Online supplement to

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"A novel branch-and-bound algorithm for the chance-constrained resource-constrained project scheduling problem

Numerical examples

We consider an instance of the problem with $n = 8$, $m = 10$ and only one resource type of availability 8. The precedence relations among activities as well as the resource consumptions are given in Figure 1.

Example 1 *An example set of realizations $\hat{\mathfrak{P}}$ and the associated matrices $\boldsymbol{\delta}$ and $\boldsymbol{\sigma}$ are given in Tables 1 to 3. In these three tables, the numbers associated with realization $\mathbf{p}^6$ are shown in bold.*

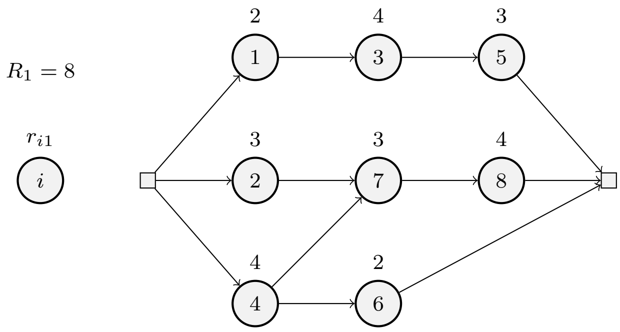

Figure 1: The graph of precedence relations among activities

i	p_i^1	p_i^2	p_i^3	p_i^4	p_i^5	p_i^6	p_i^7	p_i^8	p_i^9	p_i^{10}
1	3	3	3	2	2	**3**	3	1	1	3
2	10	5	9	6	8	**6**	7	11	11	6
3	2	3	4	3	3	**2**	5	4	5	1
4	4	3	6	6	6	**5**	4	2	4	4
5	7	7	5	9	12	**6**	6	5	9	9
6	7	9	4	6	5	**4**	9	7	9	8
7	3	4	4	2	5	**4**	6	4	6	2
8	1	1	3	3	2	**3**	3	3	2	2

Table 1: The set $\hat{\mathfrak{P}}$ of realizations for the example.

i	δ_i^1	δ_i^2	δ_i^3	δ_i^4	δ_i^5	δ_i^6	δ_i^7	δ_i^8	δ_i^9	δ_i^{10}
1	3	3	3	**3**	3	3	2	2	1	1
2	11	11	10	9	8	7	6	**6**	6	5
3	5	5	4	4	3	3	3	2	**2**	1
4	6	6	6	**5**	4	4	4	4	3	2
5	12	9	9	9	7	7	**6**	6	5	5
6	9	9	9	8	7	7	6	5	4	**4**
7	6	6	5	4	4	**4**	4	3	2	2
8	3	3	**3**	3	3	2	2	2	1	1

Table 2: The matrix $\boldsymbol{\delta}$ for the example.

i	$\sigma_{i,1}$	$\sigma_{i,2}$	$\sigma_{i,3}$	$\sigma_{i,4}$	$\sigma_{i,5}$	$\sigma_{i,6}$	$\sigma_{i,7}$	$\sigma_{i,8}$	$\sigma_{i,9}$	$\sigma_{i,10}$
1	1	2	3	**6**	7	10	4	5	8	9
2	8	9	1	3	5	7	4	**6**	10	2
3	7	9	3	8	2	4	5	1	**6**	10
4	3	4	5	**6**	1	7	9	10	2	8
5	5	4	9	10	1	2	**6**	7	3	8
6	2	7	9	10	1	8	4	5	3	**6**
7	7	9	5	2	3	**6**	8	1	4	10
8	3	4	**6**	7	8	5	9	10	1	2

Table 3: The matrix $\boldsymbol{\sigma}$ for the example.

Example 2 *Let* $\hat{\alpha} = 0.4$ *and*

$$\boldsymbol{\pi} = (\pi_1, ..., \pi_{10}) = (0.2, 0.15, 0.15, 0.1, 0.1, 0.1, 0.05, 0.05, 0.05, 0.05).$$

The set of eligible casets for the example is

$$\begin{aligned}\mathbf{C}^E = \{&C_2^1 = \{8,9\}, C_2^2 = \{1\},\\ &C_3^1 = \{7,9\}, C_3^2 = \{3,8\},\\ &C_4^1 = \{3,4,5\},\\ &C_5^1 = \{5\}, C_5^2 = \{4,9,10\},\\ &C_6^1 = \{2,7,9\}, C_6^2 = \{10\},\\ &C_7^1 = \{7,9\}, C_7^2 = \{5\}\}.\end{aligned}$$

In this case, the lead casets are $C_2^1, C_3^1, C_4^1, C_5^1, C_6^1$ *and* C_7^1*. We also compute:* $\zeta_0 = \zeta_1 = \zeta_8 = \zeta_9 = 0$*,* $\zeta_4 = 1$ *and* $\zeta_2 = \zeta_3 = \zeta_5 = \zeta_6 = \zeta_7 = 2$*.*

Example 3 *We compute the IF for activity* 5 *as follows:*

$$\psi_5 = \frac{12-9}{1} + \frac{12-7}{4} = 4.25.$$

Example 4 *For this example, $\mathcal{N}_0 = \emptyset$ represents the root node, $\mathcal{N}_1 = \{C_6^1\}$, which is branched from $\mathcal{N}_0$, is the node where only C_6^1 is excluded and $\mathcal{N}_2 = \{C_6^1, C_6^2\}$, which is branched from $\mathcal{N}_1$, represents the node where both C_6^1 and C_6^2 are excluded. The root node is the father of $\mathcal{N}_1$ and the only transitive father of $\mathcal{N}_2$. The target casets are $\Theta(\mathcal{N}_1) = C_6^1$ and $\Theta(\mathcal{N}_2) = C_6^2$ for $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively. Also,*

$$Y^{\mathcal{N}_2} = \{\mathbf{p}^1, \mathbf{p}^3, \mathbf{p}^4, \mathbf{p}^5, \mathbf{p}^6, \mathbf{p}^8\} \text{ and}$$
$$\bar{Y}^{\mathcal{N}_2} = \{\mathbf{p}^2, \mathbf{p}^7, \mathbf{p}^9, \mathbf{p}^{10}\}.$$

Example 5 *The set of effective casets for the root node and for priority rule Rule 1 consists of all lead casets ($D(\mathcal{N}_0) = \{C_6^1, C_4^1, C_5^1, C_3^1, C_7^1, C_2^1\}$) and therefore no child of the root node has a non-lead target caset. For the node $\mathcal{N}_2 = \{C_6^1, C_6^2\}$ we have:*

$$D(\mathcal{N}_2) = \{C_5^1, C_7^2, C_2^1\}.$$

Note that C_3^1 and C_7^1 are both subsets of $\bar{Y}^{\mathcal{N}_2}$ and thus the exclusions of C_3^1 and C_7^1 from $(Y^{\mathcal{N}_2}, \bar{Y}^{\mathcal{N}_2})$ are not beneficial. Additionally, the exclusions of C_3^2 and C_4^1 from $(Y^{\mathcal{N}_2}, \bar{Y}^{\mathcal{N}_2})$ are not possible. Since all effective casets must be both beneficial and possible, $D(\mathcal{N}_2)$ only consists of C_5^1, C_7^2 and C_2^1. With similar considerations, we have: $D(\mathcal{N}_4) = D(\mathcal{N}_5) = D(\mathcal{N}_6) = \emptyset$.

Example 6 *Figure 2 depicts a part of the B&B tree where branching scheme 1 is used in a depth-first mode. Each node is represented by a square. Since all information of a caset cannot be printed for each node (because of limited space), only its target caset is printed. Also, due to lack of space, the tree is not complete (the nodes with white background have not been continued and the nodes with colored background have been continued).*

The root node is branched into nodes with effective target casets C_6^1, C_4^1, C_5^1, C_3^1, C_7^1 and C_2^1. Among the children of the root node, node $\mathcal{N}_1 = \{C_6^1\}$ is branched first since its target caset's associated activity is positioned earlier in the AL (which is constructed according priority rule Rule 1 for this example). Then among the children of $\mathcal{N}_1 = \{C_6^1\}$, node $\mathcal{N}_2 = \{C_6^1, C_6^2\}$ is branched first and so on.

All gray nodes are those for which $\sum_{\mathbf{p}_l \in \bar{Y}^{\mathcal{N}}} \pi_l < \hat{\alpha}$ and all green nodes are those for which $\sum_{\mathbf{p}_l \in \bar{Y}^{\mathcal{N}}} \pi_l = \hat{\alpha}$. For example, consider node $\mathcal{N}_2 = \{C_6^1, C_6^2\}$. For this node $\bar{Y}^{\mathcal{N}_2} = \{\mathbf{p}^2, \mathbf{p}^7, \mathbf{p}^9, \mathbf{p}^{10}\}$ and $\sum_{\mathbf{p}^l \in \bar{Y}^{\mathcal{N}_2}} \pi_l = 0.3 < \hat{\alpha}$ (where $\hat{\alpha} = 0.4$), therefore its background color is gray. For node $\mathcal{N}_5 = \{C_6^1, C_6^2, C_7^2\}$, $\bar{Y}^{\mathcal{N}_5} = \{\mathbf{p}^2, \mathbf{p}^5, \mathbf{p}^7, \mathbf{p}^9, \mathbf{p}^{10}\}$ and $\sum_{\mathbf{p}^l \in \bar{Y}^{\mathcal{N}_5}} \pi_l = 0.4 = \hat{\alpha}$, therefore its background color is green. We also compute

$$\mathbf{s}^{Y^{\mathcal{N}_2}} = (0, 0, 3, 5, 0, 9, 9, 14, 19, 22) \rightarrow \mathrm{UB}^{\mathcal{N}_5} = 22 \text{ and}$$
$$\mathbf{s}^{Y^{\mathcal{N}_5}} = (0, 0, 0, 8, 3, 12, 11, 12, 18, 21) \rightarrow \mathrm{UB}^{\mathcal{N}_6} = 21.$$

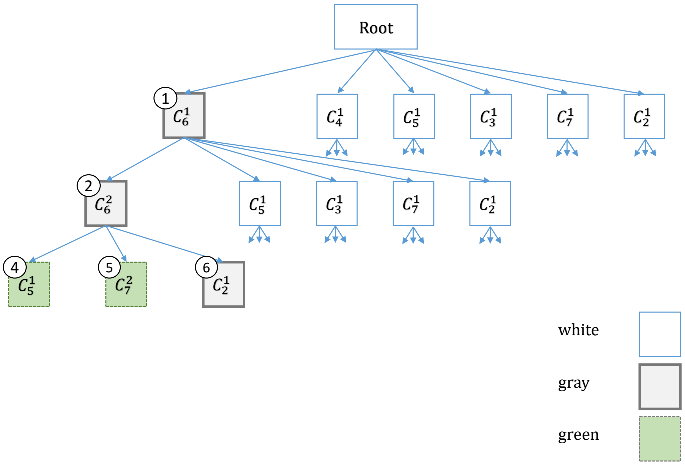

Figure 2: Branching scheme 1.

Example 7 *Consider node* $\mathcal{N}_2 = \{C_6^1, C_6^2\}$ *in* *Figure 2*. *We have:* $\bar{Y}^{\mathcal{N}_2} = \{\mathbf{p}^2, \mathbf{p}^7, \mathbf{p}^9, \mathbf{p}^{10}\}$ *and* $\sum_{\mathbf{p}^l \in \bar{Y}^{\mathcal{N}_2}} \pi_l = 0.3 < \hat{\alpha}$. *We compute* $\kappa_6 = 2, \kappa_4 = 0, \kappa_5 = 1, \kappa_3 = 1, \kappa_7 = 2$ *and* $\kappa_2 = 1$. *Therefore,*

$$\hat{\mathcal{N}}_2 = \{C_6^1, C_6^2, C_5^1, C_3^1, C_7^1, C_7^2, C_2^1\}$$

and $\bar{Y}^{\hat{\mathcal{N}}_2} = \{\mathbf{p}^2, \mathbf{p}^5, \mathbf{p}^7, \mathbf{p}^8, \mathbf{p}^9, \mathbf{p}^{10}\}$. *We also compute*

$$\mathbf{s}^{Y^{\mathcal{N}_2}} = (0, 0, 3, 5, 0, 9, 9, 14, 19, 22) \rightarrow \mathrm{UB}^{\mathcal{N}_2} = 22 \text{ and}$$
$$\mathbf{s}^{Y^{\hat{\mathcal{N}}_2}} = (0, 0, 3, 5, 0, 9, 9, 13, 17, 20) \rightarrow \mathrm{LB}^{\mathcal{N}_2} = 20.$$

Example 8 *Figure 3 depicts the B&B tree where branching scheme 2 is used in a best-first mode. Each node is represented by a square. The root node is branched into three nodes:* $\mathcal{N}_1$, $\mathcal{N}_2$ *and* $\mathcal{N}_3$. *Among these three nodes, node* $\mathcal{N}_3$ *is branched first since its lower bound is smaller than that of the other two nodes.* $\mathcal{N}_3$ *is branched into two nodes:* $\mathcal{N}_4$ *and* $\mathcal{N}_5$. *Node* $\mathcal{N}_4$ *whose lower bound is larger than the best upper bound (*UB^**) found so far is eliminated from the three, whereas* $\mathcal{N}_5$ *is branched into its children (*$\mathcal{N}_6$, $\mathcal{N}_7$ *and* $\mathcal{N}_8$*). The next node to be branched is* $\mathcal{N}_1$ *because its lower bound is smaller than the lower bounds of all other unbranched nodes. The branching continues with the same logic until no unbranched node with a lower bound smaller than* UB^* *exists in the tree.*

All white nodes with a line over them are those that are left unbranched in the tree after the branching stopped. All gray nodes are those for which $\sum_{\mathbf{p}_l \in \bar{Y}^{\mathcal{N}}} \pi_l < \hat{\alpha}$

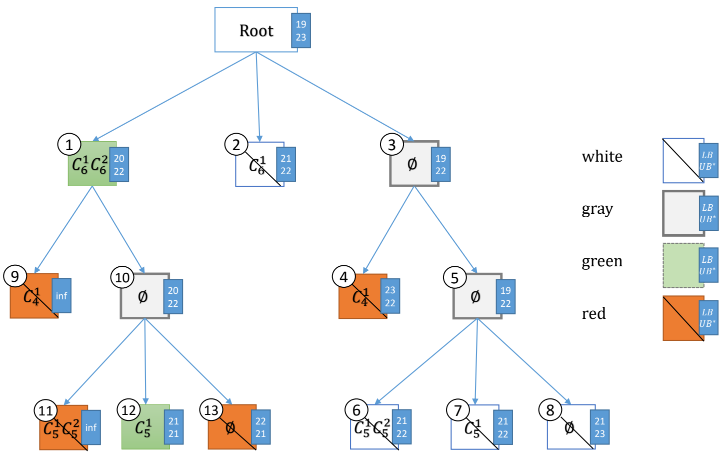

Figure 3: Branching scheme 2.

and $\mathrm{LB}^{\mathcal{N}} < \mathrm{UB}^{*} \leq \mathrm{UB}^{\mathcal{N}}$ *and thus they are branched from and not eliminated. All green nodes are those for which* $\sum_{\mathbf{p}_l \in \bar{Y}^{\mathcal{N}}} \pi_l \leq \hat{\alpha}$ *and* $\mathrm{UB}^{\mathcal{N}} < \mathrm{UB}^{*}$*. All red nodes with a line over them are those for which* $\sum_{\mathbf{p}_l \in \bar{Y}^{\mathcal{N}}} \pi_l > \hat{\alpha}$ *or* $\mathrm{LB}^{\mathcal{N}} \geq \mathrm{UB}^{*}$ *and hence they are eliminated.*