t,d^*+m-n}^{(i),\{m\}} \cdot C^{i_*-r}B^{n(k-r+1)}R^rS_{t-m+n,l-m+n}^{(i_*+1,j)}, \tag{60}$$
$$B^{i_*-r+1+n(k-r+1)}R^rQ_{t,l}^{(j),\{m\}} := B^{i_*-r+n(k-r+1)}R^rQ_{t,l}^{(j),\{m\}} -$$
$$HB^{i_*-r+n(k-r+1)}R^rQ_{t,d^*+m-n}^{\{m\}} \cdot C^{i_*-r}B^{n(k-r+1)}R^rS_{t-m+n,l-m+n}^{(i_*+1,j)}, \tag{61}$$

$$b^{i_*-r+1+n(k-r+1)}r^r s_{t,l}^{(i),\{m\}} := b^{i_*-r+n(k-r+1)}r^r s_{t,l}^{(i),\{m\}} - HB^{i_*-r+n(k-r+1)}R^r S_{t,d^*+m-n}^{(i),\{m\}} \cdot c^{i_*-r}b^{n(k-r+1)}r^r s_{t-m+n,l-m+n}^{(i_*+1)}, \tag{62}$$

$$b^{i_*-r+1+n(k-r+1)}r^r q_{t,l}^{\{m\}} := b^{i_*-r+n(k-r+1)}r^r q_{t,l}^{\{m\}} - HB^{i_*-r+n(k-r+1)}R^r Q_{t,d^*+m-n}^{\{m\}} \cdot c^{i_*-r}b^{n(k-r+1)}r^r s_{t-m+n,l-m+n}^{(i_*+1)}, \tag{63}$$

$$b^{i_*-r+1+(n-1)(k-r+1)}r^r s_{t,l}^{(i),(\{m\},\{n\})} := b^{i_*-r+(n-1)(k-r+1)}r^r s_{t,l}^{(i),(\{m\},\{n\})} - HB^{i_*-r+n(k-r+1)}R^r S_{t,d^*+m-n}^{(i),\{m\}} \cdot c^{i_*-r}b^{(n-1)(k-r+1)}r^r s_{t-m+n,l-m+n}^{(i_*+1),\{n\}},$$

$$b^{i_*-r+1+(n-1)(k-r+1)}r^r q_{t,l}^{(\{m\},\{n\})} := b^{i_*-r+(n-1)(k-r+1)}r^r q_{t,l}^{(\{m\},\{n\})} - HB^{i_*-r+n(k-r+1)}R^r Q_{t,d^*+m-n}^{\{m\}} \cdot c^{i_*-r}b^{(n-1)(k-r+1)}r^r s_{t-m+n,l-m+n}^{(i_*+1),\{n\}},$$

$$\vdots$$

$$b^{i_*-r+1}r^r s_{t,l}^{(i),(\{m\},\{n\})} := b^{i_*-r}r^r s_{t,l}^{(i),(\{m\},\{n\})} - HB^{i_*-r+n(k-r+1)}R^r S_{t,d^*+m-n}^{(i),\{m\}} \cdot c^{i_*-r}r^r s_{t-m+n,l-m+n}^{(i_*+1),\{n\}}, \tag{64}$$

$$b^{i_*-r+1}r^r q_{t,l}^{(\{m\},\{n\})} := b^{i_*-r}r^r q_{t,l}^{(\{m\},\{n\})} - HB^{i_*-r+n(k-r+1)}R^r Q_{t,d^*+m-n}^{\{m\}} \cdot c^{i_*-r}r^r s_{t-m+n,l-m+n}^{(i_*+1),\{n\}}; \tag{65}$$

for $r = 2, \ldots, k$ only, also define

$$B^{i_*-r+1+n(k-r+1)}OR^{r-1}S_{t,l}^{(i),\{m\}} := B^{i_*-r+n(k-r+1)}OR^{r-1}S_{t,l}^{(i),\{m\}} - HB^{i_*-r+n(k-r+1)}R^r S_{t,d^*+m-n}^{(i),\{m\}} \cdot C^{i_*-r}B^{n(k-r+1)}OR^{r-1}S_{t-m+n,l-m+n}^{(i_*+1)},$$

$$B^{i_*-r+1+n(k-r+1)}OR^{r-1}Q_{t,l}^{\{m\}} := B^{i_*-r+n(k-r+1)}OR^{r-1}Q_{t,l}^{\{m\}} - HB^{i_*-r+n(k-r+1)}R^r Q_{t,d^*+m-n}^{\{m\}} \cdot C^{i_*-r}B^{n(k-r+1)}OR^{r-1}S_{t-m+n,l-m+n}^{(i_*+1)}.$$

In the equations above, it is defined that

$$IB^{n(k-r+1)}R^r S_{t,u}^{(i)} := \{B^{n(k-r+1)}R^r S_{t,u}^{(i,r),\{n\}}\}\{B^{n(k-r+1)}R^r S_{t,u}^{(r+1,r),\{n\}}\}^{-1}, \tag{66}$$

followed by

$$CHB^{k-r+(n-1)(k-r+1)}R^r S_{t,l}^{(i)} := HB^{k-r+(n-1)(k-r+1)}R^r S_{t,l}^{(i),\{n\}} - IB^{n(k-r+1)}R^r S_{t,d^*}^{(i)} \cdot HB^{k-r+(n-1)(k-r+1)}R^r S_{t,l}^{(r+1),\{n\}}, \tag{67}$$

$$\vdots$$

$$CB^{k-r}HB^{(n-1)(k-r+1)}R^r S_{t,l}^{(i)} := B^{k-r}HB^{(n-1)(k-r+1)}R^r S_{t,l}^{(i),\{n\}} - IB^{n(k-r+1)}R^r S_{t,d^*}^{(i)} \cdot B^{k-r}HB^{(n-1)(k-r+1)}R^r S_{t,l}^{(r+1),\{n\}},$$

$$\vdots$$

$$CB^{k-r+(n-1)(k-r+1)}HR^r S_{t,l}^{(i)} := B^{k-r+(n-1)(k-r+1)}HR^r S_{t,l}^{(i),\{n\}} - IB^{n(k-r+1)}R^r S_{t,d^*}^{(i)} \cdot B^{k-r+(n-1)(k-r+1)}HR^r S_{t,l}^{(r+1),\{n\}}, \tag{68}$$

$$
\begin{aligned}
CB^{n(k-r+1)}R^rS^{(i,j)}_{t,l} &:= B^{n(k-r+1)}R^rS^{(i,j),\{n\}}_{t,l} - IB^{n(k-r+1)}R^rS^{(i)}_{t,d^*}\cdot \\
&\quad\ B^{n(k-r+1)}R^rS^{(r+1,j),\{n\}}_{t,l}, \qquad (69)\\
cb^{n(k-r+1)}r^rs^{(i)}_{t,l} &:= b^{n(k-r+1)}r^rs^{(i),\{n\}}_{t,l} - IB^{n(k-r+1)}R^rS^{(i)}_{t,d^*}\cdot \\
&\quad\ b^{n(k-r+1)}r^rs^{(r+1),\{n\}}_{t,l}, \qquad (70)\\
cb^{(n-1)(k-r+1)}r^rs^{(i),\{n\}}_{t,l} &:= b^{(n-1)(k-r+1)}r^rs^{(i),(\{n\},\{n-1\})}_{t,l} - IB^{n(k-r+1)}R^rS^{(i)}_{t,d^*}\cdot \\
&\quad\ b^{(n-1)(k-r+1)}r^rs^{(r+1),(\{n\},\{n-1\})}_{t,l},\ (\text{if } n=1,\ \text{omit})\\
&\ \ \vdots\\
cb^{k-r+1}r^rs^{(i),\{n\}}_{t,l} &:= b^{k-r+1}r^rs^{(i),(\{n\},\{n-1\})}_{t,l} - IB^{n(k-r+1)}R^rS^{(i)}_{t,d^*}\cdot \\
&\quad\ b^{k-r+1}r^rs^{(r+1),(\{n\},\{n-1\})}_{t,l},\ (\text{if } n=1,\ \text{omit})\\
cr^rs^{(i),\{n\}}_{t,l} &:= L^{\left((k+1)^{q-2-d}\right)}_{i-r+1}(r^rs^{\{n\}}_{t,l}) - IB^{n(k-r+1)}R^rS^{(i)}_{t,d^*}\cdot L^{\left((k+1)^{q-2-d}\right)}_2(r^rs^{\{n\}}_{t,l}); \qquad (71)
\end{aligned}
$$

for $r = 2,\dots,k$ only, also define

$$
\begin{aligned}
CB^{n(k-r+1)}OR^{r-1}S^{(i)}_{t,l} &:= B^{n(k-r+1)}OR^{r-1}S^{(i),\{n\}}_{t,l} - IB^{n(k-r+1)}R^rS^{(i)}_{t,d^*}\cdot \\
&\quad\ B^{n(k-r+1)}OR^{r-1}S^{(r+1),\{n\}}_{t,l}.
\end{aligned}
$$

It is also defined for $i_* = r+1,\dots,k$, that

$$
IC^{i_*-r}B^{n(k-r+1)}R^rS^{(i)}_{t,u} := \{C^{i_*-r}B^{n(k-r+1)}R^rS^{(i,i_*)}_{t,u}\}\{C^{i_*-r}B^{n(k-r+1)}R^rS^{(i_*+1,i_*)}_{t,u}\}^{-1} \qquad (72)
$$

folllowed by

$$
\begin{aligned}
&CIC^{i_*-r-1}B^{n(k-r+1)}R^rS^{(i)}_{t,l} := IC^{i_*-r-1}B^{n(k-r+1)}R^rS^{(i)}_{t,l} - \\
&\quad IC^{i_*-r}B^{n(k-r+1)}R^rS^{(i)}_{t,d^*}\cdot IC^{i_*-r-1}B^{n(k-r+1)}R^rS^{(i_*+1)}_{t,l}, \qquad (73)\\
&\quad \vdots\\
&C^{i_*-r}IB^{n(k-r+1)}R^rS^{(i)}_{t,l} := C^{i_*-r-1}IB^{n(k-r+1)}R^rS^{(i)}_{t,l} - \\
&\quad IC^{i_*-r}B^{n(k-r+1)}R^rS^{(i)}_{t,d^*}\cdot C^{i_*-r-1}IB^{n(k-r+1)}R^rS^{(i_*+1)}_{t,l},\\
&C^{i_*-r+1}HB^{k-r+(n-1)(k-r+1)}R^rS^{(i)}_{t,l} := C^{i_*-r}HB^{k-r+(n-1)(k-r+1)}R^rS^{(i)}_{t,l} - \\
&\quad IC^{i_*-r}B^{n(k-r+1)}R^rS^{(i)}_{t,d^*}\cdot C^{i_*-r}HB^{k-r+(n-1)(k-r+1)}R^rS^{(i_*+1)}_{t,l}, \qquad (74)\\
&\quad \vdots\\
&C^{i_*-r+1}B^{k-r+(n-1)(k-r+1)}HR^rS^{(i)}_{t,l} := C^{i_*-r}B^{k-r+(n-1)(k-r+1)}HR^rS^{(i)}_{t,l} - \\
&\quad IC^{i_*-r}B^{n(k-r+1)}R^rS^{(i)}_{t,d^*}\cdot C^{i_*-r}B^{k-r+(n-1)(k-r+1)}HR^rS^{(i_*+1)}_{t,l},\\
&C^{i_*-r+1}B^{n(k-r+1)}R^rS^{(i,j)}_{t,l} := C^{i_*-r}B^{n(k-r+1)}R^rS^{(i,j)}_{t,l} - \\
&\quad IC^{i_*-r}B^{n(k-r+1)}R^rS^{(i)}_{t,d^*}\cdot C^{i_*-r}B^{n(k-r+1)}R^rS^{(i_*+1,j)}_{t,l}, \qquad (75)
\end{aligned}
$$

$$
\begin{aligned}
c^{i_*-r+1}b^{n(k-r+1)}r^r s_{t,l}^{(i)} &:= c^{i_*-r}b^{n(k-r+1)}r^r s_{t,l}^{(i)} - IC^{i_*-r}B^{n(k-r+1)}R^r S_{t,d^*}^{(i)} \cdot \\
& \quad c^{i_*-r}b^{n(k-r+1)}r^r s_{t,l}^{(i_*+1)}, \qquad (76) \\
c^{i_*-r+1}b^{(n-1)(k-r+1)}r^r s_{t,l}^{(i),\{n\}} &:= c^{i_*-r}b^{(n-1)(k-r+1)}r^r s_{t,l}^{(i),\{n\}} - IC^{i_*-r}B^{n(k-r+1)}R^r S_{t,d^*}^{(i)} \cdot \\
& \quad c^{i_*-r}b^{(n-1)(k-r+1)}r^r s_{t,l}^{(i_*+1),\{n\}}, \\
& \vdots \\
c^{i_*-r+1}r^r s_{t,l}^{(i),\{n\}} &:= c^{i_*-r}r^r s_{t,l}^{(i),\{n\}} - IC^{i_*-r}B^{n(k-r+1)}R^r S_{t,d^*}^{(i)} \cdot c^{i_*-r}r^r s_{t,l}^{(i_*+1),\{n\}}; \\
& \qquad (77)
\end{aligned}
$$

for $r = 2, \ldots, k$ only, also define

$$
\begin{aligned}
C^{i_*-r+1}B^{n(k-r+1)}OR^{r-1}S_{t,l}^{(i)} &:= C^{i_*-r}B^{n(k-r+1)}OR^{r-1}S_{t,l}^{(i)} - IC^{i_*-r}B^{n(k-r+1)}R^r S_{t,d^*}^{(i)} \cdot \\
& \quad C^{i_*-r}B^{n(k-r+1)}OR^{r-1}S_{t,l}^{(i_*+1)}.
\end{aligned}
$$

- For the proof (of Proposition 4.1):

(a) This is none other than a specific way (based on the current definitions) of going from a multivariate AR(1) to the interested univariate AR of a higher order that is required. Next, a very short sketch of proof is provided on how this is done.

A series of inductions would normally take place (nested one inside the other), the first one concerning the d as below.

$\underline{d = 0}$

In this case, the main statement of the proposition part (i), i.e.

$$
\begin{aligned}
VS_t^{(q-2-d)} &= \sum_{l=1}^{d^*} rp_{t,l}\ RS(d)_{t,l}\ VS_{t-l}^{(q-2-d)} + \\
& \sum_{l=1}^{d^*} rp_{t,l} \left(\prod_{i=1}^{p} f_0(X_{t-i-l}) \right) rs(d)_{t,l}\ q_{t-1-l}^{(\mathbf{0})} \qquad (78)
\end{aligned}
$$

is a direct consequence of

$$
VP_t = \frac{1}{p_{t-1}} \left[L_t \cdot VP_{t-1} + \left(\prod_{l=1}^{p} f_0(X_{t-1-l}) \right) l_t \cdot q_{t-2}^{(\mathbf{0})} \right]:
$$

(as in the main text) when $RS(0)_{t,1} := L_t$ and $rs(0)_{t,1} := l_t$. In the other hand, straight from

$$
\begin{aligned}
q_{t-1}^{(\mathbf{0})} \quad &= \quad \frac{\pi_0}{p_{t-1}} \left[p_{t-1} - \sum_{m=1}^{k} S_{t-1}^{(v_m)} - \sum_{m=1}^{k} S_{t-1}^{(0,v_m)} - \ldots - \sum_{m=1}^{k} S_{t-1}^{(\mathbf{0}_{q-3},v_m)} - \right. \\
&\left. \sum_{m=1}^{k} P_{t-1}^{(\mathbf{0}_{q-2},v_m)} + \left(\prod_{l=1}^{p} f_0(X_{t-1-l}) \right) \left(f_0(X_{t-1}) - \pi_{X_{t-1}} \right) q_{t-2}^{(\mathbf{0})} \right]
\end{aligned}
$$

(as in the main text), the forms of $RQ(0)_{t,1}$ and

$$
rq(0)_{t,1} := f_0(X_{t-1}) - \pi_{X_{t-1}} = \begin{cases} 1 - \pi_0, & \text{if } X_{t-1} = 0, \\ -\pi_{v_m}, & \text{if } X_{t-1} = v_m, \ m = 1, \ldots, k \end{cases}
$$

become obvious.

$\underline{d = w}$

Write $w^* = (k+1)^w$. Accept the statements (78) and

$$
\begin{aligned}
q_{t-1}^{(\mathbf{0})} = \pi_0 \ &\left\{ \sum_{l=1}^{d^*} rp_{t,l} \ RQ(d)_{t,l} \ VS_{t-l}^{(q-2-d)} + \right. \\
&\left. \sum_{l=1}^{d^*} rp_{t,l} \left(\prod_{i=1}^{p} f_0(X_{t-i-l}) \right) \ rq(d)_{t,l} \ q_{t-1-l}^{(\mathbf{0})} \right\},
\end{aligned}
\tag{79}
$$

when $d = w$, where it can be $w = 1, \ldots, q-2$.

$\underline{d = w + 1}$

Write for convenience $VS_t^{(q-2-w)} \equiv VS_t$, $VS_t^{(q-2-(w+1))} \equiv VS_t^{((-1))}$ and $RS(w)_{t,l} \equiv RS_{t,l}$, $rs(w)_{t,l} \equiv rs_{t,l}$, $RQ(w)_{t,l} \equiv RQ_{t,l}$, $rq(w)_{t,l} \equiv rq_{t,l}$; in general omit the dimensionality w within this loop and simplify the operator $L^{\left((k+1)^{q-2-w}\right)} \equiv L$.

It holds that

$$
VS_t = \sum_{l=1}^{w^*} rp_{t,l} \ RS_{t,l} \ VS_{t-l} + \sum_{l=1}^{w^*} rp_{t,l} \left(\prod_{ii=1}^{p} f_0(X_{t-ii-l}) \right) \ rs_{t,l} \ q_{t-1-l}^{(\mathbf{0})};
$$

first there is substitution of VS_{t-1} in the equations for VS_t and $\{q_{t-1}^{(\mathbf{0})}/\pi_0\}$ to obtain $RS_{t,l}^{\{1\}}$, $rs_{t,l}^{\{1\}}$ and $RQ_{t,l}^{\{1\}}$, $rq_{t,l}^{\{1\}}$, respectively; next there is substitution of VS_{t-2}, and then of VS_{t-3} and so on, to determine $RS_{t,l}^{\{m\}}$ (as in (1)), $rs_{t,l}^{\{m\}}$ (as in (4) when $r = 0$), and $RQ_{t,l}^{\{m\}}$ (as in (5)), $rq_{t,l}^{\{m\}}$ (as in (8) when $r = 0$).

Then $vS_t^{(i)}$, $i = 1, \ldots, k$, $VS_t^{((-1))}$ and $\{q_{t-1}^{(\mathbf{0})}/\pi_0\}$ can be re-expressed from each one of these equations. Move from the 'm-step' to the 'basic step' of the proposition.

$(r = 1)$:

$< n = 0 >$: first $vS^{(1)}_{t-w^*}$ is solved from the equation for $vS^{(2)}_t$ (no m); then the result is plugged in $vS^{(i)}_t$, $i = 3, \ldots, k$ and $VS^{((-1))}_t$ (no m) yielding (29) and (30), (31) when $r = 1$; from the newly derived results, $vS^{(2)}_{t-w^*}$ is solved from the equation for $vS^{(3)}_t$ and the result is plugged in (the newly derived) $vS^{(i)}_t$, $i = 4, \ldots, k$ and $VS^{((-1))}_t$ yielding (32) and (33), (34), (35) when $r = 1$, $i_* - r = 1$. This goes on until

$$\begin{aligned} vS^{(k)}_{t-w^*} \quad = \quad & \{rp_{t,w^*}\}^{-1}\{C^{k-1}RS^{(k+1,k)}_{t,w^*}\}^{-1}\Big\{VS^{((-1))}_t - IC^{k-2}RS^{(k+1)}_{t,w^*}\ vS^{(k)}_t - \\ & CIC^{k-3}RS^{(k+1)}_{t,w^*}\ vS^{(k-1)}_t - \ldots - C^{k-2}IRS^{(k+1)}_{t,w^*}\ vS^{(2)}_t - \\ & \sum_{j=1}^{k}\sum_{l=1}^{w^*-1} rp_{t,l}\ C^{k-1}RS^{(k+1,j)}_{t,l}\ vS^{(j)}_{t-l} - \\ & \sum_{l=1}^{w^*} rp_{t,l}\ C^{k-1}RS^{(k+1,k+1)}_{t,l}\ VS^{((-1))}_{t-l} - \\ & \sum_{l=1}^{w^*} rp_{t,l}\ \left(\prod_{ii=1}^{p} f_0(X_{t-ii-l})\right)\ c^{k-1}rs^{(k+1)}_{t,l}\ q^{(\mathbf{0})}_{t-1-l}\Big\} \end{aligned}$$

is the last one that is solved from $VS^{((-1))}_t$.

Secondly for $m \in \mathbb{N}$, start from the equations for $vS^{(i)}_t$, $i = 2, \ldots, k$, $VS^{((-1))}_t$ and $\{q^{\{\mathbf{0}\}}_{t-1}/\pi_0\}$ as they were obtained from the 'm-step' and repeat a similar process: $vS^{(1)}_{t-m-w^*}$ (this is the $vS^{(1)}_{t-w^*}$ that was solved already but for $t = t - m$) is plugged in those versions yielding (15), (16) and (17), (19), (18), (20) when $r = 1$; next, in these versions of $vS^{(i)}_t$, $VS^{((-1))}_t$ and $\{q^{\{\mathbf{0}\}}_{t-1}/\pi_0\}$ that were just computed, $vS^{(2)}_{t-m-w^*}$ (i.e. the $vS^{(2)}_{t-w^*}$ from before but for $t = t-m$) is plugged in, yielding (21), (22) and (23), (25), (27), (24), (26), (28) when $r = 1$, $i_* - r = 1$. This goes on ($i_* = r+2, \ldots, k$) until it will be written in the end, that

$$\begin{aligned} vS^{(i)}_t \quad = \quad & rp_{t,m}\ HB^{k-1}RS^{(i),\{m\}}_{t,w^*+m}\ VS^{((-1))}_{t-m} + rp_{t,m}\ BHB^{k-2}RS^{(i),\{m\}}_{t,w^*+m}\ vS^{(k)}_{t-m} + \\ & rp_{t,m}\ B^2HB^{k-3}RS^{(i),\{m\}}_{t,w^*+m}\ vS^{(k-1)}_{t-m} + \ldots + rp_{t,m}\ B^{k-1}HRS^{(i),\{m\}}_{t,w^*+m}\ vS^{(2)}_{t-m} + \\ & \sum_{j=1}^{k}\sum_{l=1+m}^{w^*+m-1} rp_{t,l}\ B^kRS^{(i,j),\{m\}}_{t,l}\ vS^{(j)}_{t-l} + \\ & \sum_{l=1+m}^{w^*+m} rp_{t,l}\ B^kRS^{(i,k+1),\{m\}}_{t,l}\ VS^{((-1))}_{t-l} + \\ & \sum_{l=1+m}^{w^*+m} rp_{t,l}\ \left(\prod_{ii=1}^{p} f_0(X_{t-ii-l})\right)\ b^k rs^{(i),\{m\}}_{t,l}\ q^{(\mathbf{0})}_{t-1-l} + \\ & \sum_{l=1}^{m} rp_{t,l}\ \left(\prod_{ii=1}^{p} f_0(X_{t-ii-l})\right)\ L_i(rs^{\{m\}}_{t,l})\ q^{(\mathbf{0})}_{t-1-l},\ i = 2, \ldots, k \end{aligned} \tag{80}$$

and

$$\begin{aligned}
VS_t^{((-1))} \quad = \quad & rp_{t,m}\ HB^{k-1}RS_{t,w^*+m}^{(k+1),\{m\}}\ VS_{t-m}^{((-1))} + rp_{t,m}\ BHB^{k-2}RS_{t,w^*+m}^{(k+1),\{m\}}\ vS_{t-m}^{(k)} + \\
& rp_{t,m}\ B^2HB^{k-3}RS_{t,w^*+m}^{(k+1),\{m\}}\ vS_{t-m}^{(k-1)} + \ldots + rp_{t,m}\ B^{k-1}HRS_{t,w^*+m}^{(k+1),\{m\}}\ vS_{t-m}^{(2)} + \\
& \sum_{j=1}^{k}\sum_{l=1+m}^{w^*+m-1} rp_{t,l}\ B^kRS_{t,l}^{(k+1,j),\{m\}}\ vS_{t-l}^{(j)} + \\
& \sum_{l=1+m}^{w^*+m} rp_{t,l}\ B^kRS_{t,l}^{(k+1,k+1),\{m\}}\ VS_{t-l}^{((-1))} + \\
& \sum_{l=1+m}^{w^*+m} rp_{t,l}\ \left(\prod_{ii=1}^{p} f_0(X_{t-ii-l})\right)\ b^k rs_{t,l}^{(k+1),\{m\}}\ q_{t-1-l}^{(\mathbf{0})} + \\
& \sum_{l=1}^{m} rp_{t,l}\ \left(\prod_{ii=1}^{p} f_0(X_{t-ii-l})\right)\ L_{k+1}(rs_{t,l}^{\{m\}})\ q_{t-1-l}^{(\mathbf{0})} \qquad (81)
\end{aligned}$$

and

$$\begin{aligned}
\frac{q_{t-1}^{(\mathbf{0})}}{\pi_0} \quad = \quad & rp_{t,m}\ HB^{k-1}RQ_{t,w^*+m}^{\{m\}}\ VS_{t-m}^{((-1))} + rp_{t,m}\ BHB^{k-2}RQ_{t,w^*+m}^{\{m\}}\ vS_{t-m}^{(k)} + \\
& rp_{t,m}\ B^2HB^{k-3}RQ_{t,w^*+m}^{\{m\}}\ vS_{t-m}^{(k-1)} + \ldots + rp_{t,m}\ B^{k-1}HRQ_{t,w^*+m}^{\{m\}}\ vS_{t-m}^{(2)} + \\
& \sum_{j=1}^{k}\sum_{l=1+m}^{w^*+m-1} rp_{t,l}\ B^kRQ_{t,l}^{(j),\{m\}}\ vS_{t-l}^{(j)} + \sum_{l=1+m}^{w^*+m} rp_{t,l}\ B^kRQ_{t,l}^{(k+1),\{m\}}\ VS_{t-l}^{((-1))} + \\
& \sum_{l=1+m}^{w^*+m} rp_{t,l}\ \left(\prod_{ii=1}^{p} f_0(X_{t-ii-l})\right)\ b^k rq_{t,l}^{\{m\}}\ q_{t-1-l}^{(\mathbf{0})} + \\
& \sum_{l=1}^{m} rp_{t,l}\ \left(\prod_{ii=1}^{p} f_0(X_{t-ii-l})\right)\ rq_{t,l}^{\{m\}}\ q_{t-1-l}^{(\mathbf{0})}; \qquad (82)
\end{aligned}$$

in all (80), (81) and (82), $vS_{t-m}^{(1)}$ is not present.

$< n = 1 >$: consider (80) and (81) for $m = 1$ only and then $vS_{t-w^*}^{(1)}$ is solved from the equation for $vS_t^{(2)}$ ($m = 1$); the result is plugged in $vS_t^{(i)}$, $i = 3, \ldots, k$ and $VS_t^{((-1))}$ ($m = 1$) yielding (66) and (67), (68), (69), (70), (71) when $r = 1$ (and $n = 1$); from the newly derived results, $vS_{t-w^*}^{(2)}$ is solved from the equation for $vS_t^{(3)}$ and the result is plugged in (the newly derived) $vS_t^{(i)}$, $i = 4, \ldots, k$ and $VS_t^{((-1))}$ yielding (72) and (73), (74), (75), (76), (77) when

$r = 1$, $i_* - r = 1$ (and $n = 1$). This goes on until

$$\begin{aligned}
vS^{(k)}_{t-w^*} \quad = \quad & \{rp_{t,w^*}\}^{-1}\{C^{k-1}B^kRS^{(k+1,k)}_{t,w^*}\}^{-1}\Big\{VS^{((-1))}_t - IC^{k-2}B^kRS^{(k+1)}_{t,w^*}\ vS^{(k)}_t - \\
& CIC^{k-3}B^kRS^{(k+1)}_{t,w^*}\ vS^{(k-1)}_t - \ldots - C^{k-2}IB^kRS^{(k+1)}_{t,w^*}\ vS^{(2)}_t - \\
& rp_{t,1}\ C^{k-1}HB^{k-1}RS^{(k+1)}_{t,w^*+1}\ VS^{((-1))}_{t-1} - \ldots - rp_{t,1}\ C^{k-1}B^{k-1}HRS^{(k+1)}_{t,w^*+1}\ vS^{(2)}_{t-1} - \\
& \sum_{j=1}^{k}\sum_{l=2}^{w^*-1} rp_{t,l}\ C^{k-1}B^kRS^{(k+1,j)}_{t,l}\ vS^{(j)}_{t-l} - \\
& \sum_{l=2}^{w^*+1} rp_{t,l}\ C^{k-1}B^kRS^{(k+1,k+1)}_{t,l}\ VS^{((-1))}_{t-l} - \\
& \sum_{l=2}^{w^*+1} rp_{t,l}\left(\prod_{ii=1}^{p} f_0(X_{t-ii-l})\right)\ c^{k-1}b^krs^{(k+1)}_{t,l}\ q^{(0)}_{t-1-l} - \\
& rp_{t,1}\left(\prod_{ii=1}^{p} f_0(X_{t-ii-1})\right)\ c^{k-1}rs^{(k+1),\{1\}}_{t,1}\ q^{(\mathbf{0})}_{t-1-1}\Big\}
\end{aligned}$$

is the last one solved from $VS^{((-1))}_t$.

Next for $m = 2, 3, \ldots$, start from the equations (80), (81) and (82) as they were obtained in $< n = 0 >$ and repeat a similar process: $vS^{(1)}_{t-m+1-w^*}$ (this is the $vS^{(1)}_{t-w^*}$ that was solved already but for $t = t - m + 1$) is plugged in those versions yielding (36), (37) and (38), (40), (42), (44), (46), (48), (39), (41), (43), (45), (47), (49) when $r = 1$ (and $n = 1$); next, in these versions of $vS^{(i)}_t$, $VS^{((-1))}_t$ and $\{q^{\{\mathbf{0}\}}_{t-1}/\pi_0\}$ that were just computed, $vS^{(2)}_{t-m+1-w^*}$ (i.e. the $vS^{(2)}_{t-w^*}$ from before but for $t = t - m + 1$) is plugged in, yielding (50) and (51) and (52), (54), (56), (58), (60), (62), (64), (53), (55), (57), (59), (61), (63), (65) when $r = 1$, $i_* - r = 1$ (and $n = 1$). This goes on ($i_* = r + 2, \ldots, k$) until it will be written in the end, that

$$\begin{aligned}
vS^{(i)}_t \quad = \quad & rp_{t,m-1}\ HB^{2k-1}RS^{(i),\{m\}}_{t,w^*+m-1}\ VS^{((-1))}_{t-m+1} + \\
& rp_{t,m-1}\ BHB^{2k-2}RS^{(i),\{m\}}_{t,w^*+m-1}\ vS^{(k)}_{t-m+1} + \ldots + \\
& rp_{t,m-1}\ B^{k-1}HB^kRS^{(i),\{m\}}_{t,w^*+m-1}\ vS^{(2)}_{t-m+1} + rp_{t,m}\ B^kHB^{k-1}RS^{(i),\{m\}}_{t,w^*+m}\ VS^{((-1))}_{t-m} + \\
& \ldots + rp_{t,m}\ B^{2k-1}HRS^{(i),\{m\}}_{t,w^*+m}\ vS^{(2)}_{t-m} + \sum_{j=1}^{k}\sum_{l=1+m}^{w^*+m-2} rp_{t,l}\ B^{2k}RS^{(i,j),\{m\}}_{t,l}\ vS^{(j)}_{t-l} + \\
& \sum_{l=1+m}^{w^*+m} rp_{t,l}\ B^{2k}RS^{(i,k+1),\{m\}}_{t,l}\ VS^{((-1))}_{t-l} + \\
& \sum_{l=1+m}^{w^*+m} rp_{t,l}\left(\prod_{ii=1}^{p} f_0(X_{t-ii-l})\right)\ b^{2k}rs^{(i),\{m\}}_{t,l}\ q^{(\mathbf{0})}_{t-1-l} + \\
& rp_{t,m}\left(\prod_{ii=1}^{p} f_0(X_{t-ii-m})\right)\ b^krs^{(i),(\{m\},\{1\})}_{t,m}\ q^{(\mathbf{0})}_{t-1-m} + \\
& \sum_{l=1}^{m-1} rp_{t,l}\left(\prod_{ii=1}^{p} f_0(X_{t-ii-l})\right)\ L_i(rs^{\{m\}}_{t,l})\ q^{(\mathbf{0})}_{t-1-l},\ i = 2, \ldots, k
\end{aligned} \tag{83}$$

and

$$
\begin{aligned}
VS_t^{((-1))} \quad = \quad & rp_{t,m-1}\ HB^{2k-1}RS_{t,w^*+m-1}^{(k+1),\{m\}}\ VS_{t-m+1}^{((-1))} + \\
& rp_{t,m-1}\ BHB^{2k-2}RS_{t,w^*+m-1}^{(k+1),\{m\}}\ vS_{t-m+1}^{(k)} + \ldots + \\
& rp_{t,m-1}\ B^{k-1}HB^{k}RS_{t,w^*+m-1}^{(k+1),\{m\}}\ vS_{t-m+1}^{(2)} + rp_{t,m}\ B^{k}HB^{k-1}RS_{t,w^*+m}^{(k+1),\{m\}}\ VS_{t-m}^{((-1))} + \\
& \ldots + rp_{t,m}\ B^{2k-1}HRS_{t,w^*+m}^{(k+1),\{m\}}\ vS_{t-m}^{(2)} + \sum_{j=1}^{k}\sum_{l=1+m}^{w^*+m-2} rp_{t,l}\ B^{2k}RS_{t,l}^{(k+1,j),\{m\}}\ vS_{t-l}^{(j)} + \\
& \sum_{l=1+m}^{w^*+m} rp_{t,l}\ B^{2k}RS_{t,l}^{(k+1,k+1),\{m\}}\ VS_{t-l}^{((-1))} + \\
& \sum_{l=1+m}^{w^*+m} rp_{t,l}\ \left(\prod_{ii=1}^{p} f_0(X_{t-ii-l})\right)\ b^{2k}rs_{t,l}^{(k+1),\{m\}}\ q_{t-1-l}^{(\mathbf{0})} + \\
& rp_{t,m}\ \left(\prod_{ii=1}^{p} f_0(X_{t-ii-m})\right)\ b^{k}rs_{t,m}^{(k+1),(\{m\},\{1\})}\ q_{t-1-m}^{(\mathbf{0})} + \\
& \sum_{l=1}^{m-1} rp_{t,l}\ \left(\prod_{ii=1}^{p} f_0(X_{t-ii-l})\right)\ L_{k+1}(rs_{t,l}^{\{m\}})\ q_{t-1-l}^{(\mathbf{0})} \qquad (84)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{q_{t-1}^{(\mathbf{0})}}{\pi_0} \quad = \quad & rp_{t,m-1}\ HB^{2k-1}RQ_{t,w^*+m-1}^{\{m\}}\ VS_{t-m+1}^{((-1))} + \\
& rp_{t,m-1}\ BHB^{2k-2}RQ_{t,w^*+m-1}^{\{m\}}\ vS_{t-m+1}^{(k)} + \ldots + \\
& rp_{t,m-1}\ B^{k-1}HB^{k}RQ_{t,w^*+m-1}^{\{m\}}\ vS_{t-m+1}^{(2)} + rp_{t,m}\ B^{k}HB^{k-1}RQ_{t,w^*+m}^{\{m\}}\ VS_{t-m}^{((-1))} + \\
& \ldots + rp_{t,m}\ B^{2k-1}HRQ_{t,w^*+m}^{\{m\}}\ vS_{t-m}^{(2)} + \sum_{j=1}^{k}\sum_{l=1+m}^{w^*+m-2} rp_{t,l}\ B^{2k}RQ_{t,l}^{(j),\{m\}}\ vS_{t-l}^{(j)} + \\
& \sum_{l=1+m}^{w^*+m} rp_{t,l}\ B^{2k}RQ_{t,l}^{(k+1),\{m\}}\ VS_{t-l}^{((-1))} + \\
& \sum_{l=1+m}^{w^*+m} rp_{t,l}\ \left(\prod_{ii=1}^{p} f_0(X_{t-ii-l})\right)\ b^{2k}rq_{t,l}^{\{m\}}\ q_{t-1-l}^{(\mathbf{0})} + \\
& rp_{t,m}\ \left(\prod_{ii=1}^{p} f_0(X_{t-ii-m})\right)\ b^{k}rq_{t,m}^{(\{m\},\{1\})}\ q_{t-1-m}^{(\mathbf{0})} + \\
& \sum_{l=1}^{m-1} rp_{t,l}\ \left(\prod_{ii=1}^{p} f_0(X_{t-ii-l})\right)\ rq_{t,l}^{\{m\}}\ q_{t-1-l}^{(\mathbf{0})}; \qquad (85)
\end{aligned}
$$

in all (83), (84) and (85), $vS_{t-m+1}^{(1)}$ and $vS_{t-m}^{(1)}$ are not present;

$$\vdots$$

How this patent continues should be obvious by now. It may be accepted in $< n = \nu >$ (where $\nu = 2, \ldots, w^* - 2$) and for any $m = \nu+1, \nu+2, \ldots$, that $vS_t^{(i)}$, $VS_t^{((-1))}$ and $\{q_{t-1}^{(\mathbf{0})}/\pi_0\}$ have a certain form, in which $vS_{t-m+\nu}^{(1)}$,...,$vS_{t-m}^{(1)}$ are not present. Moving to step $< n = \nu + 1 >$,

$vS^{(1)}_{t-w^*}$ will be solved from $m=\nu+1$ (last version of $< n=\nu >$) and $i=2$ and this will be plugged in $i=3,\ldots,k+1$ (still $m=\nu+1$ and last versions of $< n=\nu >$); $vS^{(2)}_{t-w^*}$ will be solved from the version that has just been derived when $i=3$ and this will be plugged in (the just derived versions for) $i=4,\ldots,k+1$ ($m=\nu+1$);...; by the end of this $vS^{(k)}_{t-w^*}$ will have been solved (from the last derived version of $i=k+1$, i.e. $VS^{((-1))}_t$).

Still within the $< n=\nu+1 >$ loop, $vS^{(1)}_{t-m+\nu+1-w^*}$ (i.e. $vS^{(1)}_{t-w^*}$ as it was last computed and for $t=t-m+\nu+1$) is plugged in $vS^{(i)}_t$, $i=2,\ldots,k$, $VS^{((-1))}_t$, $\{q^{(\mathbf{0})}_{t-1}/\pi_0\}$, $m=\nu+2,\ldots$ (last version of $< n=\nu >$); next, $vS^{(2)}_{t-m+\nu+1-w^*}$ (again this is from $vS^{(2)}_{t-w^*}$ as it was computed and for $t=t-m+\nu+1$) is plugged in the just derived versions $vS^{(i)}_t$, $i=2,\ldots,k$, $VS^{((-1))}_t$, $\{q^{(\mathbf{0})}_{t-1}/\pi_0\}$, $m=\nu+2,\ldots$; and so on ..., until the last inclusion when $vS^{(i)}_t$, $i=2,\ldots,k$, $VS^{((-1))}_t$ and $\{q^{(\mathbf{0})}_{t-1}/\pi_0\}$, $m=\nu+2,\ldots$ are free of $vS^{(1)}_{t-m+\nu+1}$,...,$vS^{(1)}_{t-m}$.

Hence the induction argument on n when $r=1$, has been completed. Especially in the end, $n=w^*-1$ and $m=w^*,w^*+1,\ldots$, it will be written that

$$
\begin{aligned}
vS^{(i)}_t \quad = \quad & rp_{t,m-w^*+1}\ HB^{w^*k-1}RS^{(i),\{m\}}_{t,m+1}\ VS^{((-1))}_{t-m+w^*-1}+\ldots+\\
& rp_{t,m-w^*+1}\ B^{k-1}HB^{(w^*-1)k}RS^{(i),\{m\}}_{t,m+1}\ vS^{(2)}_{t-m+w^*-1}+\\
& rp_{t,m-w^*+2}\ B^{k}HB^{(w^*-1)k-1}RS^{(i),\{m\}}_{t,m+2}\ VS^{((-1))}_{t-m+w^*-2}+\ldots+\\
& rp_{t,m-w^*+2}\ B^{2k-1}HB^{(w^*-2)k}RS^{(i),\{m\}}_{t,m+2}\ vS^{(2)}_{t-m+w^*-2}+\ldots\quad\ldots+\\
& rp_{t,m}\ B^{(w^*-1)k}HB^{k-1}RS^{(i),\{m\}}_{t,w^*+m}\ VS^{((-1))}_{t-m}+\ldots+\\
& rp_{t,m}\ B^{w^*k-1}HRS^{(i),\{m\}}_{t,w^*+m}\ vS^{(2)}_{t-m}+\sum_{l=1+m}^{w^*+m} rp_{t,l}\ B^{w^*k}RS^{(i,k+1),\{m\}}_{t,l}\ VS^{((-1))}_{t-l}+\\
\\
& \sum_{l=1+m}^{w^*+m} rp_{t,l}\left(\prod_{ii=1}^{p} f_0(X_{t-ii-l})\right)\ b^{w^*k}rs^{(i),\{m\}}_{t,l}\ q^{(\mathbf{0})}_{t-1-l}+\\
& rp_{t,m}\left(\prod_{ii=1}^{p} f_0(X_{t-ii-m})\right)\ b^{(w^*-1)k}rs^{(i),(\{m\},\{w^*-1\})}_{t,m}\ q^{(\mathbf{0})}_{t-1-m}+\ldots\quad\ldots+\\
& rp_{t,m-w^*+2}\left(\prod_{ii=1}^{p} f_0(X_{t-ii-m+w^*-2})\right)\ b^{k}rs^{(i),(\{m\},\{w^*-1\})}_{t,m-w^*+2}\ q^{(\mathbf{0})}_{t-1-m+w^*-2}+\\
& \sum_{l=1}^{m-w^*+1} rp_{t,l}\left(\prod_{ii=1}^{p} f_0(X_{t-ii-l})\right)\ L_i(rs^{\{m\}}_{t,l})\ q^{(\mathbf{0})}_{t-1-l},\ i=2,\ldots,k
\end{aligned}
\tag{86}
$$

and that

$$\begin{aligned}
VS_t^{((-1))} \quad = \quad & rp_{t,m-w^*+1}\ HB^{w^*k-1}RS_{t,m+1}^{(k+1),\{m\}}\ VS_{t-m+w^*-1}^{((-1))} + \ldots + \\
& rp_{t,m-w^*+1}\ B^{k-1}HB^{(w^*-1)k}RS_{t,m+1}^{(k+1),\{m\}}\ vS_{t-m+w^*-1}^{(2)} + \\
& rp_{t,m-w^*+2}\ B^{k}HB^{(w^*-1)k-1}RS_{t,m+2}^{(k+1),\{m\}}\ VS_{t-m+w^*-2}^{((-1))} + \ldots + \\
& rp_{t,m-w^*+2}\ B^{2k-1}HB^{(w^*-2)k}RS_{t,m+2}^{(k+1),\{m\}}\ vS_{t-m+w^*-2}^{(2)} + \ldots \quad \ldots + \\
& rp_{t,m}\ B^{(w^*-1)k}HB^{k-1}RS_{t,w^*+m}^{(k+1),\{m\}}\ VS_{t-m}^{((-1))} + \ldots + \\
& rp_{t,m}\ B^{w^*k-1}HRS_{t,w^*+m}^{(k+1),\{m\}}\ vS_{t-m}^{(2)} + \sum_{l=1+m}^{w^*+m} rp_{t,l}\ B^{w^*k}RS_{t,l}^{(k+1,k+1),\{m\}}\ VS_{t-l}^{((-1))} + \\
& \sum_{l=1+m}^{w^*+m} rp_{t,l} \left(\prod_{ii=1}^{p} f_0(X_{t-ii-l})\right)\ b^{w^*k}rs_{t,l}^{(k+1),\{m\}}\ q_{t-1-l}^{(\mathbf{0})} + \\
& rp_{t,m} \left(\prod_{ii=1}^{p} f_0(X_{t-ii-m})\right)\ b^{(w^*-1)k}rs_{t,m}^{(k+1),(\{m\},\{w^*-1\})}\ q_{t-1-m}^{(\mathbf{0})} + \ldots \quad \ldots + \\
& rp_{t,m-w^*+2} \left(\prod_{ii=1}^{p} f_0(X_{t-ii-m+w^*-2})\right)\ b^{k}rs_{t,m-w^*+2}^{(k+1),(\{m\},\{w^*-1\})}\ q_{t-1-m+w^*-2}^{(\mathbf{0})} + \\
& \sum_{l=1}^{m-w^*+1} rp_{t,l} \left(\prod_{ii=1}^{p} f_0(X_{t-ii-l})\right)\ L_{k+1}(rs_{t,l}^{\{m\}})\ q_{t-1-l}^{(\mathbf{0})}
\end{aligned} \tag{87}$$

and that

$$\begin{aligned}
\frac{q_{t-1}^{(\mathbf{0})}}{\pi_0} \quad = \quad & rp_{t,m-w^*+1}\ HB^{w^*k-1}RQ_{t,m+1}^{\{m\}}\ VS_{t-m+w^*-1}^{((-1))} + \ldots + \\
& rp_{t,m-w^*+1}\ B^{k-1}HB^{(w^*-1)k}RQ_{t,m+1}^{\{m\}}\ vS_{t-m+w^*-1}^{(2)} + \\
& rp_{t,m-w^*+2}\ B^{k}HB^{(w^*-1)k-1}RQ_{t,m+2}^{\{m\}}\ VS_{t-m+w^*-2}^{((-1))} + \ldots + \\
& rp_{t,m-w^*+2}\ B^{2k-1}HB^{(w^*-2)k}RQ_{t,m+2}^{\{m\}}\ vS_{t-m+w^*-2}^{(2)} + \ldots \quad \ldots + \\
& rp_{t,m}\ B^{(w^*-1)k}HB^{k-1}RQ_{t,w^*+m}^{\{m\}}\ VS_{t-m}^{((-1))} + \ldots + \\
& rp_{t,m}\ B^{w^*k-1}HRQ_{t,w^*+m}^{\{m\}}\ vS_{t-m}^{(2)} + \sum_{l=1+m}^{w^*+m} rp_{t,l}\ B^{w^*k}RQ_{t,l}^{(k+1),\{m\}}\ VS_{t-l}^{((-1))} + \\
& \sum_{l=1+m}^{w^*+m} rp_{t,l} \left(\prod_{ii=1}^{p} f_0(X_{t-ii-l})\right)\ b^{w^*k}rq_{t,l}^{\{m\}}\ q_{t-1-l}^{(\mathbf{0})} + \\
& rp_{t,m} \left(\prod_{ii=1}^{p} f_0(X_{t-ii-m})\right)\ b^{(w^*-1)k}rq_{t,m}^{(\{m\},\{w^*-1\})}\ q_{t-1-m}^{(\mathbf{0})} + \ldots \quad \ldots + \\
& rp_{t,m-w^*+2} \left(\prod_{ii=1}^{p} f_0(X_{t-ii-m+w^*-2})\right)\ b^{k}rq_{t,m-w^*+2}^{(\{m\},\{w^*-1\})}\ q_{t-1-m+w^*-2}^{(\mathbf{0})} + \\
& \sum_{l=1}^{m-w^*+1} rp_{t,l} \left(\prod_{ii=1}^{p} f_0(X_{t-ii-l})\right)\ rq_{t,l}^{\{m\}}\ q_{t-1-l}^{(\mathbf{0})},
\end{aligned} \tag{88}$$

such that all (86), (87) and (88) do not use $vS_{t-l}^{(1)}$, $l > 0$ at all.

The cases up to $m = w^*$ will be of use only; this concludes the induction argument on n, but continues with the case ($r = 1$). Move from the 'basic step' to the 'r-step' of the proposition.

The equations (86) and (87), $m = w^*$ are put together for $i = 2, \ldots, k$ and $i = k+1$, respectively, writing

$$\begin{bmatrix} vS_t^{(2)} \\ \vdots \\ vS_t^{(k)} \\ VS_t^{((-1))} \end{bmatrix} = \sum_{l=1}^{w^*} rp_{t,l}\ R^2 S_{t,l} \begin{bmatrix} vS_{t-l}^{(2)} \\ \vdots \\ vS_{t-l}^{(k)} \\ VS_{t-l}^{((-1))} \end{bmatrix} + \sum_{l=1+w^*}^{2w^*} rp_{t,l}\ ORS_{t,l}\ VS_{t-l}^{((-1))} + \sum_{l=1}^{2w^*} rp_{t,l} \left(\prod_{ii=1}^{p} f_0(X_{t-ii-l}) \right) r^2 s_{t,l}\ q_{t-1-l}^{(\mathbf{0})}, \tag{89}$$

which yields (9), (10) and (11). Similarly, it can be written from (88), $m = w^*$, that

$$\frac{q_{t-1}^{(\mathbf{0})}}{\pi_0} = \sum_{l=1}^{w^*} rp_{t,l}\ R^2 Q_{t,l} \begin{bmatrix} vS_{t-l}^{(2)} \\ \vdots \\ vS_{t-l}^{(k)} \\ VS_{t-l}^{((-1))} \end{bmatrix} + \sum_{l=1+w^*}^{2w^*} rp_{t,l}\ ORQ_{t,l}\ VS_{t-l}^{((-1))} + \sum_{l=1}^{2w^*} rp_{t,l} \left(\prod_{ii=1}^{p} f_0(X_{t-ii-l}) \right) r^2 q_{t,l}\ q_{t-1-l}^{(0)}, \tag{90}$$

yielding (12), (13) and (14).

Move from the 'r-step' back to the 'm-step'.

It should be obvious how substitution of $[vS_{t-1}^{(2)}\ \ldots\ vS_{t-1}^{(k)}\ VS_{t-1}^{((-1))}]^\tau$ in both (89) and (90), followed by substitution of $[vS_{t-2}^{(2)}\ \ldots\ vS_{t-2}^{(k)}\ VS_{t-2}^{((-1))}]^\tau$ in both newly derived versions e.t.c., yields (2), (3) and (4), together with (6), (7) and (8).

Then for any $m \in \mathbb{N}$ (or $m = 0$ can be included before substitution as well), it can be deduced that

$$\begin{aligned} vS_t^{(i)} &= \sum_{j=2}^{k} \sum_{l=1+m}^{w^*+m} rp_{t,l}\ L_{i-1,j-1}(R^2 S_{t,l}^{\{m\}})\ vS_{t-l}^{(j)} + \sum_{l=1+m}^{w^*+m} rp_{t,l}\ L_{i-1,k}(R^2 S_{t,l}^{\{m\}})\ VS_{t-l}^{((-1))} \\ &+ \sum_{l=w^*+m+1}^{2w^*+m} rp_{t,l}\ L_{i-1,1}(ORS_{t,l}^{\{m\}})\ VS_{t-l}^{((-1))} \\ &+ \sum_{l=1}^{2w^*+m} rp_{t,l} \left(\prod_{ii=1}^{p} f_0(X_{t-ii-l}) \right) L_{i-1}(r^2 s_{t,l}^{\{m\}})\ q_{t-1-l}^{(\mathbf{0})}, \quad i = 2, \ldots, k, \end{aligned}$$

and

$$
\begin{aligned}
VS_t^{((-1))} &= \sum_{j=2}^{k}\sum_{l=1+m}^{w^*+m} rp_{t,l}\ L_{k,j-1}(R^2 S_{t,l}^{\{m\}})\ vS_{t-l}^{(j)} + \sum_{l=1+m}^{w^*+m} rp_{t,l}\ L_{k,k}(R^2 S_{t,l}^{\{m\}})\ VS_{t-l}^{((-1))} \\
&+ \sum_{l=w^*+m+1}^{2w^*+m} rp_{t,l}\ L_{k,1}(ORS_{t,l}^{\{m\}})\ VS_{t-l}^{((-1))} \\
&+ \sum_{l=1}^{2w^*+m} rp_{t,l}\ \left(\prod_{ii=1}^{p} f_0(X_{t-ii-l})\right)\ L_k(r^2 s_{t,l}^{\{m\}})\ q_{t-1-l}^{(\mathbf{0})}
\end{aligned}
$$

and that

$$
\begin{aligned}
\frac{q_{t-1}^{(\mathbf{0})}}{\pi_0} &= \sum_{j=2}^{k}\sum_{l=1+m}^{w^*+m} rp_{t,l}\ L_{j-1}(R^2 Q_{t,l}^{\{m\}})\ vS_{t-l}^{(j)} + \sum_{l=1+m}^{w^*+m} rp_{t,l}\ L_k(R^2 Q_{t,l}^{\{m\}})\ VS_{t-l}^{((-1))} \\
&+ \sum_{l=w^*+m+1}^{2w^*+m} rp_{t,l}\ ORQ_{t,l}^{\{m\}}\ VS_{t-l}^{((-1))} \\
&+ \sum_{l=1}^{2w^*+m} rp_{t,l}\ \left(\prod_{ii=1}^{p} f_0(X_{t-ii-l})\right)\ r^2 q_{t,l}^{\{m\}}\ q_{t-1-l}^{(\mathbf{0})}.
\end{aligned}
$$

The argument $(r = 1)$ is completed here.

$$\vdots$$

$(r = \rho)$ (where it might be $\rho = 2, \ldots, k-1$):

After the necessary steps have been taken, it is accepted for $m \in \mathbb{N}_0$ that

$$
\begin{aligned}
&vS_t^{(i)} = \sum_{j=\rho+1}^{k}\sum_{l=1+m}^{w^*+m} rp_{t,l}\ L_{i-\rho,j-\rho}(R^{\rho+1} S_{t,l}^{\{m\}})\ vS_{t-l}^{(j)} + \\
&\sum_{l=1+m}^{w^*+m} rp_{t,l}\ L_{i-\rho,k+1-\rho}(R^{\rho+1} S_{t,l}^{\{m\}})\ VS_{t-l}^{((-1))} + \sum_{l=w^*+m+1}^{(\rho+1)w^*+m} rp_{t,l}\ L_{i-\rho,1}(OR^{\rho} S_{t,l}^{\{m\}})\ VS_{t-l}^{((-1))} + \\
&\sum_{l=1}^{(\rho+1)w^*+m} rp_{t,l}\ \left(\prod_{ii=1}^{p} f_0(X_{t-ii-l})\right)\ L_{i-\rho}(r^{\rho+1} s_{t,l}^{\{m\}})\ q_{t-1-l}^{(\mathbf{0})}, \quad i = \rho+1, \ldots, k
\end{aligned}
$$

and that

$$
\begin{aligned}
&VS_t^{((-1))} = \sum_{j=\rho+1}^{k}\sum_{l=1+m}^{w^*+m} rp_{t,l}\ L_{k+1-\rho,j-\rho}(R^{\rho+1} S_{t,l}^{\{m\}})\ vS_{t-l}^{(j)} + \\
&\sum_{l=1+m}^{w^*+m} rp_{t,l}\ L_{k+1-\rho,k+1-\rho}(R^{\rho+1} S_{t,l}^{\{m\}})\ VS_{t-l}^{((-1))} + \sum_{l=w^*+m+1}^{(\rho+1)w^*+m} rp_{t,l}\ L_{k+1-\rho,1}(OR^{\rho} S_{t,l}^{\{m\}})\ VS_{t-l}^{((-1))} + \\
&\sum_{l=1}^{(\rho+1)w^*+m} rp_{t,l}\ \left(\prod_{ii=1}^{p} f_0(X_{t-ii-l})\right)\ L_{k+1-\rho}(r^{\rho+1} s_{t,l}^{\{m\}})\ q_{t-1-l}^{(\mathbf{0})}
\end{aligned}
$$

and

$$\begin{aligned}\frac{q^{(\mathbf{0})}_{t-1}}{\pi_0} &= \sum_{j=\rho+1}^{k}\sum_{l=1+m}^{w^*+m} rp_{t,l}\ L_{j-\rho}(R^{\rho+1}Q^{\{m\}}_{t,l})\ vS^{(j)}_{t-l} + \sum_{l=1+m}^{w^*+m} rp_{t,l}\ L_{k+1-\rho}(R^{\rho+1}Q^{\{m\}}_{t,l})\ VS^{((-1))}_{t-l} \\ &+ \sum_{l=w^*+m+1}^{(\rho+1)w^*+m} rp_{t,l}\ OR^{\rho}Q^{\{m\}}_{t,l}\ VS^{((-1))}_{t-l} + \sum_{l=1}^{(\rho+1)w^*+m} rp_{t,l}\ \left(\prod_{ii=1}^{p} f_0(X_{t-ii-l})\right)\ r^{\rho+1}q^{\{m\}}_{t,l}\ q^{(\mathbf{0})}_{t-1-l}.\end{aligned}$$

$(r=\rho+1)$: formally a new induction argument on n is required, but this will be done very quickly as it resembles the case presented already $(r=1)$;

$< n=0 >$: solve $vS^{(\rho+1)}_{t-w^*}$ from the accepted version when $m=0$, $i=\rho+2$; plug the solution in the other accepted versions $m=0$, $i=\rho+3,\ldots,k+1$, yielding the same equations (29) and (30), (31) but for $r=\rho+1$, as well as

$$COR^{\rho}S^{(i)}_{t,l} := L_{i-\rho,1}(OR^{\rho}S_{t,l}) - IR^{\rho+1}S^{(i)}_{t,w^*}\cdot L_{2,1}(OR^{\rho}S_{t,l});$$

from the just derived results, solve $vS^{(\rho+2)}_{t-w^*}$ from $i=\rho+3$ and plug the solution in the other just derived results $i=\rho+4,\ldots,k+1$, yielding the same equations (32) and (33), (34), (35) but for $r=\rho+1$, $i_*-r=1$. In fact by going on for all $i_*=\rho+2,\ldots,k$, it will be added that

$$C^{i_*-\rho}OR^{\rho}S^{(i)}_{t,l} := C^{i_*-\rho-1}OR^{\rho}S^{(i)}_{t,l} - IC^{i_*-\rho-1}R^{\rho+1}S^{(i)}_{t,w^*}\cdot C^{i_*-\rho-1}OR^{\rho}S^{(i_*+1)}_{t,l};$$

$vS^{(k)}_{t-w^*}$ will be the last to solve (from $VS^{((-1))}_{t}$).

Back to the accepted versions from $(r=\rho)$, $m\in\mathbb{N}$ for $vS^{(i)}_t$, $i=\rho+1,\ldots,k$, $VS^{((-1))}_t$ and $\{q^{(\mathbf{0})}_{t-1}/\pi_0\}$, plug in those the quantity $vS^{(\rho+1)}_{t-m-w^*}$ yielding (15), (16) and (17), (19), (18), (20) when $r=\rho+1$, as well as

$$\begin{aligned}BOR^{\rho}S^{(i),\{m\}}_{t,l} &:= L_{i-\rho,1}(OR^{\rho}S^{\{m\}}_{t,l}) - HR^{\rho+1}S^{(i),\{m\}}_{t,w^*+m}\cdot L_{2,1}(OR^{\rho}S_{t-m,l-m}),\\ BOR^{\rho}Q^{\{m\}}_{t,l} &:= OR^{\rho}Q^{\{m\}}_{t,l} - HR^{\rho+1}Q^{\{m\}}_{t,w^*+m}\cdot L_{2,1}(OR^{\rho}S_{t-m,l-m});\end{aligned}$$

next, in those versions of $vS^{(i)}_t$, $VS^{((-1))}_t$, $\{q^{(\mathbf{0})}_{t-1}/\pi_0\}$ that were just computed, $vS^{(\rho+2)}_{t-m-w^*}$ is plugged in yielding (21), (22) and (23), (25), (27), (24), (26), (28) when $r=\rho+1$, $i_*-r=1$, as well as

$$\begin{aligned}B^2OR^{\rho}S^{(i),\{m\}}_{t,l} &:= BOR^{\rho}S^{(i),\{m\}}_{t,l} - HBR^{\rho+1}S^{(i),\{m\}}_{t,w^*+m}\cdot COR^{\rho}S^{(\rho+3)}_{t-m,l-m},\\ B^2OR^{\rho}Q^{\{m\}}_{t,l} &:= BOR^{\rho}Q^{\{m\}}_{t,l} - HBR^{\rho+1}Q^{\{m\}}_{t,w^*+m}\cdot COR^{\rho}S^{(\rho+3)}_{t-m,l-m}.\end{aligned}$$

This goes on for $i_*=(\rho+1)+2,\ldots,k$ by plugging in consecutively the quantity from before and, besides the other forms that have been seen already, there will be

$$\begin{aligned}B^{i_*-\rho}OR^{\rho}S^{(i),\{m\}}_{t,l} &:= B^{i_*-\rho-1}OR^{\rho}S^{(i),\{m\}}_{t,l} - HB^{i_*-\rho-1}R^{\rho+1}S^{(i),\{m\}}_{t,w^*+m}\cdot C^{i_*-\rho-1}OR^{\rho}S^{(i_*+1)}_{t-m,l-m},\\ B^{i_*-\rho}OR^{\rho}Q^{\{m\}}_{t,l} &:= B^{i_*-\rho-1}OR^{\rho}Q^{\{m\}}_{t,l} - HB^{i_*-\rho-1}R^{\rho+1}Q^{\{m\}}_{t,w^*+m}\cdot C^{i_*-\rho-1}OR^{\rho}S^{(i_*+1)}_{t-m,l-m}.\end{aligned}$$

Especially after the last inclusion, the equations of interest become

$$vS_t^{(i)} = rp_{t,m}\ HB^{k-\rho-1}R^{\rho+1}S_{t,w^*+m}^{(i),\{m\}}\ VS_{t-m}^{((-1))} + rp_{t,m}\ BHB^{k-\rho-2}R^{\rho+1}S_{t,w^*+m}^{(i),\{m\}}\ vS_{t-m}^{(k)} +$$
$$\ldots + rp_{t,m}\ B^{k-\rho-1}HR^{\rho+1}S_{t,w^*+m}^{(i),\{m\}}\ vS_{t-m}^{(\rho+2)} + \sum_{j=\rho+1}^{k}\sum_{l=1+m}^{w^*+m-1} rp_{t,l}\ B^{k-\rho}HR^{\rho+1}S_{t,l}^{(i,j),\{m\}}\ vS_{t-l}^{(j)} +$$
$$\sum_{l=1+m}^{w^*+m} rp_{t,l}\ B^{k-\rho}HR^{\rho+1}S_{t,l}^{(i,k+1),\{m\}}\ VS_{t-l}^{((-1))} + \sum_{l=w^*+m+1}^{(\rho+1)w^*+m} rp_{t,l}\ B^{k-\rho}OR^{\rho}S_{t,l}^{(i),\{m\}}\ VS_{t-l}^{((-1))} +$$
$$\sum_{l=1+m}^{(\rho+1)w^*+m} rp_{t,l}\ \left(\prod_{ii=1}^{p} f_0(X_{t-ii-l})\right)\ b^{k-\rho}r^{\rho+1}s_{t,l}^{(i),\{m\}}\ q_{t-1-l}^{(\mathbf{0})} +$$
$$\sum_{l=1}^{m} rp_{t,l}\ \left(\prod_{ii=1}^{p} f_0(X_{t-ii-l})\right)\ L_{i-\rho}(r^{\rho+1}s_{t,l}^{\{m\}})\ q_{t-1-l}^{(\mathbf{0})},\quad i = \rho+2,\ldots,k,$$

and

$$VS_t^{((-1))} = rp_{t,m}\ HB^{k-\rho-1}R^{\rho+1}S_{t,w^*+m}^{(k+1),\{m\}}\ VS_{t-m}^{((-1))} + rp_{t,m}\ BHB^{k-\rho-2}R^{\rho+1}S_{t,w^*+m}^{(k+1),\{m\}}\ vS_{t-m}^{(k)} +$$
$$\ldots + rp_{t,m}\ B^{k-\rho-1}HR^{\rho+1}S_{t,w^*+m}^{(k+1),\{m\}}\ vS_{t-m}^{(\rho+2)} + \sum_{j=\rho+1}^{k}\sum_{l=1+m}^{w^*+m-1} rp_{t,l}\ B^{k-\rho}HR^{\rho+1}S_{t,l}^{(k+1,j),\{m\}}\ vS_{t-l}^{(j)} +$$
$$\sum_{l=1+m}^{w^*+m} rp_{t,l}\ B^{k-\rho}HR^{\rho+1}S_{t,l}^{(k+1,k+1),\{m\}}\ VS_{t-l}^{((-1))} + \sum_{l=w^*+m+1}^{(\rho+1)w^*+m} rp_{t,l}\ B^{k-\rho}OR^{\rho}S_{t,l}^{(k+1),\{m\}}\ VS_{t-l}^{((-1))} +$$
$$\sum_{l=1+m}^{(\rho+1)w^*+m} rp_{t,l}\ \left(\prod_{ii=1}^{p} f_0(X_{t-ii-l})\right)\ b^{k-\rho}r^{\rho+1}s_{t,l}^{(k+1),\{m\}}\ q_{t-1-l}^{(\mathbf{0})} +$$
$$\sum_{l=1}^{m} rp_{t,l}\ \left(\prod_{ii=1}^{p} f_0(X_{t-ii-l})\right)\ L_{k+1-\rho}(r^{\rho+1}s_{t,l}^{\{m\}})\ q_{t-1-l}^{(\mathbf{0})},$$

and

$$\frac{q_{t-1}^{(\mathbf{0})}}{\pi_0} = rp_{t,m}\ HB^{k-\rho-1}R^{\rho+1}Q_{t,w^*+m}^{\{m\}}\ VS_{t-m}^{((-1))} + rp_{t,m}\ BHB^{k-\rho-2}R^{\rho+1}Q_{t,w^*+m}^{\{m\}}\ vS_{t-m}^{(k)} +$$
$$\ldots + rp_{t,m}\ B^{k-\rho-1}HR^{\rho+1}Q_{t,w^*+m}^{\{m\}}\ vS_{t-m}^{(\rho+2)} + \sum_{j=\rho+1}^{k}\sum_{l=1+m}^{w^*+m-1} rp_{t,l}\ B^{k-\rho}HR^{\rho+1}Q_{t,l}^{(j),\{m\}}\ vS_{t-l}^{(j)} +$$
$$\sum_{l=1+m}^{w^*+m} rp_{t,l}\ B^{k-\rho}HR^{\rho+1}Q_{t,l}^{(k+1),\{m\}}\ VS_{t-l}^{((-1))} + \sum_{l=w^*+m+1}^{(\rho+1)w^*+m} rp_{t,l}\ B^{k-\rho}OR^{\rho}Q_{t,l}^{\{m\}}\ VS_{t-l}^{((-1))} +$$
$$\sum_{l=1+m}^{(\rho+1)w^*+m} rp_{t,l}\ \left(\prod_{ii=1}^{p} f_0(X_{t-ii-l})\right)\ b^{k-\rho}r^{\rho+1}q_{t,l}^{\{m\}}\ q_{t-1-l}^{(\mathbf{0})} +$$
$$\sum_{l=1}^{m} rp_{t,l}\ \left(\prod_{ii=1}^{p} f_0(X_{t-ii-l})\right)\ r^{\rho+1}q_{t,l}^{\{m\}}\ q_{t-1-l}^{(\mathbf{0})}$$

for any $m \in \mathbb{N}$; those are free of $vS_{t-m}^{(\rho+1)}$.

How this argument continues in $< n = \nu >$ and $< n = \nu + 1 >$ will not be presented here: the reader should look at the case $(r = 1)$ to complete the induction on n; once the step $r = \rho + 1$ has finished, the formulae will be verified to complete the induction on r as well.

Finally, from the special case $\rho = k-1$ of the induction for r as it has just been proven, the final equations will be of the form

$$\begin{aligned}
VS_t^{((-1))} \;=\;& rp_{t,1}\ HB^{w^*-1}R^kS_{t,w^*+1}^{(k+1),\{w^*\}}\ VS_{t-1}^{((-1))} + \\
& rp_{t,2}\ BHB^{w^*-2}R^kS_{t,w^*+2}^{(k+1),\{w^*\}}\ VS_{t-2}^{((-1))} + \ldots + \\
& rp_{t,w^*}\ B^{w^*-1}HR^kS_{t,2w^*}^{(k+1),\{w^*\}}\ VS_{t-w^*}^{((-1))} + \\
& \sum_{l=1+w^*}^{2w^*} rp_{t,l}\ B^{w^*}R^kS_{t,l}^{(k+1,k+1),\{w^*\}}\ VS_{t-l}^{((-1))} + \\
& \sum_{l=2w^*+1}^{(k+1)w^*} rp_{t,l}\ B^{w^*}OR^{k-1}S_{t,l}^{(k+1),\{w^*\}}\ VS_{t-l}^{((-1))} + \\
& \sum_{l=1+w^*}^{(k+1)w^*} rp_{t,l}\ \left(\prod_{ii=1}^{p} f_0(X_{t-ii-l})\right)\ b^{w^*}r^ks_{t,l}^{(k+1),\{w^*\}}\ q_{t-1-l}^{(\mathbf{0})} + \\
& rp_{t,w^*}\ \left(\prod_{ii=1}^{p} f_0(X_{t-ii-w^*})\right)\ b^{w^*-1}r^ks_{t,w^*}^{(k+1),(\{w^*\},\{w^*-1\})}\ q_{t-1-w^*}^{(\mathbf{0})} + \\
& \quad \vdots \\
& rp_{t,2}\ \left(\prod_{ii=1}^{p} f_0(X_{t-ii-2})\right)\ br^ks_{t,2}^{(k+1),(\{w^*\},\{w^*-1\})}\ q_{t-1-2}^{(\mathbf{0})} + \\
& rp_{t,1}\ \left(\prod_{ii=1}^{p} f_0(X_{t-ii-1})\right)\ L_2(r^ks_{t,1}^{\{w^*\}})\ q_{t-1-1}^{(\mathbf{0})}
\end{aligned}$$

and

$$\begin{aligned}
\frac{q_{t-1}^{(\mathbf{0})}}{\pi_0} \;=\;& rp_{t,1}\ HB^{w^*-1}R^kQ_{t,w^*+1}^{\{w^*\}}\ VS_{t-1}^{((-1))} + \\
& rp_{t,2}\ BHB^{w^*-2}R^kQ_{t,w^*+2}^{\{w^*\}}\ VS_{t-2}^{((-1))} + \ldots + \\
& rp_{t,w^*}\ B^{w^*-1}HR^kQ_{t,2w^*}^{\{w^*\}}\ VS_{t-w^*}^{((-1))} + \\
& \sum_{l=1+w^*}^{2w^*} rp_{t,l}\ B^{w^*}R^kQ_{t,l}^{(k+1),\{w^*\}}\ VS_{t-l}^{((-1))} + \\
& \sum_{l=2w^*+1}^{(k+1)w^*} rp_{t,l}\ B^{w^*}OR^{k-1}Q_{t,l}^{\{w^*\}}\ VS_{t-l}^{((-1))} + \\
& \sum_{l=1+w^*}^{(k+1)w^*} rp_{t,l}\ \left(\prod_{ii=1}^{p} f_0(X_{t-ii-l})\right)\ b^{w^*}r^kq_{t,l}^{\{w^*\}}\ q_{t-1-l}^{(\mathbf{0})} + \\
& rp_{t,w^*}\ \left(\prod_{ii=1}^{p} f_0(X_{t-ii-w^*})\right)\ b^{w^*-1}r^kq_{t,w^*}^{(\{w^*\},\{w^*-1\})}\ q_{t-1-w^*}^{(\mathbf{0})} + \ldots + \\
& rp_{t,2}\ \left(\prod_{ii=1}^{p} f_0(X_{t-ii-2})\right)\ br^kq_{t,2}^{(\{w^*\},\{w^*-1\})}\ q_{t-1-2}^{(\mathbf{0})} + \\
& rp_{t,1}\ \left(\prod_{ii=1}^{p} f_0(X_{t-ii-1})\right)\ r^kq_{t,1}^{\{w^*\}}\ q_{t-1-1}^{(\mathbf{0})}.
\end{aligned}$$

Following the two equations above (see that $(k+1)w^* = (k+1)(k+1)^w = (k+1)^{w+1} \equiv$

$(w+1)^*$), the notation can be rearranged to write

$$\begin{aligned} VS_t^{((-1))} &= \sum_{l=1}^{(w+1)^*} rp_{t,l}\ RS(w+1)_{t,l}\ VS_{t-l}^{((-1))} + \\ & \sum_{l=1}^{(w+1)^*} rp_{t,l}\ \left(\prod_{ii=1}^{p} f_0(X_{t-ii-l})\right)\ rs(w+1)_{t,l}\ q_{t-1-l}^{(\mathbf{0})}, \end{aligned}$$

with

$$RS(w+1)_{t,l} := \begin{cases} B^{l-1}HB^{w^*-l}R^kS_{t,w^*+l}^{(k+1),\{w^*\}}, & \text{if} \quad l=1,\ldots,w^*, \\ B^{w^*}R^kS_{t,l}^{(k+1,k+1),\{w^*\}}, & \text{if} \quad l=1+w^*,\ldots,2w^*, \\ B^{w^*}OR^{k-1}S_{t,l}^{(k+1),\{w^*\}}, & \text{if} \quad l=2w^*+1,\ldots,(w+1)^* \end{cases}$$

and

$$rs(w+1)_{t,l} := \begin{cases} L_2(r^ks_{t,l}^{\{w^*\}}), & \text{if} \quad l=1, \\ b^{l-1}r^ks_{t,l}^{(k+1),(\{w^*\},\{w^*-1\})}, & \text{if} \quad l=2,\ldots,w^*, \\ b^{w^*}r^ks_{t,l}^{(k+1),\{w^*\}}, & \text{if} \quad l=1+w^*,\ldots,(w+1)^* \end{cases},$$

as well as

$$\begin{aligned} \frac{q_{t-1}^{(\mathbf{0})}}{\pi_0} &= \sum_{l=1}^{(w+1)^*} rp_{t,l}\ RQ(w+1)_{t,l}\ VS_{t-l}^{((-1))} + \\ & \sum_{l=1}^{(w+1)^*} rp_{t,l}\ \left(\prod_{ii=1}^{p} f_0(X_{t-ii-l})\right)\ rq(w+1)_{t,l}\ q_{t-1-l}^{(\mathbf{0})}, \end{aligned}$$

with

$$RQ(w+1)_{t,l} := \begin{cases} B^{l-1}HB^{w^*-l}R^kQ_{t,w^*+l}^{\{w^*\}}, & \text{if} \quad l=1,\ldots,w^*, \\ B^{w^*}R^kQ_{t,l}^{(k+1),\{w^*\}}, & \text{if} \quad l=1+w^*,\ldots,2w^*, \\ B^{w^*}OR^{k-1}Q_{t,l}^{\{w^*\}}, & \text{if} \quad l=2w^*+1,\ldots,(w+1)^* \end{cases}$$

and

$$rq(w+1)_{t,l} := \begin{cases} r^kq_{t,l}^{\{w^*\}}, & \text{if} \quad l=1, \\ b^{l-1}r^kq_{t,l}^{(\{w^*\},\{w^*-1\})}, & \text{if} \quad l=2,\ldots,w^*, \\ b^{w^*}r^kq_{t,l}^{\{w^*\}}, & \text{if} \quad l=1+w^*,\ldots,(w+1)^* \end{cases}.$$

By comparing the four formulae to the ones concluding the statement (a) of the proposition, the proof for the induction argument on d has been completed.

(b) Straight from (78) and (79), it holds that

$$\begin{aligned} p_t &= \sum_{l=1}^{(k+1)^{q-1}} rp_{t,l}\ RS(q-1)_{t,l}\ p_{t-l} \\ &+ \sum_{l=1}^{(k+1)^{q-1}} rp_{t,l}\ \left(\prod_{ii=1}^{p} f_0(X_{t-ii-l})\right)\ rs(q-1)_{t,l}\ q_{t-1-l}^{(\mathbf{0})} \end{aligned}$$

and

$$
\begin{aligned}
q^{(\mathbf{0})}_{t-1} &= \pi_0 \left\{ \sum_{l=1}^{(k+1)^{q-1}} rp_{t,l}\ RQ(q-1)_{t,l}\ p_{t-l} \right. \\
&\left. + \sum_{l=1}^{(k+1)^{q-1}} rp_{t,l} \left(\prod_{ii=1}^{p} f_0(X_{t-ii-l}) \right) rq(q-1)_{t,l}\ q^{(\mathbf{0})}_{t-1-l} \right\},
\end{aligned}
$$

respectively, where the scalars $RS(q-1)_{t,l}$, $rs(q-1)_{t,l}$, $RQ(q-1)_{t,l}$ and $rq(q-1)_{t,l}$ have been determined in (a); it is written here $RS'_{t,l}$ and $rs'_{t,l}$ instead of $RS(q-1)_{t,l}$ and $rs(q-1)_{t,l}$, as well as it is transformed for convenience

$$
RQ'_{t,l} := \pi_0\ RQ(q-1)_{t,l} \quad \text{and} \quad rq'_{t,l} := \pi_0\ rq(q-1)_{t,l},
$$

resulting in

$$
p_t = \sum_{l=1}^{(k+1)^{q-1}} rp_{t,l}\ RS'_{t,l}\ p_{t-l} + \sum_{l=1}^{(k+1)^{q-1}} rp_{t,l} \left(\prod_{ii=1}^{p} f_0(X_{t-ii-l}) \right) rs'_{t,l}\ q^{(\mathbf{0})}_{t-1-l} \tag{91}
$$

and

$$
q^{(\mathbf{0})}_{t-1} = \sum_{l=1}^{(k+1)^{q-1}} rp_{t,l}\ RQ'_{t,l}\ p_{t-l} + \sum_{l=1}^{(k+1)^{q-1}} rp_{t,l} \left(\prod_{ii=1}^{p} f_0(X_{t-ii-l}) \right) rq'_{t,l}\ q^{(\mathbf{0})}_{t-1-l}. \tag{92}
$$

It can be seen from (91) that

$$
p_t = \sum_{l=1}^{(k+1)^{q-1}} rp_{t,l}\ RS'_{t,l}\ p_{t-l}, \text{ if } \mathbf{X}_{t-1-l} \neq \mathbf{0}_p \text{ for all } l = 1, \ldots, (k+1)^{q-1}. \tag{93}
$$

Otherwise, define $\mathcal{Y}_t := \{i = 1, \ldots, (k+1)^{q-1} : \mathbf{X}_{t-1-i} = \mathbf{0}_p\}$ and $\mathcal{N}_t := \{i = 1, \ldots, (k+1)^{q-1} : \mathbf{X}_{t-1-i} \neq \mathbf{0}_p\}$, such that (91) and (92) take the form

$$
\begin{bmatrix} p_t \\ q^{(\mathbf{0})}_{t-1} \end{bmatrix} = \sum_{i \in \mathcal{Y}_t} rp_{t,i}\ RSQ_{t,i} \begin{bmatrix} p_{t-i} \\ q^{(\mathbf{0})}_{t-1-i} \end{bmatrix} + \sum_{i \in \mathcal{N}_t} rp_{t,i}\ L_{1:2,1}(RSQ_{t,i})\ p_{t-i},
$$

where $RSQ_{t,i} := \begin{bmatrix} RS'_{t,i} & rs'_{t,i} \\ RQ'_{t,i} & rq'_{t,i} \end{bmatrix}$ and $L_{i,j}$ now operates on matrices consisting of scalars. It can be re-written that

$$
\begin{bmatrix} p_t \\ q^{(\mathbf{0})}_{t-1} \end{bmatrix} = \sum_{i \in \mathcal{Y}_t} rp_{t,i}\ RSQ_{t,i} \begin{bmatrix} p_{t-i} \\ q^{(\mathbf{0})}_{t-1-i} \end{bmatrix} + \sum_{i \in \mathcal{N}_t} rp_{t,i-1}\ L_{1:2,1}(RSQ_{t,i}). \tag{94}
$$

Write $i_1(t) < \ldots < i_{y_t}(t)$ for the members of $\mathcal{Y}_t$ and $j_1(t) < \ldots < j_{n_t}(t)$ similarly for $\mathcal{N}_t$. Then (94) becomes

$$
\begin{bmatrix} p_t \\ q^{(\mathbf{0})}_{t-1} \end{bmatrix} = \sum_{c=1}^{y_t} rp_{t,i_c(t)}\ RSQ_{t,i_c(t)} \begin{bmatrix} p_{t-i_c(t)} \\ q^{(\mathbf{0})}_{t-1-i_c(t)} \end{bmatrix} + \sum_{c=1}^{n_t} rp_{t,j_c(t)-1}\ L_{1:2,1}(RSQ_{t,j_c(t)});
$$

first, $\begin{bmatrix} p_{t-i_1(t)} \\ q^{(\mathbf{0})}_{t-1-i_1(t)} \end{bmatrix}$ should be replaced, resulting in

$$\begin{aligned}
\begin{bmatrix} p_t \\ q^{(\mathbf{0})}_{t-1} \end{bmatrix} &= \sum_{c=2}^{y_t} rp_{t,i_c(t)}\ RSQ_{t,i_c(t)} \begin{bmatrix} p_{t-i_c(t)} \\ q^{(\mathbf{0})}_{t-1-i_c(t)} \end{bmatrix} + \sum_{c=1}^{n_t} rp_{t,j_c(t)-1}\ L_{1:2,1}(RSQ_{t,j_c(t)}) + \\
&\quad \sum_{c=1}^{y_{t-i_1(t)}} rp_{t,i_1(t)+i_c(t-i_1(t))}\ RSQ_{t,i_1(t)}\ RSQ_{t-i_1(t),i_c(t-i_1(t))} \begin{bmatrix} p_{t-i_1(t)-i_c(t-i_1(t))} \\ q^{(\mathbf{0})}_{t-1-i_1(t)-i_c(t-i_1(t))} \end{bmatrix} + \\
&\quad \sum_{c=1}^{n_{t-i_1(t)}} rp_{t,i_1(t)+j_c(t-i_1(t))-1}\ RSQ_{t,i_1(t)}\ L_{1:2,1}(RSQ_{t-i_1(t),j_c(t-i_1(t))}),
\end{aligned}$$

since

$$\begin{aligned}
rp_{t,i_1(t)}\ rp_{t-i_1(t),i_c(t-i_1(t))} &= \frac{1}{p_{t-1}\cdots p_{t-i_1(t)}} \frac{1}{p_{t-i_1(t)-1}\cdots p_{t-i_1(t)-i_c(t-i_i(t))}} \\
&\equiv rp_{t,i_1(t)+i_c(t-i_1(t))}, \\
rp_{t,i_1(t)}\ rp_{t-i_1(t),j_c(t-i_1(t))-1} &= \frac{1}{p_{t-1}\cdots p_{t-i_1(t)}} \frac{1}{p_{t-i_1(t)-1}\cdots p_{t-i_1(t)-j_c(t-i_1(t))+1}} \\
&\equiv rp_{t,i_1(t)+j_c(t-i_1(t))-1}.
\end{aligned}$$

Define

$$\mathcal{Y}^{\{1\}}_t := \{i = i_1(t)+1,\ldots,i_1(t)+(k+1)^{q-1} : \mathbf{X}_{t-1-i} = \mathbf{0}_p\}$$

and observe that

$$\mathcal{Y}^{\{1\}}_t := \{i_2(t),\ldots,i_{y_t}(t)\} \cup \{i = (k+1)^{q-1}+1,\ldots,i_1(t)+(k+1)^{q-1} : \mathbf{X}_{t-1-i} = \mathbf{0}_p\};$$

write $i^{\{1\}}_1(t) < \ldots < i^{\{1\}}_{y^{\{1\}}_t}(t)$ for the members of $\mathcal{Y}^{\{1\}}_t$ with $i^{\{1\}}_c(t) \equiv i_{c+1}(t)$ for the first ones $c = 1,\ldots,y_t-1$. Note that

$$i \in \mathcal{Y}^{\{1\}}_t \ \text{ iff } \ i - i_1(t) = 1,\ldots,(k+1)^{q-1} \ \text{ and } \ \mathbf{X}_{t-i-1} \equiv \mathbf{X}_{(t-i_1(t))-(i-i_1(t))-1} = \mathbf{0}_p$$

i.e.

$$i \in \mathcal{Y}^{\{1\}}_t \ \text{ iff } \ i - i_1(t) \in \mathcal{Y}_{t-i_1(t)}. \tag{95}$$

Statement (95) implies that $i_c(t-i_1(t)) = i^{\{1\}}_c(t) - i_1(t)$ for $c = 1,\ldots,y_{t-i_1(t)} \equiv y^{\{1\}}_t$, hence

$$\begin{aligned}
&\sum_{c=1}^{y_{t-i_1(t)}} rp_{t,i_1(t)+i_c(t-i_1(t))}\ RSQ_{t,i_1(t)}\ RSQ_{t-i_1(t),i_c(t-i_1(t))} \begin{bmatrix} p_{t-i_1(t)-i_c(t-i_1(t))} \\ q^{(\mathbf{0})}_{t-1-i_1(t)-i_c(t-i_1(t))} \end{bmatrix} = \\
&\sum_{c=1}^{y^{\{1\}}_t} rp_{t,i^{\{1\}}_c(t)}\ RSQ_{t,i_1(t)}\ RSQ_{t-i_1(t),i^{\{1\}}_c(t)-i_1(t)} \begin{bmatrix} p_{t-i^{\{1\}}_c(t)} \\ q^{(\mathbf{0})}_{t-1-i^{\{1\}}_c(t)} \end{bmatrix}.
\end{aligned}$$

Also define $\mathcal{N}^{\{1\}}_t := \{i = 1,\ldots,i_1(t)+(k+1)^{q-1} : \mathbf{X}_{t-1-i} \neq \mathbf{0}_p\}$ and observe that

$$\mathcal{Y}^{\{1\}}_t \cup \mathcal{N}^{\{1\}}_t = \{1,\ldots,i_1(t)-1, i_1(t)+1,\ldots,i_1(t)+(k+1)^{q-1}\} \text{ and } \mathcal{Y}^{\{1\}}_t \cap \mathcal{N}^{\{1\}}_t = \emptyset;$$

write $j_1^{\{1\}}(t) < \ldots < j_{\{n_t^{\{1\}}\}}^{\{1\}}(t)$ for the members of $\mathcal{N}_t^{\{1\}}$ with $j_c^{\{1\}}(t) \equiv j_c(t)$ for $c = 1, \ldots, n_t \leq n_t^{\{1\}}$ since $\mathcal{N}_t \subseteq \mathcal{N}_t^{\{1\}}$. Note that

$$i \in \mathcal{N}_t^{\{1\}} \text{ iff } i - i_1(t) = 1 - i_1(t), \ldots, (k+1)^{q-1} \text{ and } \mathbf{X}_{t-i_1(t)-1-(i-i_1(t))} \neq \mathbf{0}_p,$$

i.e.

$$i \in \mathcal{N}_t^{\{1\}} \ \text{ iff } \ i - i_1(t) \in \mathcal{N}_{t-i_1(t)} \ \text{ or } \ i - i_1(t) = 1 - i_1(t), \ldots, -1. \tag{96}$$

Next to (96), it is written that

$$i \in \mathcal{N}_t^{\{1\}} \cap \{i : i > i_1(t)\} \ \text{ iff } \ i - i_1(t) \in \mathcal{N}_{t-i_1(t)}$$

and $\mathcal{N}_t^{\{1\}} \cap \{i : i > i_1(t)\} = \{i_1^+(t), \ldots, i_{n_{t-i_1(t)}}^+(t)\}$, so that it is written that $j_c(t - i_1(t)) = i_c^+(t) - i_1(t)$, $c = 1, \ldots, n_{t-i_1(t)}$, hence

$$\sum_{c=1}^{n_{t-i_1(t)}} rp_{t,i_1(t)+j_c(t-i_1(t))-1} \ RSQ_{t,i_1(t)} \ L_{1:2,1}(RSQ_{t-i_1(t),j_c(t-i_1(t))}) = \sum_{i \in \mathcal{N}_t^{\{1\}}, \ i > i_1(t)} rp_{t,i-1} \ RSQ_{t,i_1(t)} \ L_{1:2,1}(RSQ_{t-i_1(t),i-i_1(t)}).$$

Altogether now,

$$\begin{aligned}
\begin{bmatrix} p_t \\ q_{t-1}^{(\mathbf{0})} \end{bmatrix} &= \sum_{c=1}^{y_t - 1} rp_{t,i_c^{\{1\}}(t)} \ RSQ_{t,i_c^{\{1\}}(t)} \begin{bmatrix} p_{t-i_c^{\{1\}}(t)} \\ q_{t-1-i_c^{\{1\}}(t)}^{(\mathbf{0})} \end{bmatrix} + \sum_{c=1}^{n_t} rp_{t,j_c^{\{1\}}(t)-1} \ L_{1:2,1}(RSQ_{t,j_c^{\{1\}}(t)}) + \\
&\sum_{c=1}^{y_t^{\{1\}}} rp_{t,i_c^{\{1\}}(t)} \ RSQ_{t,i_1(t)} \ RSQ_{t-i_1(t),i_c^{\{1\}}(t)-i_1(t)} \begin{bmatrix} p_{t-i_c^{\{1\}}(t)} \\ q_{t-1-i_c^{\{1\}}(t)}^{(\mathbf{0})} \end{bmatrix} + \\
&\sum_{i \in \mathcal{N}_t^{\{1\}}, \ i > i_1(t)} rp_{t,i-1} \ RSQ_{t,i_1(t)} \ L_{1:2,1}(RSQ_{t-i_1(t),i-i_1(t)})
\end{aligned}$$

or

$$\begin{bmatrix} p_t \\ q_{t-1}^{(\mathbf{0})} \end{bmatrix} = \sum_{i \in \mathcal{Y}_t^{\{1\}}} rp_{t,i} \ RSQ_{t,i}^{\{1\}} \begin{bmatrix} p_{t-i} \\ q_{t-1-i}^{(\mathbf{0})} \end{bmatrix} + \sum_{i \in \mathcal{N}_t^{\{1\}}} rp_{t,i-1} \ L_{1:2,1}(RSQ_{t,i}^{\{1\}})$$

where

$$RSQ_{t,i}^{\{1\}} := \begin{cases} RSQ_{t,i} + RSQ_{t,i_1(t)} \ RSQ_{t-i_1(t),i-i_1(t)}, \\ \text{if } \ i \in \mathcal{Y}_t^{\{1\}} \cap \mathcal{Y}_t \text{ or if } \ i \in \mathcal{N}_t, \ i > i_1(t), \\ RSQ_{t,i_1(t)} \ RSQ_{t-i_1(t),i-i_1(t)}, \\ \text{if } \ i \in \mathcal{Y}_t^{\{1\}}, \ i \notin \mathcal{Y}_t \text{ or if } \ i \in \mathcal{N}_t^{\{1\}}, \ i \notin \mathcal{N}_t, \\ RSQ_{t,i}, \ \text{if } \ i = 1, \ldots, i_1(t) - 1 \end{cases}.$$

For $n = 2, \ldots$, accept that it can be written

$$\begin{bmatrix} p_t \\ q_{t-1}^{(\mathbf{0})} \end{bmatrix} = \sum_{i \in \mathcal{Y}_t^{\{n\}}} rp_{t,i} \ RSQ_{t,i}^{\{n\}} \begin{bmatrix} p_{t-i} \\ q_{t-1-i}^{(\mathbf{0})} \end{bmatrix} + \sum_{i \in \mathcal{N}_t^{\{n\}}} rp_{t,i-1} \ L_{1:2,1}(RSQ_{t,i}^{\{n\}}), \tag{97}$$

where

$$RSQ_{t,i}^{\{n\}} := \begin{cases} RSQ_{t,i}^{\{n-1\}} + RSQ_{t,i_1^{\{n-1\}}(t)}^{\{n-1\}} \; RSQ_{t-i_1^{\{n-1\}}(t),i-i_1^{\{n-1\}}(t)}, \\ \text{if } \; i \in \mathcal{Y}_t^{\{n\}} \cap \mathcal{Y}_t^{\{n-1\}} \; \text{ or if } \; i \in \mathcal{N}_t^{\{n-1\}}, \; i > i_1^{\{n-1\}}(t), \\ RSQ_{t,i_1^{\{n-1\}}(t)}^{\{n-1\}} \; RSQ_{t-i_1^{\{n-1\}}(t),i-i_1^{\{n-1\}}(t)}, \\ \text{if } \; i \in \mathcal{Y}_t^{\{n\}}, \; i \notin \mathcal{Y}_t^{\{n-1\}} \; \text{ or if } \; i \in \mathcal{N}_t^{\{n\}}, \; i \notin \mathcal{N}_t^{\{n-1\}}, \\ RSQ_{t,i}^{\{n-1\}}, \\ \text{if } \; i = 1, \ldots, i_1^{\{n-1\}}(t) - 1, \; i \neq i_1(t), i_1^{\{1\}}(t), \ldots, i_1^{\{n-2\}}(t) \end{cases} \tag{98}$$

and

$$\mathcal{Y}_t^{\{n\}} := \{i = i_1^{\{n-1\}}(t) + 1, \ldots, i_1^{\{n-1\}}(t) + (k+1)^{q-1} : \mathbf{X}_{t-1-i} = \mathbf{0}_p\}$$

with members $i_1^{\{n\}}(t) < \ldots < i_{y_t^{\{n\}}}^{\{n\}}(t)$, as well as

$$\mathcal{N}_t^{\{n\}} := \{i = 1, \ldots, i_1^{\{n-1\}}(t) + (k+1)^{q-1} : \mathbf{X}_{t-1-i} \neq \mathbf{0}_p\}$$

with members $j_1^{\{n\}}(t) < \ldots < j_{n_t^{\{n\}}}^{\{n\}}(t)$. In the second branch of (98), note that $i \in \mathcal{N}_t^{\{n\}}$, $i \notin \mathcal{N}_t^{\{n-1\}}$, implies that $\mathcal{N}_t^{\{n-1\}} \subset \mathcal{N}_t^{\{n\}}$ and $i > j_{n_t^{\{n-1\}}}^{\{n-1\}}(t)$; if $j_{n_t^{\{n-1\}}}^{\{n-1\}}(t) > i_1^{\{n-1\}}(t)$ then $i > i_1^{\{n-1\}}(t)$; if, however, $j_{n_t^{\{n-1\}}}^{\{n-1\}}(t) < i_1^{\{n-1\}}(t)$, then it has to be that $i_c^{\{n-1\}}(t) = i_{c-1}^{\{n-1\}}(t) + 1$, for all $c = 2, \ldots, y_t^{\{n-1\}} \geq 2$, such that $i > i_{y_t^{\{n-1\}}}^{\{n-1\}}(t) \geq i_1^{\{n-1\}}(t)$: either way it is implied $i > i_1^{\{n-1\}}(t)$.

Straight from (97) and using (94), it can be written that

$$\begin{bmatrix} p_t \\ q_{t-1}^{(\mathbf{0})} \end{bmatrix} = \sum_{c=2}^{y_t^{\{n\}}} rp_{t,i_c^{\{n\}}(t)} \; RSQ_{t,i_c^{\{n\}}(t)}^{\{n\}} \; \begin{bmatrix} p_{t-i_c^{\{n\}}(t)} \\ q_{t-1-i_c^{\{n\}}(t)}^{(\mathbf{0})} \end{bmatrix} + \sum_{i \in \mathcal{N}_t^{\{n\}}} rp_{t,i-1} \; L_{1:2,1}(RSQ_{t,i}^{\{n\}}) +$$

$$\sum_{c=1}^{y_{t-i_1^{\{n\}}(t)}} rp_{t,i_1^{\{n\}}(t)} \; rp_{t-i_1^{\{n\}}(t),i_c(t-i_1^{\{n\}}(t))} \; RSQ_{t,i_1^{\{n\}}(t)}^{\{n\}} \; RSQ_{t-i_1^{\{n\}}(t),i_c(t-i_1^{\{n\}}(t))} \cdot$$

$$\begin{bmatrix} p_{t-i_1^{\{n\}}(t)-i_c(t-i_1^{\{n\}}(t))} \\ q_{t-i_1^{\{n\}}(t)-1-i_c(t-i_1^{\{n\}}(t))}^{(\mathbf{0})} \end{bmatrix} +$$

$$\sum_{i \in \mathcal{N}_{t-i_1^{\{n\}}(t)}} rp_{t,i_1^{\{n\}}(t)} \; rp_{t-i_1^{\{n\}}(t),i-1} \; RSQ_{t,i_1^{\{n\}}(t)}^{\{n\}} \; L_{1:2,1}(RSQ_{t-i_1^{\{n\}}(t),i}).$$

Note that

$$rp_{t,i_1^{\{n\}}(t)} \; rp_{t-i_1^{\{n\}}(t),j} = \frac{1}{p_{t-1} \cdots p_{t-i_1^{\{n\}}(t)}} \; \frac{1}{p_{t-i_1^{\{n\}}(t)-1} \cdots p_{t-i_1^{\{n\}}(t)-j}} \equiv rp_{t,i_1^{\{n\}}(t)+j},$$

so that it is written again

$$\begin{bmatrix} p_t \\ q^{(\mathbf{0})}_{t-1} \end{bmatrix} = \sum_{c=2}^{y_t^{\{n\}}} rp_{t,i_c^{\{n\}}(t)} \; RSQ^{\{n\}}_{t,i_c^{\{n\}}(t)} \begin{bmatrix} p_{t-i_c^{\{n\}}(t)} \\ q^{(\mathbf{0})}_{t-1-i_c^{\{n\}}(t)} \end{bmatrix} + \sum_{i\in\mathcal{N}_t^{\{n\}}} rp_{t,i-1} \; L_{1:2,1}(RSQ^{\{n\}}_{t,i})$$

$$+ \sum_{c=1}^{y_{t-i_1^{\{n\}}(t)}} rp_{t,i_1^{\{n\}}(t)+i_c(t-i_1^{\{n\}}(t))} \; RSQ^{\{n\}}_{t,i_1^{\{n\}}(t)} \; RSQ_{t-i_1^{\{n\}}(t),i_c(t-i_1^{\{n\}}(t))} \begin{bmatrix} p_{t-i_1^{\{n\}}(t)-i_c(t-i_1^{\{n\}}(t))} \\ q^{(\mathbf{0})}_{t-i_1^{\{n\}}(t)-1-i_c(t-i_1^{\{n\}}(t))} \end{bmatrix}$$

$$+ \sum_{i\in\mathcal{N}_{t-i_1^{\{n\}}(t)}} rp_{t,i_1^{\{n\}}(t)+i-1} \; RSQ^{\{n\}}_{t,i_1^{\{n\}}(t)} \; L_{1:2,1}(RSQ_{t-i_1^{\{n\}}(t),i}).$$

Define $\mathcal{Y}_t^{\{n+1\}} := \{i = i_1^{\{n\}}(t)+1,\ldots,i_1^{\{n\}}(t)+(k+1)^{q-1} : \mathbf{X}_{t-1-i} = \mathbf{0}_p\}$ and write $i_1^{\{n+1\}}(t) < \ldots < i^{\{n+1\}}_{y_t^{\{n+1\}}}(t)$ for all its members in increasing order, clearly with $i_c^{\{n+1\}}(t) = i_{c+1}^{\{n\}}(t)$ for the first ones $c = 1,\ldots,y_t^{\{n\}}-1$. Note that

$$i \in \mathcal{Y}_t^{\{n+1\}} \text{ iff } i - i_1^{\{n\}}(t) = 1,\ldots,(k+1)^{q-1} \quad \text{and} \quad \mathbf{X}_{(t-i_1^{\{n\}}(t))-1-(i-i_1^{\{n\}}(t))} = \mathbf{0}_p$$

i.e.

$$i \in \mathcal{Y}_t^{\{n+1\}} \quad \text{iff} \quad i - i_1^{\{n\}}(t) \in \mathcal{Y}_{t-i_1^{\{n\}}(t)}. \tag{99}$$

Statement (99) implies that $i_c(t-i_1^{\{n\}}(t)) = i_c^{\{n+1\}}(t) - i_1^{\{n\}}(t)$ for $c = 1,\ldots,y_{t-i_1^{\{n\}}(t)} \equiv y_t^{\{n+1\}}$, hence

$$\sum_{c=1}^{y_{t-i_1^{\{n\}}(t)}} rp_{t,i_1^{\{n\}}(t)+i_c(t-i^{\{n\}})} \; RSQ^{\{n\}}_{t,i_1^{\{n\}}(t)} \; RSQ_{t-i_1^{\{n\}}(t),i_c(t-i_1^{\{n\}}(t))} \begin{bmatrix} p_{t-i_1^{\{n\}}(t)-i_c(t-i_1^{\{n\}}(t))} \\ q^{(\mathbf{0})}_{t-i_1^{\{n\}}(t)-1-i_c(t-i_1^{\{n\}}(t))} \end{bmatrix} =$$

$$\sum_{c=1}^{y_t^{\{n+1\}}} rp_{t,i_c^{\{n+1\}}(t)} \; RSQ^{\{n\}}_{t,i_1^{\{n\}}(t)} \; RSQ_{t-i_1^{\{n\}}(t),i_c^{\{n+1\}}(t)-i_1^{\{n\}}(t)} \begin{bmatrix} p_{t-i_c^{\{n+1\}}(t)} \\ q^{(\mathbf{0})}_{t-1-i_c^{\{n+1\}}(t)} \end{bmatrix}.$$

Also define $\mathcal{N}_t^{\{n+1\}} := \{i = 1,\ldots,i_1^{\{n\}}(t)+(k+1)^{q-1} : \mathbf{X}_{t-1-i} \neq \mathbf{0}_p\}$ and observe that $\mathcal{Y}_t^{\{n+1\}} \cap \mathcal{N}_t^{\{n+1\}} = \emptyset$ and that

$$\mathcal{Y}_t^{\{n+1\}} \cup \mathcal{N}_t^{\{n+1\}} = \{i : i = 1,\ldots,i_1^{\{n\}}(t)+(k+1)^{q-1},\ i \neq i_1(t), i_1^{\{1\}}(t),\ldots,i_1^{\{n\}}(t)\}.$$

Write $j_1^{\{n+1\}}(t) < \ldots < j^{\{n+1\}}_{n_t^{\{n+1\}}}(t)$ for the members of $\mathcal{N}_t^{\{n+1\}}$ with $j_c^{\{n+1\}}(t) \equiv j_c^{\{n\}}(t)$ for $c = 1,\ldots,n_t^{\{n\}} \leq n_t^{\{n+1\}}$ since $\mathcal{N}_t^{\{n\}} \subseteq \mathcal{N}_t^{\{n+1\}}$. Note that

$$\begin{aligned} i \in \mathcal{N}_t^{\{n+1\}} \ \text{ if } \quad & i - i_1^{\{n\}}(t) \in \mathcal{N}_{t-i_1^{\{n\}}(t)} \\ \text{or if } \quad & i - i_1^{\{n\}}(t) = 1 - i_1^{\{n\}}(t),\ldots,-1 \ \text{ and} \\ & i - i_1^{\{n\}}(t) \neq i_1(t) - i_1^{\{n\}}(t), i_1^{\{1\}}(t) - i_1^{\{n\}}(t),\ldots,i_1^{\{n-1\}}(t) - i_1^{\{n\}}(t). \end{aligned} \tag{100}$$

Next to (100) (and (99)), it is written

$$i \in \mathcal{N}_t^{\{n+1\}} \cap \{i : i > i_1^{\{n\}}(t)\} \ \text{ iff } \ i - i_1^{\{n\}}(t) \in \mathcal{N}_{t-i_1^{\{n\}}(t)}$$

and $\mathcal{N}_t^{\{n+1\}} \cap \{i : i > i_1^{\{n\}}(t)\} = \{i_1^{+(n+1)}(t), \ldots i_{n_{t-i_1^{\{n\}}(t)}}^{+(n+1)}(t)\}$, so that it is written that $j_c(t - i_1^{\{n\}}(t)) = i_c^{+(n+1)}(t) - i_1^{\{n\}}(t)$, $c = 1, \ldots, n_{t-i_1^{\{n\}}(t)}$, hence

$$\sum_{i\in\mathcal{N}_{t-i_1^{\{n\}}(t)}} rp_{t,i_1^{\{n\}}(t)+i-1} \ RSQ_{t,i_1^{\{n\}}(t)}^{\{n\}} \ L_{1:2,1}(RSQ_{t-i_1^{\{n\}}(t),i}) =$$
$$\sum_{i\in\mathcal{N}_t^{\{n+1\}},\ i>i_1^{\{n\}}(t)} rp_{t,i-1} \ RSQ_{t,i_1^{\{n\}}(t)}^{\{n\}} \ L_{1:2,1}(RSQ_{t-i_1^{\{n\}}(t),i-i_1^{\{n\}}(t)}).$$

Putting all the above together, it is written

$$\begin{aligned}\begin{bmatrix} p_t \\ q_{t-1}^{(\mathbf{0})} \end{bmatrix} &= \sum_{c=2}^{y_t^{\{n\}}} rp_{t,i_c^{\{n\}}(t)} \ RSQ_{t,i_c^{\{n\}}(t)}^{\{n\}} \begin{bmatrix} p_{t-i_c^{\{n\}}(t)} \\ q_{t-1-i_c^{\{n\}}(t)}^{(\mathbf{0})} \end{bmatrix} + \sum_{i\in\mathcal{N}_t^{\{n\}}} rp_{t,i-1} \ L_{1:2,1}(RSQ_{t,i}^{\{n\}}) \\ &+ \sum_{c=1}^{y_t^{\{n+1\}}} rp_{t,i_c^{\{n+1\}}(t)} \ RSQ_{t,i_1^{\{n\}}(t)}^{\{n\}} \ RSQ_{t-i_1^{\{n\}}(t),i_c^{\{n+1\}}(t)-i_1^{\{n\}}(t)} \begin{bmatrix} p_{t-i_c^{\{n+1\}}(t)} \\ q_{t-1-i_c^{\{n+1\}}(t)}^{(\mathbf{0})} \end{bmatrix} \\ &+ \sum_{i\in\mathcal{N}_t^{\{n+1\}},\ i>i_1^{\{n\}}(t)} rp_{t,i-1} \ RSQ_{t,i_1^{\{n\}}(t)}^{\{n\}} \ L_{1:2,1}(RSQ_{t-i_1^{\{n\}}(t),i-i_1^{\{n\}}(t)})\end{aligned}$$

or

$$\begin{bmatrix} p_t \\ q_{t-1}^{(\mathbf{0})} \end{bmatrix} = \sum_{i\in\mathcal{Y}_t^{\{n+1\}}} rp_{t,i} \ RSQ_{t,i}^{\{n+1\}} \begin{bmatrix} p_{t-i} \\ q_{t-1-i}^{(\mathbf{0})} \end{bmatrix} + \sum_{i\in\mathcal{N}_t^{\{n+1\}}} rp_{t,i-1} \ L_{1:2,1}(RSQ_{t,i}^{\{n+1\}}),$$

where

$$RSQ_{t,i}^{\{n+1\}} := \begin{cases} RSQ_{t,i}^{\{n\}} + RSQ_{t,i_1^{\{n\}}(t)}^{\{n\}} \ RSQ_{t-i_1^{\{n\}}(t),i-i_1^{\{n\}}(t)}, \\ \quad \text{if } \ i \in \mathcal{Y}_t^{\{n+1\}} \cap \mathcal{Y}_t^{\{n\}}, \ \text{ or if } \ i \in \mathcal{N}_t^{\{n\}}, \ i > i_1^{\{n\}}(t), \\ RSQ_{t,i_1^{\{n\}}(t)}^{\{n\}} \ RSQ_{t-i_1^{\{n\}}(t),i-i_1^{\{n\}}(t)}, \\ \quad \text{if } \ i \in \mathcal{Y}_t^{\{n+1\}}, \ i \notin \mathcal{Y}_t^{\{n\}}, \ \text{ or if } \ i \in \mathcal{N}_t^{\{n+1\}}, \ i \notin \mathcal{N}_t^{\{n\}}, \\ RSQ_{t,i}^{\{n\}}, \ \text{ if } \ i = 1, \ldots, i_1^{\{n\}}(t) - 1, \ i \neq i_1(t), i_1^{\{1\}}(t), \ldots, i_1^{\{n-1\}}(t) \end{cases},$$

which justifies an induction argument.

It may be concluded that

$$p_t = \sum_{i=1}^{(k+1)^{q-1}} rp_{t,i-1} \ L_{1,1}(RSQ_{t,i}) + \sum_{i\in\mathcal{Y}_t} rp_{t,i} \ L_{1,2}(RSQ_{t,i}) \ q_{t-1-i}^{(\mathbf{0})} \tag{101}$$

and for $n \in \mathbb{N}$, that

$$p_t = \sum_{i=1}^{i_1^{\{n-1\}}(t)+(k+1)^{q-1}} rp_{t,i-1} \ L_{1,1}(RSQ_{t,i}^{\{n\}}) + \sum_{i\in\mathcal{Y}_t^{\{n\}}} rp_{t,i} \ L_{1,2}(RSQ_{t,i}^{\{n\}}) \ q_{t-1-i}^{(\mathbf{0})}, \tag{102}$$

where

$$RSQ^{\{n\}}_{t,i} = \begin{cases} RSQ^{\{n-1\}}_{t,i} + RSQ^{\{n-1\}}_{t,i_1^{\{n-1\}}(t)}\ RSQ_{t-i_1^{\{n-1\}}(t),i-i_1^{\{n-1\}}(t)}, \\ \quad \text{if } i = i_1^{\{n-1\}}(t)+1,\dots,i_1^{\{n-2\}}(t)+(k+1)^{q-1}, \\ \quad \text{(provided that } i_1^{\{n-1\}}(t)+1 \le i_1^{\{n-2\}}(t)+(k+1)^{q-1}) \\ RSQ^{\{n-1\}}_{t,i_1^{\{n-1\}}(t)}\ RSQ_{t-i_1^{\{n-1\}}(t),i-i_1^{\{n-1\}}(t)}, \\ \quad \text{if } i = i^{\{n-2\}}(t)+(k+1)^{q-1}+1,\dots,i_1^{\{n-1\}}(t)+(k+1)^{q-1}, \\ RSQ^{\{n-1\}}_{t,i}, \\ \quad \text{if } i = 1,\dots,i_1^{\{n-1\}}(t)-1,\ i \neq i_1(t), i_1^{\{1\}}(t),\dots,i_1^{\{n-1\}}(t), \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \\ \quad \text{if } i = i_1(t), i_1^{\{1\}}(t),\dots,i_1^{\{n-1\}}(t) \end{cases}$$

and writing $RSQ^{\{0\}}_{t,i} \equiv RSQ_{t,i}$ and $i_1^{\{0\}}(t) \equiv i_1(t)$ and $i_1^{\{-1\}}(t) \equiv 0$.

It holds that $i \in \mathcal{Y}_t^{\{n\}}$ iff $i - i_1^{\{n-1\}}(t) \in \mathcal{Y}_{t-i_1^{\{n-1\}}(t)}$, so that (101) becomes

$$\begin{aligned} p_{t-i_1^{\{n-1\}}(t)} &= \sum_{i=1}^{(k+1)^{q-1}} rp_{t-i_1^{\{n-1\}}(t),i-1}\ L_{1,1}(RSQ_{t-i_1^{\{n-1\}}(t),i}) + \\ &\quad \sum_{i\in\mathcal{Y}_{t-i_1^{\{n-1\}}(t)}} rp_{t-i_1^{\{n-1\}}(t),i}\ L_{1,2}(RSQ_{t-i_1^{\{n-1\}}(t),i})\ q^{(\mathbf{0})}_{t-i_1^{\{n-1\}}(t)-1-i} \\ &\equiv \sum_{i=i_1^{\{n-1\}}(t)+1}^{i_1^{\{n-1\}}(t)+(k+1)^{q-1}} rp_{t-i_1^{\{n-1\}}(t),i-i_1^{\{n-1\}}(t)-1}\ L_{1,1}(RSQ_{t-i_1^{\{n-1\}}(t),i-i_1^{\{n-1\}}(t)}) + \\ &\quad \sum_{i\in\mathcal{Y}_t^{\{n\}}} rp_{t-i_1^{\{n-1\}}(t),i-i_1^{\{n-1\}}(t)}\ L_{1,2}(RSQ_{t-i_1^{\{n-1\}}(t),i-i_1^{\{n-1\}}(t)})\ q^{(\mathbf{0})}_{t-1-i} \end{aligned}$$

and solving with respect to $q^{(\mathbf{0})}_{t-1-i_1^{\{n\}}(t)}$, results in

$$\begin{aligned} q^{(\mathbf{0})}_{t-1-i_1^{\{n\}}(t)} &= \{rp_{t-i_1^{\{n-1\}}(t),i_1^{\{n\}}(t)-i_1^{\{n-1\}}(t)}\}^{-1}\{L_{1,2}(RSQ_{t-i_1^{\{n-1\}}(t),i_1^{\{n\}}(t)-i_1^{\{n-1\}}(t)})\}^{-1} \\ &\Bigg\{ p_{t-i_1^{\{n-1\}}(t)} - \sum_{i=i_1^{\{n-1\}}(t)+1}^{i_1^{\{n-1\}}(t)+(k+1)^{q-1}} rp_{t-i_1^{\{n-1\}}(t),i-i_1^{\{n-1\}}(t)-1}\ L_{1,1}(RSQ_{t-i_1^{\{n-1\}}(t),i-i_1^{\{n-1\}}(t)}) - \\ &\sum_{c=2}^{y_t^{\{n\}}} rp_{t-i_1^{\{n-1\}}(t),i_c^{\{n\}}(t)-i_1^{\{n-1\}}(t)}\ L_{1,2}(RSQ_{t-i_1^{\{n-1\}}(t),i_c^{\{n\}}(t)-i_1^{\{n-1\}}(t)})\ q^{(\mathbf{0})}_{t-1-i_c^{\{n\}}(t)} \Bigg\}. \end{aligned} \tag{103}$$

Next, (103) is plugged in (102), $n \in \mathbb{N}$, which gives

$$\begin{aligned}
p_t \quad = \quad & \sum_{i=1}^{i_1^{\{n-1\}}(t)+(k+1)^{q-1}} rp_{t,i-1}\ L_{1,1}(RSQ_{t,i}^{\{n\}}) + \\
& \sum_{c=2}^{y_t^{\{n\}}} rp_{t,i_c^{\{n\}}(t)}\ L_{1,2}(RSQ_{t,i_c^{\{n\}}(t)}^{\{n\}})\ q_{t-1-i_c^{\{n\}}(t)}^{(\mathbf{0})} + rp_{t,i_1^{\{n\}}(t)}\ \{L_{1,2}(RSQ_{t,i_1^{\{n\}}(t)}^{\{n\}})\} \cdot \\
& \{rp_{t-i_1^{\{n-1\}}(t),i_1^{\{n\}}(t)-i_1^{\{n-1\}}(t)}\}^{-1}\{L_{1,2}(RSQ_{t-i_1^{\{n-1\}}(t),i_1^{\{n\}}(t)-i_1^{\{n-1\}}(t)})\}^{-1} \\
& \Bigg\{ p_{t-i_1^{\{n-1\}}(t)} - \sum_{i=i_1^{\{n-1\}}(t)+1}^{i_1^{\{n-1\}}(t)+(k+1)^{q-1}} rp_{t-i_1^{\{n-1\}}(t),i-i_1^{\{n-1\}}(t)-1}\ L_{1,1}(RSQ_{t-i_1^{\{n-1\}}(t),i-i_1^{\{n-1\}}(t)}) - \\
& \sum_{c=2}^{y_t^{\{n\}}} rp_{t-i_1^{\{n-1\}}(t),i_c^{\{n\}}(t)-i_1^{\{n-1\}}(t)}\ L_{1,2}(RSQ_{t-i_1^{\{n-1\}}(t),i_c^{\{n\}}(t)-i_1^{\{n-1\}}(t)})\ q_{t-1-i_c^{\{n\}}(t)}^{(\mathbf{0})} \Bigg\}.
\end{aligned}$$

And, of course, write

$$\{rp_{t,i_1^{\{n\}}(t)}\}\ \{rp_{t-i_1^{\{n-1\}}(t),i_1^{\{n\}}(t)-i_1^{\{n-1\}}(t)}\}^{-1} = \frac{p_{t-i_1^{\{n-1\}}(t)-1} \cdots p_{t-i_1^{\{n\}}(t)}}{p_{t-1} \cdots p_{t-i_1^{\{n\}}(t)}} \equiv rp_{t,i_1^{\{n-1\}}(t)}$$

and define $HRSQ_{t,u}^{\{n\}} := \{L_{1,2}(RSQ_{t,u}^{\{n\}})\}\{L_{1,2}(RSQ_{t-i_1^{\{n-1\}}(t),u-i_1^{\{n-1\}}(t)})\}^{-1}$, so that it is re-written

$$\begin{aligned}
p_t \quad = \quad & \sum_{i=1}^{i_1^{\{n-1\}}(t)+(k+1)^{q-1}} rp_{t,i-1}\ L_{1,1}(RSQ_{t,i}^{\{n\}}) + \\
& \sum_{c=2}^{y_t^{\{n\}}} rp_{t,i_c^{\{n\}}(t)}\ L_{1,2}(RSQ_{t,i_c^{\{n\}}(t)}^{\{n\}})\ q_{t-1-i_c^{\{n\}}(t)}^{(\mathbf{0})} + \\
& rp_{t,i_1^{\{n-1\}}(t)}\ HRSQ_{t,i_1^{\{n\}}(t)}^{\{n\}}\ \Big\{ p_{t-i_1^{\{n-1\}}(t)} - \\
& \sum_{i=i_1^{\{n-1\}}(t)+1}^{i_1^{\{n-1\}}(t)+(k+1)^{q-1}} rp_{t-i_1^{\{n-1\}}(t),i-i_1^{\{n-1\}}(t)-1}\ L_{1,1}(RSQ_{t-i_1^{\{n-1\}}(t),i-i_1^{\{n-1\}}(t)}) - \\
& \sum_{c=2}^{y_t^{\{n\}}} rp_{t-i_1^{\{n-1\}}(t),i_c^{\{n\}}(t)-i_1^{\{n-1\}}(t)}\ L_{1,2}(RSQ_{t-i_1^{\{n-1\}}(t),i_c^{\{n\}}(t)-i_1^{\{n-1\}}(t)})\ q_{t-1-i_c^{\{n\}}(t)}^{(\mathbf{0})} \Bigg\},
\end{aligned}$$

or $(rp_{t,i_1^{\{n-1\}}(t)}\ rp_{t-i_1^{\{n-1\}}(t),i-i_1^{\{n-1\}}(t)} = rp_{t,i})$

$$p_t = \sum_{i=1}^{i_1^{\{n-1\}}(t)+(k+1)^{q-1}} rp_{t,i-1}\ bRSQ_{t,i}^{\{n\}} + \sum_{c=2}^{y_t^{\{n\}}} rp_{t,i_c^{\{n\}}(t)}\ BRSQ_{t,i_c^{\{n\}}(t)}^{\{n\}}\ q_{t-1-i_c^{\{n\}}(t)}^{(\mathbf{0})}, \quad (104)$$

where

$$bRSQ_{t,i}^{\{n\}} := \begin{cases} L_{1,1}(RSQ_{t,i}^{\{n\}}), \\ \text{if } i = 1, \ldots, i_1^{\{n-1\}}(t) - 1, \\ L_{1,1}(RSQ_{t,i}^{\{n\}}) + HRSQ_{t,i_1^{\{n\}}(t)}^{\{n\}} \equiv HRSQ_{t,i_1^{\{n\}}(t)}^{\{n\}}, \\ \text{if } i = i_1^{\{n-1\}}(t), \\ L_{1,1}(RSQ_{t,i}^{\{n\}}) - HRSQ_{t,i_1^{\{n\}}(t)}^{\{n\}} \cdot L_{1,1}(RSQ_{t-i_1^{\{n-1\}}(t),i-i_1^{\{n-1\}}(t)}), \\ \text{if } i = i_1^{\{n-1\}}(t) + 1, \ldots, i_1^{\{n-1\}}(t) + (k+1)^{q-1} \end{cases}$$

and $BRSQ_{t,i}^{\{n\}} := L_{1,2}(RSQ_{t,i}^{\{n\}}) - HRSQ_{t,i_1^{\{n\}}(t)}^{\{n\}} \cdot L_{1,2}(RSQ_{t-i_1^{\{n-1\}}(t),i-i_1^{\{n-1\}}(t)})$.

An induction argument is used next.

<u>$n = 1$</u>

Especially for $n = 1$, (104) becomes

$$p_t = \sum_{i=1}^{i_1(t)+(k+1)^{q-1}} rp_{t,i-1}\ bRSQ_{t,i}^{\{1\}} + \sum_{c=2}^{y_t^{\{1\}}} rp_{t,i_c^{\{1\}}(t)}\ BRSQ_{t,i_c^{\{1\}}(t)}^{\{1\}}\ q_{t-1-i_c^{\{1\}}(t)}^{(\mathbf{0})}.$$

For $n = 2, 3, \ldots$, it holds that

$$i + i_1^{\{n-2\}}(t) \in \mathcal{Y}_t^{\{n\}} \ \text{ iff } \ i \in \mathcal{Y}_{t-i_1^{\{n-2\}}(t)}^{\{1\}};$$

to understand this, observe that

$$(t - i_1^{\{n-2\}}(t)) - i_c^{\{1\}}(t - i_1^{\{n-2\}}(t)) = t - i_c^{\{n\}}(t),$$

i.e.

$$i_c^{\{1\}}(t - i_1^{\{n-2\}}(t)) = i_c^{\{n\}}(t) - i_1^{\{n-2\}}(t), \text{ for } c = 1, \ldots, y_{t-i_1^{\{n-2\}}(t)}^{\{1\}} \equiv y_t^{\{n\}}.$$

As a result, the equation p_t above for $t - i_1^{\{n-2\}}(t)$, $n = 2, 3, \ldots$, becomes

$$\begin{aligned} p_{t-i_1^{\{n-2\}}(t)} = &\sum_{i=1}^{i_1(t-i_1^{\{n-2\}}(t))+(k+1)^{q-1}} rp_{t-i_1^{\{n-2\}}(t),i-1}\ bRSQ_{t-i_1^{\{n-2\}}(t),i}^{\{1\}} + \\ &\sum_{c=2}^{y_{t-i_1^{\{n-2\}}(t)}^{\{1\}}} rp_{t-i_1^{\{n-2\}}(t),i_c^{\{1\}}(t-i_1^{\{n-2\}}(t))}\ BRSQ_{t-i_1^{\{n-2\}}(t),i_c^{\{1\}}(t-i_1^{\{n-2\}}(t))}^{\{1\}}\ q_{t-i_1^{\{n-2\}}(t)-1-i_c^{\{1\}}(t-i_1^{\{n-2\}}(t))}^{(\mathbf{0})} \\ \equiv &\sum_{i=1}^{i_1^{\{n-1\}}(t)-i_1^{\{n-2\}}(t)+(k+1)^{q-1}} rp_{t-i_1^{\{n-2\}}(t),i-1}\ bRSQ_{t-i_1^{\{n-2\}}(t),i}^{\{1\}} + \\ &\sum_{c=2}^{y_t^{\{n\}}} rp_{t-i_1^{\{n-2\}}(t),i_c^{\{n\}}(t)-i_1^{\{n-2\}}(t)}\ BRSQ_{t-i_1^{\{n-2\}}(t),i_c^{\{n\}}(t)-i_1^{\{n-2\}}(t)}^{\{1\}}\ q_{t-1-i_c^{\{n\}}(t)}^{(\mathbf{0})}, \end{aligned}$$

since $i_c(t - i_1^{\{n-2\}}(t)) = i_c^{\{n-1\}}(t) - i_1^{\{n-2\}}(t)$. Re-write the first term, i.e.

$$p_{t-i_1^{\{n-2\}}(t)} = \sum_{i=i_1^{\{n-2\}}(t)+1}^{i_1^{\{n-1\}}(t)+(k+1)^{q-1}} rp_{t-i_1^{\{n-2\}}(t),i-i_1^{\{n-2\}}(t)-1}\ bRSQ^{\{1\}}_{t-i_1^{\{n-2\}}(t),i-i_1^{\{n-2\}}(t)} + \sum_{c=2}^{y_t^{\{n\}}} rp_{t-i_1^{\{n-2\}}(t),i_c^{\{n\}}(t)-i_1^{\{n-2\}}(t)}\ BRSQ^{\{1\}}_{t-i_1^{\{n-2\}}(t),i_c^{\{n\}}(t)-i_1^{\{n-2\}}(t)}\ q^{(\mathbf{0})}_{t-1-i_c^{\{n\}}(t)}.$$

Then it can be solved that

$$\begin{aligned} q^{(\mathbf{0})}_{t-1-i_2^{\{n\}}(t)} \quad = \quad & \{rp_{t-i_1^{\{n-2\}}(t),i_2^{\{n\}}(t)-i_1^{\{n-2\}}(t)}\}^{-1}\{BRSQ^{\{1\}}_{t-i_1^{\{n-2\}}(t),i_2^{\{n\}}(t)-i_1^{\{n-2\}}(t)}\}^{-1} \\ & \left\{ p_{t-i_1^{\{n-2\}}(t)} - \sum_{i=i_1^{\{n-2\}}(t)+1}^{i_1^{\{n-1\}}(t)+(k+1)^{q-1}} rp_{t-i_1^{\{n-2\}}(t),i-i_1^{\{n-2\}}(t)-1}\ bRSQ^{\{1\}}_{t-i_1^{\{n-2\}}(t),i-i_1^{\{n-2\}}(t)} - \right. \\ & \left. \sum_{c=3}^{y_t^{\{n\}}} rp_{t-i_1^{\{n-2\}}(t),i_c^{\{n\}}(t)-i_1^{\{n-2\}}(t)}\ BRSQ^{\{1\}}_{t-i_1^{\{n-2\}}(t),i_c^{\{n\}}(t)-i_1^{\{n-2\}}(t)}\ q^{(\mathbf{0})}_{t-1-i_c^{\{n\}}(t)} \right\}. \end{aligned} \tag{105}$$

Once (105) is plugged in (104) but for $n = 2, 3, \dots$ now, these become

$$\begin{aligned} p_t \quad = \quad & \sum_{i=1}^{i_1^{\{n-1\}}(t)+(k+1)^{q-1}} rp_{t,i-1}\ bRSQ^{\{n\}}_{t,i} + \sum_{c=3}^{y_t^{\{n\}}} rp_{t,i_c^{\{n\}}(t)}\ BRSQ^{\{n\}}_{t,i_c^{\{n\}}(t)}\ q^{(\mathbf{0})}_{t-1-i_c^{\{n\}}(t)} + \\ & rp_{t,i_2^{\{n\}}(t)}\ BRSQ^{\{n\}}_{t,i_2^{\{n\}}(t)}\ \{rp_{t-i_1^{\{n-2\}}(t),i_2^{\{n\}}(t)-i_1^{\{n-2\}}(t)}\}^{-1}\{BRSQ^{\{1\}}_{t-i_1^{\{n-2\}}(t),i_2^{\{n\}}(t)-i_1^{\{n-2\}}(t)}\}^{-1} \\ & \left\{ p_{t-i_1^{\{n-2\}}(t)} - \sum_{i=i_1^{\{n-2\}}(t)+1}^{i_1^{\{n-1\}}(t)+(k+1)^{q-1}} rp_{t-i_1^{\{n-2\}}(t),i-i_1^{\{n-2\}}(t)-1}\ bRSQ^{\{1\}}_{t-i_1^{\{n-2\}}(t),i-i_1^{\{n-2\}}(t)} - \right. \\ & \left. \sum_{c=3}^{y_t^{\{n\}}} rp_{t-i_1^{\{n-2\}}(t),i_c^{\{n\}}(t)-i_1^{\{n-2\}}(t)}\ BRSQ^{\{1\}}_{t-i_1^{\{n-2\}}(t),i_c^{\{n\}}(t)-i_1^{\{n-2\}}(t)}\ q^{(\mathbf{0})}_{t-1-i_c^{\{n\}}(t)} \right\} \end{aligned}$$

or, since $\{rp_{t,i_2^{\{n\}}(t)}\}\ \{rp_{t-i_1^{\{n-2\}}(t),i_2^{\{n\}}(t)-i_1^{\{n-2\}}(t)}\}^{-1} \equiv rp_{t,i_1^{\{n-2\}}(t)}$, they become

$$p_t = \sum_{i=1}^{i_1^{\{n-1\}}(t)+(k+1)^{q-1}} rp_{t,i-1}\ b^2RSQ^{\{n\}}_{t,i} + \sum_{c=3}^{y_t^{\{n\}}} rp_{t,i_c^{\{n\}}(t)}\ B^2RSQ^{\{n\}}_{t,i_c^{\{n\}}(t)}\ q^{(\mathbf{0})}_{t-1-i_c^{\{n\}}(t)}$$

where $HBRSQ^{\{n\}}_{t,u} := \{BRSQ^{\{n\}}_{t,u}\}\ \{BRSQ^{\{1\}}_{t-i_1^{\{n-2\}}(t),u-i_1^{\{n-2\}}(t)}\}^{-1}$, and

$$b^2RSQ^{\{n\}}_{t,i} := \begin{cases} bRSQ^{\{n\}}_{t,i}, \\ \text{if } \ i = 1, \dots, i_1^{\{n-2\}}(t) - 1, \\ bRSQ^{\{n\}}_{t,i} + HBRSQ^{\{n\}}_{t,i_2^{\{n\}}(t)}, \\ \text{if } \ i = i_1^{\{n-2\}}(t), \\ bRSQ^{\{n\}}_{t,i} - HBRSQ^{\{n\}}_{t,i_2^{\{n\}}(t)} \cdot bRSQ^{\{1\}}_{t-i_1^{\{n-2\}}(t),i-i_1^{\{n-2\}}(t)}, \\ \text{if } \ i = i_1^{\{n-2\}}(t) + 1, \dots, i_1^{\{n-1\}}(t) + (k+1)^{q-1} \end{cases}$$

and $B^2RSQ_{t,i}^{\{n\}} := BRSQ_{t,i}^{\{n\}} - HBRSQ_{t,i_2^{\{n\}}(t)}^{\{n\}} \cdot BRSQ_{t-i_1^{\{n-2\}}(t),i-i_1^{\{n-2\}}(t)}^{\{1\}}$.

$\underline{n = \nu}$

For $\nu = 2, \ldots$, accept that it holds (for $n = \nu+1, \nu+2, \ldots$ only) that

$$p_t = \sum_{i=1}^{i_1^{\{n-1\}}(t)+(k+1)^{q-1}} rp_{t,i-1}\, b^{\nu+1}RSQ_{t,i}^{\{n\}} + \sum_{c=\nu+2}^{y_t^{\{n\}}} rp_{t,i_c^{\{n\}}(t)}\, B^{\nu+1}RSQ_{t,i_c^{\{n\}}(t)}^{\{n\}}\, q_{t-1-i_c^{\{n\}}(t)}^{(\mathbf{0})}, \tag{106}$$

where $B^{\nu+1}RSQ_{t,i}^{\{n\}} := B^{\nu}RSQ_{t,i}^{\{n\}} - HB^{\nu}RSQ_{t,i_{\nu+1}^{\{n\}}(t)}^{\{n\}} \cdot B^{\nu}RSQ_{t-i_1^{\{n-(\nu+1)\}}(t),i-i_1^{\{n-(\nu+1)\}}(t)}^{\{\nu\}}$,
and

$$b^{\nu+1}RSQ_{t,i}^{\{n\}} := \begin{cases} b^{\nu}RSQ_{t,i}^{\{n\}}, \\ \text{if } \ i = 1, \ldots, i_1^{\{n-(\nu+1)\}}(t) - 1, \\ b^{\nu}RSQ_{t,i}^{\{n\}} + HB^{\nu}RSQ_{t,i_{\nu+1}^{\{n\}}(t)}^{\{n\}}, \\ \text{if } \ i = i_1^{\{n-(\nu+1)\}}(t), \\ b^{\nu}RSQ_{t,i}^{\{n\}} - HB^{\nu}RSQ_{t,i_{\nu+1}^{\{n\}}(t)}^{\{n\}} \cdot b^{\nu}RSQ_{t-i_1^{\{n-(\nu+1)\}}(t),i-i_1^{\{n-(\nu+1)\}}(t)}^{\{\nu\}}, \\ \text{if } \ i = i_1^{\{n-(\nu+1)\}}(t)+1, \ldots, i_1^{\{n-1\}}(t) + (k+1)^{q-1} \end{cases}$$

as well as $HB^{\nu}RSQ_{t,u}^{\{n\}} := \{B^{\nu}RSQ_{t,u}^{\{n\}}\}\{B^{\nu}RSQ_{t-i_1^{\{n-(\nu+1)\}}(t),u-i_1^{\{n-(\nu+1)\}}(t)}^{\{\nu\}}\}^{-1}$.

$\underline{n = \nu+1}$

Straight from (106) when $n = \nu+1$, it is derived that

$$p_t = \sum_{i=1}^{i_1^{\{\nu\}}(t)+(k+1)^{q-1}} rp_{t,i-1}\, b^{\nu+1}RSQ_{t,i}^{\{\nu+1\}} + \sum_{c=\nu+2}^{y_t^{\{\nu+1\}}} rp_{t,i_c^{\{\nu+1\}}(t)}\, B^{\nu+1}RSQ_{t,i_c^{\{\nu+1\}}(t)}^{\{\nu+1\}}\, q_{t-1-i_c^{\{\nu+1\}}(t)}^{(\mathbf{0})}.$$

Nevertheless, for $n = \nu+2, \ldots$, it holds that

$$i + i_1^{\{n-(\nu+2)\}}(t) \in \mathcal{Y}_t^{\{n\}} \ \text{ iff } \ i \in \mathcal{Y}_{t-i_1^{\{n-(\nu+2)\}}(t)}^{\{\nu+1\}};$$

to be convinced about this, see that

$$(t - i_1^{\{n-(\nu+2)\}}(t)) - i_c^{\{\nu+1\}}(t - i_1^{\{n-(\nu+2)\}}(t)) = t - i_c^{\{n\}}(t),$$

i.e.

$$i_c^{\{\nu+1\}}(t - i_1^{\{n-(\nu+2)\}}(t)) = i_c^{\{n\}}(t) - i_1^{\{n-(\nu+2)\}}(t), \ c = 1, \ldots, y_{t-i_1^{\{n-(\nu+2)\}}(t)}^{\{\nu+1\}} \equiv y_t^{\{n\}}.$$

Then the equation above is used for $t - i_1^{\{n-(\nu+2)\}}(t)$, resulting in

$$p_{t-i_1^{\{n-(\nu+2)\}}(t)} = \sum_{i=1}^{i_1^{\{\nu\}}(t-i_1^{\{n-(\nu+2)\}}(t))+(k+1)^{q-1}} rp_{t-i_1^{\{n-(\nu+2)\}}(t),i-1}\ b^{\nu+1}RSQ^{\{\nu+1\}}_{t-i_1^{\{n-(\nu+2)\}}(t),i} +$$
$$\sum_{c=\nu+2}^{y^{\{\nu+1\}}_{t-i_1^{\{n-(\nu+2)\}}(t)}} rp_{t-i_1^{\{n-(\nu+2)\}}(t),i_c^{\{\nu+1\}}(t-i_1^{\{n-(\nu+2)\}}(t))}\ B^{\nu+1}RSQ^{\{\nu+1\}}_{t-i_1^{\{n-(\nu+2)\}}(t),i_c^{\{\nu+1\}}(t-i_1^{\{n-(\nu+2)\}}(t))}$$
$$q^{(\mathbf{0})}_{t-i_1^{\{n-(\nu+2)\}}(t)-1-i_c^{\{\nu+1\}}(t-i_1^{\{n-(\nu+2)\}}(t))} \equiv$$
$$\sum_{i=1}^{i_1^{\{n-1\}}(t)-i_1^{\{n-(\nu+2)\}}(t)+(k+1)^{q-1}} rp_{t-i_1^{\{n-(\nu+2)\}}(t),i-1}\ b^{\nu+1}RSQ^{\{\nu+1\}}_{t-i_1^{\{n-(\nu+2)\}}(t),i} +$$
$$\sum_{c=\nu+2}^{y_t^{\{n\}}} rp_{t-i_1^{\{n-(\nu+2)\}}(t),i_c^{\{n\}}(t)-i_1^{\{n-(\nu+2)\}}(t)}\ B^{\nu+1}RSQ^{\{\nu+1\}}_{t-i_1^{\{n-(\nu+2)\}}(t),i_c^{\{n\}}(t)-i_1^{\{n-(\nu+2)\}}(t)}\ q^{(\mathbf{0})}_{t-1-i_c^{\{n\}}(t)}$$

since $i_c^{\{\nu\}}(t - i_1^{\{n-(\nu+2)\}}(t)) = i_c^{\{n-1\}}(t) - i_1^{\{n-(\nu+2)\}}(t)$. Finally, after re-arranging the index of the first term to $i + i_1^{\{n-(\nu+2)\}}(t)$, it holds that

$$p_{t-i_1^{\{n-(\nu+2)\}}(t)} =$$
$$\sum_{i=i_1^{\{n-(\nu+2)\}}(t)+1}^{i_1^{\{n-1\}}(t)+(k+1)^{q-1}} rp_{t-i_1^{\{n-(\nu+2)\}}(t),i-i_1^{\{n-(\nu+2)\}}(t)-1}\ b^{\nu+1}RSQ^{\{\nu+1\}}_{t-i_1^{\{n-(\nu+2)\}}(t),i-i_1^{\{n-(\nu+2)\}}(t)} +$$
$$\sum_{c=\nu+2}^{y_t^{\{n\}}} rp_{t-i_1^{\{n-(\nu+2)\}}(t),i_c^{\{n\}}(t)-i_1^{\{n-(\nu+2)\}}(t)}\ B^{\nu+1}RSQ^{\{\nu+1\}}_{t-i_1^{\{n-(\nu+2)\}}(t),i_c^{\{n\}}(t)-i_1^{\{n-(\nu+2)\}}(t)}\ q^{(\mathbf{0})}_{t-1-i_c^{\{n\}}(t)},$$

which may be re-arranged to write

$$q^{(\mathbf{0})}_{t-1-i^{\{n\}}_{\nu+2}(t)} = \{rp_{t-i_1^{\{n-(\nu+2)\}}(t),i^{\{n\}}_{\nu+2}(t)-i_1^{\{n-(\nu+2)\}}(t)}\}^{-1}\{B^{\nu+1}RSQ^{\{\nu+1\}}_{t-i_1^{\{n-(\nu+2)\}}(t),i^{\{n\}}_{\nu+2}(t)-i_1^{\{n-(\nu+2)\}}(t)}\}^{-1}$$
$$\Bigg\{p_{t-i_1^{\{n-(\nu+2)\}}(t)} -$$
$$\sum_{i=i_1^{\{n-(\nu+2)\}}(t)+1}^{i_1^{\{n-1\}}(t)+(k+1)^{q-1}} rp_{t-i_1^{\{n-(\nu+2)\}}(t),i-i_1^{\{n-(\nu+2)\}}(t)-1}\ b^{\nu+1}RSQ^{\{\nu+1\}}_{t-i_1^{\{n-(\nu+2)\}}(t),i-i_1^{\{n-(\nu+2)\}}(t)} -$$
$$\sum_{c=\nu+3}^{y_t^{\{n\}}} rp_{t-i_1^{\{n-(\nu+2)\}}(t),i_c^{\{n\}}(t)-i_1^{\{n-(\nu+2)\}}(t)}\ B^{\nu+1}RSQ^{\{\nu+1\}}_{t-i_1^{\{n-(\nu+2)\}}(t),i_c^{\{n\}}(t)-i_1^{\{n-(\nu+2)\}}(t)}\ q^{(\mathbf{0})}_{t-1-i_c^{\{n\}}(t)}\Bigg\}. \tag{107}$$

Once (107) is inserted in (106) (for $n = \nu+2, \nu+3, \ldots$), this becomes

$$p_t = \sum_{i=1}^{i_1^{\{n-1\}}(t)+(k+1)^{q-1}} rp_{t,i-1}\ b^{\nu+2}RSQ^{\{n\}}_{t,i} + \sum_{c=\nu+3}^{y_t^{\{n\}}} rp_{t,i_c^{\{n\}}(t)}\ B^{\nu+2}RSQ^{\{n\}}_{t,i_c^{\{n\}}(t)}\ q^{(\mathbf{0})}_{t-1-i_c^{\{n\}}(t)},$$

where $HB^{\nu+1}RSQ_{t,u}^{\{n\}} := \{B^{\nu+1}RSQ_{t,u}^{\{n\}}\}\ \{B^{\nu+1}RSQ^{\{\nu+1\}}_{t-i_1^{\{n-(\nu+2)\}}(t),u-i_1^{\{n-(\nu+2)\}}(t)}\}^{-1}$, and

$$b^{\nu+2}RSQ_{t,i}^{\{n\}} = \begin{cases} b^{\nu+1}RSQ_{t,i}^{\{n\}}, \\ \text{if} \quad i = 1,\dots,i_1^{\{n-(\nu+2)\}}(t)-1, \\ b^{\nu+1}RSQ_{t,i}^{\{n\}} + HB^{\nu+1}RSQ^{\{n\}}_{t,i_{\nu+2}^{\{n\}}(t)}, \\ \text{if} \quad i = i_1^{\{n-(\nu+2)\}}(t), \\ b^{\nu+1}RSQ_{t,i}^{\{n\}} - HB^{\nu+1}RSQ^{\{n\}}_{t,i_{\nu+2}^{\{n\}}(t)} \cdot b^{\nu+1}RSQ^{\{\nu+1\}}_{t-i_1^{\{n-(\nu+2)\}}(t),i-i_1^{\{n-(\nu+2)\}}(t)}, \\ \text{if} \quad i = i_1^{\{n-(\nu+2)\}}(t)+1,\dots,i_1^{\{n-1\}}(t)+(k+1)^{q-1} \end{cases}$$

and $B^{\nu+2}RSQ_{t,i}^{\{n\}} := B^{\nu+1}RSQ_{t,i}^{\{n\}} - HB^{\nu+1}RSQ^{\{n\}}_{t,i_{\nu+2}^{\{n\}}(t)} \cdot B^{\nu+1}RSQ^{\{\nu+1\}}_{t-i_1^{\{n-(\nu+2)\}}(t),i-i_1^{\{n-(\nu+2)\}}(t)}$.

Hence the induction argument has been proven. Then

$$p_t = \sum_{i=1}^{i_1^{\{\nu\}}(t)+(k+1)^{q-1}} rp_{t,i-1}\, b^{\nu+1}RSQ_{t,i}^{\{\nu+1\}} + \sum_{c=\nu+2}^{y_t^{\{\nu+1\}}} rp_{t,i_c^{\{\nu+1\}}(t)}\ B^{\nu+1}RSQ^{\{\nu+1\}}_{t,i_c^{\{\nu+1\}}(t)}\ q^{(\mathbf{0})}_{t-1-i_c^{\{\nu+1\}}(t)},$$

is the special case of (106) when $n = \nu+1$.

This process will not continue indefinitely and it is determined next when to end it. Define

$$f_t^{\{0\}} := \min\{n \in \mathbb{N}_0 : y_t^{\{n\}} = 0\}.$$

To start with, it is checked whether $y_t^{\{0\}} \equiv y_t$ is equal to zero, i.e. whether $\mathbf{X}_{t-1-i} \neq \mathbf{0}_p$ for all $i = 1,\dots,(k+1)^{q-1}$. If it is zero, then $f_t^{\{0\}} := 0$; otherwise there is at least one 'lag' $i = i_1(t)$, such that $\mathbf{X}_{t-1-i_1(t)} = \mathbf{0}_p$, so that it is checked whether $\mathbf{X}_{t-i_1(t)-1-i} \neq \mathbf{0}_p$ for all $i = 1,\dots,(k+1)^{q-1}$. If they are all not zero vectors or $y_t^{\{1\}} = 0$, then $f_t^{\{0\}} := 1$. Otherwise, there is at least one 'lag' $i_1^{\{1\}}(t)$ to let us define a new set $\mathcal{Y}_t^{\{2\}}$ and check its cardinality $y_t^{\{2\}}$, and so on. According to (102), it may be written that

$$p_t = \sum_{i=1}^{i_1^{\{f_t^{\{0\}}-1\}}(t)+(k+1)^{q-1}} rp_{t,i-1}\ L_{1,1}(RSQ_{t,i}^{\{f_t^{\{0\}}\}}). \qquad (108)$$

If $f_t^{\{0\}} = 0$ ($i_1^{\{-1\}}(t) \equiv 0$) then (108) is replaced by (93) as it has already been presented.

If $y_t \geq 1$, then $\mathcal{Y}_t^{\{1\}}$ (with cardinality $y_t^{\{1\}}$) starts within the $1,\dots,(k+1)^{q-1}$ steps from $\mathcal{Y}_t$: it is meaningful to observe the random variable

$$f_t^{\{1\}} := \min\{n = 1,2,\dots : y_t^{\{n\}} = 1\}.$$

First, suppose that it is $y_t^{\{1\}} = 1$, i.e $f_t^{\{1\}} = 1$. Then suppose that it is $2 \leq y_t^{\{1\}} \leq (k+1)^{q-1}$, which means one should 'wait for' the cardinalities $y_t^{\{2\}},\dots,y_t^{\{y_t^{\{1\}}\}},\dots$ of the following $\mathcal{Y}$ to 'fall' to 1. Nevertheless, if $y_t^{\{1\}} = 0$ (implying that $f_t^{\{1\}} > 1$), which must mean that $y_t = 1$ there is no 'lag' within $\mathcal{Y}_t^{\{1\}}$ to show good sense to continue: this will be combined with $f_t^{\{0\}} \equiv 1 < f_t^{\{1\}}$ and the minimum will be picked to stop. Hence *provided that* $y_t^{\{1\}} \geq 1$ (and according to (104)), it is written that

$$p_t = \sum_{i=1}^{i_1^{\{f_t^{\{1\}}-1\}}(t)+(k+1)^{q-1}} rp_{t,i-1}\ bRSQ_{t,i}^{\{f_t^{\{1\}}\}}.$$

Similarly for any $l = 2, \ldots, (k+1)^{q-1}$, provided that $y_t^{\{2\}} \geq 2$, ..., $y_t^{\{(k+1)^{q-1}\}} = (k+1)^{q-1}$, respectively, the random variable

$$f_t^{\{l\}} := \min\{n = l, l+1, \ldots : y_t^{\{n\}} = l\}$$

becomes of interest, and it may be written that

$$p_t = \sum_{i=1}^{i_1^{\{f_t^{\{l\}}-1\}}(t)+(k+1)^{q-1}} rp_{t,i-1}\ b^l RSQ_{t,i}^{\{f_t^{\{l\}}\}}.$$

Hence, it is defined here that

$$\gamma_t := \min_{r=0,1,\ldots,(k+1)^{q-1}} \{f_t^{\{r\}}\} \tag{109}$$

together with

$$\delta_t := \text{argmin}_{r=0,1,\ldots,(k+1)^{q-1}} \{f_t^{\{r\}}\} \tag{110}$$

since it cannot be that $f_t^{\{r_1\}} = f_t^{\{r_2\}}$ for $r_1 \neq r_2$.

It may be concluded from (109) and all the above that $\mathbb{P}(\gamma_t \leq (k+1)^{q-1}) = 1$; this is something that could be seen directly during the eliminations in (b) when deterministically there would be a long string of 0s only, i.e. the opposite of the $\{y_t = 0\}$ scenario.

Thanks to the definitions (109) and (110), it can be concluded that

$$p_t = \sum_{i=1}^{i_1^{\{\gamma_t-1\}}(t)+(k+1)^{q-1}} rp_{t,i-1}\ \beta^{\delta_t+1} RSQ_{t,i}^{\{\gamma_t\}},$$

where it is written for convenience $\beta RSQ^{\{x\}} = L_{1,1}(RSQ^{\{x\}})$ and $\beta^{l+1} RSQ^{\{x\}} = b^l RSQ^{\{x\}}$, $l \geq 1$.