# Supplementary material on "Identifying Cointegration by Eigenanalysis"

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#### S.1 Proof for Section 3.1

Let

$$\Sigma_{j}^{x} = \operatorname{diag}\left[\left(\frac{1}{n}\sum_{t=1}^{n-j}(\mathbf{x}_{t+j,1} - \overline{\mathbf{x}}_{1})(\mathbf{x}_{t1} - \overline{\mathbf{x}}_{1})'\right), \left(\frac{1}{n}\sum_{t=1}^{n-j}(\mathbf{x}_{t+j,2} - \overline{\mathbf{x}}_{2})(\mathbf{x}_{t2} - \overline{\mathbf{x}}_{2})'\right)\right]$$

$$\equiv \operatorname{diag}(\Sigma_{j1}^{x}, \Sigma_{j2}^{x}),$$

 $\mathbf{W}^x = \sum_{j=0}^{j_0} \mathbf{\Sigma}_j^x (\mathbf{\Sigma}_j^x)' =: \operatorname{diag}(\mathbf{D}_1^x, \mathbf{D}_2^x)$  and  $\mathbf{\Gamma}_x$  be the  $p \times p$  orthogonal matrix such that

$$\mathbf{W}^x \mathbf{\Gamma}_x = \mathbf{\Gamma}_x \mathbf{\Lambda}_x,$$

where  $\Lambda_x$  is the diagonal matrix of eigenvalues of  $\mathbf{W}^x$ . Since  $\mathbf{x}_{t1}$  is nonstationary and  $\mathbf{x}_{t2}$  is stationary, intuitively  $\frac{1}{n} \sum_{t=1}^{n-j} (\mathbf{x}_{t+j,1} - \overline{\mathbf{x}}_1) (\mathbf{x}_{t1} - \overline{\mathbf{x}}_1)'$  and  $\frac{1}{n} \sum_{t=1}^{n-j} (\mathbf{x}_{t+j,2} - \overline{\mathbf{x}}_2) (\mathbf{x}_{t2} - \overline{\mathbf{x}}_2)'$  do not share the same eigenvalues, so  $\Gamma_x$  must be block-diagonal. Define  $\mathbf{W}^y = \mathbf{A}\mathbf{W}^x\mathbf{A}'$ , then

$$\mathbf{W}^y = \mathbf{A}\mathbf{W}^x \mathbf{A}' = \mathbf{A}\mathbf{\Gamma}_x \mathbf{\Lambda}_x \mathbf{\Gamma}'_x \mathbf{A}'.$$

This implies that the columns of  $\mathbf{A}\mathbf{\Gamma}_x$  are just the orthogonal eigenvectors of  $\mathbf{W}^y$ . Since  $\mathbf{\Gamma}_x$  is block-diagonal, it follows that  $\mathcal{M}(\mathbf{A}_2)$  is the same as the space spanned by the eigenvectors corresponding to the smallest r eigenvalues of  $\mathbf{W}^y$ . As a result, to show the distance between the cointegration space and its estimate is small, we only need to show that the space spanned by the eigenvectors of  $\mathbf{W}^y$  can be approximated by that of  $\widehat{\mathbf{W}}$ . This question is usually solved by perturbation matrix theory. In particular, let

$$\widehat{\mathbf{W}} = \mathbf{W}^y + \Delta \mathbf{W}^y, \ \Delta \mathbf{W}^y = \widehat{\mathbf{W}} - \mathbf{W}^y,$$

and

$$\operatorname{sep}(\mathbf{D}_1^x, \mathbf{D}_2^x) = \min_{\lambda \in \lambda(\mathbf{D}_1^x), \, \mu \in \lambda(\mathbf{D}_2^x)} |\lambda - \mu|,$$

where  $\lambda(\mathbf{A})$  denotes the set of eigenvalues of a matrix  $\mathbf{A}$ . When  $||\Delta \mathbf{W}^y|| = o_p(\text{sep}(\mathbf{D}_1^x, \mathbf{D}_2^x))$ , one can use the perturbation results of Golub and Loan (1996) to establish the bound of Theorems 3.1, 3.3 and 4.1, see also Lam and Yao (2012) or Chang, Guo and Yao (2017). However, in our setting  $\text{sep}(\mathbf{D}_1^x, \mathbf{D}_2^x)$  can be of smaller order than  $||\Delta \mathbf{W}^y||$ , i.e.,  $\text{sep}(\mathbf{D}_1^x, \mathbf{D}_2^x)/||\Delta \mathbf{W}^y|| \xrightarrow{p} 0$  as  $n \to \infty$  and the above method will not work.

To fix this problem, we adopt the perturbation results of Dopico, Moro and Molera (2000) instead. A similar idea was used by Chen and Hurvich (2006) to recover their fractional cointegration spaces via the periodogram matrix, using a random diagonal block matrix instead. However, because of the quadratic form of  $\mathbf{W}^x$  (=  $\sum_{j=1}^{j_0} \Sigma_j^x(\Sigma_j^x)'$ ), we cannot find a normalizing constant matrix  $\mathbf{C}_n$  such that  $\mathbf{C}_n\mathbf{W}^x\mathbf{C}_n = O_e(1)$  or  $\mathbf{C}_n\mathbf{W}^y\mathbf{C}_n = O_e(1)$ , the argument of Chen and Hurvich (2006) based on the perturbation bound of Barlow and Slapnicar (2002) cannot be used. We first establish some lemmas (i.e. Lemmas 1-4 below), which will be used to prove Theorem 3.1.

For 
$$1 \le i \le p-r$$
, set  $f_0^i(t) = W^i(t)$ ,  $f_{d_i}^i(t) = \int_0^t f_{d_i-1}^i(s) \, dt$ ,  $\mu_i = \mathbf{E} z_t^i$  and define

$$F^{i}(t) = f_{d_{i}}^{i}(t) - \int_{0}^{1} f_{d_{i}}^{i}(t) dt, \quad G_{d}(t) = \frac{\prod_{j=0}^{d-1} (t+j)}{d!}, \quad \bar{G}_{d} = \frac{1}{n} \sum_{t=1}^{n} G_{d}(t).$$

Then, we have the following weak convergence result for the sample autocovariance.

**Lemma 1.** Let  $L_d(t) = G_d(t) - \bar{G}_d$ . Suppose  $x_t^i \sim I(d_i)$ ,  $1 \le i \le p - r$ , then under Condition 1,

$$\left(\frac{x_t^i - \bar{x}^i - \mu_i L_{d_i}(t)}{n^{d_i - 1/2}}, 1 \le i \le p - r\right) \xrightarrow{d} \left(F^i(t), 1 \le i \le p - r\right) and \tag{S.1}$$

$$\left(\frac{1}{n^{d_i+1/2}}\sum_{t=1}^n (x_t^i - \bar{x}^i - \mu_i L_{d_i}(t))(x_t^j - \mathbf{E}x_t^j), i \le p - r, p - r + 1 \le j \le p\right) \xrightarrow{p} \mathbf{0}.$$
 (S.2)

*Proof.* For any  $I(d_l)$  process  $x_t^l$ , we can write

$$\nabla^{d_l} x_t^l = \mathbf{E} z_t^l + (z_t^l - \mathbf{E} z_t^l) =: \mu_l + \zeta_t^l.$$

Let  $U_t^l(0) = \zeta_t^l, V_t^l(0) = \mu_l$  and

$$U_t^l(j) = \sum_{s=1}^t U_s^l(j-1), \quad V_t^l(j) = \sum_{s=1}^t V_s^l(j-1).$$

Then

$$x_t^l = U_t^l(d_l) + V_t^l(d_l) = \sum_{j=1}^t U_j^l(d_l - 1) + \sum_{j=1}^t V_j^l(d_l - 1).$$
 (S.3)

By induction, we have

$$V_t^l(d_l) = \mu_l \prod_{j=0}^{d_l-1} (t+j)/d_l! = \mu_l G_{d_l}(t).$$
 (S.4)

On the other hand, since  $\mathrm{E}\zeta_t^l=0$ , by (i) of Condition 1 and continuous mapping theorem, it follows that

$$U_{[ns]}^l(d_l)/n^{d_l-1/2} \stackrel{J_1}{\Longrightarrow} f_{d_l}^l(s), \text{ on } D[0,1].$$
 (S.5)

Thus, by (S.3)-(S.5),

$$(x_{[ns]}^l - \mu_l G_{d_l}([ns]))/n^{d_l-1/2} \stackrel{J_1}{\Longrightarrow} f_{d_l}^l(s), \text{ on } D[0,1].$$
 (S.6)

Since  $S_n^i(t_i)$ ,  $1 \le i \le p$  converge to their limiting distribution jointly, (S.1) follows from (S.6) and the continuous mapping theorem.

As for conclusion (S.2), by the joint convergence condition (see (i) of Condition 1) and (S.1),

$$\left(\frac{x_{[nt_i]}^i - \bar{x}^i - \mu_i L_{d_i}([nt_i])}{n^{d_i - 1/2}}, \frac{1}{\sqrt{n}} \sum_{s=1}^{[nt_j]} (x_s^j - Ex_s^j), 1 \le i \le p - r, p - r + 1 \le j \le p\right)$$

$$\xrightarrow{J_1} \left(F^i(t_i), W^j(t_j)\right)_{ij},$$

on  $D[0,1] \times D[0,1]$ . (ii) of Condition 1 implies that  $E|x_s^j|^2 < \infty$ . This gives

$$\max_{1 \le s \le n} |x_s^j - \mathbf{E} x_s^j| / \sqrt{n} = o_p(1), \text{ and } \frac{1}{n} \sum_{s=1}^n |x_s^j - \mathbf{E} x_s^j| = O_p(1).$$
 (S.7)

Thus, by Theorem 3.1 of Ling and Li (1998), we have

$$\frac{1}{n^{d_i+1/2}} \sum_{t=1}^n (x_t^i - \bar{x}^i - \mu_i L_{d_i}(t)) (x_t^j - \mathbf{E} x_1^j) \stackrel{p}{\longrightarrow} 0.$$

Since p is fixed, we have (S.2) as desired.

Next, we establish a bound for the eigenvalues of  $\Sigma_j^x$  and  $\mathbf{A}'\widehat{\Sigma}_j\mathbf{A} =: \widehat{\Sigma}_j^x$ .

Without loss of generality, we assume the first  $r_1$  components of  $\mathbf{x}_{t1}$  are  $I(a_1)$ , the next  $r_2$  components are  $I(a_2)$  and the last  $r_q$  components of  $\mathbf{x}_{t1}$  are  $I(a_q)$ , that is,

$$\boldsymbol{x}_{t1} = (\overbrace{x_t^{1}, \cdots, x_t^{r_q}}^{I(a_q)}, \overbrace{x_t^{r_q+1}, \cdots, x_t^{r_q+r_{q-1}}}^{I(a_{q-1})}, \cdots, \overbrace{x_t^{\sum_{j=2}^{q} r_j + 1}, \cdots, x_t^{\sum_{j=1}^{q} r_j}}^{I(a_1)})',$$

where  $a_1 < a_2 < \dots < a_q$  are positive integers and  $\sum_{i=1}^q r_i = p - r$ . For  $1 \le i \le q$ , define  $\nu_q = 0$  and  $\nu_i = \sum_{j=i+1}^q r_j$ . Then for any  $\mathbf{x}_t(r_i) := (x_t^{\nu_i+1}, \dots, x_t^{\nu_i+r_i})'$ , if  $\boldsymbol{\mu}_i := (\mu_{\nu_i+1}, \dots, \mu_{\nu_i+r_i})' \ne 0$ ,

there must exist a  $r_i \times (r_i - 1)$  matrix  $\mathbf{P}_i$  and  $r_i \times 1$  vector  $\bar{\boldsymbol{\mu}}_i$  such that  $\mathbf{P}_i' \mathbf{P}_i = \mathbf{I}_{(r_i - 1)}$ ,  $(\mathbf{P}_i, \boldsymbol{\mu}_i)$  has full rank  $r_i$ ,  $\mathbf{P}_i' \boldsymbol{\mu}_i = 0$  and  $\bar{\boldsymbol{\mu}}_i' \boldsymbol{\mu}_i = 1$ , where  $\mathbf{I}_a$  denotes  $a \times a$  matrix. Let  $\mathbf{B}_i = (\mathbf{P}_i, n^{-1/2} \bar{\boldsymbol{\mu}}_i)'$  if  $\boldsymbol{\mu}_i \neq 0$  and  $\mathbf{B}_i = \mathbf{I}_{r_i}$  if  $\boldsymbol{\mu}_i = 0$ , and  $\boldsymbol{\Theta}_n = \text{diag}(\mathbf{B}_q, \dots, \mathbf{B}_2, \mathbf{B}_1, \mathbf{I}_r)$ . Define

$$\mathbf{D}_{n1} = \operatorname{diag}\left(\overbrace{n^{a_q - 1/2}, \cdots, n^{a_q - 1/2}}^{r_q}, \cdots, \overbrace{n^{a_1 - 1/2}, \cdots, n^{a_1 - 1/2}}^{r_1}\right), \ \mathbf{D}_{n2} = (\overbrace{1, \cdots, 1}^{r}),$$

and  $\mathbf{D}_n =: \operatorname{diag}(\mathbf{D}_{n1}, \mathbf{D}_{n2})$ . Let  $H^d(t) = t^d/d! - 1/(d+1)!$ ,  $F^i(t)$  be given as in Lemma 1,  $\mathbf{F}_i(t) = (F^{\nu_i+1}(t), \cdots, F^{\nu_i+r_i}(t))'$ ,  $\mathbf{M}_i(t) = (\mathbf{F}_i'(t)\mathbf{P}_i, H^{a_i}(t))'I(\boldsymbol{\mu}_i \neq 0) + \mathbf{F}_i(t)I(\boldsymbol{\mu}_i = 0)$ , and  $\mathbf{M}(t) = (\mathbf{M}_q'(t), \mathbf{M}_{q-1}'(t), \cdots, \mathbf{M}_1'(t))'$ . Then Lemma 2 below follows from Lemma 1 and the continuous mapping theorem.

**Lemma 2.** Let  $\Gamma_j(x) = diag\left(\frac{1}{n}\sum_{t=1}^n (\mathbf{x}_{t1} - \bar{\mathbf{x}}_1)(\mathbf{x}_{t1} - \bar{\mathbf{x}}_1)', \operatorname{Cov}(\mathbf{x}_{1+j,2}, \mathbf{x}_{1,2})\right)$ . Under Condition 1, we have

$$\mathbf{D}_n^{-1}\mathbf{\Theta}_n\mathbf{\Gamma}_j^x\mathbf{\Theta}_n'\mathbf{D}_n^{-1} \stackrel{d}{\longrightarrow} \operatorname{diag}\Big(\int_0^1 \mathbf{M}(t)\mathbf{M}'(t)\,dt, \, \operatorname{Cov}(\mathbf{x}_{1+j,2},\,\mathbf{x}_{1,2})\Big).$$

*Proof.* For any  $1 \le i \le l$ , we define  $\bar{\mathbf{x}}(s_i) = \frac{1}{n} \sum_{t=1}^n \mathbf{x}_t(s_i)$ . When  $\boldsymbol{\mu}_i = 0$ , Lemma 1 gives

$$\frac{\mathbf{B}_i(\mathbf{x}_t(s_i) - \bar{\mathbf{x}}(s_i))}{n^{a_i - 1/2}} = \frac{(\mathbf{x}_t(s_i) - \bar{\mathbf{x}}(s_i))}{n^{a_i - 1/2}} \stackrel{J_1}{\Longrightarrow} \mathbf{F}_i(t), \quad \text{on} \quad \prod_{i=1}^{s_i} D[0, 1]. \tag{S.8}$$

When  $\boldsymbol{\mu}_i \neq 0$ , by  $\bar{\boldsymbol{\mu}}_i' \boldsymbol{\mu}_i = 1$ , we have  $\mathbf{B}_i \mathbf{x}_t(s_i) = ((\mathbf{P}_i' \mathbf{x}_t(s_i))', n^{-1/2} G_{d_i}(t) + n^{-1/2} \bar{\boldsymbol{\mu}}_i' \mathbf{x}_t(s_i))'$ . Note that

$$\lim_{n \to \infty} \frac{1}{n^{a_i}} (G_{a_i}([nt]) + \bar{\boldsymbol{\mu}}_i' \mathbf{x}_{[nt]}(s_i)) \stackrel{p}{=} \lim_{n \to \infty} \frac{1}{n^{a_i}} G_{a_i}([nt]) = \lim_{n \to \infty} \frac{1}{n^{a_i}} \prod_{j=0}^{a_i-1} ([nt] + j) = \frac{t^{a_i}}{a_i!}.$$
 (S.9)

Thus, by  $\mathbf{P}'_i \boldsymbol{\mu}_i = 0$ , Lemma 1 and continuous mapping theorem, we have

$$\frac{\mathbf{B}_{i}(\mathbf{x}_{t}(s_{i}) - \bar{\mathbf{x}}(s_{i}))}{n^{a_{i}-1/2}} \stackrel{J_{1}}{\Longrightarrow} ((\mathbf{P}_{i}'\mathbf{F}_{i}(t))', H^{a_{i}}(t))' \quad \text{on} \quad \prod_{j=1}^{s_{i}} D[0, 1].$$
 (S.10)

Combining (S.8), (S.10) and the joint convergence condition ((i) of Condition 1) yields

$$\left( \left( \frac{\mathbf{B}_i(\mathbf{x}_t(s_i) - \bar{\mathbf{x}}(s_i))}{n^{a_i - 1/2}} \right)', 1 \le i \le l \right) \stackrel{J_1}{\Longrightarrow} (\mathbf{M}_i'(t), 1 \le i \le l) = \mathbf{M}'(t) \quad \text{on} \quad \prod_{j=1}^{p-r} D[0, 1]. \quad (S.11)$$

Let  $\mathbf{B} = \operatorname{diag}(\mathbf{B}_1, \mathbf{B}_2, \cdots, \mathbf{B}_l) = (b_{ij})$ . By (S.11) and continuous mapping theorem, we have

$$\mathbf{D}_{n1}^{-1}\mathbf{B}\left(\frac{1}{n}\sum_{t=1}^{n}(\mathbf{x}_{t1}-\bar{\mathbf{x}}_{1})(\mathbf{x}_{t1}-\bar{\mathbf{x}}_{1})'\right)\mathbf{B}'\mathbf{D}_{n1}^{-1} = \left(\frac{1}{n}\sum_{t=1}^{n}(\mathbf{D}_{n1}^{-1}\mathbf{B}(\mathbf{x}_{t1}-\bar{\mathbf{x}}_{1}))(\mathbf{D}_{n1}^{-1}\mathbf{B}(\mathbf{x}_{t1}-\bar{\mathbf{x}}_{1}))'\right)$$

$$\xrightarrow{d} \int_{0}^{1}\mathbf{M}(t)\mathbf{M}'(t) dt \qquad (S.12)$$

and complete the proof of Lemma 2.

Let  $F^i(t)$ ,  $1 \le i \le p-r$  be defined in Lemma 1, where  $W^i(t) = \sigma_{ii}B^i(t)$  and  $B^i(t)$ ,  $1 \le i \le p-r$  are independent Brownian motions. Let  $\mathbf{F}(t) = (F^1(t), F^2(t), \dots, F^{p-r}(t))'$ .

**Lemma 3.** Under condition 2 and  $p = o(n^{1/2-\tau})$  with  $0 < \tau < 1/2$ ,

$$\left\| \mathbf{D}_n^{-1} \mathbf{\Gamma}_j^x \mathbf{D}_n^{-1} - \operatorname{diag} \left( \int_0^1 \mathbf{F}(t) \mathbf{F}'(t) dt, \operatorname{Cov}(\mathbf{x}_{1+j,2}, \mathbf{x}_{1,2}) \right) \right\|_2 = o_p(1).$$
 (S.13)

Further,  $\int_0^1 \mathbf{F}(t)\mathbf{F}'(t) dt$  is positive definite.

*Proof.* First, we show (S.13). To this end, it is enough to show

$$\left\| \mathbf{D}_{n1}^{-1} \left( \frac{1}{n} \sum_{t=1}^{n} (\mathbf{x}_{t1} - \bar{\mathbf{x}}_1) (\mathbf{x}_{t1} - \bar{\mathbf{x}}_1)' \right) \mathbf{D}_{n1}^{-1} - \int_{0}^{1} \mathbf{F}(t) \mathbf{F}'(t) dt \right\|_{2} = o_{p}(1).$$
 (S.14)

Let  $\boldsymbol{\xi}_t = (\xi_t^1, \xi_t^2, \cdots, \xi_t^{p-r})'$  be an integrated process with components  $\xi_t^i$  satisfying

$$\nabla^{d_i} \xi_t^i = \sigma_{ii} v_t^i := \widetilde{\varepsilon}_t^i.$$

For any given  $1 \le i, j \le p - r$ ,

$$\begin{split} &\frac{1}{n^{d_i+d_j}}\sum_{t=1}^n[(x_t^i-\bar{x}^i)(x_t^j-\bar{x}^j)-(\xi_t^i-\bar{\xi}^i)(\xi_t^j-\bar{\xi}^j)]\\ &=&\frac{1}{n^{d_i+d_j}}\sum_{t=1}^n[x_t^i-\xi_t^i-(\bar{x}^i-\bar{\xi}^i)](x_t^j-\bar{x}^j)+\frac{1}{n^{d_i+d_j}}\sum_{t=1}^n(\xi_t^i-\bar{\xi}^i)[x_t^j-\xi_t^i-(\bar{x}^j-\bar{\xi}^j)]\\ &=:&r_{ij}^1+r_{ij}^2. \end{split}$$

By induction, it is easy to show that under condition (10),

$$\sup_{1 \le i \le p-r} \sup_{1 \le t \le n} \mathbb{E}[(x_t^i - \xi_t^i)/n^{d_i - 1/2}]^2 \le \sup_{1 \le i \le p-r} \sup_{1 \le t \le n} \frac{1}{n} \mathbb{E}(S_t^i - \sum_{l=1}^t \widetilde{\varepsilon}_l^i)^2 = O(n^{2\tau - 1}), \quad (S.15)$$

$$\sup_{1 \le i \le p-r} \sup_{1 \le t \le n} \mathbf{E}(\xi_t^i/n^{d_i-1/2})^2 = O(1) \quad \text{and} \quad \sup_{1 \le i \le n} \sup_{1 \le t \le n} \mathbf{E}(x_t^i/n^{d_i-1/2})^2 = O(1).$$
 (S.16)

Thus, by equations (S.15), (S.16) and the independence of the components,

$$\sum_{i,j=1}^{p-r} \left[ E(r_{ij}^1)^2 + E(r_{ij}^2)^2 \right] = O(p^2 n^{2\tau - 1}),$$

which implies

$$\left\| \mathbf{D}_{n1}^{-1} \left[ \left( \frac{1}{n} \sum_{t=1}^{n} (\mathbf{x}_{t1} - \bar{\mathbf{x}}_1) (\mathbf{x}_{t1} - \bar{\mathbf{x}}_1)' \right) - \left( \frac{1}{n} \sum_{t=1}^{n} (\boldsymbol{\xi}_t - \bar{\boldsymbol{\xi}}) (\boldsymbol{\xi}_t - \bar{\boldsymbol{\xi}})' \right) \right] \mathbf{D}_{n1}^{-1} \right\|_2 = O_p(pn^{\tau - 1/2}),$$

where  $\bar{\boldsymbol{\xi}} = (\bar{\xi}^1, \bar{\xi}^2, \cdots, \bar{\xi}^{p-r})'$ . Thus, for the proof of (S.14), it suffices to show

$$\sup_{1 \le i, j \le p-r} \left\| \frac{1}{n^{d_i + d_j}} \sum_{t=1}^n (\xi_t^i - \bar{\xi}^i)(\xi_t^j - \bar{\xi}^j) - \int_0^1 F^i(t) F^j(t) dt \right\|_2 = o_{a.s.}(n^{\tau - 1/2}). \tag{S.17}$$

Note that

$$\frac{1}{n} \sum_{t=1}^{n} \left( \frac{\xi_{t}^{i}}{n^{d_{i}-1/2}} \right) \left( \frac{\xi_{t}^{j}}{n^{d_{j}-1/2}} \right) - \sum_{t=1}^{n} \int_{(t-1)/n}^{t/n} f^{i}(a) f^{j}(a) da$$

$$= \frac{1}{n} \sum_{t=1}^{n} \left( \frac{\xi_{t}^{i}}{n^{d_{i}-1/2}} - f^{i}(t/n) \right) \left( \frac{\xi_{t}^{j}}{n^{d_{j}-1/2}} \right) + \frac{1}{n} \sum_{t=1}^{n} f^{i}(t/n) \left( \frac{\xi_{t}^{j}}{n^{d_{j}-1/2}} - f^{j}(t/n) \right) - \sum_{t=1}^{n} \int_{(t-1)/n}^{t/n} (f^{i}(a) - f^{i}(t/n)) f^{j}(a) + f^{i}(t/n) (f^{j}(a) - f^{j}(t/n)) da$$

$$=: J_{n1}(i,j) + J_{n2}(i,j) + J_{n3}(i,j).$$

From the definition of I(d) process, it is easy to deduce that if  $\xi_t \sim I(d)$  satisfying  $\nabla^d \xi_t = \varepsilon_t$ , then  $\xi_t = \sum_{s=1}^t \varepsilon_s$  when d = 1 and when  $d \geq 2$ ,

$$\xi_t = \sum_{s=1}^t \left[ \prod_{i=1}^{d-1} (t - s + i) / (d - 1)! \right] \varepsilon_s$$
 (S.18)

and  $f^{i}(t)$  can be rewritten as

$$f^{i}(t) = \int_{0}^{t} (t-s)^{d_{i}-1} dW^{i}(s)/(d_{i}-1)!.$$
 (S.19)

By (S.19) and the continuity of  $W^{i}(s)$ , it is easy to get that

$$\sup_{1 \le i, j \le p - r} \sup_{(t-1)/n \le a \le t/n} |[f^i(a) - f^i(t/n)]f^j(a)| = O_{a.s.}(n^{-1/2}\log^2 n).$$

Thus,

$$\sup_{1 \le i, j \le p-r} |J_{n3}(i,j)| = O_{a.s.}(n^{-1/2} \log^2 n).$$
(S.20)

Set  $\prod_{h=1}^{0} (t-s+h)/0! = 1$ . Using expressions (S.18) and (S.19), we have

$$\xi_t^i/n^{d_i-1/2} - f^i(t/n) = \sum_{s=1}^t \frac{\left[\prod_{h=1}^{d_i-1} (t-s+h) - (t-s)^{d_i-1}\right]}{n^{d_i-1/2} (d_i-1)!} \tilde{\varepsilon}_s^i + \left(\sum_{s=1}^t \frac{(t-s)^{d_i-1} \tilde{\varepsilon}_s^i}{n^{d_i-1/2} (d_i-1)!} - \int_0^{t/n} \frac{(t/n-s)^{d_i-1}}{(d_i-1)!} dW^i(s)\right)$$

$$=: H_{n1}^i(t) + H_{n2}^i(t).$$

It is easy to get that

$$\sup_{1 \le i \le p-r} \sup_{1 \le t \le n} |H_{n1}^i(t)| = O_{a.s.}(n^{-1/2} \log n).$$
 (S.21)

On the other hand, we have for any  $1 \le t \le n$ ,

$$H_{n2}^{i}(t) = \frac{1}{(d_{i}-1)!} \sum_{s=1}^{t} \left( \left( \frac{t}{n} - \frac{s}{n} \right)^{d_{i}-1} \frac{\widetilde{\varepsilon}_{s}^{i}}{\sqrt{n}} - \int_{(s-1)/n}^{s/n} \left( \frac{t}{n} - a \right)^{d_{i}-1} dW^{i}(a) \right)$$

$$=^{d} \frac{1}{(d_{i}-1)!} \sum_{s=1}^{t} \int_{(s-1)/n}^{s/n} \left[ \left( \frac{t}{n} - \frac{s}{n} \right)^{d_{i}-1} - \left( \frac{t}{n} - a \right)^{d_{i}-1} \right] dW^{i}(a). \tag{S.22}$$

This gives

$$\sup_{1 \le i \le p-r} \sup_{1 \le t \le n} |H_{n2}^i(t)| = nO_{a.s}(n^{-3/2} \log n) = O_{a.s.}(n^{-1/2} \log n).$$
 (S.23)

Since the normal sequences  $\{\nu_t^i\}$  are independent with respect to i, it follows that

$$\sup_{1 \le j \le p-r} \sup_{1 \le t \le n} |\xi_t^j| = O_{a.s.}(n^{d_j - 1/2} \log n). \tag{S.24}$$

Thus, by (S.21), (S.23) and (S.24), we have

$$\sup_{1 \le i, j \le p-r} |J_{n1}(i,j)| = O_{a.s.}(n^{-1/2} \log^2 n).$$
(S.25)

Similarly, we have

$$\sup_{1 \le i, j \le p-r} |J_{n2}(i,j)| = O_{a.s.}(n^{-1/2} \log^2 n).$$
 (S.26)

Using the same argument, we can show

$$\sup_{1 \le i, j \le p-r} \left| (\bar{\xi}^i / n^{d_i - 1/2}) (\bar{\xi}^j / n^{d_j - 1/2}) - \int_0^1 f^i(t) \, dt \int_0^1 f^i(t) \, dt \right| = O_{a.s.}(n^{-1/2} \log^2 n). \tag{S.27}$$

Combining equations (S.20), (S.25)–(S.27), we have (S.17) and conclude (S.13). The positive definiteness of  $\int_0^1 \mathbf{F}(t)\mathbf{F}'(t) dt$  can be shown similarly to that of Lemma 3.1.1 in Chan and Wei (1988).

**Lemma 4.** Under Condition 1, or Condition 2 and  $p = o(n^{1/2-\tau})$ , we have

$$\max_{0 \le j \le j_0} \|\mathbf{D}_n^{-1} \mathbf{\Theta}_n (\mathbf{\Sigma}_j^x - \mathbf{\Gamma}_j^x) \mathbf{\Theta}_n' \mathbf{D}_n^{-1} \|_2 \xrightarrow{p} 0 \text{ and}$$
 (S.28)

$$\max_{0 \le j \le j_0} \|\mathbf{D}_n^{-1} \mathbf{\Theta}_n(\widehat{\mathbf{\Sigma}}_j^x - \mathbf{\Gamma}_j^x) \mathbf{\Theta}_n' \mathbf{D}_n^{-1} \|_2 \xrightarrow{p} 0.$$
 (S.29)

*Proof.* We first consider the case with fixed p. To this end, we split the matrix into three parts: the nonstationary block, the cross block with elements being the product of stationary component with nonstationary component and the stationary block.

(I) As for the nonstationary block, we have for  $1 \le i, h \le p - r$ ,

$$\begin{split} \sum_{t=1}^{n-j} (x_{t+j}^i - \bar{x}^i)(x_t^h - \bar{x}^h) &- \sum_{t=1}^n (x_t^i - \bar{x}^i)(x_t^h - \bar{x}^h) \\ &= -\sum_{t=1}^j (x_t^i - \bar{x}^i)(x_t^h - \bar{x}^h) - \sum_{t=1}^{n-j} (x_{t+j}^i - \bar{x}^i)(x_{t+j}^h - x_t^h) \\ &= -\sum_{t=1}^j (x_t^i - \bar{x}^i - \mu_i L_{d_i}(t))(x_t^h - \bar{x}^h - \mu_h L_{d_h}(t)) - \sum_{t=1}^{n-j} (x_{t+j}^i - \bar{x}^i - \mu_i L_{d_i}(t))(x_{t+j}^h - x_t^h) \\ &- \mu_h \sum_{t=1}^j L_{d_h}(t)(x_t^i - \bar{x}^i - \mu_i L_{d_i}(t)) - \mu_i \sum_{t=1}^j L_{d_i}(t)(x_t^h - \bar{x}^h - \mu_h L_{d_h}(t)) \\ &- \mu_i \mu_h \sum_{t=1}^j L_{d_h}(t) L_{d_i}(t) - \mu_i \sum_{t=1}^{n-j} L_{d_i}(t)(x_{t+j}^h - x_t^h) \\ &=: \sum_{m=1}^6 \delta_{nm}(j, i, h). \end{split}$$

From (S.6), it follows that

$$\sup_{0 \le j \le j_0} \frac{|\delta_{n1}(j,i,h)|}{n^{d_i+d_h}} \le \frac{j_0}{n} \left( \sup_{1 \le t \le n} \frac{|x_t^i - \bar{x}^i - \mu_i L_{d_i}(t)|}{n^{d_i-1/2}} \right) \left( \sup_{1 \le t \le n} \frac{|x_t^h - \bar{x}^h - \mu_h L_{d_h}(t)|}{n^{d_h-1/2}} \right) (S.30)$$

$$= O_p(1/n).$$

As for  $\delta_{n2}(j,i,h)$ , we have

(i) If  $d_h=1$ , then  $x_{t+j}^h-x_t^h=\sum_{i=t+1}^{t+j}\varepsilon_i^h$ . Since  $\mathrm{E}|\varepsilon_i^h|<\infty$ , it follows that

$$\sup_{0 \le j \le j_0} \frac{|\delta_{n2}(j,i,h)|}{n^{d_i+d_h}} \le \left( \sup_{t \le n} \frac{|x_t^i - \bar{x}^i - \mu_i L_{d_i}(t)|}{n^{d_i-1/2}} \right) \left( \frac{1}{n^{3/2}} \sum_{t=1}^n \sum_{i=t+1}^{t+j_0} |\varepsilon_i^h| \right)$$

$$= O_p(1/n^{1/2}).$$
(S.31)

(ii) If  $d_h \ge 2$ , then  $x_{t+j}^h - x_t^h = \sum_{s=t+1}^{t+j} [U_s^h(d_h - 1) + V_s^h(d_h - 1)]$  (see Lemma 1), it follows that

$$\sup_{1 \le t \le n} \sup_{1 \le j \le j_0} \frac{|x_{t+j}^h - x_t^h|}{n^{d_h - 1/2}} \le \frac{j_0}{n^{1/2}} \sup_{1 \le s \le n} \frac{|U_s^h(d_h - 1) + V_s^h(d_h - 1)|}{n^{d_h - 1}} = O_p(1/n^{1/2}), \tag{S.32}$$

which implies

$$\sup_{j \le j_0} \frac{|\delta_{n2}(j, i, h)|}{n^{d_i + d_h}} \le \left( \sup_{t \le n} \frac{|x_t^i - \bar{x}^i - \mu_i L_{d_i}(t)|}{n^{d_i - 1/2}} \right) \left( \sup_{t \le n} \sup_{j \le j_0} \frac{|x_{t+j}^h - x_t^h|}{n^{d_h - 1/2}} \right)$$

$$= O_p(1/n^{1/2}).$$
(S.33)

Let  $\Delta_{nm}(j) = (\delta_{nm}(j, i, h))_{(p-r)\times(p-r)}, m = 1, 2, \dots, 6$ . Then by equations (S.30) – (S.33),

$$\sup_{1 \le j \le j_0} ||\mathbf{D}_{n1}^{-1} \mathbf{B} n^{-1} (\boldsymbol{\Delta}_{n1}(j) + \boldsymbol{\Delta}_{n2}(j)) \mathbf{B}' \mathbf{D}_{n1}^{-1}||_{2}$$

$$\le \left( \sum_{i=1}^{p} \sum_{l=1}^{p} \sum_{m=1}^{p} |b_{il} b_{im}| \right) O_{p}(1/n^{1/2})$$

$$\le \left( \sum_{i=1}^{p} \sum_{l=1}^{p} \sum_{m=1}^{p} (b_{il}^{2} + b_{im}^{2}) \right) O_{p}(1/n^{1/2}) = O_{p}(p^{2}/n^{1/2}). \tag{S.34}$$

By the definition of **B**, it is easy to see that the elements of  $\mathbf{B}\Delta_{n3}(j)$ ,  $\mathbf{B}\Delta_{n4}(j)$  are zero except in rows  $\sum_{i=1}^{j} s_i$ ,  $j=1,2,\cdots,l$  and the non-zero elements have the following forms:

$$n^{-1/2} \sum_{t=1}^{j} L_{a_h}(t) (x_t^i - \bar{x}^i - \mu_i L_{d_i}(t)).$$

Thus, by

$$\sup_{1 \le j \le j_0} \left| \frac{\sum_{t=1}^{j} L_{d_h}(t) (x_t^i - \bar{x}^i - \mu_i L_{d_i}(t))}{n^{d_h + d_i - 1/2}} \right| = O_p(1),$$

we have

$$\sup_{1 \le j \le j_0} ||\mathbf{D}_{n1}^{-1} \mathbf{B} n^{-1} (\mathbf{\Delta}_{n3}(j) + \mathbf{\Delta}_{n4}(j)) \mathbf{B}' \mathbf{D}_{n1}^{-1}||_2 = O_p(pn^{-1}).$$
 (S.35)

Similarly, by  $\sup_{1 \le j \le j_0} \left| \frac{\sum_{t=1}^j L_{d_h}(t) L_{d_i}(t)}{n^{d_i + d_h}} \right| = O_p(1)$ , we have

$$\sup_{1 \le j \le j_0} ||\mathbf{D}_{n1}^{-1} \mathbf{B} n^{-1} \mathbf{\Delta}_{n5}(j) \mathbf{B}' \mathbf{D}_{n1}^{-1}||_2 = O_p(pn^{-1}).$$
(S.36)

Further, using (S.9), similar to (S.31) and (S.32), we can show

$$\sup_{1 \le j \le j_0} \frac{1}{n^{d_i + d_h + 1/2}} \sum_{t=1}^{n-j} L_{d_i}(t) (x_{t+j}^h - x_t^h) = O_p(1/n^{1/2}).$$

Thus, for  $\Delta_{n6}(j)$  we have

$$\sup_{1 \le j \le j_0} ||\mathbf{D}_{n1}^{-1} \mathbf{B} n^{-1} \mathbf{\Delta}_{n6}(j) \mathbf{B}' \mathbf{D}_{n1}^{-1}||_2 = O_p(p/n^{1/2}).$$
(S.37)

Combining equations (S.34), (S.35), (S.36) and (S.37) gives

$$\frac{\mathbf{D}_{n1}^{-1}\mathbf{B}}{n} \left\| \sum_{t=1}^{n-j} (\mathbf{x}_{t+j,1} - \bar{\mathbf{x}}_1)(\mathbf{x}_{t1} - \bar{\mathbf{x}}_1)' - \sum_{t=1}^{n} (\mathbf{x}_{t1} - \bar{\mathbf{x}}_1)(\mathbf{x}_{t1} - \bar{\mathbf{x}}_1)' \right\|_{2} \mathbf{B}' \mathbf{D}_{n1}^{-1}$$

$$= O_{p}(p^{2}/n^{1/2}). \tag{S.38}$$

(II) As for the cross block, we first show

$$\left\| \mathbf{D}_{n1}^{-1} \mathbf{B} n^{-1} \sum_{t=1}^{n} (\mathbf{x}_{t1} - \bar{\mathbf{x}}_{1}) (\mathbf{x}_{t2} - \bar{\mathbf{x}}_{2})' \right\|_{2} + \left\| n^{-1} \sum_{t=1}^{n} (\mathbf{x}_{t2} - \bar{\mathbf{x}}_{2}) (\mathbf{x}_{t1} - \bar{\mathbf{x}}_{1})' \mathbf{B}' \mathbf{D}_{n1}^{-1} \right\|_{2}$$

$$= o_{p}(1). \tag{S.39}$$

Note that for  $1 \le i \le p-r$  and  $p-r \le h \le p$ ,

$$\sum_{t=1}^{n} (x_{t}^{i} - \bar{x}^{i})(x_{t}^{h} - \bar{x}^{h}) = \sum_{t=1}^{n} (x_{t}^{i} - \bar{x}^{i} - \mu_{i}L_{d_{i}}(t))(x_{t}^{h} - \bar{x}^{h}) + \mu_{i}\sum_{t=1}^{n} L_{d_{i}}(t)(x_{t}^{h} - \bar{x}^{h}),$$

$$=: \omega_{ih}^{1} + \omega_{ih}^{2}.$$
(S.40)

Let  $\Omega_1 = (\omega_{ih}^1)_{(p-r)\times r}$  and  $\Omega_2 = (\omega_{ih}^2)_{(p-r)\times r}$ . Then the elements of  $\mathbf{B}\Omega_1 = (e_{jh})$  have the following expression:

$$e_{jh} = \sum_{i=1}^{p-r} b_{ji} \sum_{t=1}^{n} (x_t^i - \bar{x}^i - \mu_i L_{d_i}(t))(x_t^h - \bar{x}^h) = \sum_{i=1}^{p-r} b_{ji} \omega_{ih}^1.$$

By Lemma 1, we have

$$\left| \frac{e_{jh}}{n^{d_i + 1/2}} \right| \le \frac{1}{n^{d_i + 1/2}} \sum_{i=1}^{p-r} |b_{ji}\omega_{ih}^1| = o_p(1). \tag{S.41}$$

On the other hand, by the definition of **B**, the elements of  $\mathbf{B}\Omega_2 = (d_{jh})$  can be represented as

$$d_{jh} = \frac{1}{n^{1/2}} \sum_{t=1}^{n} L_{a_i}(t) (x_t^h - \bar{x}^h) I(j = s_i), \ i = 1, 2, \dots, l.$$

It is easy to get that

$$|d_{jh}|/n^{1/2+d_i} = o_p(1). (S.42)$$

Consequently, by (S.41) and (S.42), it follows that

$$\left\| \mathbf{D}_{n1}^{-1} \mathbf{B} n^{-1} \sum_{t=1}^{n} (\mathbf{x}_{t1} - \bar{\mathbf{x}}_{1}) (\mathbf{x}_{t2} - \bar{\mathbf{x}}_{2})' \right\|_{2} \leq \left\| \left| \mathbf{D}_{n1}^{-1} \mathbf{B} n^{-1} \mathbf{\Omega}_{1} \right| \right\|_{2} + \left\| \left| \mathbf{D}_{n1}^{-1} \mathbf{B} n^{-1} \mathbf{\Omega}_{2} \right| \right\|_{2}$$

$$= o_{p}(1). \tag{S.43}$$

Similarly,

$$\left\| n^{-1} \sum_{t=1}^{n} (\mathbf{x}_{t2} - \bar{\mathbf{x}}_2) (\mathbf{x}_{t1} - \bar{\mathbf{x}}_1)' \mathbf{B}' \mathbf{D}_{n1}^{-1} \right\|_{2} = o_p(1).$$
 (S.44)

(S.39) follows from (S.43) and (S.44).

Next, we show

$$\sup_{j \le j_0} \left\| \frac{\mathbf{D}_{n1}^{-1} \mathbf{B}}{n} \left( \sum_{t=1}^{n-j} (\mathbf{x}_{t+j,1} - \bar{\mathbf{x}}_1) (\mathbf{x}_{t,2} - \bar{\mathbf{x}}_2)' - \sum_{t=1}^{n} (\mathbf{x}_{t1} - \bar{\mathbf{x}}_1) (\mathbf{x}_{t2} - \bar{\mathbf{x}}_2)' \right) \right\|_2 = o_p(1) \quad (S.45)$$

and

$$\sup_{j \le j_0} \left\| \frac{1}{n} \left( \sum_{t=1}^{n-j} (\mathbf{x}_{t,2} - \bar{\mathbf{x}}_2) (\mathbf{x}_{t+j,1} - \bar{\mathbf{x}}_1)' - \sum_{t=1}^{n} (\mathbf{x}_{t2} - \bar{\mathbf{x}}_2) (\mathbf{x}_{t1} - \bar{\mathbf{x}}_1)' \right) \mathbf{B}' \mathbf{D}_{n1}^{-1} \right\|_{2}$$
(S.46)
$$o_p(1).$$

As for (S.45), note that for any  $1 \le i \le p-r, p-r+1 \le h \le p$ 

$$\frac{1}{n^{d_{i}+1/2}} \left( \sum_{t=1}^{n-j} (x_{t+j}^{i} - \bar{x}^{i})(x_{t}^{h} - \bar{x}^{h}) - \sum_{t=1}^{n} (x_{t}^{i} - \bar{x}^{i})(x_{t}^{h} - \bar{x}^{h}) \right)$$

$$= \frac{1}{n} \sum_{t=1}^{n-j} \left( \frac{x_{t+j}^{i} - x_{t}^{i}}{n^{d_{i}-1/2}} \right) (x_{t}^{h} - Ex_{1}^{h}) - \frac{(\bar{x}^{h} - Ex_{1}^{h})}{n} \sum_{t=1}^{n-j} \left( \frac{x_{t+j}^{i} - x_{t}^{i}}{n^{d_{i}-1/2}} \right)$$

$$- \frac{1}{n^{d_{i}+1/2}} \sum_{t=n-j+1}^{n} (x_{t}^{i} - \bar{x}^{i})(x_{t}^{h} - \bar{x}^{h})$$

$$= L_{1n}(j, i, h) + L_{2n}(j, i, h) + L_{3n}(j, i, h). \tag{S.47}$$

By (S.32) and  $\frac{1}{n} \sum_{t=1}^{n} E|x_t^h| = O(1)$ , it follows that when  $d_i \geq 2$ ,

$$\sup_{0 \le j \le j_0} |L_{1n}(j, i, h)| = O_p(1/n^{1/2}). \tag{S.48}$$

When  $d_i = 1$ , by  $x_{t+j}^i - x_t^i = \sum_{s=t+1}^{t+j} \varepsilon_s^i$ , we have

$$\operatorname{E} \sup_{0 \le j \le j_0} |L_{1n}(j,i,h)| \le \max_{1 \le t \le n} \frac{1}{n^{1/2}} \sum_{s=t+1}^{t+j_0} \operatorname{E} |\varepsilon_s^i(x_t^h - \operatorname{E} x_1^h)| = O(1/n^{1/2}).$$

Thus, (S.48) also holds for  $d_i = 1$ . Similar to  $L_{1n}(j,i,h)$ , we have

$$\sup_{1 \le j \le j_0} \left| \frac{1}{n} \sum_{t=1}^{n-j} \frac{x_{t+j}^i - x_t^i}{n^{d_i - 1/2}} \right| = O_p(1/n^{1/2}).$$

This combining with Condition 1 show

$$\sup_{j \le j_0} |L_{2n}(j,i,h)| = n^{-1/2} \left( n^{1/2} |\bar{x}^h - \mathbf{E}x_1^h| \right) \left| \frac{1}{n} \sum_{t=1}^{n-j} \frac{x_{t+j}^i - x_t^i}{n^{d_i - 1/2}} \right| = O_p(1/n).$$
 (S.49)

For  $L_{3n}(j,i,h)$ , by Lemma 1 and (S.9), we have

$$\sup_{1 \le t \le n} |x_t^i - \bar{x}^i| / n^{d_i} = O_p(1),$$

thus by  $\sum_{t=n-j_0+1}^{n} \mathbf{E} |x_t^h| / n^{1/2} = O(1/n^{1/2})$ , we have

$$\sup_{j \le j_0} |L_{3n}(j, i, h)| = O_p(1/n^{1/2}). \tag{S.50}$$

Therefore, by (S.47)–(S.50),

$$\sup_{j \le j_0} \frac{1}{n^{d_i + 1/2}} \left| \sum_{t=1}^{n-j} (x_{t+j}^i - \bar{x}^i)(x_t^h - \bar{x}^h) - \sum_{t=1}^n (x_t^i - \bar{x}^i)(x_t^h - \bar{x}^h) \right| = O_p(1/n^{1/2}), \tag{S.51}$$

which shows (S.45).

For (S.46), note that for any  $1 \le i \le p-r, p-r+1 \le h \le p$ ,

$$\frac{1}{n^{d_i+1/2}} \left( \sum_{t=1}^{n-j} (x_t^i - \bar{x}^i)(x_{t+j}^h - \bar{x}^h) - \sum_{t=1}^n (x_t^i - \bar{x}^i)(x_t^h - \bar{x}^h) \right) \\
= \frac{1}{n^{d_i+1/2}} \sum_{t=1}^{n-j} (x_{t+j}^h - x_t^h)(x_t^i - \bar{x}^i) + L_{3n}(j,i,h) =: L_{4n}(j,i,h) + L_{3n}(j,i,h).$$

Let  $\mathbf{L}(j) = (L_{4n}(j, i, h))'_{(p-r)\times r}$  and decompose  $L_{4n}(j, i, h)$  into two terms as in (S.40). Using the same arguments as in (S.41) and (S.42), we can show

$$\sup_{j \le j_0} ||n^{-1} \mathbf{L}(j) \mathbf{B}' \mathbf{D}_{n1}^{-1}||_2 = o_p(1),$$
(S.52)

thus, by (S.50), we have (S.46). Combining equations (S.39) with (S.45) and (S.46) shows that the cross blocks tend to 0 in probability.

(III) As for the stationary block, let  $\Upsilon_j^x$  and  $\widehat{\Upsilon}_j^x$  be the matrixes obtained by replacing the stationary block  $\frac{1}{n} \sum_{j=1}^{n-j} (\mathbf{x}_{t+j,2} - \bar{\mathbf{x}}_2)(\mathbf{x}_{t2} - \bar{\mathbf{x}}_2)'$  in  $\Sigma_j^x$  and  $\widehat{\Sigma}_j^x$  with  $\text{Cov}(\mathbf{x}_{1+j,2}, \mathbf{x}'_{1,2})$ . By (ii) of Condition 1, we have

$$||\mathbf{\Sigma}_j^x - \mathbf{\Upsilon}_j^x||_2 = o_p(1) \text{ and } ||\widehat{\mathbf{\Sigma}}_j^x - \widehat{\mathbf{\Upsilon}}_j^x||_2 = o_p(1).$$
 (S.53)

Thus, by (S.38) and the fact that the cross blocks tend to 0 in probability (see (II)), we have

$$||\mathbf{D}_{n}^{-1}\boldsymbol{\Theta}_{n}(\boldsymbol{\Sigma}_{j}^{x}-\boldsymbol{\Gamma}_{j}^{x})\boldsymbol{\Theta}_{n}'\mathbf{D}_{n}^{-1}||_{2} \leq ||\mathbf{D}_{n}^{-1}\boldsymbol{\Theta}_{n}(\boldsymbol{\Sigma}_{j}^{x}-\boldsymbol{\Upsilon}_{j}^{x})\boldsymbol{\Theta}_{n}'\mathbf{D}_{n}^{-1}||_{2} + ||\mathbf{D}_{n}^{-1}\boldsymbol{\Theta}_{n}(\boldsymbol{\Upsilon}_{j}^{x}-\boldsymbol{\Gamma}_{j}^{x})\boldsymbol{\Theta}_{n}'\mathbf{D}_{n}^{-1}||_{2}$$

$$= o_{p}(1)$$

and

$$||\mathbf{D}_{n}^{-1}\boldsymbol{\Theta}_{n}(\widehat{\boldsymbol{\Sigma}}_{j}^{x}-\boldsymbol{\Gamma}_{j}^{x})\boldsymbol{\Theta}_{n}'\mathbf{D}_{n}^{-1}||_{2} \leq ||\mathbf{D}_{n}^{-1}\boldsymbol{\Theta}_{n}(\widehat{\boldsymbol{\Sigma}}_{j}^{x}-\widehat{\boldsymbol{\Upsilon}}_{j}^{x})\boldsymbol{\Theta}_{n}'\mathbf{D}_{n}^{-1}||_{2} + ||\mathbf{D}_{n}^{-1}\boldsymbol{\Theta}_{n}(\widehat{\boldsymbol{\Upsilon}}_{j}^{x}-\boldsymbol{\Gamma}_{j}^{x})\boldsymbol{\Theta}_{n}'\mathbf{D}_{n}^{-1}||_{2}$$

$$= o_{p}(1).$$

Hence, Lemma 4 holds for finite p.

Next, consider the case:  $p = o(n^{1/2-\tau})$ . We still split the matrix into three parts as above.

(The nonstationary block.) Since  $b_1 \leq \lim_{n \to \infty} \operatorname{Var}(\sum_{s=1}^n z_s^i / \sqrt{n}) \equiv \sigma_{ii}^2 \leq b_2$  for all i, it follows that as  $n \to \infty$ ,

$$\max_{1 \le t \le n} \operatorname{Var}\left(\sum_{s=1}^{t} z_s^i / \sqrt{n}\right) \le b_2 \quad \text{and} \quad \max_{1 \le t \le n} \operatorname{Var}\left(x_t^i / n^{d_i - 1/2}\right) \le b_2. \tag{S.54}$$

Let  $\delta_{n1}(j,i,h), \delta_{n2}(j,i,h)$  be defined as above with  $\mu_i = \mu_h = 0$ . Note that the components of  $\{\mathbf{z}_t\}$  are independent, by (S.54) and some elementary computation, we can show

$$E\left[\sum_{i,h=1}^{p-r} \left( \sup_{j \le j_0} \frac{|\delta_{n1}(j,i,h)|}{n^{d_i+d_h}} \right)^2 \right] = O(j_0^2 p^2/n^2)$$

and

$$E\left[\sum_{i,h=1}^{p-r} \left(\sup_{j \le j_0} \frac{|\delta_{n2}(j,i,h)|}{n^{d_i+d_h}}\right)^2\right] = O(j_0^2 p^2/n).$$

Combining the above two equations yields

$$\mathbf{D}_{n1}^{-1} n^{-1} \left\| \sum_{t=1}^{n-j} (\mathbf{x}_{t+j,1} - \bar{\mathbf{x}}_1) (\mathbf{x}_{t1} - \bar{\mathbf{x}}_1)' - \sum_{t=1}^{n} (\mathbf{x}_{t,1} - \bar{\mathbf{x}}_1) (\mathbf{x}_{t1} - \bar{\mathbf{x}}_1)' \right\|_{2} \mathbf{D}_{n1}^{-1}$$

$$= O_{p}(pn^{-\frac{1}{2}}). \tag{S.55}$$

(The cross block.) Let  $\omega_{ih}$  be defined as in (S.40) with  $\mu_i = 0$ . Since  $\mathbf{z}_t$  and  $\mathbf{x}_{t2}$  are independent, it follows from (S.54) that

$$\begin{split} \mathbf{E} \Big[ \sum_{i=1}^{p-r} \sum_{h=p-r+1}^{p} \omega_{ih}^2 \Big] &= \sum_{i=1}^{p-r} \sum_{h=p-r+1}^{p} \mathbf{E}(\omega_{ih}^2) \\ &= \sum_{i=1}^{p-r} \sum_{h=p-r+1}^{p} n^{-2} \sum_{t,t'=1}^{n} \mathbf{E} \Big[ \frac{(x_t^i - \bar{x}^i)}{n^{d_i - 1/2}} \frac{(x_{t'}^i - \bar{x}^i)}{n^{d_i - 1/2}} \Big] \mathbf{E} [(x_t^h - \bar{x}^h)] [(x_{t'}^h - \bar{x}^h)] \\ &= O\left( \sum_{i=1}^{p-r} \sum_{h=p-r+1}^{p} n^{-2} \sum_{t,t'=1}^{n} |\mathbf{E} [(x_t^h - \bar{x}^h)] [(x_{t'}^h - \bar{x}^h)] | \right) = O(p^2/n^{1-2\tau}) \end{split}$$

by (iii) of Condition 2. Thus,

$$\left\| \mathbf{D}_{n1}^{-1} n^{-1} \sum_{t=1}^{n} (\mathbf{x}_{t1} - \bar{\mathbf{x}}_1) (\mathbf{x}_{t2} - \bar{\mathbf{x}}_2)' \right\|_{2} = \left\| \left| \mathbf{D}_{n1}^{-1} n^{-1} \mathbf{\Omega}_1 \right| \right\|_{2} = O_p(pn^{-1/2 + \tau}).$$
 (S.56)

Similarly,

$$\left\| n^{-1} \sum_{t=1}^{n} (\mathbf{x}_{t2} - \bar{\mathbf{x}}_2) (\mathbf{x}_{t1} - \bar{\mathbf{x}}_1)' \mathbf{D}_{n1}^{-1} \right\|_2 = O_p(pn^{-1/2+\tau}).$$
 (S.57)

Further, by some elementary computation, it is easy to show

$$E\left[\sum_{i=1}^{p-r} \sum_{j=p-r+1}^{p} \sum_{l=1}^{3} (L_{ln}(j,i,h))^{2}\right] = O(pr/n^{1-2\tau}),$$

which gives

$$\sup_{j \le j_0} \left\| \frac{\mathbf{D}_{n1}^{-1}}{n} \left( \sum_{t=1}^{n-j} (\mathbf{x}_{t+j,1} - \bar{\mathbf{x}}_1) (\mathbf{x}_{t+j,2} - \bar{\mathbf{x}}_2)' - \sum_{t=1}^{n} (\mathbf{x}_{t1} - \bar{\mathbf{x}}_1) (\mathbf{x}_{t2} - \bar{\mathbf{x}}_2)' \right) \right\|_{2}$$

$$= O_p(pn^{-\frac{1}{2} + \tau}). \tag{S.58}$$

Note that  $L_{4n}(j,i,h) = \sum_{t=n-j+1}^{n} (x_t^i - \bar{x}^i) x_t^h - \sum_{t=1}^{j} (x_t^i - \bar{x}^i) x_t^h - \sum_{t=1}^{n-j} (x_{t+j}^i - x_t^i) x_{t+j}^h$ , it is easy to show that  $\mathbb{E}\left[\sum_{i=1}^{p-r} \sum_{j=p-r+1}^{p} (L_{4n}(j,i,h))^2\right] = O(pr/n^{1-2\tau})$  too. As a result, we have

$$\sup_{j \le j_0} \left\| \frac{1}{n} \left( \sum_{t=1}^{n-j} (\mathbf{x}_{t+j,2} - \bar{\mathbf{x}}_2) (\mathbf{x}_{t+j,1} - \bar{\mathbf{x}}_1)' - \sum_{t=1}^{n} (\mathbf{x}_{t2} - \bar{\mathbf{x}}_2) (\mathbf{x}_{t1} - \bar{\mathbf{x}}_1)' \right) \mathbf{D}_{n1}^{-1} \right\|_{2}$$

$$= O_p(pn^{-\frac{1}{2} + \tau}). \tag{S.59}$$

Consequently, by equations (S.56)–(S.59), we get that the norms of the cross blocks are  $O_p(pn^{-\frac{1}{2}+\tau})$ . (THE STATIONARY BLOCK.) By (ii) of Condition 2, we also have (S.53). Thus, by (S.55) and the bound of the cross blocks (see above), we have if  $p = o(n^{1/2-\tau})$ ,

$$||\mathbf{D}_{n}^{-1}(\mathbf{\Sigma}_{j}^{x} - \mathbf{\Gamma}_{j}^{x})\mathbf{D}_{n}^{-1}||_{2} \leq ||\mathbf{D}_{n}^{-1}(\mathbf{\Sigma}_{j}^{x} - \mathbf{\Upsilon}_{j}^{x})\mathbf{D}_{n}^{-1}||_{2} + ||\mathbf{D}_{n}^{-1}(\mathbf{\Upsilon}_{j}^{x} - \mathbf{\Gamma}_{j}^{x})\mathbf{D}_{n}^{-1}||_{2}$$
$$= o_{p}(1) + O_{p}(pn^{-1/2}) = o_{p}(1)$$

and

$$||\mathbf{D}_{n}^{-1}(\widehat{\boldsymbol{\Sigma}}_{j}^{x} - \boldsymbol{\Gamma}_{j}^{x})\mathbf{D}_{n}^{-1}||_{2} \leq ||\mathbf{D}_{n}^{-1}(\widehat{\boldsymbol{\Sigma}}_{j}^{x} - \widehat{\boldsymbol{\Upsilon}}_{j}^{x})\mathbf{D}_{n}^{-1}||_{2} + ||\mathbf{D}_{n}^{-1}(\widehat{\boldsymbol{\Upsilon}}_{j}^{x} - \boldsymbol{\Gamma}_{j}^{x})\mathbf{D}_{n}^{-1}||_{2} = o_{p}(1).$$

Hence, Lemma 4 holds also for large p with  $p = o(n^{1/2-\tau})$ .

# Proof of Theorem 3.1. Since

$$\{D(\widehat{\mathcal{M}}(\mathbf{A}_2), \mathcal{M}(\mathbf{A}_2))\}^2 = \frac{1}{r} \{ \operatorname{tr}[\mathbf{A}_2'(I_p - \widehat{\mathbf{A}}_2 \widehat{\mathbf{A}}_2') \mathbf{A}_2] \}$$

$$\leq ||\mathbf{A}_2'(\mathbf{A}_2 \mathbf{A}_2' - \widehat{\mathbf{A}}_2 \widehat{\mathbf{A}}_2') \mathbf{A}_2||_2 \leq 2||\widehat{\mathbf{A}}_2 - \mathbf{A}_2||_2^2,$$

it follows from Theorem I.5.5 of Stewart and Sun (1990) (see also Proposition 2.1 of Vu and Lei (2013)) that

$$D(\widehat{\mathcal{M}}(\mathbf{A}_2), \mathcal{M}(\mathbf{A}_2)) \le \sqrt{2}||\widehat{\mathbf{A}}_2 - \mathbf{A}_2||_2 \le \sqrt{2}||\widehat{\mathbf{A}}_2 - \mathbf{A}_2||_F \le 2\sqrt{2}||\sin\Theta(\widehat{\mathbf{A}}_2, \mathbf{A}_2)||_F, \quad (S.60)$$

where  $\Theta(\widehat{\mathbf{A}}_2, \mathbf{A}_2) = \arccos[(\mathbf{A}_2'\widehat{\mathbf{A}}_2\widehat{\mathbf{A}}_2\mathbf{A}_2)^{1/2}]$  is the canonical angle between the column spaces of  $\widehat{\mathbf{A}}_2$  and  $\mathbf{A}_2$ . Let  $\eta = \min_{\lambda \in \lambda(\mathbf{D}_1^x), \, \mu \in \lambda(\widetilde{\mathbf{D}}_2^x)} |\lambda - \mu| / \sqrt{\lambda \mu}$ , where  $\lambda(\widetilde{\mathbf{D}}_2^x)$  consists of the r smallest eigenvalues of  $\mathbf{A}'\widehat{\mathbf{W}}\mathbf{A} =: \widehat{\mathbf{W}}^x$ . By Theorem 2.4 of Dopico, Moro and Molera (2000), we have

$$||\sin\Theta(\widehat{\mathbf{A}}_2, \mathbf{A}_2)||_F \le ||(\mathbf{W}^y)^{-1/2} \Delta \mathbf{W}^y(\widehat{\mathbf{W}})^{-1/2}||_F / \eta.$$
(S.61)

Note that

$$(\mathbf{W}^y)^{-1/2} \Delta \mathbf{W}^y (\widehat{\mathbf{W}})^{-1/2} = (\mathbf{W}^y)^{-1/2} (\widehat{\mathbf{W}})^{1/2} - (\mathbf{W}^y)^{1/2} (\widehat{\mathbf{W}})^{-1/2}.$$
 (S.62)

Thus, by equations (S.60), (S.61) and (S.62), we have

$$D(\widehat{\mathcal{M}}(\mathbf{A}_2), \mathcal{M}(\mathbf{A}_2)) \le (||(\mathbf{W}^y)^{-1/2}(\widehat{\mathbf{W}})^{1/2}||_F + ||(\mathbf{W}^y)^{1/2}(\widehat{\mathbf{W}})^{-1/2}||_F)/\eta.$$

Next, we show that  $||(\mathbf{W}^y)^{-1/2}(\widehat{\mathbf{W}})^{1/2}||_F = O_p(1)$ , which is equivalent to

$$||(\mathbf{W}^x)^{-1/2}(\widehat{\mathbf{W}}^x)^{1/2}||_F = O_p(1).$$
 (S.63)

Note that

$$0 \le \Sigma_0^x \le (\mathbf{W}^x)^{1/2} \le \sum_{j=0}^{j_0} \{ \Sigma_j^x (\Sigma_j^x)' \}^{1/2} \text{ and } 0 \le \widehat{\Sigma}_0^x \le (\widehat{\mathbf{W}}^x)^{1/2} \le \sum_{j=0}^{j_0} \{ \widehat{\Sigma}_j^x (\widehat{\Sigma}_j^x)' \}^{1/2}.$$
 (S.64)

It follows from (S.64) that

$$||(\mathbf{W}^x)^{-1/2}(\widehat{\mathbf{W}}^x)^{1/2}||_F \le \sum_{i=0}^{j_0} ||(\mathbf{\Sigma}_0^x)^{-1}\{\widehat{\mathbf{\Sigma}}_j^x(\widehat{\mathbf{\Sigma}}_j^x)'\}^{1/2}||_F.$$

Thus, for (S.63), it is enough to show the eigenvalues of  $(\Sigma_0^x)^{-1} \sum_{j=0}^{j_0} \{\widehat{\Sigma}_j^x(\widehat{\Sigma}_j^x)'\}^{1/2}$  are  $O_p(1)$ , which is equivalent to

the solutions 
$$\lambda$$
 of  $|\{\widehat{\boldsymbol{\Sigma}}_{i}^{x}(\widehat{\boldsymbol{\Sigma}}_{i}^{x})'\}^{1/2} - \lambda \boldsymbol{\Sigma}_{0}^{x}| = 0 \text{ are } O_{p}(1).$  (S.65)

Since diag  $\left(\int_0^1 \mathbf{M}(t)\mathbf{M}'(t) dt, \operatorname{Var}(\mathbf{x}_{1,2})\right) > 0$ , by Lemma 4 the solutions  $(\lambda)$  of equation

$$|\mathbf{D}_n^{-1}\mathbf{\Theta}_n\{\widehat{\boldsymbol{\Sigma}}_j^x(\widehat{\boldsymbol{\Sigma}}_j^x)'\}^{1/2}\mathbf{\Theta}_n'\mathbf{D}_n^{-1} - \lambda\mathbf{D}_n^{-1}\mathbf{\Theta}_n\boldsymbol{\Sigma}_0^x\mathbf{\Theta}_n'\mathbf{D}_n^{-1}| = 0$$
(S.66)

are bounded in probability. Thus, we have (S.65) and (S.63) as desired.

Similarly, we can show

$$||(\mathbf{W}^y)^{1/2}(\widehat{\mathbf{W}})^{-1/2}||_F = ||(\mathbf{W}^x)^{1/2}(\widehat{\mathbf{W}}^x)^{-1/2}||_F = O_p(1).$$
 (S.67)

Using equations (S.64) and (S.67), the remainder of the proof of Theorem 3.1 consists of showing that there exist two positive constants  $c_1, c_2$  such that in probability  $\eta \geq c_1 n^{2a_1-1}/\sqrt{j_0}$  provided  $|I_0| \geq 2$  or  $|I_0| = 1$  and  $Ez_t^{I_0} = 0$  and  $\eta \geq c_2 n^{2a_1}/\sqrt{j_0}$  provided  $|I_0| = 1$  and  $Ez_t^{I_0} \neq 0$ .

Define  $\lambda_i(\mathbf{A})$  to be the *i*-th eigenvalue of a matrix **A**. Note that

diag 
$$\left(\int_0^1 \mathbf{M}(t)\mathbf{M}'(t) dt, \operatorname{Var}(\mathbf{x}_{1,2})\right) > 0.$$

By Lemmas 2 and 4, it follows that when  $|I_0| \ge 2$  or  $|I_0| = 1$  and  $\mathbb{E}z_t^{I_0} = 0$ ,  $\lambda_{p-r}(\Sigma_j^x) = O_e(n^{2a_1-1})$  and  $\lambda_{p-r+1}(\widehat{\Sigma}_j^x) = O_e(1)$ . Thus, there exist two positive constants  $c_3, c_4$  such that in probability

$$\lambda_{p-r}(\mathbf{W}^x) \ge \lambda_{p-r}(\mathbf{\Sigma}_0^x(\mathbf{\Sigma}_0^x)') \ge c_3 n^{2(2a_1-1)}$$
(S.68)

and

$$c_3 \le \lambda_{p-r+1}(\widehat{\boldsymbol{\Sigma}}_0^x(\widehat{\boldsymbol{\Sigma}}_0^x)') \le \lambda_{p-r+1}(\widehat{\mathbf{W}}^x) \le \left[\lambda_{p-r+1}\left(\sum_{j=0}^{j_0} \{\widehat{\boldsymbol{\Sigma}}_j^x(\widehat{\boldsymbol{\Sigma}}_j^x)'\}^{1/2}\right)\right]^2 \le c_4 j_0^2.$$
 (S.69)

Hence, in probability

$$\eta \ge |c_3 n^{2(2a_1-1)} - c_4 j_0^2| / \sqrt{c_3 n^{2(2a_1-1)} c_4 j_0^2} \ge c' n^{2a_1-1} / j_0.$$

Similarly, we have  $|I_0| = 1$  and  $Ez_t^{I_0} \neq 0$ , then in probability,

$$\eta \ge c' n^{2a_1} / j_0. \tag{S.70}$$

Since  $j_0$  is fixed, combining (S.63), (S.70) and (S.70), we complete the proof of (i) and (ii). Conclusion (iii) can be shown similarly by treating  $A_{1i}$  as the role of  $A_2$ , see also the proof of Theorem 1 of Chen and Hurvich (2006), we omit the details here.

Let 
$$\mathbf{A}_{1,0} = \mathbf{A}_2$$
 and  $\widehat{\mathbf{B}}_{1i} = (\widehat{\gamma}_{\nu_i+1}, \cdots, \widehat{\gamma}_{\nu_i+r_i})$  for  $i = 1, \cdots, q$  and  $\widehat{\mathbf{B}}_{10} = (\widehat{\gamma}_{p-r+1}, \cdots, \widehat{\gamma}_p)$ .

Lemma 5. Under Condition 1, we have

$$\|\mathbf{B}_{1,l}\mathbf{A}_{1,h}\|_F = O_p(n^{-2|a_h - a_l|}), \text{ for } l \neq h.$$

*Proof.* Let  $\eta(\mathbf{B}_{1,l}, \mathbf{A}_{1,h})$  be defined as  $\eta$  above, i.e.,

$$\eta(\mathbf{B}_{1,l}, \mathbf{A}_{1,h}) = \min_{\lambda \in \{\widehat{\lambda}_{\nu_l+1}, \cdots, \widehat{\gamma}_{\nu_l+r_l}\}, \, \mu \in \{\lambda_{\nu_h+1}, \cdots, \lambda_{\nu_h+r_h}\}} |\lambda - \mu| / \sqrt{\lambda \mu}.$$

By Lemmas 2 and 4, using the same arguments as in Theorem 3.1, we have

$$\eta(\mathbf{B}_{1,l}, \mathbf{A}_{1,h}) \ge cn^{2|a_h - a_l|} \tag{S.71}$$

for some c > 0. It has been shown in Theorem 1 that  $||(\mathbf{W}^y)^{-1/2}\Delta \mathbf{W}^y(\widehat{\mathbf{W}})^{-1/2}||_F = O_p(1)$ , thus by Theorem 2.4 of Dopico, Moro and Molera (2000) (see also Theorem 4.1 of Barlow and Slapničar (2000)), we have

$$\|\mathbf{B}_{1,l}\mathbf{A}_{1,h}\|_{F} \le \|(\mathbf{W}^{y})^{-1/2}\Delta\mathbf{W}^{y}(\widehat{\mathbf{W}})^{-1/2}\|_{F}/\eta(\mathbf{B}_{l},\mathbf{A}_{h})$$
  
=  $O_{p}(n^{-2|a_{h}-a_{l}|}).$ 

This completes the proof of Lemma 5.

**Proof of Theorem 3.2.** First, we prove the consistency of  $\hat{r}$ . For any  $1 \le i \le p$ ,

$$\widehat{\mathbf{x}}_{t}^{i} = \widehat{\gamma}_{t}' \mathbf{y}_{t} = (\widehat{\gamma}_{t}' \mathbf{A}_{1q} \mathbf{x}_{t1q}, \cdots, \widehat{\gamma}_{t}' \mathbf{A}_{11} \mathbf{x}_{t11}, \widehat{\gamma}_{t}' \mathbf{A}_{2} \mathbf{x}_{t2}). \tag{S.72}$$

Let  $\nu_i$  be defined as that after Lemma 1 and  $r_0=r$ . By Lemma 5, when  $\nu_l+1\leq i\leq \nu_l+r_l,\ l\neq h,$ 

$$\widehat{\gamma}_i' \mathbf{A}_{1h} = O_p(n^{-2|a_h - a_l|}).$$

Thus, by  $\sup_{1 \le t \le n} |\mathbf{x}_{t1h}| = O_p(n^{a_h-1/2})$  for  $h \ge 1$  (see Lemma 1), we have

$$\widehat{\gamma}_{l}' \mathbf{A}_{1h} \mathbf{x}_{t1h} = O_{p}(n^{-a_{h}+2a_{l}-1/2}) I(h > l) + O_{p}(n^{-2a_{l}+3a_{h}-1/2}) I(1 \le h < l).$$

As a result, by (S.72), it follows that for any  $\nu_l + 1 \le i \le \nu_l + r_l$ ,

$$\widehat{x}_{t}^{i} = \widehat{\gamma}_{i}' \mathbf{A}_{1l} \mathbf{x}_{t1l} + O_{p} \left( \sum_{h=l+1}^{q} n^{-a_{h}+2a_{l}-1/2} + \sum_{h=1}^{l} n^{-2a_{l}+3a_{l-1}-1/2} \right)$$

$$= \widehat{\gamma}_{i}' \mathbf{A}_{1l} \mathbf{x}_{t1l} + O_{p} \left( n^{-a_{l+1}+2a_{l}-1/2} + n^{-2a_{l}+3a_{l-1}-1/2} \right),$$

where  $\mathbf{x}_{t10} = \mathbf{x}_{t2}$ . Thus, for any given m, we have

$$\sum_{k=1}^{m} \left( \frac{1}{n-k} \sum_{t=1}^{n-k} (\widehat{x}_{t+k}^{i} - \overline{\widehat{x}}^{i}) (\widehat{x}_{t,i} - \overline{\widehat{x}}^{i}) \right)$$

$$= \frac{\widehat{\gamma}_{i}' \mathbf{A}_{1l}}{n-k} \sum_{k=1}^{m} \sum_{t=1}^{n-k} (\mathbf{x}_{t+k,1l} - \overline{\mathbf{x}}_{1l}) (\mathbf{x}_{t1l} - \overline{\mathbf{x}}_{1l})' \mathbf{A}_{1l}' \widehat{\gamma}_{i} (1 + o_{p}(1)). \tag{S.73}$$

By (S.73), we have that for any  $\nu_l + 1 \le i \le \nu_l + r_l$ ,  $l = 1, \dots, q$ 

$$\sum_{k=1}^{m} \left( \frac{1}{n-k} \sum_{t=1}^{n-k} (\widehat{x}_{t+k}^{i} - \overline{\widehat{x}}^{i}) (\widehat{x}_{t,i} - \overline{\widehat{x}}^{i}) \right)$$

$$= m\widehat{\gamma}_{i}' \mathbf{A}_{1l} \left( \frac{1}{n} \sum_{t=1}^{n} (\mathbf{x}_{t1l} - \overline{\mathbf{x}}_{1l}) (\mathbf{x}_{t1l} - \overline{\mathbf{x}}_{1l})' \right) \mathbf{A}_{1l}' \widehat{\gamma}_{i} (1 + o_{p}(1)) = O_{e}(mn^{2a_{l}-1}). \quad (S.74)$$

On the other hand, by (S.73) and  $\|\sum_{k=1}^m \frac{1}{n-k} \sum_{t=1}^{n-k} (\mathbf{x}_{t+k,2} - \overline{\mathbf{x}}_2) (\mathbf{x}'_{t,2} - \overline{\mathbf{x}}_2)\| \le C$  in probability, it follows that for  $p - r + 1 \le i \le p$ ,

$$\sum_{k=1}^{m} \left( \frac{1}{n-k} \sum_{t=1}^{n-k} (\widehat{x}_{t+k}^{i} - \overline{\widehat{x}}^{i}) (\widehat{x}_{t,i} - \overline{\widehat{x}}^{i}) \right) = O_{p}(1). \tag{S.75}$$

Equation (S.74) together with (S.75) yields the conclusion of Theorem 3.2 as desired.  $\Box$ 

# S.2 Proofs for Section 3.2

**Proof of Theorems 3.3 and 3.4.** Theorem 3.3 can be shown similarly to Theorem 3.1 by using Lemma 3 instead of Lemma 2, except that when  $p \to \infty$ ,

$$||(\mathbf{\Sigma}_0^x)^{-1} \{\widehat{\mathbf{\Sigma}}_j^x (\widehat{\mathbf{\Sigma}}_j^x)'\}^{1/2}||_F = O_p\left(\left(\sum_{i=1}^p (\widetilde{\lambda}_i)^2\right)^{1/2}\right) = O_p(p^{1/2}),$$

where  $\tilde{\lambda}_i, 1 \leq i \leq p$  are solutions of (S.65). As a result, (S.63) should be replaced by

$$||(\mathbf{W}^y)^{-1/2}(\widehat{\mathbf{W}})^{1/2}||_F = O_p(p^{1/2}) \text{ and } ||(\mathbf{W}^x)^{-1/2}(\widehat{\mathbf{W}}^x)^{1/2}||_F = O_p(p^{1/2}).$$
 (S.76)

Theorem 3.4 can be shown similarly to Theorem 3.2. We omit the details.

#### S.3 Proofs for Section 4

To prove Theorems 4.1 and 4.2, we first introduce some notations. Let  $k_{ni} = n^{d_i - 1/2}I(d_i > 1/2) + n^{d_i + 1/2}I(d_i < 1/2)$  and  $\lambda_i(t - s) = (t - s)^{d_i - 1}/\Gamma(d_i)I(d_i > 1/2) + (t - s)^{d_i}/\Gamma(d_i + 1)I(d_i < 1/2)$ . Define  $\mathbf{K}_n = \operatorname{diag}(k_{n1}, \dots, k_{np})$ ,  $\mathbf{\Lambda}(t, s) = \operatorname{diag}(\lambda_1(t - s), \dots, \lambda_p(t - s))$  and

$$\mathbf{B}_0 = 0, \ \mathbf{B}_t = (B_t^1, \cdots, B_t^p)' = \int_0^t \mathbf{\Lambda}(t, s) \, d\mathbf{W}_s, \ \mathbf{U}_t = \mathbf{B}_t - \int_0^1 \mathbf{B}_t \, dt,$$

where  $\mathbf{W}_s$  is given in (ii) of Condition 3. Let  $\nabla^{d_l} v_t^l = \mu_l$ ,  $I_1^c = \{i : d_i > 1/2\}$  and  $\mathbf{x}_{t,I} = (x_t^i, i \in I)'$  and  $\mathbf{v}_{t,I} = (v_t^i, i \in I)'$ 

**Lemma 6.** Let  $\mathbf{Z}_n(t) = ((\mathbf{x}_{[nt],I_1^c} - \mathbf{v}_{[nt],I_1^c})', \sum_{j=1}^{[nt]} (\mathbf{x}_{j,I_1} - \mathbf{v}_{j,I_1})')'$ . Under (ii) of Condition 3,

$$\mathbf{K}_n^{-1}\mathbf{Z}_n(t) \stackrel{J_1}{\Longrightarrow} \mathbf{B}_t, \text{ on } D[0,1]^p. \tag{S.77}$$

Proof. Let  $d_{I_1} = \{d_i : i \in I_1\}$ , then  $\sum_{j=1}^{[nt]} \mathbf{x}_{j,I_1}$  is an integrated fractional process with order  $d_{I_1} + 1$ , and each of its components has order larger than 1/2. We can show this lemma similarly to Theorem 1 of Marinucci and Robinson (2000) by replacing their Lemma 2 with Condition 3(ii).

Let  $\Theta_n$  and  $\mathbf{M}_i(t)$  be defined as that after Lemma 1 by using  $H^d(t) = t^d/\Gamma(d_i+1) - 1/\Gamma(d_i+2)$  and  $\mathbf{F}_i(t) = (U^{\nu_i+1}(t), \dots, U^{\nu_i+r_i}(t))'$ , where  $U^i_t$  be the *i*-th component of  $\mathbf{U}_t$ . Let  $\mathbf{L}_n = \operatorname{diag}(l_{n1}, \dots, l_{np}), \ l_{ni} = n^{d_i-1/2}I(d_i > 1/2) + I(d_i < 1/2)$ . Similar to Lemma 2, by Lemma 6 and the continuous mapping theorem, we have the following lemma.

**Lemma 7.** Let the conditions of Theorem 5 hold. Then the following assertions hold for any  $0 \le j \le j_0$ .

(i) If  $\delta > 1/2$ , then

$$\mathbf{L}_{n}^{-1}\boldsymbol{\Theta}_{n}\widehat{\boldsymbol{\Sigma}}_{j}^{x}\boldsymbol{\Theta}_{n}'\mathbf{L}_{n}^{-1} \stackrel{d}{\longrightarrow} \int_{0}^{1} (\mathbf{M}_{t}', \mathbf{U}_{t2}')'(\mathbf{M}_{t}', \mathbf{U}_{t2}') dt, \quad and$$

$$\mathbf{L}_{n}^{-1}\boldsymbol{\Theta}_{n}\boldsymbol{\Sigma}_{j}^{x}\boldsymbol{\Theta}_{n}'\mathbf{L}_{n}^{-1} \stackrel{d}{\longrightarrow} \operatorname{diag}\left(\int_{0}^{1} \mathbf{M}_{t}\mathbf{M}_{t}' dt, \int_{0}^{1} \mathbf{U}_{t,2}\mathbf{U}_{t,2}' dt\right),$$

where  $\mathbf{U}_{t,2}$  is corresponding to the last p components of  $\mathbf{U}_t$ .

(ii) If  $\delta < 1/2$ , then

$$\mathbf{L}_{n}^{-1}\boldsymbol{\Theta}_{n}\widehat{\boldsymbol{\Sigma}}_{j}^{x}\boldsymbol{\Theta}_{n}'\mathbf{L}_{n}^{-1} \stackrel{d}{\longrightarrow} \operatorname{diag}\left(\int_{0}^{1} \mathbf{M}_{t}\mathbf{M}_{t}' dt, \operatorname{Cov}(\mathbf{x}_{t+j,I_{1}}\mathbf{x}_{t,I_{1}})\right), \tag{S.78}$$

and

$$\mathbf{L}_{n}^{-1}\boldsymbol{\Theta}_{n}\boldsymbol{\Sigma}_{j}^{x}\boldsymbol{\Theta}_{n}'\mathbf{L}_{n}^{-1} \stackrel{d}{\longrightarrow} \operatorname{diag}\left(\int_{0}^{1} \mathbf{M}_{t}\mathbf{M}_{t}' dt, \operatorname{Cov}(\mathbf{x}_{t+j,I_{1}}\mathbf{x}_{t,I_{1}})\right). \tag{S.79}$$

*Proof.* We only give the proof for  $\mu_j = 0$ ,  $j = 1, \dots, p$  in details, other cases can be proved similarly. By Lemma 6 and the continuous mapping theorem, it follows that (i) holds for j = 0. Thus, for the proof of (i), it suffices to show for any  $1 \le i, h \le p$ ,

$$\sup_{1 \le j \le j_0} \left| \frac{1}{n^{d_i + d_h}} \left( \sum_{t=1}^{n-j} (x_{t+j}^i - \bar{x}^i)(x_t^h - \bar{x}^h) - \sum_{t=1}^n (x_t^i - \bar{x}^i)(x_t^h - \bar{x}^h) \right) \right| = o_p(1). \tag{S.80}$$

Observe that

$$\sum_{t=1}^{n-j} (x_{t+j}^i - \bar{x}^i)(x_t^h - \bar{x}^h) - \sum_{t=1}^n (x_t^i - \bar{x}^i)(x_t^h - \bar{x}^h)$$

$$= \sum_{t=1}^{n-j} (x_{t+j}^i - x_t^i)(x_t^h - \bar{x}^h) - \sum_{t=n-j+1}^n (x_t^i - \bar{x}^i)(x_t^h - \bar{x}^h) =: \alpha_{n1}(j, i, h) + \alpha_{n2}(j, i, h).$$

By Lemma 5, it follows that for any  $1 \le i, h \le p$ ,

$$\left( (x_{[nt]}^i - \bar{x}^i) / n^{d_i - 1/2}, (x_{[ns]}^h - \bar{x}^h) / n^{d_h - 1/2} \right) \stackrel{J_1}{\Longrightarrow} (U^i(t), U^h(s)) \text{ on } D[0, 1]^2.$$
 (S.81)

This gives

$$\sup_{0 \le j \le j_0} |\alpha_{n2}(j, i, h)| / n^{d_i + d_h} = O_p(1/n).$$
(S.82)

Further, for any  $\varepsilon > 0$ , then

$$\lim_{n \to \infty} P\{ \sup_{1 \le j \le j_0} \sup_{1 \le t \le n-j} |x_{t+j}^i - x_t^i| / n^{d_i - 1/2} > \varepsilon \} = 0.$$
 (S.83)

Thus,

$$\sup_{0 \le j \le j_0} |\alpha_{n1}(j, i, h)| / n^{d_i + d_h} = o_p(1).$$
(S.84)

Combining (S.82) and (S.84) gives (S.80) as desired.

By (i) of Condition 3, it follows that

$$\frac{1}{n} \sum_{t=1}^{n} (\mathbf{x}_{t+j,I_1} - \bar{\mathbf{x}}_{I_1}) (\mathbf{x}_{t,I_1} - \bar{\mathbf{x}}_{I_1}) \stackrel{p}{\longrightarrow} \operatorname{Cov}(\mathbf{x}_{t+j,I_1} \mathbf{x}_{t,I_1}).$$

Thus, by (i) of Lemma 6, we have (S.79).

As for (S.78), it is enough to show for any  $i \in I_1^c$  and  $h \in I_1$ ,

$$\frac{1}{n^{d_{i}+1/2}} \sum_{i=1}^{n-j} (x_{t+j}^{i} - \bar{x}^{i})(x_{t}^{h} - \bar{x}^{h}) = \frac{\alpha_{n1}(j, i, h)}{n^{d_{i}+1/2}} + \frac{\alpha_{n2}(j, i, h)}{n^{d_{i}+1/2}} + \frac{1}{n^{d_{i}+1/2}} \sum_{i=1}^{n} (x_{t}^{i} - \bar{x}^{i})(x_{t}^{h} - \bar{x}^{h})$$

$$\stackrel{p}{\longrightarrow} 0, \tag{S.85}$$

holds for all  $0 \le j \le j_0$ . Similar to Lemma 1, we can show

$$\frac{1}{n^{d_i+1/2}} \sum_{i=1}^n (x_t^i - \bar{x}^i)(x_t^h - \bar{x}^h) \xrightarrow{p} 0.$$
 (S.86)

By (S.81) and  $n^{-1} \sum_{t=n-j_0}^n \mathbf{E} |x_t^h - \bar{x}^h| = O(j_0 n^{-1})$ , we have

$$\sup_{j \le j_0} \alpha_{n2}(j, i, h) / n^{d_i + 1/2} = O_p(1/n). \tag{S.87}$$

By (S.83) and  $\frac{1}{n}\sum_{t=1}^{n} \mathbf{E}|x_t^h - \bar{x}^h| = O(1)$ , we have

$$\sup_{j \le j_0} \alpha_{n1}(j, i, h) / n^{d_i + 1/2} = o_p(1). \tag{S.88}$$

(S.85) follows by equations (S.86)–(S.88).

**Proof of Theorems 4.1 and 4.2.** By Lemma 7, Theorems 4.1 and 4.2 can be established in a similar manner as to Theorems 3.1 and 3.2. Therefore we omit the detailed proofs.

### S.4 Proofs for Remarks 3.4 and 3.5

**Proof of Remark 3.4.** (i) By the martingale version of the Skorokhod representation theorem (Strassen 1967, Hall and Heyde 1980, and Wu 2007), we have for all i, on a richer probability space, there exists a standard Brownian motion  $\{W(t)\}$  and a non-negative stopping times  $\{\tau_j^i\}$  such that for  $t \geq 1$ ,

$$S_t^i = W(\sum_{j=1}^t \tau_j^i) \quad \text{and} \quad \mathrm{E}[\tau_t^i | \mathcal{F}_{t-1}(i)] = \mathrm{E}[(\varepsilon_t^i)^2 | \mathcal{F}_{t-1}(i)], \tag{S.89}$$

where  $\mathcal{F}_t(i)$  is the  $\sigma$ -algebra generated by  $\{\varepsilon_s^i, s \leq t\}$ . This implies that

$$E|S_t^i - W(\sigma_{ii}t)|^2 = E\left|\sum_{j=1}^t \tau_j^i - \sigma_{ii}t\right|$$

$$\leq E\left|\sum_{j=1}^t (\tau_j^i - E(\tau_j^i|\mathcal{F}_{t-1}(i)))\right| + E\left|\sum_{j=1}^t [(\varepsilon_j^i)^2 - E((\varepsilon_j^i)^2|\mathcal{F}_{t-1}(i))]\right| + E\left|\sum_{j=1}^t (\varepsilon_j^i)^2 - \sigma_{ii}t\right|.$$

Since both  $\{\tau_j^i - \mathrm{E}(\tau_j^i | \mathcal{F}_{t-1}(i))\}$  and  $\{(\varepsilon_j^i)^2 - \mathrm{E}((\varepsilon_j^i)^2 | \mathcal{F}_{t-1}(i))\}$  are martingale difference and  $\mathrm{E}|\varepsilon_t^i|^q < \infty$ , it follows that

$$\sup_{1 \le t \le n} \mathbf{E} \left| \sum_{j=1}^{t} (\tau_j^i - \mathbf{E}(\tau_j^i | \mathcal{F}_{t-1}(i))) \right| = O\left( \mathbf{E} \left| \sum_{j=1}^{n} [\tau_j^i - \mathbf{E}(\tau_j^i | \mathcal{F}_{t-1}(i))] \right| \right) = O(n^{2/q^*}).$$

Similarly,  $\sup_{1 \le t \le n} \mathbf{E} \left| \sum_{j=1}^t [(\varepsilon_j^i)^2 - \mathbf{E}((\varepsilon_j^i)^2 | \mathcal{F}_{t-1}(i))] \right| = O(n^{2/q^*})$ . Further, condition  $\mathbf{E} \left| \sum_{k=1}^n [(\varepsilon_t^i)^2 - \sigma_{ii}^2] \right| = O(n^{2/q^*})$  implies that  $\sup_{1 \le t \le n} \mathbf{E} \left| \sum_{k=1}^t [(\varepsilon_t^i)^2 - \sigma_{ii}^2] \right| = O(n^{2/q^*})$ . Thus, Condition 2(i) holds for any  $\tau > 1/q^*$ . If  $p = o(n^{1/2})$ , Condition 2(ii) holds. Since the components of  $\varepsilon_t$  are independent, Condition 2(iii) follows with  $\sup_j \sum_{s,t=1}^n \mathbf{E} |\varepsilon_s^j \varepsilon_t^j| = O(n)$ .

(ii) By the proof of Theorem 9.3.1 of Lin and Lu (1996), we know that there exists a martingale difference sequence  $\{m_t^i\}$  such that  $R_t = S_t^i - M_t^i$  satisfying  $E|R_t|^q = O(1)$ , where  $M_t^i = \sum_{j=1}^t m_j^i$ . Further,

$$E\left|\sum_{j=1}^{n} [(m_j^i)^2 - E(m_j^i)^2]\right|^{q/2} \le Cn\log n.$$
 (S.90)

As a result, Condition 2(i) holds for any  $\tau > 1/q$ . Similarly, Condition 2(iii) can be easily obtained by basic inequality for mixing processes, see Lemma 1.2.2 of Lin and Lu (1996). Note that for any given j,

$$\mathbb{E} \left\| \frac{1}{n} \sum_{t=1}^{n-j} (\mathbf{x}_{t+j,2} - \bar{\mathbf{x}}_2) (\mathbf{x}_{t2} - \bar{\mathbf{x}}_2)' - \operatorname{Cov}(\mathbf{x}_{1+j,2}, \mathbf{x}_{1,2}) \right\|_{2} \\
\leq \sum_{i,j=p-r+1}^{p} \mathbb{E} \left( \frac{1}{n} \sum_{t=1}^{n} [(x_t^i - \bar{x}^i)(x_t^j - \bar{x}^j) - \operatorname{Cov}(x_t^i, x_t^j)] \right)^2 = O(p^2/n) \to 0$$

as  $p = o(n^{1/2})$ . Condition 2(ii) holds too.

(iii) By Beveridge-Nelson decomposition,  $\varepsilon_t^i$  can be represented as

$$\varepsilon_t^i = \left(\sum_{j=0}^{\infty} c_{ij}\right) \eta_t^i - (\epsilon_t - \epsilon_{t-1}), \ \epsilon_t = \sum_{j=0}^{\infty} \widetilde{c}_{ij} \eta_{t-j}^i, \ \widetilde{c}_{ij} = \sum_{h=j+1}^{\infty} c_h.$$

Let  $R_t^i = S_t^i - \left(\sum_{j=0}^{\infty} c_{ij}\right) \sum_{j=1}^t \eta_j^i = \epsilon_t - \epsilon_0$ . Then

$$\sup_{1 \le t \le n} \left| S_t^i - \left( \sum_{j=0}^\infty c_{ij} \right) \sum_{j=1}^t \eta_j^i \right|^2 = O(1)$$
 (S.91)

Since  $\{\eta_t^i\}$  is an i.i.d sequence with  $\mathrm{E}|\eta_j^i|^q < \infty$ , by Theorem 4.3 of Strassen (1967), the stopping  $\{\tau_t^i\}$  defined as in (S.89) is an independent sequence with  $\mathrm{E}\tau_t^i = \mathrm{E}(\eta_j^i)^2$  and  $\mathrm{E}|\tau_t^i|^{q/2} < \infty$ . Thus,  $\sup_{1 \le t \le n} \mathrm{E}|\sum_{j=1}^t [\tau_j^i - \mathrm{E}(\eta_j^i)^2]|^{q/2} = O(n + n^{q/4})$ . As a result, we have for  $q_0 = \min\{q, 4\}$ ,

$$\sup_{1 \le t \le n} E \left| \sum_{j=1}^{t} [\tau_j^i - E(\eta_j^i)^2] \right| = O(n^{2/q_0}).$$

Let  $a_i = \mathrm{E}(\eta_j^i)^2$ , then on a richer space there exist a standard Brownian motion W(t) such that

$$E(\sum_{i=1}^{t} \eta_j^i - W(a_i t))^2 = O(n^{2/q_0}).$$
 (S.92)

Thus, by (S.91), (S.92), Condition 2(i) holds for  $\tau = 1/q_0$ . It is easy to show that  $\sup_{p-r < j \le p} \sum_{s,t=1}^{n} |\mathrm{E}(\varepsilon_t^j \varepsilon_s^j)|$  = O(n). Thus, Condition 2(iii) follows by the independence of the components. Condition 2(ii) can be shown similarly to Remark 3.4(ii).

**Proof of Remark 3.5.** It is easy to get  $\lambda^* = O_e(1)$  when p - r is fixed. We only show the case when  $m := p - r \to \infty$  as  $n \to \infty$ . Let  $d = d_{\min}$ ,  $\xi_t^i$  be I(1) process defined as in Lemma 3,  $\bar{\xi}^i = (\bar{\xi}^i, \dots, \bar{\xi}^i)'$  and  $\mathbf{e} = (1, \dots, 1)'$  be two n dimensional vectors. Let  $\mathbf{E}_n$  and  $\mathbf{\Pi}_n$  be  $n \times n$  matrices given by

$$\mathbf{E}_n = \mathbf{I}_n - \frac{1}{n}\mathbf{e}\mathbf{e}'$$
 and  $\mathbf{\Pi}_n = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}$ ,

then for any  $1 \le i, j \le m$ ,

$$\boldsymbol{\xi}^{i} = (\xi_{1}^{i}, \cdots, \xi_{n}^{i})' = \sigma_{ii} \boldsymbol{\Pi}_{n}^{-d} (v_{1}^{i}, v_{2}^{i}, \cdots, v_{n}^{i})' =: \sigma_{ii} \boldsymbol{\Pi}_{n}^{-d} \mathbf{V}^{i}, \text{ and}$$

$$(\boldsymbol{\xi}^{i} - \bar{\boldsymbol{\xi}}^{i})' (\boldsymbol{\xi}^{j} - \bar{\boldsymbol{\xi}}^{j}) = \sigma_{ii} \sigma_{jj} (\mathbf{V}^{i})' (\boldsymbol{\Pi}_{n}^{-d})' \mathbf{E}_{n} \mathbf{E}'_{n} \boldsymbol{\Pi}_{n}^{-d} \mathbf{V}^{j}.$$

Let  $\delta_1 \leq \cdots \leq \delta_n$  and  $\gamma_1 \geq \cdots \geq \gamma_n$  be the eigenvalues of  $\Pi_n \Pi'_n$  and  $(\Pi_n^{-d})' \mathbf{E}_n \mathbf{E}'_n \Pi_n^{-d}$  respectively. Since  $\lambda_1(\mathbf{E}_n \mathbf{E}'_n) = \cdots = \lambda_{n-1}(\mathbf{E}_n \mathbf{E}'_n) = 1$ , by Theorem 9 of Merikoski and Kumar (2004), it follows that

$$\delta_{i+1}^{-d} = \lambda_{i+1}((\mathbf{\Pi}_n^{-d})'\mathbf{\Pi}_n^{-d})\lambda_{n-1}(\mathbf{E}_n\mathbf{E}_n') \le \gamma_i \le \lambda_i((\mathbf{\Pi}_n^{-d})'\mathbf{\Pi}_n^{-d})\lambda_1(\mathbf{E}_n\mathbf{E}_n') = \delta_i^{-d}.$$
 (S.93)

Further,  $\delta_k = 2 - 2\cos(2k\pi/(2n+1))$ ,  $k = 1, 2, \dots, n$  (see Yueh (2005)), which implies

$$\delta_k \sim 4(k\pi/(2n+1))^2$$
, as  $k/n \to 0$ . (S.94)

Let **U** be an orthogonal matrix with row vectors  $\mathbf{u}_1, \dots, \mathbf{u}_n$  such that  $(\mathbf{\Pi}_n^{-d})' \mathbf{E}_n \mathbf{E}_n' \mathbf{\Pi}_n^{-d} = \mathbf{U} \operatorname{diag}(\gamma_1, \dots, \gamma_n) \mathbf{U}'$  and let  $\mathbf{\Omega} = (\mathbf{V}^1, \dots, \mathbf{V}^m)$ . For  $\mathbf{x} \in \mathcal{R}^m$  with  $\mathbf{x}' \mathbf{x} = 1$ , define  $\mathbf{U} \mathbf{\Omega} \mathbf{x} = (b_{1\mathbf{x}}, \dots, b_{n\mathbf{x}})' = \mathbf{b}_{\mathbf{x}} \in \mathcal{R}^n$ . By (S.93), we have

$$\lambda_{\min} \left( \frac{1}{n^{2d}} \sum_{t=1}^{n} (\boldsymbol{\xi}_{t} - \bar{\boldsymbol{\xi}}) (\boldsymbol{\xi}'_{t} - \bar{\boldsymbol{\xi}}') \right) \\
= \lambda_{\min} \left( \frac{1}{n^{2d}} (\boldsymbol{\xi}^{1} - \bar{\boldsymbol{\xi}}^{1}, \dots, \boldsymbol{\xi}^{m} - \bar{\boldsymbol{\xi}}^{m})' (\boldsymbol{\xi}^{1} - \bar{\boldsymbol{\xi}}^{1}, \dots, \boldsymbol{\xi}^{m} - \bar{\boldsymbol{\xi}}^{m}) \right) \\
\geq \left\{ \min_{i} (\sigma_{ii}) \right\}^{2} \min_{\mathbf{x}} \frac{1}{n^{2d}} \mathbf{x}' (\mathbf{U} \mathbf{\Omega})' \operatorname{diag}(\gamma_{1}, \dots, \gamma_{n}) (\mathbf{U} \mathbf{\Omega}) \mathbf{x} \\
= \left\{ \min_{i} (\sigma_{ii}) \right\}^{2} \min_{\mathbf{x}} \frac{(\mathbf{b}'_{\mathbf{x}} \mathbf{b}_{\mathbf{x}})}{n^{2d}} \frac{\mathbf{b}'_{\mathbf{x}} (\gamma_{1}, \dots, \gamma_{n}) \mathbf{b}_{\mathbf{x}}}{\mathbf{b}'_{\mathbf{x}} \mathbf{b}_{\mathbf{x}}} \\
\geq \left\{ \min_{i} (\sigma_{ii}) \right\}^{2} n^{1-2d} \lambda_{\min} (\mathbf{\Omega}' \mathbf{\Omega}/n) \min_{\mathbf{x}} \left( \sum_{l=1}^{k} \delta_{k+1}^{-d} b_{l\mathbf{x}}^{2} / \sum_{l=1}^{n} b_{l\mathbf{x}}^{2} \right) \\
\geq \left\{ \min_{i} (\sigma_{ii}) \right\}^{2} (k/n^{2d}) [\lambda_{\min} (\mathbf{\Omega}' \mathbf{\Omega}/n) / \lambda_{\max} (\mathbf{\Omega}' \mathbf{\Omega}/n)] \delta_{k+1}^{-d} \min_{\mathbf{x}} \frac{1}{k} \sum_{l=1}^{k} b_{l\mathbf{x}}^{2} \\
= O_{e}(k^{1-2d}) \lambda_{\min} \{ [\mathbf{\Omega}' (\mathbf{u}'_{1}, \dots, \mathbf{u}'_{k})] [(\mathbf{u}'_{1}, \dots, \mathbf{u}'_{k})' \mathbf{\Omega}] \}, \tag{S.95}$$

where  $\lambda_{\min}$ ,  $\lambda_{\max}$  denote the smallest and largest eigenvalues of a matrix respectively, the last equation follows by (S.94) and the fact that there exist two positive constants  $C_1, C_2$  such that  $C_1 < \lambda_{\min}(\Omega'\Omega/n) \le \lambda_{\max}(\Omega'\Omega/n) < C_2$  in probability when  $p/n^{1/2} \to 0$ . Since U is orthogonal and the elements of  $\Omega$  are independent standard normal variables, it follows that the elements of  $(\mathbf{u}'_1, \dots, \mathbf{u}'_k)'\Omega$  are independent and standard normal variables, thus by Theorem 2 of Bai and Yin (1993), we have if  $m/k \in (0,1)$ ,

$$\lambda_{\min}\{[\mathbf{\Omega}'(\mathbf{u}_1',\cdots,\mathbf{u}_k')][(\mathbf{u}_1',\cdots,\mathbf{u}_k')'\mathbf{\Omega}]\} = (1-\sqrt{m/k})^2, \text{ a.s.}$$
 (S.96)

Taking k = 2m, then by (S.95) and (S.96), we have

$$\lambda_{\min}\left(\frac{1}{n^{2d}}\sum_{t=1}^{n}(\boldsymbol{\xi}_{t}-\bar{\boldsymbol{\xi}})(\boldsymbol{\xi}'_{t}-\bar{\boldsymbol{\xi}}')\right) \geq Cm^{1-2d}.$$
 (S.97)

Since  $||\frac{1}{n^{2d}}\sum_{t=1}^{n}(\boldsymbol{\xi}_{t}-\bar{\boldsymbol{\xi}})(\boldsymbol{\xi}'_{t}-\bar{\boldsymbol{\xi}}')-\int_{0}^{1}\mathbf{F}(t)\mathbf{F}'(t)\,dt||_{2}=o_{p}(1)$  (see Lemma 3), Remark 3.5 follows from (S.97).

# S.5 Numerical studies for $j_0$ and m

To evaluate the impact of the choice of  $j_0$  and m, we report some of our simulation results in this subsection.

We first present a numerical result for the choice  $j_0$ . We let  $\mathbf{x}_{t2}$  in model (1) consist of r stationary AR(1) processes with coefficients generated independently from U(-0.8, 0.8),  $\mathbf{x}_{t1}$  be p-r ARIMA(1,1,1) processes with coefficients generated from U(0,0.8) and U(-0.8,0.8). Let the elements of  $\mathbf{A}$  be generated independently from U(-3,3). We estimate the cointegration rank r by the ACF unit-root test defined in (5). For each setting, we replicate the exercise 500 times with sample size n = 300, 500, 1000 and  $j_0$  ranging from 5 to 90. The relative frequencies for the occurrence of events  $\{\hat{r} = r\}$  and the average distance between the true cointegrating space and its true space are reported in Table S.1. It is shown from Table S.1 that the performance is stable with respect to the choice of  $j_0$ , especially for small p and large n.

Table S.1: Relative frequencies (RF) of the occurrence of event  $\{\hat{r} = r\}$  and average distance  $D_1$  with  $j_0$  ranging over (5,90) and 500 replications.

(p,r)	n		j=5	10	20	30	40	50	60	70	80	85	90
		RF	.950	.928	.938	.960	.940	.924	.928	.948	.934	.926	.926
	300	D1	.062	.069	.057	.047	.065	.065	.070	.053	.064	.068	.068
		RF	.982	.982	.986	.978	.984	.974	.978	.980	.978	.976	.988
(4, 2)	500	D1	.029	.024	.023	.029	.027	.036	.028	.030	.029	.030	.024
		RF	.992	.994	.996	.996	.994	.998	1.00	1.00	.994	.998	.998
	1000	D1	.013	.012	.013	.012	.011	.014	.008	.009	.014	.012	.009
		RF	.822	.794	.834	.810	.802	.826	.828	.794	.834	.816	.812
	300	D1	.128	.137	.115	.131	.136	.121	.122	.136	.120	.124	.129
		RF	.934	.948	.946	.938	.960	.962	.964	.970	.952	.960	.958
(6, 3)	500	D1	.061	.053	.053	.062	.052	.049	.047	.045	.050	.045	.050
		RF	.988	.990	.994	.994	.976	.984	.992	.994	.988	.994	.994
	1000	D1	.024	.018	.018	.018	.026	.022	.017	.017	.021	.020	.020
		RF	.562	.564	.578	.628	.612	.648	.620	.646	.592	.598	.610
	300	D1	.230	.224	.223	.204	.211	.198	.209	.194	.217	.213	.213
		RF	.874	.886	.858	.908	.884	.920	.910	.934	.900	.898	.914
(8, 4)	500	D1	.093	.078	.101	.078	.085	.077	.077	.067	.081	.081	.078
		RF	.966	.978	.986	.980	.984	.986	.986	.986	.988	.988	.990
	1000	D1	.046	.031	.028	.032	.030	.030	.027	.029	.031	.028	.028

Next table is reported for the choice of m. In this simulation,  $\mathbf{x}_{t1}$  and  $\mathbf{x}_{t2}$  are generated from model (1) as the previous example. We also replicate the exercise 500 times in each setting with sample size n=300,500 and 1000 and the lags number m is taken from 5 to 90. The corresponding relative frequencies for the occurrence of events  $\{\hat{r}=r\}$  and the average distance between the true cointegrating space and its true space are reported in Table S.2. It is shown from Table S.2 that a relatively small m always works well for the estimation of the cointegrating space. On the contrary, if m is selected too large, the performance is relatively poor, especially

when the sample size n is relatively small. This is reasonable, because from Remark 2.1, we know that only when  $n/m \to \infty$ ,  $\sum_{k=1}^{m} \widehat{\rho}(k)/m - 1 \to 0$ , otherwise, it is difficult to distinguish the integrated process from the stationary process, which means that m could not be selected too large, especially when n is relatively small. This simulation also confirms that m = 20 is usually good enough for the procedure.

In our simulation, we also use a data driven procedure in selecting m, which is given by

$$\widehat{m} = \operatorname{argmin}_m \{ f(m) \vee f^{-1}(m) \}$$

where  $T(m) = \left\{\frac{n}{m(m+1)}\sum_{i=1}^{p}\sum_{k=1}^{m}(1-\widehat{\rho}_{i}(k))\right\}$  and f(m) = T(m)/T(m+1). It also work reasonably, for example, when (p,r) = (6,3), the corresponding relative correct frequencies and average distance is (0.776,0.119) for n=300, (0.888,0.062) for n=500 and (0.970,0.0229) for n=1000.

Table S.2: Relative frequencies (RF) of the occurrence of event  $\{\hat{r} = r\}$  and average distance  $D_1$  with m ranging over (5,90) and 500 replications.

(p,r)	n		j=5	10	20	30	40	50	60	70	80	85	90
		RF	.872	.976	.980	.968	.936	.882	.816	.744	.644	.602	.586
	300	D1	.016	.021	.025	.031	.048	.078	.115	.156	.211	.235	.243
		RF	.864	.980	.998	.998	.996	.986	.968	.942	.914	.896	.866
(3, 2)	500	D1	.008	.009	.010	.010	.011	.017	.026	.041	.056	.067	.084
		RF	.874	.980	.996	.996	.996	.994	.994	.992	.992	.992	.992
	1000	D1	.006	.005	.006	.006	.006	.007	.007	.008	.008	.008	.008
		RF	.850	.942	.916	.818	.702	.560	.398	.250	.140	.102	.070
	300	D1	.043	.050	.064	.101	.149	.211	.283	.349	.403	.422	.441
		RF	.848	.966	.972	.958	.916	.854	.762	.660	.568	.518	.462
(6, 4)	500	D1	0.024	.026	.029	.034	.051	.076	.115	.160	.200	.223	.248
		RF	.842	.974	.996	.996	.992	.984	.976	.968	.942	.928	.906
	1000	D1	.011	.012	.013	.013	.014	.017	.020	.023	.034	.039	.049
		RF	.802	.848	.732	.510	.290	.118	.040	.004	0	0	0
	300	D1	.083	.095	.128	.203	.286	.363	.414	.452	.476	.483	.494
		RF	.800	.944	.930	.896	.792	.638	.438	.294	.186	.132	.104
(9, 6)	500	D1	.039	.045	.052	.063	.098	.154	.227	.286	.334	.359	.375
		RF	.856	.984	.990	.980	.980	.964	.946	.914	.862	.834	.786
	1000	D1	.018	.019	.019	.021	.021	.026	.032	.044	.062	.072	.089

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