1 Reduced models of short term synaptic plasticity

Let c, p, and x represent the intracellular concentration of Ca^{2+} at the presynaptic terminal, the proportion of released vesicles per unit time, and the normalized readily releasable vesicles. Assume that the dynamics are given by

$$\begin{aligned}
\partial_t c &= \frac{c_\infty - c}{\tau_c} + f(t) &(1)\\
\partial_t p &= \alpha\, c\,(1-p) - \beta p &(2)\\
\partial_t x &= x\left(\frac{x_\infty - x}{\tau_x}\right) - x\, p &(3)
\end{aligned}$$

where $f(c)$ represents the flux of Ca^{2+}into the terminal.

1.1 Dynamics for c

Assume that the flux of calcium is a Dirac comb given by

$$f(t) = h\sum_{k=1}^{n} \delta\left(t - t_k\right) \tag{4}$$

where t_0, ..., t_n represent stimulus times. Suppose that, at the kth pulse time, the value of $c(t)$ changes to $c_k = c(t_k) + h$. Assuming that $c(t) = c_\infty$ for $t \leq t_0$, then $c_0 = c_\infty + h$ and

$$\begin{aligned}
c(t) &= c_\infty - (c_\infty - c_0)\exp\left(\frac{t_0 - t}{\tau_c}\right)\\
&= c_\infty - (c_\infty - c_\infty - h)\exp\left(\frac{t_0 - t}{\tau_c}\right)\\
&= c_\infty + h\exp\left(\frac{t_0 - t}{\tau_c}\right)
\end{aligned}$$

for $t \in (t_0, t_1)$. At $t = t_1$ the calcium concentration changes again, to $c_1 = c(t_1) + h$

$$\begin{aligned}
c(t) &= c_\infty - (c_\infty - c_1)\exp\left(\frac{t_1 - t}{\tau_c}\right)\\
&= c_\infty - \left(c_\infty - c_\infty - h - h\exp\left(\frac{t_0 - t_1}{\tau_c}\right)\right)\exp\left(\frac{t_1 - t}{\tau_c}\right)\\
&= c_\infty + h\left(1 + \exp\left(\frac{t_0 - t_1}{\tau_c}\right)\right)\exp\left(\frac{t_1 - t}{\tau_c}\right)\\
&= c_\infty + h\left(\exp\left(\frac{t_1 - t}{\tau_c}\right) + \exp\left(\frac{t_0 - t}{\tau_c}\right)\right),
\end{aligned}$$

for $t \in (t_1, t_2)$. In general, for $t \in (t_{n-1}, t_n)$,

$$c(t) = c_\infty + h\sum_{k=0}^{n-1}\exp\left(\frac{t_k - t}{\tau_c}\right)$$

And at t_n the value of c

$$c(t_n) = c_\infty + h\sum_{k=0}^{n}\exp\left(\frac{t_k - t_n}{\tau_c}\right)$$

If pulses are periodic, with $d = t_{k+1} - t_k$, for all $k \in \{1, ..., n\}$ then $t_n - t_k = (n-k)d$. In this case, let

$$u = \exp\left(-\frac{d}{\tau_c}\right), \tag{5}$$

so that

$$\begin{aligned} c(t_n) &= c_\infty + h \sum_{k=0}^{n} u^{(n-k)}, \\ &= c_\infty + h\left(\frac{1-u^n}{1-u}\right). \end{aligned}$$

Explicitly,

$$c(t_n) = c_\infty + h\left(\frac{1-\exp\left(-n\frac{d}{\tau_c}\right)}{1-\exp\left(-\frac{d}{\tau_c}\right)}\right). \tag{6}$$

The asymptotic behavior as $n \to \infty$ is then

$$c_* = c_\infty + h\left(\frac{1}{1-\exp\left(-\frac{d}{\tau_c}\right)}\right). \tag{7}$$

1.2 p dynamics and the subsystem $c - p$

The dynamics of the subsystem c, p are given by

$$\partial_t c = \frac{c_\infty - c}{\tau_c} + f(t) \tag{8}$$

$$\partial_t p = \alpha\, c\,(1-p) - \beta p \tag{9}$$

The time constant and steady state for the p are

$$\tau_p(c) = \frac{1}{\alpha\, c + \beta} \tag{10}$$

$$p_\infty(c) = \frac{\alpha\, c}{\alpha\, c + \beta} = \frac{c}{c + \frac{\beta}{\alpha}} \tag{11}$$

c increases as action potentials arrive, then $\tau_p(c)$ decreases and $p_\infty(c)$ increases. So p increases with c, but the dynamics for p become faster as that happens.

if $\tau_c >> \tau_p$, then the dynamics for p are fast enough to substitute p with $p_\infty(c)$. Explicitly, it would be required that

$$\tau_c > \frac{1}{\alpha\, c + \beta}.$$