Appendix A

Proofs related to the sign of $\partial U_R/\partial\rho$

By definition, all the variables and parameters are positive, $\gamma \in (-\infty, 1)$ and the term $\left\{\bar{L} - 1/\left[\phi\left(1/\rho + 1\right)\right]\right\}$ is also positive.

Let

$$f\left(\rho; \bar{L}, k, \phi, \gamma\right) = k\left[\bar{L} - \frac{1}{\phi}\left(\frac{1}{\rho} + 1\right)\right]^{\gamma-1}\frac{1}{\phi\rho^2}$$

and

$$g\left(\rho; Q_R, m, \gamma\right) = mQ_P^{\gamma}\left(\frac{1}{1+\rho}\right)^{\gamma+1}.$$

Equation (7) then becomes

$$\frac{\partial U_R}{\partial\rho} = X\left(\rho; Q_R, Q_P, \bar{L}, m, k, \rho, \gamma\right)\cdot\left[f\left(\rho; \bar{L}, k, \phi, \gamma\right) - g\left(\rho; Q_P, m, \gamma\right)\right]. \tag{8}$$

A.1 Initial Value of $\partial U_R/\partial\rho$

When ρ is relatively small, in which case the home country's tourism development is in its early stage, let ρ be $1/\left(\phi\bar{L} - 1 - \triangle\right)$ where $\triangle$ is an extremely small number.

$$X\left(\rho \to \frac{1}{\phi\bar{L}-1}; Q_R, Q_P, \bar{L}, m, k, \phi, \gamma\right) = \left\{Q_R^{\gamma} + k\left(\frac{\triangle}{\phi}\right)^{\gamma} + m\left[\frac{Q_P\left(\phi\bar{L} - 1 - \triangle\right)}{\phi\bar{L} - \triangle}\right]^{\gamma}\right\}^{\frac{1-\gamma}{\gamma}},$$

$$f\left(\rho \to \frac{1}{\phi\bar{L}-1}; \bar{L}, k, \phi, \gamma\right) = k\left(\frac{\triangle}{\phi}\right)^{\gamma-1}\frac{\left(\phi\bar{L} - 1 - \triangle\right)^2}{\phi},$$

and

$$g\left(\rho \to \frac{1}{\phi\bar{L}-1}; Q_P, m, \gamma\right) = mQ_P^{\gamma}\left(\frac{\phi\bar{L}-1-\triangle}{\phi\bar{L}-\triangle}\right)^{\gamma+1}.$$

When $\gamma \in (-\infty, 0]$, as $\triangle \to 0$, it can be show that

$$\lim_{\rho \to \frac{1}{\phi\bar{L}-1}} X\left(\rho; Q_R, Q_P, \bar{L}, m, k, \phi, \gamma\right) = 0^+,$$

$$\lim_{\rho \to \frac{1}{\phi\bar{L}-1}} f\left(\rho; \bar{L}, k, \phi, \gamma\right) = +\infty,$$

and

$$\lim_{\rho \to \frac{1}{\phi\bar{L}-1}} g\left(\rho; Q_P, m, \gamma\right) = mQ_P^{\gamma}\left(\frac{\phi\bar{L}-1}{\phi\bar{L}}\right)^{\gamma+1} > 0.$$

When $\gamma \in (0, 1)$, as $\triangle \to 0$, it can be show that

$$\lim_{\rho \to \frac{1}{\phi\bar{L}-1}} X\left(\rho; Q_R, Q_P, \bar{L}, m, k, \phi, \gamma\right) = \left\{Q_R^{\gamma} + m\left[\frac{Q_P\left(\phi\bar{L}-1\right)}{\phi\bar{L}}\right]^{\gamma}\right\}^{\frac{1-\gamma}{\gamma}} > 0,$$

$$\lim_{\rho \to \frac{1}{\phi\bar{L}-1}} f\left(\rho; \bar{L}, k, \phi, \gamma\right) = +\infty,$$

and

$$\lim_{\rho \to \frac{1}{\phi\bar{L}-1}} g\left(\rho; Q_P, m, \gamma\right) = mQ_P^{\gamma}\left(\frac{\phi\bar{L}-1}{\phi\bar{L}}\right)^{\gamma+1} > 0.$$

Therefore, across the domain of γ, we have

$$\lim_{\rho \to \frac{1}{\phi\bar{L}-1}} \frac{\partial U_R}{\partial \rho} > 0.$$

A.2 Infinity approximation of $\partial U_R/\partial \rho$

With the development of home country's tourism industry, the number of tourists, ρ, increases. As ρ grow large and approaches infinity, when $\gamma \in (-\infty, -1]$,

$$\lim_{\rho \to +\infty} X\left(\rho; Q_R, Q_P, \bar{L}, m, k, \phi, \gamma\right) = 0^+,$$
$$\lim_{\rho \to +\infty} f\left(\rho; \bar{L}, k, \phi, \gamma\right) = 0^+,$$

and

$$\lim_{\rho \to +\infty} g\left(\rho; Q_P, m, \gamma\right) = +\infty.$$

In terms of convergence speed, we have

$$X\left(\rho; Q_R, Q_P, \bar{L}, m, k, \phi, \gamma\right) = \mathcal{O}\left(\frac{1}{\rho^{1-\gamma}}\right),$$
$$f\left(\rho; \bar{L}, k, \phi, \gamma\right) = \mathcal{O}\left(\frac{1}{\rho^2}\right),$$

and

$$\frac{1}{g\left(\rho; Q_P, m, \gamma\right)} = \mathcal{O}\left(\frac{1}{\rho^{1+\gamma}}\right).$$

Therefore, we have

$$\lim_{\rho \to +\infty} X \cdot f = 0^+,$$

and

$$\lim_{\rho \to +\infty} (-X \cdot g) = 0^-,$$

with

$$X \cdot f = \mathcal{O}\left(\frac{1}{\rho^{2-2\gamma}}\right),$$

and

$$(-X \cdot g) = \mathcal{O}\left(\frac{1}{\rho^{-2\gamma}}\right).$$

Since $2 - 2\gamma > -2\gamma > 0$, $X \cdot f$ converges to zero faster than $(-X \cdot g)$. That is, $(X \cdot f - X \cdot g)$ becomes negative before it converges to zero.

When $\gamma \in (-1, 0]$, we have

$$\lim_{\rho \to +\infty} X\left(\rho; Q_R, Q_P, \bar{L}, m, k, \phi, \gamma\right) = 0^+,$$
$$\lim_{\rho \to +\infty} f\left(\rho; \bar{L}, k, \phi, \gamma\right) = 0^+,$$

and

$$\lim_{\rho \to +\infty} g\left(\rho; Q_P, m, \gamma\right) = 0^+,$$

with

$$X\left(\rho; Q_R, Q_P, \bar{L}, m, k, \phi, \gamma\right) = \mathcal{O}\left(\frac{1}{\rho^{1-\gamma}}\right),$$
$$f\left(\rho; \bar{L}, k, \phi, \gamma\right) = \mathcal{O}\left(\frac{1}{\rho^{2}}\right),$$

and

$$g\left(\rho; Q_P, m, \gamma\right) = \mathcal{O}\left(\frac{1}{\rho^{1+\gamma}}\right).$$

We therefore have

$$\lim_{\rho \to +\infty} X \cdot f = 0^{+},$$

and

$$\lim_{\rho \to +\infty} (-X \cdot g) = 0^{-},$$

with

$$X \cdot f = \mathcal{O}\left(\frac{1}{\rho^{2(1-2\gamma)}}\right),$$

and

$$(-X \cdot g) = \mathcal{O}\left(\frac{1}{\rho^{(1-\gamma)(1+\gamma)}}\right).$$

Since $2(1-\gamma) > (1-\gamma)(1+\gamma) > 0$, $X \cdot f$ converges to zero faster than $(-X \cdot g)$. That is, once again, $(X \cdot f - X \cdot g)$ becomes negative before it converges to zero.

When $\gamma \in (0, 1)$,

$$\lim_{\rho \to +\infty} X\left(\rho; Q_R, Q_P, \bar{L}, m, k, \phi, \gamma\right) = \left\{Q_R^{\gamma} + k\left[\bar{L} - \frac{1}{\phi}\right]^{\gamma}\right\}^{\frac{1-\gamma}{\gamma}} > 0,$$
$$\lim_{\rho \to +\infty} f\left(\rho; \bar{L}, k, \phi, \gamma\right) = 0^{+},$$

and

$$\lim_{\rho \to +\infty} g\left(\rho; Q_P, m, \gamma\right) = 0^{+},$$

with

$$f\left(\rho; \bar{L}, k, \phi, \gamma\right) = \mathcal{O}\left(\frac{1}{\rho^2}\right),$$

and

$$g\left(\rho; Q_P, m, \gamma\right) = \mathcal{O}\left(\frac{1}{\rho^{1+\gamma}}\right).$$

We therefore have

$$\lim_{\rho \to +\infty} X \cdot f = 0^+,$$

and

$$\lim_{\rho \to +\infty} (-X \cdot g) = 0^-,$$

with

$$X \cdot f = \mathcal{O}\left(\frac{1}{\rho^2}\right),$$

and

$$(-X \cdot g) = \mathcal{O}\left(\frac{1}{\rho^{(1+\gamma)}}\right).$$

Since $2 > (1+\gamma) > 0$, $X \cdot f$ converges to zero faster than $(-X \cdot g)$. That is, similar to previous results, $(X \cdot f - X \cdot g)$ becomes negative before it converges to zero.

In general, across the domain of γ,

$$\lim_{\rho \to +\infty} \frac{\partial U_R}{\partial \rho} = 0^-.$$

A.3 Proof of the existence and uniqueness of ρ^*

The solution of setting Equation (8) to zero determines the value of ρ^*. Since $X\left(\rho; Q_R, Q_P, \bar{L}, m, k, \phi, \gamma\right)$ is strictly positive, the equation simplifies into

$$f\left(\rho^*; \bar{L}, k, \phi, \gamma\right) = g\left(\rho^*; Q_P, m, \gamma\right),$$

or

$$k\left[\bar{L} - \frac{1}{\phi}\left(\frac{1}{\rho^*} + 1\right)\right]^{\gamma-1} \frac{1}{\phi \rho^{*2}} = mQ_p^\gamma \left(\frac{1}{1+\rho^*}\right)^{\gamma+1}. \tag{9}$$

Since both sides of Equation (9) are positive, take natural logarithm on both sides, we have

$$\ln\left(\frac{k}{\phi}\right) + (\gamma-1)\ln\left[\bar{L} - \frac{1}{\phi}\left(\frac{1}{\rho^*} + 1\right)\right] - 2\ln\rho^* = \ln\left(mQ_P^\gamma\right) - (\gamma+1)\ln\left(1+\rho^*\right).$$

Rearranging terms, we have

$$(\gamma-1)\ln\left[\bar{L} - \frac{1}{\phi}\left(\frac{1}{\rho^*} + 1\right)\right] - 2\ln\rho^* + (\gamma+1)\ln\left(1+\rho^*\right) + \ln\left(\frac{k}{m\phi Q_P^\gamma}\right) = 0.$$

Define

$$F(\rho) = (\gamma-1)\ln\left[\bar{L} - \frac{1}{\phi}\left(\frac{1}{\rho} + 1\right)\right] - 2\ln\rho + (\gamma+1)\ln\left(1+\rho\right) + \ln\left(\frac{k}{m\phi Q_p^\gamma}\right).$$

We can derive

$$\frac{\partial F(\rho)}{\partial \rho} = \frac{\frac{(\gamma-1)(1+\rho)}{\phi\rho} + \left[(\gamma-1)\rho - 2\right]\left[\bar{L} - \frac{1}{\phi}\left(\frac{1}{\rho}+1\right)\right]}{\left[\bar{L} - \frac{1}{\phi}\left(\frac{1}{\rho}+1\right)\right]\rho\left(1+\rho\right)}.$$

Since

$$\left[\bar{L} - \frac{1}{\phi}\left(\frac{1}{\rho} + 1\right)\right] > 0,$$

$$(\gamma - 1) < 0,$$

and

$$\rho\,(1 + \rho) > 0.$$

We can conclude that

$$\frac{\partial F\,(\rho)}{\partial \rho} < 0.$$

When ρ is relatively small,

$$\lim_{\rho \to \frac{1}{\phi \bar{L} - 1}} F\,(\rho) = (\gamma - 1)\ln(0) - 2\ln\left(\frac{1}{\phi \bar{L} - 1}\right) + (\gamma + 1)\ln\left(\frac{\phi \bar{L}}{\phi \bar{L} - 1}\right) + \ln\left(\frac{k}{m\phi Q_P^{\gamma}}\right).$$

Since $\lim_{\triangle \to 0} \ln(\triangle) = -\infty$ and $(\gamma - 1) < 0$, we have

$$\lim_{\rho \to \frac{1}{\phi \bar{L} - 1}} F\,(\rho) = +\infty.$$

Rearranging $F\,(\rho)$, we can have

$$F\,(\rho) = (\gamma - 1)\left\{\ln\left[\bar{L} - \frac{1}{\phi}\left(\frac{1}{\rho} + 1\right)\right] + \ln(1 + \rho)\right\} + 2\ln\left(\frac{1}{\rho} + 1\right) + \ln\left(\frac{k}{m\phi Q_P^{\gamma}}\right).$$

When ρ approaches positive infinity,

$$\lim_{\rho \to +\infty} F\,(\rho) = (\gamma - 1)\left\{\ln\left(\bar{L} - \frac{1}{\phi}\right) + \ln(\infty)\right\} + 2\ln(1) + \ln\left(\frac{k}{m\phi Q_P^{\gamma}}\right).$$

Since $\ln(\infty) \to +\infty$ and $(\gamma - 1) < 0$, we have

$$\lim_{\rho \to +\infty} F(\rho) = -\infty.$$

According to intermediate value theorem, since $F(\rho)$ is continuous and monotonic decreasing, $\lim_{\rho \to \frac{1}{\phi \bar{L} - 1}} F(\rho) = +\infty$, and $\lim_{\rho \to +\infty} F(\rho) = -\infty$, there must exist one unique solution to the equation $F(\rho) = 0$. That is, there exists one unique critical value of ρ, ρ^*.

With the derivative first being positive, equaling to zero at ρ^*, and approaching zero from negative side when ρ is large, we can derive a utility representation as illustrated in Figure 1.

Appendix B

Proofs of the Influencing Factors of ρ^*

B.1 Change of ρ^* with respect to Q_P

Assume Equation (8) equals to zero and take natural logarithm on both sides, rearranging terms, we can have

$$\ln Q_P = \frac{1}{\gamma} + \frac{\gamma - 1}{\gamma} \ln \left[\bar{L} - \frac{1}{\phi}\left(\frac{1}{\rho^*} + 1\right)\right] - \frac{2}{\gamma} \ln \rho^* + \frac{\gamma + 1}{\gamma} \ln \left(1 + \rho^*\right).$$

It can be shown that,

$$\frac{\partial Q_P}{\partial \rho^*} = e^{\ln Q_P} \cdot \frac{\partial \ln Q_P}{\partial \rho^*} = e^{\ln Q_P} \frac{1}{\gamma} \frac{\partial F\left(\rho^*\right)}{\partial \rho^*}.$$

Appendix A.3 shows that $\left(\partial F\left(\rho^*\right) / \partial \rho^*\right) < 0$. That is, when $\gamma < 0$, we have $\left(\partial Q_P / \partial \rho^*\right) > 0$, and when $\gamma \in (0, 1)$, we have $\left(\partial Q_P / \partial \rho^*\right) < 0$. Since Q_P is a monotonic function of ρ^*, by chain rule, we can conclude that $\left(\partial \rho^* / \partial Q_P\right) > 0$ when $\gamma < 0$, and $\left(\partial \rho^* / \partial Q_P\right) < 0$ when $\gamma \in (0, 1)$.

In the special case of $\gamma = 0$, the CES utility function will become Cobb-Douglas type,

$$U_R = Q_R \left(\bar{L} - L\right)^k \left(\frac{Q_P}{q + \rho}\right)^m.$$

The optimal level of ρ, ρ^*, is therefore,

$$\rho^* = \frac{1 + \sqrt{1 + 4\phi}}{2\phi},$$

which is not a function of Q_P. We therefore have $\left(\partial \rho^* / \partial Q_P\right) = 0$ when $\gamma = 0$.

In general, $\left(\partial \rho^* / \partial Q_P\right) > 0$ when $\gamma < 0$, $\left(\partial \rho^* / \partial Q_P\right) = 0$ when $\gamma = 0$, and

$(\partial\rho^*/\partial Q_P) < 0$ when $\gamma \in (0, 1)$.

B.2 Change of ρ^* with respect to ϕ

Rearranging terms of $F(\rho)$ from Appendix A.3, we can have

$$F(\rho) = (\gamma - 1)\ln\left[\bar{L} - \frac{1}{\phi}\left(\frac{1}{\rho} + 1\right)\right] - 2\ln\rho + (\gamma + 1)\ln(1 + \rho) + \ln\left(\frac{k}{m}\right) - \ln\phi - \gamma\ln Q_P.$$

As proved in Appendix A.3,

$$\frac{\partial F(\rho^*, \phi, Q_P, \gamma)}{\partial\rho^*} < 0.$$

It can also be shown that

$$\frac{\partial F(\rho^*, \phi, Q_P, \gamma)}{\partial\phi} = -\frac{1}{\phi}\left[\frac{\bar{L} - \gamma\frac{1}{\phi}\left(\frac{1}{\rho^*} + 1\right)}{\bar{L} - \frac{1}{\phi}\left(\frac{1}{\rho^*} + 1\right)}\right] < 0.$$

Therefore, by chain rule,

$$\frac{\partial\rho^*}{\partial\phi} = -\frac{\partial F(\rho^*, \phi, Q_P, \gamma)/\partial\phi}{\partial F(\rho^*, \phi, Q_P, \gamma)/\partial\rho^*} < 0.$$

In the special case of $\gamma = 0$,

$$\frac{\partial\rho^*}{\partial\phi} = -\left(\frac{1 + 2\phi}{\sqrt{1 + 4\phi}} + 1\right)\frac{1}{2\phi^2} < 0.$$

We therefore have $(\partial\rho^*/\partial\phi) < 0$ for $\gamma \in (-\infty, 1)$.