THE ELEMENTARY CONSTRUCTION OF FORMAL ANAFUNCTORS

DAVID MICHAEL ROBERTS

Abstract. These notes give an elementary and formal 2-categorical construction of the bicategory of anafunctors, starting from a 2-category equipped with a family of covering maps that are fully faithful.

1. Introduction

Anafunctors were introduced by Makkai [Mak96] as new 1-arrows in the 2-category **Cat** to talk about category theory in the absence of the axiom of choice. The aim was to make functorial those constructions that are only defined by some universal property, rather than by some specified operation. One also recovers the characterisation of equivalences of categories as essentially surjective, fully faithful 1-arrows. The construction by Bartels [Bar06] of the analogous bicategory $\mathbf{Cat}_{ana}(S, J)$, whose 1-arrows are anafunctors, starting from the 2-category $\mathbf{Cat}(S)$ of internal categories was extended in [Rob12] to variable full sub-2-categories $\mathbf{Cat}'(S) \hookrightarrow \mathbf{Cat}(S)$. The canonical inclusion 2-functor $\mathbf{Cat}'(S) \hookrightarrow \mathbf{Cat}'_{ana}(S, J)$ was there shown to be a 2-categorical localisation in the sense of Pronk [Pro96] at the fully faithful functors which are locally weakly split in the given pretopology J.

In these notes I show that given a 2-cateory K equipped with a strict subcanonical singleton pretopology J whose elements are fully faithful arrows, one can construct an analogue K_J of the bicategory $\mathbf{Cat}'_{ana}(S, J)$. The 1-arrows of K_J are formal 2-categorical versions of anafunctors, here dubbed J-*fractions*. The construction of K_J is elementary in the sense of only needing the first-order theory of 2-categories, and the construction is Choice-free. The original 2-category K is a wide and locally full sub-bicategory of K_J and the inclusion 2-functor $A_\mathsf{J} \colon K \hookrightarrow K_\mathsf{J}$ is a bicategorical localisation; this result uses Pronk's comparison theorem from [Pro96], but it should be possible to prove directly using the construction given here.

The following quote from [Sim06] should be kept in mind when reading the elementary calculations in these notes, as no such details have fully appeared in the literature, let alone at the level of generality here:

> Nonetheless, it is interesting to note the prevalence of formulations leaving "to the reader" parts of the proofs of details of the localization constructions. ... Another interesting reference is Pronk's paper on localization of 2-categories [21][1], pointed out to me by I. Moerdijk. This paper constructs the localization of a 2-category by a subset of 1-morphisms satisfying a generalization of the right fraction condition. ... the full set of details for the coherence relations on the level of 2-cells is still too much, so the paper ends with:
>
> [21 , p. 302:] "It is left to the reader to verify that the above defined isomorphisms a, l and r are natural in their arguments and satisfy the identity coherence axioms."

One pleasant feature of the current approach, at least for the author, is that one could take the opposite 2-category everywhere in the current notes and everything will still work fine, so one could also localise suitable 2-categories using cospans, rather than spans, for instance 2-categories whose

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[1][Pro96] in the References of these notes.

objects are more algebraic in nature, rather than geometric, like Hopf algebroids. It is not clear that for alternative presentations of the localisation, say one using of left-principal bibundles, such an approach would still work, or what should play the rôle of the 1-arrows when one is working with a general 2-category and not a 2-category of structured or internal groupoids.

2. Preliminaries

We refer to [Lei98], or the original [Bén67], for background on bicategories.

Definition 2.1. If $\mathcal{P}$ is a property of functors, then we say that a 1-arrow $f\colon x \to y$ in a 2-category K is *representably* $\mathcal{P}$ if for all objects z of K we have that $f_*\colon K(z,x) \to K(z,y)$ has property $\mathcal{P}$.

The most important case for the present paper is the property 'fully faithful', and we will abbreviate 'representably fully faithful' to ff, and will denote by ff the class of ff 1-arrows in a 2-category. Note however that Definition 2.1 can be rewritten as a first-order property of a 1-arrow in a 2-category.

Definition 2.2. A 1-arrow $f\colon x \to y$ in a 2-category is ff if for all $g,h\colon w \to x$ and $\widetilde{a}\colon f\circ g \Rightarrow f \circ h$ there is a unique $a\colon g \Rightarrow h$ such that $\widetilde{a} = 1_f \circ a$.

It is easy to see that any equivalence in a 2-category is ff. More generally, if f has a pseudo-retract, in the sense that there is an arrow $g\colon y \to x$ such that $g \circ f \simeq \mathrm{id}_x$, then $f \in \mathsf{ff}$. Note that this definition and these simple examples work fine in any bicategory as well, as does the following lemma.

Lemma 2.3. *If $f\colon y \to z \in \mathsf{ff}$ and $g\colon x \to y$ is any other arrow then $g \in \mathsf{ff}$ if and only if $f \circ g \in \mathsf{ff}$. If $h\colon y \to z$ is another arrow that is isomorphic to f in $K(y,z)$ then $h \in \mathsf{ff}$. Moreover, if f is isomorphic to $f'\colon y' \to z'$ (in the arrow 2-category) $f' \in \mathsf{ff}$.*

The following lemma will be a major workhorse in the construction below.

Lemma 2.4. *Let $u \to a$ be an ff 1-arrow in a 2-category K. Then for any 1-arrow $f\colon b \to a$ and any two lifts $k,l\colon b \to u$, there is a unique 2-arrow $k \Rightarrow l$ covering the identity 2-arrow on f.*

The proof of this lemma follows almost immediately from the definition of ff.

Example 2.5. Given a commutative triangle

$$\begin{tikzcd} u \arrow[rr, "\phi"] \arrow[dr] & & v \arrow[dl] \\ & x & \end{tikzcd}$$

with $v \to x$ ff, there is the equality

$$\begin{tikzcd} u \times_x u \arrow[r, bend left, "\mathrm{pr}_1"] \arrow[r, bend right, "\mathrm{pr}_2"'] \arrow[r, phantom, "\Downarrow"] & u \arrow[r, "\phi"] & v \end{tikzcd} \quad = \quad \begin{tikzcd} u \times_x u \arrow[r, "\phi\times\phi"] & v \times_x v \arrow[r, bend left, "\mathrm{pr}_1"] \arrow[r, bend right, "\mathrm{pr}_2"'] \arrow[r, phantom, "\Downarrow"] & v \end{tikzcd}$$

In particular, if $\phi = \mathrm{id}_u$, there is an invertible 2-arrow $\mathrm{pr}_1 \Rightarrow \mathrm{pr}_2\colon u \times_x u \to u$.

Example 2.6. A more complicated example is

$$\begin{tikzcd} & & u \times_x u \arrow[dr, bend left, "\mathrm{pr}_1"] \arrow[dr, bend right, "\mathrm{pr}_2"'] \arrow[dr, phantom, "\Downarrow"] & \\ u \times_x u \arrow[r, "\Delta\times\mathrm{id}_u"] & u \times_x u \times_x u \arrow[ur, bend left, "\mathrm{pr}_{12}"] \arrow[dr, bend right, "\mathrm{pr}_{23}"'] & & u \\ & & u \times_x x \arrow[ur, bend left, "\mathrm{pr}_1"] \arrow[ur, bend right, "\mathrm{pr}_2"'] \arrow[ur, phantom, "\Downarrow"] & \end{tikzcd} \quad = \quad \begin{tikzcd} u \times_x u \arrow[r, bend left, "\mathrm{pr}_1"] \arrow[r, bend right, "\mathrm{pr}_2"'] \arrow[r, phantom, "\Downarrow"] & u \end{tikzcd}$$

The structure of a site on a 2-category is not a common notion so we need to specify what we mean. There are at least two different ways to describe this in the 1-categorical case, namely using sieves and using pretopologies, and it is not clear a priori that they generalise to the same thing for 2-categories. Our definition will be as follows, as this paper only deals with unary sites.

Definition 2.7. A *singleton strict pretopology* on a 2-category K is a class J of 1-arrows which contains all identity arrows, is closed under composition and the strict pullback an element of J exists and is again in J. We will assume that *specified* strict pullbacks are given—rather than merely assuming they exist—and that the pullback of an identity 1-arrow is again an identity 1-arrow.

Since this is the same thing as a singleton pretopology on the 1-category underlying the 2-category, we refrain from placing the prefix '2-' in the name. If one merely asks for existence of pullbacks, then one may use a global axiom of choice to make the pullback of a cover an operation.

Example 2.8. Let K be a 2-category which admits specified strict pullbacks. Then ff is a singleton strict pretopology.

This is in some sense a degenerate example. The following is more of interest.

Example 2.9. Let S be a finitely complete category with specified limits and J_0 a singleton pretopology on S. Then we have the 2-categories $\mathbf{Cat}(S)$ and $\mathbf{Gpd}(S)$ of internal categories and groupoids. Let J denote the class of internal functors either of those 2-categories whose object component is an arrow in J_0. Then J is a singleton strict pretopology on both $\mathbf{Cat}(S)$ and $\mathbf{Gpd}(S)$.

In addition, we need to consider a 2-categorical version of subcanonicity, and here we cannot avoid involving the 2-arrows. This makes the notion essentially 2-categorical, and not just a structure on the underlying 1-category as is the case for definition 2.7.

Definition 2.10. A singleton strict pretopology J is called a *totally strict subcanonical singleton pretopology* if for every $j\colon u \to x$ in J, and every object y

$$j^*\colon K(x,y) \to \mathrm{StrDesc}(u,y)$$

is a fully faithful functor.

Here $\mathrm{StrDesc}(u,y)$ (strict descent data with values in y) is the equaliser of $K(u,y) \rightrightarrows K(u^{[2]},y)$, where the two arrows are induced by the projections. This can be described as the subcategory of $K(u,y)$ with objects those 1-arrows $f\colon u \to y$ in K whose precompositions with the two projections $u \times_x u \to u$ are equal, and as arrows those 2-arrows $a\colon f \Rightarrow g$ in K whose whiskerings with the two projections are equal.

Unpacking this definition, we have the following elementary definition.

Definition 2.11. An arrow $p\colon u \to x$ in a 2-category is *strictly 2-regular* if for every pair of arrows $f, g\colon x \to y$ and every 2-arrow $\widetilde{a}\colon f \circ p \Rightarrow g \circ p\colon u \to y$ satisfying

$$\begin{tikzcd} u \times_x u \arrow[r, "\mathrm{pr}_1"] & u \arrow[r, bend left, "f\circ p"] \arrow[r, bend right, "g\circ p"'] \arrow[r, phantom, "\Downarrow\widetilde{a}"] & y \end{tikzcd} \quad = \quad \begin{tikzcd} u \times_x u \arrow[r, "\mathrm{pr}_2"] & u \arrow[r, bend left, "f\circ p"] \arrow[r, bend right, "g\circ p"'] \arrow[r, phantom, "\Downarrow\widetilde{a}"] & y \end{tikzcd} \tag{1}$$

there is a *unique* 2-arrow $a\colon f \Rightarrow g\colon x \to y$ such that

$$\begin{tikzcd} u \arrow[r, bend left, "f\circ p"] \arrow[r, bend right, "g\circ p"'] \arrow[r, phantom, "\Downarrow\widetilde{a}"] & y \end{tikzcd} \quad = \quad \begin{tikzcd} u \arrow[r, "p"] & x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow a"] & y \end{tikzcd}$$

We can swap out the condition for on a when $p \in \mathsf{ff}$, using Lemma 2.4, which is ultimately the case we are interested in, by the following lemma.

Lemma 2.12. *Given a 1-arrow $j \in \mathsf{ff}$ in a 2-category K, it is strictly 2-regular if and only if $j^*\colon K(x,y) \to K(u,y)$ is fully faithful. Or, in elementary terms, an ff 1-arrow is strictly 2-regular precisely when the condition in Definition 2.11 holds for* any *2-arrow $\widetilde{a}\colon f \Rightarrow g$, not just those satsifying the condition (1).*

Proof. If condition that $j^*\colon K(x,y) \to K(u,y)$ is fully faithful holds, then for a 2-arrow $\widetilde{a}$ as in Definition 2.11, we can forget the fact that it satisfies equation (1), and still get the unique descended 2-arrow. Conversely, assuming Definition 2.11 holds for $j\colon u \to x$ and $j \in \mathsf{ff}$, consider an arbitrary 2-arrow $\widetilde{a}\colon f \circ j \Rightarrow g \circ j\colon u \to y$. Recall from Lemma 2.4 that there is a canonical 2-arrow $\mathrm{pr}_1 \Rightarrow \mathrm{pr}_2\colon u \times_x u \to u$ such that right whiskering this with j gives the identity 2-arrow on $u \times_x u \to x$. Then:

$$
\begin{tikzcd}
u \times_x u \arrow[r, "\mathrm{pr}_1"] & u \arrow[r, bend left, "f\circ j"] \arrow[r, bend right, "g\circ j"'] \arrow[r, phantom, "\Downarrow\widetilde{a}"] & y
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & & x \arrow[dr, "f"] & \\
u \times_x u \arrow[urr, bend left, "j\circ \mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & u \arrow[ur, "j"] \arrow[rr, bend right, "g\circ j"'] \arrow[rr, phantom, "\Downarrow\widetilde{a}"] & & y
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
u \times_x u \arrow[r, bend left, "\mathrm{pr}_1"] \arrow[r, bend right, "\mathrm{pr}_2"'] \arrow[r, phantom, "\Downarrow"] & u \arrow[r, bend left, "f\circ j"] \arrow[r, bend right, "g\circ j"'] \arrow[r, phantom, "\Downarrow\widetilde{a}"] & y
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
u \times_x u \arrow[r, "\mathrm{pr}_1"] \arrow[drr, bend right, "j\circ \mathrm{pr}_2"'] & u \arrow[rr, bend left, "f\circ j"] \arrow[dr, "j"] \arrow[rr, phantom, "\Downarrow\widetilde{a}"] & & y \\
 & & x \arrow[ur, "g"'] &
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
u \times_x u \arrow[r, "\mathrm{pr}_2"] & u \arrow[r, bend left, "f\circ j"] \arrow[r, bend right, "g\circ j"'] \arrow[r, phantom, "\Downarrow\widetilde{a}"] & y
\end{tikzcd}
\; .
$$

Hence as we are assuming j is strictly 2-regular, there is a unique 2-arrow $a\colon f \Rightarrow g$ that is the descent of $\widetilde{a}$ along j, as required. □

The main object of study of this paper are 2-categories K with a choice of totally strict subcanonical singleton pretopology J, which we shall call *strict subcanonical unary 2-sites.* We shall just refer to these as 2-sites for brevity. In fact we only work with those 2-sites for which $\mathsf{J} \subset \mathsf{ff}$: our constructions rely on this property.

The next example partly recovers the examples that were used in [Rob12, §8]; variants on this definition will give all examples from *loc. cit.*

Example 2.13. Continuing example 2.9, let $\widetilde{\mathsf{J}}_{\mathsf{ff}} = \widetilde{\mathsf{J}} \cap \mathsf{ff}$ for a site (S, J) with J subcanonical. Then $\mathbf{Cat}(S)$ and $\mathbf{Gpd}(S)$ are strict subcanonical unary 2-sites taking $\widetilde{\mathsf{J}}_{\mathsf{ff}}$ for our pretopology.

3. The bicategory of J-fractions

We are aiming to localise a 2-category, and in time-honoured tradition we shall call the arrows in the localised 2-category fractions. Fractions are defined relative to a strict pretopology.

Definition 3.1. Let K be a 2-category and J a totally strict subcanonical singleton pretopology on K. A J-*fraction* is a span $x \xleftarrow{j} u \xrightarrow{f} y$ in K where $j \in \mathsf{J}$, to be denoted (j, f).

For example, given any 1-arrow $f\colon x \to y$ in K, we have the fraction (id_x, f), where x covers itself by the identity arrow. In particular, we have for any object a the *identity fraction*, which is $(\mathrm{id}_x, \mathrm{id}_x)$

Definition 3.2. A *map of* J*-fractions* $(j, f) \Rightarrow (k, g)$ is a diagram of the form

$$\begin{tikzcd} x & u \times_a v \arrow[l] \arrow[r, bend left, "f \circ \mathrm{pr}_1"] \arrow[r, bend right, "g \circ \mathrm{pr}_2"'] \arrow[r, phantom, "\Downarrow"] & y \end{tikzcd},$$

There are certain maps of fractions which are easier to describe and to compose, and the coherence maps of the bicategory we are going to define all turn out to be examples, so we shall spend some time detailing these.

Definition 3.3. A *renaming map* r from the fraction (j, f) to the fraction (k, g) is a map of spans in a 2-category of the form:

$$\begin{tikzcd} & u \arrow[dl, "j"'] \arrow[dr, "f"] \arrow[dd, "r"] & \\ x & \arrow[r, phantom, "\Downarrow a_r"] & y \\ & v \arrow[ul, "k"] \arrow[ur, "g"'] & \end{tikzcd}$$

We can compose renaming maps and so get a category $K_{\mathsf{J}}^R(x, y)$ with objects the fractions from x to y and arrows the renaming maps.

As we shall see, we will also have a category with objects the J-fractions and arrows the maps of fractions, and a functor invcluding $K_{\mathsf{J}}^R(x, y)$ into this latter category. For now we will be content with giving the definition of the arrow component of this functor, without proving functoriality; namely, a renaming map $r\colon (u, f) \to (v, g)$ is sent to the map $\iota(r)$ of fractions,

(2)
$$\begin{tikzcd} & & u \arrow[dr, "f"] \arrow[dd, "r"] & \\ x & v \times_x u \arrow[l] \arrow[ur, "\mathrm{pr}_1"] \arrow[dr, "\mathrm{pr}_2"'] \arrow[r, phantom, "\Downarrow"] & \arrow[r, phantom, "\Downarrow a_r"] & y \\ & & v \arrow[ur, "g"'] & \end{tikzcd}.$$

where the 2-arrow on the left is the canonical lift of the identity 2-arrow on the 1-arrow $u \times_x v \to x$ in the diagram

$$\begin{tikzcd} u \times_x v \arrow[r, "r \times \mathrm{id}_v"] \arrow[drr, bend right] & v \times_x v \arrow[r, bend left, "\mathrm{pr}_1"] \arrow[r, bend right, "\mathrm{pr}_2"'] \arrow[r, phantom, "\Downarrow"] & v \arrow[d, "j"] \\ & & x \end{tikzcd}$$

using Lemma 2.4.

Definition 3.4. The identity map $1\colon (j, f) \to (j, f)$ on a J-fraction $x \xleftarrow{j} u \xrightarrow{f} y$ is given by $\iota(\mathrm{id}_u)$.

The (vertical) composition of maps of J-fractions proceeds as follows. Given

$$\begin{aligned} t_1 &\colon (j_1, f_1) \to (j_2, f_2) \\ t_2 &\colon (j_2, f_2) \to (j_3, f_3) \end{aligned}$$

where $x \xleftarrow{j_i} u_i \xrightarrow{f_i} b$, consider the 2-arrow $t_1 \oplus t_2$ filling the diagram

$$
\begin{tikzcd}
 & u_1 \times_x u_2 \arrow[r] \arrow[dr] & u_1 \arrow[dr, "f_1"] \arrow[d, phantom, "\Downarrow t_1"] & \\
u_1 \times_x u_2 \times_x u_3 \arrow[ur] \arrow[dr] & & u_2 \arrow[r, "f_2"] \arrow[d, phantom, "\Downarrow t_2"] & y \\
 & u_2 \times_x u_3 \arrow[r] \arrow[ur] & u_3 \arrow[ur, "f_3"'] &
\end{tikzcd}
\quad , \tag{3}
$$

Which we shall call the *precomposition* of t_1 and t_2. We need to show that this 2-arrow descends along the arrow $u_1 \times_x u_2 \times_x u_3 \xrightarrow{\mathrm{pr}_{13}} u_1 \times_a u_3 \in \mathsf{J}$. But pr_{13} is strictly 2-regular, and the source and target of $t_1 \oplus t_2$ factor as $u_1 \times_x u_2 \times_x u_3 \xrightarrow{\mathrm{pr}_{13}} u_1 \times_x u_3 \to u_i \xrightarrow{f_i} y$ for $i = 1$ and $i = 3$ respectively, hence we can apply Lemma 2.12 to $t_1 \oplus t_2$. Thus $t_1 \oplus t_2$ descends uniquely, and we call this descended 2-arrow $t_1 + t_2$ (note that $+$ is *not* a commutative operation!), and it is a map of J-fractions $(u_1, f_1) \to (u_3, f_3)$.

Remark 3.5. If $u_1 \times_x u_2 \times_x u_3 \to u_1 \times_x u_3$ has a section, then the vertical composition $t_1 + t_2$ is the whiskering of $t_1 \oplus t_2$ on the left with this section.

Proposition 3.6. *We have a category $K_{\mathsf{J}}(x, y)$ with objects the J-fractions from x to y and arrows the maps of J-fractions.*

Proof. We first show that $1_{(j,f)}$ is the identity arrow for $x \xleftarrow{j} u \xrightarrow{f} y$. Consider the map of J-fractions

$$
\begin{tikzcd}
x & u \times_x v \arrow[l] \arrow[r, bend left, "f \circ \mathrm{pr}_1"] \arrow[r, bend right, "g \circ \mathrm{pr}_2"'] \arrow[r, phantom, "\Downarrow t"] & y
\end{tikzcd}
\ .
$$

Then $1_{(j,f)} \oplus t$ looks like

$$
\begin{tikzcd}
 & u \times_x u \arrow[dr, bend left, "\mathrm{pr}_1"] \arrow[dr, bend right, "\mathrm{pr}_2"'] \arrow[dr, phantom, "\Downarrow"] & & \\
u \times_x u \times_x v \arrow[ur, "\mathrm{pr}_{12}"] \arrow[dr, "\mathrm{pr}_{23}"'] & & u \arrow[r, "f"] \arrow[d, phantom, "\Downarrow"] & y \\
 & u \times_x v \arrow[ur] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}
\tag{4}
$$

and we want to show that

$$
\begin{tikzcd}
u \times_x u \times_x v \arrow[r, "\mathrm{pr}_{13}"] & u \times_x v \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow t"] & y
\end{tikzcd}
$$

is equal to (4), since then the 2-arrow in (4) which descends to be $1_{(j,f)} + t$, actually descends to be t. Note that by Lemma 2.4 we have

$$
\begin{tikzcd}
 & u \times_x u \arrow[dr, bend left, "\mathrm{pr}_1"] \arrow[dr, bend right, "\mathrm{pr}_2"'] \arrow[dr, phantom, "\Downarrow"] & \\
u \times_x u \times_x v \arrow[ur, "\mathrm{pr}_{12}"] \arrow[dr, "\mathrm{pr}_{23}"'] & & u \\
 & u \times_x v \arrow[ur] &
\end{tikzcd}
=
\begin{tikzcd}
u \times_x u \times_x v \arrow[r, bend left, "\mathrm{pr}_{13}"] \arrow[r, bend right, "\mathrm{pr}_{23}"'] \arrow[r, phantom, "\Downarrow"] & u \times_x v \arrow[r, "\mathrm{pr}_1"] & u
\end{tikzcd}
\ .
$$

Hence (4) is equal to

$$
\begin{tikzcd}
u \times_x u \times_x v \arrow[r, bend left, "\mathrm{pr}_{13}"] \arrow[r, bend right, "\mathrm{pr}_{23}"'] \arrow[r, phantom, "\Downarrow"] & u \times_x v \arrow[r, bend left, "\tilde{f}"] \arrow[r, bend right, "\tilde{g}"'] \arrow[r, phantom, "\Downarrow t"] & y
\end{tikzcd} . \tag{5}
$$

But note that we can decompose this into the vertical composition of t whiskered on the left by pr_{13} and the canonical 2-arrow whiskered on the right by $\tilde{g}$. This latter 2-arrow is

$$
\begin{tikzcd}
u \times_x u \times_x v \arrow[r, bend left, "\mathrm{pr}_{13}"] \arrow[r, bend right, "\mathrm{pr}_{23}"'] \arrow[r, phantom, "\Downarrow"] & u \times_x v \arrow[r, "\mathrm{pr}_2"] & v \arrow[r, "g"] & y
\end{tikzcd}
$$

which is the identity 2-arrow on $\mathrm{pr}_3 \colon u \times_x u \times_x v \to v$ whiskered on the right with g. Thus (5) is equal to the desired 2-arrow, and $1_{(j,f)}$ is a left identity for vertical composition. A symmetric argument will show that it is also a right identity.

We now need to show composition is associative. Consider the diagram

$$
\begin{tikzcd}
 & & \boldsymbol{u_{12}} \arrow[rr] \arrow[dr] & & \boldsymbol{u_1} \arrow[dd, "f_1"] \arrow[dl, phantom, "\Downarrow t_1"] \\
 & \boldsymbol{u_{123}} \arrow[ur] \arrow[dr] & & \boldsymbol{u_2} \arrow[dr, "f_2"] & \\
u_{1234} \arrow[ur] \arrow[dr] & & \boldsymbol{u_{23}} \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\Downarrow t_2"] & & \boldsymbol{y} \\
 & u_{234} \arrow[ur] \arrow[dr] & & \boldsymbol{u_3} \arrow[ur, "f_3"'] & \\
 & & u_{34} \arrow[ur] \arrow[rr] & & u_4 \arrow[uu, "f_4"'] \arrow[ul, phantom, "\Downarrow t_3"]
\end{tikzcd}
$$

We will show that the composite

$$
\begin{tikzcd}
u_{234} \arrow[r] & u_{14} \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_4"'] \arrow[r, phantom, "\Downarrow a"] & y
\end{tikzcd}
$$

is equal to (3) for both $a = (t_1 + t_2) + t_3$ and $a = t_1 + (t_2 + t_3)$. First consider $(t_1 + t_2) + t_3$:

$$
\begin{tikzcd}
u_{1234} \arrow[r] & u_{14} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow (t_1+t_2)+t_3"] & y
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
u_{1234} \arrow[r] & u_{134} \arrow[r] & u_{14} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow (t_1+t_2)+t_3"] & y
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & & & u_1 \arrow[ddr] \arrow[dd, phantom, "\Downarrow t_1+t_2"] & \\
 & & & u_{13} \arrow[ur] \arrow[dr] & & \\
= & u_{1234} \arrow[r] & u_{134} \arrow[ur] \arrow[dr] & & u_3 \arrow[r] \arrow[dd, phantom, "\Downarrow t_3"] & y \\
 & & & u_{34} \arrow[ur] \arrow[dr] & & \\
 & & & & u_4 \arrow[uur] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & & & \boldsymbol{u_1} \arrow[ddr] \arrow[dd, phantom, "\Downarrow t_1+t_2"] & \\
 & & \boldsymbol{u_{123}} \arrow[r] & \boldsymbol{u_{13}} \arrow[ur] \arrow[dr] & & \\
= & u_{1234} \arrow[r] \arrow[ur] & u_{134} \arrow[ur] \arrow[dr] & & \boldsymbol{u_3} \arrow[r] \arrow[dd, phantom, "\Downarrow t_3"] & \boldsymbol{y} \\
 & & & u_{34} \arrow[ur] \arrow[dr] & & \\
 & & & & u_4 \arrow[uur] &
\end{tikzcd}
$$

ommitting some of the labels on the 1-arrows for clarity. Now the whiskered 2-arrow in the subdiagram on the bold symbols above is equal to the composite 2-arrow in the subdiagram of (3) on the bold symbols, hence the whole diagram equals (3). A symmetric argument shows that $t_1+(t_2+t_3)\circ 1_{u_{1234}\to u_{14}}$ is also equal to (3). By uniqueness of descent, composition of maps of J-spans is associative, and $K_{\mathsf{J}}(x,y)$ is a category. □

3.1. **Defining the bicategory K_{J}.** Now we want to show that $K_{\mathsf{J}}(x,y)$ is the hom-category of a bicategory, so we need a composition functor. Composing 1-arrows is easy:

Definition 3.7. The composition of J-fractions is the composite span

$$
\begin{tikzcd}
 & & u \times_y v \arrow[dl] \arrow[dr] & & \\
 & u \arrow[dl] \arrow[dr] & & v \arrow[dl] \arrow[dr] & \\
x & & y & & z
\end{tikzcd}
$$

where recall we are assuming we have specified pullbacks of 1-arrows in J, so this is well-defined.

We shall define the composition in the bicategory K_{J} by defining left and right whiskering functors and proving the interchange law as outlined in [Mak96, pp 126-127][2] for the case where $K = \mathbf{Cat}$ and J is the class of fully faithful, surjective-on-objects functors. Let t be a map of fractions from $x \leftarrow u \xrightarrow{f} y$ to $x \leftarrow v \xrightarrow{g} y$.

[2]Makkai says, helpfully, "Next, we need to verify that thus we have defined functors. . . we leave the task to the reader." [*ibid.* page 127]

Definition 3.8. The *right whiskering* of t by the J-fraction $y \xleftarrow{l} w \xrightarrow{h} z$ is given by

$$\begin{tikzcd}
 & & & w \times_{y,f} u \arrow[dr, "\widetilde{f}"] \arrow[dd, phantom, "\Downarrow \rho_{(w,h)}t"] & & \\
x & u \times_x v \arrow[l] & w \times_{y,f} (u \times_x v) \times_{g,y} w \arrow[l] \arrow[ur] \arrow[dr] & & w \arrow[r, "h"] & z \\
 & & & v \times_{g,y} w \arrow[ur, "\widetilde{g}"'] & &
\end{tikzcd}$$

where the 2-arrow $\rho_{(w,h)}t \colon \mathrm{pr}_1 \Rightarrow \mathrm{pr}_4$ is the unique lift through $l \colon w \to y$ of

$$\begin{tikzcd}
 & & u \arrow[dr, "f"] \arrow[dd, phantom, "\Downarrow t"] & \\
w \times_{y,f} (u \times_x v) \times_{g,y} w \arrow[r] & u \times_x v \arrow[ur] \arrow[dr] & & y \ . \\
 & & v \arrow[ur, "g"'] &
\end{tikzcd}$$

Proposition 3.9. *Right whiskering with $y \leftarrow w \xrightarrow{h} z$ is a functor $K_{\mathsf{J}}(x,y) \to K_{\mathsf{J}}(x,z)$.*

Proof. First, let us show right whiskering preserves identity 2-arrows. That is, the horizontal composition of a pair of identity 2-arrows is the identity 2-arrow of the composition of the 1-arrows. Let $x \xleftarrow{j} u \xrightarrow{f} y$ be a pair of fraction and consider the right whiskering of the map $\mathrm{id}_{(u,f)}$ by (w,h). This is the map of fractions given by

$$\begin{tikzcd}
x & w \times_y u^{[2]} \times_y w \arrow[l] \arrow[r, bend left, "\mathrm{pr}_1"] \arrow[r, bend right, "\mathrm{pr}_4"'] \arrow[r, phantom, "\Downarrow"] & w \arrow[r, "h"] & z
\end{tikzcd} \tag{6}$$

where the 2-arrow is the unique lift of

$$\begin{tikzcd}
w \times_y u^{[2]} \times_y w \arrow[r] & u^{[2]} \arrow[r, bend left, "\mathrm{pr}_1"] \arrow[r, bend right, "\mathrm{pr}_2"'] \arrow[r, phantom, "\Downarrow"] & u \arrow[r, "f"] & y \ ,
\end{tikzcd}$$

the unlabelled maps being the obvious projections. But we have the equality

$$\begin{tikzcd}
w \times_y u^{[2]} \times_y w \arrow[r, bend left, "\mathrm{pr}_1"] \arrow[r, bend right, "\mathrm{pr}_4"'] \arrow[r, phantom, "\Downarrow"] & w
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
w \times_y u^{[2]} \times_y w \arrow[r, bend left, "\mathrm{pr}_{12}"] \arrow[r, bend right, "\mathrm{pr}_{34}"'] \arrow[r, phantom, "\Downarrow"] & u \times_x y \arrow[r, "\tilde{f}"] & w
\end{tikzcd}$$

hence (6) is

$$\begin{tikzcd}
x & w \times_y u^{[2]} \times_y w \arrow[l] \arrow[r, bend left, "\mathrm{pr}_{12}"] \arrow[r, bend right, "\mathrm{pr}_{34}"'] \arrow[r, phantom, "\Downarrow"] & u \times_y w \arrow[r, "h \circ \tilde{f}"] & z
\end{tikzcd}
\quad = \quad \mathrm{id}_{x \leftarrow u \times_x w \to z} \ .$$

Thus whiskering is unital.

Now to prove that right whiskering preserves composition we will again use uniqueness of descent, and prove equal a pair of 2-arrows with 0-source a cover of the 0-source of the 2-arrows we are interested in. Without loss of generality, we can right whisker by the fraction $y \leftarrow w \xrightarrow{\mathrm{id}_w} w$, as the component of the fraction pointing the 'correct' way plays no role in what is to follow.

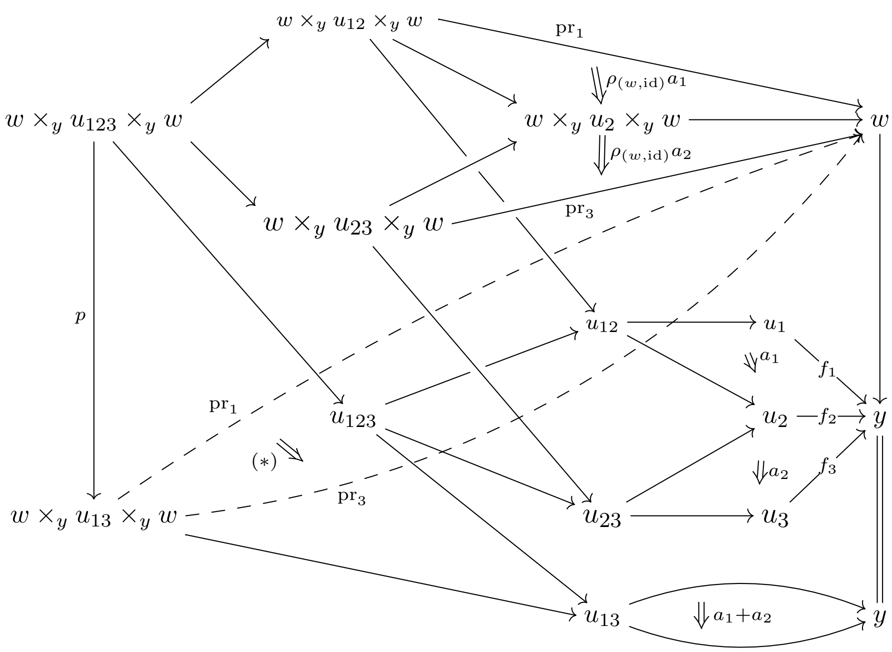

Figure 1. Right whiskering is functorial

Consider the composable pair of maps of fractions

$$\begin{tikzcd} x & u_1 \times_x u_2 \arrow[l] \arrow[r, bend left, "\tilde{f}_1"] \arrow[r, bend right, "\tilde{f}_2"'] \arrow[r, phantom, "\Downarrow a_1"] & y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} x & u_2 \times_x u_3 \arrow[l] \arrow[r, bend left, "\tilde{f}_2"] \arrow[r, bend right, "\tilde{f}_3"'] \arrow[r, phantom, "\Downarrow a_2"] & y \end{tikzcd} .$$

Let $u_{123} := u_1 \times_x u_2 \times_x u_3$ and similarly for u_{12}, u_{23} and consider the diagram

$$\begin{tikzcd} & u_{12} \arrow[r] \arrow[dr] & u_1 \arrow[dr, "f_1"] \arrow[d, phantom, "\Downarrow a_1"] & & & \\ u_{123} \arrow[ur] \arrow[dr] & & u_2 \arrow[r, "f_2"] \arrow[d, phantom, "\Downarrow a_2"] & y & w \times_y w \arrow[l] \arrow[r, bend left, "\mathrm{pr}_1"] \arrow[r, bend right, "\mathrm{pr}_2"'] \arrow[r, phantom, "\Downarrow"] & w \\ & u_{23} \arrow[r] \arrow[ur] & u_3 \arrow[ur, "f_3"'] & & & \end{tikzcd} .$$

We need to prove equal the pair of 2-cells $(\rho_{(w,\mathrm{id})}a_1) + (\rho_{(w,\mathrm{id})}a_2)$ and $\rho_{(w,\mathrm{id})}(a_1 + a_2)$ between the two 1-cells $(w \times_y u_1) \times_x (u_3 \times_y w) \simeq w \times_y u_{13} \times_y w \xrightarrow{\mathrm{pr}_i} w$, for $i = 1, 3$.

In Figure 1 the sub-diagram consisting of just the solid arrows together with the 2-arrows between them 2-commutes, so the precomposition $(\rho_{(w,\mathrm{id})}a_1) \oplus (\rho_{(w,\mathrm{id})}a_2)$ is given by the top layer of the diagram, namely

$$\begin{tikzcd}
 & w \times_y u_{12} \times_y w \arrow[dr] \arrow[drr, "\mathrm{pr}_1"] \arrow[dr, phantom, shift left=6, "\Downarrow \rho_{(w,\mathrm{id})} a_1"] & & \\
w \times_y u_{123} \times_y w \arrow[ur] \arrow[dr] & & w \times_y u_2 \times_y w \arrow[r] & w \\
 & w \times_y u_{23} \times_y w \arrow[ur] \arrow[urr, "\mathrm{pr}_3"'] \arrow[ur, phantom, shift right=6, "\Downarrow \rho_{(w,\mathrm{id})} a_2"] & &
\end{tikzcd}$$

and $(\rho_{(w,\mathrm{id})}a_1)+(\rho_{(w,\mathrm{id})}a_2)$ is given by the unique descent of this 2-arrow along p. The 2-arrow marked $(*)$ is the whiskering $\rho_{(w,\mathrm{id})}(a_1+a_2)$, and forms a 2-commuting diagram with a_1+a_2 and the 1-arrows $w \times_y u_{13} \times_y w \to u_{13}$ and $w \to y$. The 2-cell

$$\begin{tikzcd}
w \times_y u_{123} \times_y w \arrow[r] & w \times_y u_{13} \times_y w \arrow[r, bend left=50, dashed] \arrow[r, bend right=50, dashed] \arrow[r, phantom, "\Downarrow (*)"] & w
\end{tikzcd}$$

in Figure 1 is, by uniqueness of lifts through p and $w \to y$ (both in J) equal to $(\rho_{(w,\mathrm{id})}a_1) \oplus (\rho_{(w,\mathrm{id})}a_2)$. Thus the descent of $(\rho_{(w,\mathrm{id})}a_1) \oplus (\rho_{(w,\mathrm{id})}a_2)$ along p is just $\rho_{(w,\mathrm{id})}(a_1+a_2)$, which is what we needed to prove. $\square$

The definition of the left whiskering is slightly more complicated, as it is such that it doesn't permit us to ignore half of the span as we can for right whiskering. What we shall do is define left whiskering by a general J-span $x \leftarrow ju \xrightarrow{f} y$ in two cases, using the factorisation $(j,f) = (\mathrm{id}_u, f) \circ (j, \mathrm{id}_u)$.

3.1.1. *Case I: left whiskering by* (id_u, f). Let a be a map of fractions from $y \leftarrow v_1 \xrightarrow{g} z$ to $y \leftarrow v_2 \xrightarrow{h} z$, and $f\colon u \to y$ an arrow in K. The whiskered 2-arrow will be a morphism of fractions from $u \xleftarrow{\mathrm{pr}_1} u \times_y v_1 \xrightarrow{g \circ \mathrm{pr}_2} z$ to $u \xleftarrow{\mathrm{pr}_1} u \times_y v_2 \xrightarrow{h \circ \mathrm{pr}_2} z$, and so the desired 2-arrow in K will be of the form

$$\left(u \times_y v_{12} \xrightarrow{g \circ \mathrm{pr}_2} z\right) \Rightarrow \left(u \times_y v_{12} \xrightarrow{h \circ \mathrm{pr}_3} z\right),$$

where $v_{12} := v_1 \times_y v_2$.

Definition 3.10. The left whiskering of the map a by $u \xleftarrow{\mathrm{id}_u} u \xrightarrow{f} y$ is given by the 2-arrow $\lambda^{I}_{(\mathrm{id}_u,f)}a$, defined as

$$\begin{tikzcd}
u \times_y v_{12} \arrow[r] & v_{12} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "a\Downarrow"] & z
\end{tikzcd} .$$

3.1.2. *Case II: left whiskering by* (u, id_u). Let $V_{12} := v_1 \times_x v_2$, and similarly, $V_{ij\ldots} := v_i \times_x v_j \times_x \ldots$. There is a canonical map $v_{12} \to V_{12}$ (and similarly $v_{ij\ldots} \to V_{ij\ldots}$). Notice also that there is a trivial factorisation of $\mathrm{pr}_i\colon v_{12} \to v_i$ as $v_{12} \to V_{12} \xrightarrow{\mathrm{pr}_i} v_i$.

Definition 3.11. The left whiskering of the map a by $x \xleftarrow{j} u \xrightarrow{\mathrm{id}_u} u$ is given by the 2-arrow $\lambda^{II}_{(j,\mathrm{id}_u)}a$ in K defined via unique descent along $v_{12} \times_x V_{12} \to V_{12}$ by the equation

$$\begin{tikzcd}
v_{12} \times_x V_{12} \arrow[r, "\mathrm{pr}_2"] & V_{12} \arrow[r, bend left=50, "g \circ \mathrm{pr}_1"] \arrow[r, bend right=50, "h \circ \mathrm{pr}_2"'] \arrow[r, phantom, "\lambda^{II}_{(j,\mathrm{id}_u)}a \Downarrow"] & z
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & & V_{12} \arrow[dr, "g \circ \mathrm{pr}_1"] & \\
v_{12} \times_x V_{12} \arrow[r] \arrow[urr, bend left, "\mathrm{pr}_2"] \arrow[drr, bend right, "\mathrm{pr}_2"'] & v_{12} \arrow[ur] \arrow[dr] \arrow[u, phantom, "\Downarrow"] \arrow[d, phantom, "\Downarrow"] & \arrow[r, phantom, "a \Downarrow"] & z \\
 & & V_{12} \arrow[ur, "h \circ \mathrm{pr}_2"'] &
\end{tikzcd} .$$

That this definition works uses Lemma 2.12. Left whiskering by an arbitrary fraction $x \xleftarrow{j} u \xrightarrow{f} y$ will then be the composite of the two (putative) functors given by cases I and II.

Proposition 3.12. *Left whiskering with $x \xleftarrow{j} u \xrightarrow{f} y$ is a functor $K_{\mathsf{J}}(y,z) \to K_{\mathsf{J}}(x,z)$.*

The proof that left whiskering preserves (vertical) composition will be deferred to appendix A, as it is a sizable calculation.

Proof. (Left whiskering is unital) We want to do the whiskering

$$\begin{tikzcd} x & u \arrow[l, "j"'] \arrow[r, "f"] & y & v \times_y v \arrow[l] \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow a"] & v \arrow[r, "g"] & z \end{tikzcd} .$$

Note that without loss of generality we can assume $g = \mathrm{id}_v$, the general case follows exactly the same argument merely with g right whiskered onto all the 2-cells involved. We treat case I and case II of the definition of left whiskering separately.

Case I. Note that v_{12} in this case is $v \times_y v$. The left whiskering of the identity map on $y \leftarrow v \xrightarrow{\mathrm{id}} v$ has 2-cell component

$$\begin{tikzcd} u \times_y (v \times_y v) \arrow[r] & v \times_y v \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow a"] & v \end{tikzcd}$$

but $u \times_y (v \times_y v) \simeq (u \times_y v) \times_u (u \times_y v)$, and by Lemma 2.4 this is equal to

$$\begin{tikzcd} (u \times_y v) \times_u (u \times_y v) \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow a"] & u \times_y v \arrow[r] & v \end{tikzcd}$$

and this is the identity map on the composite $u \leftarrow u \times_y v \to v$, as required.

Case II. Again, in this case, $v_{12} = v \times_u v$, which for now will be denoted $v^{[2]}$ and $V_{12} = v \times_x v$. Recall that the 2-cell component of the whiskered identity map will be the unique 2-cell $\lambda := \lambda^{II}_{(j,\mathrm{id}_u)} a$ in the diagram

$$\begin{tikzcd} v^{[2]} \times_x (v \times_x v) \arrow[r] & v \times_x v \arrow[r, bend left, "\mathrm{pr}_1"] \arrow[r, bend right, "\mathrm{pr}_2"'] \arrow[r, phantom, "\Downarrow \lambda"] & v \end{tikzcd} = \begin{tikzcd} & & v \times_x v \arrow[dr, "\mathrm{pr}_1"] \arrow[dd, phantom, "\Downarrow"] & \\ v^{[2]} \times_x (v \times_x v) \arrow[urr, bend left] \arrow[drr, bend right] \arrow[r] \arrow[r, phantom, shift left=4, "\Downarrow"] \arrow[r, phantom, shift right=4, "\Downarrow"] & v^{[2]} \arrow[ur] \arrow[dr] & & v \\ & & v \times_x v \arrow[ur, "\mathrm{pr}_2"'] & \end{tikzcd} ,$$

which exists by descent along the J-cover $v^{[2]} \times_x (v \times_x v) \to v \times_x v$. However, by Lemma 2.4 the canonical 2-arrow $\mathrm{pr}_1 \Rightarrow \mathrm{pr}_2 \colon v \times_x v \to v$ fits into such an equation of 2-arrows, and this is none other than the 2-cell component of the identity 2-arrow on the composite fraction $x \leftarrow v \xrightarrow{\mathrm{id}_v} v$.

Putting case I and case II together, we have that left whiskering $\lambda_{(j,f)} \colon K_{\mathsf{J}}(y,z) \to K_{\mathsf{J}}(x,z)$ preserves identity maps. □

With Propositions 3.12 and 3.9 we can, by virtue of [Lan71, Proposition II.3.1], define a composition functor

$$K_{\mathsf{J}}(x,y) \times K_{\mathsf{J}}(y,z) \to K_{\mathsf{J}}(x,z).$$

In order for this to be the composition functor for a bicategory we just need to now show that it is coherently associative and unital. In fact, by virtue of Definition 2.7, this composition is *strictly* unital, since the composition of any fraction with the identity fraction of its source or target is unchanged.

Definition 3.13. The *associator* for the 3-tuple of fractions

$$\begin{tikzcd} & u \arrow[dl] \arrow[dr] & & v \arrow[dl] \arrow[dr] & & w \arrow[dl] \arrow[dr] & \\ x_1 & & x_2 & & x_3 & & x_4 \end{tikzcd}$$

is the map of J-fractions $\iota(a_{uvw})$ where a_{uvw} is the renaming map associated to the canonical isomorphism

$$(u \times_{x_2} v) \times_{x_3} w \simeq u \times_{x_2} (v \times_{x_3} w)$$

over x_1, and the appropriate identity 2-arrow.

We can thus check that the associator satisfies the necessary coherence diagrams in the bicategory of fractions and renaming maps, since it will then hold in the bicategory of fractions and maps of fractions. In fact, since the renaming map in question is the associator for products in the strict slice K/x_1 (i.e. strict pullbacks in K), it satisfies coherence by the universal property of pullbacks.

Remark 3.14. If we do not assume that pullbacks of identity arrows are again identity arrows, then we do get nontrivial unitors, but they are, like the associator, renaming maps, and one can check they are coherent.

We have thus proved:

Proposition 3.15. *There is a bicategory K_J with the same objects as K, fractions as 1-arrows and maps of fractions as 2-arrows.*

We now define an identity-on-objects strict 2-functor $A_\mathsf{J} \colon K \to K_\mathsf{J}$ as follows. For a 1-arrow $f \colon x \to y$ of K, let $A_\mathsf{J}(f)$ be the fraction $x \xleftarrow{\mathrm{id}_x} x \xrightarrow{f} y$. Given a 2-arrow $a \colon f \Rightarrow g \colon x \to y$ in K, let $A_\mathsf{J}(a)$ be the map of fractions

$$\begin{tikzcd} x & x \arrow[l, "\mathrm{id}_x"'] \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow a"] & y \end{tikzcd} ,$$

where, recall, $x \times_x x = x$ by assumption. To check that A_J is a strict 2-functor, we need to check first that it is functorial for vertical composition of 2-arrows. In the definition of vertical composition of 2-cells, the diagram (3) in the case of maps of fractions in the image of A_J collapses as all objects u_i and their fibre products reduce to x, with all arrows between them identity arrows. The descended 2-arrow is then just the vertical composite in K, and so A_J preserves vertical composition. It is also simple to show that A_J preserves identity 2-arrows.

Secondly, we need to show that A_J is functorial for horizontal compsition. Identity 1-arrows are preserved strictly, as is composition of 1-arrows, so it is just a matter of checking that horizontal composition of 2-cells is preserved. Since horizontal composition is defined via left and right whiskering, we need to check that whiskering a map of fractions in the image of A_J by a fraction in the image of A_J is of the same form. The right whiskering of $A_\mathsf{J}(a \colon f_1 \Rightarrow f_2)$ where $f_1, f_2 \colon x \to y$ by $y \xleftarrow{\mathrm{id}_y} y \xrightarrow{g} z$ involves a 2-cell $\rho_{(\mathrm{id}_y, g)} a$ (see Definition 3.8). Since our fractions are in the image of A_J, the diagram again collapses so that all appearances of $u \times_x v$ are equal to x, and $w = y$, so that $\rho_{(\mathrm{id}_y, g)} a = a$, and the final result has the 2-cell component the right whiskering of a by g. The left whiskering we need is case I, so we consider Definition 3.10. Consider the map of fractions $A_\mathsf{J}(a \colon g_1 \Rightarrow g_2)$ where $g_1, g_2 \colon y \to z$ and whisker it by $x \xleftarrow{\mathrm{id}_x} x \xrightarrow{f} y$. Now in the definition of the 2-cell $\lambda^I_{(\mathrm{id}_x, f)} a$, we have $v_{12} = v_1 = v_2 = y$, the maps between them are identity maps, $u = x$, and $u \times_y v_{12} \to v_{12}$ is just f. Thus the whiskered map of fractions is again in the image of A_J, and we have proved that A_J is a strict 2-functor.

Lemma 3.16. *The 2-functor A_J is locally fully faithful, that is, $K(x, y) \to K_\mathsf{J}(x, y)$ is fully faithful for all objects x and y of K.*

Proof. A map of J-spans $(\mathrm{id}_x, f) \to (\mathrm{id}_x, g)$ is precisely the same data as a 2-arrow $f \Rightarrow g$ in K. □

Definition 3.17. Given J, a 1-arrow in $q\colon x \to y$ in K is J-*locally split* if there is an arrow $u \to y$ in J and a diagram of the form

$$\begin{tikzcd} & x \arrow[d, "q"] \\ u \arrow[ur, "s"] \arrow[r, "u"'] \arrow[r, phantom, shift left=3, "\Downarrow"] & y \end{tikzcd}$$

with the 2-arrow invertible. A 1-arrow in K is a *weak equivalence* if it is ff and J-locally split. Denote the class of weak equivalences by W_{J}.

Clearly $\mathsf{J} \subset W_{\mathsf{J}}$ as we are assuming all arrows in J are ff, and every arrow in J is trivially J-locally split.

Proposition 3.18. *Let f be a 1-arrow of K. Then $A_{\mathsf{J}}(f)$ an equivalence if and only if $f \in W_{\mathsf{J}}$.*

Proof. Let $f\colon x \to y$ be a 1-arrow in K such that (id_x, f) is an equivalence in K_{J}, i.e. there is a J-span $y \xleftarrow{j} u \xrightarrow{g} x$ such that

1. $(\mathrm{id}_x, f) \circ (j, g) \xrightarrow{\sim} (\mathrm{id}_y, \mathrm{id}_y)$
2. $(\mathrm{id}_x, \mathrm{id}_x) \xrightarrow{\sim} (j, g) \circ (\mathrm{id}_x, f)$

Point 1 implies that we have an isomorphism of J-spans

$$\begin{tikzcd} & u \arrow[dl, "j"'] \arrow[d, equal] \arrow[dr, "fg"] & \\ y \arrow[dr, equal] & u \arrow[d, "j"] \arrow[r, phantom, "\simeq\Downarrow"] & y \arrow[dl, equal] \\ & y & \end{tikzcd}$$

The right hand half of this diagram means that f is J-locally split. The second point implies that (id_x, f) is a ff arrow *in the bicategory* K_{J}, by the existence of the pseudo-retract (j, g) to (id_x, f). Since A_{J} is locally fully faithful it reflects ff 1-arrows, hence f is an ff arrow in K. □

For a number of diverse examples of weak equivalences in practice, see [Rob12, §8].

3.2. K_{J} **as a localisation.** Given a 2-category (or bicategory) B with a class W of 1-arrows, we say that a 2-functor $Q\colon B \to \widetilde{B}$ is a *localisation of B at W* if it sends the 1-arrows in W to equivalences in $\widetilde{B}$ and is universal with this property. This latter means that for any bicategory A precomposition with Q,

$$Q^*\colon \mathbf{Bicat}(\widetilde{B}, A) \to \mathbf{Bicat}_W(B, A),$$

is an equivalence of hom-bicategories, with $\mathbf{Bicat}_W$ meaning the full sub-bicategory on those 2-functors sending arrows in W to equivalences.

Theorem 3.19. *A strict 2-site (K, J) admits a bicategory of fractions for W_{J}, and the inclusion 2-functor $A_J\colon K \to K_{\mathsf{J}}$ is a localisation at the class W_{J} of weak equivalences.*

Proof. That (K, J) admits a bicategory of fractions for W_{J} is [Rob16, Theorem 6]. The proof that A_{J} is a localisation proceeds via Pronk's comparison theorem [Pro96, Proposition 24], the conditions of which imply that the canonical 2-functor $B[W^{-1}] \to A$ is an equivalence of bicategories. Here $B[W^{-1}]$ is the bicategory of fractions constructed by Pronk.

Let us show the conditions in [Pro96, Proposition 24] hold. To begin with, the 2-functor A_{J} sends weak equivalences to equivalences by Proposition 3.18.

EF1 A_{J} is the identity on objects, and hence surjective on objects.

EF2 This is equivalent to showing that for any J-fraction $x \xleftarrow{j} u \xrightarrow{f} y$ there are 1-arrows w, g in K such that w is in W_{J} and

$$(j, f) \overset{\simeq}{\Rightarrow} A_{\mathsf{J}}(g) \circ \overline{A_{\mathsf{J}}(w)}$$

where $\overline{A_{\mathsf{J}}(w)}$ is some pseudoinverse for $A_{\mathsf{J}}(w)$. We can take $w = j$ and $g = f$, since by the proof of Proposition 3.18, (j, id_u) is a pseudoinverse for (id_u, j), and the composite span of (j, id_u) and (id_u, f) is just (j, f).

EF3 This holds by Lemma 3.16.

Thus A_{J} is a localisation of K at W_{J}. □

As a last remark, one would like to know if the localisation of K at the weak equivalences is locally essentially small. This can be assured by the following result, where we have used the condition WISC from [Rob12], which states that every object x of K has a set of covers that are weakly initial in the subcategory of K/x on the J-covers.

Proposition 3.20. *If the locally essentially small strict 2-site (K, J) satisfies WISC, then K_{J} is locally essentially small, and hence so is any localisation of K at W_{J}.*

Notice that local essential smallness in *not* automatic, as there are well-pointed toposes with a natural numbers object, otherwise very nice categories, for which the 2-category of internal categories fails the hypothesis of Proposition 3.20. For example the toposes of material sets in models of ZF as given by Gitik (see [vdBM14]) and Karagila [Kar14], or the well-pointed topos of structural sets in [Rob15].

Finally, note that nothing in the above relies on K being a (2,1)-category, namely a 2-category with only invertible 2-arrows. This is usually assumed for results subsumed by Theorem 3.19, but is unnecessary in the framework presented here.

Appendix A. Proof that left whiskering in K_{J} preserves vertical composition

The definition of left whiskering in K_{J} is slightly more complicated, as it is such that it doesn't permit us to ignore half of the span as we can for right whiskering. Revall that we define left whiskering by a general J-fraction $x \xleftarrow{j} u \xrightarrow{f} y$ in two cases, using the factorisation $(j, f) = (\mathrm{id}_u, f) \circ (j, \mathrm{id}_u)$.

A.0.1. *Case I: left whiskering by* (id_u, f). Recall the definition of case I of left whiskering.

Definition A.1. The left whiskering of $a\colon (j, g) \to (k, h)$ by (id_u, f) is given by the 2-arrow $\lambda^I_{(\mathrm{id}_u, f)} a$, defined as

$$\begin{tikzcd} u \times_y v_{12} \arrow[r] & v_{12} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "a\Downarrow"] & z \end{tikzcd} .$$

Proposition A.2. *Left whisking by* (id_u, f) *preserves composition.*

Proof. In the following, let $\lambda^I(-) = \lambda^I_{(\mathrm{id}, f)}(-)$

$$\begin{tikzcd} u \times_y v_{123} \arrow[d] & \\ u \times_y v_{13} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\lambda^I(a_1+a_2)\Downarrow"] & z \end{tikzcd} = \begin{tikzcd} u \times_y v_{123} \arrow[d] & \\ u \times_y v_{13} \arrow[d] & \\ v_{13} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "a_1+a_2\Downarrow"] & z \end{tikzcd}$$

$$
= \quad
\begin{tikzcd}
u \times_y v_{123} \arrow[d] \\
v_{123} \arrow[d] \\
v_{13} \arrow[rr, bend left] \arrow[rr, bend right] \arrow[rr, phantom, "a_1+a_2\Downarrow"] && z
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
u \times_y v_{123} \arrow[dd] \\
& v_{12} \arrow[r] \arrow[dr] & v_1 \arrow[dr] \arrow[d, phantom, "a_1\Downarrow"] \\
v_{123} \arrow[ur] \arrow[dr] & & v_2 \arrow[r] \arrow[d, phantom, "a_2\Downarrow"] & z \\
& v_{23} \arrow[r] \arrow[ur] & v_3 \arrow[ur]
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
& u \times_y v_{12} \arrow[r] \arrow[dr] \arrow[dd] & u \times_y v_1 \arrow[dr] \arrow[d, phantom, "\lambda^I a_1\Downarrow"] \\
u \times_y v_{123} \arrow[ur] \arrow[dd] & & u \times_y v_2 \arrow[r] \arrow[dd] & z \arrow[dd, equal] \\
& v_{12} \arrow[dr] \\
v_{123} \arrow[ur] \arrow[dr] & & v_2 \arrow[r] \arrow[d, phantom, "a_2\Downarrow"] & z \\
& v_{23} \arrow[r] \arrow[ur] & v_3 \arrow[ur]
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
& u \times_y v_{12} \arrow[r] \arrow[dr] & u \times_y v_1 \arrow[dr] \arrow[d, phantom, "\lambda^I a_1\Downarrow"] \\
u \times_y v_{123} \arrow[ur] \arrow[dr] & & u \times_y v_2 \arrow[r] \arrow[d, phantom, "\lambda^I a_2\Downarrow"] & z \\
& u \times_y v_{23} \arrow[r] \arrow[ur] & u \times_y v_2 \arrow[ur]
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
u \times_y v_{123} \arrow[d] & \\
u \times_y v_{13} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\lambda^I a_1 + \lambda^I a_2 \Downarrow"] & z
\end{tikzcd}
$$

By uniqueness of descent, $\lambda^I(a_1 + a_2) = \lambda^I a_1 + \lambda^I a_2$.

∎

A.0.2. *Case II: left whiskering by* (j, id_u). Recall the notations $V_{12} := v_1 \times_x v_2$, $V_{ij\ldots} := v_i \times_x v_j \times_x \ldots$. and the canonical maps $v_{ij\ldots} \to V_{ij\ldots}$.

Definition A.3. The left whiskering of $a\colon (j, g) \to (k, h)$ by (j, id) is given by the 2-arrow $\lambda^{II}_{(j,\mathrm{id})} a$ in K defined via unique descent by the equation

$$
\begin{tikzcd}
v_{12} \times_x V_{12} \arrow[r, "\mathrm{pr}_2"] & V_{12} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\lambda_{(j,\mathrm{id})} a \Downarrow"] & z
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & & V_{12} \arrow[dr] & \\
v_{12} \times_x V_{12} \arrow[urr, bend left, "\mathrm{pr}_2"] \arrow[drr, bend right, "\mathrm{pr}_2"'] \arrow[r] & v_{12} \arrow[ur] \arrow[dr] \arrow[u, phantom, "\Downarrow"] \arrow[d, phantom, "\Downarrow"] & \arrow[r, phantom, "a \Downarrow"] & z \\
 & & V_{12} \arrow[ur] &
\end{tikzcd}
.
$$

Proposition A.4. *Left whiskering by* (j, id_u) *preserves composition.*

Proof. In the following, let $\lambda^{II}(-) := \lambda^{II}_{(j,\mathrm{id}_u)}(-)$

$$
\begin{tikzcd}
v_{123} \times_x V_{123} \arrow[d] & \\
v_{13} \times_x V_{13} \arrow[d] & \\
V_{13} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\lambda^{II}(a_1+a_2) \Downarrow"] & z
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
v_{123} \times_x V_{123} \arrow[dd] & & & \\
 & & V_{13} \arrow[dr] & \\
v_{13} \times_x V_{13} \arrow[urr, bend left] \arrow[drr, bend right] \arrow[r] & v_{13} \arrow[ur] \arrow[dr] \arrow[u, phantom, "\Downarrow"] \arrow[d, phantom, "\Downarrow"] & \arrow[r, phantom, "a_1+a_2 \Downarrow"] & z \\
 & & V_{13} \arrow[ur] &
\end{tikzcd}
$$

$$
=\quad
\begin{tikzcd}
 & & V_{123} \arrow[ddd, bend right] & \\
v_{123}\times_x V_{123} \arrow[urr, bend left] \arrow[r] \arrow[drr, bend right] \arrow[r, phantom, shift left=3ex, "\Downarrow"] \arrow[r, phantom, shift right=3ex, "\Downarrow"] & v_{123} \arrow[ur] \arrow[dr] \arrow[ddd] & & \\
 & & V_{123} \arrow[ddd, bend left] & \\
 & & V_{13} \arrow[dr] \arrow[dd, Rightarrow, "a_1+a_2"'] & \\
 & v_{13} \arrow[ur] \arrow[dr] & & z \\
 & & V_{13} \arrow[ur] &
\end{tikzcd}
$$

$$
=\quad
\begin{tikzcd}
 & & V_{123} \arrow[dr] \arrow[dd, Rightarrow, "a_1\oplus a_2"'] & \\
v_{123}\times_x V_{123} \arrow[urr, bend left] \arrow[r] \arrow[drr, bend right] \arrow[r, phantom, shift left=3ex, "\Downarrow"] \arrow[r, phantom, shift right=3ex, "\Downarrow"] & v_{123} \arrow[ur] \arrow[dr] & & z \\
 & & V_{123} \arrow[ur] &
\end{tikzcd}
$$

$$
=\quad
\begin{tikzcd}
 & V_{123} \arrow[rr] & & V_{12} \arrow[ddr] & \\
 & & v_{12} \arrow[ur] \arrow[dr] & & \\
v_{123}\times_x V_{123} \arrow[ur] \arrow[r] \arrow[dr] \arrow[r, phantom, shift left=3ex, "\Downarrow"] \arrow[r, phantom, shift right=3ex, "\Downarrow"] & v_{123} \arrow[u] \arrow[d] \arrow[ur] \arrow[dr] & & v_2 \arrow[r] \arrow[r, phantom, shift left=3ex, "a_1\Downarrow"] \arrow[r, phantom, shift right=3ex, "a_2\Downarrow"] & z \\
 & & v_{23} \arrow[ur] \arrow[dr] & & \\
 & V_{123} \arrow[rr] & & V_{23} \arrow[uur] &
\end{tikzcd}
$$

$$
= \quad \begin{tikzcd}
 & V_{123} \arrow[rr] & & V_{12} \arrow[ddrr] & & \\
 & & \boldsymbol{v_{12}} \arrow[ur] \arrow[r] & \boldsymbol{V_{12}} \arrow[dr] \arrow[r, phantom, "\Downarrow a_1"] & & \\
\boldsymbol{v_{123} \times_x V_{123}} \arrow[uur] \arrow[r] \arrow[ddr] \arrow[r, phantom, shift left=4, "\Downarrow"] \arrow[r, phantom, shift right=4, "\Downarrow"] & \boldsymbol{v_{123}} \arrow[uu] \arrow[dd] \arrow[ur] \arrow[dr] & & & \boldsymbol{v_2} \arrow[r] & z \\
 & & \boldsymbol{v_{23}} \arrow[r] \arrow[dr] & \boldsymbol{V_{23}} \arrow[ur] \arrow[r, phantom, "\Downarrow a_2"] & & \\
 & V_{123} \arrow[rr] & & V_{23} \arrow[uurr] & &
\end{tikzcd}
$$

$$
= \quad \begin{tikzcd}
 & & & V_{12} \arrow[dddrr] & & \\
 & \boldsymbol{v_{12} \times_x V_{12}} \arrow[urr, bend left] \arrow[r] \arrow[drr, bend right] \arrow[r, phantom, shift left=4, "\Downarrow"] \arrow[r, phantom, shift right=4, "\Downarrow"] & \boldsymbol{v_{12}} \arrow[ur] \arrow[dr] & & & \\
 & & & \boldsymbol{V_{12}} \arrow[dr] \arrow[r, phantom, "a_1 \Downarrow"] & & \\
\boldsymbol{v_{123} \times_x V_{123}} \arrow[uur] \arrow[ddr] & & & & \boldsymbol{v_2} \arrow[r] & z \\
 & & & \boldsymbol{V_{23}} \arrow[ur] \arrow[r, phantom, "a_2 \Downarrow"] & & \\
 & \boldsymbol{v_{23} \times_x V_{23}} \arrow[urr, bend left] \arrow[r] \arrow[drr, bend right] \arrow[r, phantom, shift left=4, "\Downarrow"] \arrow[r, phantom, shift right=4, "\Downarrow"] & \boldsymbol{v_{23}} \arrow[ur] \arrow[dr] & & & \\
 & & & V_{23} \arrow[uuurr] & &
\end{tikzcd}
$$

(where the subdiagrams on the bold symbols are equal)

$$
= \quad \begin{tikzcd}
 & v_{12} \times_x V_{12} \arrow[r] & V_{12} \arrow[r] \arrow[dr] & v_1 \arrow[dr] \arrow[d, phantom, "\lambda^{II} a_1 \Downarrow"] & \\
v_{123} \times_x V_{123} \arrow[ur] \arrow[dr] & & & v_2 \arrow[r] \arrow[d, phantom, "\lambda^{II} a_2 \Downarrow"] & z \\
 & v_{23} \times_x V_{23} \arrow[r] & V_{23} \arrow[r] \arrow[ur] & v_3 \arrow[ur] &
\end{tikzcd}
$$

$$
= \quad \begin{tikzcd}
v_{123} \times_x V_{123} \arrow[d] & V_{12} \arrow[r] \arrow[dr] & v_1 \arrow[dr] \arrow[d, phantom, "\lambda^{II} a_1 \Downarrow"] & \\
V_{123} \arrow[ur] \arrow[dr] & & v_2 \arrow[r] \arrow[d, phantom, "\lambda^{II} a_2 \Downarrow"] & z \\
 & V_{23} \arrow[r] \arrow[ur] & v_3 \arrow[ur] &
\end{tikzcd}
$$

$$
= \quad \begin{tikzcd}
v_{123} \times_x V_{123} \arrow[d] & \\
V_{123} \arrow[d] & \\
V_{13} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\lambda^{II}a_1+\lambda^{II}a_2 \Downarrow"] & z
\end{tikzcd}
$$

By uniqueness of descent, we have $\lambda^{II}(a_1 + a_2) = \lambda^{II}a_1 + \lambda^{II}a_2$, so left whiskering preserves composition. □

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E-mail address: `david.roberts@adelaide.edu.au`

School of Mathematical Sciences, University of Adelaide, SA, 5005, Australia