Capital Ownership under Market Incompleteness: Does it matter?*

EVA CARCELES POVEDA [†] AND DANIELE COEN-PIRANI [‡]

This version March 2005

ABSTRACT. In the macroeconomic literature, the implications of a context with household heterogeneity and incomplete financial markets have been mostly studied under the assumption that households own the physical capital and undertake the intertemporal investment decision. Further, firms rent capital and labor from the households to maximize period profits. The present paper provides the conditions under which this assumption is still irrelevant when markets are incomplete. It shows that, if firms own the physical capital and undertake the investment decision to maximize their asset value, in the sense that they discount future cash flows with positive state price processes that are consistent with security prices, the equilibrium allocations are the same as in the standard setting with static firms. On the other hand, the firm valuation of future cash flows only coincides with the valuation of the unconstrained shareholders. Given this, value maximization may still lead to shareholder disagreement in the presence of effectively binding portfolio restrictions.

Keywords: Incomplete Markets, Shareholder Disagreement, Firm Objectives

JEL Classification: D52, E44, G12, L20

1. INTRODUCTION

Following Bewley (1986), an extensive literature has studied the implications of models with heterogeneous agents and incomplete financial markets. Among others, Aiyagari (1994), Huggett (1997), Storesletten, Telmer and Yaron (2001a, 2001b, 2004) and Krusell and Smith (1997, 1998) have analyzed the effects of such a framework on the aggregate savings rate, the shape of the wealth distribution, the portfolio asset returns, and the welfare costs of business cycles.

An important assumption shared by the previous models is that households are the owners of the capital stock and undertake the intertemporal investment decision. Further, firms simply rent capital and labor from the households to maximize profits period by period. Whereas this assumption is innocuous under complete markets, things are very different when markets are incomplete. In particular, if one assumes instead that firms own the physical capital and decide on the optimal investment level, the usual share value maximization

*We are very grateful to Herakles Polemarchakis, Yair Tauman, Tom Muench and Harald Uhlig for fruitful discussions on the topic. The paper has also benefited from comments of conference and seminar participants at the NBER Summer Institute (2003), the Econometric Society Summer Meetings (2003), the CEF (2003), CEMFI (2002), Duke (2003), Nova de Lisboa (2003), the University of Bilbao (2003), North Carolina State (2004) and Arizona State (2004).

[†] *Correspondence:* Department of Economics, State University of New York, Stony Brook NY-11794-4384. *Email:* ecarcelespov@notes.cc.sunysb.edu. *Webpage:* <http://ms.cc.sunysb.edu/~ecarcelespov/>.

[‡] *Correspondence:* Tepper School of Business, Carnegie Mellon University, Pittsburgh, PA 15217. *Email:* coenp@andrew.cmu.edu. *Webpage:* <http://www.andrew.cmu.edu/user/coenp/res.htm>.

objective is no longer well defined. This is due to the fact that the available markets do not provide sufficient information to value future streams of resources unambiguously. In addition, shareholders' evaluations of different investment alternatives might differ from those of the firms, and shareholder disagreement may result in equilibrium.¹

This paper extends to an incomplete markets setting the result, well known under complete markets, that capital ownership is irrelevant. In other words, it shows that equilibria are the same whether the agents own the capital and rent it to the firm, or whether the firm, considered as an infinitely lived corporation that is owned by the agents as shareholders, holds the capital stock. Apart from identifying the particular firm objective that yields the same equilibrium as in the standard setting, we discuss the conditions under which this objective can be supported by the shareholders, contributing to a large strand of literature on shareholder unanimity that has been mostly developed in a static or two period setup.

The framework for our analysis is an infinite horizon economy with one good, aggregate uncertainty, idiosyncratic risk and sequential trading subject to general portfolio restrictions. With respect to the primitives of the economy, our results require constant returns to scale in production and the standard assumption that households are impatient, which implies convergence of present value sums and rules out price bubbles in equilibrium. To prove this last result in our setting, we follow Santos and Woodford (1997), who analyze the existence of price bubbles in exchange economies. In addition, we require value maximization from the part of the firms when making the investment decision.

As stated earlier, the main problem with the objective of the firm when markets are incomplete arises due to the lack of sufficient information to value future streams of resources. In particular, a model with incomplete markets and value maximization must incorporate a firm valuation function that reflects its beliefs about future prices. The approach taken by authors like DeMarzo (1988, 1993) and Duffie and Shaffer (1986b) is to discount future cash flows with a present value process that is consistent with security prices, in the sense that they satisfy a no arbitrage relation between the security prices and their one period payoffs. On the other hand, recognizing that value maximization according to consistent present value prices might lead to shareholder disagreement, a second group of authors (see e.g. Dreze (1974, 1985), DeMarzo (1993), and Grossmann and Hart (1979)) have suggested to discount future cash flows with a weighted sum of the marginal rates of substitution of the different shareholders, where the weights reflect the general shareholder composition. Finally, a third group of authors (see e.g. Radner (1972a), Sandmo (1972), Sondermann (1974) or Leland (1972)) have simply assumed the existence of a utility function for the firm defined exogenously over profits.

In the present paper, we first assume a general objective for the firm that nests the different approaches as particular cases. Further, a competitive equilibrium is defined as an allocation where both households and firms optimize. We then show that, under value maximization, the equilibrium allocations are the same irrespective of who (households or firms) owns the physical capital stock. In other words, if the discount factor of the firm belongs to the set of consistent present value prices, a value maximizing competitive equilibrium is also an equilibrium in the standard setting where firms maximize period profits. Conversely, an equilibrium in the standard setting is a value maximizing equilibrium in a setting where

¹Note that this problem does not arise if one assumes privately owned firms, as in Angeletos (2005) and Angeletos and Calvet (2003, 2004, 2005).

firms make the intertemporal investment decision. Moreover, these results can be extended to a more general framework that allows for heterogeneous firms and external financing of the investment level.

Several remarks are worth noting. First, if the set of consistent present value processes that constitutes an equilibrium with value maximization is not a singleton, the economy with value maximizing firms might generate indeterminacy with respect to the firm discount factor. On the other hand, since our previous results imply that the dimension of the set of equilibria with value maximization is the same as in the standard setting, we can establish that the equilibrium will be invariant with respect to the firm's discount factor if the equilibrium with one period lived firms is unique.

Second, the previous equilibrium concept does not take into account the ownership structure of the firm. Given this, we define a value maximizing competitive equilibrium with shareholder agreement as one where the valuation of future cash flows of the firm coincides with the valuation of the shareholders. We show that only the shareholders who are unconstrained every period agree with a value maximizing production plan, since their marginal rates of substitution (or valuation of future profits) also belong to the set of consistent present value processes. In our companion paper (Carceles-Poveda and Coen-Pirani, 2005), instead of postulating a particular objective for the dynamic firm, we explicitly address the question of unanimity among shareholders. We do so, by first deriving the investment decision that an arbitrary shareholder would make if he were the unique owner of the firm. We then demonstrate that, if portfolio restrictions on shareholders do not bind, all of the firm's shareholders would agree on its optimal level of investment.

Shareholder disagreement with respect to value maximization opens the possibility for assuming other objectives for the firm that could potentially have different quantitative implications to the ones established in the literature. As an example, authors like Aiyagari (1994), Huggett (1997) or Krusell and Smith (1997, 1998) have shown that imperfect risk sharing in the standard setting with one period lived firms can lead to an increase in the aggregate capital stock due to precautionary savings motives. Our results suggest, however, that the aggregate household savings in an economy where firms do not maximize their market value need not be the same as the aggregate capital stock. This is due to the fact that, in such a setting, the equivalence between the stock of physical capital and the market value of firm is broken, and precautionary savings are not necessarily reflected on the aggregate capital stock. A quantitative assessment of the importance of capital ownership in such a setting is an important issue that we leave for further research.

Our work is also related to the existing general equilibrium literature on incomplete markets with an infinite horizon, where authors like Hernandez and Santos (1996), Levine (1989), Magill and Quinzii (1994a, 1994b) or Levine and Zame (1996) have established the existence of an equilibrium in exchange economies. In addition, Duffie and Shaffer (1986b) have analyzed shareholder agreement and the existence and optimality of equilibrium in a general incomplete markets framework with production. An analysis of the existence and optimality of the equilibrium in the presence of production is outside the scope of the present paper. We believe, however, that our work does contribute to the understanding of production in infinite horizon economies with incomplete financial markets, while establishing a bridge between the general equilibrium macro-finance literature and the existing shareholder unanimity literature mentioned above.

The rest of the paper is organized as follows. The following section presents the model

and section three discusses the main equivalence results. These are further discussed in section four, and section five summarizes and concludes.

2. THE MODEL ECONOMIES

This section presents the different model economies. We first introduce the common general environment. Next, we discuss two production economies where the investment level is optimally determined depending on the ownership of the physical capital stock.² Throughout the section, we mostly adopt the notation of Santos and Woodford (1997).

2.1. The General Environment. We consider an infinite horizon economy with aggregate uncertainty, idiosyncratic income shocks and sequential trading. Time is discrete and indexed by $t = 0, 1, 2, \dots$. Further, the resolution of uncertainty is represented by an information structure or event-tree N . Each node or date-state $s^t \in N$, summarizing the history of the environment through and including date t , has a finite number $S(s^t)$ of immediate successors. We use the notation $s^r | s^t$ with $r \geq t$ to indicate that node s^r belongs to the sub-tree with root s^t . Further, with the exception of the unique root node s^0 dated at $t = 0$, each node has a unique predecessor dated at $t - 1$, which we denote by s^{t-1} . The probability of date-event s^t at period zero is denoted by $\pi(s^t)$, with $\pi(s^0) = 1$, since the initial realization s^0 is given. In addition, $\pi(s^r | s^t)$ denotes the probability of s^r given s^t . Throughout the section, we let $x = \{x(s^t)\}_{s^t \in N}$.

Technology. At each node $s^t \in N$, there exists a spot market for a single consumption good $y(s^t)$, produced with the following aggregate technology:

$$y(s^t) = f(z(s^t), k(s^{t-1}), n(s^t)) \quad (1)$$

where $k(s^{t-1}) \in \mathbb{R}_+$ and $n(s^t) \in \mathbb{R}_+$ denote the aggregate physical capital and labor, $z(s^t)$ is an aggregate productivity shock, and the initial stock of capital $k(s^{-1}) \in \mathbb{R}_{++}$ is given. We make the following assumptions.

(A.1) The technology shock follows a stationary (Markov) process with state space $S_z = \{z_m : m \in M_z, z_m \in [\underline{z}, \bar{z}]\}$, where M_z is a finite set of integers, $0 < \underline{z} < \bar{z} < +\infty$, and the initial realization $z(s^0)$ is given.

(A.2) Given z , the production function $f(z, \cdot, \cdot) : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ is continuously differentiable on the interior of its domain, strictly increasing, strictly quasiconcave, and homogeneous of degree one in the two arguments. We also assume that $f(z, 0, n) = 0$, $f_k(z, k, n) > 0$ and $f_n(z, k, n) > 0$ for all $k > 0$ and $n > 0$. Further, $\lim_{k \rightarrow 0} f_k(z, k, n) = \infty$ and $\lim_{k \rightarrow \infty} f_k(z, k, n) = 0$ for all $n > 0$.

The previous two assumptions are standard in the macroeconomic literature. Assumption (A.1) models the technology shock as a stationary Markov chain. Here, it is important to note that our results only require that the shock takes a finite number of positive values, and we can therefore relax the Markov assumption. Further, assumption (A.2) imposes standard conditions on the production process. In particular, the homogeneity assumption implies

²Whereas the analysis assumes the existence of a representative firm (or a large number of identical firms) and no external financing of the investment level, our results can also be extended to the cases where firms are heterogeneous and where investment is financed with external funds. This will be further discussed in a later section.

that $f(z, k, n) = f_k(z, k, n)k + f_n(z, k, n)n$ via Euler's theorem. As we will see later, this last property is crucial to obtain our results.

The aggregate capital stock depreciates at the rate $\delta \in (0, 1)$, and we denote the total supply of goods available from production at s^t including undepreciated capital by:

$$F(z(s^t), k(s^{t-1}), n(s^t)) = f(z(s^t), k(s^{t-1}), n(s^t)) + (1 - \delta)k(s^{t-1}). \quad (2)$$

Financial Markets. At each date-state s^t , there exist spot markets for a finite number L of securities. The first is a claim to productive activity that is indexed by $l = 1$. The rest are financial assets whose returns are denominated in units of the consumption good.

A security $l \in L$ traded at s^t is defined by its current price $q^l(s^t) \in \mathbb{R}_+$ and by the payoffs it promises to deliver at future nodes. Holding a portfolio of securities $a(s^{t-1}) \in \mathbb{R}^L$ at the end of period $t-1$ entitles the owner to a one period payoff of $R(s^t)' = [d(s^t) + q(s^t)'] a(s^{t-1})$ if date-state s^t is realized, where $q(s^t) = (q^1(s^t), \dots, q^L(s^t))'$ and $d(s^t) = (d^1(s^t), \dots, d^L(s^t))'$ are the vectors of prices and dividends respectively.³

A security traded at s^t is of *finite maturity* if there exists a date T such that $R^l(s^r|s^t) = 0$ for all $s^r|s^t$ with $r \geq T$. Otherwise, the security is *infinitely lived*. Further, security markets are one period *complete* at node s^t if the rank of the matrix defined by $[R(s^{t+1})']_{s^{t+1}|s^t}$, where one row corresponds to $R(s^{t+1})'$ for each node $s^{t+1}|s^t$, is equal to $S(s^t)$. Moreover, markets are complete if they are one period complete at every date-state. We make the following assumptions.

(A.3) For all $s^t \in N$, $d(s^t) \in \mathbb{R}_+$.

(A.4) $L \leq S(s^t)$ at all $s^t \in N$.

Assumption (A.3) requires that dividends are nonnegative, and it makes sense with nonnegative security prices, implied by the free disposal of securities. As we will see later, the dividends on the productive claim in the two production economies will be equal to the residual of output net of investment and labor income, given by the marginal product of labor multiplied by the total labor supply. Given this, assumption (A.3) will impose the following restriction on the aggregate investment plan at every node:

$$k(s^t) \leq F(z(s^t), k(s^{t-1}), n(s^t)) - f_n(z(s^t), k(s^{t-1}), n(s^t)) n(s^t). \quad (3)$$

Clearly, a necessary condition for markets to be complete is that $L \geq S(s^t)$ at all $s^t \in N$. On the other hand, since we are particularly interested in the case where markets are incomplete, assumption (A.4) limits the number of available assets at each node.

No Arbitrage Pricing. The security price process q is *arbitrage free* at s^t if there does not exist a portfolio $a(s^t) \in \mathbb{R}^L$ such that $R(s^{t+1})'a(s^t) \geq 0$ for all $s^{t+1}|s^t$ and $q(s^t)'a(s^t) \leq 0$, with at least one strict inequality. In other words, arbitrage free prices have to be such that it is not possible to construct a portfolio with non-positive value and nonnegative payoffs at every successor node. While this must be the case in equilibrium, the presence of no arbitrage at date-state s^t implies the existence of positive *present value prices* $\lambda(s^t) > 0$ and

³Note that we are abstracting from securities that pay bundles of other securities and from trade in different securities at different date-states, although this can be easily incorporated at the expense of additional notation. See Santos and Woodford (1997) or Hernandez and Santos (1996).

$\{\lambda(s^{t+1})\}$ with $\lambda(s^{t+1}) > 0$ for each $s^{t+1}|s^t$, such that:

$$q(s^t)' = \sum_{s^{t+1}|s^t} \frac{\lambda(s^{t+1})}{\lambda(s^t)} R(s^{t+1})' \quad (4)$$

Given (q, d) , the absence of arbitrage at each date-state s^t allows us to define processes $\lambda = \{\lambda(s^t)\}_{s^t \in N}$ for the entire information structure such that the previous no arbitrage equation holds. In what follows, we denote the set of such processes for the sub-tree with root s^t by $Q_{s^t}(q, d)$. Further, we define $\lambda_t^{s^r} = \lambda(s^r)/\lambda(s^t)$ for $s^r|s^t$ to simplify notation. Note that the present value ratios $\lambda_t^{s^{t+1}}$ that are consistent with security prices are uniquely determined by (4) if markets are complete. On the other hand, the number of linearly independent equations is not sufficient to uniquely determine the ratios when markets are incomplete.

The previous present value prices can be used to evaluate future streams of consumption goods. In particular, for a non-negative stream x that specifies $x(s^t) \in \mathbb{R}_+$ for all $s^t \in N$, the *present value* at s^t of the subsequent stream with respect to some $\lambda \in Q_{s^t}(q, d)$ can be defined as:

$$v_x(s^t, \lambda) = \sum_{r=1}^{\infty} \sum_{s^{t+r}|s^t} \lambda_t^{s^{t+r}} x(s^{t+r})$$

Similarly, we can define the fundamental value $v_{dl}(s^t, \lambda)$ of security l with respect to some $\lambda \in Q_{s^t}(q, d)$. In addition, using some algebra, the bubble component of the security with respect to $\lambda \in Q_{s^t}(q, d)$ can be expressed as:

$$\sigma^l(s^t, \lambda) = q^l(s^t) - v_{dl}(s^t, \lambda) = \lim_{r \rightarrow \infty} \sum_{s^{t+r}|s^t} \lambda_t^{s^{t+r}} q^l(s^{t+r})$$

As shown by Santos and Woodford (1987), if a security price is non-negative, its fundamental value $v_{dl}(s^t, \lambda)$ satisfies $0 \leq v_{dl}(s^t, \lambda) \leq q^l(s^t)$ for all $\lambda \in Q_{s^t}(q, d)$. Further, whereas the fundamental value need not be the same for all state prices satisfying equation (4), the authors show that it must lie between the finite bounds⁴ $\underline{v}_{dl}(s^t, \lambda) = \inf_{\lambda} v_{dl}(s^t, \lambda)$ and $\bar{v}_{dl}(s^t, \lambda) = \sup_{\lambda} v_{dl}(s^t, \lambda)$. Note that there unambiguously exists no bubble for security $l \in L$ if $\underline{v}_{dl}(s^r, \lambda) = \bar{v}_{dl}(s^r, \lambda) = q^l(s^r)$. In this case, the fundamental value is uniquely defined for all $\lambda \in Q_{s^t}(q, d)$.

Households. The economy is populated by a countable set of infinitely lived households I . Households' preferences $\succsim = (\succsim_i)_{i \in I}$ over consumption plans c_i satisfy the following assumption.

(A.5) For every $i \in I$, $\succsim_i$ can be represented by the following function:

$$U_i(c_i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta_i^t \pi(s^t) u_i(c_i(s^t)) \quad (5)$$

⁴These results follow from Propositions 2.1 and 2.2 in Santos and Woodford (1997), which can be directly applied to our setup.

where $\beta_i \in (0, 1)$ is the individual discount factor, and the period utility function $u_i : \mathbb{R}_+ \rightarrow \mathbb{R}$ is strictly increasing, strictly concave and continuously differentiable in the interior of its domain, with $\lim_{c_i \rightarrow 0} u'_i(c_i) = \infty$ and $\lim_{c_i \rightarrow \infty} u'_i(c_i) = 0$.

Assumption (A.5) is standard in the macroeconomic literature, and it implies that households have a sufficient degree of impatience. In other words, they are willing to give up a fraction of their future consumption to have more consumption today. The presence of a sufficient degree of impatience has been assumed by several authors studying infinite horizon exchange economies with incomplete markets, such as Santos and Woodford (1997), Hernandez and Santos (1996), Levine and Zame (1996) or Magill and Quinzii (1994a) and (1994b). As we will see later, it will be crucial to establish the absence of price bubbles in the present setup.

Each household $i \in I$ enters the markets at $t = 0$ with a finite initial endowment $a_i^l(s^{-1})$ of each security, whose sum across households determines the net supply of the security at each node, which we denote by $A^l = \sum_{i \in I} a_i^l(s^{-1})$. Without loss of generality, the supply of the productive claim and of the rest of securities is normalized to one and zero respectively, and we let $A = (A^1, \dots, A^L)'$. At each date-state $s^t \in N$, households are also endowed with one unit of time that is entirely allocated to labor, and which they can transform into $\epsilon_i(s^t)$ efficiency labor units that will be used to produce output in exchange of wages. Given this, the labor income of the household at s^t is given by $w_i(s^t) = w(s^t)\epsilon_i(s^t)$, where $w(s^t)$ is the fraction of output allocated to labor payments. We make the following assumptions.

(A.6) For all $i \in I$, $a_i(s^{-1}) \in \mathbb{R}_+$.

(A.7) The labor income shock ϵ_i follows a stationary (Markov) process with state space $S_\epsilon = \{\epsilon_{im} : m \in M_\epsilon, \epsilon_{im} \in [\underline{\epsilon}, \bar{\epsilon}]\}$, where M_ϵ is a finite set of integers, $0 < \underline{\epsilon} < \bar{\epsilon} < 1$, and the initial realization $\epsilon_i(s^0)$ is given.

Assumption (A.6) guarantees that the supply of each security is non-negative. Further, assumption (A.7) models the labor income shock as a discrete state Markov chain. As before, our results only require that the shock takes a finite number of positive values, and we can therefore relax the Markov assumption. The aggregate and idiosyncratic shocks could potentially be correlated, and we denote their joint transition matrix by Π in what follows.

At each node s^t , household $i \in I$ chooses consumption $c_i(s^t) \in \mathbb{R}_+$ and a portfolio of securities $a_i(s^t) \in \mathbb{R}^L$ subject to the following constraints:

$$c_i(s^t) + q(s^t)'a_i(s^t) \leq \omega_i(s^t) \quad (6)$$

$$\omega_i(s^{t+1}) = w_i(s^{t+1}) + R(s^{t+1})'a_i(s^t) \quad (7)$$

$$q(s^t)'a_i(s^t) \geq B_i(s^t) \quad (8)$$

equation (6) is the standard budget constraint with sequential markets and equation (7) is the law of motion of the individual wealth $\omega_i(s^t)$. At $t = 0$, equation (7) takes the same form with $\omega_i(s^0) = q'(s^0)a_i(s^{-1}) + d^1(s^0)a_i^1(s^{-1}) + w_i(s^0)$, where we have used the fact that $d^l(s^0) = 0$ for $l \geq 2$. Finally, to avoid Ponzi schemes, equation (8) imposes a finite limit of $B_i(s^t)$ on the total amount that households can borrow at every node⁵.

⁵ Note that we could also impose constraints on the individual asset holdings, since market clearing implies that in every asset market and every node there exists at least one household that is unconstrained.

A particular trading restriction that one can impose is the *present value constraint*, which is effectively never binding at any finite date. In particular, it is the tightest borrowing limit such that the portfolio holdings satisfy the budget constraint with $c_i(s^t) \in \mathbb{R}_+$ for all $s^t \in N$, and wealth is always non-negative after a finite date. As shown by Santos and Woodford (1997), this constraint can be formally specified as:

$$B_i(s^t) = -\underline{v}_{w_i}(s^t, \lambda) \text{ where } \underline{v}_{w_i}(s^t, \lambda) = \inf_{\lambda \in Q_{s^t}(q, d)} v_{w_i}(s^t, \lambda) \quad (9)$$

In essence, the restriction implies that households can borrow at most the lowest present value of their individual endowments in order to be solvent. The two production economies are described in what follows.

2.2. The k -economy. In the k -economy, we make the usual assumption in the macro-finance literature, implying that households are the owners of the physical capital stock and make the inter-temporal investment decision. In this case, the problem of the firm is particularly simple. At each date-state s^t , after observing the realization of the productivity shock z , the firm chooses capital and labor to maximize period profits. Thus, it solves a sequence of static problems:

$$\text{Max}_{\{k, n\}} F(z(s^t), k(s^{t-1}), n(s^t)) - w(s^t)n(s^t) - r(s^t)k(s^{t-1}) \quad (10)$$

leading to the following necessary and sufficient first order conditions:

$$w(s^t) = F_n(z(s^t), k(s^{t-1}), n(s^t)) = f_n(z(s^t), k(s^{t-1}), n(s^t)) \quad (11)$$

$$r(s^t) = F_k(z(s^t), k(s^{t-1}), n(s^t)) = f_k(z(s^t), k(s^{t-1}), n(s^t)) + 1 - \delta \quad (12)$$

where $w(s^t) \in \mathbb{R}_+$ and $r(s^t) \in \mathbb{R}_+$ are the competitively determined wage and gross capital rental rates respectively. Further, each household $i \in I$ maximizes the preferences in (A.5) subject to the following constraints:

$$c_i(s^t) + k_i(s^t) + \sum_{l \geq 2} q^l(s^t) a_i^l(s^t) \leq \omega_i(s^t) \quad (13)$$

$$\omega_i(s^{t+1}) = w_i(s^{t+1}) + r(s^{t+1})k_i(s^t) + \sum_{l \geq 2} R^l(s^{t+1}) a_i^l(s^t) \quad (14)$$

$$k_i(s^t) + \sum_{l \geq 2} q^l(s^t) a_i^l(s^t) \geq B_i(s^t) \quad (15)$$

In the previous equations, $k_i(s^t)$ is the amount of physical capital held by the household at the end of period t , illustrating the fact that households make the inter-temporal investment decision. If we denote by $k_i(s^{-1})$ and $a_i(s^{-1})$ the initial asset holdings of i at $t = 0$, the period zero budget constraint takes the same form with $\omega_i(s^0) = w_i(s^0) + r(s^0)k_i(s^{-1}) + q(s^0)a_i(s^{-1})$.

A k -economy is specified by a set of preferences $\succsim$, a transition matrix Π , initial values $(k_0, a_0, z_0, \epsilon_0) = \{k(s^{-1}), (a_i(s^{-1}), \epsilon_i(s^0))_{i \in I}, z(s^0)\}$, security processes $d^a = (d^l)_{l \geq 2}$ and borrowing limits $B = (B_i)_{i \in I}$, where $a_i^1(s^{-1}) = k_i(s^{-1})/k(s^{-1})$ represents the initial endowment of capital shares of household $i \in I$. A k -economy is therefore described by

$$E_k = \{\succsim, (k_0, a_0, z_0, \epsilon_0), \Pi, d^a, B\}.$$

Definition 2.1. *The vector of processes $\{(c_i, k_i, (a_i^l)_{l \geq 2})_{i \in I}, (q^l)_{l \geq 2}, (w, r)\}$ is a CE for $E_k = \{\succsim, (k_0, a_0, z_0, \epsilon_0), \Pi, d^a, B\}$ if (i) for each $i \in I$ and for each $s^t \in N$, $\{c_i, k_i, (a_i^l)_{l \geq 2}\}$ is optimal under the preferences $\succsim$ given $(q^l)_{l \geq 2}, (w, r), (k_0, a_0, z_0, \epsilon_0), \Pi, d^a$ and B (ii) (w, r) satisfies the firm's optimality conditions (iii) all markets clear, i.e., for all $s^t \in N$, $n(s^t) = \sum_{i \in I} \epsilon_i(s^t)$, $\sum_{i \in I} k_i(s^t) = k(s^t)$, $\sum_{i \in I} a_i^l(s^t) = A^l$ for $l \geq 2$ and $\sum_{i \in I} c_i(s^t) = w_c(s^t) = w(s^t)n(s^t) + d(s^t)A$.*

Before discussing the framework with dynamic firms, it is important to note that the constraints faced by the household sector in the k -economy can be directly mapped into the framework of the general environment if we define the shares of physical capital held by household i at date-state s^t as $a_i^1(s^t) = k_i(s^t)/k(s^t)$. With this normalization, the total supply of shares is positive and equal to $A^1 = 1$. Further, $q^1(s^t) = k(s^t)$, $R^1(s^{t+1}) = r(s^{t+1})k(s^t)$ and $d^1(s^t) = r(s^t)k(s^{t-1}) - k(s^t) = F(z(s^t), k(s^{t-1}), n(s^t)) - w(s^t)n(s^t) - k(s^t)$.

The e -Economy. In the e -economy, we assume that the firm owns the entire stock of capital and undertakes the inter-temporal investment decision by solving a dynamic optimization problem. Further, households are entitled to the future dividend payments through their ownership of a perfectly divisible equity share in the firm that is traded at price $q^1(s^t)$.

At each node s^t , households maximize the preferences in (A.5) subject to constraints (6)-(8). Further, the firm produces output, pays wages to the total labor employed and decides on the amount of investment. Investment is entirely financed with retained earnings, and the residual of gross profits (output net of labor payments) and investment is paid out as dividends to the firm equity owners, i.e.,

$$d^1(s^t) = F(z(s^t), k(s^{t-1}), n(s^t)) - w(s^t)n(s^t) - k(s^t) = N_f(s^t) \quad (16)$$

where $N_f(s^t)$ is the net cash flow of the firm. Unfortunately, the definition of an appropriate firm objective is more complicated than before, since the standard approach, that firms maximize their share value, is not well specified under market incompleteness. The reason is that the available markets do not provide sufficient information to value future dividend streams unambiguously. To see this, consider the case of effective complete markets and let $m_t^{s^{t+r}}$ be the $t+r$ -period ahead pricing kernel. Note that $m_t^{s^{t+r}}$ represents the period t price of one unit of time $t+r$ consumption, contingent on the economy being at date-state $s^{t+r}|s^t$. Since all the shareholders will agree on the pricing kernel under complete markets, the objective of the firm at date-state s^t can then be naturally specified as follows:

$$\text{Max}_{\{k, n\}} \sum_{r=0}^{\infty} \sum_{s^{t+r}} m_t^{s^{t+r}} N_f(s^{t+r}) \text{ where } m_t^{s^{t+r}} = \beta_i^r \frac{\pi(s^{t+r}|s^t) u'_i(c_i(s^{t+r}))}{u'_i(c_i(s^t))} \text{ for all } i \in I$$

As usual when markets are complete, the firm maximizes the present discounted value of its net cash flows, using as a discount factor the unique present value process of its shareholders, which is also the only element of the set $Q_{s^t}(q, d)$. In addition, both the agents and the firm value future output in each state identically, and all shareholders will therefore agree with the investment choice made by the firm. On the other hand, since a unique present value process that is consistent with market prices is not necessarily available under market incompleteness, the previous objective is no longer well defined, and shareholder

disagreement may result in equilibrium.

In what follows, we briefly discuss some of the approaches that have been proposed in the literature following the seminal paper of Diamond (1967). Moreover, a discussion of shareholder agreement in the present setup is provided in a later section⁶

Value Maximization. As noted by DeMarzo (1993), a natural generalization of the previous Arrow Debreu firm objective to an incomplete markets setup is to require firms to maximize the value of their output according to some consistent present value prices, in the sense that they satisfy the no arbitrage condition in (4). The two period value maximizing firm objective postulated by the author can therefore be expressed in our multi-period setup as follows:

$$U_f(N_f) = \text{Max}_{\{k,n\}} \sum_{r=0}^{\infty} \sum_{s^{t+r}} \lambda_t^{s^{t+r}} N_f(s^{t+r}) \text{ for some } \lambda \in Q_{s^t}(q, d) \quad (17)$$

This approach has also been followed by DeMarzo (1988) and Duffie and Shaffer (1986b), who study the validity of the Modigliani Miller theorem and the existence of equilibrium and shareholder agreement in a general incomplete markets context. As noted by the authors, one could alternatively assume that the firm maximizes its share price according to some valuation function that assigns a price process to a given stream of cash flows. Further, as long as this valuation does not predict security prices that allow for arbitrage opportunities, there exist some positive present value prices $\lambda \in Q_{s^t}(q, d)$ such that the valuation conjectured by the firm is equal to the objective function above.

The optimization problem under value maximization can be characterized by the following necessary and sufficient first order conditions:

$$w(s^t) = f_n(z(s^t), k(s^{t-1}), n(s^t)) \quad (18)$$

$$1 = \sum_{s^{t+1}|s^t} \lambda_t^{s^{t+1}} [f_k(z(s^{t+1}), k(s^t), n(s^{t+1})) + 1 - \delta] \quad (19)$$

The first equation determines the equilibrium aggregate wage rate. Further, the second equation determines the optimal production plan and it illustrates that the inter-temporal investment decision in this economy is made by the firm.

The e -economy is specified by a set of preferences $\succsim$, initial values $(k_0, a_0, z_0, \epsilon_0) = \{k(s^{-1}), (a_i(s^{-1}), \epsilon_i(s^0))_{i \in I}, z(s^0)\}$, a transition matrix Π , security processes $d^a = (d^l)_{l \geq 2}$ and limits $B = (B_i)_{i \in I}$. The e -economy is then described by $E_e = \{\succsim, (k_0, a_0, z_0, \epsilon_0), \Pi, d^a, B\}$.

Definition 2.2. *The vector of processes $\{(c_i, a_i)_{i \in I}, q, w, k\}$ is a value maximizing CE (VM CE) for $E_e = \{\succsim, (k_0, a_0, z_0, \epsilon_0), \Pi, d^a, B\}$ if (i) for each $i \in I$ and for each $s^t \in N$, (c_i, a_i) is optimal under the preferences $\succsim$ given (q, w) , $(k_0, a_0, z_0, \epsilon_0)$, Π , d^a and B (ii) (w, k) satisfies the firm's optimality conditions for some $\lambda \in Q_{s^t}(q, d)$ (iii) all markets clear, i.e.,*

⁶ A discussion of unanimity under value maximization and an excellent survey on the existing unanimity results is provided in Grossman and Stiglitz (1977, 1980). Further, see Duffie and Shafer (1986) for a discussion of unanimity under value maximization in a general multiperiod setup, and Carceles-Poveda and Coen-Pirani (2005) for a discussion of unanimity in a finite period version of the present setting.

for all $s^t \in N$, $n(s^t) = \sum_{i \in I} \epsilon_i(s^t)$, $\sum_{i \in I} a_i^l(s^t) = A^l$ for $l \in L$ and $\sum_{i \in I} c_i(s^t) = w_c(s^t) = w(s^t)n(s^t) + d(s^t)A$.

Several remarks are worth noting. First, the previous equilibrium definition implies that the set of allowable present value processes $Q_{s^t}(q, d)$ that the firm can use to discount its net cash flows has to satisfy a fixed point problem in the following sense. When the firm discounts profits with some λ that belongs to the set of admissible present value prices $Q_{s^t}(q, d)$, its production choice $k(\lambda)$ generates a new asset structure $(q(\lambda), d(\lambda))$ and a new set of admissible present value prices $Q_{s^t}(q(\lambda), d(\lambda))$ to which the original λ has to belong. Thus, if we define a mapping from the admissible set of present value prices to the set of present value prices that it generates, the equilibrium set of discount factors can be seen as a fixed point of this mapping. Moreover, if the set satisfying the previous fixed point problem is not single valued, the presence of incomplete financial markets might generate indeterminacy of equilibria with respect to the firm discount factor (see Duffie and Shaffer (1986b)). Second, since the state process λ can be interpreted as the discount factor used by the firm to value future net cash flows, value maximization will generate shareholder disagreement if λ does not agree with the valuation of the controllers of the firm. This will be further discussed in a later section.

Alternative Firm Objective Criteria. Recognizing that value maximization according to some consistent present value prices might generate shareholder disagreement, a group of authors have proposed other criteria concerning the discount factor of the firm. In particular, in a two period context, Dreze (1974) has suggested to use as a discount factor a weighted average of the marginal rates of substitution of the different shareholders, with the weights reflecting the final shareholdings. Further, Grossmann and Hart (1979) have extended this idea to a multi-period setting, arguing that the weights should correspond to the initial shareholder composition of each firm. In addition, Dreze (1985) and DeMarzo (1993) have attempted to introduce a control mechanism to decide among alternative production plans, implying that the firm should discount profits with some weighted average of the marginal rate of substitution of the controllers of the firm.

The previous objectives imply that the discount factor for the firm at node $s^{t+r}|s^t$ is given by $\lambda_t^{s^{t+r}} = \sum_{i \in I} \mu_i m_{it}^{s^{t+r}}$, where $\mu_i \geq 0$, $\sum_{i \in I} \mu_i = 1$ and $m_{it}^{s^{t+r}}$ is the marginal rate of substitution of shareholder $i \in I$. Given this, they will be denoted as weighted value maximization in what follows.⁷ Finally, recognizing that weighted value maximization requires shareholder consultation that can be time consuming and costly, a second group of authors simply have assumed the existence of a utility function for the firm, defined exogenously over state distribution of profits. This objective will be denoted by utility maximization in what follows, and important contributions here are due to Radner (1972), Sandmo (1972), Sondermann (1974), and Leland (1972) among others.

Competitive Equilibrium with Shareholder Agreement. All the decision criteria discussed above can be nested by the following objective function:

$$U_f(N_f) \equiv \sum_{r=0}^{\infty} \sum_{s^{t+r}} \lambda_t^{s^{t+r}} u_f(N_f(s^{t+r})) \quad (20)$$

⁷ Note that these objectives will only be equivalent to value maximization in the absence of binding portfolio restrictions.

In particular, value maximization assumes that $u_f(N_f) = N_f$ and $\lambda \in Q_{s^t}(q, d)$. Weighted value maximization assumes that $u_f(N_f) = N_f$, and λ is given by the weighted average of the shareholders' present value factors. Finally, utility maximization assumes that λ is fixed and u_f is strictly increasing and concave. The next definition extends the previous e -economy equilibrium concept to the general class of firm objectives defined above, which include value maximization as a particular case.

Definition 2.3. *The vector of processes $\{(c_i, a_i)_{i \in I}, q, w, k\}$ is a CE for the economy specified by $E_e = \{\succsim, (k_0, a_0, z_0, \epsilon_0), \Pi, d^a, B\}$ relative to U_f if (i) for each $i \in I$ and for each $s^t \in N$, (c_i, a_i) is optimal under the preferences $\succsim$ given (q, w) , $(k_0, a_0, z_0, \epsilon_0)$, Π , d^a and B (ii) (w, k) satisfy the optimality conditions of the firm under U_f (iii) all markets clear, i.e., for all $s^t \in N$, $n(s^t) = \sum_{i \in I} \epsilon_i(s^t)$, $\sum_{i \in I} a_i^l(s^t) = A^l$ for $l \in L$ and $\sum_{i \in I} c_i(s^t) = w_c(s^t) = w(s^t)n(s^t) + d(s^t)A$.*

It is important to note that the equilibrium concepts in definitions 2.2 and 2.3 do not take into account the relationship between the investment decision of the firm and its ownership structure. In other words, the investment decisions can potentially be made without taking into account the preferences of the shareholders. To address this issue, we follow Duffie and Shaffer (1986b) and assume that a shareholder agrees on the production choice unilaterally chosen by the firm if it maximizes his or her own utility. Formally, let the value of the problem of the households be defined as follows:

$$\overline{U}_i(k) = \max_{\{c_i, a_i\}} U_i(c_i) \text{ s.t. (6)-(8)}$$

A feasible production plan k chosen by the firm is optimal for shareholder $i \in I$ if it solves the problem:

$$\max_k \overline{U}_i(k) \quad (21)$$

Let $I^c \subseteq I$ be the subset of shareholders that have control over the firm. Note that we can now extend the previous equilibrium definitions to an equilibrium concept with shareholder agreement, in the sense that all shareholders in the control group I^c support the production plan chosen by the firm. To do this, we can replace condition (ii) in definition 2.3 with (ii)' (w, k) satisfy the optimality conditions of the firm under $U_f(N_f)$ and k solves (21) for all $i \in I^c$. Further, if the firm has a value maximizing objective, we can replace condition (ii) in definition 2.2 with (ii)' (w, k) satisfies the firm's optimality conditions for some $\lambda \in Q_{s^t}(q, d)$ that coincides with the valuation of future cash flows of all $i \in I^c$.

3. EQUIVALENCE OF THE PRODUCTION ALLOCATIONS

This section shows the equivalence of the set of equilibria in the two production economies under value maximization. Throughout the section, we assume that the assumptions of section two are satisfied. Further, we restrict our attention to equilibria where the aggregate investment plan satisfies a stronger version of the capital restriction in (3). In particular, we assume that there exists some $\varepsilon > 0$ such that $k(s^t) + \varepsilon \leq F(z(s^t), k(s^{t-1}), n(s^t)) - w(s^t)n(s^t)$ at all nodes. Note that this implies that productive dividends are bounded away from zero. Alternatively, the assumption can be reformulated as the aggregate resources $w_c(s^t) = d^1(s^t) + w(s^t)n(s^t)$ being higher than the aggregate labor payments $w_n(s^t) = w(s^t)n(s^t)$ at every node. Whereas this is typically satisfied in standard calibrated economies, it will rule out price bubbles in the present setup.

We start by stating several results that we will use to prove the main theorems, and relegate most of the proofs to the appendix. To distinguish the allocations, the caret bearing variables always denote k -economy allocations.

Proposition 3.1 *Consider a CE for E_k or E_e . For each node $s^t \in N$ and for each security $l \in L$ traded at s^t that is either (i) of finite maturity or (ii) in positive supply, we have that $q^l(s^t) = v_{d^l}(s^t, \lambda)$ for all $\lambda \in Q_{s^t}(q, d)$. In addition, $\lim_{r \rightarrow \infty} \sum_{s^{t+r}|s^t} \lambda_t^{s^{t+r}} k(s^{t+r}) = 0$ for all $\lambda \in Q_{s^t}(q, d)$.*

Proposition 3.1 establishes the absence of price bubbles for securities that are of finite maturity or in positive supply. Further, it implies that the discounted value of the aggregate capital converges to zero as time goes to infinity. These results can be established by showing that our preference assumptions imply that bubbles cannot exist for any security that is finitely lived or in positive supply and for any present value process $\lambda \in Q_{s^t}(q, d)$ such that the present value of the aggregate labor endowment $v_{w_n}(s^t, \lambda)$ is finite when this state price process is used. This part directly follows from Santos and Woodford (1997), and we only include the proof for completeness. Second, we show that the presence of trade in a claim to productive activity implies that the present value of the aggregate labor endowment is finite for any consistent present value process. To do this, we rely on the definition of the productive dividend payments. Clearly, the previous two results imply that bubbles cannot exist for any asset that is finitely lived or in positive supply⁸. The next lemma shows that the aggregate capital stock chosen by a value maximizing firm in the absence of price bubbles is equal to the ex-dividend firm value q^1 .

Lemma 3.1 *If the e -economy firm discounts its net cash flows with some $\lambda \in Q_{s^t}(q, d)$, the equilibrium investment plan satisfies $k(s^t) = q^1(s^t) = v_{N_f}(s^t, \lambda)$ for all $s^t \in N$.*

To prove the lemma, recall that the first order conditions of the firm's problem under value maximization imply that:

$$1 = \sum_{s^{t+1}|s^t} \lambda_t^{s^{t+1}} [f_k(z(s^{t+1}), k(s^t), n(s^t)) + 1 - \delta]$$

where $\lambda \in Q_{s^t}(q, d)$. Multiplying the previous expression with $k(s^t)$, adding and subtracting $k(s^{t+1})$ on the right hand side, and using the homogeneity condition of the production function, we obtain:

$$k(s^t) = \sum_{s^{t+1}|s^t} \lambda_t^{s^{t+1}} [N_f(s^{t+1}) + k(s^{t+1})]$$

Further, substituting iteratively for $k(s^{t+r})$ for $1 \leq r \leq T$, we have that:

$$k(s^t) = \sum_{r=1}^T \sum_{s^{t+r}|s^t} \lambda_t^{s^{t+r}} [N_f(s^{t+r})] + \sum_{s^{t+T}|s^t} \lambda_t^{s^{t+T}} k(s^{t+T})$$

⁸ As briefly discussed by Santos and Woodford (1997), who study the existence of price bubbles in exchange economies, bubbles can be ruled out in the presence of a claim to productive activity if one assumes that $d^1(s^t) \geq \phi w_c(s^t)$ for some $\phi > 0$. In the present setup, we assume instead that $d^1(s^t)$ is bounded away from zero. Further, we provide a direct proof of the absence of bubbles that just relies on the definition of the productive dividend payments.

The first term on the right-hand side of the previous equation has a well defined limit, and the second term converges to zero as T goes to infinity by proposition 3.1. Thus, taking limits of the previous equation as T goes to infinity, the aggregate capital stock can be expressed as:

$$k(s^t) = \sum_{r=1}^{\infty} \sum_{s^{t+r}|s^t} \lambda_t^{s^{t+r}} [N_f(s^{t+r})] = v_{N_f}(s^t, \lambda)$$

On the other hand, the ex-dividend firm value is equal to the value of equity $q^1(s^t)$, whose dividends are given by $d^1(s^t) = N_f(s^t)$ at all $s^t \in N$. Further, since equity is in positive supply, proposition 3.1 implies that $q^1(s^t) = v_{N_f}(s^t, \lambda)$ for all $\lambda \in Q_{s^t}(q, d)$, establishing the result. ■

The result of the previous lemma is crucial to establish the equivalence of the equilibrium allocations in the two production economies. In particular, as shown by lemma 3.2 below, it implies that the set of budget feasible allocations is the same across the two production economies as long as they are characterized by the same preferences, initial values, transition matrices for the shocks, financial asset structure, portfolio constraints and production plans.

Lemma 3.2 *Consider optimal allocations in the k - and e -economies. Further, assume that $(k_0, a_0, z_0, \epsilon_0)$, Π , $(d^l, q^l)_{l \geq 2}$, B and k are the same. If the firm in the e -economy has a value maximizing objective, the set of budget feasible allocations is the same in the two production economies.*

Lemma 3.2 shows that households in the k -economy can achieve the same consumption allocation as in the e -economy (and vice-versa) by choosing the same portfolio of assets $(a_i^l)_{l \geq 2}$ and a physical capital investment k_i that is equal to the total equity investment $q^1 a_i^1$ they would choose in the e -economy. If the production plan k is the same across the two production economies, this asset choice generates the same financial wealth, implying that a consumption plan that is feasible in one economy is also feasible in the other. As before, the result relies on the homogeneity of the production function, since it requires that $k(s^t) = q^1(s^t)$. We are now ready to state our main results.

Theorem 3.1 *Let $\{(c_i, a_i)_{i \in I}, q, w, k\}$ be a VM CE for $E_e = \{\succsim, (k_0, a_0, z_0, \epsilon_0), \Pi, d^a, B\}$. Then, there exist processes $\hat{k}_i$ and $\hat{r}$ such that $\{(c_i, \hat{k}_i, (a_i^l)_{l \geq 2})_{i \in I}, (q^l)_{l \geq 2}, w, \hat{r}\}$ is a CE for $E_k = \{\succsim, (k_0, a_0, z_0, \epsilon_0), \Pi, d^a, B\}$. In particular, $\hat{k}_i(s^t) = q^1(s^t) a_i^1(s^t)$ and $\hat{r}(s^t) = R^1(s^t)/q^1(s^{t-1})$ for all $s^t \in N$.*

Theorem 3.1 asserts that a value maximizing equilibrium in the e -economy is also an equilibrium in a k -economy with the same characteristics. The argument of the proof is very simple. We first note that the k -economy aggregate capital stock $\hat{k}$ is equal to the value of the firm q^1 in the e -economy, which is in turn equal to the e -economy capital stock k by lemma 3.1. Given this, the returns on labor and capital defined by $(w, \hat{r})$ satisfy the firm's optimality conditions in the k -economy, and lemma 3.2 implies that an optimal household allocation in the e -economy is also optimal in the k -economy. Finally, market clearing in the latter economy follows from market clearing in the first. Theorem 3.2 below states that the reverse is also true.

Theorem 3.2 *Let $\{(\hat{c}_i, \hat{k}_i, (\hat{a}_i^l)_{l \geq 2})_{i \in I}, (\hat{q}^l)_{l \geq 2}, \hat{w}, r\}$ be a CE for the economy specified by $E_k = \{\succsim, (\hat{k}_0, \hat{a}_0, z_0, \epsilon_0), \Pi, \hat{d}^a, \hat{B}\}$. Then, there exist processes for a_i^1 , and q^1 such that*

$\left\{ \left(\hat{c}_i, a_i^1, (\hat{a}_i^l)_{l \geq 2} \right)_{i \in I}, q^1, (\hat{q}^l)_{l \geq 2}, \hat{w}, \hat{k} \right\}$ is a VM CE for $E_e = \left\{ \succsim, (\hat{k}_0, \hat{a}_0, z_0, \epsilon_0), \Pi, \hat{d}^a, \hat{B} \right\}$.
 In particular, $a_i^1(s^t) = \frac{\hat{k}_i(s^t)}{\hat{k}(s^t)}$ and $q^1(s^t) = \hat{k}(s^t) = \sum_{i \in I} \hat{k}_i(s^t) = \hat{q}^1(s^t)$ for all $s^t \in N$.

Theorem 3.2 can be proved using similar arguments. In particular, since the aggregate capital is the same in the two economies, the fact that $q^1 = \hat{q}^1$ implies that both economies have the same asset structure and the same set of consistent present value prices. Given this, the k -economy aggregate wage rate $\hat{w}$ and aggregate capital stock $\hat{k}$, which satisfies the no arbitrage pricing condition in (4), satisfy the firm's optimality conditions in the e -economy for some $\lambda \in Q_{s^t}(q, d)$. Finally, lemma 3.2 implies that an optimal household allocation in the k -economy is also optimal in the e -economy, and market clearing in the latter economy directly follows from market clearing in the first.

The previous two theorems imply that value maximization leads to the same dimension of the set of equilibria in the two production economies. On the other hand, the equilibrium in the e -economy might depend on the particular firm discount factor $\lambda \in Q_{s^t}(q, d)$ (see Duffie and Shaffer (1986b)). As stated by the following theorem, however, if the equilibrium with one period lived firms exists and is unique, the equilibrium under dynamic firms is independent from the discount factor of the firm.

Theorem 3.3 *If a CE in the k -economy exists and is unique, the e -economy VM CE is invariant with respect to the firm discount factor $\lambda \in Q_{s^t}(q, d)$.*

The proof of this theorem follows from the previous two. In particular, theorems 3.1 and 3.2 establish that the set of equilibria has the same dimension in the two production economies. Thus, if we consider two equilibria in the e -economy that just differ in the discount factor chosen by the firm, the theorems show that they are also equilibria in the k -economy. On the other hand, if the k -economy equilibrium is unique, the two e -economy equilibria will clearly result in the same allocations.

4. DISCUSSION

Theorems 3.1 and 3.2 extend to an incomplete markets setting with general portfolio restrictions the result, well known under complete markets, that capital ownership is irrelevant under value maximization. In other words, equilibria are the same whether the agents own the capital and rent it to the firm, or whether the firm, considered as an infinitely lived corporation that is owned by the agents as shareholders, holds the capital stock. Several remarks are worth noting.

First, in addition to the assumption of constant returns to scale in production, the results require the standard assumption that agents are impatient, which implies convergence of the present value calculations in equilibrium. To get some intuition for the previous findings and for why these assumptions are needed, consider the investment decision in the two production economies. In a k -economy with portfolio restrictions, the aggregate capital stock is determined by the unconstrained households, whose marginal rate of substitution between periods t and $t + 1$ is denoted by $m_t^{s^{t+1}}$. If we substitute for $r(s^{t+1})$ and multiply with $k(s^t)$ the first order condition of an unconstrained household, it follows that the capital stock in this economy has to satisfy the following Euler condition:

$$k(s^t) = \sum_{s^{t+1}|s^t} m_t^{s^{t+1}} [F_k(z(s^{t+1}), k(s^t), n(s^{t+1}))k(s^t)] \quad (22)$$

Similarly, consider an analogous e -economy where the investment decision is made by a value maximizing firm. In the absence of price bubbles, a result that requires a sufficient degree of impatience (see Proposition 3.1), the homogeneity of $f(z, \cdot, \cdot) : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ implies that the stock of a value maximizing firm is equal to the aggregate capital stock, $k(s^t) = q^1(s^t)$, as shown by Lemma 3.1. Given this, the aggregate investment plan in the e -economy has to satisfy the following Euler condition for some $\bar{\lambda} \in Q_{s^t}(q, d)$:

$$k(s^t) = \sum_{s^{t+1}|s^t} \bar{\lambda}_t^{s^{t+1}} [d^1(s^{t+1}) + q^1(s^{t+1})] = q^1(s^t) \quad (23)$$

In essence, the results of section 3 can be seen as the equivalence of equations (22) and (23). Note that this implies that a value maximizing firm in the e -economy will choose the same capital stock as if households were making the inter-temporal investment decision in the k -economy, and viceversa. Further, to see that our assumptions guarantee that this is the case, consider the process $m \in Q_{s^t}(q, d)$, where $m_t^{s^{t+r}}$ is defined by the marginal rate of substitution of the particular household that is unconstrained at each node between s^t and $s^{t+r}|s^t$, and which may or may not be the same household every period. First, the absence of price bubbles implies that we can substitute $\bar{\lambda}$ with m in equation (23). This is due to the fact that $q^1(s^t)$ is equal to its fundamental value $v_{d^1}(s^t, \lambda)$, which is uniquely defined for all $\lambda \in Q_{s^t}(q, d)$. Second, under constant returns to scale, Euler's Theorem implies that $d^1(s^t) + q^1(s^t) = F_k(z(s^{t+1}), k(s^t), n(s^{t+1}))k(s^t)$. It therefore follows that equations (22) and (23) are equivalent. In turn, this also implies that both economies will have the same budget sets (see Lemma 3.2) and the same equilibrium allocations (see Theorems 3.1 and 3.2). On the other hand, since the assumptions needed are relatively standard in the macro-finance literature, our results can be applied to a wide class of economies. Two important extensions are briefly discussed in what follows.

Second, theorems 3.1-3.3 can be easily extended to economies with (i) external finance and (ii) heterogeneous firms. In the first case, we assume that firms in the two production settings can raise capital by issuing different assets. In the second case, we assume that firms differ in their productivity process. In addition, the e -economy firms can also differ in their discount factors as long as they belong to the set of consistent present value prices, which is common across firms. Using similar arguments to the ones in Proposition 3.1, price bubbles can be ruled out in these two cases. In the presence of external financing, we can then show that the ex-dividend firm value in the e -economy, which is equal to the market value of the assets in its capital structure, is equal to the economy wide capital stock. In the economy with heterogeneous firms, we can then show that each firm j will set its investment level to its market value, $k^j(s^t) = q^j(s^t) = v_{d^j}(s^t, \lambda^j)$ where $\lambda^j \in Q_{s^t}(q, d)$. Given this, the results of Lemma 3.2 and Theorems 3.1-3.3 follow through in both cases.⁹

Third, in spite of the fact that a value maximizing firm in the e -economy chooses the same production plan as if households were making the intertemporal investment decision, this does not necessarily imply that the plan is unanimously approved by the shareholders that belong to the control group $I^c \subseteq I$ of the firm. In particular, a plan that is chosen by a value maximizing firm will only be approved by the shareholders if the discount factor λ

⁹The proofs of these results can be found in the technical appendix accompanying the paper, which can be provided by the authors upon request.

coincides with the future cash flow valuation of the controllers of the firm. On the other hand, equation (23) implies that the valuation of the firm only coincides with the valuation of future profits of shareholders that are unconstrained every period. To see this, note that the equation can be rewritten as:

$$k(s^t) = \sum_{r=1}^{\infty} \sum_{s^{t+r}|s^t} \bar{\lambda}_t^{s^{t+r}} d^1(s^{t+r}) = \sum_{r=1}^{\infty} \sum_{s^{t+r}|s^t} m_t^{s^{t+r}} d^1(s^{t+r}) \quad (24)$$

The first equality represents that fundamental value of equity with respect to $\bar{\lambda}$. Further, since this is uniquely defined for all consistent present value processes, the fact that $m \in Q_{s^t}(q, d)$ implies that $v_{d^1}(s^t, \bar{\lambda}) = v_{d^1}(s^t, m)$. Here, it is important to note that only the marginal rate of substitution of a shareholder that is unconstrained every period belongs to the set of consistent present value prices $Q_{s^t}(q, d)$. Given this, only if the shareholders that belong to the control group of the firm are unconstrained, unanimity will obtain with respect to value maximization. As an example, this would happen if the controllers of the firm were subject to the present value constraint in (9), since it is effectively never binding. On the other hand, this implies that an equilibrium with shareholder agreement might not exist in the presence of effectively binding portfolio restrictions.

Note that this could have important quantitative implications related to the existing literature, since it opens the possibility for other firm objectives that differ from value maximization. In particular, several authors have shown that imperfect risk sharing in a k -economy setting can lead to an increase in the aggregate capital stock with respect to its value under complete markets due to precautionary savings (see e.g. Aiyagari (1994), Huggett (1997) and Krusell and Smith (1997,1998)). On the other hand, our results suggest that firms and households may make different investment decisions in the absence of value maximization, since the aggregate household savings, which are equal to the ex-dividend market value of the firm, will differ in general from the aggregate capital stock. A quantitative assessment of the importance of the ownership of physical capital in such a framework is an important issue that we leave for further research.

5. CONCLUSIONS

The present paper provides the conditions under which the irrelevance of physical capital ownership that arises under complete markets still holds in an incomplete markets environment with general portfolio restrictions and possibly external financing. It is shown that, if firms undertake the inter-temporal investment decision to maximize their market value, the equilibrium allocations are the same as in the standard setup where households make the investment decision and firms rent capital and labor to maximize profits period by period. The result requires that production functions are homogeneous of degree one in the two inputs and that preferences exhibit a sufficient degree of impatience. Since these two assumptions are relatively standard in the macro-finance literature, however, they can be applied to a relatively general setup.

Whereas the ownership of physical capital is irrelevant under value maximization, this objective might still lead to shareholder disagreement within the firm in the presence of effectively binding portfolio restrictions. The reason is that only the shareholders that are unconstrained every period unanimously agree on the value maximizing production plan chosen by the firm, opening the possibility for further analysis of shareholder agreement

issues in the present context. Finally, it is also shown that the quantitative implications of the absence of value maximization might differ from the standard static firm setup. This is due to the fact that the equivalence between the aggregate capital stock and the value of the firm is broken in this case. On the other hand, since this seems to be more in line with the data, the absence of value maximization might be a promising avenue towards the explanation of several facts, such as the behavior of capital versus stock returns.

APPENDIX

Proof of proposition 3.1

(a) We first show that, if there exists a state prices process $\lambda \in Q_{st}(q, d)$ such that the present value of the aggregate labor income $v_{w_n}(s^t, \lambda)$ is finite for every date-state s^t , then $q^l(s^r) = v_{d^l}(s^r, \lambda)$ for all $s^r | s^t$ with $r \geq t$ and each security traded at date-state s^r that is either (i) of finite maturity or (ii) in positive net supply. To prove this, note that the preferences defined by $\succsim$ and satisfying assumption (A.6) have the following property. For each $i \in I$, there exists a $0 \leq \gamma_i < 1$ such that for any date state $s^t \in N$,

$$(c_i^-(s^t), c_i(s^t) + w_c(s^t), \gamma_i c_i^+(s^t)) \succ (c_i^-(s^t), c_i(s^t), c_i^+(s^t)) \quad (1)$$

for all consumption plans satisfying $c_i(s^r) \leq w_c(s^r)$ at each $s^r \in N$ and all $\gamma \geq \gamma_i$. Here, $\succ_i$ denotes strict preference, $c_i^-(s^t)$ denotes the consumption coordinates at all nodes other than the sub-tree nodes $s^r \in N$ such that $s^r | s^t$, and $c_i^+(s^t)$ denotes the consumption coordinates at the nodes $s^r \in N$ such that $s^r | s^t$ and $r > t$. Given this, if the plan (c_i, a_i) is optimal at q , we have that, for all s^t :

$$(1 - \gamma_i)q(s^t)'a_i(s^t) \leq w_c(s^t) \quad (2)$$

To see that equation (2) is true, suppose that $(1 - \gamma_i)q(s^t)'a_i(s^t) > w_c(s^t)$ for some s^t . Household i could then choose the alternative plan $(\hat{c}_i, \hat{a}_i)$:

$$\begin{aligned} (\hat{c}_i^-(s^t), \hat{c}_i(s^t), \hat{c}_i^+(s^t)) &= (c_i^-(s^t), c_i(s^t) + w_c(s^t), \gamma_i c_i^+(s^t)) \\ (\hat{a}_i^-(s^t), \hat{a}_i(s^t), \hat{a}_i^+(s^t)) &= (a_i^-(s^t), \gamma_i a_i(s^t), \gamma_i a_i^+(s^t)) \end{aligned}$$

which is feasible and would be preferred to (c_i, a_i) by equation (1), contradicting the fact that (c_i, a_i) is optimal. Given this, equation (2) must hold. Next, we show that, for all s^t :

$$\sum_{r=1}^{\infty} \sum_{s^{t+r} | s^t} \lambda(s^{t+r}) c_i(s^{t+r}) \geq \sum_{r=1}^{\infty} \sum_{s^{t+r} | s^t} \lambda(s^{t+r}) w_i(s^{t+r}) + \lambda(s^t) q(s^t)' a_i(s^t) \quad (3)$$

To see that this is the case, we can multiply the date-state s^t budget constraint of consumer i , satisfied with equality for each date-state given our assumptions on preferences, with some $\lambda \in Q_{st}(q, d)$ for which $v_w(s^t, \lambda) < +\infty$. Further, summing over all date-states s^{t+r} , with dates $1 \leq$

$r \leq T$, we obtain:

$$\begin{aligned} & \sum_{r=1}^T \sum_{s^{t+r}|s^t} \lambda(s^{t+r})c_i(s^{t+r}) + \sum_{s^{t+T}|s^t} \lambda(s^{t+T})q'(s^{t+T})a_i(s^{t+T}) \\ &= \sum_{r=1}^T \sum_{s^{t+r}|s^t} \lambda(s^{t+r})w_i(s^{t+r}) + \lambda(s^t)q(s^t)'a_i(s^t) \end{aligned}$$

Substituting equation (2) and taking the limit of the previous equation as T goes to infinity, we have that:

$$\begin{aligned} & \lim_{T \rightarrow \infty} \sum_{r=1}^T \sum_{s^{t+r}|s^t} \lambda(s^{t+r})c_i(s^{t+r}) + \lim_{T \rightarrow \infty} (1 - \gamma_i)^{-1} \sum_{s^{t+T}|s^t} \lambda(s^{t+T})w_c(s^{t+T}) \\ & \geq \lim_{T \rightarrow \infty} \sum_{r=1}^T \sum_{s^{t+r}|s^t} \lambda(s^{t+r})w_i(s^{t+r}) + \lambda(s^t)q(s^t)'a_i(s^t) \end{aligned}$$

Since $v_{w_n}(s^t, \lambda) < +\infty$ by assumption, it follows that $v_{w_i}(s^t, \lambda) < +\infty$ for all i , and the right hand side of the previous equation has a finite limit equal to $\lambda(s^t)v_{w_i}(s^t, \lambda) < +\infty$. Since $w_n(s^t) + d(s^t)A = w_c(s^t)$ and $v_{dA}(s^t, \lambda) \leq q(s^t)'A < +\infty$, $v_{w_n}(s^t, \lambda) < +\infty$ also implies that $v_{w_c}(s^t, \lambda) < +\infty$, and it follows that $v_{c_i}(s^t, \lambda) < +\infty$ for all $i \in I$. Given this, the first term on the left hand side of the previous equation also has a well defined and finite limit equal to $\lambda(s^t)v_{c_i}(s^t, \lambda) < +\infty$. Finally, since $v_{w_c}(s^t, \lambda) < +\infty$ and $w_c(s^{t+T}) \geq 0$ for all date-states $s^{t+T}|s^t$, we have that $\lim_{T \rightarrow \infty} (1 - \gamma_i)^{-1} \sum_{s^{t+T}|s^t} \lambda(s^{t+T})w_c(s^{t+T}) = 0$, which establishes the inequality in equation (3). Summing the inequality over households, we obtain:

$$\sum_{r=1}^{\infty} \sum_{s^{t+r}|s^t} \lambda(s^{t+r})c(s^{t+r}) \geq \sum_{r=1}^{\infty} \sum_{s^{t+r}|s^t} \lambda(s^{t+r})w_n(s^{t+r}) + \lambda(s^t)q(s^t)'A \quad (4)$$

Finally, substituting for $c(s^t) = w(s^t)n(s^t) + d(s^t)A$, we have that:

$$\sum_{r=1}^{\infty} \sum_{s^{t+r}|s^t} \lambda_t^{s^{t+r}} d(s^{t+r})A \geq q(s^t)'A \quad (5)$$

On the other hand, the fact that $v_{d^l}(s^t, \lambda) \leq q^l(s^t)$ for all $l \in L$ implies that

$$\sum_{r=1}^{\infty} \sum_{s^{t+r}|s^t} \lambda_t^{s^{t+r}} d(s^{t+r})A \leq q(s^t)'A \quad (6)$$

Therefore, $\sigma'(s^t, \lambda)A = 0$, where $\sigma'(s^t, \lambda) = (\sigma^1(s^t, \lambda), \dots, \sigma^L(s^t, \lambda))'$, and $\sigma^l(s^t, \lambda) = 0$ if $A^l \in \mathbb{R}_{++}$. Note that, for finite maturity securities, the result directly follows from their definition.

(b) We now show that $v_{w_n}(s^t, \lambda) < +\infty$ and $\lim_{r \rightarrow \infty} \sum_{s^{t+r}|s^t} \lambda_t^{s^{t+r}} k(s^{t+r}) = 0$ for all s^t and all $\lambda \in Q_{s^t}(q, d)$. To prove this, note that, in equilibrium,

$$d^1(s^t) = F(z(s^t), k(s^{t-1}), n(s^t)) - w(s^t)n(s^t) - k(s^t) \geq 0 \quad (7)$$

Given that $v_{d^1}(s^t, \lambda) \leq q^1(s^t) < +\infty$ for all $s^t \in N$ and all $\lambda \in Q_{s^t}(q, d)$, we have that, for all $\lambda \in Q_{s^t}(q, d)$:

$$\sum_{r=1}^{\infty} \sum_{s^{t+r}|s^t} \lambda_t^{s^{t+r}} [F(z(s^{t+r}), k(s^{t+r-1}), n(s^{t+r})) - w(s^{t+r})n(s^{t+r}) - k(s^{t+r})] < +\infty \quad (8)$$

Suppose now that $v_{w_n}(s^t, \lambda) = +\infty$ for some $\lambda \in Q_{s^t}(q, d)$. Since the previous inequality holds for every $\lambda \in Q_{s^t}(q, d)$, this would imply that:

$$\sum_{r=1}^{\infty} \sum_{s^{t+r}|s^t} \lambda_t^{s^{t+r}} [F(z(s^{t+r}), k(s^{t+r-1}), n(s^{t+r})) - k(s^{t+r})] = +\infty \quad (9)$$

On the other hand, equation (7) implies that the equity dividends can be expressed as a fraction $\phi(s^t)$ of output net of investment, i.e., $d^1(s^t) = \phi(s^t)x(s^t)$, where $x(s^t) = [F(z(s^t), k(s^{t-1}), n(s^t)) - k(s^t)]$. Let $\phi = \inf_{s^t} \phi(s^t) > 0$, where the last inequality follows from the fact that the productive dividend payments are bounded away from zero. Given this, we have that:

$$d^1(s^t) \geq \phi[F(z(s^t), k(s^{t-1}), n(s^t)) - k(s^t)]$$

implying that

$$\phi \sum_{r=1}^{\infty} \sum_{s^{t+r}|s^t} \lambda_t^{s^{t+r}} [F(z(s^{t+r}), k(s^{t+r-1}), n(s^{t+r})) - k(s^{t+r})] \leq v_{d^1}(s^t, \lambda) < +\infty$$

which contradicts equation (9). Therefore, it follows that $v_{w_n}(s^t, \lambda) < +\infty$ for all $\lambda \in Q_{s^t}(q, d)$. Finally, to see that

$$\lim_{n \rightarrow \infty} \sum_{s^{t+r}|s^t} \lambda_t^{s^{t+r}} k(s^{t+r}) = 0$$

for all $\lambda \in Q_{s^t}(q, d)$, note that this directly follows from (a) in the k -economy. Further, to show that this is also the case in the e -economy, note that $w(s^t)n(s^t) + d^1(s^t) + k(s^t) = F(z(s^t), k(s^{t-1}), n(s^t))$. Given this, we can use the same arguments as above to show that for some $\phi > 0$, we have that $w(s^t)n(s^t) \geq \phi[F(z(s^t), k(s^{t-1}), n(s^t))]$. This clearly implies that the first infinite sum in equation (9) is finite for every $\lambda \in Q_{s^t}(q, d)$. Therefore, for the total sum to be finite, it must be the case that:

$$\sum_{r=1}^{\infty} \sum_{s^{t+r}|s^t} \lambda_t^{s^{t+r}} [k(s^{t+r})] < +\infty$$

implying that $\lim_{r \rightarrow \infty} \sum_{s^{t+r}|s^t} \lambda_t^{s^{t+r}} k(s^{t+r}) = 0$ for every $\lambda \in Q_{s^t}(q, d)$. ■

Proof of lemma 3.2

To prove the lemma, let $\widehat{F}_i(s^t)$ and $F_i(s^t)$ be the set of budget feasible allocations at s^t in the two production economies. Note first that $\widehat{c}_i(s^t) \in \widehat{F}_i(s^t)$ if there exists a set of portfolio strategies $\{\widehat{k}_i, (\widehat{a}_i^l)_{l \geq 2}\}_{i \in I}$ such that, for all $s^t \in N$ and all $i \in I$:

$$\widehat{c}_i(s^t) + \widehat{k}_i(s^t) + \sum_{l \geq 2} \widehat{q}^l(s^t) \widehat{a}_i^l(s^t) \leq \widehat{\omega}_i(s^t)$$

$$\begin{aligned}\widehat{\omega}_i(s^{t+1}) &= \widehat{w}_i(s^{t+1}) + [f_k(z(s^{t+1}), \widehat{k}(s^t), \widehat{n}(s^t)) + 1 - \delta] \widehat{k}_i(s^t) + \sum_{l \geq 2} \widehat{R}^l(s^{t+1}) \widehat{a}_i^l(s^t) \\ \widehat{k}_i(s^t) + \sum_{l \geq 2} \widehat{q}^l(s^t) \widehat{a}_i^l(s^t) &\geq \widehat{B}_i(s^t)\end{aligned}$$

where we have substituted for the equilibrium values of $\widehat{r}(s^{t+1}) = f_k(z(s^{t+1}), \widehat{k}(s^t), \widehat{n}(s^t)) + 1 - \delta$. Similarly, $c_i(s^t) \in F_i(s^t)$ if there exists a set of portfolio strategies $(a_i^l)_{l \geq 1}$ such that, for all s^t and all $i \in I$:

$$\begin{aligned}c_i(s^t) + q^1(s^t) a_i^1(s^t) + \sum_{l \geq 2} q^l(s^t) a_i^l(s^t) &\leq \omega_i(s^t) \\ \omega_i(s^{t+1}) &= w_i(s^{t+1}) + [f_k(z(s^{t+1}), k(s^t), n(s^t)) + 1 - \delta] q^1(s^t) a_i^1(s^t) + \sum_{l \geq 2} R^l(s^{t+1}) a_i^l(s^t) \\ q^1(s^t) a_i^1(s^t) + \sum_{l \geq 2} q^l(s^t) a_i^l(s^t) &\geq B_i(s^t)\end{aligned}$$

where we have used homogeneity of the production function and the fact that $q^1(s^t) = k(s^t)$ by lemma 3.1, implying that:

$$q^1(s^{t+1}) + d^1(s^{t+1}) = (f_k(z(s^{t+1}), k(s^t), n(s^t)) + 1 - \delta) k(s^t)$$

Let $\widehat{c}_i(s^t) \in \widehat{F}_i(s^t)$ and assume that the hypothesis of the lemma are satisfied. We now show that a plan setting $c_i(s^t) = \widehat{c}_i(s^t)$ at each node is feasible in the e -economy. To see this, consider any date-state $s^t \in N$. If $\omega_i(s^t) = \widehat{\omega}_i(s^t)$, households can choose the portfolio $a_i^l(s^t) = \widehat{a}_i^l(s^t)$ for $l \geq 2$ and $q^1(s^t) a_i^1(s^t) = \widehat{k}_i(s^t)$, implying that:

$$\begin{aligned}\widehat{c}_i(s^t) + q^1(s^t) a_i^1(s^t) + \sum_{l \geq 2} q^l(s^t) a_i^l(s^t) &= \widehat{c}_i(s^t) + \widehat{k}_i(s^t) + \sum_{l \geq 2} \widehat{q}^l(s^t) \widehat{a}_i^l(s^t) \leq \widehat{\omega}_i(s^t) = \omega_i(s^t) \\ q^1(s^t) a_i^1(s^t) + \sum_{l \geq 2} q^l(s^t) a_i^l(s^t) &= \widehat{k}_i(s^t) + \sum_{l \geq 2} \widehat{q}^l(s^t) \widehat{a}_i^l(s^t) \geq \widehat{B}_i(s^t) = B_i(s^t)\end{aligned}$$

Further, if household $i \in I$ chooses this portfolio, his wealth at the beginning of next period will be equal to:

$$\omega_i(s^{t+1}) = \widehat{w}_i(s^{t+1}) + [f_k(z(s^{t+1}), \widehat{k}(s^t), \widehat{n}(s^t)) + (1 - \delta)] \widehat{k}_i(s^t) + \sum_{l \geq 2} \widehat{R}^l(s^{t+1}) \widehat{a}_i^l(s^t) = \widehat{\omega}_i(s^{t+1})$$

where we have used the fact that $w_i(s^{t+1}) = \widehat{w}_i(s^{t+1})$ and $n(s^t) = \widehat{n}(s^t)$. Therefore, $c_i(s^{t+1}) = \widehat{c}_i(s^{t+1})$ is also feasible in the e -economy at date-state $s^{t+1}|s^t$. Finally, if the initial values are the same, implying that $k(s^{-1}) a_i^1(s^{-1}) = \widehat{k}_i(s^{-1})$, the period zero wealth of household $i \in I$ in the e -economy is given by:

$$\begin{aligned}\omega_i(s^0) &= \sum_{l \geq 2} q^l(s^0) a_i^l(s^{-1}) + w_i(s^0) + [(f_k(z(s^0), k(s^{-1}), n(s^0)) + (1 - \delta)) k(s^{-1})] a_i^1(s^{-1}) \\ &= \sum_{l \geq 2} \widehat{q}^l(s^0) \widehat{a}_i^l(s^{-1}) + \widehat{w}_i(s^0) + [f_k(z(s^0), \widehat{k}(s^{-1}), \widehat{n}(s^0)) + (1 - \delta)] \widehat{k}_i(s^{-1}) = \widehat{\omega}_i(s^0)\end{aligned}$$

Since $\omega_i(s^0) = \widehat{\omega}_i(s^0)$, it follows that $c_i(s^0) = \widehat{c}_i(s^0)$ is feasible, implying that $\widehat{c}_i(s^t) \in F_i(s^t)$ at all $s^t \in N$. Conversely, assume that $c_i(s^t) \in F_i(s^t)$ and consider any date-state $s^t \in N$. If $\widehat{\omega}_i(s^t) = \omega_i(s^t)$, households in the k -economy can choose the portfolio $\widehat{a}_i^l(s^t) = a_i^l(s^t)$ for $l \geq 2$ and $\widehat{k}_i(s^t) = q^1(s^t)a_i^1(s^t)$, achieving the same consumption allocation as in the e -economy at date-state s^t , since:

$$c_i(s^t) + \widehat{k}_i(s^t) + \sum_{l \geq 2} \widehat{q}^l(s^t) \widehat{a}_i^l(s^t) = c_i(s^t) + q^1(s^t)a_i^1(s^t) + \sum_{l \geq 2} q^l(s^t)a_i^l(s^t) \leq \omega_i(s^t) = \widehat{\omega}_i(s^t)$$

$$\widehat{k}_i(s^t) + \widehat{q}^1(s^t) \widehat{a}_i^1(s^t) = q^1(s^t)a_i^1(s^t) + q^1(s^t)a_i^1(s^t) \geq B_i(s^t) = \widehat{B}_i(s^t)$$

Further, since $\widehat{w}_i(s^{t+1}) = w_i(s^{t+1})$ and $\widehat{n}(s^t) = n(s^t)$, this will lead to the same wealth next period, i.e.,

$$\widehat{\omega}_i(s^{t+1}) = w_i(s^{t+1}) + [f_k(z(s^{t+1}), k(s^t), n(s^t)) + (1 - \delta)]q^1(s^t)a_i^1(s^t) + \sum_{l \geq 2} R^l(s^{t+1})a_i^l(s^t)$$

Since $\widehat{\omega}_i(s^{t+1}) = \omega_i(s^{t+1})$, we again have that $c_i(s^{t+1}) = \widehat{c}_i(s^{t+1})$ is feasible in the k -economy at date state $s^{t+1}|s^t$. Finally, since $\widehat{\omega}_i(s^0) = \omega_i(s^0)$, it follows that $c_i(s^0) = \widehat{c}_i(s^0)$ is feasible, and $c_i(s^t) \in \widehat{F}_i(s^t)$ at all nodes. ■

Proof of Theorem 3.1

Let $\{(c_i, a_i)_{i \in I}, q, w, k\}$ be a VM CE for $E_e = \{\succsim, (k_0, a_0, z_0, \epsilon_0), \Pi, d^a, B\}$. To show that $\{c_i, \widehat{k}_i, (a_i^l)_{l \geq 2}\}_{i \in I}, (q^l)_{l \geq 2}, w, \widehat{r}\}$ with $\widehat{r}(s^t) = R^1(s^t)/q^1(s^{t-1})$ and $\widehat{k}_i(s^t) = q^1(s^t)a_i^1(s^t)$ for all $s^t \in N$ is a CE for $E_k = \{\succsim, (k_0, a_0, z_0, \epsilon_0), \Pi, d^a, B\}$, note first that the aggregate capital in the k -economy is given by:

$$\widehat{k}(s^t) = \sum_{i \in I} \widehat{k}_i(s^t) = \sum_{i \in I} q^1(s^t)a_i^1(s^t) = q^1(s^t) = k(s^t)$$

where the last equality holds by lemma 3.1. Further, we have used the fact that a_i^1 generates market clearing in the e -economy. Given this, the two factor prices:

$$w(s^t) = f_n(z(s^t), k(s^{t-1}), n(s^t))$$

$$\widehat{r}(s^t) = R^1(s^t)/q^1(s^{t-1}) = (d^1(s^t) + k(s^t))/k(s^{t-1}) = f_k(z(s^t), k(s^{t-1}), n(s^t)) + 1 - \delta$$

satisfy the firm's optimality conditions in the k -economy, where we have substituted for the labor market clearing conditions and have again used lemma 3.1. Second, since $(k_0, a_0, z_0, \epsilon_0), \Pi, d^a, B, (q^l)_{l \geq 2}$ and k are the same across the two production economies, lemma 3.2 implies that $\widehat{F}_i(s^t) = F_i(s^t)$ for all $i \in I$ and all $s^t \in N$. Thus, the fact that c_i is optimal for each $i \in I$ in the e -economy implies that it is also optimal for each $i \in I$ in the k -economy. In addition, the portfolio strategies achieving this allocation, given by $\widehat{a}_i^l(s^t) = a_i^l(s^t)$ for $l \geq 2$ and $\widehat{k}_i(s^t) = q^1(s^t)a_i^1(s^t)$, are optimal. To see that they satisfy the portfolio constraints, note that:

$$\sum_{l \geq 2} \widehat{q}^l(s^t) \widehat{a}_i^l(s^t) + \widehat{k}_i(s^t) = \sum_{l \geq 2} q^l(s^t)a_i^l(s^t) + q^1(s^t)a_i^1(s^t) \geq B_i(s^t) = \widehat{B}_i(s^t)$$

Finally, the fact that (c_i, a_i) generates market clearing in the e -economy implies that the allocations still clear the markets in the k -economy. To see this, note that:

$$\sum_{i \in I} \hat{c}_i(s^t) = \sum_{i \in I} c_i(s^t) = F(z(s^t), k(s^{t-1}), n(s^t)) - k(s^t) + \sum_{l \geq 2} d^l(s^t) A^l = w_c(s^t) = \hat{w}_c(s^t)$$

$$\sum_{i \in I} \hat{a}_i^l(s^t) = \sum_{i \in I} a_i^l(s^t) = A^l = \hat{A}^l \text{ for } l \geq 2$$

$$\sum_{i \in I} \hat{k}_i(s^t) = \sum_{i \in I} q^1(s^t) a_i^1(s^t) = q^1(s^t) = k(s^t) = \hat{k}(s^t)$$

This establishes the result. ■

Proof of Theorem 3.2

Let $\left\{ \left(\hat{c}_i, \hat{k}_i, (\hat{a}_i^l)_{l \geq 2} \right)_{i \in I}, (\hat{q}^l)_{l \geq 2}, \hat{w}, \hat{r} \right\}$ be a CE for $E_k = \left\{ \succsim, (\hat{k}_0, \hat{a}_0, z_0, \epsilon_0), \Pi, \hat{d}^a, \hat{B} \right\}$. We now show that $\left\{ \left(\hat{c}_i, a_i^1, (\hat{a}_i^l)_{l \geq 2} \right)_{i \in I}, q^1, (\hat{q}^l)_{l \geq 2}, \hat{w}, \hat{k} \right\}$ with $a_i^1(s^t) = \frac{\hat{k}_i(s^t)}{\hat{k}(s^t)}$ and $q^1(s^t) = \hat{k}(s^t) = \hat{q}^1(s^t)$ is a VE CE for $E_e = \left\{ \succsim, (\hat{k}_0, \hat{a}_0, z_0, \epsilon_0), \Pi, \hat{d}^a, \hat{B} \right\}$. To prove this, note first that the absence of arbitrage implies that the aggregate capital stock in the k -economy satisfies the following condition:

$$\hat{k}(s^t) = \sum_{s^{t+1}|s^t} \hat{\lambda}_t^{t+1} [f_k(z(s^{t+1}), \hat{k}(s^t), \hat{n}(s^t)) \hat{k}(s^t) + (1 - \delta) \hat{k}(s^t)]$$

for some $\hat{\lambda} \in Q_{s^t}(\hat{q}, \hat{d})$, where we have substituted for:

$$\hat{R}^1(s^t) = \hat{r}(s^t) \hat{k}(s^{t-1}) = f_k(z(s^t), \hat{k}(s^{t-1}), \hat{n}(s^t)) \hat{k}(s^{t-1}) + (1 - \delta) \hat{k}(s^{t-1})$$

Since $q^1(s^t) = \hat{q}^1(s^t)$ and $d^1(s^t) = \hat{d}^1(s^t)$ due to the fact that the aggregate capital stock is the same in the two economies, it follows that $Q_{s^t}(q, d) = Q_{s^t}(\hat{q}, \hat{d})$. Therefore, the following values of $k(s^t)$ and $w(s^t)$ satisfy the firm's optimality conditions in the e -economy for some $\lambda \in Q_{s^t}(q, d)$:

$$k(s^t) = \hat{k}(s^t) = \sum_{s^{t+1}|s^t} \hat{\lambda}_t^{t+1} [f_k(z(s^{t+1}), \hat{k}(s^t), \hat{n}(s^t)) \hat{k}(s^t) + (1 - \delta) \hat{k}(s^t)] \hat{k}(s^t)$$

$$w(s^t) = \hat{w}(s^t) = f_n(z(s^t), \hat{k}(s^{t-1}), \hat{n}(s^t))$$

Second, since $(\hat{k}_0, \hat{a}_0, z_0, \epsilon_0), \Pi, \hat{d}^a, \hat{B}, \hat{q}^a, \hat{k}$ are the same in the two economies, lemma 3.2 implies that $F_i(s^t) = \hat{F}_i(s^t)$ for all $i \in I$ and all $s^t \in N$. Therefore, since $\hat{c}_i$ is optimal for each $i \in I$ in the k -economy, it is also optimal for each $i \in I$ in the e -economy. In addition, this also implies that the portfolio strategies achieving this allocation $a_i^l(s^t) = \hat{a}_i^l(s^t)$ for $l \geq 2$ and $a_i^1(s^t) = \hat{k}_i(s^t)/\hat{k}(s^t)$, implying that $q^1(s^t) a_i^1(s^t) = \hat{k}_i(s^t)$, are optimal, and they also satisfy the portfolio constraints, since:

$$\sum_{l \geq 2} q^l(s^t) a_i^l(s^t) + q^1(s^t) a_i^1(s^t) = \sum_{l \geq 2} \hat{q}^l(s^t) \hat{a}_i^l(s^t) + \hat{k}_i(s^t) \geq \hat{B}_i(s^t) = B_i(s^t)$$

Finally, the fact that $\{\hat{c}_i, k_i, (\hat{a}_i)_{l \geq 2}\}$ generate market clearing in the k -economy implies that the

allocations also clear the markets in the e -economy. To see this note that:

$$\sum_{i \in I} c_i(s^t) = \sum_{i \in I} \widehat{c}_i(s^t) = F(z(s^t), \widehat{k}(s^{t-1}), \widehat{n}(s^t)) - \widehat{k}(s^t) + \sum_{l \geq 2} \widehat{d}^l(s^t) \widehat{A}^l = \widehat{w}_c(s^t) = w_c(s^t)$$

$$\sum_{l \geq 2} a_i^l(s^t) = \sum_{i \in I} \widehat{a}_i^l(s^t) = \widehat{A}^l = A^l \text{ for } l \geq 2$$

$$\sum_{i \in I} a_i^1(s^t) = \sum_{i \in I} \frac{\widehat{k}_i(s^t)}{\widehat{k}(s^t)} = 1 = A^1$$

This establishes the result. ■

REFERENCES

- Aiyagari D. R. (1994): Uninsured Idiosyncratic Risk and Aggregate Saving, *The Quarterly Journal of Economics*, 109 (3), 659-684
- Angeletos. G. M., (2005): Uninsured Idiosyncratic Investment Risk and Aggregate Saving, manuscript
- Angeletos. G. M. and Calvet L.(2003): Idiosyncratic Production Risk, Growth and the Business Cycle, *NBER Working Paper* 9764, forthcoming in *Journal of Monetary Economics*.
- Angeletos. G. M. and Calvet L. (2005), "Incomplete Market Dynamics in a Neoclassical Growth Economy", *Journal of Mathematical Economics* (41), 407-438.
- Bewley, T. F. (1986): Stationary Monetary Equilibrium with a Continuum of Independently Fluctuating Consumers, in Hildebrand, W. and A. Mas-Colell (eds.), *Contributions of Mathematical Economics in Honor of Gerard Debreu*, Amsterdam: North-Holland, 79-102.
- Bewley, T. F. (1977): The Permanent Income Hypothesis: A Theoretical Formulation, *Journal of Economic Theory*, 16 (2), 252-292.
- Carceles-Poveda E. and Coen-Pirani, D., (2005): Shareholders Unanimity with Incomplete Markets, manuscript.
- DeMarzo, P.M. (1993): Majority Voting and Corporate Control: The Rule of the Dominant Shareholder, *Review of Economic Studies*, 60, 713-734.
- DeMarzo, P.M. (1988): An Extension of the Modigliani-Miller Theorem to Stochastic Economies with Incomplete Markets and Interdependent Securities, *Journal of Economic Theory*, 45, 353-369.
- Diamond, P. A. (1967): The Role of a Stock Market in a General Equilibrium Model with Technological Uncertainty, *American Economic Review*, 57, 759-76.
- Drèze, J. H. (1985): (Uncertainty and) the Firm in General Equilibrium Theory, *Economic Journal*, 95, 1-20.
- Drèze, J. H. (1974): Investment under Private Ownership: Optimality, Equilibrium and Stability, in *Allocation under Uncertainty: Equilibrium and Optimality*, ed. by Drèze, J., NY, Macmillan.
- Duffie, D., and W. Shaffer (1986a): Equilibrium in Incomplete Markets II: Generic Existence for Stochastic Economies, *Journal of Mathematical Economics* 15, 199-216.
- Duffie, D., and W. Shaffer (1986b), Equilibrium and the Role of the Firm in Incomplete Markets, manuscript.
- Duffie, D., and W. Shaffer (1985), Equilibrium in Incomplete Markets I: A Basic Model of Generic Existence, *Journal of Mathematical Economics* 14, 285-300.
- Grossmann, S. J. and O. D. Hart (1979): A Theory of Competitive Equilibrium in Stock Market Economies, *Econometrica*, 47 (2).
- Grossmann, S. J., and J., E. Stiglitz (1980): Stockholder Unanimity in Making Production and Financial Decisions, *The Quarterly Journal of Economics*, 94 (3), 543-566.
- Grossmann, S. J., and J., E. Stiglitz (1977), On Value Maximization and Alternative Objectives for the Firm, *The Journal of Finance*, 32 (2), 389-402.
- Hernandez, A., and M. Santos (1996), Competitive Equilibria for Infinite Horizon Economies, *Journal of Economic Theory*, 71, 102-130.
- Huggett, M., (1997): The One-Sector Growth Model with Idiosyncratic Shocks: Steady States and Dynamics, *Journal of Monetary Economics*, 39, 385- 403.
- Krusell, P. and A. Smith (1998): Income and Wealth Heterogeneity in the Macroeconomy, *Journal of Political Economy*, 106 (5).

Krusell, P. and A. Smith (1997): Income and Wealth Heterogeneity, Portfolio Choice, and Equilibrium Asset Returns, *Macroeconomic Dynamics*, 1 (2), 387-422.

Leland, H., E. (1972): Theory of the Firm Facing Uncertain Demand, *The American Economic Review*, 279-291.

Levine, D., (1989): Infinite Horizon Equilibrium with Incomplete Markets, *Journal of Mathematical Economics*, 18, 357-376.

Levine, D., and W. Zame (1996): Debt Constraints and Equilibrium in Infinite Horizon Equilibrium with Incomplete Markets, *Journal of Mathematical Economics*, 26, 103-131.

Magill, M. and M. Quinzii (1996): *Theory of Incomplete Markets*, vol. 1, The MIT Press, Cambridge, Massachusetts.

Magill, M. and M. Quinzii (1994a): Incomplete Markets over an Infinite Horizon, Long Lived Securities and Speculative Bubbles, *Journal of Mathematical Economics*, 26, 133-170.

Magill, M. and M. Quinzii (1994b): Infinite Horizon Incomplete Markets, *Econometrica*, 62 (4), 853-888.

Radner, R. (1972a): New ideas in pure theory, Problems in the Theory of Markets under Uncertainty, *American Economic Review*, 454-460.

Radner, R. (1972b), Existence of Equilibrium of Plans, Prices, and Price Expectations in a Sequence of Markets, *Econometrica*, 40 (2), 289-303.

Sandmo, A. (1972): On the Theory of the Competitive Firm Under Price Uncertainty, *The American Economic Review*, 65-73.

Santos, M., and M. Woodford (1997): Rational Asset Pricing Bubbles, *Econometrica*, 65, (1), 19-57.

Sondermann, D. (1974): Temporary Competitive Equilibrium Under Uncertainty, in *Allocation under Uncertainty: Equilibrium and Optimality*, ed. by Drèze, J., NY, Macmillan.

Storesletten, K., Telmer, C., and A. Yaron (2004): Consumption and Risk Sharing over the Life Cycle, *Journal of Monetary Economics*, 51, 609-633.

Storesletten, K., Telmer, C., and A. Yaron (2001a): The Welfare Costs of Business Cycles Revisited: Finite Lives and Cyclical Variation in Idiosyncratic Risk, *European Economic Review*, 45 (7), 1311-1339.

Storesletten, K., Telmer, C., and A. Yaron (2001b): Asset Pricing with Idiosyncratic Risk and Overlapping Generations, manuscript.