Online Supplement for *Integration of Sparse Singular Vector Decomposition and Statistical Process Control for Traffic Monitoring and Quality of Service Improvement in Mission-Critical Communication Networks* by *Kun Wang, Jing Li*

Appendix I: Brief Introduction to SVD

SVD is a matrix factorization method. Let $\mathbf{Z}$ be a $I \times J$ matrix. The SVD of $\mathbf{Z}$ is:

$$\mathbf{Z} = \mathbf{USV}^T = \sum_{k=1}^{r} s_k \mathbf{u}_k \mathbf{v}_k^T, \tag{A-1}$$

where r is the rank of $\mathbf{Z}$, $\mathbf{U} = (\mathbf{u}_1, \dots, \mathbf{u}_r)$ is a matrix consisting of orthonormal left singular vectors, $\mathbf{V} = (\mathbf{v}_1, \dots, \mathbf{v}_r)$ is a matrix consisting of orthonormal right singular vectors, $\mathbf{S}$ is a diagonal matrix with positive singular values $s_1 \geq \cdots \geq s_r$ on its diagonal. $\mathbf{U}$, $\mathbf{V}$, and $\mathbf{S}$ can be obtained by eigendecomposition. (A-1) indicates that SVD decomposes $\mathbf{Z}$ into a summation of r rank-one matrices, $\mathbf{Z}_k = s_k \mathbf{u}_k \mathbf{v}_k^T$, $k = 1, \dots, r$. It has been shown that $\mathbf{Z}_1$ is the closest rank-one approximation to $\mathbf{Z}$ in terms of minimizing the square Frobenius norm. This means that s_1, $\mathbf{u}_1$, $\mathbf{v}_1$ can be obtained by solving the following optimization problem:

$$(\hat{s}_1, \hat{\mathbf{u}}_1, \hat{\mathbf{v}}_1) = arg \min_{s_1, \mathbf{u}_1, \mathbf{v}_1} \left\| \mathbf{Z} - s_1 \mathbf{u}_1 \mathbf{v}_1^T \right\|_F^2,$$

$$s.t. \|\mathbf{u}_1\|_2 = 1, \|\mathbf{v}_1\|_2 = 1, s_1 \geq 0.$$

Appendix II: Proof of Proposition I

It is easy to show that $E(\mathbf{X}) - np_0 \mathbf{1}_{q \times m}$ is a rank-one matrix. Therefore,

$$E(\mathbf{X}) - np_0 \mathbf{1}_{q \times m} = s_1 \mathbf{u}_1 \mathbf{v}_1^T. \tag{A-2}$$

In other words, $s_k = 0$ for $k > 1$. Furthermore, let $\{E(\boldsymbol{X}) - np_0 \mathbf{1}_{q \times m}\}_{ij,t}$ be the element of matrix $E(\mathbf{X}) - np_0 \mathbf{1}_{q \times m}$ at the row corresponding to the sender-receiver pair (i, j) and the t-th column. We know from the definition of $E(\mathbf{X}) - np_0 \mathbf{1}_{q \times m}$ that

$$\{E(\boldsymbol{X}) - np_0 \mathbf{1}_{q \times m}\}_{ij,t} = \begin{cases} 0 & , for\ (i,j) \notin \boldsymbol{F} \\ -n\delta(t) & , for\ (i,j) \in \boldsymbol{F} \end{cases}. \tag{A-3}$$

Using (A-2), we can further write (A-3) into

$$s_1 u_{1,ij} v_{1,t} = \begin{cases} 0 & , for\ (i,j) \notin \boldsymbol{F} \\ -n\delta(t) & , for\ (i,j) \in \boldsymbol{F} \end{cases}, \tag{A-4}$$

where $u_{1,ij}$ is the element of $\mathbf{u}_1$ corresponding to the sender-receiver pair (i,j) and $v_{1,t}$ is the the t-th element of $\mathbf{v}_1$.

Let (i',j') be a sender-receiver pair that is affected by the fault and $(\tilde{\imath},\tilde{\jmath})$ be one that is not, i.e., $(i',j') \in \boldsymbol{F}$ and $(\tilde{\imath},\tilde{\jmath}) \notin \boldsymbol{F}$. Then, according to (A-4),

$$s_1 u_{1,\tilde{\imath}\tilde{\jmath}} v_{1,t} = 0 \text{ and } s_1 u_{1,i'j'} v_{1,t} \neq 0. \tag{A-5}$$

$s_1 \neq 0$ by the definition of SVD. Then, (A-5) becomes

$$u_{1,\tilde{\imath}\tilde{\jmath}} v_{1,t} = 0 \text{ and } u_{1,i'j'} v_{1,t} \neq 0.$$

The sufficient and necessary condition for the above simultaneous equations to hold is $v_{1,t} \neq 0$, $u_{1,\tilde{\imath}\tilde{\jmath}} = 0$, and $u_{1,i'j'} \neq 0$. Next, we derive the formula for $u_{1,i'j'}$, $v_{1,t}$, and s_1.

Let (i'',j'') be another sender-receiver pair that is affected by the fault, i.e., $(i'',j'') \in \boldsymbol{F}$. According (A-4), $s_1 u_{1,i'j'} v_{1,t} = s_1 u_{1,i''j''} v_{1,t} = -n\delta(t)$, i.e., $u_{1,i'j'} = u_{1,i''j''}$. Furthermore, because $\mathbf{u}_1$ is orthonormal, we have $\sum_{(i,j)\in\boldsymbol{F}} u_{1,ij}^2 = |\boldsymbol{F}| \times u_{1,ij}^2 = 1$. Solving this equation gives $u_{1,ij} = \frac{1}{\sqrt{|\boldsymbol{F}|}}$ for $\forall (i,j) \in \boldsymbol{F}$.

To derive the formula for $v_{1,t}$ and s_1, focus on a sender-receiver pair $(i,j) \in \boldsymbol{F}$. Then, $s_1 u_{1,ij} v_{1,t} = s_1 \frac{1}{\sqrt{|\boldsymbol{F}|}} v_{1,t} = -n\delta(t)$, i.e.,

$$v_{1,t} = \frac{-n\delta(t)\sqrt{|\boldsymbol{F}|}}{s_1}. \tag{A-6}$$

Using the property that $\mathbf{v}_1$ is orthonormal, we have $\sum_t v_{1,t}^2 = 1$. Combining this with (A-6), we get $s_1 = \sqrt{\sum_{t=1}^{m} [n\delta(t)]^2} \times \sqrt{|\mathbf{F}|}$. Inserting this into (A-6), we get $v_{1,t} = \frac{-\delta(t)}{\sqrt{\sum_{t=1}^{m} \delta(t)^2}}$. Δ