Supplementary Material for "Statistical Inference in a Directed Network Model with Covariates"

This supplementary material contains the histograms of the estimated out-degree and in-degree parameters for the Sino Weibo data, the proofs for Theorems 1 and 3, and the proof of Lemma 4 in the main text. Section 2 presents the preliminaries that will be used in the proofs. The proofs of Theorems 1 and 3 are in sections 3 and 4, respectively. Section 5 presents the proof of Lemma 4.

1 Histograms of estimates of degree parameters fitted in the Sino Weibo data

Figure 1 provides the histograms of $\hat{\alpha}_i$'s and $\hat{\beta}_j$'s for the Sino Weibo data with 2242 nodes.
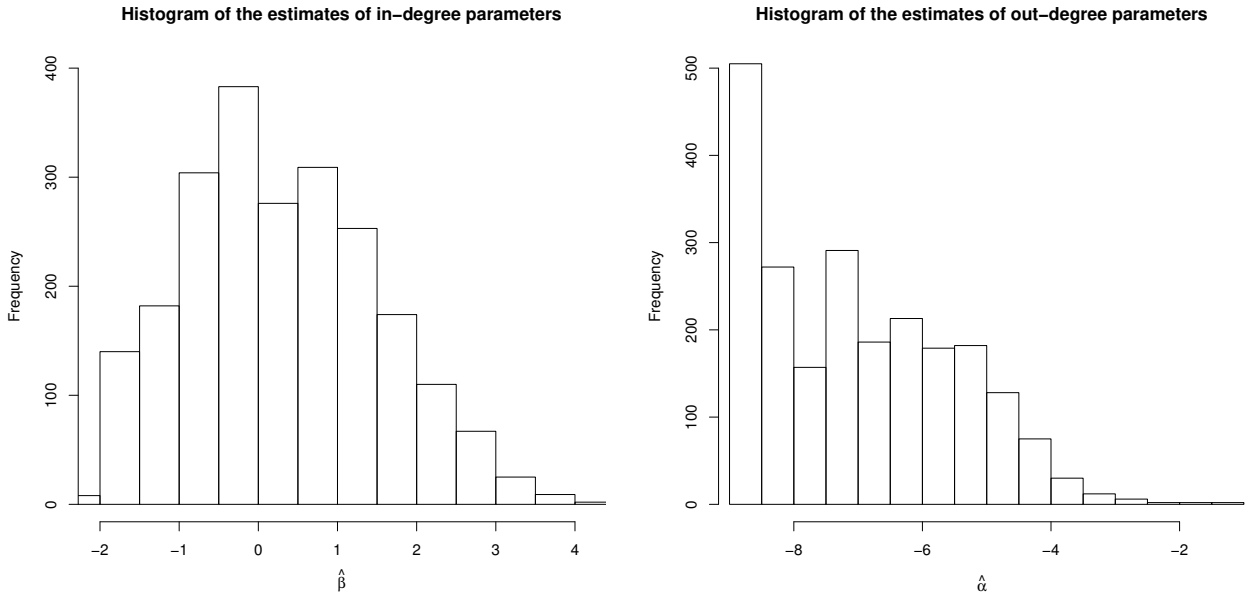

Figure 1: The histogram of the estimates of the in-degree (left) and out-degree (right) parameters in the Sino Weibo data.

2 Preliminaries

We first restate Lemma 1 giving the upper bound of the approximate error using S to approximate the inverse of the matrix V in the main text here for clarity.

Lemma 5. *If $V \in \mathcal{L}_n(m, M)$ with $M/m = o(n)$, then for large enough n,*

$$\|V^{-1} - S\| \leq \frac{c_1 M^2}{m^3 (n-1)^2},$$

where c_1 is a constant that does not depend on M, m and n, and $\|A\| := \max_{i,j} |a_{i,j}|$ for a general matrix $A = (a_{i,j})$.

Let D be an open convex subset of $\mathbb{R}^{2n-1}$, $\Omega(\mathbf{x}, r)$ denote the open ball $\{\mathbf{y} \in \mathbb{R}^{2n-1} : \|\mathbf{x} - \mathbf{y}\|_\infty < r\}$ and $\overline{\Omega(\mathbf{x}, r)}$ be its closure, where $\mathbf{x} \in \mathbb{R}^{2n-1}$. In order to characterize the rate of convergence of the Newton's iterative sequence for the function defined in equation (7) in the main text, we quote the theorem 7 from Yan et al. (2016), stated as one lemma below.

Lemma 6 (Yan et al. (2016)). *Define a system of equations:*

$$F_i(\boldsymbol{\theta}) = d_i - \sum_{k=1, k \neq i}^{n} f(\alpha_i + \beta_k), \quad i = 1, \ldots, n,$$

$$F_{n+j}(\boldsymbol{\theta}) = b_j - \sum_{k=1, k \neq j}^{n} f(\alpha_k + \beta_j), \quad j = 1, \ldots, n-1,$$

$$F(\boldsymbol{\theta}) = (F_1(\boldsymbol{\theta}), \ldots, F_n(\boldsymbol{\theta}), F_{n+1}(\boldsymbol{\theta}), \ldots, F_{2n-1}(\boldsymbol{\theta}))^\top,$$

where $f(\cdot)$ is a continuous function with the third derivative. Let $D \subset \mathbb{R}^{2n-1}$ be a convex set and assume for any $\mathbf{x}, \mathbf{y}, \mathbf{v} \in D$, we have

$$\|[F'(\mathbf{x}) - F'(\mathbf{y})]\mathbf{v}\|_\infty \leq K_1 \|\mathbf{x} - \mathbf{y}\|_\infty \|\mathbf{v}\|_\infty, \tag{1}$$

$$\max_{i=1,\ldots,2n-1} \|F_i'(\mathbf{x}) - F_i'(\mathbf{y})\|_\infty \leq K_2 \|\mathbf{x} - \mathbf{y}\|_\infty, \tag{2}$$

where $F'(\boldsymbol{\theta})$ is the Jacobin matrix of F on $\boldsymbol{\theta}$ and $F_i'(\boldsymbol{\theta})$ is the gradient function of F_i on $\boldsymbol{\theta}$. Consider $\boldsymbol{\theta}^{(0)} \in D$ with $\Omega(\boldsymbol{\theta}^{(0)}, 2r) \subset D$, where $r = \|[F'(\boldsymbol{\theta}^{(0)})]^{-1} F(\boldsymbol{\theta}^{(0)})\|_\infty$. For any $\boldsymbol{\theta} \in \Omega(\boldsymbol{\theta}^{(0)}, 2r)$, we assume that $F'(\boldsymbol{\theta}) \in \mathcal{L}_n(m, M)$ or $-F'(\boldsymbol{\theta}) \in \mathcal{L}_n(m, M)$. For $k = 1, 2, \ldots,$ define the Newton iterates $\boldsymbol{\theta}^{(k+1)} = \boldsymbol{\theta}^{(k)} - [F'(\boldsymbol{\theta}^{(k)})]^{-1} F(\boldsymbol{\theta}^{(k)})$. Let

$$\rho = \frac{c_1 (2n-1) M^2 K_1}{2m^3 n^2} + \frac{K_2}{(n-1)m}. \tag{3}$$

If $\rho r < 1/2$, then $\boldsymbol{\theta}^{(k)} \in \Omega(\boldsymbol{\theta}^{(0)}, 2r)$, $k = 1, 2, \ldots$, are well-defined and satisfy

$$\|\boldsymbol{\theta}^{(k+1)} - \boldsymbol{\theta}^{(0)}\|_{\infty} \leq r/(1 - \rho r). \tag{4}$$

Further, $\lim_{k \to \infty} \boldsymbol{\theta}^{(k)}$ exists and the limiting point is precisely the solution of $F(\boldsymbol{\theta}) = 0$ in the range of $\boldsymbol{\theta} \in \Omega(\boldsymbol{\theta}^{(0)}, 2r)$.

Regarding the asymptotic normality of $g_i - \mathbb{E}(g_i)$, we note that both $d_i = \sum_{k \neq i} a_{i,k}$ and $b_j = \sum_{k \neq j} a_{k,j}$ are sums of $n - 1$ independent Bernoulli random variables. By the central limit theorem for the bounded case in Loéve (1977, p.289), we know that $v_{i,i}^{-1/2}(d_i - \mathbb{E}(d_i))$ and $v_{n+j,n+j}^{-1/2}(b_j - \mathbb{E}(b_j))$ are asymptotically standard normal if $v_{i,i}$ diverges. Since we assume that Z_{ij}'s lie in a compact subset of $\mathbb{R}^p$ and the parameter space Θ of covariate parameters is compact, we have for all $i \neq j$,

$$\max_{\boldsymbol{\gamma} \in \Theta} |Z_{ij}^{\top} \boldsymbol{\gamma}| \leq \kappa, \tag{5}$$

where κ is a constant. Since $e^x/(1 + e^x)^2$ is a decreasing function on x when $x \geq 0$ and an increasing function when $x \leq 0$, we have

$$\frac{(n-1)e^{2\|\boldsymbol{\theta}^*\|_{\infty}+\kappa}}{(1 + e^{2\|\boldsymbol{\theta}^*\|_{\infty}+\kappa})^2} \leq v_{i,i} = \sum_{j \neq i} \frac{e^{Z_{ij}^{\top}\boldsymbol{\gamma}^*+\alpha_i^*+\beta_j^*}}{(1 + e^{Z_{ij}^{\top}\boldsymbol{\gamma}^*+\alpha_i^*+\beta_j^*})^2} \leq \frac{n-1}{4}, \quad i = 1, \ldots, 2n. \tag{6}$$

When $\|\boldsymbol{\theta}^*\|_{\infty} \leq \tau \log n$ for $\tau < 1/24$, both the lower and upper bounds go to ∞ as $n \to \infty$. Thus, we have the following proposition.

Proposition 1. *Assume that $A \sim \mathbb{P}_{\boldsymbol{\gamma}^*, \boldsymbol{\theta}^*}$ with $\boldsymbol{\gamma}^* \in \Gamma$. If $e^{\|\boldsymbol{\theta}^*\|_{\infty}} = o(n^{1/2})$, then for any fixed $k \geq 1$, as $n \to \infty$, the vector consisting of the first k elements of $S\{\mathbf{g} - \mathbb{E}(\mathbf{g})\}$ is asymptotically multivariate normal with mean zero and covariance matrix given by the upper left $k \times k$ block of S.*

3 Proofs for Theorem 1

The proof is similar to the proof of Theorem 1 in Yan et al. (2016) and we only give the different steps here. Recall the definition of $F_{\gamma}(\boldsymbol{\theta})$ in equation (7) in the main text. For notation convenience, we suppress the subscript γ in $F_{\gamma}(\boldsymbol{\theta})$ here. Then the Jacobin matrix $F'(\boldsymbol{\theta})$ of $F(\boldsymbol{\theta})$ can be calculated as follows. For $i = 1, \ldots, n$,

$$\frac{\partial F_i}{\partial \alpha_l} = 0, \quad l = 1, \ldots, n, \ l \neq i; \quad \frac{\partial F_i}{\partial \alpha_i} = -\sum_{k=1; k \neq i}^{n} \frac{e^{Z_{ij}^{\top}\boldsymbol{\gamma}+\alpha_i+\beta_k}}{(1 + e^{\alpha_i+\beta_k})^2},$$

$$\frac{\partial F_i}{\partial \beta_j} = -\frac{e^{Z_{ij}^\top \gamma + \alpha_i + \beta_j}}{(1 + e^{Z_{ij}^\top \gamma + \alpha_i + \beta_j})^2}, \quad j = 1, \ldots, n-1, \ j \neq i; \quad \frac{\partial F_i}{\partial \beta_i} = 0$$

and for $j = 1, \ldots, n-1$,

$$\frac{\partial F_{n+j}}{\partial \alpha_l} = -\frac{e^{Z_{ij}^\top \gamma + \alpha_l + \beta_j}}{(1 + e^{Z_{ij}^\top \gamma + \alpha_l + \beta_j})^2}, \quad l = 1, \ldots, n, \ l \neq j; \quad \frac{\partial F_{n+j}}{\partial \alpha_j} = 0,$$

$$\frac{\partial F_{n+j}}{\partial \beta_j} = -\sum_{k=1; k \neq j}^{n} \frac{e^{Z_{ij}^\top \gamma + \alpha_k + \beta_j}}{(1 + e^{Z_{ij}^\top \gamma + \alpha_k + \beta_j})^2}, \quad \frac{\partial F_{n+j}}{\partial \beta_l} = 0, \quad l = 1, \ldots, n-1.$$

It is easily verified that $-F'(\boldsymbol{\theta}) \in \mathcal{L}_n(m, M)$. Thus Lemmas 5 and 6 can be applied. Note that $\boldsymbol{\gamma}^*$ and $\boldsymbol{\theta}^*$ denote the true parameter vector. For every $\boldsymbol{\gamma} \in \Theta$, the constants K_1, K_2 and r in Lemma 6 are given in the following.

Lemma 7. *Take $D = R^{2n-1}$ and $\boldsymbol{\theta}^{(0)} = \boldsymbol{\theta}^*$ in Lemma 6. Assume that $\boldsymbol{\gamma} \in \Theta$ and*

$$\max\{\max_{i=1,\ldots,n} |d_i - \mathbb{E}(d_i)|, \max_{j=1,\ldots,n} |b_j - \mathbb{E}(b_j)|\} \leq \sqrt{(n-1)\log(n-1)}. \tag{7}$$

Then we can choose the constants K_1, K_2 and r in Lemma 6 as

$$K_1 = n-1, \quad K_2 = \frac{n-1}{2}, \quad r \leq \frac{(\log n)^{1/2}}{n^{1/2}} \left(c_{11} e^{6\|\boldsymbol{\theta}^*\|_\infty} + c_{12} e^{2\|\boldsymbol{\theta}^*\|_\infty} \right),$$

where c_{11} and c_{12} are constants.

Proof. The proof is similar to the proof of Lemma 2 in Yan et al. (2016). Note that $-F'(\boldsymbol{\theta}^*) \in \mathcal{L}_{2n-1}(m_*, M_*)$, where

$$M_* = \frac{1}{4}, \quad m_* = \frac{e^{2\|\boldsymbol{\theta}^*\|_\infty + \kappa}}{(1 + e^{2\|\boldsymbol{\theta}^*\|_\infty + \kappa})^2}.$$

The left proof only requires verification of the fact that all the steps hold when we replace $F_i'(\boldsymbol{\theta})$ in Yan et al. (2016) with the new expression here.

$\square$

Since the out- and in-degree of each node are a sequence of independent Bernoulli random variables, Lemma 3 in Yan et al. (2016) assures that condition (7) holds with a large probability.

Lemma 8 (Lemma 3 in Yan et al. (2016))**.** *With probability at least $1 - 4n/(n-1)^2$, we have*

$$\max\{\max_i |d_i - \mathbb{E}(d_i)|, \max_j |b_j - \mathbb{E}(b_j)|\} \leq \sqrt{(n-1)\log(n-1)}.$$

Combining the above two lemmas, we have the result of consistency.

Proof of Theorem 1. Assume that $\boldsymbol{\gamma} \in \Theta$ and condition (7) holds. Recall the Newton's iterates $\boldsymbol{\theta}^{(k+1)} = \boldsymbol{\theta}^{(k)} - [F'(\boldsymbol{\theta}^{(k)})]^{-1} F(\boldsymbol{\theta}^{(k)})$ with $\boldsymbol{\theta}^{(0)} = \boldsymbol{\theta}^*$. Let $r = O((\log n)^{1/2} e^{6\|\boldsymbol{\theta}^*\|_\infty} n^{-1/2})$ given in Lemma 6. If $\boldsymbol{\theta} \in \Omega(\boldsymbol{\theta}^*, 2r)$, then $-F'(\boldsymbol{\theta}) \in \mathcal{L}_n(m, M)$ with

$$M = \frac{1}{4}, \quad m = \frac{e^{2(\|\boldsymbol{\theta}^*\|_\infty + 2r + \kappa)}}{(1 + e^{2(\|\boldsymbol{\theta}^*\|_\infty + 2r + \kappa)})^2}.$$

Then similar to derive the bound of ρr in the proof of Theorem 1 in Yan et al. (2016), by Lemma 7 and condition (7), we have for sufficiently small r:

$$\rho r = O\left(\frac{(\log n)^{1/2} e^{12\|\boldsymbol{\theta}^*\|_\infty}}{n^{1/2}}\right).$$

Therefore, if $\|\boldsymbol{\theta}^*\|_\infty \leq \tau \log n$, then $\rho r \to 0$ as $n \to \infty$. Consequently, by Lemma 6, $\lim_{n\to\infty} \widehat{\boldsymbol{\theta}}^{(n)}$ exists. Denote the limit as $\widehat{\boldsymbol{\theta}}$, then it satisfies

$$\|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*\|_\infty \leq 2r = O\left(\frac{(\log n)^{1/2} e^{8\|\boldsymbol{\theta}^*\|_\infty}}{n^{1/2}}\right) = o(1).$$

By Lemma 8, condition (7) holds with probability approaching one, thus the above inequality also holds with probability approaching one. Here, $\widehat{\boldsymbol{\theta}}(\boldsymbol{\gamma})$ depends on $\boldsymbol{\gamma}$ and the above inequality holds for every $\boldsymbol{\gamma} \in \Theta$. Since $\widehat{\boldsymbol{\gamma}} \in \Theta$, it shows the consistency of the MLE $\widehat{\boldsymbol{\theta}}$. Since the likelihood is convex, if $\widehat{\boldsymbol{\theta}}$ exists, then it is unique. $\qquad\square$

4 Proofs for Theorem 3

The method of the proof of Theorem 3 follows the proof of Theorem 2 in Yan et al. (2016). Before proving Theorem 3, we first establish two lemmas.

Lemma 9. *Let $R = V^{-1} - S$ and $U = Cov[R\{\mathbf{g} - \mathbb{E}\mathbf{g}\}]$. Then*

$$\|U\| \leq \|V^{-1} - S\| + \frac{3(1 + e^{2\|\boldsymbol{\theta}^*\|_\infty + \kappa})^4}{4e^{4\|\boldsymbol{\theta}^*\|_\infty + 2\kappa}(n-1)^2}. \tag{8}$$

Proof. Note that
$$U = WVW^\top = (V^{-1} - S) - S(I - VS),$$

where I is a $(2n-1) \times (2n-1)$ diagonal matrix, and by the inequality (C3) in Yan et al. (2016), we have
$$|\{S(I - VS)\}_{i,j}| \leq \frac{3(1 + e^{2\|\boldsymbol{\theta}^*\|_\infty + \kappa})^4}{4e^{4\|\boldsymbol{\theta}^*\|_\infty + 2\kappa}(n-1)^2}.$$

Thus,

$$\|U\| \le \|V^{-1} - S\| + \|S(I_{2n-1} - VS)\| \le \|V^{-1} - S\| + \frac{3(1 + e^{2\|\boldsymbol{\theta}^*\|_\infty + \kappa})^4}{4e^{4\|\boldsymbol{\theta}^*\|_\infty + 2\kappa}(n-1)^2}.$$

$\square$

Lemma 10. *Assume that the conditions in Theorem 1 hold. If $\|\boldsymbol{\theta}^*\|_\infty \le \tau \log n$ and $\tau < 1/40$, then for any i,*

$$\widehat{\theta}_i - \theta_i^* = ([V(\widehat{\boldsymbol{\gamma}})]^{-1}\{\mathbf{g} - \mathbb{E}(\mathbf{g})\})_i + o_p(n^{-1/2}), \tag{9}$$

where $V(\widehat{\boldsymbol{\gamma}})$ is a matrix by replacing $\boldsymbol{\gamma}$ in V with its estimator $\widehat{\boldsymbol{\gamma}}$.

Proof. The proof follows the same line of arguments as the proof of Lemma 9 in Yan et al. (2016). The main difference is that the Taylor expansion for $\mathbf{g} - \mathbb{E}(\mathbf{g})$ contains one more term since it has the covariate term. We only present the different steps here. Let $\widehat{\xi}_{i,j} = \widehat{\alpha}_i + \widehat{\beta}_j - \alpha_i^* - \beta_j^*$. Specially, by Taylor's expansion, for any $1 \le i \ne j \le n$,

$$\frac{e^{Z_{ij}^\top \widehat{\boldsymbol{\gamma}} + \widehat{\alpha}_i + \widehat{\beta}_j}}{1 + e^{Z_{ij}^\top \widehat{\boldsymbol{\gamma}} + \widehat{\alpha}_i + \widehat{\beta}_j}} - \frac{e^{Z_{ij}^\top \widehat{\boldsymbol{\gamma}} + \alpha_i^* + \beta_j^*}}{1 + e^{Z_{ij}^\top \widehat{\boldsymbol{\gamma}} + \alpha_i^* + \beta_j^*}} = \frac{e^{Z_{ij}^\top \widehat{\boldsymbol{\gamma}} + \alpha_i^* + \beta_j^*}}{(1 + e^{Z_{ij}^\top \widehat{\boldsymbol{\gamma}} + \alpha_i^* + \beta_j^*})^2}\widehat{\xi}_{i,j} + h_{i,j},$$

where

$$h_{i,j} = \frac{e^{Z_{ij}^\top \widehat{\boldsymbol{\gamma}} + \alpha_i^* + \beta_j^* + \phi_{i,j}\widehat{\xi}_{i,j}}(1 - e^{Z_{ij}^\top \widehat{\boldsymbol{\gamma}} + \alpha_i^* + \beta_j^* + \phi_{i,j}\widehat{\xi}_{i,j}})}{(1 + e^{Z_{ij}^\top \widehat{\boldsymbol{\gamma}} + \alpha_i^* + \beta_j^* + \phi_{i,j}\widehat{\xi}_{i,j}})^3}\widehat{\xi}_{i,j}^2,$$

and $0 \le \phi_{i,j} \le 1$. Let

$$t_{i,j} := \frac{e^{Z_{ij}^\top \widehat{\boldsymbol{\gamma}} + \alpha_i + \beta_j}}{1 + e^{Z_{ij}^\top \widehat{\boldsymbol{\gamma}} + \alpha_i + \beta_j}} - \frac{e^{Z_{ij}^\top \boldsymbol{\gamma}^* + \alpha_i^* + \beta_j^*}}{1 + e^{Z_{ij}^\top \boldsymbol{\gamma}^* + \alpha_i^* + \beta_j^*}} = \frac{Z_{ij}^\top e^{Z_{ij}^\top \widetilde{\boldsymbol{\gamma}} + \alpha_i^* + \beta_j^*}}{(1 + e^{Z_{ij}^\top \widetilde{\boldsymbol{\gamma}} + \alpha_i^* + \beta_j^*})^2}(\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}^*),$$

where $\widetilde{\boldsymbol{\gamma}}$ lies in between $\widehat{\boldsymbol{\gamma}}$ and $\boldsymbol{\gamma}^*$. In the above equation, the second equation is due to the mean value theorem. By the likelihood equation (3) in the main paper, we have

$$\mathbf{g} - \mathbb{E}(\mathbf{g}) = V(\widehat{\boldsymbol{\gamma}})(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*) + \mathbf{h} + \mathbf{t},$$

where $\mathbf{h} = (h_1, \ldots, h_{2n-1})^\top$, $\mathbf{t} = (t_1, \ldots, t_{2n-1})^\top$ and,

$$h_i = \sum_{k=1, k \ne i}^n h_{i,k}, \quad i = 1, \ldots, n, h_{n+i} = \sum_{k=1, k \ne i}^n h_{k,i}, \quad i = 1, \ldots, n-1,$$
$$t_i = \sum_{k=1, k \ne i}^n t_{i,k}, \quad i = 1, \ldots, n, t_{n+i} = \sum_{k=1, k \ne i}^n t_{k,i}, \quad i = 1, \ldots, n-1.$$

Equivalently,

$$\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}^* = [V(\widehat{\boldsymbol{\gamma}})]^{-1}(\mathbf{g} - \mathbb{E}(\mathbf{g})) + [V(\widehat{\boldsymbol{\gamma}})]^{-1}\mathbf{h} + [V(\widehat{\boldsymbol{\gamma}})]^{-1}\mathbf{t}. \tag{10}$$

The upper bound for $|[V(\widehat{\boldsymbol{\gamma}})]^{-1}\mathbf{t})_i|$ can be derived as follows. By noting that $e^x/(1+e^x) \le 1/4$ and Z_{ij} is bounded by a constant, we have

$$\|t_{i,j}\|_\infty \le \frac{c}{4}\|\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}^*\|_\infty,$$

where c is a constant. Therefore,

$$|[V(\widehat{\boldsymbol{\gamma}})]^{-1}\mathbf{t})_i| \le |[S(\widehat{\boldsymbol{\gamma}})\mathbf{t}]_i| + |[R(\widehat{\boldsymbol{\gamma}})\mathbf{t}]_i| \le \frac{|t_i|}{\hat{v}_{ii}} + \frac{|t_{2n}|}{\hat{v}_{2n,2n}} + \|R\|_\infty \times [(2n-1)\max_i |t_i|] = o(n^{-1/2}).$$

$\square$

Proof of Theorem 3. Note that equation (10) holds. By Lemmas 9 and 10, if $\|\boldsymbol{\theta}^*\|_\infty \le \tau \log n$ and $\tau < 1/44$, then

$$(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})_i = [S(\widehat{\boldsymbol{\gamma}})\{\mathbf{g} - \mathbb{E}(\mathbf{g})\}]_i + o_p(n^{-1/2}).$$

Since $[S(\widehat{\boldsymbol{\gamma}})]_{r \times r}$ is a consistent estimator of $S_{r \times r}$ for a fixed r, Theorem 3 follows directly from Proposition 1.

$\square$

5   Proof for Lemma 4

Since a_{ij}'s for $1 \le i \ne j \le n$ are independent random variables and $s^*_{\gamma_{ij}}(\boldsymbol{\gamma}^*, \boldsymbol{\theta}^*)$ is only associated with the random variable a_{ij}, the $n(n-1)$ random variables $s^*_{\gamma_{ij}}(\boldsymbol{\gamma}^*, \boldsymbol{\theta}^*)$ ($1 \le i \ne j \le n$) are also independent. Next, we will show that $s^*_{\gamma_{ij}}(\boldsymbol{\gamma}^*, \boldsymbol{\theta}^*)$ is a bounded random vector. Since $s_{\gamma_{ij}}(\boldsymbol{\gamma}^*, \boldsymbol{\theta}^*)$ is a bounded random vector, it is sufficient to show that $H_{\boldsymbol{\gamma\theta}}H_{\boldsymbol{\theta\theta}}^{-1}s_{\theta_{ij}}(\boldsymbol{\gamma}^*, \boldsymbol{\theta}^*)$ is bounded.

By calculations, we have

$$(-H_{\boldsymbol{\gamma\theta}}S)_{,i} = \frac{\sum_{j\ne i} p_{ij}(1-p_{ij})Z_{ij}}{v_{ii}} + \frac{\sum_{i\ne n} p_{in}(1-p_{in})Z_{in}}{v_{2n,2n}}, \quad i=1,\ldots,n,$$

$$(-H_{\boldsymbol{\gamma\theta}}S)_{,n+j} = \frac{\sum_{i\ne j} p_{ij}(1-p_{ij})Z_{ij}}{v_{n+j,n+j}} - \frac{\sum_{i\ne n} p_{in}(1-p_{in})Z_{in}}{v_{2n,2n}}, \quad i=1,\ldots,n.$$

Therefore,

$$(-H_{\boldsymbol{\gamma\theta}}S)T_{ij} = \frac{\sum_{j\ne i} p_{ij}(1-p_{ij})Z_{ij}}{v_{ii}} + \frac{\sum_{i\ne j} p_{ij}(1-p_{ij})Z_{ij}}{v_{n+j,n+j}}. \tag{11}$$

By lemma (5), we have

$$\|H_{\boldsymbol{\gamma\theta}}W\| \leq (2n-1) \times \frac{c_1 M^2}{m^3(n-1)^2} \times \frac{1}{4}\|Z_{ij}\|. \tag{12}$$

Combining (11) and (12), we have

$$
\begin{aligned}
&|[(-H_{\boldsymbol{\gamma\theta}})(-H_{\boldsymbol{\theta\theta}})^{-1}T_{ij}]_k| \\
=\ & |(-H_{\boldsymbol{\gamma\theta}}ST_{ij})_k - (H_{\boldsymbol{\gamma\theta}}WT_{ij})_k| \\
\leq\ & \frac{\sum_{j\neq i}p_{ij}(1-p_{ij})Z_{ij,k}}{v_{ii}} + \frac{\sum_{i\neq j}p_{ij}(1-p_{ij})Z_{ij,k}}{v_{n+j,n+j}} + 2(2n-1) \times \frac{c_1 M^2}{m^3(n-1)^2} \times \frac{1}{4}\|Z_{ij,k}\| \\
=\ & O(1).
\end{aligned}
$$

Since

$$(-H_{\boldsymbol{\gamma\theta}})(-H_{\boldsymbol{\theta\theta}})^{-1}s_{\theta_{ij}}(\boldsymbol{\gamma}^*,\boldsymbol{\theta}^*) = (-H_{\boldsymbol{\gamma\theta}})(-H_{\boldsymbol{\theta\theta}})^{-1}T_{ij}\left(a_{ij} - \frac{e^{Z_{ij}^\top\boldsymbol{\gamma}+\alpha_i+\beta_j}}{1 + e^{Z_{ij}^\top\boldsymbol{\gamma}+\alpha_i+\beta_j}}\right),$$

it shows that each element of $H_{\boldsymbol{\gamma\theta}}H_{\boldsymbol{\theta\theta}}^{-1}s_{\theta_{ij}}(\boldsymbol{\gamma}^*,\boldsymbol{\theta}^*)$ is bounded. It can be checked that

$$
\begin{aligned}
&Var\left(\sum_i\sum_{j\neq i}s_{\gamma_{ij}}^*(\boldsymbol{\gamma}^*,\boldsymbol{\theta}^*)\right) \\
=\ & \sum_i\sum_{j\neq i}p_{ij}(1-p_{ij})(Z_{ij}Z_{ij}^\top - 2Z_{ij}(H_{\boldsymbol{\gamma\theta}}H_{\boldsymbol{\theta\theta}}^{-1}T_{ij})^\top + H_{\boldsymbol{\gamma\theta}}H_{\boldsymbol{\theta\theta}}^{-1}T_{ij}T_{ij}^\top H_{\boldsymbol{\theta\theta}}^{-1}H_{\boldsymbol{\gamma\theta}}^\top \\
=\ & NI_n(\boldsymbol{\gamma}^*).
\end{aligned}
$$

Then Lemma 4 follows by the central limit theorem for the bounded case in Loéve (1977, p.289).

References

Lang, S. (1993). Real and Functional Analysis. Springer.

Loéve, M. (1977). *Probability theory I*. 4th ed. Springer, New York.

Yan, T., Leng, C. and Zhu, J. (2016). Asymptotics in directed exponential random graph models with an increasing bi-degree sequence. *The Annals of Statistics*, **44**, 31–57.