Singular Invariant Markov Equilibrium in Stochastic Overlapping Generations Models*

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This paper examines the invariant Markov distribution associated with the rational expectations equilibrium in multi-period stochastic overlapping generations models under pure exchange. We prove that a simple linear iterated function system can generate the invariant distribution in these models under small discrete aggregate shocks. We provide sufficient conditions for the linear iterated function system to generate a singular invariant measure. We also characterize the set of economies which satisfy the sufficient conditions. The attractors of the singular invariant measure in our models exhibit fractal patterns. The existence of the linear iterated function system implies that an algorithm based on its structure can allow computing equilibria in long-period stochastic overlapping generations models with heterogeneity.

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1 Introduction

The multi-period stochastic overlapping generations (SOLG) model, extending the work of Samuelson (1958), has been widely used in macroeconomics, finance, and policy-making as an important workhorse model. Despite its importance, theoretical results in this model have been restricted to the existence, uniqueness, and stability of the stochastic steady state equilibria in the literature (see Grandmont and Hildenbrand 1974; Laitner 1981; Spear and Srivastava 1986; Wang 1993; Duffie et al. 1994; Wang 1994; Morand and Reffett 2007 and many others).

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The steady state in a stochastic model generates an invariant distribution over a fixed subset of the space of state variables. The deterministic OLG model, as a special case by collapsing a shock to a certain state, has the simplest invariant measure which concentrates on a single point with probability one.[1] Due to their simplicity, it is acceptable to limit attention to the properties described above when examining the deterministic steady states.

In contrast to deterministic OLG models, the steady state of a SOLG model is more complicated in the sense that the support of the invariant measure can be an uncountable subset of the state space even under a small aggregate shock.[2] In very simple stochastic models, such as that originally examined by Lucas (1972), one can generally find equilibria in which prices only depend on the contemporaneous aggregate shock, which we call strongly stationary equilibria. Spear (1985), however, showed that these strongly stationary, or short memory equilibria do not exist in SOLG models where two-period lived agents trade multiple goods and have intertemporally non-separable preferences.[3] Citanna and Siconolfi (2007) and Henriksen and Spear (2012) obtain the same result for two-period lived OLG models with time-separable utilities and cohort heterogeneity, and three-period lived OLG models populated by representative agents, respectively.

For general SOLG models without these kinds of stochastic equilibria, Duffie et al. (1994) showed that there will exist stationary Markov equilibria once one includes an appropriate set of lagged endogenous state variables. Citanna and Siconolfi (2010) showed that it was generically sufficient to consider recursive equilibria, in which the endogenous state variables are taken as the wealth distribution.

The exclusion of the memoryless equilibria and the existence of the Markov equilibria imply uncountable support for an invariant measure in general SOLG models. However, neither of these papers provides any characterizations of the nature of the continuous invariant measure. Therefore, an important question arises as to the nature of the invariant measure of the Markov equilibria as the stochastic steady-state equilibria.

By the Lebesgue decomposition theorem, measures on a Euclidean space can be decomposed into the sum of two measures, one absolutely continuous with respect to the Lebesgue measure, and the other singular. The *singular continuous* distribution is defined as a probability measure which assigns probability one to a Lebesgue measure zero set. A probability measure is *absolutely continuous* with respect to the Lebesgue measure if it assigns a positive probability to Lebesgue measure non-zero sets. It is well-known that a simple shock process such as a symmetric Bernoulli distribution can produce either a singular or an absolutely continuous invariant measure in a stochastic model (see Mitra et al. 2003 for instance).

To complete the theoretical characterizations of the steady-state equilibria in SOLG models, we study the continuity properties of their invariant distributions. Examining this feature is theoretically interesting since there exists a bifurcation into the two types of the measure continuity, depending on the parameters of the models for small deviations from a deterministic

[1] Benhabib and Day (1982) discuss the possibility of deterministic OLG models generating continuous invariant measures if erratic paths exist. However, the invariant measure of interest in this paper is one generated by iterating a dynamical system where the contraction mapping principle is applied. Such an invariant measure in deterministic OLG models should be a degenerate probability measure on a stationary point.

[2] In our model, aggregate uncertainty has a finite number of states. A small aggregate shock, in this case, means a short distance between the realized values of stochastic fundamentals in each state.

[3] The short memory equilibria depend on a finite history of shocks. As a special case of the equilibria, the strongly stationary equilibria is only affected by the current exogenous shock. Therefore, the cardinality of the support of the short-memory equilibria is finite.

model. Knowing the continuity features of an invariant measure can help determine the existence of its density function. Specifically, an absolutely continuous measure has a density with respect to the Lebesgue measure whereas a singular continuous measure does not. One needs only know the parameters identifying its density function to describe an absolutely continuous measure. On the other hand, representing a singular continuous measure requires a large set of information: the value of the distribution for every point in its support.

Despite its significance, there has been little attention, in the literature, paid to the problem of characterizing the continuity features of invariant measures in stochastic models. (Exceptions are Mitra et al. (2003) and Mitra and Privileggi (2009).) A key reason for the paucity of research is that researchers lack knowledge of the equilibrium process in these stochastic models. Even in models where one makes specific assumptions about preferences and other primitives, characterizing the equilibrium law of motion for the stochastic economy can be daunting. Furthermore, even if one is aware of the functional form of an equilibrium mapping, it can be difficult to determine under what conditions singular or absolutely continuous measures arise, and how they depend on economic environments unless the mapping is of a simple form.

Therefore, another main goal of this paper is to study whether there exists a simple form for the equilibrium mappings in SOLG models. We then provide a sufficient condition for the invariant measure of the equilibrium mapping to be singular. We also investigate under what economic parameter values the equilibrium mapping generates a singular invariant measure by satisfying the sufficient condition.

We approach the study of continuity properties by working from a simple OLG model to a general one, step by step. For analytical tractability, we first study a monetary model where agents live three periods, have logarithmic preferences, consume a single good, and save via accumulations of money holdings. There is an independent and identically distributed (I.I.D.) shock to the total amount of money which follows a simple Bernoulli process. This tractable model yields a closed form equilibrating process. One noticeable point is that the law of motion for the state variable, the money holding of the young, is linear in the lagged state variable as long as the aggregate shock is sufficiently small. The price of the single good is also linear in the state variable. Therefore, a linear iterated function system (LIFS) can represent the dynamics of Markov equilibria (ME).[4]

The closed form solution, in this case, states that the slope parameter of the LIFS is only affected by the endowment ratio between ages, not the total amount of endowment, the size of money shock or the probability of shocks. The Lipschitz constant of the LIFS belongs to $\left(0, \frac{1}{2}\right)$. A one-dimensional LIFS with this scaling factor satisfies the so-called "no-overlap" property. A non-overlapped LIFS creates a Cantor-like invariant distribution whose support is a Cantor-like invariant set. The Cantor-like invariant distribution is singular with respect to the Lebesgue measure if generated by a LIFS and the Cantor-like attractor shows fractal self-affinity.[5] Therefore, the log-linear monetary model has a singular continuous Markov measure under small aggregate shocks. This example shows that a multi-period SOLG model can generate fractal patterns in the rational expectation equilibrium.

[4] The Markov equilibria of interest in this paper are also called 'simple Markov equilibria' to contrast with generalized Markov equilibria where the state space adds variables such as the last period's marginal utility or a Lagrange multiplier (Kydland and Prescott, 1980), asset prices and individual consumption (Duffie et al., 1994), and continuation utilities (Feng et al., 2014). We avoid the possibility of multiple solutions issue given the minimal state variables by restricting to the equilibria around the deterministic steady state of interest instead of expanding the state space.

[5] A self-affine object is invariant under an anisotropic transformation (non-uniform scaling).

In a three-period SOLG model with a simple Lucas tree asset and general preferences, we cannot obtain a closed-form solution as in the log-linear monetary model. Hence, we apply the implicit function theorem (IFT) to infer that a LIFS can generate ME around a deterministic steady state for small shocks. We show that the slope parameters of the LIFS correspond to the eigenvalues of the Jacobian matrix for the locally linearized price dynamics evaluated at the deterministic steady state as long as the aggregate shock is small enough. Based on this relationship, we numerically study how structural parameters affect the Lipschitz constant and thus determine an open set of economies where the no-overlap property holds and hence, a singular invariant measure arises. We find that a model with low risk-aversion and small dividend income share can generate a singular Markov measure. This result highlights why there can exist a unique singular measure in the monetary model above, which is characterized by a zero dividend and risk-aversion set at one.

Finally, we extend the model to a more realistic version by allowing agents to live many-period lives, introducing permanent heterogeneity and letting the number of states of nature be arbitrary but finite. As in the simpler models, we show the existence of a simple but high-dimensional LIFS in a neighborhood of the deterministic steady state as long as the aggregate shocks are sufficiently small. We provide a condition under which the LIFS has identical affine matrices across states. As we did in the three-period model, we find a relationship between the affine matrices of the LIFS and the stable eigenvalues of the locally linearized price dynamics for both determinate and indeterminate equilibria. We give a weak sufficient condition for the model to generate the singular invariant measure. This condition is closely related to the size of the spectral radius of the affine matrices. We numerically check a similar relation between the largest eigenvalue and structural parameters as in the three-period model. Lastly, we produce some examples where multi-dimensional singular Markov measures appear when satisfying our weak sufficient condition for singularity.

The empirical relevance of our theoretical results can be found in the literature on fractal phenomena observed in financial data. The self-affinity aspect of stock prices was one of the topics studied in both the non-linear dynamics and time-series econometrics literature for over a decade between the mid-1980's and '90's. In the former literature, earlier researchers analyzed low-dimensional deterministic dynamic systems as data generating processes since the attractors of the chaotic dynamic systems can have self-affinity features. They claimed to have found evidence for the existence of low-dimensional chaotic attractors in the time-series stock prices data (see Baumol and Benhabib 1989; Scheinkman and Lebaron 1989). However, later work with large data sets concluded that the previous results were misderived due to the paucity of data (see Vassilicos et al. 1993). The consensus now is that there is very little evidence for low-dimensional deterministic systems as stock price generating processes (see Vassilicos et al. 1993; LeBaron 1994). These results, then, require (at a minimum) high-dimensional chaotic systems or stochastic systems to account for the empirical features of stock prices.

In the econometrics literature, many stochastic models have been developed based on the theory of fractal measure and fractional Brownian motion processes pioneered by Mandelbrot (1963) and Mandelbrot (1967). The time-series econometrics models capture the self-affinity properties observed in the financial time-series data. However, they lack an economic mechanism behind the data-generating processes. To fill this gap, our paper shows how stochastic economic models can generate the self-affine features in their rational expectation equilibria via high-dimensional recursive policy functions.

This paper contributes to the literature on several dimensions. On the theoretical side, as far as we know, this is the first paper to study the continuity property of an invariant measure in multi-period SOLG models by providing mechanisms and conditions for singular measures to appear. This paper also identifies the set of economies satisfying these conditions. This, in turn, extends the theoretical characterization of SOLG models beyond the well-known results on existence, uniqueness, and convergence.

There are a few papers in the literature which also examine singular invariant distributions in dynamic general equilibrium models. Mitra et al. (2003) characterizes the invariant Markov distribution in terms of singularity versus absolute continuity in a one-sector stochastic growth model with logarithmic preferences and Cobb-Douglas production functions. Thus, the stylized model has an explicit law of motion. On the other hand, we study not only a tractable monetary OLG model and but also a general model without preference specifications. Hence, we have to deal with the equilibrating process described by a system of implicit difference equations. We show simple linear policy functions can generate ME under certain conditions via an application of the IFT.

Mitra and Privileggi (2009) shows similar results with ours in the sense that they provide the characteristics of structural parameters to generate a singular invariant measure without parametric assumptions. One main difference is their paper analyzes a one-sector growth model whereas this paper deals with SOLG models. More importantly, we specify the high-dimensional law of motion explicitly even for general preferences and shock processes, so that we examine the continuity property of a multi-dimensional invariant measure.

Gardini et al. (2009) shows the existence of a Cantor-like limiting distribution of forward-looking equilibria in OLG models. However, their paper concentrates on two-period deterministic OLG models and thus, the invariant measure of interest in their paper is essentially for sunspot equilibria when multiple equilibria exist due to a backward bending offer curve. The nature of the equilibria in this paper is different in terms of dimensionality and the cause of stochasticity. This paper treats high-dimensional stochastic steady state generated by an intrinsic shock such as a shock to the endowment, dividend and asset quantity.

Another theoretical contribution of the paper is to explicitly show that the LIFS takes the eigenvalues of its corresponding full price dynamic system as its own in multi-period OLG models with cohort heterogeneity. Thus, given information on the eigenvalues of price dynamics, one can infer the contractivity and singularity property of the LIFS. To the best of our knowledge, this is the first paper to clearly show how the two systems are related via eigenvalues in general OLG models.

The results in this paper also have computational implications from the existence of a LIFS as the equilibrating process for SOLG models with heterogeneous agents. Researchers can simply adopt the structure of the LIFS and iterate this functional specification to convergence to find Markov policy rules. Since the LIFS requires only the first-order term for each endogenous state variable in the approximated policy functions, it will make computing very long-period lived SOLG models feasible by reducing a lot of unknown coefficients.

Since we prove the existence of a LIFS for an arbitrary shock process, a linear form approximation for the equilibrium process can be accurate even under an imperfectly correlated shock between labor and capital incomes. Under this type of shock, quasi-aggregation methods such as Krusell and Smith (1998) might generate large approximation errors since the marginal propensity to save can vary greatly by age. Therefore, our results offer the foundation of an alternative algorithm to deal with the limitations of existing ones in computing equilibrium in SOLG models.

2 Log-Linear Monetary Model

We first study a standard three-period SOLG model with fiat money under logarithmic preferences. Time is discrete labeled by $t = 0, 1, 2, \ldots$. Agents live three periods: youth, middle-aged and retired denoted by y, m, and r respectively. They consume a single good and can save via accumulations of the fiat money.

We assume an aggregate shock has a two states of support and follows an I.I.D. process with the probability of state s occurring equally to π^s where $0 < \pi^s < 1$ for $s \in \{h, l\}$ and $\pi^h + \pi^l = 1$. The history of the aggregate shocks given the initial shock realization, s_0, is represented by $S^t = \{s_1, s_2, \ldots, s_t \mid s_0\} \in \Sigma^t = \{h, l\}^t$ where we let l represent the low shock and h the high shock. We let $\left(S^t, s_{t+1}\right)_{s_{t+1} \in \{h,l\}}$ denote the set of nodes for one period after the histroy S^t. We make $\left(S^t, s_{t+1}, s_{t+2}\right)_{s_{t+1}, s_{t+2} \in \{h,l\}^2}$ denote the set of nodes for two periods after the histroy S^t. The total money supply in time t is stochastic in the amount M^{s_t} which depends only on the realization of the current shock.

If the economy moves from the low state to the high state, the total money supply increases. In the opposite case, the total money supply decreases. In these changes of the total money supply, we assume the government sells lump sums of the new money or buys lump sums of the old money by trading with the single good. If the government increases or decreases the money holding of existing agents proportionately, we recover the strong stationary equilibrium in parallel to the results in Lucas (1972). To have history dependent equilibria, we need the lump sum transaction mechanism which generates a real effect as above.

Agents are endowed with non-stochastic consumption goods in the amounts $(\omega_y, \omega_m, 0)$. The consumption plan of the representative household born in time t and history S^t is denoted by $c\left(S^t\right) = \left(c_y\left(S^t\right), \left(c_m\left(S^t, s_{t+1}\right)\right)_{s_{t+1} \in \{h,l\}}, \left(c_r\left(S^t, s_{t+1}, s_{t+2}\right)\right)_{s_{t+1}, s_{t+2} \in \{h,l\}^2}\right)$ where $c_y\left(S^t\right)$ is consumption when young in node S^t, $c_m\left(S^t, s_{t+1}\right)$ is consumption when middle-aged in node $\left(S^t, s_{t+1}\right)$, and $c_r\left(S^t, s_{t+1}, s_{t+2}\right)$ is the old period consumption in node $\left(S^t, s_{t+1}, s_{t+2}\right)$. Similarly, the money holdings of the household is given by $m\left(S^t\right) = \left(m_y\left(S^t\right), \left(m_m\left(S^t, s_{t+1}\right)\right)_{s_{t+1} \in \{h,l\}}\right)$ where $m_y\left(S^t\right)$ and $m_m\left(S^t, s_{t+1}\right)$ denote money holdings when young and middle-aged in node S^t and $\left(S^t, s_{t+1}\right)$, respectively.

Lifetime preferences are additively time-separable logarithmic given by a von Neumann-Morgenstern utility function $U : \mathbb{R}^7_+ \to \mathbb{R}$. U is specified by:

$$(1)\quad E_t U\left(c\left(S^t\right)\right) = \ln\left(c_y\left(S^t\right)\right) + \sum_{s_{t+1} \in \{h,l\}} \pi^{s_{t+1}} \left\{ \ln\left(c_m\left(S^t, s_{t+1}\right)\right) + \sum_{s_{t+2} \in \{h,l\}} \pi^{s_{t+2}} \ln\left(c_r\left(S^t, s_{t+1}, s_{t+2}\right)\right) \right\}$$

where the discount factor sets at one, $\beta = 1$, for analytical tractability. Sequential budget constraints are given by:

$$(2)\quad \begin{aligned} c_y\left(S^t\right) &= \omega_y - \tilde{p}\left(S^t\right) m_y\left(S^t\right) \\ c_m\left(S^t, s_{t+1}\right) &= \omega_m + \tilde{p}\left(S^t, s_{t+1}\right) m_y\left(S^t\right) - \tilde{p}\left(S^t, s_{t+1}\right) m_m\left(S^t, s_{t+1}\right) \text{ for } s_{t+1} \in \{h, l\} \\ c_r\left(S^t, s_{t+1}, s_{t+2}\right) &= \tilde{p}\left(S^t, s_{t+1}, s_{t+2}\right) m_m\left(S^t, s_{t+1}\right) \text{ for } (s_{t+1}, s_{t+2}) \in \{h, l\}^2 \end{aligned}$$

where $\tilde{p}\left(S^t\right)$, $\tilde{p}\left(S^t, s_{t+1}\right)$ and $\tilde{p}\left(S^t, s_{t+1}, s_{t+2}\right)$ are the price of the fiat money in terms of the single good in node S^t, $\left(S^t, s_{t+1}\right)$ and $\left(S^t, s_{t+1}, s_{t+2}\right)$, respectively. Note that we do not specify the problems of the initial middle-aged and old generations in time 0 since we focus on the stochastic steady state in all models analyzed in this paper.

Agents maximize the expected utility subject to the sequential budget constraints. This yields the following three first-order conditions (FOC):

$$\text{(3)} \qquad \frac{\tilde{p}\left(S^t\right)}{c_y\left(S^t\right)} = \frac{\pi^h \tilde{p}\left(S^t, h\right)}{c_m\left(S^t, h\right)} + \frac{\pi^l \tilde{p}\left(S^t, l\right)}{c_m\left(S^t, l\right)}$$

and

$$\text{(4)} \qquad \frac{\tilde{p}\left(S^t, s_{t+1}\right)}{c_m\left(S^t, s_{t+1}\right)} = \frac{\pi^h \tilde{p}\left(S^t, s_{t+1}, h\right)}{c_r\left(S^t, s_{t+1}, h\right)} + \frac{\pi^l \tilde{p}\left(S^t, s_{t+1}, l\right)}{c_r\left(S^t, s_{t+1}, l\right)}$$

for $s_{t+1} \in \{h, l\}$.

The money market clearing condition requires:[6]

$$\text{(5)} \qquad M^{s_t} = m_y\left(S^t\right) + m_m\left(S^t\right)$$

for $s_t \in \{h, l\}$.

Since there is a change in the total money supply between periods, the current money supply can be rewritten with the previous money supply as:

$$\text{(6)} \qquad M^{s_t} = m_y\left(S^t\right) + m_m\left(S^t\right) = M^{s_{t-1}} + \triangle M_{t-1,t} = m_y\left(S^{t-1}\right) + m_m\left(S^{t-1}\right) + \triangle M_{t-1,t}$$

where $M^{s_{t-1}}$ is the total money supply in time $t-1$ under state s_{t-1}, M^{s_t} is the one in time t under state s_t, and $\triangle M_{t-1,t}$ denotes a change in the total money supply between time $t-1$ and t.

With the notations above, we define two equilibrium concepts for this monetary model: the competitive equilibrium and the recursive ME.

Definition 1. The competitive equilibrium in the monetary model is a sequence of the money holdings, consumption and asset prices in all nodes starting in time 0: $\left\{m\left(S^t\right), c\left(S^t\right), \tilde{p}\left(S^t\right)\right\}$ for $\forall S^t$ and $t \geq 0$. The competitive equilibrium requires:

- Individuals maximize their expected utility under budget constraints given the sequence of the price of the fiat money.
- The money market clears and the aggregate resource constraint holds.

The existence of a competitive equilibrium can be verified using a standard truncation method used in Balasko and Shell (1981).

Definition 2. The recursive ME is defined by time-homogeneous policy functions for money holdings, consumption and asset prices: $\left\{m_y(\chi), m_m(\chi), c_y(\chi), c_m(\chi), c_r(\chi), \widetilde{p}(\chi)\right\}$ which solve the household problem and clear the money and consumption markets. $\chi = \left[m_{y,-1}, s\right] \in \hat{\Sigma} \subset \mathbb{R}^2$ represents the minimal state variables: the lagged money holdings of the young and the realization of the current aggregate uncertainty.

[6] By Walras's law, we can ignore the market clearing for the consumption good.

For the recursive ME, we can re-write the equilibrium conditions as follows:

$$\frac{\tilde{p}\left(m_{y,t-1}, s_t\right)}{c_y\left(m_{y,t-1}, s_t\right)} = \frac{\pi^h \tilde{p}\left(m_{y,t}, h\right)}{c_m\left(m_{y,t}, h\right)} + \frac{\pi^l \tilde{p}\left(m_{y,t}, l\right)}{c_m\left(m_{y,t}, l\right)} \tag{7}$$

$$\frac{\tilde{p}\left(m_{y,t}, s_{t+1}\right)}{c_m\left(m_{y,t}, s_{t+1}\right)} = \frac{\pi^h \tilde{p}\left(m_{y,t+1}, h\right)}{c_r\left(m_{y,t+1}, h\right)} + \frac{\pi^l \tilde{p}\left(m_{y,t+1}, l\right)}{c_r\left(m_{y,t+1}, l\right)} \tag{8}$$

for $s_{t+1} \in \{h, l\}$ and

$$M^{s_t} = m_y\left(m_{y,t-1}, s_t\right) + m_m\left(m_{y,t-1}, s_t\right) \tag{9}$$

We impose the following two linear functions for prices and money holdings that the agents use to forecast future variables:

$$\begin{aligned} m_y\left(m_{y,t-1}, s_t\right) &= \bar{m}^{s_t} - \gamma^{s_t}\left(m_{y,t-1} - \bar{m}^{s_t}\right) \\ q\left(m_{y,t-1}, s_t\right) &= \bar{q}^{s_t} + \rho^{s_t}\left(m_{y,t-1} - \bar{m}^{s_t}\right) \end{aligned} \tag{10}$$

for $s_t \in \{h, l\}$ where $q = \frac{1}{\tilde{p}}$, i.e. the price of the consumption good in terms of the fiat money. The first and second equations in (10) are the forecast functions for the money holding of the young and the price of the single good respectively given the lagged money holding of the young and the realization of the current shock. By plugging these forecast functions into the FOCs and combining them with the market clearing condition, we can solve this monetary model and get a LIFS generating ME as long as the size of the aggregate shock measured by $M^h - M^l$ is sufficiently small. We summarize these results in the following proposition.

Proposition 1. *In a three-period monetary OLG model with $u(c) = ln(c)$ and $\beta = 1$, there will be ME for the model generated by a LIFS given by (10) for sufficiently small shocks. The slope parameters in (10) are parallel between states : $\gamma = \gamma^h = \gamma^l$ and $\rho = \rho^h = \rho^l$. A closed form solution for γ and ρ is given by:*

$$\gamma = -\frac{1}{2} - \frac{\omega_y}{\omega_m} + \sqrt{\left(\frac{1}{2} + \frac{\omega_y}{\omega_m}\right)^2 + \frac{\omega_y}{\omega_m}} \tag{11}$$

and

$$\rho = \frac{2\gamma - 1}{\omega_m} \tag{12}$$

Proof. *See Appendix A*

The policy functions for the money holding for the young determine the LIFS in the monetary model. The closed form solution for the Lipschitz constant of the LIFS, γ, is only affected by the endowment ratio between the young and the middle-aged, not the total amount of endowment. On the other hand, the slope of the price function is influenced by the total endowment because there is ω_m in the denominator of the closed form solution for ρ. Likewise, the state-contingent stationary points for the money holding function, $\{\bar{m}^s\}$, are independent of the total amount of endowment but dependent on the endowment ratio while the ones for the price function, $\{\bar{q}^s\}$, are affected by the total endowment. (see the proof of Proposition 1 in Appendix A.)

The intuitions behind these results are as follows. Since the logarithmic utility function is inter-temporally homothetic, optimal consumption allocations are proportional to the wealth of agents. When the total endowment changes but the endowment ratio between generations remains constant, the equilibrium price will move in proportion to the total endowment variation. Thus, the same money holding across generations can attain the optimal consumption choice. However, the money holding demands will alter if there are any variations in the endowment ratio between generations since some generations want more money and others not. Therefore, only the ratio of endowment matters for the money holding demand whereas both the total endowment and its ratio affect the price of consumption.

The slope parameters defined by (11) and (12) do not depend on the total money supply in each state and the probability distribution of shocks at least if the shock process is I.I.D. The money supply and probabilities do, however, affect the state-contingent intercept parameters. These findings imply that the rates of variations in the current money holding for the young and the price in response to variations in the lagged money holding for the young are constant whatever the size of the stochastic money supply and its probability distribution. These results accord closely with the findings in Mitra et al. (2003) that the amplitude of an exogenous shock and its probability distribution do not have an effect on the Lipschitz constant of the LIFS in a one-sector stochastic growth model with a logarithmic utility function and a Cobb-Douglas production function.

With the closed form solution above, we know the slope of the LIFS is increasing in the endowment ratio, $\frac{\omega_y}{\omega_m}$. The slope converges to 0 as the endowment ratio goes to 0 and it converges to $\frac{1}{2}$ as the endowment ratio goes to infinity. Under this range of the Lipschitz constant, the images of two parallel functions in the LIFS on the smallest open interval containing its attractor have an empty intersection set. Hence, the law of motion in the monetary model always satisfies the no-overlap property. This result is presented in the following corollary.

Corollary 1. *The Lipschitz constant of the LIFS, γ, is between 0 and $\frac{1}{2}$ for the entire set of economies in the monetary model. Hence, both contractivity and no-overlap property hold always so that there exists a unique Markov measure which is a Cantor-like distribution.*

Proof. *From the well-known results for the one-dimensional homogeneous LIFS with two states.*

A one-dimensional non-overlapped LIFS creates a unique Cantor-like invariant distribution of which support is a unique Cantor-like invariant set. The Cantor-like invariant distribution is singular with respect to the Lebesgue measure if it is generated by a LIFS because the images of the iterates of the LIFS decreases proportionally by a factor of $(1-2\gamma)$ and the limiting set has Lebesgue measure zero. Therefore, there exists a unique singular Markov measure for the entire economies in the log-linear monetary model under a small aggregate shock. Moreover, the monetary example here shows a multi-period SOLG model can generate fractal patterns in the rational expectation equilibrium since the Cantor-like invariant set or attractor has a self-affine structure. (see Appendix A to see how a one-dimensional non-overlapped generates a singular invariant distribution in detail.)

Finally, the fact that money holdings are a sufficient statistic for the history of shocks is particularly apparent here given that the Cantor set is itself homeomorphic to the space of the history of shocks from the minus infinity to the present. We denote this space of infinite histories by Σ. From the results in Woodford (1986), we also know the price and savings will be given by a unique function of the histories of shocks at least for small shocks. One can define a function, known as the coding map, $\pi(S)$ where $S \in \Sigma$, which maps infinite histories into the realizations of allocations. By the results on the contractive LIFS in Atkins et al. (2010), one can also show that the lagged asset holdings in the model will remain sufficient statistics for the history of shocks even in cases where the LIFS doesn't satisfy the no-overlap condition and the invariant measure is absolutely continuous. Their result implies that the coding map is point fiber when a LIFS is contractive and thus, there is a unique history of shocks leading to points in the attractor. This in turn tells us that the prices generated in the stochastic equilibrium can be written as $\pi_q(S_{-1}s) = q(m_{-1}, s) = q(\pi_m(S_{-1}), s)$ where π_q and π_m are coding maps for the price and money holdings respectively, the money holdings coding map is point fiber, S_{-1} is the history of shocks up to the previous period and s denote the state of the current shock. (see Appendix A for the definitions of the coding map and fiber in detail.)

3 Lucas-Tree Model

In this section, we replace the fiat money with a Lucas-tree to study an invariant Markov measure in a more general setting. Thus, agents save via accumulations of equity shares in a tree asset. We keep the assumption that aggregate uncertainty is given by two states of nature, $s \in \{h, l\}$. However, we assume here that the shock follows a first-order Markov process with a transition probability matrix $\prod = \begin{bmatrix} \pi^{hh} & \pi^{hl} \\ \pi^{lh} & \pi^{ll} \end{bmatrix}$ where $\pi^{ss'} \geq 0$ for $(s, s') \in \{h, l\}^2$ and $\pi^{sh} + \pi^{sl} = 1$ for $s \in \{h, l\}$. The tree asset in time t delivers dividends stochastically in the amount δ^{s_t} which depend only on the current shock.

Agents born in time t and history S^t are endowed with stochastic consumption goods in the amounts $\left(\omega_y^{s_t}, \omega_m^{s_{t+1}}, \omega_r^{s_{t+2}}\right)$ which also depends only on the current shock. The lifetime asset portfolio of the representative households is given by $e\left(S^t\right) = \left(e_y\left(S^t\right), \left(e_m\left(S^t, s_{t+1}\right)\right)_{s_{t+1} \in \{h,l\}}\right)$ where $e_y\left(S^t\right)$ is equity holding when young in node S^t and $e_m\left(S^t, s_{t+1}\right)$ is equity holding when middle-aged in node $\left(S^t, s_{t+1}\right)$.

Lifetime preferences are additively time-separable given by a von Neumann-Morgenstern utility function $U : \mathbb{R}_+^7 \to \mathbb{R}$. U is specified by:

(13)

$$E_t U\left(c\left(S^t\right)\right) = u\left(c_y\left(S^t\right)\right) + \beta \sum_{s_{t+1} \in \{h,l\}} \pi^{s_t s_{t+1}} \left\{ u\left(c_m\left(S^t, s_{t+1}\right)\right) + \beta \sum_{s_{t+2} \in \{h,l\}} \pi^{s_{t+1} s_{t+2}} u\left(c_r\left(S^t, s_{t+1}, s_{t+2}\right)\right) \right\}$$

where $\beta \in (0, 1]$. We assume utility functions $u(\cdot)$ satisfy regular conditions: $u'(c) > 0$, $u''(c) < 0$, and $u'(0) = +\infty$.

Sequential budget constraints are given by:

$$\begin{aligned} c_y\left(S^t\right) &= \omega_y^{s_t} - p\left(S^t\right) e_y\left(S^t\right) \\ c_m\left(S^t, s_{t+1}\right) &= \omega_m^{s_{t+1}} + \left(p\left(S^t, s_{t+1}\right) + \delta^{s_{t+1}}\right) e_y\left(S^t\right) - p\left(S^t, s_{t+1}\right) e_m\left(S^t, s_{t+1}\right) \text{ for } s_{t+1} \in \{h, l\} \\ c_r\left(S^t, s_{t+1}, s_{t+2}\right) &= \omega_r^{s_{t+2}} + \left(p\left(S^t, s_{t+1}, s_{t+2}\right) + \delta^{s_{t+2}}\right) e_m\left(S^t, s_{t+1}\right) \text{ for } \left(s_{t+1}, s_{t+2}\right) \in \{h, l\}^2 \end{aligned} \quad (14)$$

where $p\left(S^t\right)$, $p\left(S^t, s_{t+1}\right)$ and $p\left(S^t, s_{t+1}, s_{t+2}\right)$ are the price of the equity in terms of the single good in node S^t, $\left(S^t, s_{t+1}\right)$ and $\left(S^t, s_{t+1}, s_{t+2}\right)$, respectively. Agents maximize expected utility subject to the sequential budget constraints. This yields the following three first-order conditions conditional on the shock history:

$$p\left(S^t\right) u'\left(c_y\left(S^t\right)\right) = \beta \sum_{s_{t+1}\in\{h,l\}} \pi^{s_t s_{t+1}} \left(p\left(S^t, s_{t+1}\right) + \delta^{s_{t+1}}\right) u'\left(c_m\left(S^t, s_{t+1}\right)\right) \tag{15}$$

and

$$p\left(S^t, s_{t+1}\right) u'\left(c_m\left(S^t, s_{t+1}\right)\right) = \beta \sum_{s_{t+2}\in\{h,l\}} \pi^{s_{t+1} s_{t+2}} \left(p\left(S^t, s_{t+1}, s_{t+2}\right) + \delta^{s_{t+2}}\right) u'\left(c_r\left(S^t, s_{t+1}, s_{t+2}\right)\right) \tag{16}$$

for $s_{t+1} \in \{h, l\}$.

Under standard assumptions on preferences and endowments, Eq. (15) and (16) yield asset demand functions:

$$e_y\left(S^t\right) = e_y\left(\boldsymbol{P}_t^{t+2}\left(S^t\right)\right) \tag{17}$$

and

$$e_m\left(S^t, s_{t+1}\right) = e_m\left(\boldsymbol{P}_t^{t+2}\left(S^t\right); s_{t+1}\right) \tag{18}$$

for $s_{t+1} \in \{h, l\}$ where $\boldsymbol{P}_t^{t+2}\left(S^t\right) = \left\{p\left(S^t\right), \left(p\left(S^t, s_{t+1}\right)\right)_{s_{t+1}\in\{h,l\}}, \left(p\left(S^t, s_{t+1}, s_{t+2}\right)\right)_{s_{t+1}, s_{t+2}\in\{h,l\}^2}\right\}$.

The demands for the equity are the functions of $\boldsymbol{P}_t^{t+2}\left(S^t\right)$. We let $\left(p\left(S^t, s_{t+1}\right)\right)_{s_{t+1}\in\{h,l\}}$ denote the set of equity prices that agents should expect over all possible paths of shock histories one period after S^t. We make $\left(p\left(S^t, s_{t+1}, s_{t+2}\right)\right)_{s_{t+1}, s_{t+2}\in\{h,l\}^2}$ denote the one two periods after S^t. Thus, $\boldsymbol{P}_t^{t+2}\left(S^t\right)$ is the set of all equity prices that agents born in time t and node S^t have to forecast over their lifetime. The equity demand functions are indexed by the possible history of shocks realized after the first age in the lifetime. For example, we index $e_m\left(\boldsymbol{P}_t^{t+2}\left(S^t\right); s_{t+1}\right)$ by s_{t+1}.

The market-clearing condition then requires that:[7]

$$e_y\left(\boldsymbol{P}_t^{t+2}\left(S^t\right)\right) + e_m\left(\boldsymbol{P}_{t-1}^{t+1}\left(S^{t-1}\right); s_t\right) = 1 \tag{19}$$

When convenient, we will also allow the total number of assets to vary stochastically, so that the market-clearing condition becomes:

$$e_y\left(\boldsymbol{P}_t^{t+2}\left(S^t\right)\right) + e_m\left(\boldsymbol{P}_{t-1}^{t+1}\left(S^{t-1}\right); s_t\right) = \bar{a}^{s_t} \tag{20}$$

With the notations above, we define two equilibrium concepts as in the monetary model.

Definition 3. The competitive equilibrium in the Lucas-tree model is a sequence of the equity holdings, consumptions and asset prices in all nodes starting in time 0, $\left\{e\left(S^t\right), c\left(S^t\right), p\left(S^t\right)\right\}$ for $\forall S^t$ and $t \geq 0$, such that:

- Individuals maximize their expected utility under budget constraints given the sequence of the price of the equity.

[7] By Walras's law, we also ignore the market clearing for the consumption good here.

- The asset market clears and the aggregate resource constraint holds.

We can show the existence of a competitive equilibrium using a standard method as stated in the monetary model.

Definition 4. The recursive ME is defined by time-homogeneous policy functions for the equity holdings, consumptions and asset prices: $\{e_y(\chi), e_m(\chi), c_y(\chi), c_m(\chi), c_r(\chi), p(\chi)\}$ which solve the household problem and clear both the asset and consumption markets. $\chi = [e_{y,-1}, s] \in \hat{\Sigma} \subset \mathbb{R}^2$ represents the minimal state variables: the lagged equity holdings of the young and the realization of the current aggregate uncertainty.

For the recursive ME, we write the following equilibrium conditions:

$$
\begin{aligned}
e_y\left(\boldsymbol{P}_t^{t+2}\left(S^t\right)\right) &= e_y\left(e_{y,t-1}, s_t\right) \\
1 &= e_y\left(e_{y,t-1}, s_t\right) + e_m\left(\boldsymbol{P}_{t-1}^{t+1}\left(S^{t-1}\right); s_t\right)
\end{aligned} \tag{21}
$$

where the first equation represents the optimality condition for the household problem and the second one is the asset market clearing condition.

The analysis we are undertaking hereafter is parallel to that of Woodford (1986), particularly with respect to Theorem 2 of his paper. This paper looks at a two-period lived model with multiple commodities and shows, via a functional application of the IFT that for a SOLG economy with small shocks, there is a unique equilibrium price function which depends on the infinite history of shocks to endowments. We will use a similar functional application of the IFT to show the existence of a LIFS as the ME mapping where the lagged endogenous state variables are, in fact, sufficient statistics for the shock history around the deterministic steady-states under small aggregate shocks.

For this analysis, we impose the following linear forecast functions:

$$
\begin{aligned}
e_y\left(e_{y,t-1}, s_t\right) &= \bar{e}^{s_t} - \gamma\left(e_{y,t-1} - \bar{e}^{s_t}\right) = G\left(e_{y,t-1}, s_t\right) \\
p\left(e_{y,t-1}, s_t\right) &= \bar{p}^{s_t} + \rho\left(e_{y,t-1} - \bar{e}^{s_t}\right) = H\left(e_{y,t-1}, s_t\right)
\end{aligned} \tag{22}
$$

for $s_t \in \{h, l\}$. Note that we let the affine coefficients in (22) be independent of the state of the current shock according to the result in Proposition 1.

To show that the recursive ME can be implemented by (22), we will proceed two steps. First, we show that the linear forecast functions in a three-period deterministic OLG model will be the steady-state values of the equity holding for the young and the equity price when evaluated at the steady states. This is the content of the following lemma.

Lemma 1. *In a three-period deterministic OLG model with a single long-lived asset, the linear forecast functions will be the steady-state values of e_y and p when $e_{y,-1} = \bar{e}$. , i.e. $G\left(e_{y,-1}\right) = \bar{e} - \gamma\left(e_{y,-1} - \bar{e}\right) = \bar{e}$ and $H\left(e_{y,-1}\right) = \bar{p} + \rho\left(e_{y,-1} - \bar{e}\right) = \bar{p}$ at $e_{y,-1} = \bar{e}$.*

Proof. *See Appendix A*

With the result in Lemma 1, we apply the IFT to show the existence of a LIFS in this three-period SOLG model in a neighborhood of the deterministic steady state if the aggregate shock is sufficiently small. This result is presented in the following proposition.

Proposition 2. *In a three-period SOLG model with a single long-lived asset, there will be ME generated by a LIFS given by (22) in a neighborhood of the deterministic steady state for sufficiently small shocks.*

Proof. *See Appendix A*

To prove Proposition 2, we do not specify any particular deterministic steady states in applying the IFT. Thus, there exists a LIFS around all possible deterministic steady states. Proposition 2 does not essentially need any specific assumptions on structural parameters such as preferences, endowments, dividends, total asset quantities and the Markov shock processes other than the size of the shock measured by $\max\left[\left\{\left|\omega_i^h - \omega_i^l\right|\right\}_{i\in\{y,m,r\}}, \left|\delta^h - \delta^l\right|\right]$. The transversal density theorem in the proof of Proposition 2 indicates that the result in this proposition holds for almost all asset quantities. Thus, the results in Proposition 2 apply for a fairly broad set of economies in a three-period SOLG model with a single long-lived asset.

To study the continuity property of an invariant Markov measure, we first have to examine the characteristics of reduced form parameters in the linear forecast functions, γ and ρ. By the IFT results in Proposition 2, the slope parameters of a LIFS in a stochastic model are close to the corresponding ones in a deterministic model by continuity as long as the size of an exogenous shock is sufficiently small. Here, we show that the slopes of the LIFS correspond to the stable eigenvalues of the Jacobian matrix at the steady state for the underlying price dynamics in the deterministic model. We show this relationship by constructing a first-order forecast function. (as in Kehoe and Levine (1985) which restricts the forward dynamics to the stable manifold of the steady-state equilibrium.) This will then guarantee that the LIFS derived in Proposition 2 is contractive and that the resulting equilibrium stochastic process is not transient.

To demonstrate this relationship, we first note that, generically, the underlying price dynamics for the three-period deterministic OLG model in a neighborhood of the steady state will take the form:

$$p_{t+1} = z\left(p_t, p_{t-1}, p_{t-2}\right) \tag{23}$$

This is determined from the solutions for p_{t+1} in the following equilibrium conditions by applying the IFT under the generic assumption that $\frac{\partial e_y}{\partial p_{t+1}} \neq 0$:

$$e_y\left(p_{t-1}, p_t, p_{t+1}\right) + e_m\left(p_{t-2}, p_{t-1}, p_t\right) = 1 \tag{24}$$

Letting $\hat{q}_t = \left(p_t, p_{t-1}, p_{t-2}\right)$, we can write the third-order equilibrium law of motion for the prices as the first-order vector system:

$$\hat{q}_{t+1} = \hat{z}\left(\hat{q}_t\right) = \begin{bmatrix} z\left(p_t, p_{t-1}, p_{t-2}\right) \\ p_t \\ p_{t-1} \end{bmatrix} \tag{25}$$

The Jacobian matrix for this system at the steady-state, $\bar{q} = [\bar{p}, \bar{p}, \bar{p}]$, takes the form:

$$D_{\hat{q}}\hat{z} = Z = \begin{bmatrix} z_1 & z_2 & z_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{26}$$

where $z_i = \frac{\partial z}{\partial p_{t+1-i}}\left|_{\hat{q}_t=\bar{q}}\right.$ for $\forall i \in \{1,2,3\}$. The local stability of the steady state (equivalently, the determinacy of the steady-state) depends on the eigenvalues of the matrix Z.

We obtain the first-order forecast function following Kehoe and Levine (1985) by requiring that:

$$p_{t+1} = f\left(p_t\right) \tag{27}$$

Given such a forecast function, we let:

$$\hat{e}_y(p_t) = e_y(p_t, f(p_t), f \circ f(p_t)) \tag{28}$$

Then, we define:

$$\hat{e}'_y = \frac{\partial e_y}{\partial p_t} + \frac{\partial e_y}{\partial p_{t+1}} f' + \frac{\partial e_y}{\partial p_{t+2}} f'^2 \,|_{p_t=\bar{p}} \tag{29}$$

With these notations, we can derive the model-consistent specification of the values of $-\gamma$ and ϱ which we summarize in the following proposition.

Proposition 3. *In a three-period deterministic OLG model with a single long-lived asset, the slope parameter of the LIFS, $-\gamma$, coincides with the stable eigenvalue of Z at the corresponding steady-states. The slope parameter of the equity price function, ρ, is defined by:*

$$\varrho = \frac{-\gamma}{\hat{e}'_y} \tag{30}$$

Proof. *See Appendix A*

The main implication of Proposition 3 is that the stable eigenvalue of the Jacobian matrix for the price dynamics system determines the slope parameter of the LIFS in the neighborhood of the steady-state.

According to the conditions in Blanchard and Kahn (1980), one can characterize the local determinacy and stability of equilibrium dynamics around steady-states. In a three-period OLG model, there are three eigenvalues for Z and one predetermined price variable. Thus, if the number of the stable eigenvalues of Z, whose moduli lie inside the unit circle, is exactly one, then the equilibrium dynamics converging to the steady state is locally determinate. This locally unique equilibrium is called saddle-path stable. Locally indeterminate equilibrium arises when there are more than one stable eigenvalues for Z. In this case, there are a continuum of equilibria converging to the steady-states.

For the determinate case, there is one contractive LIFS and thus there exists a unique invariant measure around the deterministic steady states by the contraction mapping theorem as long as the size of a shock is small. On the other hand, for the indeterminate case, there can be multiple contractive LIFSs taking different stable eigenvalues as their Lipschitz constant. Since each contractive LIFS generates a unique invariant measure, this result implies there are possibly multiple invariant measures for an indeterminate steady state under a small shock.

If we don't impose the stability restriction to the LIFS, we can end up generating an equilibrium stochastic process which is transient in the sense that under repeated shocks, the forward equilibrium trajectory eventually leaves any open neighborhood of the deterministic steady-state.

If there are no stable eigenvalues, then the equilibrium is called explosive in the sense that it diverges from the deterministic steady-states unless it starts at the stationary points. In this case with potential bubbles, there can be a stable equilibrium trajectory via the backward dynamics of the model rather than the forward dynamics.

Since the stable eigenvalues of Z determine the no-overlap property of the one-dimensional LIFS as well, we now use eigenvalues information on Z to classify economies with the Lucas tree as having singular or absolutely continuous measures instead of finding a LIFS explicitly.

We also find a relationship between the structural parameters in this model and the slope of the LIFS. Since this problem is not analytically tractable, we instead use a numerical analysis under certain parametric specifications. We introduce a constant relative risk aversion (CRRA) utility function, $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, where σ is the coefficient of relative risk aversion. We normalize $\omega_y + \omega_m + \delta$ to be 1. Lastly, we assume a symmetric I.I.D. shock with the two states, i.e. $\pi^h = \pi^l = 0.5$.

In this three-period Lucas-tree OLG model with the parametric assumptions above, we numerically find that there exists a unique steady state where the equity price is positive in the set of parameters of interest. For the unique steady-state, there exist one real eigenvalue inside the unit circle and two complex eigenvalues outside the unit circle. Following the Blanchard-Kahn eigenvalue condition, we can classify this unique steady state as a determinate or locally saddle-path stable equilibrium. We restrict the LIFS to the real eigenvalue of the unique steady state since we want the equilibrium mapping to generate a stable unique invariant measure.

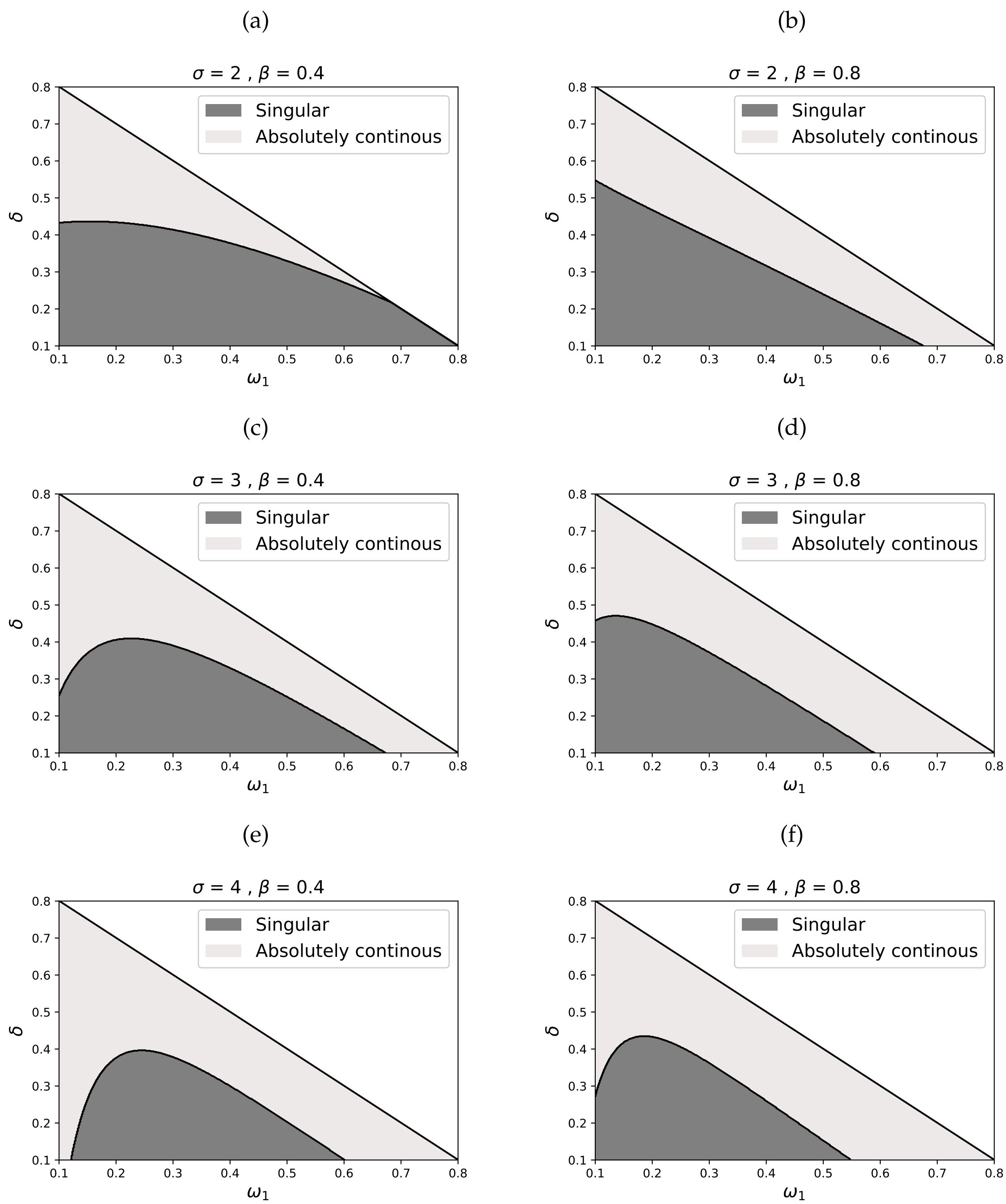
For the case where the aggregate shock has two states with equal probability, the no-overlap condition will hold if $\gamma < 1/2$. Since $-\gamma$ sets equal to the real eigenvalue of Z, we classify the set of economies where the absolute value of the real eigenvalue is less than $1/2$ as singularly continuous and the rest of economies as absolutely continuous. We should emphasize that there can exist a singular continuous measure even when the no-overlap property is not satisfied, i.e. $\gamma \in (1/2, 1)$. The set of γ values in $(1/2, 1)$ generating such essentially singular measure has Lebesgue measure zero relative to $\mathbb{R}$. These results justify classifying the economies where $\gamma \in (1/2, 1)$ as being absolutely continuous in our numerical analysis. (We refer the interested readers to Appendix A for more details about the essentially singular measure.)

Figure 1 shows the classification of economies into the two types. From all panels in the figure, we observe that γ increases as δ increases. This means that the current young's equity holding drops more sharply as the lagged equity holding rises under a higher dividend share. As δ and $e_{y,t-1}$ rise, the change in capital income for the middle-aged can be seen in $(p_t + \delta + \triangle\delta)\left(e_{y,t-1} + \triangle e_{y,t-1}\right)$ from the middle-aged budget constraint. The interaction term $\triangle\delta \cdot \triangle e_{y,t-1}$ indicates the current middle-aged takes a larger income share out of the total resource as $e_{y,t-1}$ increases under a higher δ. Thus, the middle-aged asset demand increases which makes the slope of the equity policy function steeper.

Figure 1 also implies γ increases as ω_1 rises when ω_1 is high enough. This result can be explained by a price change accompanied by a change in the endowment structure. As ω_1 increases, the aggregate demand for equity increases and thus, the equity prices will rise. Unlike the hump-shaped endowment structure where young agents might want to borrow, both the young and the middle-aged will save under the decreasing endowment structure with $\omega_1 > \omega_2 > \omega_3 = 0$. From the budget constraint for the middle-aged, an increment in the equity price can be reflected in this way: $(p_t + \triangle p_t + \delta)\left(e_{y,t-1} + \triangle e_{y,t-1}\right)$. The interaction term $\triangle p_t \cdot \triangle e_{y,t-1}$ implies the current middle-aged get a larger income share in the decreasing endowment structure due to the price hike for their previous equity holdings. Thus, γ increases, just as it does for an increase in δ.

Another interesting result is that an increase in ω_1 rather leads to a decline in γ when ω_1 is low enough. In this hump-shaped endowment structure, the young agents can borrow via short-selling the equity asset, assuming there are no borrowing constraints. As above, an increase in ω_1 still leads to a price hike as the demand for the equity by the middle-aged increases whereas the young short-sells the equity less. Under a higher equity price, the middle-aged's income share rises more sharply as their previous equity holdings increase by $\triangle e_{y,t-1}$ and thus,

FIGURE 1. The invariant Markov measure for different parameter configurations



they are willing to buy more assets. However, the current equilibrium asset holdings of the middle-aged might not grow faster, since the young agents short-sell the equity less under a higher ω_1. The higher equity price further reduces the amount of short-sales by the young since they can consume as much as they want even with a lower amount of short-sales under that price. Therefore, γ decreases even when ω_1 increases if ω_1 is low enough.

By the comparison of two panels with the same relative risk aversion but different time discount factors, we see that the area for the singular measure contracts from the right but expands to the left. For the contraction, the mechanism is quite close to the one in the case with an increase in ω_1 when ω_1 is already large because both the young and the middle-aged are willing to save more as β goes up. Thus, the equity price will rise, making the middle-aged wealthier as their previous equity holdings increase. For the expansion, we can borrow the intuition from the case with an increase in ω_1 when ω_1 sets low enough since the young will borrow less whereas the middle-aged will demand more in a higher β. The equity price will rise as well in this case, which raises the value of the previous equity holdings of the middle-aged and thus they demand more assets. However, they cannot buy more assets in equilibrium because the young short-sells the asset less when β is higher. Hence, γ goes up as β increases when ω_1 is high enough whereas it is in reverse when ω_1 is sufficiently low.

As the relative risk aversion increases, the set of parameters for the singular invariant measure shrinks from both the left and right sides. For the right side contraction, as σ goes up, both the young and the middle-aged demand the equity more because of a strong consumption smoothing motivation. Thus, the equity price will be higher relative to the case with a lower σ under the same ω_1. The wealthier middle-aged will increase its demand for the asset resulting in a higher γ. For the left side contraction, as σ goes up, the young will borrow more to smooth consumption, requiring more short-sales. This allows the middle-aged to purchase more equity in equilibrium and thus γ becomes higher.

In summary, we find that a low δ and σ expand the area for the singular measure. This result sheds light on why the unique Markov measure in the monetary model is singular since the dividend and relative risk aversion degenerate to 0 and 1, respectively. An increase in ω_1 and β converts a singular measure into an absolutely continuous measure when ω_1 is high enough. On the other hand, when ω_1 is low enough, a rise in ω_1 and β converts an absolutely continuous measure into a singular measure. The U-shape of γ in ω_1 means the singular measure is likely to arise under a hump-shaped endowment structure.

We run a simulation for a three-period SOLG model with the Lucas tree to verify the implications of the results in Figure 1. For this simulation, we approximate the policy functions with high-order Chebyshev polynomials. As the numerical analysis above, we use the CRRA utility function and set $\sigma = 2$ following Henriksen and Spear (2012). We let $\beta = 0.54 = 0.97^{60/3}$ where a one-year time-discount factor is 0.97 as commonly used in the applied macroeconomic literature and we regard one period in this model as 20 years. In the deterministic version of the model, we assume the total endowment is 1. The labor's share of the total endowment, ω, is $\frac{2}{3}$ and the ratio of labor income between the young and the middle-aged $\frac{\omega_y}{\omega_m} = \frac{3}{5}$. Hence, $\delta = \frac{1}{3}$, $\omega_y = \frac{1}{4}$ and $\omega_m = \frac{5}{12}$. The total asset quantity is assumed to be 1. We introduce shocks on the dividend and the total asset quantity which follow a symmetric I.I.D. Bernoulli process with two states. The realizations on the two stochastic parts are perfectly correlated as next: $\{\delta^l, \delta^h\} = \{0.99, 1.01\}$ and $\{a^l, a^h\} = \{0.99, 1.01\}$[8].

Proposition 2 and Figure 1 imply that the invariant Markov distribution can be generated

[8] These shocks are multiplicative. For example, the realized total dividend is $1.01 \times \frac{1}{3} = 0.3367$ in the high state.

by a LIFS under a small shock and it should be singular under these parameter values. Figure 2 graphically shows that the LIFS is indeed the equilibrium mapping under the small shock we set above. Its slope in this example is less than 1/2. Therefore, the ergodic distribution of this numerical model is a Cantor-like distribution as seen in Figure 3.

FIGURE 2. The equity policy function in the three-period SOLG economy with a single asset

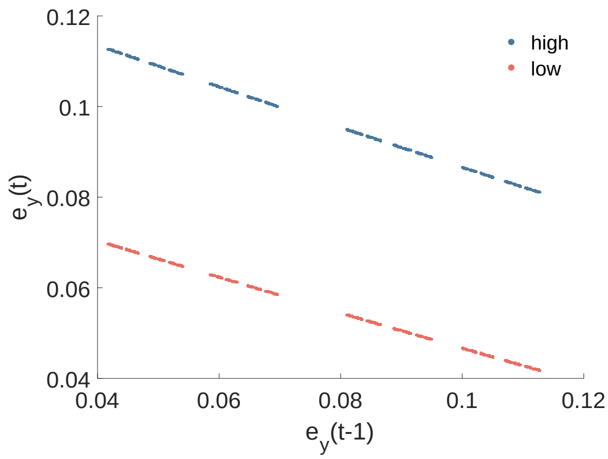
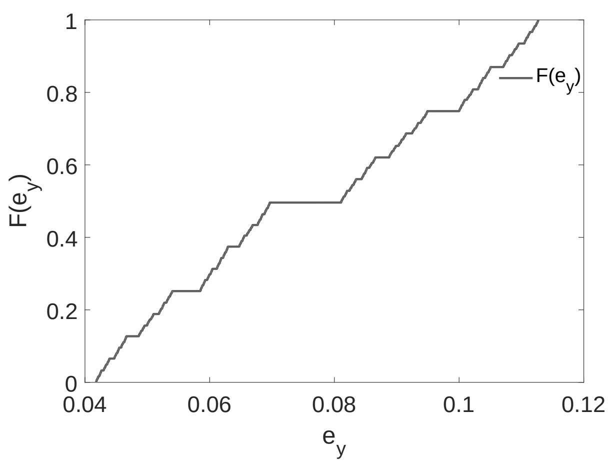


FIGURE 3. The invariant measure of the three-period SOLG economy with a single asset



Finally, it is natural to ask if there is an intuitive reason why the ME should be implementable via a LIFS. From the mathematics of the model, it is clear that the LIFS gives us enough variables to solve the model's equilibrium equations. It is also clear that the LIFS will generate the correct equilibrium relationships between the history of shocks and prices/allocations. Furthermore, any equivalent (possibly non-linear) policy functions would necessarily have to deliver the same relationships, since the results in Woodford (1986) imply that equilibrium prices as functions of the histories of shocks are locally unique for small shocks around the steady state. Thus, in this sense, the linear system is the simplest mechanism for allowing the lagged endogenous state variables to act as sufficient statistics for the infinite histories. In the simple exchange setting, the perturbations of asset holdings generated by the equilibrium will have no effect on the overall resources of the economy given the state, and so will only have an impact on welfare via allocative effects. Since we already know that the competitive equilibria in these models are not Pareto optimal (see Henriksen and Spear 2012), there would seem to be nothing to be gained by working with more general policy functions. This is indeed what comes out of the simulations of the model when we allow for flexible functional forms.

4 Generalized Model

In this section, we extend the model to be more realistic by allowing an arbitrary lifetime, heterogeneity, and a general shock process. The goal of this section is to show the existence of a simple but high dimensional LIFS and then study the invariant Markov measure in the complicated model with the help of the LIFS.

Agents live L-period lives (say $L = 70$ as in Ríos-Rull (1996)) from the youngest age 1 to the oldest age L, with $M \geq 1$ different types of agents born each period who differ in terms of preferences and endowments. We assume there is a continuum of each type of agent, and thus they take prices as given.

For generality, there are $S > 1$ states of exogenous shocks, $s_t \in \{z_1, \ldots, z_S\}$. The aggregate shock follows the first-order Markov process with a stationary Markov chain $\prod$ with support $s \in \{z_1, \ldots, z_S\}$ under which $\pi(s' \mid s) \geq 0$ for $(s, s') \in \{z_1, \ldots, z_S\}^2$ and $\sum_{s' \in \{z_1, \ldots, z_S\}} \pi(s' \mid s) = 1$ for $\forall s \in \{z_1, \ldots, z_S\}$. $\pi(s' \mid s)$ denotes the probability that state s' occurs given state s in the previous time. $\pi(S^\tau \mid S^t)$ implies the probability that the node S^τ occurs in time τ conditional on the node S^t in time t.

We maintain the assumption that there is only one asset, either money or a tree asset. Individuals consume a single good, and can save via accumulations of the single asset. The total asset quantity varies stochastically in the amount $\bar{a}^{s_t}$ for $s_t \in \{z_1, \ldots, z_S\}$. The single asset delivers dividends stochastically in the amount δ^{s_t} for $s_t \in \{z_1, \ldots, z_S\}$ if the single asset is the Lucas tree. Note that both the total asset quantity and the dividend depend only on the realization of the current shock.

The single asset delivers dividends stochastically in the amount δ^{s_t} for $s_t \in \{z_1, \ldots, z_S\}$ if the single asset is the Lucas tree.

Type-j agents born in time t are endowed with stochastic consumption goods in the amounts $\omega_j = \left\{\omega_{i,j}^{s_{t+i-1}}\right\}_{i=1}^{L}$ where $\omega_{i,j}^{s_{t+i-1}}$ is the endowment of the type-j agent in age i in time $t+i-1$ which depends only on the current shock realization s_{t+i-1}. The consumption stream of a type-j agent born in time t and history S^t is denoted by $c_j(S^t) = \left\{\left(c_{i,j}\left(S^t, S_{t+1}^{t+i-1}\right)\right)_{S_{t+1}^{t+i-1}}\right\}_{i=1}^{L}$ where $c_{i,j}\left(S^t, S_{t+1}^{t+i-1}\right)$ is the consumption of a type-j agent in age i given a path of shocks for $(i-1)$ periods after the history S^t, $S_{t+1}^{t+i-1} = (s_{t+1}, \ldots, s_{t+i-1})$ for $i \geq 1$. When $i = 1$, we define $c_{1,j}(S^t, S_{t+1}^t) = c_{1,j}(S^t)$. Similarly, the lifetime portfolio of a type-j household is denoted by $e_j(S^t) = \left\{\left(e_{i,j}\left(S^t, S_{t+1}^{t+i-1}\right)\right)_{S_{t+1}^{t+i-1}}\right\}_{i=1}^{L-1}$ where $e_{i,j}\left(S^t, S_{t+1}^{t+i-1}\right)$ is the equity holding of a type-j agent in age i in node S^{t+i-1}. As above, we define $e_{1,j}(S^t, S_{t+1}^t) = e_{1,j}(S^t)$. Note that households do not save in the last age, L.

The lifetime expected utility for a type-j individual is given by a von Neumann-Morgenstern utility function $U : \mathbb{R}_+^{\left(S^L-1\right)/(S-1)} \to \mathbb{R}$:

$$E_t U_j \left(c_j \left(S^t\right)\right) = E_t U_j \left(\left\{ \left(c_{i,j} \left(S^t, S_{t+1}^{t+i-1} \right) \right)_{S_{t+1}^{t+i-1}} \right\}_{i=1}^{L} \right) \tag{31}$$

where the certainty utility function satisfies the following regularity conditions, $U_j'(\cdot) > 0$, $U_j''(\cdot) < 0$, and $U_j'(\cdot) = +\infty$ for all arguments.

Type-j agents maximize the lifetime expected utility subject to a sequence of budget constraints as follows:

$$\begin{aligned} c_{1,j}\left(S^t\right) &= \omega_{1,j}^{s_t} - p\left(S^t\right) e_{1,j}\left(S^t\right) \\ c_{i,j}\left(S^t, S_{t+1}^{t+i-1}\right) &= \omega_{i,j}^{s_{t+i-1}} + \left(p\left(S^t, S_{t+1}^{t+i-1}\right) + \delta^{s_{t+i-1}}\right) e_{(i-1),j}\left(S^t, S_{t+1}^{t+i-2}\right) \\ &\quad - p\left(S^t, S_{t+1}^{t+i-1}\right) e_{i,j}\left(S^t, S_{t+1}^{t+i-1}\right) \text{ for } \forall S_{t+1}^{t+i-1} \text{ and } i \in \{2, \ldots, L-1\} \\ c_{L,j}\left(S^t, S_{t+1}^{t+L-1}\right) &= \omega_{L,j}^{s_{t+L-1}} + \left(p\left(S^t, S_{t+1}^{t+L-1}\right) + \delta^{s_{t+L-1}}\right) e_{(L-1),j}\left(S^t, S_{t+1}^{t+L-2}\right) \text{ for } \forall S_{t+1}^{t+L-1} \end{aligned} \tag{32}$$

where $p\left(S^t, S_{t+1}^{t+i-1}\right) = p\left(S^{t+i-1}\right)$ is the price of the equity in terms of the single good in node S^{t+i-1}.

Solving the optimization problem for type-j agents yields asset demand functions:

$$e_{i,j}\left(S^t, S_{t+1}^{t+i-1}\right) = e_{i,j}\left(\boldsymbol{P}_t^{t+L-1}\left(S^t\right); S_{t+1}^{t+i-1}\right) \tag{33}$$

for $\forall S_{t+1}^{t+i-1}$ and $i \in \{1, \ldots, L-1\}$ where $\boldsymbol{P}_t^{t+L-1}\left(S^t\right) = \left\{ p\left(S^t\right), \ldots, \left(p\left(S^t, S_{t+1}^{t+L-1}\right)\right)_{S_{t+1}^{t+L-1}} \right\}$. The equity demand functions are indexed by the history of shocks realized after the first period in the lifetime, S_{t+1}^{t+i-1} for $i > 1$. When $i = 1$, we define $e_{1,j}\left(S^t\right) = e_{1,j}\left(\boldsymbol{P}_t^{t+L-1}\left(S^t\right)\right)$.

For the extended model, the asset market-clearing at time t and node S^t requires that:[9]

$$\sum_{i=1}^{L-1} \sum_{j=1}^{M} e_{i,j}\left(\boldsymbol{P}_{t+1-i}^{t+L-i}\left(S^{t+1-i}\right); S_{t+2-i}^{t}\right) = \bar{a}^{s_t} \tag{34}$$

With the notations above, we define two equilibrium concepts as in models above.

Definition 5. The competitive equilibrium in the generalized model is a sequence of the asset holdings and consumptions for all types and asset prices in all nodes starting in time 0, $\left\{ \left\{ e_j\left(S^t\right), c_j\left(S^t\right) \right\}_j, p\left(S^t\right) \right\}$ for $\forall S^t$ and $t \geq 0$, satisfying:

- Individuals maximize their expected utility under budget constraints given the sequence of asset prices.
- The asset market clears and the aggregate resource constraint holds.

[9] By Walras's law, we ignore the market clearing for the consumption good here as well.

We can show the existence of a competitive equilibrium for this generalized model using a standard truncation method as well.

For the recursive ME that we will define below, we take all but one of the asset demand quantities as the lagged endogenous state variables via the asset market clearing condition. For specificity, we exclude the asset demand of the type-M agent in the second oldest cohort with age $L-1$ and let the set of the lagged state variables in time t:

$$\xi_{t-1} = \left\{ \left\{ e_{i,j,t-1} \right\}_{j=1,\ldots,M} \right\}_{i=1,\ldots,L-1} \setminus \left\{ e_{(L-1),M,t-1} \right\} \tag{35}$$

With an abuse of notation, we also denote ξ_{t-1} as the $((L-1)M-1)$ vector of the lagged endogenous state variables in time t.

Definition 6. The recursive ME is defined by time-homogeneous policy functions for the asset holdings, consumptions and asset prices: $\left\{ \left\{ e_j(\chi), c_j(\chi) \right\}_j, p(\chi) \right\}$ which solve the household problem and clear both the asset and consumption markets. $\chi = [\xi_{-1}, s] \in \hat{\Sigma} \subset \mathbb{R}^{(L-1)M}$ represents the minimal state variables: the lagged asset holdings distribution and the realization of the current aggregate uncertainty.

For the recursive ME, we re-write the equilibrium conditions as follows:

$$\begin{aligned} \boldsymbol{E}\left(S^t\right) &= \xi\left(\xi_{t-1}, s_t\right) \\ \bar{a}^{s_t} &= \iota^T \xi\left(\xi_{t-1}, s_t\right) + e_{(L-1),M}\left(\boldsymbol{P}_{t-L+2}^{t+1}\left(S^{t-L+2}\right); S_{t-L+3}^t\right) \end{aligned} \tag{36}$$

where ι is an $((L-1)M-1)$ vector of ones to sum the asset holdings distribution except for the demand by the type-M and age-$(L-1)$ agent and

$$\boldsymbol{E}\left(S^t\right) = \begin{bmatrix} \left\{ e_{(L-1),j}\left(\boldsymbol{P}_{t-L+2}^{t+1}\left(S^{t-L+2}\right); S_{t-L+3}^t\right) \right\}_{j=1,\ldots,M-1} \\ \left\{ e_{(L-2),j}\left(\boldsymbol{P}_{t-L+3}^{t+2}\left(S^{t-L+3}\right); S_{t-L+4}^t\right) \right\}_{j=1,\ldots,M} \\ \vdots \\ \left\{ e_{1,j}\left(\boldsymbol{P}_{t}^{t+L-1}\left(S^{t}\right)\right) \right\}_{j=1,\ldots,M} \end{bmatrix} \tag{37}$$

is the $((L-1)M-1)$ vector of demand functions for all but the second oldest type-M agents in node S^t.

As we did in Section 3, we show the existence of a LIFS as the ME mapping around the deterministic steady-states under small aggregate shocks in this generalized model via a functional application of the IFT.

For this analysis, we specify the law of motion for the endogenous state variables as:

$$\xi\left(\xi_{t-1}, s_t\right) = \bar{\xi}^{s_t} + \Gamma^{s_t}\left(\xi_{t-1} - \bar{\xi}^{s_t}\right) = G\left(\xi_{t-1}, s_t\right) \tag{38}$$

for $s_t \in \{z_1, \ldots, z_S\}$ where Γ^{s_t} is a $((L-1)M-1) \times ((L-1)M-1)$ coefficient matrix given state s_t in time t with $\rho\left(\Gamma^{s_t}\right)$ – the spectral radius of Γ^{s_t} – less than one by focusing on either determinate or indeterminate case as discussed later.

Next, we let the law of motion for prices be:

$$p\left(\xi_{t-1}, s_t\right) = \bar{p}^{s_t} + \left(\Lambda^{s_t}\right)^T\left(\xi_{t-1} - \bar{\xi}^{s_t}\right) = H\left(\xi_{t-1}, s_t\right) \tag{39}$$

for $s_t \in \{z_1, \dots, z_S\}$ where Λ^{s_t} is a $((L-1)M-1)$ coefficient vector given state s_t in time t. Note that we allow the affine matrices, Γ^{s_t} and Λ^{s_t}, to vary over the state since we allow an arbitrary number of states which can be more than two in this generalized model.

As the analysis in the three-period SOLG model above, we first show in Lemma 2 that when the shocks are zero, the linear forecast functions satisfy the equilibrium conditions for the extended model at the steady states.

Lemma 2. *In a general deterministic OLG model with a single long-lived asset, the linear forecast functions will be the steady-state values of ξ and p when $\xi_{-1} = \bar{\xi}$, i.e. $G(\xi_{-1}) = \bar{\xi} + \Gamma\left(\xi_{-1} - \bar{\xi}\right) = \bar{\xi}$ and $H(\xi_{-1}) = \bar{p} + \Lambda^T\left(\xi_{-1} - \bar{\xi}\right) = \bar{p}$ at $\xi_{-1} = \bar{\xi}$.*

Proof. *See Appendix A*

With Lemma 2, we apply the functional version of the IFT to show the existence of a high-dimensional LIFS in this extended SOLG model in a neighborhood of the deterministic steady state under a small aggregate shock. This is the main content of the following proposition.

Proposition 4. *In a general SOLG model with a single long-lived asset, there will be ME generated by a LIFS given by (38) and (39) in a neighborhood of the deterministic steady state for sufficiently small shocks.*

Proof. *See Appendix A*

As in the three-period model, we do not specify the deterministic steady states to prove Proposition 4. Thus, there exists a LIFS around all possible deterministic steady states. Proposition 4 also does not require any specific assumptions on structural parameters other than the size of the shock.[10] Therefore, the results in Proposition 4 apply to an extensive set of economies in a general SOLG model with a single long-lived asset.

We now numerically check the assertion of Proposition 4. For this analysis, we simulate four different multi-period SOLG models with high-degree Chebyshev polynomials: four-period lived agents model, five-period lived agents model, six-period lived agents model and four-period lived and two types of agents model. These four different models have two, three, four and five lagged endogenous state variables, respectively.

In these simulations, we keep using the CRRA preference. $\sigma = 5$ and $\beta = 0.97^{60/L}$ for one period. There is an I.I.D. imperfectly correlated multiplicative shock between endowment and dividend as follows: $\{\delta^{z_1}, \omega^{z_1}\} = \{0.95, 0.95\}$, $\{\delta^{z_2}, \omega^{z_2}\} = \{1.05, 0.95\}$, $\{\delta^{z_3}, \omega^{z_3}\} = \{0.95, 1.05\}$, $\{\delta^{z_4}, \omega^{z_4}\} = \{1.05, 1.05\}$ where the probability $\pi^s = 0.25$ for $s \in \{z_1, z_2, z_3, z_4\}$. Since each model has a different number of overlapping generations and types within a cohort, we set a different distribution of endowment shares across ages and types for each model. The endowment distributions are summarized in Table 1.

It is worth noting that for the four-period lived and two types of agents model, we let the endowment ratio between types in a cohort changes as the cohort ages to reflect the kind of income mobility observed in data. Otherwise, a constant endowment ratio between types over ages gives rise to linear dependence between the asset holdings of heterogeneous agents in each generation because of the inter-temporally homothetic utility function. Under this type of preference, consumption allocations are proportional to the agents' endowment shares, and thus excess good demands as well. Since excess good supplies imply the agents' saving via holding assets, the agents' asset holdings are also in proportion to their endowment shares.

[10] The size of a shock is measured by $\max_{(s,s')}\left[\left\{\left|\omega_{i,j}^{z_s} - \omega_{i,j}^{z_{s'}}\right|\right\}_{(i,j)}, |\delta^{z_s} - \delta^{z_{s'}}|, |\bar{a}^{z_s} - \bar{a}^{z_{s'}}|\right]$.

Table 1. Distributions of endowment shares in simulated models

	(1) 4pd1type age	(2) 5pd1type age	(3) 6pd1type age	(4) 4pd2type age	type
ω_1	0.1833	0.1487	0.1233	0.1833	$(0.5609, 0.4391)$
ω_2	0.2400	0.1753	0.1493	0.2400	$(0.5716, 0.4284)$
ω_3	0.2433	0.1847	0.1560	0.2433	$(0.5363, 0.4637)$
ω_4	0	0.1580	0.1253	0	-
ω_5	-	0	0.1127	-	-
ω_6	-	-	0	-	-
Sum	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	

- The subscript numbers in the leftmost column indicate age.
- The endowment profiles set for each model to be hump-shaped following data.
- The parentheses denote the endowment ratio between types in each age.
- The endowments across ages sum up to the labor's share of the total endowment, $\frac{2}{3}$.

Therefore, the asset holdings as lagged endogenous state variables are linearly dependent on each other under the constant endowment ratio between types across ages if preferences are described by the CRRA utility function.

To check the linearity property of the equilibrium laws of motion, we run a standard ordinary least square (OLS) regression on the simulated data from the four different models. For each model, we simulate the economy for 21,000 periods and ignore the first 1000 periods to avoid the effect of initial conditions on the results and make sure the data used in the regression analysis lies on the equilibrium attractor. Tables 2 – 5 summarize the results by regressing the equilibrium allocations on the lagged endogenous state variables for each state of nature in the four distinct models with the corresponding simulated data. Tables 2 and 3 show respectively the R-squared value and the standard deviation of the error from the regression of the equity holdings of the age-2 group on the lagged endogenous state variables.[11] Tables 4 and 5 display R^2 and $\hat{\sigma}$ respectively from the regression of the equity price on the lagged state variables. In the four tables, rows and columns represent states and models, respectively. As Proposition 4 indicates, these numerical results show that the LIFS can represent the recursive ME in a neighborhood of a deterministic steady state because R^2 and $\hat{\sigma}$ are almost 1 and 0, respectively.

Table 2. R^2 from equity holdings of age-2 group on endogenous state variables

$\{\delta, \omega\}$	(1) 4pd1type	(2) 5pd1type	(3) 6pd1type	(4) 4pd2type
$\{0.95, 0.95\}$	1.0000	1.0000	1.0000	1.0000
$\{1.05, 0.95\}$	1.0000	1.0000	1.0000	1.0000
$\{0.95, 1.05\}$	1.0000	1.0000	1.0000	1.0000
$\{1.05, 1.05\}$	1.0000	1.0000	1.0000	1.0000

Endogenous variables are all significant with the p value of 0.001.

[11] One can get similar regression results for the equity holdings of other age groups in all models examined here.

Table 3. $\hat{\sigma}$ from equity holdings of age-2 group on endogenous state variables

$\{\delta, \omega\}$	(1) 4pd1type	(2) 5pd1type	(3) 6pd1type	(4) 4pd2type
$\{0.95, 0.95\}$	0.0000	0.0000	0.0000	0.0000
$\{1.05, 0.95\}$	0.0001	0.0000	0.0000	0.0000
$\{0.95, 1.05\}$	0.0000	0.0000	0.0000	0.0000
$\{1.05, 1.05\}$	0.0000	0.0000	0.0000	0.0000

Endogenous variables are all significant with the p value of 0.001.

Table 4. R^2 from equity price on endogenous state variables

$\{\delta, \omega\}$	(1) 4pd1type	(2) 5pd1type	(3) 6pd1type	(4) 4pd2type
$\{0.95, 0.95\}$	0.9997	0.9996	0.9995	0.9997
$\{1.05, 0.95\}$	0.9997	0.9996	0.9995	0.9997
$\{0.95, 1.05\}$	0.9997	0.9996	0.9995	0.9997
$\{1.05, 1.05\}$	0.9997	0.9996	0.9994	0.9997

Endogenous variables are all significant with the p value of 0.001.

Table 5. $\hat{\sigma}$ from equity price on endogenous state variables

$\{\delta, \omega\}$	(1) 4pd1type	(2) 5pd1type	(3) 6pd1type	(4) 4pd2type
$\{0.95, 0.95\}$	0.0001	0.0001	0.0001	0.0001
$\{1.05, 0.95\}$	0.0001	0.0001	0.0001	0.0001
$\{0.95, 1.05\}$	0.0001	0.0001	0.0001	0.0001
$\{1.05, 1.05\}$	0.0001	0.0001	0.0002	0.0001

Endogenous variables are all significant with the p value of 0.001.

Another purpose of the numerical analysis is to check the extent of an aggregate shock under which an economy can have a system of linear functions as the equilibrating process. We stress that structural parameters can affect the bounds of the shock support needed to achieve the linearity result. A higher σ generates a smaller R^2 fixing the size of the shock due to imposing more curvatures in the problem. In other words, a smaller size of shock is required for a model with a high σ to achieve the same level of R^2 in a model with a low σ. Similarly, a lower β generates a smaller R^2 fixing the shock size. These results imply that the extent of an aggregate shock enlarges in models with a lower σ and a higher β.

Our numerical results indicate that we can observe an acceptable linear law of motion determined by an OLS regression in all the four different models with a high relative risk aversion – $\sigma = 5$ – and a moderate size of imperfectly correlated shock. In the applied macro literature, widely used values of the relative risk aversion are between 1 and 5. Existing estimates using micro-level data support this range since they report the elasticity of intertemporal substitution lies between 0.2 and 2 which implies the relative risk aversion is between 0.5 and 5 assuming the CRRA utility function (see Havránek 2013). We indeed find many examples adopting a small aggregate shock and $\sigma \leq 5$ for calibration in the macro literature (see Krusell and Smith 1998; Storesletten et al. 2007; Hasanhodzic and Kotlikoff 2013).

Thus, one can adopt the algorithm based on the structure of the LIFS to compute equilibria in a broad set of SOLG models with an aggregate shock with a finite support.[12] The algorithm has some important advantages. First, it will generate small approximation errors under a moderate size of imperfectly correlated shock between labor and capital incomes as the linearity results hold under this type of shock. Moreover, the algorithm will make computing a very long-period lived SOLG model with heterogeneity feasible since it needs to include only the first-order term for each endogenous state variable in approximating functions, not any higher orders or interaction terms between endogenous variables.[13]

Therefore, this algorithm will resolve the issue that arises in applying the algorithm of Krusell and Smith (1998) to the long-period lived SOLG models with heterogeneity where generations are affected disparately by an imperfectly correlated shock. Their algorithm exploits low-order moments including aggregate wealth or the mean of wealth distribution over age and type to predict future prices. However, the low-order moments are not the sufficient statistics to summarize agents' choice when marginal propensities to save are distinct across generations and types due to different effects on labor and capital incomes from the imperfectly correlated shock. Thus, the forecasts based on the low-order moments deviate substantially from the actual equilibrium in this case so that the Krusell and Smith algorithm might generate relatively large approximation errors (see Krueger and Kubler (2004)).

Our findings show that very accurate forecasts might require information about the wealth for each age and type since different generations and types exhibit heterogeneous saving behaviors under the imperfectly correlated shock. Tracing the asset holdings of all ages and types seems to resurrect the curse of dimensionality issue. However, this issue can be partly avoided by the fact that the ME can be generated by the simple LIFS, and thus it is enough to perform the first-order approximation in the projection method. We stress that our results do not contradict the Krusell and Smith method, but rather complement their work by providing a possible alternative in certain circumstances under which their algorithm does not provide a good approximation.

We now analyze the condition under which there exists a homogeneous LIFS as seen in the three-period model with the Lucas-tree. This is the main argument of the following proposition.

Corollary 2. *In a general SOLG model with a single long-lived asset, there will be ME generated by a homogeneous LIFS with $\Lambda^s = \Lambda$ and $\Gamma^s = \Gamma$ for $\forall s$ in a neighborhood of the deterministic steady state for sufficiently small shocks, if $S \leq (L-1)\,M\,((L-1)\,M-1)+1$.*

Proof. *See Appendix A*

[12] Note that there are many papers in the insurance, asset pricing, and social security literature which use SOLG models with a discrete shock (see Ríos-Rull 1994; Ríos-Rull 1996; Storesletten et al. 2007; Krueger and Kubler 2006 and others).

[13] To show this point, one would examine how the projection method based on the structure of the LIFS reduces the number of unknown polynomial coefficients compared to the standard one with tensor products. The standard projection method requires $(L-1)\,MS\,((L-1)\,M-1)^{d+1}$ unknowns where d is the degree of the approximating polynomials, whereas the LIFS structure requires only $((L-1)\,M)^2\,S$ number of unknowns. As an example, think about a SOLG model where a representative agent lives10 periods and there are two states. In this case, the LIFS algorithm and the standard one yield 162 and 9216 numbers of unknowns respectively, assuming $d = 2$.

According to Corollary 2, one can find a LIFS having constant affine matrices over states if $S \leq (L-1)\, M\, ((L-1)\, M - 1) + 1$. In the three-period lived SOLG model with $s \in \{h, l\}$ above, $L = 3$ and $M = 1$, and thus $S = 2 < (L-1)\, M\, ((L-1)\, M - 1) + 1 = 3$. Hence, there exists a homogeneous LIFS in this model.

Next, we study the continuity property of an invariant Markov measure in this generalized model. We first examine the characteristics of reduced form parameters in the linear law of motion. As we did for the three-period model, we restrict our attention to the case where a homogeneous LIFS exists. By the IFT results in Proposition 4 and Corollary 2, the affine matrices of a LIFS in a stochastic model are close to the corresponding ones in a deterministic model by continuity if the size of an aggregate shock is sufficiently small. In the deterministic model, we show that the affine matrices take a subset of the stable eigenvalues of the Jacobian matrix of the price dynamics at the steady state as their own.

As with the three-period model, the key to showing the dynamic consistency of the ME is the construction of a forecast function which is lower order than the full price dynamic forecast. We first obtain the full price dynamic system by applying the IFT to the asset market clearing conditions in a neighborhood of the steady state for the generalized deterministic OLG model:

$$p_{t+1} = z\,(\hat{q}_t) \tag{40}$$

where $\hat{q}_t = (p_t, \ldots, p_{t-2L+4})$.

Then, we can write this as the first-order vector system:

$$\hat{q}_{t+1} = \hat{z}\,(\hat{q}_t) = \begin{bmatrix} z\,(\hat{q}_t) \\ p_t \\ \vdots \\ p_{t-2L+5} \end{bmatrix} \tag{41}$$

where $\hat{z} : \mathbb{R}^{2L-3}_{++} \to \mathbb{R}^{2L-3}_{++}$.

We consider general forecasts which depend on the $(L-2)$ predetermined price variables at time $t+1$, $q_t = (p_t, \ldots, p_{t-L+3})$. We denote these forecast functions as:

$$p_{t+1} = f\,(q_t) \tag{42}$$

These forecasts restrict the forward dynamics of the prices to the stable manifold of the steady state as we will show later. Similar to $\hat{z}\,(\hat{q}_t)$, we define:

$$q_{t+1} = \hat{f}\,(q_t) = \begin{bmatrix} f\,(q_t) \\ p_t \\ \vdots \\ p_{t-L+4} \end{bmatrix} \tag{43}$$

where $\hat{f} : \mathbb{R}^{L-2}_{++} \to \mathbb{R}^{L-2}_{++}$.

Let Z and F be given by:

$$Z = \begin{bmatrix} Dz \\ J_{(2L-3)} \end{bmatrix} \text{ and } F = \begin{bmatrix} Df \\ J_{(L-2)} \end{bmatrix} \tag{44}$$

where Z is the Jacobian matrix of $\hat{z}\,(\hat{q}_t)$ with respect to $\hat{q}_t^T$ and F is the Jacobian matrix of $\hat{f}\,(q_t)$ with respect to q_t^T evaluated at the steady state. $Dz = \frac{\partial z(\hat{q}_t)}{\partial \hat{q}_t^T}$ and $Df = \frac{\partial f(q_t)}{\partial q_t^T}$ at the steady state. $J_n = \begin{bmatrix} I_{n-1} & \mathbf{0} \end{bmatrix}$ where I_{n-1} is a $(n-1)$ dimensional identity matrix and $\mathbf{0}$ is a $(n-1)$ dimensional zero column vector.

For the model without heterogeneity, let the vector of the asset holdings for ages from 1 to $(L-2)$ in time $t+1$ be:

$$\xi\left(p_{t+L}, \ldots, p_{t+1}, p_t, \ldots, p_{t-L+4}\right) = \xi\left(\hat{q}_{t+L}\right) \tag{45}$$

Given the general forecast function, one can write this asset holdings vector as:

$$\hat{\xi}\left(q_{t+1}\right) = \xi\left(f\left(q_{t+L-1}\right), \ldots, f\left(q_{t+1}\right), q_{t+1}\right) \tag{46}$$

We define the derivative of (46) with respect to q_{t+1}^T evaluated at the steady state:

$$\hat{\Xi} = \frac{\partial \hat{\xi}\left(q_{t+1}\right)}{\partial q_{t+1}^T} = \frac{\partial \xi\left(\hat{q}_{t+L}\right)}{\partial \hat{q}_{t+L}^T} \frac{\partial \hat{q}_{t+L}}{\partial q_{t+1}^T} = \Xi K \tag{47}$$

where $\hat{\Xi}$ is the derivative of $\hat{\xi}\left(q_{t+1}\right)$ with respect to q_{t+1}^T, Ξ is the derivative of $\xi\left(\hat{q}_{t+L}\right)$ with respect to $\hat{q}_{t+L}^T$, and K is the derivative of $\hat{q}_{t+L}$ with respect to q_{t+1}^T evaluated at the steady state. These matrices are given by:

$$\Xi = \begin{bmatrix} \frac{\partial \xi_1}{\partial p_{t+L}} & \frac{\partial \xi_1}{\partial p_{t+L-1}} & \cdots & \frac{\partial \xi_1}{\partial p_{t-L+4}} \\ \frac{\partial \xi_2}{\partial p_{t+L}} & \frac{\partial \xi_2}{\partial p_{t+L-1}} & \cdots & \frac{\partial \xi_2}{\partial p_{t-L+4}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \xi_{L-2}}{\partial p_{t+L}} & \frac{\partial \xi_{L-2}}{\partial p_{t+L-1}} & \cdots & \frac{\partial \xi_{L-2}}{\partial p_{t-L+4}} \end{bmatrix} \text{ and } K = \begin{bmatrix} DfF^{L-2} \\ \vdots \\ DfF \\ Df \\ I_{(L-2)} \end{bmatrix} \tag{48}$$

(see Kim and Spear (2017) for the derivation of the matrix K in detail.)

For the model with heterogeneity, one can define $\hat{\Xi}$ similar with the one without heterogeneity but it is a $((L-1)M-1) \times (L-2)$ matrix in the heterogeneity case since there are $((L-1)M-1)$ lagged asset holdings as the endoegenous state variables.

With these notations, we can examine a relationship between the eigenvalues of Γ and Z matrices for both the determinate and indeterminate cases in the general model. We also find a functional relationship between Γ and Λ. These results are summarized in the following proposition.

Proposition 5. *In a general deterministic OLG model with a single long-lived asset with and without cohort heterogeneity, Γ takes a subset of the stable eigenvalues for the Z matrix as a part of its eigenvalues. There is a functional relationship between Γ and Λas:*

$$\Lambda^T = DfF^{-1}\hat{\Xi}^{-1}\Gamma \tag{49}$$

for the model without heterogeneity and

$$DfF = \Lambda^T \Gamma \hat{\Xi} \tag{50}$$

for the model with heterogeneity.

Proof. *See Appendix A*

As we noted in the three-period model, we can also classify the general model as locally determinate, indeterminate and explosive cases based on the Blanchard and Kahn condition. In the determinate case, there are the same number of stable eigenvalues for Z with the predetermined variables. One can find stable eigenvalues for Z more than the number of the predetermined variables in the indeterminate case. Lastly, the explosive case lacks the stable eigenvalues and thus its number is less than the number of the predetermined variables.

For the model without heterogeneity, the eigenvalues of Γ can exactly match the stable eigenvalues of Z in the determinate case since Γ is a $(L-2) \times (L-2)$ matrix and there are $(L-2)$ stable eigenvalues for Z. Thus, there is one contractive LIFS which generates a unique invariant measure around the deterministic steady states by the contraction mapping theorem as long as the size of a shock is small.

For the indeterminate case in the model without heterogeneity, the number of the stable eigenvalues of Z is more than $(L-2)$. Γ can take a subset of the stable eigenvalues as its own. Thus, one can construct multiple contractive LIFSs. This result indicates there are possibly multiple invariant measures for an indeterminate steady state under a small shock.

Finally, in the explosive case, there are fewer than $(L-2)$ stable eigenvalues for Z. One cannot find a LIFS of which Γ has all eigenvalues inside the unit circle. In this case, one can find a stable equilibrium trajectory by working in the backward dynamics of the model, as we noted in the three-period model.

For the model with heterogeneity, the stable eigenvalues of Z determine only a subset of the eigenvalues of Γ in all cases since the cohort heterogeneity expands the dimension of Γ.

A strong sufficient condition for a high-dimensional LIFS in a general model to generate a singular invariant distribution is that the spectral radii of the affine matrices $\{\Gamma^s\}_s$ are sufficiently close to zero. Figure 2 provides an intuition behind how the no-overlap property can be satisfied when the spectral radius or the Lipschitz constant is small enough in a one-dimensional LIFS. As the spectral radius goes to zero, the linear maps in Figure 2 become flatter and the images of the LIFS are less likely to be overlapped. Analogously, for the high-dimensional LIFS, as the spectral radiuses of $\{\Gamma^s\}_s$ converge to zero, open sets containing the images of individual functions in the LIFS shrink and degenerate to $\{\bar{\xi}^s\}_s$ in the limit case. Therefore, the high-dimensional LIFS will be non-overlapped.

However, although the images of individual functions in the high-dimensional LIFS have both overlap and gaps, its attractor can be a Lebesgue measure zero set as long as there are gaps in the images of the LIFS when first iterating on an open set containing its attractor. Through iterations, the gaps fill out the open set and thus, the attractor of the LIFS will be a Lebesgue measure zero set and its invariant measure will be singular in the limit (see Jorgensen et al. 2007).

Based on this result, we study a relatively weak sufficient condition for a high-dimensional LIFS to generate a singular invariant measure. For this analysis, we assume that Γ^s is diagonalizable with linearly independent eigenvectors, $\{e_i^s\}_{i=1}^{(L-1)M-1}$, for $\forall s$. With this assumption, we can transform the high-dimensional LIFS, $G_s : \mathbb{R}^{(L-1)M-1} \to \mathbb{R}^{(L-1)M-1}$ for $s \in \{z_1, \ldots, z_S\}$, into the following system:

$$\sum_{i=1}^{(L-1)M-1} \theta_i e_i^s = \sum_{i=1}^{(L-1)M-1} \left(\theta_{-1,i} - \bar{\theta}_i^s\right) \lambda_i^s e_i^s + \sum_{i=1}^{(L-1)M-1} \bar{\theta}_i^s e_i^s \tag{51}$$

where λ_i^s is the i-th eigenvalue of Γ^s for $i \in \{1, \ldots, (L-1)\, M-1\}$.

Since we focus on the set of orthogonal eigenvectors, the coefficients of each eigenvector in both sides of (51) should be equivalent. Thus, the system in (51) reduces to:

$$\boldsymbol{\theta} = \boldsymbol{\lambda}^s \left[\boldsymbol{\theta}_{-1} - \bar{\boldsymbol{\theta}}^s\right] + \bar{\boldsymbol{\theta}}^s \tag{52}$$

where $\boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_{(L-1)M-1} \end{bmatrix}$, $\boldsymbol{\theta}_{-1} = \begin{bmatrix} \theta_{-1,1} \\ \theta_{-1,2} \\ \vdots \\ \theta_{-1,(L-1)M-1} \end{bmatrix}$, $\bar{\boldsymbol{\theta}}^s = \begin{bmatrix} \bar{\theta}^s_1 \\ \bar{\theta}^s_2 \\ \vdots \\ \bar{\theta}^s_{(L-1)M-1} \end{bmatrix}$ and $\boldsymbol{\lambda}^s$ is a diagonal matrix with eigenvalues as entries. For the homogeneous LIFS, $\Gamma^s = \Gamma$ for $\forall s$ and thus, the superscript for the eigenvalues drops out, i.e. $\lambda^s_i = \lambda_i$ for $\forall s$ and $\forall i$.

Lastly, let us define S_i as the number of distinct i-th row elements in $\left\{\bar{\boldsymbol{\theta}}^s\right\}_s$. For example, $\bar{\boldsymbol{\theta}}^{z_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,$\bar{\boldsymbol{\theta}}^{z_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,$\bar{\boldsymbol{\theta}}^{z_3} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ when $S = 3$. In this case, $S_1 = 2$ - $\{0,1\}$ - and $S_2 = 3$ - $\{0,1,2\}$.

With these notations, we can define a weak sufficient condition for the system of maps $\{G_s\}_s$ to generate a singular invariant measure. We summarize this condition in the following proposition.

Proposition 6. *The attractor of a high-dimensional LIFS is a Lebesgue measure zero set and its corresponding invariant measure is singular with respect to the Lebesgue measure if the following conditions are satisfied: i) $S \leq (L-1)\,M - 1$ or ii) if $S > (L-1)\,M - 1$, there exists i such that $\max_s \left\{\lambda^s_i\right\} < \frac{1}{S_i}$ for the heterogeneous LIFS and there exists i such that$\lambda_i < \frac{1}{S_i}$ for the homogeneous LIFS.*

Proof. *See Appendix A*

As for the three-period model, we analyze a relationship between the structural parameters and the existence of a singular invariant measure in this general model via the weak sufficient condition in Proposition 6. The weak sufficient condition also implies that the spectral radii of the affine matrices are the single most important measure for determining singularity, as the affine coefficient is in the three-period model. Thus, we examine the effects of the structural parameters on the spectral radiuses which we can find by rather calculating the stable eigenvalue of Z in the general model due to the result in Proposition 5.

In this general model, it is hard to classify entire economies into having singular or absolutely continuous measures even when assuming the CRRA preference since a longer lifetime increases the number of endowment parameters which enlarges the dimension of the parameters to be analyzed. Instead, we select a subset of economies enough to check the relationship between the structural parameters and the spectral radii.

Via the numerical analysis of this sample, we find a similar relationship as in the three-period model. Specifically, an increase in δ increases the spectral radius. The spectral radius is small under a hump-shaped endowment profile no matter what the lifetime length is. A lower σ leads to a smaller maximal eigenvalue. Hence, a singular measure will arise in a general model with a low δ and σ under a hump-shaped endowment because the LIFS in the model can satisfy the weak sufficient condition in Proposition 6 which needs small spectral radii.

We provide an intuition for how a high δ leads to a large maximal eigenvalue as follows. One can find similar intuitions for how other parameters affect the spectral radius. Additional lagged equity holdings for other ages reduce an individual's current asset holdings more sharply under a higher δ since other ages become wealthier and can purchase more assets. On the other hand, additional lagged equity holdings for an agent rather raises her current asset holdings more sharply in this case because she now gets wealthier and can buy more assets. Thus, a high δ increases the absolute values of all elements in the affine matrices which makes the spectral radius larger.

Now, we study invariant measures in two four-period lived representative agent models with the CRRA utility functions to illustrate the results in Proposition 6. The first model has two states of nature whereas the second one has four states of nature. Both models satisfy the condition in Corollary 2 because $(L-1)\,M\,((L-1)\,M-1)+1 = 7$. Thus, there exists a homogenous LIFS in all these models. The first model has a singular invariant measure since the number of states is two which is equal to the dimension of the affine matrix. We call this measure a trivial singular measure. On the other hand, the second model has a singular measure by satisfying the eigenvalues condition in Proposition 6 given that the number of states is greater than the dimension of the affine matrix. We call this measure a non-trivial singular measure.

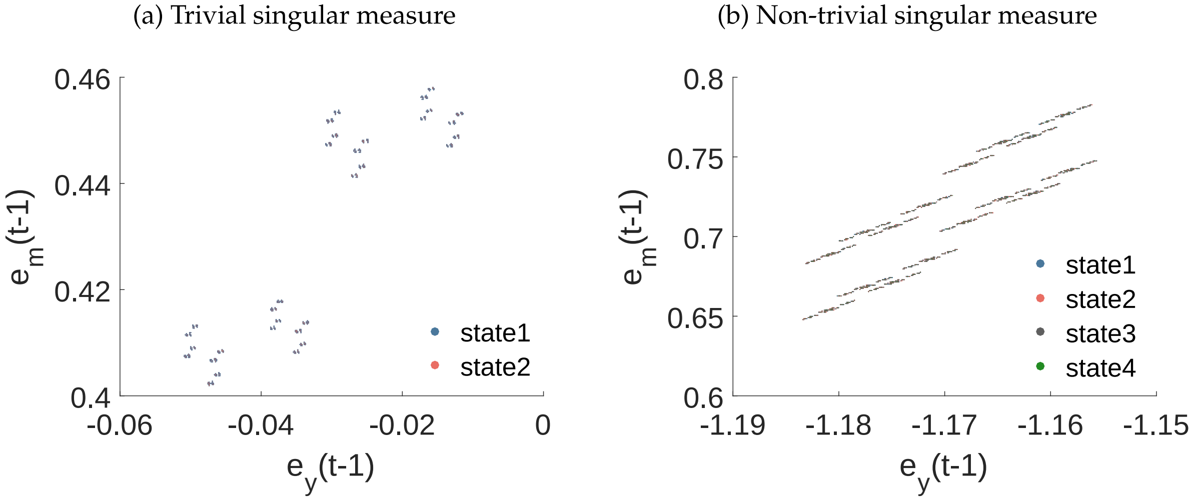
For the model generating a trivial singular measure, we set $\sigma = 2$, $\beta = 0.97^{60/4}$, $\delta = \frac{1}{3}$ and $\omega = \frac{2}{3}$. The lifetime endowment stream is denoted by $\{\omega_1, \omega_2, \omega_3, \omega_4\} = \omega \times \left\{\frac{1}{4}, \frac{1}{2}, \frac{1}{4}, 0\right\}$. For this model, we assume an I.I.D. shock with two states having an equal probability. This multiplicative shock affects the dividend and total asset quantity and its realizations in each state are given by $\{\delta^{z_1}, a^{z_1}\} = \{0.95, 0.95\}$ and $\{\delta^{z_2}, a^{z_2}\} = \{1.05, 1.05\}$.

For the model generating a non-trivial singular measure, we use the logarithmic preference, set the dividend share very small – $\delta = \frac{1}{10}$ – and make the endowment profile quite hump-shaped – $\{\omega_1, \omega_2, \omega_3, \omega_4\} = \omega \times \left\{\frac{1}{16}, \frac{10}{16}, \frac{5}{16}, 0\right\}$ – to obtain a low spectral radius which can satisfy the eigenvalues condition in Proposition 6. We maintain the same time discount factor with the first model. In this model, we introduce an I.I.D. shock with four states having an equal probability. This shock is also multiplicative and affects the dividend and total asset quantity. Its realizations in each state are defined by $\{\delta^{z_1}, a^{z_1}\} = \{0.95, 0.95\}$, $\{\delta^{z_2}, a^{z_2}\} = \{1.05, 0.95\}$, $\{\delta^{z_3}, a^{z_3}\} = \{0.95, 1.05\}$, $\{\delta^{z_4}, a^{z_4}\} = \{1.05, 1.05\}$.

Figures 4a and 4b show the attractor set for each trivial and non-trivial singular measure in the two four period-lived SOLG models.[14] The attractor sets in these figures are sparse in the two-dimensional state space for the lagged asset holdings because the attractor of any singular measures is a Lebesgue measure zero set. As seen in the three-period model, the attractors in these models also exhibit fractal self-affinity.

[14] To draw these attractor sets, we plot the simulation data from the two different models into the state space ignoring the first 1000 periods of observations. Based on the ergodic theorem, the set of time-series data generated by equilibrium mappings represents their invariant set.

FIGURE 4. The attractor sets of two four-period lived SOLG models



5 Conclusion

This paper studies the invariant Markov distribution associated with the rational expectations equilibrium in diverse SOLG models under pure exchange: a log-linear monetary model, a three-period model with a Lucas tree asset, and a generalized model with cohort heterogeneity. We are especially interested in the singular measure in these models since the support of this distribution exhibits self-affinity which is consistent with the fractal patterns observed in macro-finance data. We prove that the local ME in all the models of this paper can be generated by a LIFS under arbitrary shock processes if the aggregate shock is sufficiently small. Therefore, we examine the continuity property of the invariant measure by characterizing the sufficient conditions on the LIFS for its distribution to be singular with respect to the Lebesgue measure.

For the log-linear monetary model, we derive a closed form solution for the slope parameters of the LIFS under a small shock. This form implies that only the endowment ratio between ages matters and the absolute value of the Lipschitz constant is less than $1/2$ for the entire parameter space. Thus, the LIFS in this model under small aggregate shocks is non-overlapped and generates a Cantor-like invariant distribution all the times. The attractor of the LIFS is a Cantor-like invariant set showing fractal self-affinity.

In the three-period model with a long-lived asset and general preferences, we demonstrate that the slope parameters of the LIFS correspond to the stable eigenvalues of the Jacobian matrix for the price dynamics evaluated at the deterministic steady state if the aggregate shock is small. Based on this relationship, we numerically examine how structural parameters affect the stable eigenvalues of the price system and under what parameters values they have an absolute value less than $1/2$. Assuming a CRR preference, we find that a decrease in risk-aversion and dividend income share decreases the absolute value of the stable eigenvalues. Thus, the LIFS will be non-overlapped and its invariant Markov distribution can be a Cantor-like distribution for a model inhabited by low risk-averse agents with a low dividend share.

We extend the results above to the general model with a longer lifespan and heterogeneity. Similar to the three-period model, we show that the affine matrices of the LIFS take, as their own eigenvalues, the stable eigenvalues of the Jacobian matrix for the price dynamics evaluated at the steady states under the small aggregate shocks. We provide a sufficient condition for the high-dimensional LIFS to generate a singular measure which is closely related to the spectral radius of the affine matrices. Thus, information on the stable eigenvalues of the price systems can allow one to infer the singularity property of the invariant measures generated by the LIFS. The sufficient condition implies that we can find a trivial singular measure if the dimension of the essential state space for policy functions is larger than or equal to the number of states of nature. To obtain a non-trivial singular measure, the number of the shock states should be larger than the dimension of the state space and the eigenvalues of the affine matrices should be small enough. We numerically find a similar relationship between the largest eigenvalue and structural parameters in this general model as in the three-period model. Thus, it requires a low risk-aversion and a low dividend share in total income to produce a non-trivial singular measure in the general model.

The existence of the LIFS implies that an algorithm based on this structure allows computing equilibria in a very long-period SOLG models with heterogeneous agents at least for small aggregate shocks, thus avoiding the well-known curse of dimensionality. Since we prove the existence of the LIFS under arbitrary shock processes, an approximation adopting the LIFS structure might work well under an imperfectly correlated shock between labor and capital incomes as an alternative to quasi-aggregation methods which might generate large approximation errors in this case.

Finally, we should note that our results have some limitations that merit further research. First, we do not provide a necessary and sufficient condition for a singular measure to arise from a linear stochastic equilibrium mapping. Studying such conditions for a high-dimensional LIFS is a very difficult problem well-known in the dynamic system literature. As next, the invariant Markov measure of interest in this paper is one generated by the models under small aggregate shocks. To examine the continuity property of the invariant measure under a large size shock, one should study a possibly non-linear IFS since the existence of a LIFS as the equilibrium mapping will not hold in this case. We deal with the possibility of multiple solutions issue given the minimal state variables by restricting to the equilibria around the deterministic steady state of interest. To avoid such restriction, one can study sunspot-like equilibria where a sunspot variable picks one of the multiple solutions. Otherwise, one should provide conditions for the equilibria given the minimal state space to be unique. We also leave the question to develop and test the algorithm based on the implication of the LIFS. Since our analytical findings hold under small aggregate shocks, it is worth analyzing the size and types of shocks where algorithms adopting the LIFS structure generate acceptable approximation errors.

A Supplementary Material

One can find the proofs of propositions, lemmas, and corollaries, background on iterated function system, and numerical algorithm for computing equilibria in the models of this paper in supplementary material related to this article.

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