STEPHEN WIGGINS

SOLUTIONS TO THE EXERCISES IN ELEMENTARY CLASSICAL MECHANICS

SCHOOL OF MATHEMATICS
UNIVERSITY OF BRISTOL

Contents

Preface 7

1 *Solutions for Problem Set 1* 9

2 *Solutions for Problem Set 2* 15

3 *Solutions for Problem Set 3* 21

4 *Solutions for Problem Set 4* 25

5 *Solutions for Problem Set 5* 27

6 *Solutions for Problem Set 6* 35

7 *Solutions for Problem Set 7* 41

8 *Solutions for Problem Set 8* 47

9 *Solutions for Problem Set 9* 51

10 *Solutions for Problem Set 10* 55

11 Solutions for Problem Set 11 61

List of Figures

1.1 The resultants are denoted by the dashed vectors. 9
1.2 $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$. 10
1.3 Addition and subtraction laws of vectors using the parallelogram law. 10
1.4 Scalar multiplication laws of vectors using the parallelogram law. 11
1.5 Scalar multiplication laws of vectors using the parallelogram law. 11
1.6 Scalar multiplication and vector addition laws using the parallelogram law. 12
1.7 Graphical illustration of the algebra of the dot product. 13
1.8 Geometry of vector addition and subtraction and the relationship to a parallelogram. 13
1.9 Relationship between the cross product and the area of a parallelogram. 14

9.1 Graph of the potential energy function and the phase portrait. 51
9.2 Graph of the potential energy function and the phase portrait. 52
9.3 Graph of the potential energy function and the phase portrait. 53

10.1 Graph of the potential energy function and the phase portrait. 56
10.2 Graph of the potential energy function and the phase portrait. 57
10.3 Graph of the potential energy function and the phase portrait. 57
10.4 Graph of the potential energy function and the phase portrait. 58

Preface

This book contains the solutions to all of the problems in my book *Elementary Classical Mechanics*, also available on figshare (`https://doi.org/10.6084/m9.figshare.5309851.v4`).

This book was typeset with the Tufte latex package.

1

Solutions for Problem Set 1

1. The graphical constructions are given in Fig. 1.1. The resultants in (a) and (b) have equal length and opposite direction. The resultants in (c) and (d) have the same direction, but the resultant in (c) has twice the length of the resultant in (d).

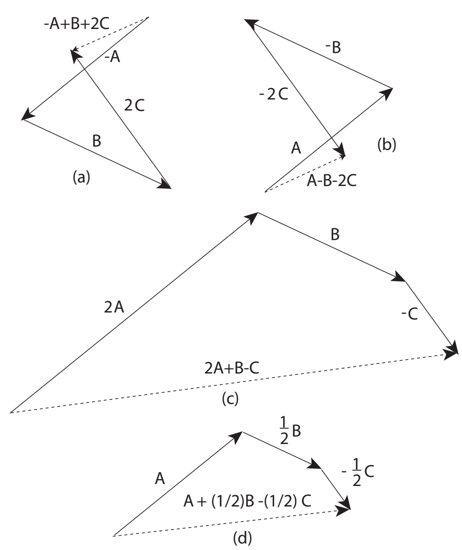


Figure 1.1: The resultants are denoted by the dashed vectors.

2. The same vectors from Fig. 1 of the problems are used here.

   (a) This is shown in Fig. 1.2

   (b) Referring to Fig. 1.3, in (i) $\mathbf{A}+\mathbf{B}$ is constructed, and in (ii) $\mathbf{C}$ is added to $\mathbf{A}+\mathbf{B}$. In (iii) $\mathbf{B}+\mathbf{C}$ is constructed and in (iv) $\mathbf{B}+\mathbf{C}$ is added to $\mathbf{A}$. It follows that (ii) = (iv).

   (c) This is illustrated in Fig. 1.4, where the scalar a is taken to be $\frac{1}{2}$, and the scalar b is taken to be $-\frac{3}{2}$.

   (d) This is illustrated in Fig. 1.5, where, as above, the scalar a is taken to be $\frac{1}{2}$, and the scalar b is taken to be $-\frac{3}{2}$.

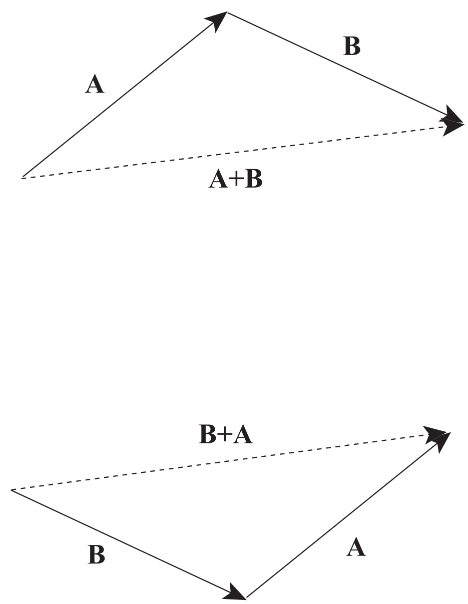


Figure 1.2: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.

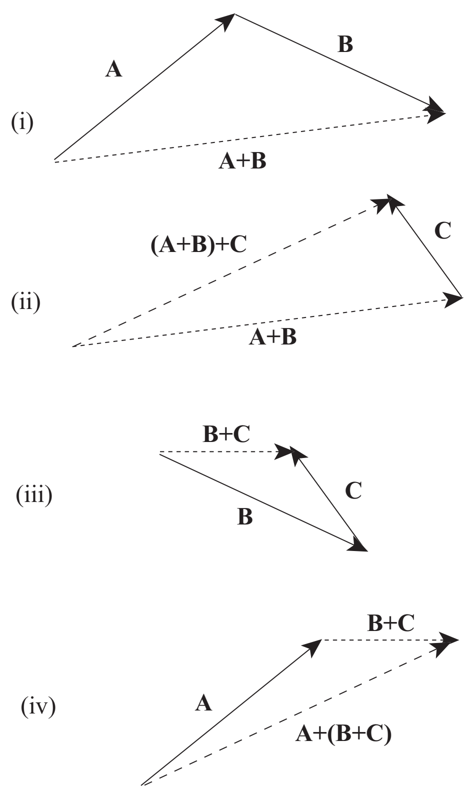


Figure 1.3: Addition and subtraction laws of vectors using the parallelogram law.

(e) This is illustrated in Fig. 1.6, where, as above, the scalar a is taken to be $\frac{1}{2}$.

3.(a) The main idea here is that scalar multiplication is commutative.

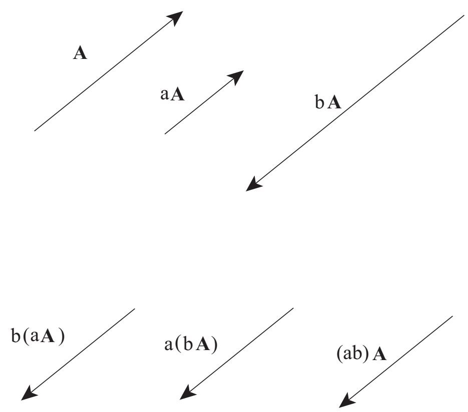


Figure 1.4: Scalar multiplication laws of vectors using the parallelogram law.

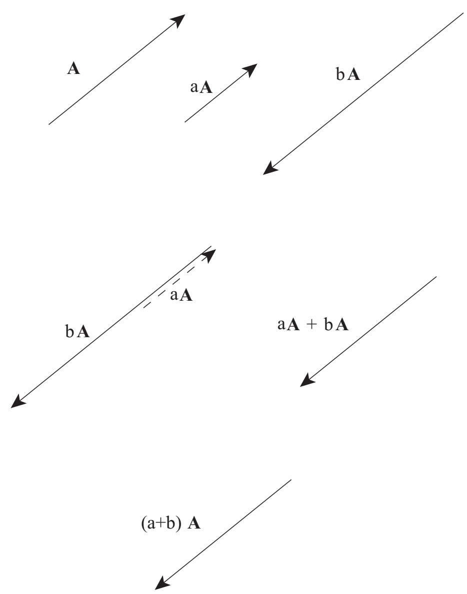


Figure 1.5: Scalar multiplication laws of vectors using the parallelogram law.

Then we have:

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}|\,|\mathbf{B}| \cos\theta = |\mathbf{B}|\,|\mathbf{A}| \cos\theta = \mathbf{B} \cdot \mathbf{A},$$

where it is assumed evident that "the angle between **A** and **B**" is the same as the "the angle between **B** and **A**".

(b) In order to prove this we use the idea of the projection of one vector on another. We will take it as evident that the projection of $\mathbf{B} + \mathbf{C}$ on **A** is the projection of **B** on **A** plus the projection of **C** on **A**. Let $\mathbf{a} \equiv \frac{\mathbf{A}}{|\mathbf{A}|}$. Then the previous statement is mathematically written as:

$$(\mathbf{B} + \mathbf{C}) \cdot \mathbf{a} = \mathbf{B} \cdot \mathbf{a} + \mathbf{C} \cdot \mathbf{a}.$$

Multiplying this expression by $|\mathbf{A}|$ gives:

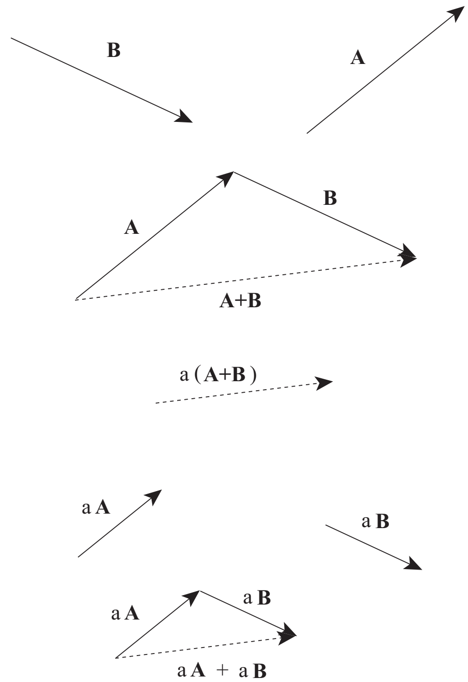


Figure 1.6: Scalar multiplication and vector addition laws using the parallelogram law.

$$(\mathbf{B}+\mathbf{C})\cdot|\mathbf{A}|\mathbf{a} = \mathbf{B}\cdot|\mathbf{A}|\mathbf{a} + \mathbf{C}\cdot|\mathbf{A}|\mathbf{a},$$

or

$$(\mathbf{B}+\mathbf{C})\cdot\mathbf{A} = \mathbf{B}\cdot\mathbf{A} + \mathbf{C}\cdot\mathbf{A}.$$

Using commutativity of the dot product gives:

$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{B}\cdot\mathbf{A} + \mathbf{C}\cdot\mathbf{A}.$$

(c) The main idea here is that scalar multiplication is commutative. First, if $a \geq 0$ the result follows easily from applying the definition of the dot product. From Fig. 1.7 it follows that:

$$-\mathbf{A}\cdot\mathbf{B} = -|\mathbf{A}|\,|\mathbf{B}|\cos\theta.$$

Using this result, the result obtained for $a > 0$ can be easily obtained for $a < 0$ following exactly the same reasoning.

4. We have

$$(\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}-\mathbf{B}) = \mathbf{A}\cdot\mathbf{A} - \mathbf{A}\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{A} - \mathbf{B}\cdot\mathbf{B} = |\mathbf{A}|^2 - |\mathbf{B}|^2.$$

This expression is zero if and only if $|\mathbf{A}| = \pm|\mathbf{B}|$. Since the magnitude of a vector is never negative, the minus sign is impossible.

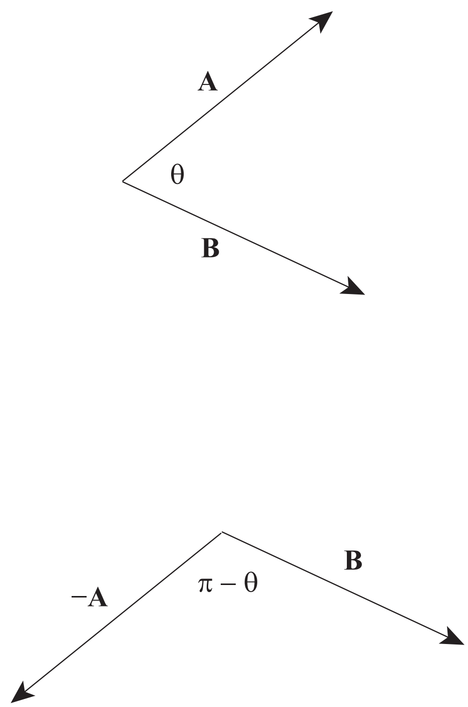


Figure 1.7: Graphical illustration of the algebra of the dot product.

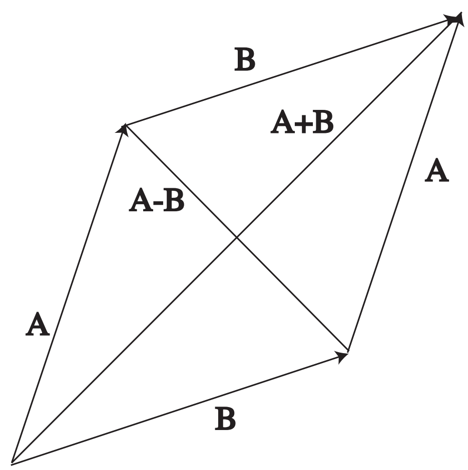


Figure 1.8: Geometry of vector addition and subtraction and the relationship to a parallelogram.

5. See Fig. 1.9. We have:

$$
\begin{aligned}
\text{Area of parallelogram} &= h\,|\mathbf{B}|, \\
&= |\mathbf{A}\,|\sin\theta\,|\mathbf{B}|, \\
&= |\mathbf{A} \times \mathbf{B}|
\end{aligned}
$$

It should be clear from Fig. 1.9 and the calculation above that the area of the triangle with sides **A** and **B** is given by $\frac{1}{2}|\mathbf{A} \times \mathbf{B}|$.

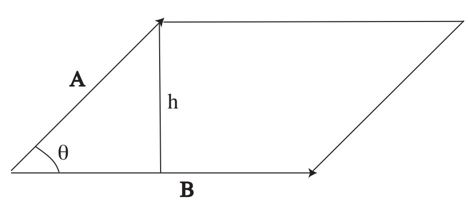


Figure 1.9: Relationship between the cross product and the area of a parallelogram.

2

Solutions for Problem Set 2

1. Let

$$\begin{aligned}
\mathbf{A} &= A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}, \\
\mathbf{B} &= B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}, \\
\mathbf{C} &= C_1\mathbf{i} + C_2\mathbf{j} + C_3\mathbf{k}, \\
\mathbf{D} &= D_1\mathbf{i} + D_2\mathbf{j} + D_3\mathbf{k}.
\end{aligned}$$

(a) We have

$$\mathbf{A} \times \mathbf{B} = (A_2B_3 - A_3B_2)\mathbf{i} + (A_3B_1 - A_1B_3)\mathbf{j} + (A_1B_2 - A_2B_1)\mathbf{k},$$

and

$$\mathbf{B} \times \mathbf{A} = (B_2A_3 - B_3A_2)\mathbf{i} + (B_3A_1 - B_1A_3)\mathbf{j} + (B_1A_2 - B_2A_1)\mathbf{k}.$$

By inspection, you see that $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$.

(b) We have:

$$\begin{aligned}
(\mathbf{A}+\mathbf{B}) \times (\mathbf{C}+\mathbf{D}) &= ((A_2+B_2)(C_3+D_3) - (A_3+B_3)(C_2+D_2))\,\mathbf{i} \\
&+ ((A_3+B_3)(C_1+D_1) - (A_1+B_1)(C_3+D_3))\,\mathbf{j} \\
&+ ((A_1+B_1)(C_2+D_2) - (A_2+B_2)(C_1+D_1))\,\mathbf{k}.
\end{aligned}$$

Also,

$$\mathbf{A} \times \mathbf{C} = (A_2C_3 - A_3C_2)\mathbf{i} + (A_3C_1 - A_1C_3)\mathbf{j} + (A_1C_2 - A_2C_1)\mathbf{k},$$

$$\mathbf{B} \times \mathbf{C} = (B_2C_3 - B_3C_2)\mathbf{i} + (B_3C_1 - B_1C_3)\mathbf{j} + (B_1C_2 - B_2C_1)\mathbf{k}.$$

$$\mathbf{A} \times \mathbf{D} = (A_2 D_3 - A_3 D_2)\mathbf{i} + (A_3 D_1 - A_1 D_3)\mathbf{j} + (A_1 D_2 - A_2 D_1)\mathbf{k},$$

$$\mathbf{B} \times \mathbf{D} = (B_2 D_3 - B_3 D_2)\mathbf{i} + (B_3 D_1 - B_1 D_3)\mathbf{j} + (B_1 D_2 - D_2 C_1)\mathbf{k}.$$

By inspection of these quantities it follows that the equality holds.

(c) Once $\mathbf{A} \times \mathbf{B}$ has been computed in components, the result becomes trivial as it just involves commutation of scalar multiplication.

2. $\mathbf{A} \times \mathbf{B} = -23\mathbf{i} + 10\mathbf{j} + 31\mathbf{k}$, $\mathbf{A} \cdot \mathbf{B} = 11$.

3. The two vectors are parallel (one is a scalar multiple of the other), so $\mathbf{A} \times \mathbf{B} = 0$.

4.(a) $\mathbf{a} \cdot \mathbf{b} = 1$, $\mathbf{c} \cdot \mathbf{d} = 1$, hence $\frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{c} \cdot \mathbf{d}} = 1$.

(b) This expression is not mathematically valid.

(c) The vector operations are all fine. However, the denominator is zero since $\mathbf{b} \cdot \mathbf{c} = 0$.

(d) Dividing by a vector is not defined.

(e) $(\mathbf{a} \cdot \mathbf{b})\, \mathbf{d} = \mathbf{i} + \mathbf{j}$ and $|\mathbf{i} + \mathbf{j}| = \sqrt{2}$.

(f) $|\mathbf{c} - \mathbf{d}| = |-\mathbf{i}| = 1$, therefore $\mathbf{a}\, |\mathbf{c} - \mathbf{d}| = \mathbf{a} = \mathbf{i} - \mathbf{j}$,

(g) This would be much clearer if there were parentheses in appropriate places in the expression. The usual rules of algebra require that multiplications are performed first, then addition. The same holds in vector algebra. In this case we compute the dot product of $\mathbf{b}$ and $\mathbf{c}$ first (which is a scalar), then add it to $\mathbf{a}$. However, adding a scalar to a vector is not defined.

(h) $\mathbf{a} \times \mathbf{b} = (\mathbf{i} - \mathbf{j}) \times \mathbf{i} = \mathbf{k}$,

(i) This is ambiguous since $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$, is not equal to $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$, and therefore you don't know which cross product to compute first,

(j) This expression describes the addition of a vector and a scalar, which is not a mathematically valid operation.

(k) This expression might appear ambiguous since it is not clear whether to evaluate the cross product first or the dot product, and evaluating the dot product first would give nonsense. However, in the literature there is a convention for such expressions involving cross products and dot products. It is $\mathbf{a} \times \mathbf{b} \cdot \mathbf{d} \equiv (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d}$. In this case we have $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d} = \mathbf{k} \cdot (\mathbf{i} \cdot \mathbf{j}) = 0$,

(l) This expression is not mathematically valid since you cannot compute the cross product of a vector with a scalar.

5. Let **PQ** denote the vector starting at the point P and ending at the point Q.

$$\mathbf{A} + \mathbf{PQ} = \mathbf{B}.$$

Then

$$\begin{aligned} \mathbf{PQ} &= \mathbf{B} - \mathbf{A}, \\ &= (B_1 - A_1)\mathbf{i} + (B_2 - A_2)\mathbf{j} + (B_3 - A_3)\mathbf{k}, \end{aligned}$$

and

$$|\mathbf{PQ}| = \sqrt{(B_1 - A_1)^2 + (B_2 - A_2)^2 + (B_3 - A_3)^2},$$

6. We will omit denoting the explicit dependence of the functions on t for the sake of a less cumbersome notation.

(a) $a\mathbf{A} = aA_1\mathbf{i} + aA_2\mathbf{j} + aA_3\mathbf{k}$, then

$$\begin{aligned} \frac{d}{dt}(a\mathbf{A}) &= \left(\frac{da}{dt}A_1 + a\frac{dA_1}{dt}\right)\mathbf{i} + \left(\frac{da}{dt}A_2 + a\frac{dA_2}{dt}\right)\mathbf{j} + \left(\frac{da}{dt}A_3 + a\frac{dA_3}{dt}\right)\mathbf{k}, \\ &= \frac{da}{dt}A_1\mathbf{i} + \frac{da}{dt}A_2\mathbf{j} + \frac{da}{dt}A_3\mathbf{k} + a\frac{dA_1}{dt}\mathbf{i} + a\frac{dA_2}{dt}\mathbf{j} + a\frac{dA_3}{dt}\mathbf{k}, \\ &= \frac{da}{dt}\left(A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}\right) + a\left(\frac{dA_1}{dt}\mathbf{i} + \frac{dA_2}{dt}\mathbf{j} + \frac{dA_3}{dt}\mathbf{k}\right), \\ &= \frac{da}{dt}\mathbf{A} + a\frac{d\mathbf{A}}{dt}. \end{aligned}$$

(b) $\mathbf{A} \cdot \mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3$, then

$$\begin{aligned} \frac{d}{dt}\left(\mathbf{A} \cdot \mathbf{B}\right) &= \left(\frac{dA_1}{dt}B_1 + A_1\frac{dB_1}{dt}\right) + \left(\frac{dA_2}{dt}B_2 + A_2\frac{dB_2}{dt}\right) + \left(\frac{dA_3}{dt}B_3 + A_3\frac{dB_3}{dt}\right), \\ &= \frac{dA_1}{dt}B_1 + \frac{dA_2}{dt}B_2 + \frac{dA_3}{dt}B_3 + A_1\frac{dB_1}{dt} + A_2\frac{dB_2}{dt} + A_3\frac{dB_3}{dt}, \\ &= \frac{d\mathbf{A}}{dt} \cdot \mathbf{B} + \mathbf{A} \cdot \frac{d\mathbf{B}}{dt}. \end{aligned}$$

(c) $\mathbf{A} \times \mathbf{B} = (A_2B_3 - A_3B_2)\mathbf{i} + (A_3B_1 - A_1B_3)\mathbf{j} + (A_1B_2 - A_2B_1)\mathbf{k}$,
then

$$\begin{aligned}
\frac{d}{dt}(\mathbf{A} \times \mathbf{B}) &= \left(\frac{dA_2}{dt}B_3 + A_2\frac{dB_3}{dt} - \frac{dA_3}{dt}B_2 - A_3\frac{dB_2}{dt}\right)\mathbf{i} \\
&+ \left(\frac{dA_3}{dt}B_1 + A_3\frac{dB_1}{dt} - \frac{dA_1}{dt}B_3 - A_1\frac{dB_3}{dt}\right)\mathbf{j} \\
&+ \left(\frac{dA_1}{dt}B_2 + A_1\frac{dB_2}{dt} - \frac{dA_2}{dt}B_1 - A_2\frac{dB_1}{dt}\right)\mathbf{k}, \\
&= \left(\frac{dA_2}{dt}B_3 - \frac{dA_3}{dt}B_2\right)\mathbf{i} + \left(\frac{dA_3}{dt}B_1 - \frac{dA_1}{dt}B_3\right)\mathbf{j} + \left(\frac{dA_1}{dt}B_2 - \frac{dA_2}{dt}B_1\right)\mathbf{k} \\
&+ \left(A_2\frac{dB_3}{dt} - A_3\frac{dB_2}{dt}\right)\mathbf{i} + \left(A_3\frac{dB_1}{dt} - A_1\frac{dB_3}{dt}\right)\mathbf{j} + \left(A_1\frac{dB_2}{dt} - A_2\frac{dB_1}{dt}\right)\mathbf{k}, \\
&= \frac{d\mathbf{A}}{dt} \times \mathbf{B} + \mathbf{A} \times \frac{d\mathbf{B}}{dt}.
\end{aligned}$$

7.(a)

$$\begin{aligned}
&\int_2^3 4(t-1)dt\mathbf{i} - (2t+3)dt\mathbf{j} + 6t^2dt\mathbf{k}, \\
&= \left(2t^2 - 4t\right)\Big|_2^3\mathbf{i} - (t^2+3t)\Big|_2^3 dt\mathbf{j} + 2t^3\Big|_2^3\mathbf{k}, \\
&= 6\mathbf{i} - 8\mathbf{j} + 38\mathbf{k}.
\end{aligned}$$

(b)

$$\int_1^2 (t\mathbf{i} - 2\mathbf{k}) \cdot \mathbf{A}(t)dt = \int_1^2 \left(4(t^2-t) - 12t^2\right)dt = -\int_1^2 \left(8t^2 + 4t\right)dt = -\frac{74}{3}.$$

8. We have $x(t) = e^{-t}\cos t$, $y(t) = e^{-t}\sin t$, $z(t) = e^{-t}$, then:

$$\begin{aligned}
&\dot{x} = -e^{-t}\cos t - e^{-t}\sin t, \quad \dot{y} = -e^{-t}\sin t + e^{-t}\cos t, \quad \dot{z} = -e^{-t}, \\
&\ddot{x} = e^{-t}\cos t + e^{-t}\sin t + e^{-t}\sin t - e^{-t}\cos t = 2e^{-t}\sin t, \\
&\ddot{y} = e^{-t}\sin t - e^{-t}\cos t - e^{-t}\cos t - e^{-t}\sin t = -2e^{-t}\cos t, \\
&\ddot{z} = e^{-t}.
\end{aligned}$$

Then we have:

$$|\mathbf{v}| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} = e^{-t}\sqrt{(\cos t + \sin t)^2 + (\cos t - \sin t)^2 + 1} = \sqrt{3}e^{-t}.$$

Similarly,

$$|\mathbf{a}| = \sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2} = e^{-t}\sqrt{(2\sin t)^2 + (-2\cos t)^2 + 1} = \sqrt{5}e^{-t}.$$

9. We have $\mathbf{r} = a\cos\omega t\mathbf{i} + a\sin\omega t\mathbf{j} + bt^2\mathbf{k}$, then:

$$\begin{aligned}\mathbf{v} &= -a\omega\sin\omega t\mathbf{i} + a\omega\cos\omega t\mathbf{j} + 2bt\mathbf{k},\\ \mathbf{a} &= -a\omega^2\cos\omega t\mathbf{i} - a\omega^2\sin\omega t\mathbf{j} + 2b\mathbf{k}.\end{aligned}$$

and

$$\begin{aligned}|\mathbf{v}| &= \sqrt{a^2\omega^2(\sin^2\omega t + \cos^2\omega t) + (2bt)^2} = \sqrt{a^2\omega^2 + (2bt)^2},\\ |\mathbf{a}| &= \sqrt{a^2\omega^4(\cos^2\omega t + \sin^2\omega t) + (2b)^2} = \sqrt{a^2\omega^4 + (2b)^2}.\end{aligned}$$

The particle follows the path of a helix in three dimensions.

3
Solutions for Problem Set 3

1. We have

$$\mathbf{r} = 3\cos 2t\mathbf{i} + 3\sin 2t\mathbf{j} + (8t-4)\mathbf{k},$$

then

$$\mathbf{v} = -6\sin 2t\mathbf{i} + 6\cos 2t\mathbf{j} + 8\mathbf{k},$$

and

$$\frac{ds}{dt} = v = |\mathbf{v}| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{36+64} = 10.$$

(a)
$$\mathbf{T} = \frac{\mathbf{v}}{v} = -\frac{3}{5}\sin 2t\mathbf{i} + \frac{3}{5}\cos 2t\mathbf{j} + \frac{4}{5}\mathbf{k}.$$

(b) This should be clear.

(c) $\kappa = \left|\frac{d\mathbf{T}}{ds}\right|$. We have:

$$\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{dt}\frac{dt}{ds} = \left(-\frac{6}{5}\cos 2t\mathbf{i} - \frac{6}{5}\sin 2t\mathbf{j}\right)\frac{1}{10}.$$

Therefore

$$\kappa = \frac{3}{25}.$$

(d) $R = \frac{1}{\kappa} = \frac{25}{3}$.

(e) $\mathbf{N} = R\frac{d\mathbf{T}}{ds} = \frac{25}{3}\left(-\frac{6}{5}\cos 2t\mathbf{i} - \frac{6}{5}\sin 2t\mathbf{j}\right)\frac{1}{10}$.

2. We have:

$$\mathbf{r} = \cos\omega t\mathbf{i} + \sin\omega t\mathbf{j},$$

from which it follows that:

$$\mathbf{v} = -\omega \sin \omega t \mathbf{i} + \omega \cos \omega t \mathbf{j},$$

and

$$\mathbf{a} = -\omega^2 \cos \omega t \mathbf{i} - \omega^2 \sin \omega t \mathbf{j}.$$

(a) This is a trivial calculation.

(b) $\mathbf{a} = -\omega^2 \mathbf{r}$

(c) $\mathbf{r} \times \mathbf{v} = \omega \mathbf{k}$.

3. $m\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}}$.

4. $\nabla V(\mathbf{r}) \cdot \dot{\mathbf{r}}$.

5. Use the previous problem:

$$\begin{aligned} \frac{dV}{dt} &= (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot (-\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}), \\ &= (\cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}) \cdot (-\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}) = t. \end{aligned}$$

6. First, you should have verified that the two points are on the curve. Then recall the definition of arclength, s:

$$\frac{ds}{dt} \equiv \sqrt{\left(\frac{dx}{dt}(t)\right)^2 + \left(\frac{dy}{dt}(t)\right)^2 + \left(\frac{dz}{dt}(t)\right)^2}.$$

So for this problem we have:

$$\text{length} = \int_0^{\frac{\pi}{2}} \sqrt{2}dt = \frac{\pi}{\sqrt{2}}.$$

It is crucial that you understand the reason for the choice of the limits in the integral.

7. Use the indefinite integral from the previous problem to compute arclength as a function of t:

$$s = \int_0^s ds = \int_0^t \sqrt{2}dt = \sqrt{2}t.$$

(Why were the limits on the integrals chosen as above?) Then we have:

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k} = \cos \frac{1}{\sqrt{2}}s\mathbf{i} + \sin \frac{1}{\sqrt{2}}s\mathbf{j} + \frac{1}{\sqrt{2}}s\mathbf{k} = \mathbf{r}(t(s)) = \mathbf{r}(s).$$

(Make sure you understand what is meant by the four equality signs in the expression above.)

8.
$$\frac{d\mathbf{r}}{dt} = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}.$$

$$\frac{d\mathbf{r}}{ds} = \frac{1}{\sqrt{2}}\left(-\sin\frac{1}{\sqrt{2}}s\mathbf{i} + \cos\frac{1}{\sqrt{2}}s\mathbf{j} + \mathbf{k}\right).$$

$$\frac{ds}{dt} = \sqrt{2}.$$

4
Solutions for Problem Set 4

1. First,

$$\begin{aligned}\int_C \mathbf{A}\cdot d\mathbf{r} &= \int_C \Big((3x^2-6yz)\mathbf{i}+(2y+3xz)\mathbf{j}+(1-4xyz^2)\mathbf{k}\Big)\cdot(dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k}),\\ &= \int_C (3x^2-6yz)dx+(2y+3xz)dy+(1-4xyz^2)dz.\end{aligned}$$

(a) If $x = t$, $y = t^2$, $z = t^3$, then the points $(0,0,0)$ and $(1,1,1)$ correspond to $t = 0$ and $t = 1$, respectively. Then we have

$$\begin{aligned}\int_C \mathbf{A}\cdot d\mathbf{r} &= \int_{t=0}^{t=1} (3t^2-6t^5)dt+(2t^2+3t^4)d(t^2)+(1-4t^9)d(t^3),\\ &= \int_{t=0}^{t=1} (3t^2-6t^5)dt+(4t^3+6t^5)dt+3(t^2-4t^{11})dt,\\ &= \Big(t^3-t^6+t^4+t^5+t^3-t^{12}\Big)\Big|_0^1 = 2.\end{aligned}$$

(b) Along the straight line joining $(0,0,0)$ to $(1,1,1)$ we have $x = t$, $y = t$, $z = t$. Then since $dx = dy = dz = dt$, we have:

$$\begin{aligned}\int_C \mathbf{A}\cdot d\mathbf{r} &= \int_C (3x^2-6yz)dx+(2y+3xz)dy+(1-4xyz^2)dz,\\ &= \int_0^1 (3t^2-6t^2)dt+(2t+3t^2)dt+(1-4t^4)dt,\\ &= (-t^3+t^2+t^3+t-\frac{4}{5}t^5)\Big|_0^1 = \frac{6}{5}.\end{aligned}$$

2.

$$\frac{\partial\phi}{\partial x} = 3x^2+y+y\cos xy-\frac{2x}{z}\sin\frac{x^2}{z}.$$

$$\frac{\partial\phi}{\partial y} = z+x+x\cos xy.$$

$$\frac{\partial \phi}{\partial z} = y + \frac{x^2}{z^2} \sin \frac{x^2}{z}.$$

3. Show that $\nabla \times \mathbf{A} = 0$.

$$\frac{\partial A_1}{\partial x} = 2y, \ \frac{\partial A_1}{\partial y} = 2x, \ \frac{\partial A_1}{\partial z} = 3z^2.$$

$$\frac{\partial A_2}{\partial x} = 2x, \ \frac{\partial A_2}{\partial y} = 2, \ \frac{\partial A_2}{\partial z} = 0.$$

$$\frac{\partial A_3}{\partial x} = 3z^2, \ \frac{\partial A_3}{\partial y} = 0, \ \frac{\partial A_3}{\partial z} = 6xz.$$

   Now it is easy to verify that:

$$\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} = \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} = 0.$$

4. First, note the each vector field is the gradient of a scalar valued function, $\mathbf{A} = \nabla V$. Therefore, the line integral of the vector along a path between two points is the difference of the scalar valued function evaluated at the two points.

   (a) $V = \sin x \sin y \sin z$. $V(1,1,1) - V(0,0,0) = (\sin 1)^3$.

   (b) $V = xyz$. $V(1,1,1) - V(0,0,0) = 1$.

   (c) $V = \frac{z^2}{2}$. $V(1,1,1) - V(0,0,0) = \frac{1}{2}$.

5
Solutions for Problem Set 5

1. The answer to both parts of this question involve an understanding of Newton's first law: velocity doesn't change unless there is a force, and the velocity changes in the direction of the force.

   (a) The initial velocity vector and the vector defining the gravitational force define a plane, which we label as the $y-z$ plane. Motion must occur in this plane since there is no force acting "out of the plane."

   (b) There is no force in the y direction. Therefore the initial velocity in the y direction never changes.

2. The starting point is Newton's equations, which are given by:

$$m\frac{d^2x}{dt^2}\mathbf{i} = F\mathbf{i}, \quad x(0) = 0, \dot{x}(0) = v_0.$$

We integrate once (with respect to time) to get speed and velocity:

$$\int_0^t \frac{d}{d\tau}\left(\frac{dx}{d\tau}(\tau)\right) d\tau\mathbf{i} = \frac{F}{m}\int_0^t d\tau\mathbf{i},$$

or

$$\frac{dx}{dt}(t)\mathbf{i} = \left(v_0 + \frac{F}{m}t\right)\mathbf{i},$$

which gives the velocity as a function of time. The speed is the magnitude of velocity:

$$\frac{dx}{dt}(t) = v_0 + \frac{F}{m}t. \tag{5.1}$$

To get distance we integrate the expression for velocity (with respect to time):

$$\int_0^t \frac{dx}{d\tau}(\tau)d\tau = \int_0^t \left(v_0 + \frac{F}{m}\tau\right) d\tau,$$

or

$$x(t) = v_0 t + \frac{F}{2m}t^2. \tag{5.2}$$

Finally, we solve for speed as a function of position. Start with (5.2). This is a quadratic equation for t that we can solve for t:

$$t = -\frac{mv_0}{F} \pm \frac{m}{F}\sqrt{v_0^2 + \frac{2Fx(t)}{m}}.$$

There are two choices of sign here. Which one do we take? Now t is positive (we start from $t = 0$ and t increases). The constants m, F, and v_0 are all positive, which implies that $x(t)$ is positive (look at (5.2)). Therefore for t positive we must have:

$$t = -\frac{mv_0}{F} + \frac{m}{F}\sqrt{v_0^2 + \frac{2Fx(t)}{m}}.$$

Substituting this into (5.1) (and writing $\frac{dx}{dt}(t) = v(t)$) gives:

$$v(t) = \sqrt{v_0^2 + \frac{2Fx(t)}{m}}, \tag{5.3}$$

or

$$(v(t))^2 = v_0^2 + \frac{2Fx(t)}{m}.$$

3. We denote the position vector of the object by $\mathbf{r} = z\mathbf{k}$. The Newton's equations become:

$$m\frac{d^2z}{dt^2}\mathbf{k} = -mg\mathbf{k}, \quad z(0) = 0, \dot{z}(0) = v_0 > 0,$$

or,

$$\frac{d^2z}{dt^2}\mathbf{k} = -g\mathbf{k}, \quad z(0) = 0, \dot{z}(0) = v_0 > 0,$$

Now these equations are identical to those of the previous problem with

$$\frac{F}{m} = -g.$$

So, using (5.2), we have

$$z(t) = v_0 t - \frac{g}{2}t^2. \tag{5.4}$$

Next we need to compute the time taken to reach the highest point. We must ask ourselves, "what characterizes the highest point"? The object goes up, stops "instantaneously", and falls back down. So the highest point is reached at the time when the speed vanishes.

Using (5.1), we have:

$$\frac{dz}{dt}(t) = v_0 - gt. \tag{5.5}$$

Setting the left-hand-side of this equation to zero gives:

$$t = \frac{v_0}{g}.$$

What is the maximum height? We merely substitute this time into (5.4) to get:

$$z_{\max} = \frac{v_0^2}{2g}.$$

To get the speed as a function of distance from the origin we use (5.3 to obtain:

$$v(t) = \sqrt{v_0^2 - 2gz(t)}. \tag{5.6}$$

4. First, we write down Newton's equations:

$$m\frac{d^2z}{dt^2}\mathbf{k} = -mg\mathbf{k} - \beta\frac{dz}{dt}\mathbf{k}, \quad z(0) = h,\, \dot{z}(0) = 0,$$

or

$$\dot{w} + \frac{\beta}{m}w = -g, \quad w(0) = 0, \tag{5.7}$$

where $w \equiv \frac{dz}{dt}$. As discussed in class, this is a *linear, inhomogeneous first order equation for* w. We solve for w, then integrate the result to get the height.

To find the general solution of (5.7), we find a solution to the homogeneous equation:

$$\dot{w} + \frac{\beta}{m}w = 0,$$

a *particular solution* to the inhomogeneous equation:

$$\dot{w} + \frac{\beta}{m} w = -g. \tag{5.8}$$

then add the two together, and evaluate the unknown constant in the homogeneous solution by satisfying the initial condition.

The solution to the homogeneous equation is given by:

$$w(t) = Ce^{-\frac{\beta}{m}t},$$

where C is a constant.

Now we need to obtain a particular solution to the inhomogeneous problem. There is a general method for this. But this problem has a particular structure that makes it simple. Look at the right-hand-side of (5.8). It is a constant. The derivative of a constant is zero. Now look at the left hand side of (5.8). It has a term that is a derivative of w, plus a constant times w. Hence, it follows that we can find a solution of the form $w =$ constant. In this case:

$$w_p = -\frac{mg}{\beta}.$$

Then the general solution is:

$$w(t) = Ce^{-\frac{\beta}{m}t} - \frac{mg}{\beta}.$$

Now $w(0) = 0$, so we have:

$$w(0) = C - \frac{mg}{\beta} = 0,$$

or

$$C = \frac{mg}{\beta},$$

and therefore:

$$w(t) = \frac{mg}{\beta} e^{-\frac{\beta}{m}t} - \frac{mg}{\beta}.$$

or

$$\frac{dz}{dt} = w(t) = \frac{mg}{\beta}\left(e^{-\frac{\beta}{m}t} - 1\right).$$

This gives the speed as a function of time. We easily see that there is a limiting speed since:

$$\lim_{t\to\infty} w(t) = \lim_{t\to\infty} \frac{mg}{\beta}\left(e^{-\frac{\beta}{m}t} - 1\right) = -\frac{mg}{\beta}.$$

We could quickly get the acceleration as a function of time by differentiating the expression for the velocity as a function of time:

$$\ddot{z} = -ge^{-\frac{\beta}{m}t}.$$

To obtain the position as a function of time we integrate the expression for the velocity as a function of time:

$$\int_0^t \frac{dz}{d\tau}(\tau)d\tau = \int_0^t \left(\frac{mg}{\beta}\left(e^{-\frac{\beta}{m}\tau} - 1\right)\right)d\tau,$$

which gives:

$$z(t) = h - \frac{mg}{\beta}t - \frac{m^2 g}{\beta^2}\left(e^{-\frac{\beta}{m}t} - 1\right).$$

5. Substitute the proposed solution into the ODE and see if it indeed satisfies the ODE.

(a) We need to show that:

$$m\frac{d^2(k_1 s_1)}{dt^2} - (a_0 + a_1(t))(k_1 s_1) - (b_0 + b_1(t))\frac{d}{dt}(k_1 s_1) = 0.$$

but this is the same as:

$$k_1\left(m\frac{d^2 s_1}{dt^2} - (a_0 + a_1(t))s_1 - (b_0 + b_1(t))\dot{s_1}\right) = 0.$$

and we know that the expression in parentheses is zero since $s_1(t)$ is a solution.

(b) We need to show that:

$$m\frac{d^2(k_1 s_1 + k_2 s_2)}{dt^2} - (a_0 + a_1(t))(k_1 s_1 + k_2 s_2)) - (b_0 + b_1(t))\frac{d}{dt}(k_1 s_1 + k_2 s_2) = 0.$$

but this is the same as:

$$k_1\left(m\frac{d^2 s_1}{dt^2} - (a_0 + a_1(t))s_1 - (b_0 + b_1(t))\dot{s_1}\right) + k_2\left(m\frac{d^2 s_2}{dt^2} - (a_0 + a_1(t))s_2 - (b_0 + b_1(t))\dot{s_2}\right) = 0.$$

and we know that the expressions in parentheses are zero since $s_1(t)$ and $s_2(t)$ are solutions.

(c) No.

(d) What do you know? You know that $\tilde{s}(t)$ is a solution of Newton's equations satisfying $\tilde{s}(0) = 12$ and $\frac{d\tilde{s}}{dt}(0) = 0$. What we would like to find is a solution of Newton's equations, $\hat{s}(t)$ satisfying $\hat{s}(0) = 24$ and $\frac{d\hat{s}}{dt}(0) = 0$. It is very tempting to appeal to the linear properties of the equation by setting $\hat{s}(t) \equiv 2\tilde{s}(t)$,then $\hat{s}(0) = 2\tilde{s}(0) = 24$ and $\frac{d\hat{s}}{dt}(0) = 2\frac{d\tilde{s}}{dt}(0) = 0$. However, this is not correct since these properties only apply to linear *homogeneous* equations. So we have:

$$
\begin{aligned}
\tilde{s}(t) &= \tilde{s}(0) - \frac{1}{2}gt^2 \\
\hat{s}(t) &= \hat{s}(0) - \frac{1}{2}gt^2.
\end{aligned}
$$

The term $\frac{1}{2}gt^2$ is due to the inhomogeneous term in Newton's equations.

6.(a) linear,

(b) nonlinear,

(c) linear,

(d) linear,

(e) nonlinear.

7. In class we showed that the general solution of Newton's equation in one dimension for a constant force is:

$$s(t) = s_0 + v_0(t - t_0) + \frac{F}{2m}(t - t_0)^2.$$

So for this problem we have:

$$s(t) = s_0 + \frac{g}{2m}t^2.$$

8. In class we showed that the general solution of Newton's equation in one dimension for a purely time-dependent force is:

$$s(t) = s_0 + v_0(t - t_0) + \frac{1}{m}\int_{t_0}^{t}\int_{t_0}^{\tau'} F(\tau)d\tau d\tau'.$$

So for this problem we have:

$$s(t) = s_0 + \frac{1}{m}(t - \sin t).$$

Does this result make sense? The force is bounded, and it's average value is zero. Yet, according to the solution for the position as a function of time, the particle moves to infinity as $t \to \infty$.

9. From the lecture in class we define:

$$V(s) = -\int_c^s (s' - s'^2) ds',$$

where c is a "conveniently chosen" constant. Choosing $c = 0$ we have:

$$V(s) = -\frac{s^2}{2} + \frac{s^3}{3}.$$

Then we showed that all solutions must satisfy:

$$\frac{m}{2}\dot{s}^2 - \frac{s^2}{2} + \frac{s^3}{3} = \text{constant}.$$

What do we mean by "all solutions"? Where are the initial conditions? You will see plenty of this later in the course.

10. With $t = \sqrt{m}\tau$ we have:

$$\frac{d}{dt} = \frac{d}{d\tau}\frac{d\tau}{dt} = \frac{1}{\sqrt{m}}\frac{d}{d\tau},$$

and

$$\frac{d^2}{dt^2} = \frac{1}{m}\frac{d^2}{d\tau^2}$$

from which the result easily follows.

11. No. This should be a trivial calculation. In general, superposition does NOT hold for nonlinear ODE's. This is ONE of the major differences. (However, there are certain exceptional situations where nonlinear ODEs can be said to obey a superposition principle.)

6

Solutions for Problem Set 6

1. We did this example in class *without* friction. In this case, in addition to the forces $\mathbf{W}$ and $\mathbf{N}$ acting on P, there is a frictional force $\mathbf{f}$ directed up the incline (in a direction opposite to the motion) and with magnitude:

$$\begin{aligned} \mu N &= \mu m g \cos\alpha, \quad \text{or} \\ \mathbf{f} &= -\mu m g \cos\alpha \mathbf{e}_1. \end{aligned}$$

Using this to modify Newton's equations that we derived in class, you should readily see that:

$$\begin{aligned} m\frac{d^2(s\mathbf{e}_1)}{dt^2} &= \mathbf{W} + \mathbf{N} + \mathbf{f}, \\ &= mg\sin\alpha\mathbf{e}_1 - \mu m g\cos\alpha\mathbf{e}_1. \end{aligned}$$

The acceleration is given by:

$$\frac{d^2 s}{dt^2}\mathbf{e}_1 = g(\sin\alpha - \mu\cos\alpha)\mathbf{e}_1, \tag{6.1}$$

where, recall, s is the distance from the top of the incline. It should be clear that we must have $\sin\alpha > \mu\cos\alpha$ or the frictional force is so great that the particle does not move at all.

Next we compute the velocity. Replacing $\frac{d^2s}{dt^2}$ by $\frac{dv}{dt}$ in (6.1), using the fact that the particle starts from rest (i.e., $v(0) = 0$), and integrating from 0 to t gives the velocity:

$$v\mathbf{e}_1 = g(\sin\alpha - \mu\cos\alpha)t\mathbf{e}_1.$$

Finally, we compute the distance traveled after time t. Replacing v with $\frac{ds}{dt}$ in the above equation, using $s(0) = 0$, and integrating from 0 to t gives:

$$s = \frac{g}{2}(\sin\alpha - \mu\cos\alpha)t^2,$$

where we have dropped $\mathbf{e}_1$ since we are only interested in displacement.

2. In class, we solved for the motion of the projectile in the absence of the incline. We found that the position vector at any time t was given by:

$$\mathbf{r} = (v_0\cos\beta)t\mathbf{j} + \left((v_0\sin\beta)t - \frac{g}{2}t^2\right)\mathbf{k},$$

or, in components,

$$y = (v_0\cos\beta)t, \quad z = (v_0\sin\beta)t - \frac{g}{2}t^2. \tag{6.2}$$

The equation for the incline (which is a line in the $y - z$ plane) is given by:

$$z = y\tan\alpha. \tag{6.3}$$

Substituting (6.2) into (6.3), it follows that the projectiles path and the incline intersect at those values of t for which:

$$(v_0\sin\beta)t - \frac{g}{2}t^2 = ((v_0\cos\beta)t)\tan\alpha,$$

i.e.,

$$t = 0, \quad \text{or} \quad t = \frac{2v_0(\sin\beta\cos\alpha - \cos\beta\sin\alpha)}{g\cos\alpha} = \frac{2v_0\sin(\beta-\alpha)}{g\cos\alpha}.$$

The value $t = 0$ gives the intersection point O. The second value of t gives point A, which is the required point. Using this value of t in the first equation of (6.2), the range of the projectile up the incline is given by:

$$R = y\sec\alpha = (v_0\cos\beta)\left(\frac{2v_0\sin(\beta-\alpha)}{g\cos\alpha}\right)\sec\alpha = \frac{2v_0^2\sin(\beta-\alpha)\cos\beta}{g\cos^2\alpha}.$$

3. Three forces are acting on the object: the weight, $\mathbf{W} = -mg\mathbf{k}$, the normal force $\mathbf{N}$ of the surface on the object, and the frictional force $\mathbf{f}$. Hence, Newton's equations have the form:

$$m\frac{dv}{dt}\mathbf{i} = \mathbf{W} + \mathbf{N} + \mathbf{f}.$$

But $\mathbf{N} = -\mathbf{W}$, and the magnitude of $\mathbf{f}$ is $f = \mu N = \mu mg$ so that $\mathbf{f} = -\mu mg\mathbf{i}$. Then Newton's equations are written as:

$$m\frac{dv}{dt}\mathbf{i} = -\mu mg\mathbf{i}, \quad \text{or} \quad \frac{dv}{dt} = -\mu g. \tag{6.4}$$

Integrating this equation, and using $v = v_0$ at $t = 0$ gives:

$$v = v_0 - \mu gt, \quad \text{or} \quad \frac{dx}{dt} = v_0 - \mu gt. \tag{6.5}$$

Integrating again, using $x = 0$ at $t = 0$ gives:

$$x = v_0 t - \frac{1}{2}\mu g t^2. \tag{6.6}$$

From (6.5), we see that the object comes to rest (i.e. $v = 0$) when:

$$v_0 - \mu gt = 0 \quad \text{or} \quad t = \frac{v_0}{\mu g}.$$

Substituting this time into (6.6), and noting that $x = x_0$ at this time gives:

$$x_0 = \frac{v_0^2}{\mu g} - \frac{1}{2}\mu g\left(\frac{v_0}{\mu g}\right)^2,$$

or

$$\mu = \frac{v_0^2}{2gx_0}.$$

4. $z(t)$ negative is a perfectly valid solution of the differential equation governing the dynamics of the projectile. However, a difficulty arises if we want to use the differential equation to model a particular physical situation. For example, if $z = 0$ is the ground (i.e. the "flat Earth") then we cannot consider situations in which $z(t)$ becomes negative.

5. Using the expression for the position of z as a function of time from the first example of the Week 18 Notes, we have:

$$-H = (v_0 \sin\alpha)t - \frac{g}{2}t^2,$$

or

$$t^2 - \frac{2v_0 \sin\alpha}{g}t - \frac{2H}{g} = 0.$$

Solving this quadratic equation for t gives:

$$t = \frac{v_0 \sin \alpha}{g} \pm \frac{1}{2}\sqrt{\frac{4v_0^2 \sin^2 \alpha}{g^2} + \frac{8H}{g}}.$$

Of the two roots, we take the "plus sign" since the other is negative (and the minus sign, which is perfectly valid from the point of view of the differential equation, is not valid for the physical situation we are modelling):

$$t = \frac{v_0 \sin \alpha}{g} + \sqrt{\frac{v_0^2 \sin^2 \alpha}{g^2} + \frac{2H}{g}}.$$

Now there is a detail we need to check. If the projectile is to go over the "side of the cliff" (and therefore hit the bottom at $z = -H$), the horizontal distance that it travels must be larger than d.

6.(a) $t = \frac{d}{v_0 \cos \alpha}$.

(b) The height that it reaches after this time is:

$$T = d \tan \alpha - \frac{gd^2}{2v_0^2 \cos^2 \alpha}.$$

For the correct physical interpretation, the right hand side of this expression must be positive, i.e.. we must have

$$\tan \alpha > \frac{gd}{2v_0^2 \cos^2 \alpha},$$

or

$$d < \frac{2v_0^2 \sin \alpha \cos \alpha}{g}.$$

(c) The equation to solve for d is:

$$T = d \tan \alpha - \frac{gd^2}{2v_0^2 \cos^2 \alpha}.$$

Using the values of the parameters given in the statement of the problem, we have:

$$10 = d - 0.00098\, d^2,$$

There are two possible values:

$$d = 10.2m \quad \text{and} \quad d = 1010.308m.$$

Does it make sense for there to be two possible values of d? If you think about the shape of the path of the projectile (a parabola) it does make sense, but only if the two values for d are smaller than the range (without the wall being present). Using the parameters given, and the formula for the range from the lecture notes, we compute that $R = 1020.4$m.

7
Solutions for Problem Set 7

1. **F** and the displacement, **r** would be proportional, i.e. lie along the same line, if **r** and $\frac{d^2\mathbf{r}}{dt^2}$ were proportional. However, we know that this is not *generally* the case (although it could be true in special cases, see problem 4 below).

2. A force of this particular form does no work since it is perpendicular to the velocity(think about this in the context of the question above).

3. An issue with both of these questions is how to translate "common language" into mathematical formulae.

   (a) We have proven that the work done by the net forces acting on a particle of constant mass m in moving a particle from a point P_1 to a point P_2 is the kinetic energy of the particle at P_2 minus the kinetic energy of the particle at P_1.

   (b) If we equate motion to nonzero velocity then if there is no motion, there is no velocity (of the particle), and therefore it has no kinetic energy, and therefore no change in kinetic energy is possible.

4.(a) The position vector is:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} = a\cos\omega t\mathbf{i} + b\sin\omega t\mathbf{j},$$

or

$$x = a\cos\omega t, \quad y = b\sin\omega t,$$

which are just the parametric equations of an ellipse having semi-major axis of length a and semi-minor axis of length b. Alternately, since

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \cos^2\omega t + \sin^2\omega t = 1,$$

we also obtain the "other" equation for an ellipse that we usually learn:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

(b) Assuming that the particle has constant mass, the force acting on it is:

$$\begin{aligned}
\mathbf{F} &= m\frac{d^2\mathbf{r}}{dt^2} = m\frac{d^2}{dt^2}\left(a\cos\omega t\mathbf{i} + b\sin\omega t\mathbf{j}\right), \\
&= m\left(-\omega^2 a\cos\omega t\mathbf{i} - \omega^2 b\sin\omega t\mathbf{j}\right), \\
&= -m\omega^2\left(a\cos\omega t\mathbf{i} + b\sin\omega t\mathbf{j}\right) = -m\omega^2\mathbf{r},
\end{aligned}$$

from which it follows immediately that the force is directed towards the origin.

(c) The velocity is given by:

$$\mathbf{v} = -\omega a\sin\omega t\mathbf{i} + \omega b\cos\omega t\mathbf{j}.$$

Therefore the kinetic energy is given by:

$$\frac{1}{2}m\mathbf{v}\cdot\mathbf{v} = \frac{1}{2}m\left(\omega^2 a^2\sin^2\omega t + \omega^2 b^2\cos^2\omega t\right).$$

So we have:

$$\text{Kinetic energy at A (where } \cos\omega t = 1,\ \sin\omega t = 0) = \frac{1}{2}m\omega^2 b^2.$$

$$\text{Kinetic energy at B (where } \cos\omega t = 0,\ \sin\omega t = 1) = \frac{1}{2}m\omega^2 a^2.$$

(d)

$$\begin{aligned}
\text{Work done} &= \int_A^B \mathbf{F}\cdot d\mathbf{r}, \\
&= \int_0^{\frac{\pi}{2\omega}}\left(-m\omega^2\left(a\cos\omega t\mathbf{i} + b\sin\omega t\mathbf{j}\right)\right)\cdot\left(-\omega a\sin\omega t\mathbf{i} + \omega b\cos\omega t\mathbf{j}\right)dt, \\
&= \int_0^{\frac{\pi}{2\omega}} m\omega^3(a^2 - b^2)\sin\omega t\cos\omega t dt, \\
&= \frac{1}{2}m\omega^2(a^2 - b^2)\sin^2\omega t\Big|_0^{\frac{\pi}{2\omega}} = \frac{1}{2}m\omega^2(a^2 - b^2).
\end{aligned}$$

(e) Using the previous two results:

$$\begin{aligned}
\text{Work done} &= \frac{1}{2}m\omega^2(a^2 - b^2) = \frac{1}{2}m\omega^2 a^2 - \frac{1}{2}m\omega^2 b^2, \\
&= \text{kinetic energy at B} - \text{kinetic energy at A.}
\end{aligned}$$

(f) Using the result above from (d), in making a complete circuit around the ellipse we go from $t = 0$ to $t = \frac{2\pi}{\omega}$. Therefore:

$$\begin{aligned}\text{Work done} &= \int_0^{\frac{2\pi}{\omega}} m\omega^3(a^2 - b^2)\sin\omega t\cos\omega t dt, \\ &= \frac{1}{2}m\omega^2(a^2-b^2)\sin^2\omega t\bigg|_0^{\frac{2\pi}{\omega}} = 0.\end{aligned}$$

(g) The force was obtained in b). A direct calculation shows that $\nabla \times \mathbf{F} = 0$.

(h) Since the force is conservative there exists a potential V such that:

$$\mathbf{F} = -m\omega^2 x\mathbf{i} - m\omega^2 y\mathbf{j} = -\nabla V = -\frac{\partial V}{\partial x}\mathbf{i} - \frac{\partial V}{\partial y}\mathbf{j} - \frac{\partial V}{\partial z}\mathbf{k}.$$

Then we have:

$$m\omega^2 x = \frac{\partial V}{\partial x}, \quad m\omega^2 y = \frac{\partial V}{\partial y}, \quad \frac{\partial V}{\partial z} = 0.$$

Solving these equations (and setting the integration constant to zero) gives the potential:

$$V = \frac{1}{2}m\omega^2x^2 + \frac{1}{2}m\omega^2y^2 = \frac{1}{2}m\omega^2(x^2+y^2) = \frac{1}{2}m\omega^2r^2.$$

(i)

$$\begin{aligned}\text{Potential at point A (where r=a)} &= \frac{1}{2}m\omega^2a^2, \\ \text{Potential at point B (where r=b)} &= \frac{1}{2}m\omega^2b^2.\end{aligned}$$

Then we have

$$\begin{aligned}\text{Work done from A to B} &= \text{Potential at A} - \text{Potential at B}, \\ &= \frac{1}{2}m\omega^2a^2 - \frac{1}{2}m\omega^2b^2,\end{aligned}$$

which agrees with the result obtained in d).

5. First we collect together the relevant results from the example that have already been computed in the notes.

$$\begin{aligned}\mathbf{F} &= -mg\mathbf{k}, \\ \mathbf{v} &= v_0\cos\alpha\mathbf{j} + (v_0\sin\alpha - gt)\,\mathbf{k}, \\ \mathbf{r} &= (v_0\cos\alpha)t\mathbf{j} + \left((v_0\sin\alpha)t - \frac{1}{2}gt^2\right)\mathbf{k},\end{aligned}$$

and, if the particle is launched at $t = 0$, the time required for the particle to reach its highest point is:

$$t_h = \frac{v_0 \sin\alpha}{g}.$$

(a) We have:

$$\begin{aligned}
\int_0^{t_h} \mathbf{F}\cdot d\mathbf{r} &= \int_0^{t_h} \mathbf{F}\cdot \frac{d\mathbf{r}}{dt}dt \\
&= \int_0^{t_h} -mg(v_0\sin\alpha - gt)dt, \\
&= -(mg\sin\alpha)t + \frac{mg^2}{2}t^2\Big|_0^{\frac{v_0\sin\alpha}{g}}, \\
&= -\frac{m}{2}v_0^2\sin^2\alpha.
\end{aligned}$$

(b) Let P_1 denote the point where the projectile is launched (the origin) and P_2 denote the highest point of the projectile. Then we have:

$$T_{P_2} - T_{P_1} = \frac{1}{2}mv_0^2\cos^2\alpha - \frac{1}{2}mv_0^2 = -\frac{1}{2}mv_0^2\sin^2\alpha.$$

6. First we collect together the relevant results from the Problem Set 6 Solutions (Problem 1):

$$\begin{aligned}
\mathbf{F} &= (mg\sin\alpha - \mu mg\cos\alpha)\mathbf{e_1}, \\
\mathbf{v} &= (g\sin\alpha - \mu g\cos\alpha)t\,\mathbf{e_1}, \\
s &= \frac{g}{2}\left(\sin\alpha - \mu\cos\alpha\right)t^2,
\end{aligned}$$

The particle starts at rest from the top of the incline. If the incline is of length L, then the time to reach the bottom is obtained by solving:

$$L = \frac{g}{2}\left(\sin\alpha - \mu\cos\alpha\right)t^2,$$

or

$$t_b = \sqrt{\frac{2L}{g(\sin\alpha - \mu\cos\alpha)}}.$$

(a) We have:

$$
\begin{aligned}
\int_0^{t_b} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{t_b} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt, \\
&= \int_0^{t_b} mg^2 (\sin\alpha - \mu\cos\alpha)^2 \, t dt, \\
&= \frac{mg^2}{2} (\sin\alpha - \mu\cos\alpha)^2 \left(\sqrt{\frac{2L}{g(\sin\alpha - \mu\cos\alpha)}} \right)^2, \\
&= mg (\sin\alpha - \mu\cos\alpha) L.
\end{aligned}
$$

(b) The particle starts from rest at the top of the incline, so at $t = 0$ we have $T_{top} = 0$. At the bottom of the incline the velocity is given by:

$$
\mathbf{v}_b = (g\sin\alpha - \mu g\cos\alpha) \sqrt{\frac{2L}{g(\sin\alpha - \mu\cos\alpha)}} \, \mathbf{e_1}
$$

Then the kinetic energy at the bottom of the incline, T_b, is given by:

$$
T_b = \frac{1}{2} m \left((g\sin\alpha - \mu g\cos\alpha) \sqrt{\frac{2L}{g(\sin\alpha - \mu\cos\alpha)}} \right)^2 = mg (\sin\alpha - \mu\cos\alpha) L.
$$

8

Solutions for Problem Set 8

1. First, we write down some preliminary quantities. The initial velocity is given by:

$$\mathbf{v}_0 = v_0 \cos\alpha \mathbf{j} + v_0 \sin\alpha \mathbf{k}. \tag{8.1}$$

We will denote the (unknown) time-dependent position and velocity vectors by:

$$\begin{aligned} \mathbf{r}(t) &= y(t)\mathbf{j} + z(t)\mathbf{k}, & (8.2) \\ \mathbf{v}(t) &= \dot{y}(t)\mathbf{j} + \dot{z}(t)\mathbf{k}. & (8.3) \end{aligned}$$

Now, here is an important point to understand. The only force acting on the projectile is the gravitational force in the vertical direction, i.e., there are *no* forces in the horizontal direction. Now recall *Newton's First Law*:

Every particle persists in a state of rest or of uniform motion in a straight line (i.e., with constant velocity) unless acted upon by a force.

It follows from this that the horizontal component of velocity does not change in time. Hence,

$$\dot{y} = v_0 \cos\alpha.$$

This can be integrated immediately (using $y(0) = 0$) to give:

$$y(t) = (v_0 \cos\alpha)t.$$

Now we return to finding the maximum height reached. We choose the reference point for the gravitational potential energy so that it is zero for $z = 0$. The maximum height is characterized as the point where zero vertical velocity is attained (and we know that the horizontal velocity component is the same as it was initially). So we have:

$$\begin{array}{ccccccc} \text{P.E. at O} & + & \text{K. E. at O} & = & \text{P.E. at max. height} & + & \text{K. E. at max. height} \\ 0 & + & \frac{1}{2}mv_0^2 & = & mgz_{max} & + & \frac{1}{2}m(v_0\cos\alpha)^2, \end{array}$$

So, a bit of easy algebra gives:

$$z_{max} = \frac{v_0^2 \sin^2\alpha}{2g}.$$

Now to finish the problem we need to compute the position vector. Since we have already computed $y(t)$, we need only find $z(t)$. How do we do this using energy? The same way. The total energy at an arbitrary point along the path is $\frac{1}{2}m(\dot{y}^2 + \dot{z}^2) + mgz$. Since energy is conserved we can equation this to the total energy at the origin:

$$\frac{1}{2}m(\dot{y}^2 + \dot{z}^2) + mgz = \frac{1}{2}mv_0^2.$$

Substituting $\dot{y}$ from above gives:

$$\frac{1}{2}m((v_0\cos\alpha)^2 + \dot{z}^2) + mgz = \frac{1}{2}mv_0^2.$$

This gives us an equation for $\dot{z}$ which we can integrate to get $z(t)$:

$$\dot{z} = \sqrt{v_0^2\sin^2\alpha - 2gz},$$

or

$$\int_0^z \frac{dz'}{\sqrt{v_0^2\sin^2\alpha - 2gz'}} = \int_0^t dt' = t.$$

I did this integral in the Week 20 Notes:

$$\int_0^z \frac{dz'}{\sqrt{v_0^2\sin^2\alpha - 2gz'}} = \left.\frac{-2\sqrt{v_0^2\sin^2\alpha - 2gz'}}{2g}\right|_0^z = t.$$

Working through the algebra, you will find:

$$z = (v_0\sin\alpha)t - \frac{1}{2}gt^2.$$

2. The energy at the top of the incline is solely potential energy given by:

$$mg\ell \sin\alpha.$$

The energy of the particle at an arbitrary point on the incline is:

$$\frac{1}{2}m\dot{s}^2 + mg(\ell - s)\sin\alpha.$$

Equating these (because energy is conserved) gives:

$$mg\ell \sin\alpha = \frac{1}{2}m\dot{s}^2 + mg(\ell - s)\sin\alpha.$$

Hence,

$$\dot{s} = \sqrt{2gs\ \sin\alpha}.$$

This can be integrated to find $s(t)$:

$$\int_0^s \frac{ds'}{2s'\ \sin\alpha} = \left.\frac{2s'}{\sqrt{2gs'\ \sin\alpha}}\right|_0^s = \int_0^t dt' = t.$$

After some algebra you get:

$$s = \frac{1}{2}g\sin\alpha t^2.$$

3.(a) Since there is no net force acting on the particles we use conservation of momentum and conclude that:

$$mv_0 + MV_0 = mv + MV. \tag{8.4}$$

This answer is correct. However, we are overlooking a point that deserves further thought. Certainly when the particles are not in contact, by assumption, there are no forces acting on either particle. But what about at the instant of contact? Certainly each particle exerts a force on the other (or else the momentum of each particle could not change, even though the total momentum is unchanged). The key here is to consider Newtons third law of motion.

(b) Using (8.4), we have:

$$\dot{\xi} = \frac{m\dot{x} + M\dot{X}}{m+M} = \frac{mv + MV}{m+M} = \frac{mv_0 + MV_0}{m+M} = \text{constant}.$$

(c) We will use momentum conservation and energy conservation to obtain two (linear) equations to solve for the two unknowns v and V. We first rewrite

$$\frac{1}{2}mv_0^2 + \frac{1}{2}MV_0^2 = \frac{1}{2}mv^2 + \frac{1}{2}MV^2. \tag{8.5}$$

as

$$m(v^2 - v_0^2) = M(V_0^2 - V^2), \tag{8.6}$$

and (8.4) as:

$$m(v - v_0) = M(V_0 - V). \tag{8.7}$$

Dividing (8.6) by (8.7) gives:

$$v + v_0 = V_0 + V. \tag{8.8}$$

Therefore (8.4) and (8.8) give the following pair of equations to solve for the unknowns v and V:

$$\begin{aligned} mv + MV &= mv_0 + MV_0, \\ v - V &= -v_0 + V_0. \end{aligned} \tag{8.9}$$

These two equations can easily be solved for v and V to obtain the result:

$$\begin{aligned} v &= \frac{m-M}{m+M}v_0 + \frac{2M}{m+M}V_0, \\ V &= \frac{M-m}{m+M}V_0 + \frac{2m}{m+M}v_0. \end{aligned} \tag{8.10}$$

(d) Using (8.10), we see that for $m = M$ it follows that $v = V_0$ and $V = v_0$. In other words, after the collision the mass on the left moves with the initial velocity of the mass on the right, and the mass on the right moves with the initial velocity of the mass on the left.

(e) We compute the limit as $M \to \infty$ to obtain:

$$\begin{aligned} v &= -v_0 + 2V_0, \\ V &= V_0. \end{aligned} \tag{8.11}$$

If $V_0 = 0$ we see that the small mass "bounces off" the large mass and reverses its direction with the same speed, but velocity in the opposite direction.

9
Solutions for Problem Set 9

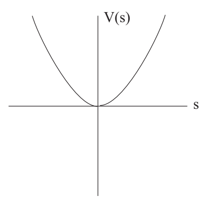
1.(a)

$$\begin{aligned}\dot{s} &= v, \\ \dot{v} &= -s.\end{aligned}$$

(b) $V(s) = \frac{s^2}{2}$ sketched in Fig. 9.1.

(c) Stable equilibria at $(s, v) = (0, 0)$ (relative minima of the potential).

(d) Phase portrait sketched in Fig. 9.1.



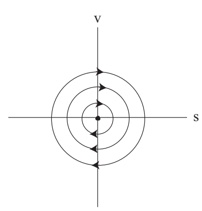


Figure 9.1: Graph of the potential energy function and the phase portrait.

(e) Using the expression derived from the energy integral in class:

$$\int_0^s \frac{ds'}{\sqrt{2E - s'^2}} = \sin^{-1} \frac{s'}{\sqrt{2E}}\bigg|_0^s = t.$$

or

$$s = \sqrt{2E} \sin t, \quad \text{and} \quad v = \dot{s}.$$

2.(a)

$$\begin{aligned}\dot{s} &= v, \\ \dot{v} &= s - s^3.\end{aligned}$$

(b) $V(s) = -\frac{s^2}{2} + \frac{s^4}{4}$ sketched in Fig. 9.2.

(c) Stable equilibria at $(s, v) = (\pm 1, 0)$ (relative minima of the potential), a saddle point at $(s, v) = (0, 0)$ (relative maxima of the potential).

(d) Phase portrait sketched in Fig. 9.2.

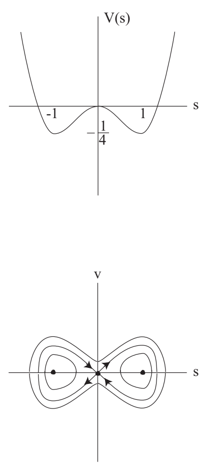


Figure 9.2: Graph of the potential energy function and the phase portrait.

3.(a)

$$\begin{aligned}\dot{s} &= v, \\ \dot{v} &= s - s^2.\end{aligned}$$

(b) $V(s) = -\frac{s^2}{2} + \frac{s^3}{3}$ sketched in Fig. 9.3.

(c) Stable equilibrium at $(s, v) = (1, 0)$ (relative minima of the potential), a saddle point at $(s, v) = (0, 0)$ (relative maxima of the potential).

(d) Phase portrait sketched in Fig. 9.3.

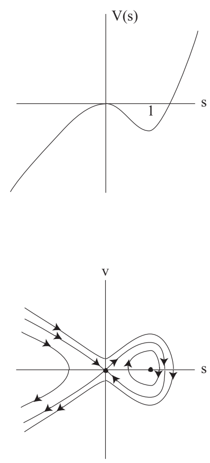


Figure 9.3: Graph of the potential energy function and the phase portrait.

10

Solutions for Problem Set 10

1. We have

$$v = \pm\sqrt{\frac{2}{m}}\sqrt{E - V(s)}.$$

It follows that:

$$\frac{dv}{ds} = \mp\sqrt{\frac{1}{2m\,(E - V(s))}}\frac{dV}{ds}(s).$$

At the crossing point (or, as I referred to it in class, turning point), $v = 0$, $E = V(s)$. However, we need to take into account both branches of this function, i.e, the $\pm$ signs. Look at each branch separately and consider the limiting behaviour of the slope as the s axis is approached.

The question could also be answered by implict differentiation. A level set of the energy is given by:

$$H(s, v) = \frac{1}{2}mv^2 + V(s) = E.$$

Implictly differentiating with respect to s gives:

$$mv\frac{dv}{ds} + \frac{dV}{ds} = 0,$$

or

$$\frac{dv}{ds} = -\frac{1}{mv}\frac{dV}{ds}$$

which is infinite at $v = 0$. (In order to draw this conclusion, do we have to say something about $\frac{dV}{ds}$ at the turning point?)

Make sure you understand what it means, and what we are assuming, when we implicitly differentiate the energy function. If you don't, ask for an explanation.

2. The phase portrait does not change at all in the sense of the geometry of the level sets of the energy function. However, the *value* of the energy for the different level sets changes according to the value of the constant.

3. Yes. Substitute this (constant) function into the left hand side, and right had side, of Newton's equations and show equality. What do the left and right hand sides equal?

4. See Fig. 10.1. There are four equilibria: two stable and two unstable.

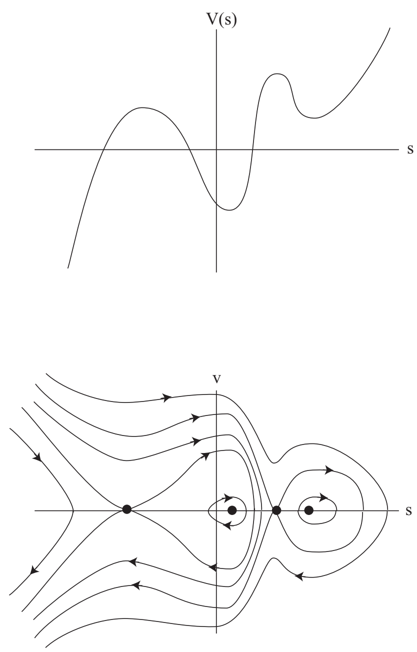


Figure 10.1: Graph of the potential energy function and the phase portrait.

5. See Fig. 10.2. There are three equilibria: two unstable and one stable.

6. See Fig. 10.3. There are no equilibria.

7. See Fig. 10.4. There is one unstable equilibrium.

8.
$$\mathbf{\Lambda} = \frac{d\mathbf{\Omega}}{dt} = 12t\mathbf{i} - 2\mathbf{j} + (36t^2 - 16t)\mathbf{k}.$$

evaluating this expression at $t = 1$ gives:

$$12\mathbf{i} - 2\mathbf{j} + 20\mathbf{k}.$$

9.(a) From Newton's second law:

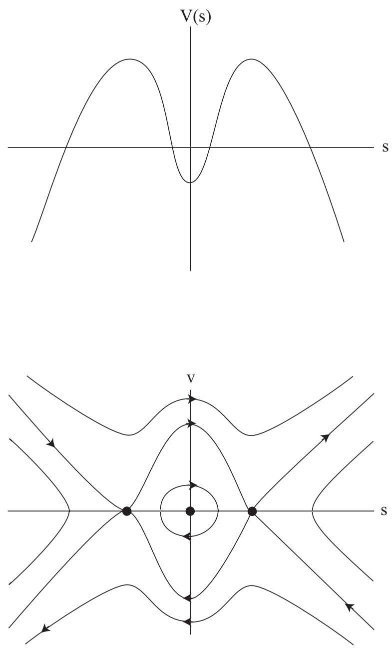


Figure 10.2: Graph of the potential energy function and the phase portrait.

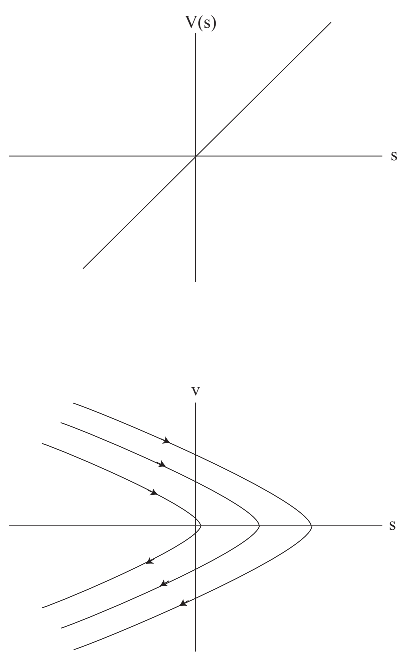


Figure 10.3: Graph of the potential energy function and the phase portrait.

$$2\frac{d\mathbf{v}}{dt} = 24t^2\mathbf{i} + (36t - 16)\mathbf{j} - 12t\mathbf{k},$$

or,

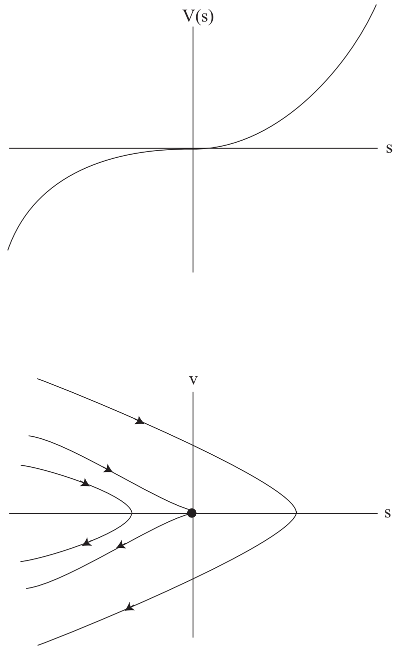


Figure 10.4: Graph of the potential energy function and the phase portrait.

$$\frac{d\mathbf{v}}{dt} = 12t^2\mathbf{i} + (18t - 8)\mathbf{j} - 6t\mathbf{k}.$$

Integrating this expression with resepct to t gives:

$$\mathbf{v} = 4t^3\mathbf{i} + (9t^2 - 8t)\mathbf{j} - 3t^2\mathbf{k} + \mathbf{c}_1.$$

At $t = 0$, $\mathbf{v} = \mathbf{v}_0$, and therefore:

$$\mathbf{v}_0 = 6\mathbf{i} + 15\mathbf{j} - 8\mathbf{k} = \mathbf{c}_1,$$

and we have:

$$\mathbf{v} = (4t^3 + 6)\mathbf{i} + (9t^2 - 8t + 15)\mathbf{j} - (3t^2 + 8)\mathbf{k}.$$

(b) Using the previous result:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (4t^3 + 6)\mathbf{i} + (9t^2 - 8t + 15)\mathbf{j} - (3t^2 + 8)\mathbf{k}.$$

Integrating this expression gives:

$$\mathbf{r} = (t^4 + 6t)\mathbf{i} + (3t^3 - 4t^2 + 15t)\mathbf{j} - (t^3 + 8t)\mathbf{k} + \mathbf{c}_2.$$

At $t = 0$, $\mathbf{r} = \mathbf{r}_0$, and therefore:

$$\mathbf{r} = \mathbf{r}_0 = 3\mathbf{i} - \mathbf{j} + 4\mathbf{k} = \mathbf{c}_2,$$

and we have:

$$\mathbf{r} = (t^4 + 6t + 3)\mathbf{i} + (3t^3 - 4t^2 + 15t - 1)\mathbf{j} - (t^3 + 8t - 4)\mathbf{k}.$$

(c)

$$\begin{aligned}
\mathbf{\Lambda} &= \mathbf{r} \times \mathbf{F} = \Big((t^4 + 6t + 3)\mathbf{i} + (3t^3 - 4t^2 + 15t - 1)\mathbf{j} - (t^3 + 8t - 4)\mathbf{k}\Big) \times \Big(24t^2\mathbf{i} + (36t - 16)\mathbf{j} - 12t\mathbf{k}\Big), \\
&= \Big(32t^3 + 108t^2 - 260t + 64\Big)\,\mathbf{i} - \Big(12t^5 + 192t^3 - 168t^2 - 36t\Big)\,\mathbf{j} \\
&- \Big(36t^5 - 80t^4 + 360t^3 - 240t^2 - 12t + 48\Big)\,\mathbf{k}
\end{aligned}$$

(d)

$$\begin{aligned}
\mathbf{\Omega} &= \mathbf{r} \times (m\mathbf{v}) = m(\mathbf{r} \times \mathbf{v}) = 2\left((t^4 + 6t + 3)\mathbf{i} + (3t^3 - 4t^2 + 15t - 1)\mathbf{j} - (t^3 + 8t - 4)\mathbf{k}\right) \\
&\times \left((4t^3 + 6)\mathbf{i} + (9t^2 - 8t + 15)\mathbf{j} - (3t^2 + 8)\mathbf{k}\right), \\
&= (8t^4 + 36t^3 - 130t^2 + 64t - 104)\mathbf{i} - (2t^6 + 48t^4 - 56t^3 - 18t^2 - 96)\mathbf{j} \\
&- (6t^6 - 16t^5 + 90t^4 - 80t^3 - 6t^2 + 48t - 102)\mathbf{k}
\end{aligned}$$

10. The time rate of change of the angular momentum about the origin is given by the tirque about the origin. Therefore we only need to show that the torque about the origin is zero. This is a trivial computation:

$$\mathbf{\Lambda} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times f(r)\frac{\mathbf{r}}{r} = \frac{f(r)}{r}(\mathbf{r} \times \mathbf{r}) = 0.$$

11. It follows from the previous problem that:

$$\mathbf{r} \times \mathbf{F} = 0,$$

and therefore

$$\mathbf{r} \times m\frac{d\mathbf{v}}{dt} = 0,$$

or

$$\mathbf{r} \times \frac{d\mathbf{v}}{dt} = 0,$$

which is the same as (why?)

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{v}) = 0.$$

Integrating this equation with respect to time gives:

$$\mathbf{r} \times \mathbf{v} = \mathbf{h}, \tag{10.1}$$

where $\mathbf{h}$ is a constant vector. Now $\mathbf{r} \times \mathbf{v}$ is perpendicular to $\mathbf{r}$ (why?). Therefore taking the dot product of both sides of (10.1) with $\mathbf{r}$ gives:

$$\mathbf{r} \cdot (\mathbf{r} \times \mathbf{v}) = 0 = \mathbf{r} \cdot \mathbf{h}.$$

Therefore the position vector is always perpendicular to the constant vector $\mathbf{h}$, so that the motion is always in a plane.

11

Solutions for Problem Set 11

1. The magnitude of the areal velocity is given by $\frac{1}{2}|\mathbf{r} \times \mathbf{v}|$. Hence, we need to compute $\mathbf{r} \times \mathbf{v}$ in cartesian coordinates, and then compute the magnitude of the resulting vector.

$$\mathbf{r} \times \mathbf{v} = (x\mathbf{i} + y\mathbf{j}) \times (\dot{x}\mathbf{i} + \dot{y}\mathbf{j}) = x\dot{y}\mathbf{k} - y\dot{x}\mathbf{k}.$$

   Then

$$|\mathbf{r} \times \mathbf{v}| = x\dot{y} - y\dot{x}.$$

2. We start with the equation derived in class:

$$\ddot{r} - \frac{h^2}{r^3} = \frac{f(r)}{m}. \tag{11.1}$$

   We need two preliminary relations. From $r^2\dot{\theta} = h$ we have:

$$\dot{\theta} = \frac{h}{r^2}. \tag{11.2}$$

   Differentiating $r^2\dot{\theta} = h$ with respect to t gives:

$$2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = 0,$$

   or

$$\ddot{\theta} = -\frac{2\dot{\theta}}{r}\dot{r} = -\frac{2h}{r^3}\dot{r}. \tag{11.3}$$

   Now we use the chain rule:

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{dr}{d\theta} = \frac{h}{r^2}\frac{dr}{d\theta}. \tag{11.4}$$

$$
\begin{aligned}
\frac{d^2r}{dt^2} &= \left(\frac{d}{dt}\left(\frac{dr}{d\theta}\right)\right)\dot{\theta} + \frac{dr}{d\theta}\ddot{\theta}, \\
&= \frac{d^2r}{d\theta^2}\dot{\theta}^2 + \frac{dr}{d\theta}\ddot{\theta}, \\
&= \frac{h^2}{r^4}\frac{d^2r}{d\theta^2} - \frac{2h^2}{r^5}\left(\frac{dr}{d\theta}\right)^2, \quad \text{where we have used (11.2), (11.3) and (11.4)} \qquad (11.5)
\end{aligned}
$$

Now substituting (11.5) into (11.1) gives the result.

3. This result uses conservation of energy. From class we derived the following equation that expresses conservation of energy for a particle moving in a central force field:

$$\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \int f(r)dr = E.$$

Substituting $\dot{\theta} = \frac{h}{r^2}$ into this equation gives:

$$\frac{1}{2}m\left(\dot{r}^2 + \frac{h^2}{r^2}\right) - \int f(r)dr = E,$$

or

$$\dot{r}^2 = \frac{2E}{m} + \frac{2}{m}\int f(r)dr - \frac{2h^2}{mr^2} \equiv G(r).$$

From this expression we obtain:

$$\frac{dr}{dt} = \sqrt{G(r)},$$

or

$$t = \int \frac{1}{\sqrt{G(r)}}dr.$$

The second equation follows by writing $\dot{\theta} = \frac{h}{r^2}$ as:

$$dt = \frac{1}{h}r^2 d\theta.$$

4.(a) The potential is given by:

$$V(r) = \int \frac{K}{r^2}dr = -\frac{K}{r}.$$

(b) The work done is given by:

$$V(r=a) - V(r=b) = \frac{K}{b} - \frac{K}{a} = \frac{K(a-b)}{ab}.$$

5. From

$$r^2\dot{\theta} = h = \text{constant},$$

we derive the quadrature:

$$\int d\theta = h \int \frac{dt}{r(t)^2}.$$

6. In the lectures we showed that:

$$\mathbf{r} \times \mathbf{v} = r^2\dot{\theta}\mathbf{k}.$$

Hence, $mr^2\dot{\theta}\mathbf{k}$ is the angular momentum of the particle about O.