Online Appendix for ‘Variational Bayes Estimation of Discrete-Margined Copula Models with Application to Time Series’ by Loaiza-Maya and Smith.

This online appendix has two parts:

Part A: Provides additional empirical results for the VBDA fits two our multivariate and univariate time series examples.

Table A1: Copula parameter estimates for the trivariate Homicide examples using VBDA with VA1.

Table A2: Copula parameter estimates for the trivariate Homicide examples using VBDA with VA2.

Table A3: Copula parameter estimates for the bivariate Bankruptcy/VIX example using VBDA.

Figure A1: Comparison of pair-copula density estimates for the Auto-Logistic example using VA1, VA2 and VA3.

Figure A2: Comparison of posterior parameter estimates from the MCMC and VBDA method with VA1.

Figure A3: Comparison of posterior parameter estimates from the MCMC and VBDA method with VA3.

Figure A4: Pair-copula density estimates for the bivariate Bankruptcy/VIX example.

Part B: Explains how to use the MATLAB functions and provides examples.

Part A: Additional Empirical Results

	D-Vine copula: C^{MIX} with Convex Gumbel Components					
Parameters	$\tau^a > 0$	δ^a	$\tau^b > 0$	δ^b	w	Spearman
$\boldsymbol{\theta}^{(0)}_{1,2}$	0.200	0.591	0.323	0.500	0.917	0.167 (0.109,0.229)
$\boldsymbol{\theta}^{(0)}_{1,3}$	0.078	0.428	0.114	0.496	0.604	0.014 (-0.030,0.071)
$\boldsymbol{\theta}^{(0)}_{2,3}$	0.130	0.689	0.167	0.473	0.782	0.077 (0.025,0.139)
$\boldsymbol{\theta}^{(1)}_{1,1}$	0.097	0.349	0.165	0.488	0.605	0.009 (-0.055,0.071)
$\boldsymbol{\theta}^{(1)}_{1,2}$	0.165	0.486	0.122	0.479	0.663	0.072 (0.011,0.151)
$\boldsymbol{\theta}^{(1)}_{1,3}$	0.106	0.434	0.124	0.491	0.503	0.005 (-0.065,0.051)
$\boldsymbol{\theta}^{(1)}_{2,1}$	0.195	0.734	0.176	0.481	0.854	0.149 (0.088,0.216)
$\boldsymbol{\theta}^{(1)}_{2,2}$	0.336	0.690	0.193	0.489	0.924	0.299 (0.230,0.369)
$\boldsymbol{\theta}^{(1)}_{2,3}$	0.133	0.521	0.105	0.513	0.531	0.017 (-0.037,0.086)
$\boldsymbol{\theta}^{(1)}_{3,1}$	0.126	0.489	0.095	0.468	0.497	0.010 (-0.040,0.069)
$\boldsymbol{\theta}^{(1)}_{3,2}$	0.132	0.496	0.145	0.490	0.698	0.064 (0.008,0.135)
$\boldsymbol{\theta}^{(1)}_{3,3}$	0.139	0.564	0.124	0.501	0.564	0.026 (-0.029,0.096)

Table A1: The VBDA posterior means of the pair-copula parameters for the D-Vine copula fitted to the three-dimensional crime series using approximation VA1 and $K = 15$. The posterior mean and 90% probability intervals are also given for the Spearman's rho of each pair copula. Murder, Attempted Murder and Manslaughter counts are denoted as series 1, 2 and 3, respectively.

	D-Vine copula: C^{MIX} with Convex Gumbel Components					
Parameters	$\tau^a > 0$	δ^a	$\tau^b > 0$	δ^b	w	Spearman
$\boldsymbol{\theta}_{1,2}^{(0)}$	0.201	0.546	0.349	0.510	0.927	0.169 (0.111,0.230)
$\boldsymbol{\theta}_{1,3}^{(0)}$	0.088	0.390	0.126	0.496	0.619	0.020 (-0.027,0.079)
$\boldsymbol{\theta}_{2,3}^{(0)}$	0.147	0.655	0.186	0.501	0.804	0.093 (0.039,0.153)
$\boldsymbol{\theta}_{1,1}^{(1)}$	0.103	0.353	0.163	0.480	0.594	0.008 (-0.057,0.073)
$\boldsymbol{\theta}_{1,2}^{(1)}$	0.175	0.475	0.119	0.487	0.661	0.076 (0.013,0.158)
$\boldsymbol{\theta}_{1,3}^{(1)}$	0.105	0.412	0.142	0.505	0.513	-0.006 (-0.070,0.054)
$\boldsymbol{\theta}_{2,1}^{(1)}$	0.201	0.706	0.175	0.479	0.852	0.153 (0.094,0.217)
$\boldsymbol{\theta}_{2,2}^{(1)}$	0.339	0.625	0.213	0.469	0.924	0.301 (0.232,0.372)
$\boldsymbol{\theta}_{2,3}^{(1)}$	0.139	0.502	0.112	0.529	0.545	0.023 (-0.037,0.094)
$\boldsymbol{\theta}_{3,1}^{(1)}$	0.135	0.480	0.093	0.483	0.505	0.012 (-0.038,0.073)
$\boldsymbol{\theta}_{3,2}^{(1)}$	0.176	0.415	0.131	0.483	0.695	0.085 (0.024,0.159)
$\boldsymbol{\theta}_{3,3}^{(1)}$	0.148	0.558	0.139	0.503	0.563	0.027 (-0.037,0.107)

Table A2: The VBDA posterior means of the pair-copula parameters for the D-Vine copula fitted to the three-dimensional crime series using approximation VA3 and $K = 15$. The posterior mean and 90% probability intervals are also given for the Spearman's rho of each pair copula. Murder, Attempted Murder and Manslaughter counts are denoted as series 1, 2 and 3, respectively.

Parameters	D-Vine copula: C^{MIX} with Convex Gumbel Components $\tau^a > 0$	δ^a	$\tau^b > 0$	δ^b	w	Spearman
$\boldsymbol{\theta}_{1,2}^{(0)}$	0.134	0.207	0.102	0.421	0.713	0.071
						(0.030,0.112)
$\boldsymbol{\theta}_{1,1}^{(1)}$	0.360	0.870	0.326	0.491	0.940	0.321
						(0.259,0.384)
$\boldsymbol{\theta}_{1,2}^{(1)}$	0.122	0.458	0.121	0.544	0.608	0.032
						(-0.017,0.101)
$\boldsymbol{\theta}_{2,1}^{(1)}$	0.191	0.232	0.107	0.470	0.814	0.139
						(0.095,0.184)
$\boldsymbol{\theta}_{2,2}^{(1)}$	0.719	0.693	0.241	0.509	0.979	0.699
						(0.662,0.734)
$\boldsymbol{\theta}_{1,1}^{(2)}$	0.152	0.641	0.135	0.478	0.801	0.103
						(0.042,0.176)
$\boldsymbol{\theta}_{1,2}^{(2)}$	0.112	0.414	0.100	0.483	0.703	0.056
						(0.007,0.127)
$\boldsymbol{\theta}_{2,1}^{(2)}$	0.095	0.515	0.084	0.477	0.585	0.022
						(-0.022,0.083)
$\boldsymbol{\theta}_{2,2}^{(2)}$	0.235	0.272	0.174	0.575	0.436	-0.009
						(-0.079,0.057)

Table A3: The VBDA posterior means of the pair-copula parameters for the D-Vine copula fitted to the bankruptcy counts and VIX using approximation VA2 and $K = 15$. The posterior mean and 90% probability intervals are also given for the Spearman's rho of each pair copula. Bankruptcy counts and VIX are denoted as series 1 and 2, respectively.

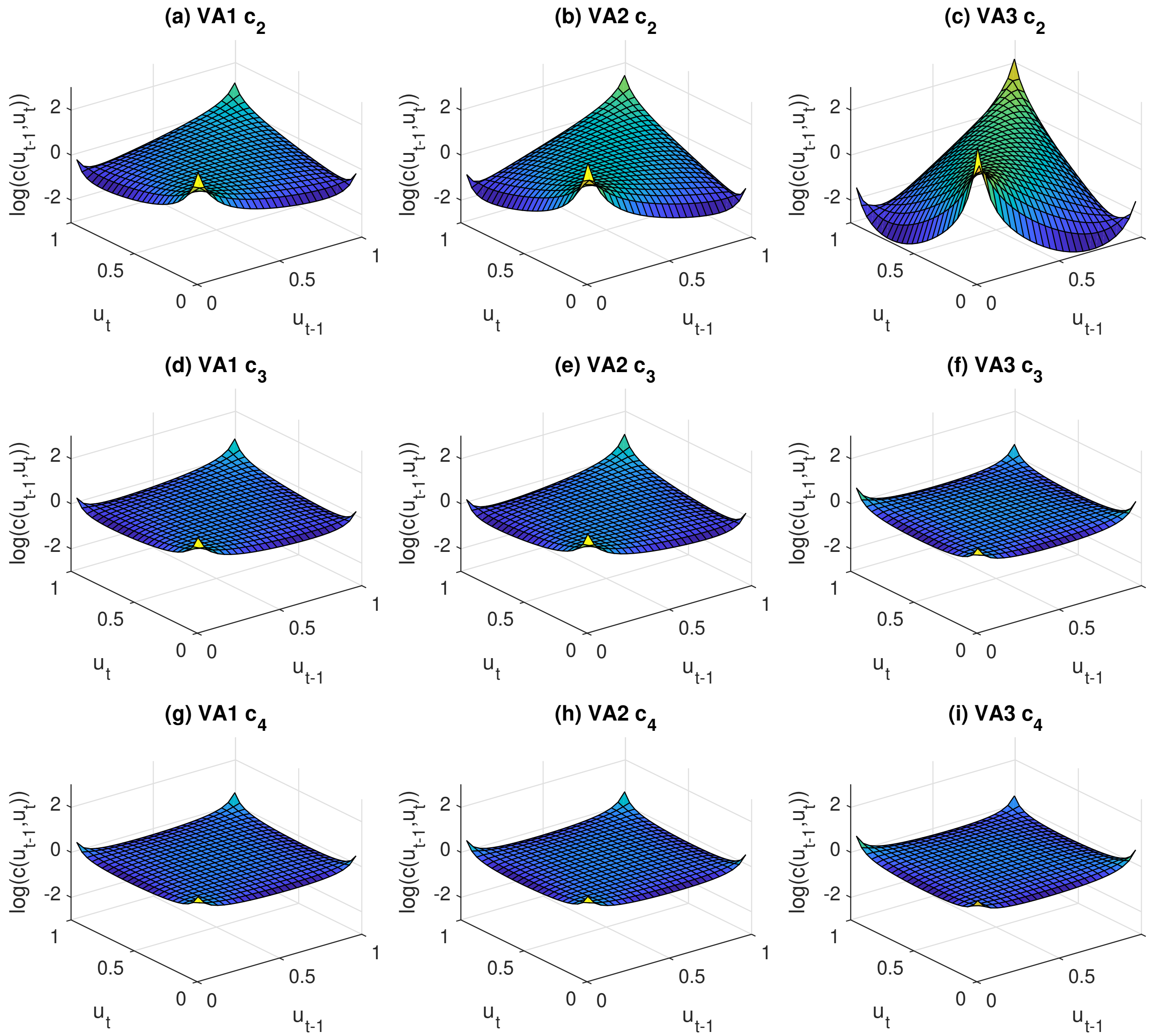

Figure A1: Log-density function of the fitted pair-copulas at the variational posterior mean parameter values for the Auto-Logistic model. Rows one to three correspond to the pair-copulas c_2, c_3 and c_4 respectively. Columns one to three correspond to the approximations, VA1, VA2 and VA3, respectively.

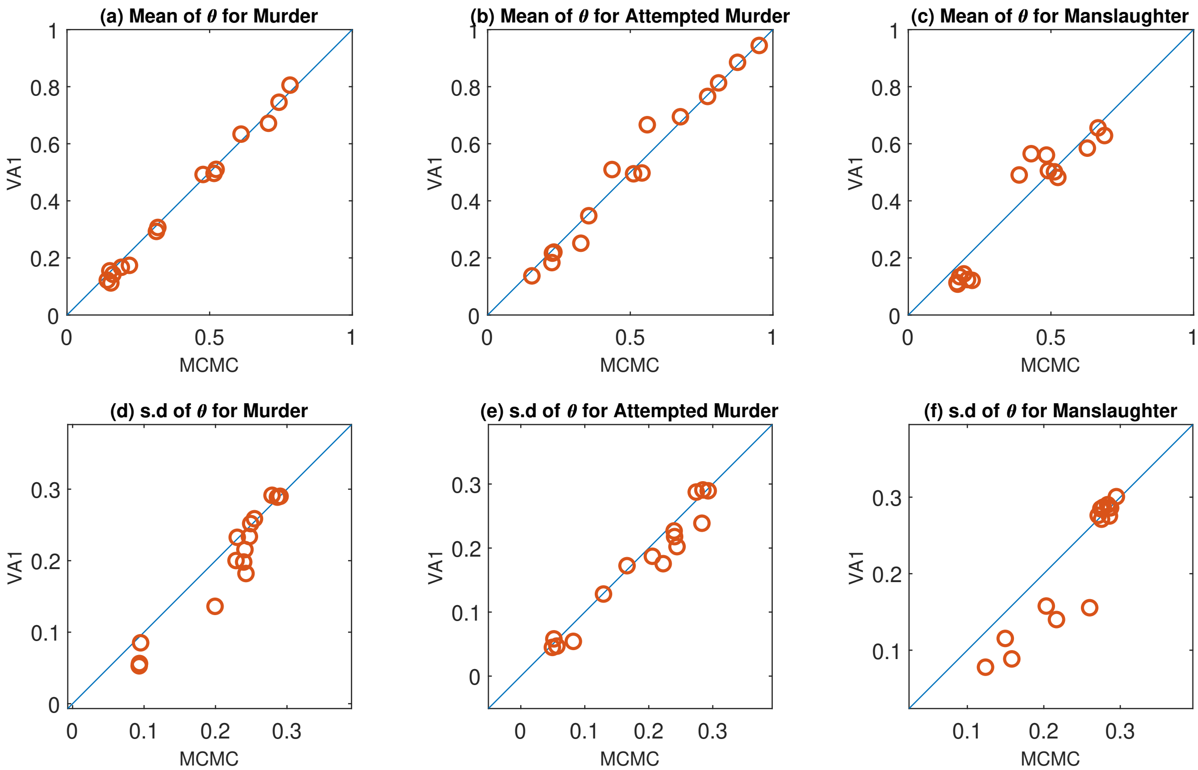

Figure A2: Comparison of the parameter posterior means and standard deviations from the MCMC and VBDA (with VA1) methods for the copula models of each crime count time series. The first row compares the posterior means, while the second row the posterior standard deviations. Each column corresponds to a different crime series.

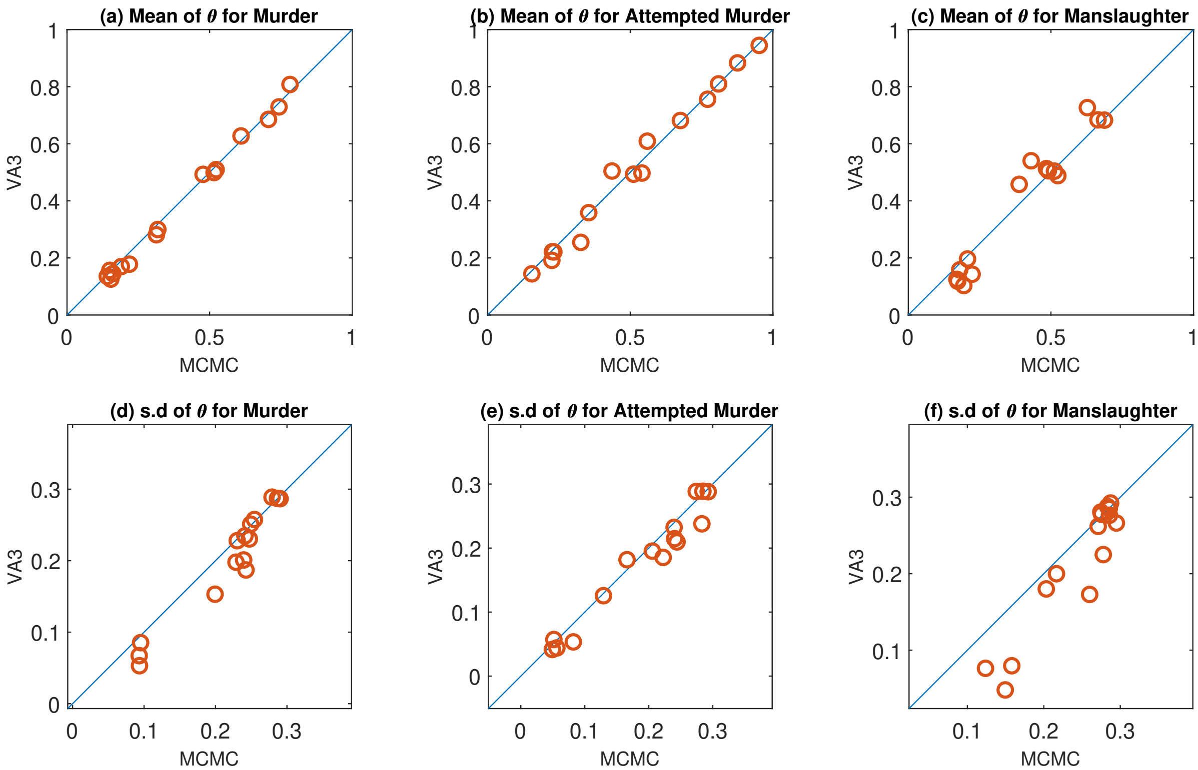

Figure A3: Comparison of the parameter posterior means and standard deviations from the MCMC and VBDA (with VA3) methods for the copula models of each crime count time series. The first row compares the posterior means, while the second row the posterior standard deviations. Each column corresponds to a different crime series.

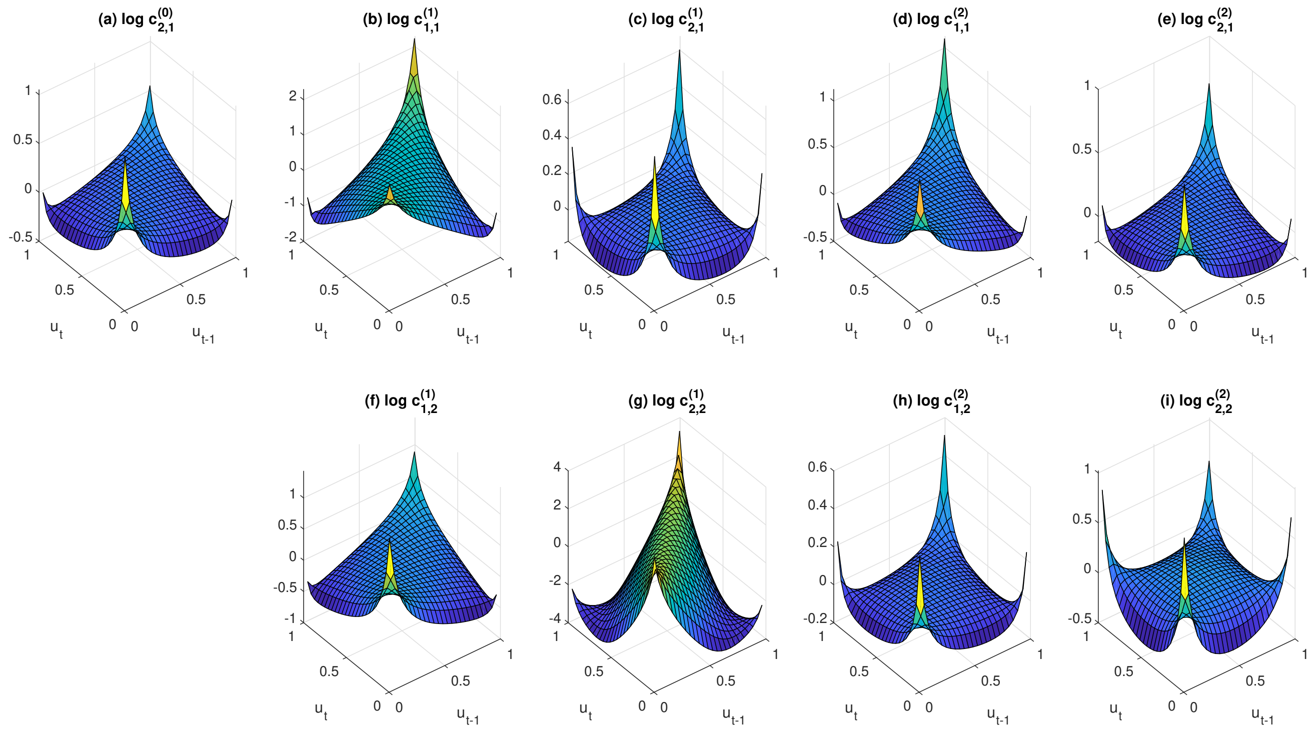

Figure A4: Log-density function of the fitted pair-copulas at the VBDA (with VA2 and $K = 15$) estimates of the posterior mean parameter values for the bivariate bankruptcy model. The superscript indicates the lagging in the second argument of the pair-copula, while the subscripts 1 and 2 correspond to the Bankruptcy and VIX variables, respectively.

Part B: MATLAB functions and examples

Main Functions

`VBDAfit_uni` *Estimates copula models for univariate time series data using VBDA*

Inputs:

`y`	Data vector of dimension $T \times 1$.
`mdist`	MATLAB probability distribution object fitted to `y`.
`family`	Pair-copula family to use for D-Vine (only 'gumbel_mix' is available).
`p`	Markov order imposed on the D-Vine copula.
`k`	Number of factors in variational approximation.
`S`	Number of iterates used to compute gradient.
`nVB`	Total number of variational Bayes steps to be taken for inference.
`VA`	Selection of variational approximation. Set to 1 for VA1, to 2 for VA2 and to 3 for VA3.
`filename`	Optional MATLAB file name to export VBDA estimates. `filename` must end with the extension `.mat` (`filename`='example.mat'). If empty, no `.mat` file is printed.

Outputs: The function produces an object called `VBDAobj`. To call each argument use the format `VBDAobj.arg`, where `arg` can be either any of the input names, or the following:

`gamma_mean`	The variational posterior mean of the D-Vine copula parameters.
`gamma_sd`	The variational standard deviation of the D-Vine copula parameters.
`mu`	Mean vector of approximation to the copula parameters distribution.
`B`	Factor matrix of the VB approximation covariance matrix.
`D`	Diagonal matrix of the VB approximation covariance matrix.
`muz`	For `VA=1` is empty. For `VA={2,3}` is the vector with the means of the variational approximation.
`logsigmaz`	For `VA={1,3}` is empty. For `VA=2` is the vector with the logarithm of the standard deviations of the variational approximation.

`C`	For `VA={1,2}` is empty. For `VA=2` is the vector with the logarithm of the standard deviations of the variational approximation.
`lambda`	Vector with all the variational parameters of the VBDA method.
`LB`	Lower bound at each VB step.

`VBDAfit_multi` *Estimates copula models for multivariate time series data using VBDA*

Inputs:

`y`	Data matrix of dimension $T \times r$.
`type`	A MATLAB cell of size $1 \times r$. Each element indicates if corresponding column in `y` is discrete or continuous. Set `type` to `'discrete'` or `'continuous'`.
`mdist`	A MATLAB cell of size $1 \times r$, each element is a distribution object fitted to corresponding column in `y`.
`family`	Pair-copula family to use for D-Vine (only 'gumbel_mix' is available).
`p`	Markov order imposed on the D-Vine copula.
`k`	Number of factors in variational approximation.
`S`	Number of iterates used to compute gradient.
`nVB`	Total number of variational Bayes steps to be taken for inference.
`VA`	Selection of variational approximation. Set to 1 for VA1, to 2 for VA2 and to 3 for VA3.
`filename`	Optional MATLAB file name to export VBDA estimates. `filename` must end with the extension `.mat` (`filename`='example.mat'). If empty, no `.mat` file is printed.

Outputs: The function produces an object called `VBDAobj`. To call each argument use the format `VBDAobj.arg`, where `arg` can be either any of the input names, or the following:

`gamma_mean`	The variational posterior mean of the D-Vine copula parameters.
`gamma_sd`	The variational standard deviation of the D-Vine copula parameters.
`mu`	Mean vector of approximation to the copula parameters distribution.
`B`	Factor matrix of the VB approximation covariance matrix.
`D`	Diagonal matrix of the VB approximation covariance matrix.
`muz`	For `VA=1` is empty. For `VA={2,3}` is the vector with the means of the

variational approximation.

`logsigmaz`	For `VA={1,3}` is empty. For `VA=2` is the vector with the logarithm of the standard deviations of the variational approximation.
`C`	For `VA={1,2}` is empty. For `VA=2` is the vector with the logarithm of the standard deviations of the variational approximation.
`lambda`	Vector with all the variational parameters of the VBDA method.
`LB`	Lower bound at each VB step.

`VBDApredict` *Generation from predictive densities of copula model*

Inputs:

`VBDAobj`	Object as produced by either function `VBDAfit_uni` or `VBDAfit_multi`.
`h`	Predictive horizon.
`n`	Number of draws.

Outputs:

`y_pred`	A matrix of dimension $h \times r \times n$ with predictions from the copula model.

`VBDAspearman_uni` *Spearman's correlations from fitted copula model for univariate times series*

Inputs:

`VBDAobj`	Object as produced by function `VBDAfit_uni`.

Outputs:

`rhoSmean`	Posterior mean of pairwise Spearman's correlations.
`rhoSquant`	5% and 95% quantiles of pairwise Spearman's correlations.

`VBDAspearman_multi` *Spearman's correlations from fitted copula model for multivariate times series*

Inputs:

`VBDAobj` Object as produced by function `VBDAfit_multi`.

Outputs:

`rhoSmean` Posterior mean of pairwise Spearman's correlations.

`rhoSquant` 5% and 95% quantiles of pairwise Spearman's correlations.

`VBDAplots` *Summary plots from VBDA estimation*

Inputs:

`VBDAobj` Object as produced by either function `VBDAfit_uni` or `VBDAfit_multi`.

Outputs: This function produces three plots. 1) The data and the relative frequencies; 2) All the pair-copulas from the fitted D-Vine model; 3) The lower bound.

Examples

Example 1: Fitting Auto-Logistic example with binomial margin

```
%-------------  Using VBDA on a univariate time series  ------------------------%
clear
family = 'gumbel_mix';     %Copula family for pair-copula components
p = 3;                     %D-Vine Markov order
k = 3;                     %Number of factors for covariance matrix
S = 500;                   %Number of evaluations to compute gradient
nVB = 5000;                %Number of VB steps
y = pickdata('Logit');     %Ordinal time series, change to your own data
mdist = fitdist(y,'Binomial'); %Change to your own marginal density
VA = 2;                    %Variational Bayes approach for estimation
VBDAobj = VBDAfit_uni(y,mdist,family,p,k,S,nVB,VA);%Using VBDA
VBDAplots(VBDAobj)         %Producing plots from object VBDAobj
[rhoSmean,rhoSquant,rhoS] = VBDAspearman_uni(VBDAobj); %Spearman's rhos
h = 8; n = 10;
y_pred = VBDApredict(VBDAobj,h,n); %Predicting from VBDAobj h steps ahead
```

Example 2: Fitting Homicide example with negative binomial margins

```
%---------------  Using VBDA on multivariate time series  ----------------------%
clear
family = 'gumbel_mix';         %Copula family for pair-copula components
p = 1;                         %D-Vine Markov order
k = 15;                        %Number of factors for covariance matrix
S = 500;                       %Number of evaluations to compute gradient
nVB = 5000;                    %Number of VB steps
VA = 2;                        %Variational Bayes approach for estimation
[y,margintype] = pickdata('Homicide');    %Ordinal time series, change to your own data
r = size(y,2);
mdist = cell(r,1);            %Marginal distribution of y
for i = 1:r                   %Change to your margins of choice
    mdist{i} = fitdist(y(:,i),'NegativeBinomial');
end
VBDAobj = VBDAfit_multi(y,margintype,mdist,family,p,k,S,nVB,VA);%Using VBDA
[rsMean,rsQuant,rhoS] = VBDAspearman_multi(VBDAobj); %Spearman's rhos
VBDAplots(VBDAobj) %Producing plots from object VBDAobj
h = 8;
n = 10;
y_pred = VBDApredict(VBDAobj,h,n); %Predicting from VBDAobj h steps ahead
```