# Online Supplementary Materials for "A Mallows-type Model Averaging Estimator for the Varying-Coefficient Partially Linear Model"

The online supplemental file contains the proofs of Lemmas 1 and 2 and detailed proofs of Theorems 1 and 2. The main challenge confronting our proofs is twofold. First, our proofs require cumbersome matrix operations of complex combinations of random variables; e.g., the quantity $\frac{1}{n}D_t^{\mathrm{T}}W_{(m)t}D_t$ to appear in Lemma 1 contains not only $Z$ and $\Psi$, but also the kernel function $K_{h_m}(\cdot)$ and the bandwidth $h_m$. In our proof, we rewrite each of these elements as a non-random quantity plus a term bounded in probability. Second, for the VCPLM, the project matrix $P_{(m)}$ in Equation (9) is neither symmetric nor idempotent as in a linear model. Thus, the usual technique for decomposing $P_{(m)}$ is inapplicable to VCPLM. We overcome this difficulty by writing $P_{(m)} = \hat{P}_{(m)}(I_n - A_{(m)}) + A_{(m)}$, where $A_{(m)} = BC_{(m)}E_{(m)}F_{(m)}H_{(m)}$. See the proof of Lemma 2.

## S.1    Preliminary results

The proofs of Theorems 1 and 2 require the following lemmas.

**Lemma 1.** *Let Conditions (C.3)-(C.5) hold. Then for all $m = 1, 2, ..., M$ and $t \in \Delta$, we have*

$$\frac{1}{n}D_t^{\mathrm{T}}W_{(m)t}D_t = \begin{pmatrix} f(t) + O_{up}(h_m^2) & \mu_2(K)h_m f'(t) + O_{up}(h_m^2) \\ \mu_2(K)h_m f'(t) + O_{up}(h_m^2) & \mu_2(K)f(t) + O_{up}(h_m^2) \end{pmatrix} \otimes C_z \qquad \text{(S.1)}$$

*and*

$$\left\{\frac{1}{n}D_t^{\mathrm{T}}W_{(m)t}D_t\right\}^{-1} = \begin{pmatrix} f^{-1}(t) + O_{up}(h_m^2) & -h_m f'(t)f^{-2}(t) + O_{up}(h_m^2) \\ -h_m f'(t)f^{-2}(t) + O_{up}(h_m^2) & \mu_2^{-1}(K)f^{-1}(t) + O_{up}(h_m^2) \end{pmatrix} \otimes C_z^{-1} \text{(S.2)}$$

*where $\mu_2(K) = \int_{v \in supp(K)} K(v)v^2 dv$, and if a function $g(t) = O_{up}(b_m^2)$, then $g(t)/b_m^2$ is bounded in probability uniformly for any $t$ within the interior of $\Delta$.*

**Proof**  Note that

$$\left\{\frac{1}{n}D_t^{\mathrm{T}}W_{(m)t}D_t\right\}^{-1}$$

$$= \begin{pmatrix} \frac{1}{n}\sum_{l=1}^{n} z_l z_l^{\mathrm{T}} K_{h_m}(t_l - t) & \frac{1}{n}\sum_{l=1}^{n} z_l z_l^{\mathrm{T}} K_{h_m}(t_l - t)\frac{t_l - t}{h_m} \\ \frac{1}{n}\sum_{l=1}^{n} z_l z_l^{\mathrm{T}} K_{h_m}(t_l - t)\frac{t_l - t}{h_m} & \frac{1}{n}\sum_{l=1}^{n} z_l z_l^{\mathrm{T}} K_{h_m}(t_l - t)\left(\frac{t_l - t}{h_m}\right)^2 \end{pmatrix}^{-1}$$

$$\equiv \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}^{-1}. \tag{S.3}$$

Therefore, we have to show

$$J_{11} = \frac{1}{n} \sum_{l=1}^{n} z_l z_l^{\mathrm{T}} K_{h_m}(t_l - t) = C_z f(t) + O_{up}(h_m^2 I_R), \tag{S.4}$$

$$J_{12} = \frac{1}{n} \sum_{l=1}^{n} z_l z_l^{\mathrm{T}} K_{h_m}(t_l - t) \frac{t_l - t}{h_m} = C_z \mu_2(K) h_m f'(t) + O_{up}(h_m^2 I_R), \tag{S.5}$$

$$J_{21} = J_{12}^{\mathrm{T}} = C_z \mu_2(K) h_m f'(t) + O_{up}(h_m^2 I_R), \tag{S.6}$$

and

$$J_{22} = \frac{1}{n} \sum_{l=1}^{n} z_l z_l^{\mathrm{T}} K_{h_m}(t_l - t) \left( \frac{t_l - t}{h_m} \right)^2 = C_z \mu_2(K) f(t) + O_{up}(h_m^2 I_R). \tag{S.7}$$

Let us consider $z_{li} z_{lk}$ and $J_{11,ik}$, which are the $ik$th element of $z_l z_l^{\mathrm{T}}$ and $J_{11}$ respectively. Let $C_{z,ik}$ be the $ik$th element of $C_z$. By Conditions (C.3)-(C.5), we have

$$
\begin{aligned}
&E(J_{11,ik}) \\
=~& E\left\{ \frac{1}{n} \sum_{l=1}^{n} z_{li} z_{lk} K_{h_m}(t_l - t) \right\} \\
=~& E\{ z_{1i} z_{1k} K_{h_m}(t_1 - t) \} \\
=~& E\{ z_{1i} z_{1k} \} E\{ K_{h_m}(t_1 - t) \} \\
=~& C_{z,ik} \int_{t_1 \in \Delta} \frac{1}{h_m} K\left( \frac{t_1 - t}{h_m} \right) f(t_1) dt_1 \\
=~& C_{z,ik} \int_{v \in supp(K)} \frac{1}{h_m} K(v) f(t + h_m v) h_m dv \\
=~& C_{z,ik} \int_{v \in supp(K)} K(v) \left\{ f(t) + f'(t) h_m v + \frac{1}{2} f''(t^*) h_m^2 v^2 \right\} dv \\
=~& C_{z,ik} \left\{ f(t) + \frac{1}{2} h_m^2 \int_{v \in supp(K)} f''(t^*) K(v) v^2 dv \right\},
\end{aligned}
\tag{S.8}
$$

with $t^*$ lying between $t$ and $t + h_m v$, (the notation $t^*$ is used in subsequent formulae, although the bounds for $t^*$ may not be the same everywhere), and

$$var(J_{11,ik})$$

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$$= var\left\{\frac{1}{n}\sum_{l=1}^{n}z_{li}z_{lk}K_{h_m}(t_l-t)\right\}$$

$$= \frac{1}{n}var\left\{z_{1i}z_{1k}K_{h_m}(t_1-t)\right\}$$

$$= \frac{1}{n}\{E\{z_{1i}z_{1k}K_{h_m}(t_1-t)\}^2-[E\{z_{1i}z_{1k}K_{h_m}(t_1-t)\}]^2\}$$

$$= \frac{1}{n}[E\{(z_{1i}z_{1k})^2\}E\{K^2_{h_m}(t_1-t)\}-E^2(z_{1i}z_{1k})E^2\{K_{h_m}(t_1-t)\}]$$

$$\leq \frac{1}{n}\kappa_z\int_{t_1\in\Delta}\frac{1}{h_m^2}K^2\left(\frac{t_1-t}{h_m}\right)f(t_1)dt_1-\frac{1}{n}C^2_{z,ik}\left\{f(t)+\frac{1}{2}h_m^2\int_{v\in supp(K)}f''(t^*)K(v)v^2dv\right\}^2$$

$$= \frac{1}{n}\kappa_z\int_{v\in supp(K)}\frac{1}{h_m^2}K^2(v)f(t+h_mv)h_mdv-\frac{1}{n}C^2_{z,ik}\left\{f(t)+\frac{1}{2}h_m^2\int_{v\in supp(K)}f''(t^*)K(v)v^2dv\right\}^2$$

$$= \frac{1}{n}\kappa_z\int_{v\in supp(K)}\frac{1}{h_m^2}K^2(v)\left\{f(t)+f'(t)h_mv+\frac{1}{2}f''(t^*)h_m^2v^2\right\}h_mdv$$
$$-\frac{1}{n}C^2_{z,ik}\left\{f(t)+\frac{1}{2}h_m^2\int_{v\in supp(K)}f''(t^*)K(v)v^2dv\right\}^2$$

$$= \frac{1}{n}\kappa_z\left\{f(t)\frac{1}{h_m}\int_{v\in supp(K)}K^2(v)dv+f'(t)\int_{v\in supp(K)}K^2(v)vdv+\frac{1}{2}h_m\int_{v\in supp(K)}K^2(v)f''(t^*)v^2dv\right\}$$
$$-\frac{1}{n}C^2_{z,ik}\left\{f(t)+\frac{1}{2}h_m^2\int_{v\in supp(K)}f''(t^*)K(v)v^2dv\right\}^2$$

$$= \frac{1}{n}\kappa_z\left\{f(t)\frac{1}{h_m}\int_{v\in supp(K)}K^2(v)dv+\frac{1}{2}h_m\int_{v\in supp(K)}K^2(v)f''(t^*)v^2dv\right\}$$
$$-\frac{1}{n}C^2_{z,ik}\left\{f(t)+\frac{1}{2}h_m^2\int_{v\in supp(K)}f''(t^*)K(v)v^2dv\right\}^2$$

$$= \frac{1}{nh_m}\kappa_zf(t)R(K)+O\left(\frac{1}{n}\right), \tag{S.9}$$

where $R(K)=\int_{v\in supp(K)}K^2(v)dv$. As $f(\cdot)$ and $f''(\cdot)$ are bounded in $\Delta$, and $K(v)$ is bounded in $supp(K)$ and $R(K)<\infty$, we have

$$J_{11}$$
$$= C_z\left\{f(t)+\frac{1}{2}h_m^2\int_{v\in supp(K)}f''(t^*)K(v)v^2dv\right\}+O_p\left(\sqrt{\frac{1}{nh_m}\kappa_zf(t)R(K)+O\left(\frac{1}{n}\right)}I_R\right)$$
$$= C_zf(t)+O_{up}(h_m^2I_R), \tag{S.10}$$

which is (S.4).

Similarly,

$$E(J_{12,ik})$$

$$= E\left\{\frac{1}{n}\sum_{l=1}^{n}z_{li}z_{lk}K_{h_m}(t_l-t)\frac{t_l-t}{h_m}\right\}$$

$$= E\left\{z_{1i}z_{1k}K_{h_m}(t_1-t)\frac{t_1-t}{h_m}\right\}$$

$$= E(z_{1i}z_{1k})E\left\{K_{h_m}(t_1-t)\frac{t_1-t}{h_m}\right\}$$

$$= E(z_{1i}z_{1k})\int_{t_1\in\Delta}\frac{1}{h_m}K\left(\frac{t_1-t}{h_m}\right)\frac{t_1-t}{h_m}f(t_1)dt_1$$

$$= C_{z,ik}\int_{v\in supp(K)}\frac{1}{h_m}K(v)vf(t+h_mv)h_mdv$$

$$= C_{z,ik}\int_{v\in supp(K)}\frac{1}{h_m}K(v)v\left\{f(t)+f'(t)h_mv+\frac{1}{2}f''(t^*)h_m^2v^2\right\}h_mdv$$

$$= C_{z,ik}\left\{f'(t_j)h_m\int_{v\in supp(K)}K(v)v^2dv+\frac{1}{2}h_m^2\int_{v\in supp(K)}f''(t^*)K(v)v^3dv\right\}$$

$$= C_{z,ik}\left\{f'(t)h_m\mu_2(K)+\frac{1}{2}h_m^2\int_{v\in supp(K)}f''(t^*)K(v)v^3dv\right\}, \tag{S.11}$$

and

$$var(J_{12,ik})$$

$$= var\left\{\frac{1}{n}\sum_{l=1}^{n}z_{li}z_{lk}K_{h_m}(t_l-t)\frac{t_l-t}{h_m}\right\}$$

$$= \frac{1}{n}var\left\{z_{1i}z_{1k}K_{h_m}(t_1-t)\frac{t_1-t}{h_m}\right\}$$

$$= \frac{1}{n}\left\{E\left[\left\{z_{1i}z_{1k}K_{h_m}(t_1-t)\frac{t_1-t}{h_m}\right\}^2\right]-\left[E\left\{z_{1i}z_{1k}K_{h_m}(t_1-t)\frac{t_1-t}{h_m}\right\}\right]^2\right\}$$

$$= \frac{1}{n}E\{(z_{1i}z_{1k})^2\}E\left[\left\{K_{h_m}(t_1-t)\frac{t_1-t}{h_m}\right\}^2\right]$$

$$-\frac{1}{n}C_{z,ik}^2\left\{f'(t)h_m\mu_2(K)+\frac{1}{2}h_m^2\int_{v\in supp(K)}f''(t^*)K(v)v^3dv\right\}^2$$

$$\leq \frac{1}{n}\kappa_z\int_{t_1\in\Delta}\frac{1}{h_m^2}K^2\left(\frac{t_1-t}{h_m}\right)\left(\frac{t_1-t}{h_m}\right)^2f(t_1)dt_1$$

$$-\frac{1}{n}C_{z,ik}^2\left\{f'(t)h_m\mu_2(K)+\frac{1}{2}h_m^2\int_{v\in supp(K)}f''(t^*)K(v)v^3dv\right\}^2$$

$$= \frac{1}{n}\kappa_z\int_{v\in supp(K)}\frac{1}{h_m^2}K^2(v)v^2f(t+h_mv)h_mdv$$

$$-\frac{1}{n}C_{z,ik}^2\left\{f'(t)h_m\mu_2(K)+\frac{1}{2}h_m^2\int_{v\in supp(K)}f''(t^*)K(v)v^3dv\right\}^2$$

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$$
= \frac{1}{n}\kappa_z \int_{v\in supp(K)} \frac{1}{h_m^2} K^2(v) v^2 \left\{ f(t) + f'(t)h_m v + \frac{1}{2}f''(t)h_m^2 v^2 + \frac{1}{6}f'''(t^*)h_m^3 v^3 \right\} h_m dv
$$

$$
- \frac{1}{n}C_{z,ik}^2 \left\{ f'(t)h_m \mu_2(K) + \frac{1}{2}h_m^2 \int_{v\in supp(K)} f''(t^*)K(v)v^3 dv \right\}^2
$$

$$
= \frac{1}{n}\kappa_z \left\{ \frac{1}{h_m}f(t)\int_{v\in supp(K)} K^2(v)v^2 dv + \frac{1}{2}h_m \int_{v\in supp(K)} f''(t^*)K^2(v)v^4 dv \right\}
$$

$$
- \frac{1}{n}C_{z,ik}^2 \left\{ f'(t)h_m \mu_2(K) + \frac{1}{2}h_m^2 \int_{v\in supp(K)} f''(t^*)K(v)v^3 dv \right\}^2
$$

$$
= \frac{1}{nh_m}\kappa_z f(t)\int_{v\in supp(K)} K^2(v)v^2 dv + O\left(\frac{h_m}{n}\right). \tag{S.12}
$$

Similar to the derivation of (S.4) above, noting that $f(\cdot)$ and $f''(\cdot)$ are bounded in $\Delta$, and $K(v)$ is bounded in $supp(K)$ and $h_m = O(n^{-1/5})$, we obtain

$$
\begin{aligned}
J_{12} &= C_z \left\{ f'(t)h_m \mu_2(K) + \frac{1}{2}h_m^2 \int_{v\in supp(K)} f''(t^*)K(v)v^3 dv \right\} \\
&\quad + O_p\left(\sqrt{\frac{1}{nh_m}\kappa_z f(t)\int_{v\in supp(K)} K^2(v)v^2 dv + O\left(\frac{h_m}{n}\right)} I_R\right) \\
&= C_z \mu_2(K)h_m f'(t) + O_{up}(h_m^2 I_R), \tag{S.13}
\end{aligned}
$$

and

$$
J_{21} = J_{12}^{\mathrm{T}} = C_z \mu_2(K)h_m f'(t) + O_{up}(h_m^2 I_R), \tag{S.14}
$$

which are (S.5) and (S.6) respectively.

Now,

$$
\begin{aligned}
&E(J_{22,ik}) \\
&= E\left\{ \frac{1}{n}\sum_{l=1}^{n} z_{li}z_{lk}K_{h_m}(t_l - t)\left(\frac{t_l - t}{h_m}\right)^2 \right\} \\
&= E\left\{ z_{1i}z_{1k}K_{h_m}(t_1 - t)\left(\frac{t_1 - t}{h_m}\right)^2 \right\} \\
&= E(z_{1i}z_{1k})E\left\{ K_{h_m}(t_1 - t)\left(\frac{t_1 - t}{h_m}\right)^2 \right\} \\
&= E(z_{1i}z_{1k})\int_{t_1\in\Delta} \frac{1}{h_m}K\left(\frac{t_1 - t}{h_m}\right)\left(\frac{t_1 - t}{h_m}\right)^2 f(t_1)dt_1 \\
&= C_{z,ik}\int_{v\in supp(K)} \frac{1}{h_m}K(v)v^2 f(t + h_m v)h_m dv
\end{aligned}
$$

$$= C_{z,ik} \int_{v \in supp(K)} K(v)v^2 \left\{ f(t) + f'(t)h_m v + \frac{1}{2}f''(t^*)h_m^2 v^2 \right\} dv$$

$$= C_{z,ik} \left\{ f(t) \int_{v \in supp(K)} K(v)v^2 dv + \frac{1}{2}h_m^2 \int_{v \in supp(K)} f''(t^*)K(v)v^4 dv \right\}$$

$$= C_{z,ik} \left\{ f(t)\mu_2(K) + \frac{1}{2}h_m^2 \int_{v \in supp(K)} f''(t^*)K(v)v^4 dv \right\}, \tag{S.15}$$

and

$$var(J_{22,ik})$$

$$= var\left\{ \frac{1}{n}\sum_{l=1}^{n} z_{li}z_{lk}K_{h_m}(t_l - t)\left(\frac{t_l - t}{h_m}\right)^2 \right\}$$

$$= \frac{1}{n}var\left\{ z_{1i}z_{1k}K_{h_m}(t_1 - t)\left(\frac{t_1 - t}{h_m}\right)^2 \right\}$$

$$= \frac{1}{n}\left\{ E\left[ \left\{ z_{1i}z_{1k}K_{h_m}(t_1 - t)\left(\frac{t_1 - t}{h_m}\right)^2 \right\}^2 \right] - \left[ E\left\{ z_{1i}z_{1k}K_{h_m}(t_1 - t)\left(\frac{t_1 - t}{h_m}\right)^2 \right\} \right]^2 \right\}$$

$$= \frac{1}{n}E\{(z_{1i}z_{1k})^2\}E\left\{ K_{h_m}^2(t_1 - t)\left(\frac{t_l - t}{h_m}\right)^4 \right\}$$

$$\quad -\frac{1}{n}C_{z,ik}^2 \left\{ f(t)\mu_2(K) + \frac{1}{2}h_m^2 \int_{v \in supp(K)} f''(t^*)K(v)v^4 dv \right\}^2$$

$$\leq \frac{1}{n}\kappa_z \int_{t_1 \in \Delta} \frac{1}{h_m^2}K^2\left(\frac{t_1 - t}{h_m}\right)\left(\frac{t_1 - t}{h_m}\right)^4 f(t_1)dt_1$$

$$\quad -\frac{1}{n}C_{z,ik}^2 \left\{ f(t)\mu_2(K) + \frac{1}{2}h_m^2 \int_{v \in supp(K)} f''(t^*)K(v)v^4 dv \right\}^2$$

$$= \frac{1}{n}\kappa_z \int_{v \in supp(K)} \frac{1}{h_m^2}K^2(v)v^4 f(t + h_m v)h_m dv$$

$$\quad -\frac{1}{n}C_{z,ik}^2 \left\{ f(t)\mu_2(K) + \frac{1}{2}h_m^2 \int_{v \in supp(K)} f''(t^*)K(v)v^4 dv \right\}^2$$

$$= \frac{1}{n}\kappa_z \int_{v \in supp(K)} \frac{1}{h_m^2}K^2(v)v^4 \left\{ f(t) + f'(t)h_m v + \frac{1}{2}f''(t^*)h_m^2 v^2 \right\} h_m dv$$

$$\quad -\frac{1}{n}C_{z,ik}^2 \left\{ f(t)\mu_2(K) + \frac{1}{2}h_m^2 \int_{v \in supp(K)} f''(t^*)K(v)v^4 dv \right\}^2$$

$$= \frac{1}{nh_m}\kappa_z f(t) \int_{v \in supp(K)} K^2(v)v^4 dv + \frac{h_m}{2n} \int_{v \in supp(K)} f''(t^*)K^2(v)v^6 dv$$

$$\quad -\frac{1}{n}C_{z,ik}^2 \left\{ f(t)\mu_2(K) + \frac{1}{2}h_m^2 \int_{v \in supp(K)} f''(t^*)K(v)v^4 dv \right\}^2$$

$$= \frac{1}{nh_m}\kappa_z f(t) \int_{v \in supp(K)} K^2(v)v^4 dv + O\left(\frac{1}{n}\right). \tag{S.16}$$

Again, given that $f(\cdot)$ and $f''(\cdot)$ are bounded in $\Delta$, and $K(v)$ is bounded in $supp(K)$ and $h_m = O(n^{-1/5})$, we have

$$
\begin{aligned}
& J_{22} \\
= & C_z \left\{ f(t)\mu_2(K) + \frac{1}{2}h_m^2 \int_{v \in supp(K)} f''(t^*)K(v)v^4 dv \right\} \\
& + O_p\left( \sqrt{\frac{1}{nh_m}\kappa_z f(t) \int_{v \in supp(K)} K^2(v)v^4 dv + O\left(\frac{1}{n}\right)} I_R \right) \\
= & C_z \mu_2(K) f(t) + O_{up}(h_m^2 I_R),
\end{aligned}
\tag{S.17}
$$

which is (S.7). Thus, by (S.4)-(S.7), we have

$$
\begin{aligned}
& \frac{1}{n}D_t^{\mathrm{T}}W_{(m)t}D_t \\
\equiv & \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \\
= & \begin{pmatrix} f(t) + O_{up}(h_m^2) & \mu_2(K)h_m f'(t) + O_{up}(h_m^2) \\ \mu_2(K)h_m f'(t) + O_{up}(h_m^2) & \mu_2(K)f(t) + O_{up}(h_m^2) \end{pmatrix} \otimes C_z,
\end{aligned}
\tag{S.18}
$$

which is (S.1).

Note that

$$
\begin{aligned}
& \begin{pmatrix} f(t) + O_{up}(h_m^2) & \mu_2(K)h_m f'(t) + O_{up}(h_m^2) \\ \mu_2(K)h_m f'(t) + O_{up}(h_m^2) & \mu_2(K)f(t) + O_{up}(h_m^2) \end{pmatrix}^{-1} \\
= & \begin{pmatrix} f^{-1}(t) + O_{up}(h_m^2) & -h_m f'(t)f^{-2}(t) + O_{up}(h_m^2) \\ -h_m f'(t)f^{-2}(t) + O_{up}(h_m^2) & \mu_2^{-1}(K)f^{-1}(t) + O_{up}(h_m^2) \end{pmatrix}.
\end{aligned}
\tag{S.19}
$$

Hence

$$
\begin{aligned}
& \left\{ \frac{1}{n}D_t^{\mathrm{T}}W_{(m)t}D_t \right\}^{-1} \\
= & \left\{ \begin{pmatrix} f(t) + O_{up}(h_m^2) & \mu_2(K)h_m f'(t) + O_{up}(h_m^2) \\ \mu_2(K)h_m f'(t) + O_{up}(h_m^2) & \mu_2(K)f(t) + O_{up}(h_m^2) \end{pmatrix} \otimes C_z \right\}^{-1} \\
= & \begin{pmatrix} f(t) + O_{up}(h_m^2) & \mu_2(K)h_m f'(t) + O_{up}(h_m^2) \\ \mu_2(K)h_m f'(t) + O_{up}(h_m^2) & \mu_2(K)f(t) + O_{up}(h_m^2) \end{pmatrix}^{-1} \otimes C_z^{-1} \\
= & \begin{pmatrix} f^{-1}(t) + O_{up}(h_m^2) & -h_m f'(t)f^{-2}(t) + O_{up}(h_m^2) \\ -h_m f'(t)f^{-2}(t) + O_{up}(h_m^2) & \mu_2^{-1}(K)f^{-1}(t) + O_{up}(h_m^2) \end{pmatrix} \otimes C_z^{-1},
\end{aligned}
\tag{S.20}
$$

which is (S.2).

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**<u>Lemma</u> 2.** *Assume that Conditions (C.3)-(C.5) are satisfied. Then we have*

$$\max_{1\leq m\leq M}\left|tr(A_{(m)})\right|=O_p(h^{-1}\tilde{r}), \tag{S.21}$$

$$\max_{1\leq m\leq M}\bar{\lambda}(A_{(m)})=O_p(\tilde{r}^{1/2}), \tag{S.22}$$

*and*

$$\max_{1\leq m\leq M}\bar{\lambda}(P_{(m)})=O_p(\tilde{r}^{1/2}). \tag{S.23}$$

**Proof** Let $e_i=(0,...,1,...,0)^{\mathrm{T}}$ be an $n\times 1$ vector in which the $i$th element is one and the other elements are zeros, and $E_i$ be an $n\times n$ matrix with the $i$th diagonal element being one and other elements being zeros. By (7), (S.2) and Conditions (C.3)-(C.5), we have

$$
\begin{aligned}
&\max_{1\leq m\leq M}\left|tr(A_{(m)})\right|\\
={}&\max_{1\leq m\leq M}\left|tr\begin{pmatrix}(z_{(m)1}^{\mathrm{T}},0^{\mathrm{T}})\{D_{(m)t_1}^{\mathrm{T}}W_{(m)t_1}D_{(m)t_1}\}^{-1}D_{(m)t_1}^{\mathrm{T}}W_{(m)t_1}\\ \vdots\\ (z_{(m)n}^{\mathrm{T}},0^{\mathrm{T}})\{D_{(m)t_n}^{\mathrm{T}}W_{(m)t_n}D_{(m)t_n}\}^{-1}D_{(m)t_n}^{\mathrm{T}}W_{(m)t_n}\end{pmatrix}\right|\\
={}&\max_{1\leq m\leq M}\left|\sum_{i=1}^{n}(z_{(m)i}^{\mathrm{T}},0^{\mathrm{T}})\{D_{(m)t_i}^{\mathrm{T}}W_{(m)t_i}D_{(m)t_i}\}^{-1}D_{(m)t_i}^{\mathrm{T}}W_{(m)t_i}e_i\right|\\
={}&\max_{1\leq m\leq M}\left|\sum_{i=1}^{n}tr\left\{\left(\frac{1}{n}D_{(m)t_i}^{\mathrm{T}}W_{(m)t_i}D_{(m)t_i}\right)^{-1}\frac{1}{n}D_{(m)t_i}^{\mathrm{T}}W_{(m)t_i}E_iD_{(m)t_i}\right\}\right|\\
={}&\max_{1\leq m\leq M}\left|\sum_{i=1}^{n}tr\left[\left\{\begin{pmatrix}f^{-1}(t_i)+O_{up}(h_m^2)&-h_mf'(t_i)f^{-2}(t_i)+O_{up}(h_m^2)\\ -h_mf'(t_i)f^{-2}(t_i)+O_{up}(h_m^2)&\mu_2^{-1}(K)f^{-1}(t_i)+O_{up}(h_m^2)\end{pmatrix}\otimes(\Pi_{2m}C_z\Pi_{2m}^{\mathrm{T}})^{-1}\right\}\right.\right.\\
&\left.\left.\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix}\otimes\frac{1}{n}K_{h_m}(0)z_{(m)i}z_{(m)i}^{\mathrm{T}}\right\}\right]\right|\\
={}&\max_{1\leq m\leq M}\left|\sum_{i=1}^{n}tr\left\{\begin{pmatrix}f^{-1}(t_i)+O_{up}(h_m^2)&0\\ -h_mf'(t_i)f^{-2}(t_i)+O_{up}(h_m^2)&0\end{pmatrix}\otimes(\Pi_{2m}C_z\Pi_{2m}^{\mathrm{T}})^{-1}\frac{1}{n}K_{h_m}(0)z_{(m)i}z_{(m)i}^{\mathrm{T}}\right\}\right|\\
={}&\max_{1\leq m\leq M}\left|\sum_{i=1}^{n}tr\left\{\begin{pmatrix}f^{-1}(t_i)+O_{up}(h_m^2)&0\\ -h_mf'(t_i)f^{-2}(t_i)+O_{up}(h_m^2)&0\end{pmatrix}\right\}tr\left\{(\Pi_{2m}C_z\Pi_{2m}^{\mathrm{T}})^{-1}\frac{1}{n}K_{h_m}(0)z_{(m)i}z_{(m)i}^{\mathrm{T}}\right\}\right|\\
\leq{}&\max_{1\leq m\leq M}\frac{1}{n}K_{h_m}(0)\sum_{i=1}^{n}\left|f^{-1}(t_i)+O_{up}(h_m^2)\right|\left|z_{(m)i}^{\mathrm{T}}(\Pi_{2m}C_z\Pi_{2m}^{\mathrm{T}})^{-1}z_{(m)i}\right|\\
\leq{}&\max_{1\leq m\leq M}\frac{1}{n}K_{h_m}(0)\sum_{i=1}^{n}\{f^{-1}(t_i)+O_{up}(h_m^2)\}\bar{\lambda}(\Pi_{2m}C_z\Pi_{2m}^{\mathrm{T}})^{-1}z_{(m)i}^{\mathrm{T}}z_{(m)i}
\end{aligned}
$$

$$\leq \max_{1 \leq m \leq M} \frac{1}{nh_m} K(0) \sum_{i=1}^{n} \{f^{-1}(t_i) + O_{up}(h_m^2)\} \underline{c}_z^{-1} z_{(m)i}^{\mathrm{T}} z_{(m)i}$$

$$\leq \frac{1}{nh} K(0) \sum_{i=1}^{n} \{f^{-1}(t_i) + O_{up}(h_m^2)\} \underline{c}_z^{-1} O_p(\tilde{r})$$

$$= O_P\left(h^{-1}\tilde{r}\right), \tag{S.24}$$

which implies (S.21) is true.

Denote

$$B = \mathrm{diag}\left(e_1^{\mathrm{T}}, e_2^{\mathrm{T}}, ..., e_n^{\mathrm{T}}\right)_{n \times n^2}, \tag{S.25}$$

$$C_{(m)} = \mathrm{diag}\left(D_{(m)t_1}, D_{(m)t_2}, ..., D_{(m)t_n}\right)_{n^2 \times 2r_m n}, \tag{S.26}$$

$$E_{(m)} = \mathrm{diag}\left\{\left(\frac{1}{n}D_{(m)t_1}^{\mathrm{T}}W_{(m)t_1}D_{(m)t_1}\right)^{-1}, \left(\frac{1}{n}D_{(m)t_2}^{\mathrm{T}}W_{(m)t_2}D_{(m)t_2}\right)^{-1}, ..., \left(\frac{1}{n}D_{(m)t_n}^{\mathrm{T}}W_{(m)t_n}D_{(m)t_n}\right)^{-1}\right\}_{2r_m n \times 2r_m n} \tag{S.27}$$

$$F_{(m)} = \mathrm{diag}\left(\frac{1}{\sqrt{n}}D_{(m)t_1}^{\mathrm{T}}W_{(m)t_1}^{1/2}, \frac{1}{\sqrt{n}}D_{(m)t_2}^{\mathrm{T}}W_{(m)t_2}^{1/2}, ..., \frac{1}{\sqrt{n}}D_{(m)t_n}^{\mathrm{T}}W_{(m)t_n}^{1/2}\right)_{2r_m n \times n^2}, \tag{S.28}$$

and

$$H_{(m)} = \begin{pmatrix} \frac{1}{\sqrt{n}}W_{(m)t_1}^{1/2} \\ \vdots \\ \frac{1}{\sqrt{n}}W_{(m)t_n}^{1/2} \end{pmatrix}_{n^2 \times n}. \tag{S.29}$$

It is easily seen that

$$\begin{aligned}
A_{(m)} \\
= \begin{pmatrix} (z_{(m)1}^{\mathrm{T}}, 0^{\mathrm{T}})\{D_{(m)t_1}^{\mathrm{T}}W_{(m)t_1}D_{(m)t_1}\}^{-1}D_{(m)t_1}^{\mathrm{T}}W_{(m)t_1} \\ \vdots \\ (z_{(m)n}^{\mathrm{T}}, 0^{\mathrm{T}})\{D_{(m)t_n}^{\mathrm{T}}W_{(m)t_n}D_{(m)t_n}\}^{-1}D_{(m)t_n}^{\mathrm{T}}W_{(m)t_n} \end{pmatrix}_{n \times n} \\
= BC_{(m)}E_{(m)}F_{(m)}H_{(m)},
\end{aligned} \tag{S.30}$$

and

$$\begin{aligned}
& \max_{1 \leq m \leq M} \bar{\lambda}(A_{(m)}A_{(m)}^{\mathrm{T}}) \\
= & \max_{1 \leq m \leq M} \bar{\lambda}(BC_{(m)}E_{(m)}F_{(m)}H_{(m)}H_{(m)}^{\mathrm{T}}F_{(m)}^{\mathrm{T}}E_{(m)}^{\mathrm{T}}C_{(m)}^{\mathrm{T}}B^{\mathrm{T}})
\end{aligned}$$

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$$\leq \max_{1\leq m\leq M} \bar{\lambda}(H_{(m)}H_{(m)}^{\mathrm{T}})\bar{\lambda}(E_{(m)}F_{(m)}F_{(m)}^{\mathrm{T}}E_{(m)}^{\mathrm{T}})\bar{\lambda}(BC_{(m)}C_{(m)}^{\mathrm{T}}B^{\mathrm{T}}). \tag{S.31}$$

To prove (S.22), we only need to verify that

$$\max_{1\leq m\leq M}\bar{\lambda}(H_{(m)}H_{(m)}^{\mathrm{T}}) = O_p(1), \tag{S.32}$$

$$\max_{1\leq m\leq M}\bar{\lambda}(E_{(m)}F_{(m)}F_{(m)}^{\mathrm{T}}E_{(m)}^{\mathrm{T}}) = O_p(1), \tag{S.33}$$

and

$$\max_{1\leq m\leq M}\bar{\lambda}(BC_{(m)}C_{(m)}^{\mathrm{T}}B^{\mathrm{T}}) = O_p(\tilde{r}). \tag{S.34}$$

Let us first consider (S.32). By (S.29) and Conditions (C.3)-(C.5), we have

$$
\begin{aligned}
&\max_{1\leq m\leq M}\bar{\lambda}(H_{(m)}H_{(m)}^{\mathrm{T}})\\
=\;&\max_{1\leq m\leq M}\bar{\lambda}(H_{(m)}^{\mathrm{T}}H_{(m)})\\
=\;&\max_{1\leq m\leq M}\bar{\lambda}\left(\frac{1}{n}\sum_{i=1}^{n}W_{(m)t_i}\right)\\
=\;&\max_{1\leq m\leq M}\bar{\lambda}\left(\frac{1}{n}\sum_{i=1}^{n}\mathrm{diag}\left\{K_{h_m}(t_1-t_i),K_{h_m}(t_2-t_i),...,K_{h_m}(t_n-t_i)\right\}\right)\\
=\;&\max_{1\leq m\leq M}\max_{1\leq j\leq n}\frac{1}{n}\sum_{i=1}^{n}K_{h_m}(t_j-t_i)\\
=\;&\max_{1\leq m\leq M}\max_{1\leq j\leq n}\left\{f(t_j)+O_{up}(h_m^2)\right\}\\
=\;&O_p(1). 
\end{aligned} \tag{S.35}
$$

Next, consider (S.33). By (S.27), (S.28) and Lemma 1, we can show that

$$
\begin{aligned}
&\max_{1\leq m\leq M}\bar{\lambda}(E_{(m)}F_{(m)}F_{(m)}^{\mathrm{T}}E_{(m)}^{\mathrm{T}})\\
=\;&\max_{1\leq m\leq M}\bar{\lambda}(E_{(m)})\\
=\;&\max_{1\leq m\leq M}\bar{\lambda}\left(\mathrm{diag}\left\{\left(\frac{1}{n}D_{(m)t_1}^{\mathrm{T}}W_{(m)t_1}D_{(m)t_1}\right)^{-1},\left(\frac{1}{n}D_{(m)t_2}^{\mathrm{T}}W_{(m)t_2}D_{(m)t_2}\right)^{-1},...,\right.\right.\\
&\left.\left.\left(\frac{1}{n}D_{(m)t_n}^{\mathrm{T}}W_{(m)t_n}D_{(m)t_n}\right)^{-1}\right\}\right)\\
=\;&\max_{1\leq m\leq M}\max_{1\leq j\leq n}\bar{\lambda}\left\{\left(\frac{1}{n}D_{(m)t_j}^{\mathrm{T}}W_{(m)t_j}D_{(m)t_j}\right)^{-1}\right\}
\end{aligned}
$$

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$$
= \max_{1\leq m\leq M} \max_{1\leq j\leq n} \bar{\lambda}\left(\left\{(I_2\otimes\Pi_{2m})\frac{1}{n}D_{t_j}^{\mathrm{T}}W_{(m)t_j}D_{t_j}(I_2\otimes\Pi_{2m}^{\mathrm{T}})\right\}^{-1}\right)
$$

$$
= \max_{1\leq m\leq M} \max_{1\leq j\leq n} \bar{\lambda}\left(\left[(I_2\otimes\Pi_{2m})\left\{\begin{pmatrix} f(t_j)+O_{up}(h_m^2) & \mu_2(K)h_mf'(t_j)+O_{up}(h_m^2) \\ \mu_2(K)h_mf'(t_j)+O_{up}(h_m^2) & \mu_2(K)f(t_j)+O_{up}(h_m^2) \end{pmatrix}\otimes C_z\right\}\right.\right.
$$
$$
\left.\left.(I_2\otimes\Pi_{2m}^{\mathrm{T}})\right]^{-1}\right)
$$

$$
= \max_{1\leq m\leq M} \max_{1\leq j\leq n} \bar{\lambda}\left(\left\{\begin{pmatrix} f(t_j)+O_{up}(h_m^2) & \mu_2(K)h_mf'(t_j)+O_{up}(h_m^2) \\ \mu_2(K)h_mf'(t_j)+O_{up}(h_m^2) & \mu_2(K)f(t_j)+o_p(1) \end{pmatrix}\otimes(\Pi_{2m}C_z\Pi_{2m}^{\mathrm{T}})\right\}^{-1}\right)
$$

$$
= \max_{1\leq m\leq M} \max_{1\leq j\leq n} \bar{\lambda}\left\{\begin{pmatrix} f(t_j)+O_{up}(h_m^2) & \mu_2(K)h_mf'(t_j)+O_{up}(h_m^2) \\ \mu_2(K)h_mf'(t_j)+O_{up}(h_m^2) & \mu_2(K)f(t_j)+O_{up}(h_m^2) \end{pmatrix}^{-1}\otimes(\Pi_{2m}C_z\Pi_{2m}^{\mathrm{T}})^{-1}\right\}
$$

$$
= \max_{1\leq m\leq M} \max_{1\leq j\leq n} \bar{\lambda}\left\{\begin{pmatrix} f^{-1}(t_j)+O_{up}(h_m^2) & -h_mf'(t_j)f^{-2}(t_j)+O_{up}(h_m^2) \\ -h_mf'(t_j)f^{-2}(t_j)+O_{up}(h_m^2) & \mu_2^{-1}(K)f^{-1}(t_j)+O_{up}(h_m^2) \end{pmatrix}\otimes(\Pi_{2m}C_z\Pi_{2m}^{\mathrm{T}})^{-1}\right\}
$$

$$
= \max_{1\leq m\leq M} \max_{1\leq j\leq n} \bar{\lambda}\left\{\begin{pmatrix} f^{-1}(t_j)+O_{up}(h_m^2) & -h_mf'(t_j)f^{-2}(t_j)+O_{up}(h_m^2) \\ -h_mf'(t_j)f^{-2}(t_j)+O_{up}(h_m^2) & \mu_2^{-1}(K)f^{-1}(t_j)+O_{up}(h_m^2) \end{pmatrix}\right\}
$$
$$
\max_{1\leq m\leq M} \bar{\lambda}\left((\Pi_{2m}C_z\Pi_{2m}^{\mathrm{T}})^{-1}\right)
$$

$$
\leq \max_{1\leq m\leq M} \max_{1\leq j\leq n}\left(\frac{f^{-1}(t_j)+\mu_2^{-1}(K)f^{-1}(t_j)+O_{up}(h_m^2)}{2}\right.
$$
$$
\left.+\frac{\sqrt{\{f^{-1}(t_j)-\mu_2^{-1}(K)f^{-1}(t_j)+O_{up}(h_m^2)\}^2+4\{-h_mf'(t_j)f^{-2}(t_j)+O_{up}(h_m^2)\}^2}}{2}\right)\frac{1}{\underline{c}_z}
$$

$$
= \max_{1\leq m\leq M} \max_{1\leq j\leq n}\left\{f^{-1}(t_j)+O_{up}(h_m^2)\right\}\frac{1}{\underline{c}_z}
$$
$$
= O_p(1). \tag{S.36}
$$

Hence (S.33) is correct. We now consider (S.34). By Condition (C.4), (S.25) and (S.26), we have

$$
\max_{1\leq m\leq M} \bar{\lambda}(BC_{(m)}C_{(m)}^{\mathrm{T}}B^{\mathrm{T}})
$$
$$
= \max_{1\leq m\leq M} \bar{\lambda}\left\{\mathrm{diag}\left(e_1^{\mathrm{T}}, e_2^{\mathrm{T}}, ..., e_n^{\mathrm{T}}\right)\mathrm{diag}\left(D_{(m)1}, D_{(m)2}, ..., D_{(m)n}\right)\right.
$$
$$
\left.\mathrm{diag}\left(D_{(m)1}^{\mathrm{T}}, D_{(m)2}^{\mathrm{T}}, ..., D_{(m)n}^{\mathrm{T}}\right)\mathrm{diag}\left(e_1, e_2, ..., e_n\right)\right\}
$$
$$
= \max_{1\leq m\leq M} \bar{\lambda}\left\{\mathrm{diag}\left(e_1^{\mathrm{T}}D_{(m)1}D_{(m)1}^{\mathrm{T}}e_1, e_2^{\mathrm{T}}D_{(m)2}D_{(m)2}^{\mathrm{T}}e_2, ..., e_n^{\mathrm{T}}D_{(m)n}D_{(m)n}^{\mathrm{T}}e_n\right)\right\}
$$
$$
= \max_{1\leq m\leq M} \bar{\lambda}\left\{\mathrm{diag}\left(z_{(m)1}^{\mathrm{T}}z_{(m)1}, z_{(m)2}^{\mathrm{T}}z_{(m)2}, ..., z_{(m)n}^{\mathrm{T}}z_{(m)n}\right)\right\}
$$
$$
= \max_{1\leq m\leq M} \max_{1\leq j\leq n} z_{(m)j}^{\mathrm{T}}z_{(m)j}
$$
$$
= O_p(\tilde{r}). \tag{S.37}
$$

By (S.31)-(S.34), we obtain

$$
\begin{aligned}
& \max_{1\leq m\leq M} \bar{\lambda}(A_{(m)}) \\
= & \max_{1\leq m\leq M} \bar{\lambda}^{1/2}(A_{(m)}A_{(m)}^{\mathrm{T}}) \\
= & \max_{1\leq m\leq M} \bar{\lambda}^{1/2}(BC_{(m)}E_{(m)}F_{(m)}H_{(m)}H_{(m)}^{\mathrm{T}}F_{(m)}^{\mathrm{T}}E_{(m)}^{\mathrm{T}}C_{(m)}^{\mathrm{T}}B^{\mathrm{T}}) \\
\leq & \max_{1\leq m\leq M} \bar{\lambda}^{1/2}(H_{(m)}H_{(m)}^{\mathrm{T}})\bar{\lambda}^{1/2}(E_{(m)}F_{(m)}F_{(m)}^{\mathrm{T}}E_{(m)}^{\mathrm{T}})\bar{\lambda}^{1/2}(BC_{(m)}C_{(m)}^{\mathrm{T}}B^{\mathrm{T}}) \\
\leq & \max_{1\leq m\leq M} \bar{\lambda}^{1/2}(H_{(m)}H_{(m)}^{\mathrm{T}}) \max_{1\leq m\leq M} \bar{\lambda}^{1/2}(E_{(m)}F_{(m)}F_{(m)}^{\mathrm{T}}E_{(m)}^{\mathrm{T}}) \max_{1\leq m\leq M} \bar{\lambda}^{1/2}(BC_{(m)}C_{(m)}^{\mathrm{T}}B^{\mathrm{T}}) \\
= & O_p(\tilde{r}^{1/2}), \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{S.38})
\end{aligned}
$$

which implies that (S.22) is true. As $\hat{P}_{(m)}$ is an idempotent matrix, $\bar{\lambda}(\hat{P}_{(m)}) \leq 1$. Now, by (S.22),

$$
\begin{aligned}
& \max_{1\leq m\leq M} \bar{\lambda}(P_{(m)}) \\
\leq & \max_{1\leq m\leq M} \bar{\lambda}(\hat{P}_{(m)}(I_n - A_{(m)}) + A_{(m)}) \\
\leq & \max_{1\leq m\leq M} \bar{\lambda}(\hat{P}_{(m)})\{1 + \bar{\lambda}(A_{(m)})\} + \bar{\lambda}(A_{(m)}) \\
\leq & 1 + 2\max_{1\leq m\leq M} \bar{\lambda}(A_{(m)}) \\
= & O_p(\tilde{r}^{1/2}), \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{S.39})
\end{aligned}
$$

which implies (S.23).

### S.2  Proof of Theorem 1

Note that the $C_n(w)$ is

$$
\begin{aligned}
C_n(w) & = \|Y - \hat{\mu}(w)\|^2 + 2tr(P(w)\Omega) \\
& = L_n(w) + \|\epsilon\|^2 - 2\epsilon^{\mathrm{T}}(P(w) - I_n)\mu - 2\{\epsilon^{\mathrm{T}}P(w)\epsilon - tr(P(w)\Omega)\}, \quad (\text{S.40})
\end{aligned}
$$

where $\|\epsilon\|^2$ is independent of $w$. Hence to prove Theorem 1, we need only to verify that

$$
\sup_{w\in\mathscr{H}_n} \left| \frac{\epsilon^{\mathrm{T}}(P(w) - I_n)\mu}{R_n(w)} \right| \xrightarrow{p} 0, \qquad\qquad\qquad\qquad\qquad (\text{S.41})
$$

$$
\sup_{w\in\mathscr{H}_n} \left| \frac{\epsilon^{\mathrm{T}}P(w)\epsilon - tr(P(w)\Omega)}{R_n(w)} \right| \xrightarrow{p} 0, \qquad\qquad\qquad\qquad (\text{S.42})
$$

and

$$
\sup_{w\in\mathscr{H}_n} \left| \frac{L_n(w)}{R_n(w)} - 1 \right| \xrightarrow{p} 0. \qquad\qquad\qquad\qquad\qquad (\text{S.43})
$$

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We observe that for any $\delta > 0$,

$$P\left(\sup_{w \in \mathscr{H}_n} \left|\frac{\epsilon^{\mathrm{T}}(P(w) - I_n)\mu}{R_n(w)}\right| > \delta \Big| X, Z, \Psi\right)$$

$$\leq P\left(\sup_{w \in \mathscr{H}_n} \sum_{m=1}^{M} w_m \left|\epsilon^{\mathrm{T}}(P_{(m)} - I_n)\mu\right| > \delta\xi_n \Big| X, Z, \Psi\right)$$

$$= P\left(\max_{1 \leq m \leq M} \left|\epsilon^{\mathrm{T}}(P_{(m)} - I_n)\mu\right| > \delta\xi_n \Big| X, Z, \Psi\right)$$

$$= P\left(\{|\epsilon^{\mathrm{T}}(P(w_1^0) - I_n)\mu| > \delta\xi_n\} \cup \{|\epsilon^{\mathrm{T}}(P(w_2^0) - I_n)\mu| > \delta\xi_n\}\right.$$

$$\left. \cup \ldots \cup \{|\epsilon^{\mathrm{T}}(P(w_M^0) - I_n)\mu| > \delta\xi_n\} \Big| X, Z, \Psi\right)$$

$$\leq \sum_{m=1}^{M} P\left(\{|\epsilon^{\mathrm{T}}(P(w_m^0) - I_n)\mu| > \delta\xi_n\} \Big| X, Z, \Psi\right)$$

$$\leq \sum_{m=1}^{M} \frac{E\left(|\epsilon^{\mathrm{T}}(P(w_m^0) - I_n)\mu|^{2G} \Big| X, Z, \Psi\right)}{\delta^{2G}\xi_n^{2G}}$$

$$\leq \sum_{m=1}^{M} \frac{c_1 \|(P(w_m^0) - I_n)\mu\|^{2G}}{\delta^{2G}\xi_n^{2G}}$$

$$\leq \sum_{m=1}^{M} \frac{c_1 \{R_n(w_m^0)\}^G}{\delta^{2G}\xi_n^{2G}}$$

$$= \frac{c_1}{\delta^{2G}} \cdot \frac{\sum_{m=1}^{M} \{R_n(w_m^0)\}^G}{\xi_n^{2G}}$$

$$= o_p(1), \tag{S.44}$$

where $c_1$ is a positive constant, the second inequality follows from the Bonferroni's inequality, the third inequality is obtained by an extension of Chebyshev's inequality, the fourth inequality is guaranteed by Theorem 2 of Whittle (1960) and Condition (C.1), the fifth inequality follows from (10), and the last equality is from Condition (C.2).

Denote $A_{n1}(X, Z, \Psi) = P\left(\sup_{w \in \mathscr{H}_n} \left|\frac{\epsilon^{\mathrm{T}}(P(w) - I_n)\mu}{R_n(w)}\right| > \delta \Big| X, Z, \Psi\right)$. Let $F(X, Z, \Psi)$ be the joint cumulative distribution function of $X, Z$ and $\Psi$, and $c_2, \cdots, c_9$ be a sequence of positive constants. Note that $A_{n1}(X, Z, \Psi)$ is the conditional probability density given $X, Z$ and $\Psi$ and thus $|A_{n1}(X, Z, \Psi)| \leq 1$. From (S.44), for any $\varsigma > 0$, we obtain $P(|A_{n1}(X, Z, \Psi)| \geq \varsigma) \to 0$. Then we have

$$P\left(\sup_{w \in \mathscr{H}_n} \left|\frac{\epsilon^{\mathrm{T}}(P(w) - I_n)\mu}{R_n(w)}\right| > \delta\right)$$

$$= E\left\{P\left(\sup_{w \in \mathscr{H}_n} \left|\frac{\epsilon^{\mathrm{T}}(P(w) - I_n)\mu}{R_n(w)}\right| > \delta \Big| X, Z, \Psi\right)\right\}$$

$$
\begin{aligned}
&= E\left\{A_{n1}(X, Z, \Psi)\right\} \\
&= \int_{|A_{n1}(X,Z,\Psi)|\geq\varsigma} A_{n1}(X, Z, \Psi)dF(X, Z, \Psi) + \int_{|A_{n1}(X,Z,\Psi)|<\varsigma} A_{n1}(X, Z, \Psi)dF(X, Z, \Psi) \\
&\leq P(|A_{n1}(X, Z, \Psi)| \geq \varsigma) + \varsigma \to 0,
\end{aligned}
\tag{S.45}
$$

which implies that (S.41) is true.

Similar to the proof of (S.41), we have

$$
\begin{aligned}
&P\left( \sup_{w\in\mathscr{H}_n} \left| \frac{\epsilon^{\mathrm{T}}P(w)\epsilon - tr(P(w)\Omega)}{R_n(w)} \right| > \delta \Big| X, Z, \Psi \right) \\
&\leq P\left( \sup_{w\in\mathscr{H}_n} \sum_{m=1}^{M} w_m \left| \epsilon^{\mathrm{T}}P_{(m)}\epsilon - tr(P_{(m)}\Omega) \right| > \delta\xi_n \Big| X, Z, \Psi \right) \\
&= P\left( \max_{1\leq m\leq M} \left| \epsilon^{\mathrm{T}}P_{(m)}\epsilon - tr(P_{(m)}\Omega) \right| > \delta\xi_n \Big| X, Z, \Psi \right) \\
&\leq \sum_{m=1}^{M} P\left( \left| \epsilon^{\mathrm{T}}P(w_m^0)\epsilon - tr(P(w_m^0)\Omega) \right| > \delta\xi_n \Big| X, Z, \Psi \right) \\
&\leq \sum_{m=1}^{M} \frac{E\left( \left| \epsilon^{\mathrm{T}}P(w_m^0)\epsilon - tr(P(w_m^0)\Omega) \right|^{2G} \Big| X, Z, \Psi \right)}{(\delta\xi_n)^{2G}} \\
&\leq \sum_{m=1}^{M} \frac{c_2 tr^G\{P^{\mathrm{T}}(w_m^0)P(w_m^0)\Omega\}}{(\delta\xi_n)^{2G}} \\
&\leq \sum_{m=1}^{M} \frac{c_2 \{R_n(w_m^0)\}^G}{(\delta\xi_n)^{2G}} \\
&= \frac{c_2}{\delta^{2G}} \cdot \frac{\sum_{m=1}^{M} \{R_n(w_m^0)\}^G}{\xi_n^{2G}} \\
&= o_p(1).
\end{aligned}
\tag{S.46}
$$

Applying the steps in (S.45) to the above equation, we can obtain (S.42).

Now, note that

$$
\sup_{w\in\mathscr{H}_n} \left| \frac{L_n(w)}{R_n(w)} - 1 \right| = \sup_{w\in\mathscr{H}_n} \left| \frac{2\epsilon^{\mathrm{T}}P^{\mathrm{T}}(w)(P(w) - I_n)\mu + \|P(w)\epsilon\|^2 - tr(P^{\mathrm{T}}(w)P(w)\Omega)}{R_n(w)} \right|.
$$

To prove (S.43), it is sufficient to show that

$$
\sup_{w\in\mathscr{H}_n} \left| \frac{\epsilon^{\mathrm{T}}P^{\mathrm{T}}(w)(P(w) - I_n)\mu}{R_n(w)} \right| \xrightarrow{p} 0,
\tag{S.47}
$$

and

$$
\sup_{w\in\mathscr{H}_n} \left| \frac{\|P(w)\epsilon\|^2 - tr(P^{\mathrm{T}}(w)P(w)\Omega)}{R_n(w)} \right| \xrightarrow{p} 0.
\tag{S.48}
$$

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By Lemma 1 and Conditions (C.1) and (C.2), we have

$$P\left(\sup_{w\in\mathscr{H}_n}\left|\frac{\epsilon^{\mathrm{T}}P^{\mathrm{T}}(w)(P(w)-I_n)\mu}{R_n(w)}\right|>\delta\Big|X,Z,\Psi\right)$$

$$\leq P\left(\sup_{w\in\mathscr{H}_n}\sum_{t=1}^{M}\sum_{m=1}^{M}w_tw_m\left|\epsilon^{\mathrm{T}}P_{(t)}^{\mathrm{T}}(P_{(m)}-I_n)\mu\right|>\delta\xi_n\Big|X,Z,\Psi\right)$$

$$\leq P\left(\max_{1\leq t\leq M}\max_{1\leq m\leq M}\left|\epsilon^{\mathrm{T}}P_{(t)}^{\mathrm{T}}(P_{(m)}-I_n)\mu\right|>\delta\xi_n\Big|X,Z,\Psi\right)$$

$$\leq \sum_{t=1}^{M}\sum_{m=1}^{M}P\left(\left|\epsilon^{\mathrm{T}}P^{\mathrm{T}}(w_t^0)(P(w_m^0)-I_n)\mu\right|>\delta\xi_n\Big|X,Z,\Psi\right)$$

$$\leq \sum_{t=1}^{M}\sum_{m=1}^{M}\frac{E\left(\left|\epsilon^{\mathrm{T}}P^{\mathrm{T}}(w_t^0)(P(w_m^0)-I_n)\mu\right|^{2G}\Big|X,Z,\Psi\right)}{(\delta\xi_n)^{2G}}$$

$$\leq \sum_{t=1}^{M}\sum_{m=1}^{M}\frac{c_3\left\|P^{\mathrm{T}}(w_t^0)(P(w_m^0)-I_n)\mu\right\|^{2G}}{(\delta\xi_n)^{2G}}$$

$$\leq \sum_{t=1}^{M}\sum_{m=1}^{M}\frac{c_3\bar{\lambda}^{2G}(P(w_t^0))\left\|(P(w_m^0)-I_n)\mu\right\|^{2G}}{(\delta\xi_n)^{2G}}$$

$$\leq \sum_{t=1}^{M}\sum_{m=1}^{M}\frac{c_3\bar{\lambda}^{2G}(P(w_t^0))\{R_n(w_m^0)\}^{G}}{(\delta\xi_n)^{2G}}$$

$$= \frac{c_4}{\delta^{2G}}\cdot\frac{MO_p(\tilde{r}^G)\sum\limits_{m=1}^{M}\{R_n(w_m^0)\}^{G}}{\xi_n^{2G}}$$

$$= o_p(1). \tag{S.49}$$

Equation (S.47) can be obtained by applying the steps in (S.45) to (S.49).

Furthermore,

$$P\left(\sup_{w\in\mathscr{H}_n}\left|\frac{\|P(w)\epsilon\|^2-tr(P^{\mathrm{T}}(w)P(w)\Omega)}{R_n(w)}\right|>\delta\Big|X,Z,\Psi\right)$$

$$\leq P\left(\sup_{w\in\mathscr{H}_n}\sum_{t=1}^{M}\sum_{m=1}^{M}w_tw_m\left|\epsilon^{\mathrm{T}}P_{(t)}^{\mathrm{T}}P_{(m)}\epsilon-tr(P_{(t)}^{\mathrm{T}}P_{(m)}\Omega)\right|>\delta\xi_n\Big|X,Z,\Psi\right)$$

$$\leq P\left(\max_{1\leq t\leq M}\max_{1\leq m\leq M}\left|\epsilon^{\mathrm{T}}P_{(t)}^{\mathrm{T}}P_{(m)}\epsilon-tr(P_{(t)}^{\mathrm{T}}P_{(m)}\Omega)\right|>\delta\xi_n\Big|X,Z,\Psi\right)$$

$$\leq \sum_{t=1}^{M}\sum_{m=1}^{M}P\left(\left|\epsilon^{\mathrm{T}}P^{\mathrm{T}}(w_t^0)P(w_m^0)\epsilon-tr(P^{\mathrm{T}}(w_t^0)P(w_m^0)\Omega)\right|>\delta\xi_n\Big|X,Z,\Psi\right)$$

$$\leq \sum_{t=1}^{M}\sum_{m=1}^{M}\frac{E\left(\left|\epsilon^{\mathrm{T}}P^{\mathrm{T}}(w_t^0)P(w_m^0)\epsilon-tr(P^{\mathrm{T}}(w_t^0)P(w_m^0)\Omega)\right|^{2G}\Big|X,Z,\Psi\right)}{(\delta\xi_n)^{2G}}$$

$$\leq \sum_{t=1}^{M}\sum_{m=1}^{M} \frac{c_5 tr^G\{\Omega^{1/2}P^{\mathrm{T}}(w_t^0)P(w_m^0)\Omega P^{\mathrm{T}}(w_m^0)P(w_t^0)\Omega^{1/2}\}}{(\delta\xi_n)^{2G}}$$

$$\leq \sum_{t=1}^{M}\sum_{m=1}^{M} \frac{c_5 \bar{\lambda}^{2G}(P(w_m^0))\bar{\lambda}^G(\Omega) tr^G\{P^{\mathrm{T}}(w_t^0)P(w_t^0)\Omega\}}{(\delta\xi_n)^{2G}}$$

$$\leq \sum_{t=1}^{M}\sum_{m=1}^{M} \frac{c_5 \bar{\lambda}^{2G}(P(w_m^0))\bar{\lambda}^G(\Omega) R_n^G(w_t^0)}{(\delta\xi_n)^{2G}}$$

$$= \frac{c_6}{\delta^{2G}} \cdot \frac{MO_p(\tilde{r}^G)\sum_{t=1}^{M}\{R_n(w_t^0)\}^G}{\xi_n^{2G}}$$

$$= o_p(1). \tag{S.50}$$

Equation (S.48) is obtained by applying the derivation steps in (S.45) to (S.50).

### S.3    Proof of Theorem 2

When $\Omega$ is replaced by $\hat{\Omega}$, $C_n(w)$ under the known $\Omega$ case is correspondingly changed to

$$\hat{C}_n(w) = C_n(w) + 2\{tr(P(w)\hat{\Omega}) - tr(P(w)\Omega)\}.$$

From the result of Theorem 1, it suffices to prove that,

$$\sup_{w\in\mathscr{H}_n} \left| tr(P(w)\hat{\Omega}) - tr(P(w)\Omega) \right| / R_n(w) = o_p(1). \tag{S.51}$$

Let $Q_{(m)} = \mathrm{diag}(\rho_{11}^{(m)}, ..., \rho_{nn}^{(m)})$ and $Q(w) = \sum_{m=1}^{M} w_m Q_{(m)}$. It is readily seen that

$$\sup_{w\in\mathscr{H}_n} \left| tr(P(w)\hat{\Omega}) - tr(P(w)\Omega) \right| / R_n(w)$$

$$= \sup_{w\in\mathscr{H}_n} \left| (Y - P_{(M^*)}Y)^{\mathrm{T}}Q(w)(Y - P_{(M^*)}Y) - tr(Q(w)\Omega) \right| / R_n(w)$$

$$= \sup_{w\in\mathscr{H}_n} \left| (\epsilon + \mu)^{\mathrm{T}}(I_n - P_{(M^*)})^{\mathrm{T}}Q(w)(I_n - P_{(M^*)})(\epsilon + \mu) - tr(Q(w)\Omega) \right| / R_n(w)$$

$$\leq \sup_{w\in\mathscr{H}_n} \left| \epsilon^{\mathrm{T}}(I_n - P_{(M^*)})^{\mathrm{T}}Q(w)(I_n - P_{(M^*)})\epsilon \right.$$
$$\left. - tr\{(I_n - P_{(M^*)})^{\mathrm{T}}Q(w)(I_n - P_{(M^*)})\Omega\} \right| / R_n(w)$$

$$+2\sup_{w\in\mathscr{H}_n} \left| \epsilon^{\mathrm{T}}(I_n - P_{(M^*)})^{\mathrm{T}}Q(w)(I_n - P_{(M^*)})\mu \right| / R_n(w)$$

$$+\sup_{w\in\mathscr{H}_n} \left| \mu^{\mathrm{T}}(I_n - P_{(M^*)})^{\mathrm{T}}Q(w)(I_n - P_{(M^*)})\mu \right| / R_n(w)$$

$$+\sup_{w\in\mathscr{H}_n} \left| tr\{P_{(M^*)}^{\mathrm{T}}Q(w)P_{(M^*)}\Omega\} \right| / R_n(w)$$

$$+2\sup_{w\in\mathscr{H}_n} \left| tr\{P_{(M^*)}^{\mathrm{T}}Q(w)\Omega\} \right| / R_n(w)$$

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$$\equiv \; \Xi_1 + \Xi_2 + \Xi_3 + \Xi_4 + \Xi_5.$$

Now, define $\rho = \max\limits_{1 \le m \le M} \max\limits_{1 \le i \le n} |\rho_{ii}^{(m)}|$. From Lemma 2 and Condition (C.6), we have

$$
\begin{aligned}
\rho \;\le\;& cn^{-1} \max_{1 \le m \le M} |tr(P_{(m)})| \\
=\;& cn^{-1} \max_{1 \le m \le M} |tr(\hat{P}_{(m)}(I_n - A_{(m)}) + A_{(m)})| \\
\le\;& cn^{-1} \max_{1 \le m \le M} |tr(\hat{P}_{(m)})| + cn^{-1} \max_{1 \le m \le M} |tr(\hat{P}_{(m)} A_{(m)})| + cn^{-1} \max_{1 \le m \le M} |tr(A_{(m)})| \\
=\;& cn^{-1}\tilde{k} + cn^{-1}2^{-1} \max_{1 \le m \le M} |tr(\hat{P}_{(m)} A_{(m)} + A_{(m)}^{\mathrm{T}} \hat{P}_{(m)})| + cn^{-1} \max_{1 \le m \le M} |tr(A_{(m)})| \\
\le\;& cn^{-1}\tilde{k} + cn^{-1}2^{-1} \max_{1 \le m \le M} |\bar{\lambda}(\hat{P}_{(m)} A_{(m)} + A_{(m)}^{\mathrm{T}} \hat{P}_{(m)}) rank(\hat{P}_{(m)} A_{(m)} + A_{(m)}^{\mathrm{T}} \hat{P}_{(m)})| \\
& + cn^{-1} \max_{1 \le m \le M} |tr(A_{(m)})| \\
\le\;& cn^{-1}\tilde{k} + cn^{-1}2 \max_{1 \le m \le M} |\bar{\lambda}(\hat{P}_{(m)}) \bar{\lambda}(A_{(m)}) k_m| + cn^{-1} \max_{1 \le m \le M} |tr(A_{(m)})| \\
=\;& cn^{-1}\tilde{k} + c_7 n^{-1} O_p(\tilde{r}^{1/2})\tilde{k} + c_8 n^{-1} O_p(h^{-1}\tilde{r}) \\
=\;& O_p(n^{-1}\tilde{r}^{1/2}\tilde{k} + n^{-1}h^{-1}\tilde{r}). \tag{S.52}
\end{aligned}
$$

It follows from (10) and Condition (C.2) that

$$\xi_n^{-1} = o_p(1), \quad M\xi_n^{-2G}\tilde{r}^G = o_p(1) \quad \text{and} \quad \xi_n^{-2}\tilde{r}\|P_{(M^*)}\mu - \mu\|^2 = o_P(1). \tag{S.53}$$

Using (S.23), (S.52), Chebyshev's inequality, and Theorem 2 of Whittle (1960), we can obtain, for any $\delta > 0$, that

$$
\begin{aligned}
& P\left(\Xi_1 > \delta \Big| X, Z, \Psi\right) \\
\le\;& \sum_{m=1}^{M} P\left(\Big| \epsilon^{\mathrm{T}}(I_n - P_{(M^*)})^{\mathrm{T}} Q_{(m)}(I_n - P_{(M^*)})\epsilon \right. \\
& \left. - tr\left\{(I_n - P_{(M^*)})^{\mathrm{T}} Q_{(m)}(I_n - P_{(M^*)})\Omega\right\}\Big| > \delta\xi_n \Big| X, Z, \Psi\right) \\
\le\;& \delta^{-2G}\xi_n^{-2G} \sum_{m=1}^{M} E\left\{[\epsilon^{\mathrm{T}}(I_n - P_{(M^*)})^{\mathrm{T}} Q_{(m)}(I_n - P_{(M^*)})\epsilon \right. \\
& \left. - tr\{(I_n - P_{(M^*)})^{\mathrm{T}} Q_{(m)}(I_n - P_{(M^*)})\Omega\}]^{2G} \Big| X, Z, \Psi\right\} \\
\le\;& c_9 \delta^{-2G}\xi_n^{-2G} \sum_{m=1}^{M} tr^G\{\Omega^{1/2}(I_n - P_{(M^*)})^{\mathrm{T}} Q_{(m)}(I_n - P_{(M^*)})\Omega(I_n - P_{(M^*)})^{\mathrm{T}} Q_{(m)}(I_n - P_{(M^*)})\Omega^{1/2}\} \\
\le\;& c_9 \delta^{-2G}\xi_n^{-2G} \sum_{m=1}^{M} \bar{\lambda}^{4G}(I_n - P_{(M^*)}) \bar{\lambda}^{2G}(\Omega) \bar{\lambda}^{2G}(Q_{(m)}) n^G \\
\le\;& c_9 \delta^{-2G}\xi_n^{-2G} M\{\bar{\lambda}(I_n) + \bar{\lambda}(P_{(M^*)})\}^{4G} \bar{\lambda}^{2G}(\Omega) n^G \rho^{2G}
\end{aligned}
$$

$$
\begin{aligned}
&= c_9 \delta^{-2G} \xi_n^{-2G} M \{1 + O_p(\tilde{r}^{1/2})\}^{4G} O(1) n^G \rho^{2G} \\
&= \xi_n^{-2G} M O_p(\tilde{r}^{2G}(n^{-1}\tilde{r}^{1/2}\tilde{k} + n^{-1}h^{-1}\tilde{r})^{2G} n^G) \\
&= \xi_n^{-2G} M \tilde{r}^G O_p((n^{-1}\tilde{r}^2\tilde{k}^2 + n^{-1}h^{-2}\tilde{r}^3)^G), \quad\quad\quad\quad\quad\quad (\text{S.54})
\end{aligned}
$$

where $G$ is the integer defined in Condition (C.1). It follows from (S.53), (S.54) and Condition (C.8) that $P\left(\Xi_1 > \delta \big| X, Z, \Psi\right) = o_p(1)$, which, along with the derivation steps in (S.45), implies that $\Xi_1 = o_p(1)$.

By (S.23), (S.52), (S.53) and Condition (C.1), we have

$$
\begin{aligned}
\Xi_2 &\le 2\xi_n^{-1} \|(I_n - P_{(M^*)})\mu\| \sup_{w \in \mathscr{H}_n} \left\| Q(w)(I_n - P_{(M^*)})\epsilon \right\| \\
&\le 2\xi_n^{-1} \|(I_n - P_{(M^*)})\mu\| \rho\bar{\lambda}(I_n - P_{(M^*)}) \|\epsilon\| \\
&\le 2\xi_n^{-1} \|(I_n - P_{(M^*)})\mu\| \rho\{1 + \bar{\lambda}(P_{(M^*)})\} \|\epsilon\| \\
&= \xi_n^{-1} O_p(\tilde{r}^{1/2}) \|(I_n - P_{(M^*)})\mu\| \rho O_p(n^{1/2}) \\
&= o_p(1) O_p(n^{-1/2}\tilde{r}^{1/2}\tilde{k} + n^{-1/2}h^{-1}\tilde{r}), \quad\quad\quad\quad\quad\quad (\text{S.55})
\end{aligned}
$$

which, along with Condition (C.8), implies that $\Xi_2 = o_p(1)$. Using (S.23), (S.52), (S.53) and Condition (C.7), we obtain

$$
\begin{aligned}
\Xi_3 &\le \xi_n^{-1} \rho \|(I_n - P_{(M^*)})\mu\|^2 \\
&\le \xi_n^{-1} \|(I_n - P_{(M^*)})\mu\| \rho\{1 + \bar{\lambda}(P_{(M^*)})\} \|\mu\| \\
&= \xi_n^{-1} O_p(\tilde{r}^{1/2}) \|(I_n - P_{(M^*)})\mu\| \rho O(n^{1/2}) \\
&= o_p(1) O_p(n^{-1/2}\tilde{r}^{1/2}\tilde{k} + n^{-1/2}h^{-1}\tilde{r}), \quad\quad\quad\quad\quad\quad (\text{S.56})
\end{aligned}
$$

which, along with Condition (C.8), implies that $\Xi_3 = o_p(1)$. By (S.23) and (S.52), we can show that

$$
\begin{aligned}
&\Xi_4 + \Xi_5 \\
&\le \xi_n^{-1} \sup_{w \in \mathscr{H}_n} \bar{\lambda}(P_{(M^*)}^{\mathrm{T}} Q(w) P_{(M^*)} \Omega) n + 2\xi_n^{-1} \sup_{w \in \mathscr{H}_n} \bar{\lambda}(P_{(M^*)}^{\mathrm{T}} Q(w)\Omega) n \\
&\le \xi_n^{-1} \bar{\lambda}^2(P_{(M^*)}^{\mathrm{T}}) \sup_{w \in \mathscr{H}_n} \bar{\lambda}(Q(w))\bar{\lambda}(\Omega) n + 2\xi_n^{-1} \bar{\lambda}(P_{(M^*)}^{\mathrm{T}}) \sup_{w \in \mathscr{H}_n} \bar{\lambda}(Q(w))\bar{\lambda}(\Omega) n \\
&\le \xi_n^{-1} \bar{\lambda}^2(P_{(M^*)}^{\mathrm{T}}) \rho\bar{\lambda}(\Omega) n + 2\xi_n^{-1} \bar{\lambda}(P_{(M^*)}^{\mathrm{T}}) \rho\bar{\lambda}(\Omega) n \\
&= \xi_n^{-1} O_p(\tilde{r}) O_p(n^{-1}\tilde{r}^{1/2}\tilde{k} + n^{-1}h^{-1}\tilde{r}) n \\
&= O_p(\xi_n^{-1} n^{1/2}\tilde{r}^{1/2}) O_p(n^{-1/2}\tilde{r}\tilde{k} + n^{-1/2}h^{-1}\tilde{r}^{3/2}), \quad\quad\quad\quad (\text{S.57})
\end{aligned}
$$

which, along with Condition (C.9), implies that $\Xi_4 + \Xi_5 = o_p(1)$. Hence we obtain (S.51). This completes the proof of Theorem 2.

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