Dynamic Versions of Figures from Tristan Needham's *Visual Complex Analysis*

This document contains finished code to generate interactive versions of the pictures in *Visual Complex Analysis*. Educational modules could be made that are centered around understanding the mathematics underlying this code and constructing the code.

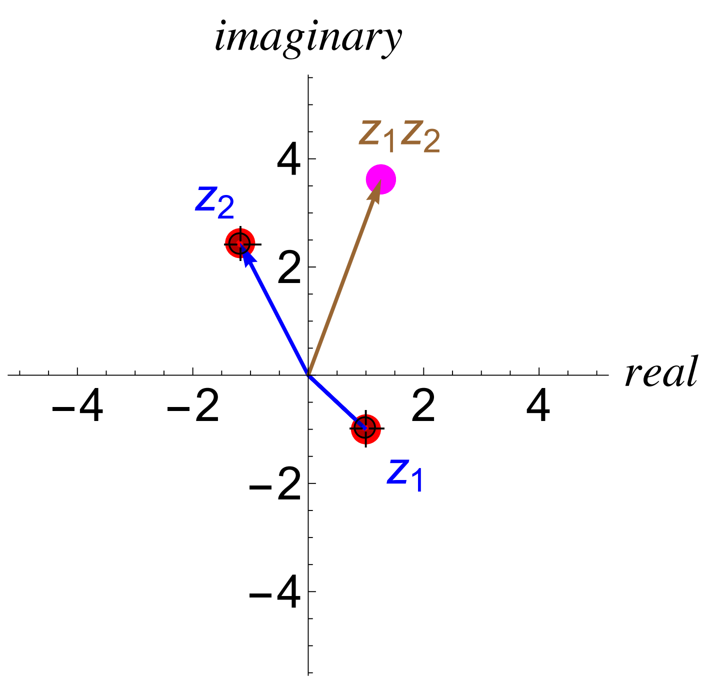
Geometric Intepretation of Complex Multiplication

(Chapter 1, Figures 1 and 2 (pages 2 and 3))

```
In[2]:= Manipulate[ProductPt = {Re[(pt1[[1]] + I * pt1[[2]]) * (pt2[[1]] + I * pt2[[2]])],
    Im[(pt1[[1]] + I * pt1[[2]]) * (pt2[[1]] + I * pt2[[2]])]};
  Grid[{{Show[ListPlot[{pt1, pt2}, PlotStyle → {{Red, PointSize[.05]}}],
      ListPlot[{{Re[(pt1[[1]] + I * pt1[[2]]) * (pt2[[1]] + I * pt2[[2]])], Im[
          (pt1[[1]] + I * pt1[[2]]) * (pt2[[1]] + I * pt2[[2]])]}}, PlotStyle → {{Magenta,
          PointSize[.05]}}], Graphics[{Thick, Blue, Arrow[{{0, 0}, pt1, {0, 0}, pt2}],
        Style[Text["z₁", {pt1[[1]], pt1[[2]]} + (R / 5) * {pt1[[1]], pt1[[2]]} / Norm[{pt1[[1]], pt1[[2]]}]], Large],
        Style[Text["z₂", {pt2[[1]], pt2[[2]]} + (R / 5) * {pt2[[1]], pt2[[2]]} / Norm[{pt2[[1]], pt2[[2]]}]],
         Large]}], Graphics[{Thick, Brown, Arrow[{{0, 0}, ProductPt}],
        Style[Text["z₁z₂", ProductPt + (R / 5) * ProductPt / Norm[ProductPt]], Large]}],
      PlotRange → R, AspectRatio → 1, AxesOrigin → {0, 0}, AxesLabel →
       {Text[Style["real", Large, Italic]], Text[Style["imaginary", Large, Italic]]},
      ImageSize → Medium, TicksStyle → Large], , , Column[{Row[{Style[Text["z₁"],
          Large], " = ", Item[NumberForm[Style[pt1[[1]] + ⅈ * pt1[[2]], Large],
           {6, 4}], Frame → True]}], Row[{Style[Text["z₂"], Large], " = ", Item[
          NumberForm[Style[pt2[[1]] + ⅈ * pt2[[2]], Large], {6, 4}], Frame → True]}],
       Row[{Style[Text["z₁z₂"], Large], " = ", Item[NumberForm[
           Style[ProductPt[[1]] + ⅈ * ProductPt[[2]], Large], {6, 4}], Frame → True]}],
       Row[{Style[Text["|z₁|"], Large], " = ", Item[NumberForm[Style[N[Norm[pt1]],
             Large], {6, 4}], Frame → True]}], Row[{Style[Text["|z₂|"], Large], " = ",
          Item[NumberForm[Style[N[Norm[pt2]], Large], {6, 4}], Frame → True]}],
       Row[{Style[Text["|z₁z₂|"], Large], " = ",
          Item[NumberForm[Style[N[Norm[ProductPt]], Large], {6, 4}], Frame → True]}],
       Row[{Style[Text["Arg(z₁)  (degrees)"], Large], " = ", Item[NumberForm[Style[
             N[ArcTan[pt1[[1]], pt1[[2]]] * 180 / π], Large], {6, 4}], Frame → True]}],
       Row[{Style[Text["Arg(z₂)  (degrees)"], Large], " = ", Item[NumberForm[Style[
             N[ArcTan[pt2[[1]], pt2[[2]]] * 180 / π], Large], {6, 4}], Frame → True]}],
       Row[{Style[Text["Arg(z₁z₂)  (degrees)"], Large], " = ", Item[NumberForm[
           Style[N[(180 / π) * ArcTan[ProductPt[[1]], ProductPt[[2]]]], Large],
           {6, 4}], Frame → True]}], Row[{Style[Text["Arg(z₁)+Arg(z₂)"], Large],
         " = ", Item[NumberForm[Style[N[ArcTan[pt1[[1]], pt1[[2]]] * 180 / π +
              ArcTan[pt2[[1]], pt2[[2]]] * 180 / π], Large], {6, 4}], Frame → True]}],
       Row[{Style[Text["|Arg(z₁)+Arg(z₂)-Arg(z₁z₂)|"], Large], " = ",
         Item[NumberForm[Style[Round[Abs[ArcTan[pt1[[1]], pt1[[2]]] * 180 / π +
                ArcTan[pt2[[1]], pt2[[2]]] * 180 / π - (180 / π) * ArcTan[ProductPt[[1]],
                  ProductPt[[2]]]]], Large], {6, 4}], Frame → True]}]}]}}],
  {{pt1, {1, -1}}, Locator}, {{pt2, {1, 1}}, Locator}, {{R, 5}, .1, 200},
  LabelStyle → Large]
```

Out[2]=

R



z_1 = 1.0000 − i

z_2 = −1.1800 + 2.4

$z_1 z_2$ = 1.2600 + 3.6

$|z_1|$ = 1.4142

$|z_2|$ = 2.7104

$|z_1 z_2|$ = 3.8330

Arg(z_1) (degrees) = −4

Arg(z_2) (degrees) = 11

Arg($z_1 z_2$) (degrees)
= 70.8087

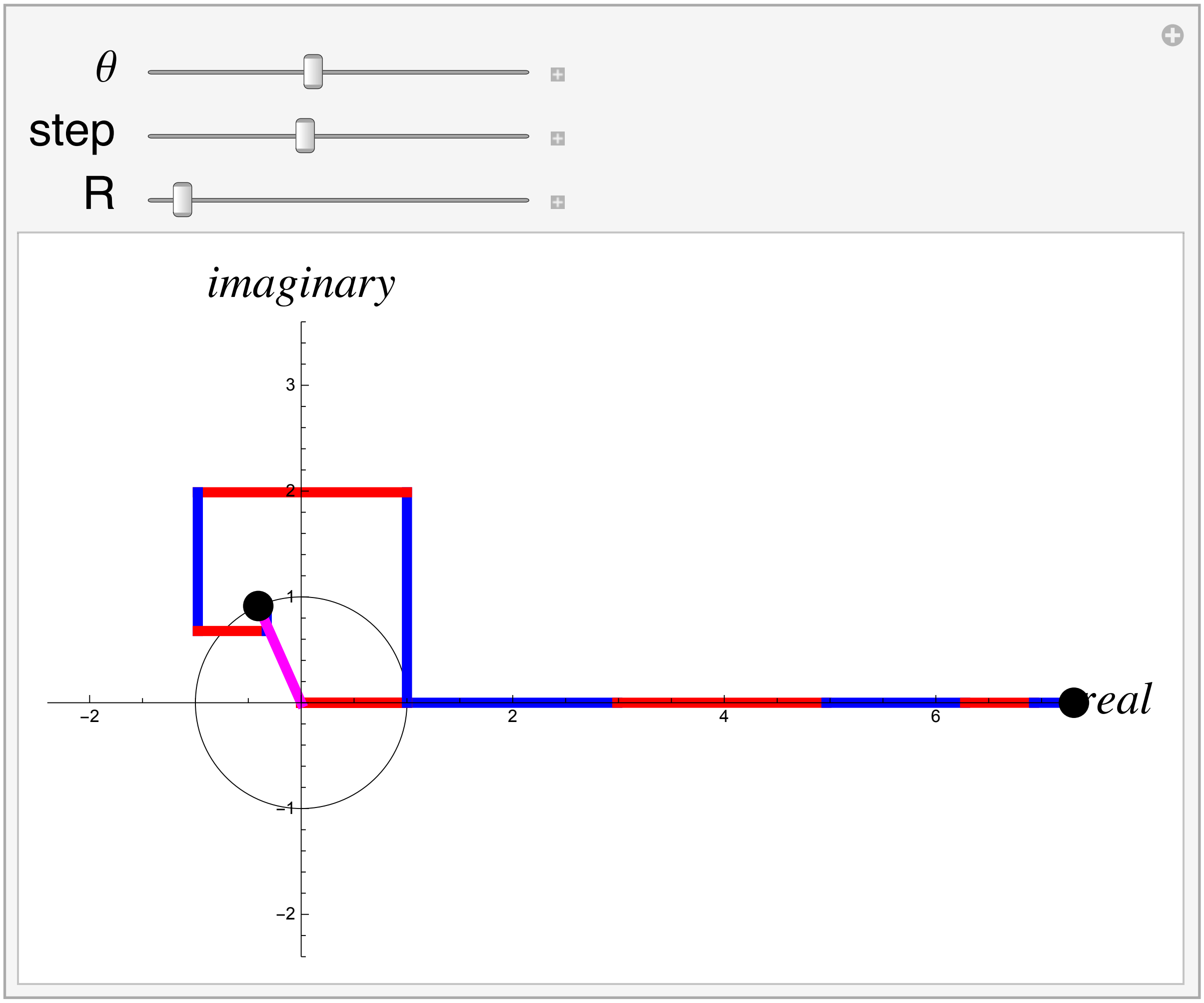
Arg(z_1)+Arg(z_2) = 70

|Arg(z_1)+Arg(z_2)−Arg(
= 0.0000

The Geometry of Euler’s Identity (Chapter 1, Figure 9 (page 13))

In[3]:=
```
P₀[x_] := 1; Pₙ_[x_] := 1 + Sum[x^(k) / k!, {k, 1, n}]; Manipulate[Show[Graphics[{Black, Circle[]}],
  Table[Graphics[{Thickness[.01], Red, Line[{{0, 0}, {1, 0}}], Thickness[.01],
     If[OddQ[i], Red, Blue], Line[{{Pᵢ[θ], 0}, {Pᵢ₊₁[θ], 0}}]}], {i, 0, step}],
  Table[Graphics[{Thickness[.01], If[OddQ[i], Red, Blue],
     Line[{{Re[Pᵢ[I*θ]], Im[Pᵢ[I*θ]]}, {Re[Pᵢ₊₁[I*θ]], Im[Pᵢ₊₁[I*θ]]}}]}], {i, 0,
    step}], Graphics[{Thickness[.01], Magenta, Line[{{0, 0}, {Cos[θ], Sin[θ]}}]}],
  ListPlot[{{Cos[θ], Sin[θ]}, {E^(θ), 0}}, PlotStyle → {Black, PointSize[.03]}],
  PlotRange → {{-R / 3, R}, {-R / 3, R / 2}}, Axes → True, AxesLabel →
   {Text[Style["real", Large, Italic]], Text[Style["imaginary", Large, Italic]]},
  ImageSize → Large], {{θ, Pi / 12}, 0, 3 π / 2},
 {step, 0, 10, 1}, {{R, 4}, 2, 120}, LabelStyle → Large]
```

Out[3]=

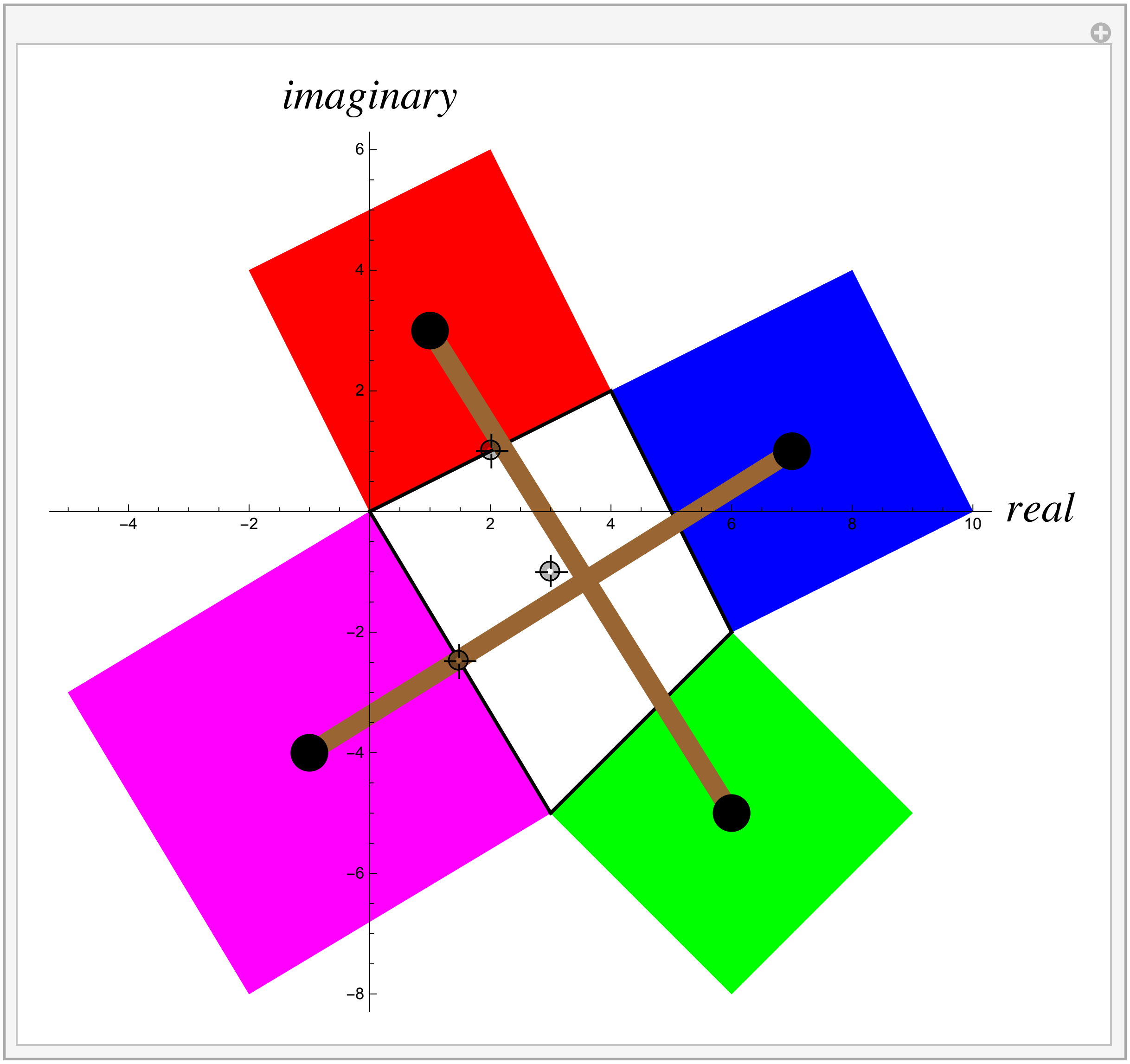


Line Segments between Centers of Opposite Squares Constructed on the Sides of a Quadrilateral are Perpendicular and of

Equal Length (Chapter 1, Figure 12 (page 16))

In[4]:=
```
Manipulate[A = a[[1]] + a[[2]] * I;
 B = (b[[1]] + b[[2]] * I) - A;
 Ce = (c[[1]] + c[[2]] * I) - (b[[1]] + b[[2]] * I);
 De = - (c[[1]] + c[[2]] * I);
 p = a + {Re[A * I], Im[A * I]};
 q = 2 a + {Re[B], Im[B]} + {Re[B * I], Im[B * I]};
 r = 2 a + {Re[2 B], Im[2 B]} + {Re[Ce], Im[Ce]} + {Re[Ce * I], Im[Ce * I]};
 s = 2 a + {Re[2 B], Im[2 B]} +
   {Re[2 Ce], Im[2 Ce]} + {Re[De], Im[De]} + {Re[De * I], Im[De * I]};
 Show[Graphics[{Red, Polygon[{{0, 0}, 2 a, 2 a + {Re[2 A * I], Im[2 A * I]},
       2 a + {Re[2 A * I], Im[2 A * I]} - {Re[2 A], Im[2 A]}}]}], Graphics[
   {Blue, Polygon[{2 a, 2 b, 2 b + {Re[2 B * I], Im[2 B * I]}, 2 b + {Re[2 B * I], Im[2 B * I]} -
        {Re[2 B], Im[2 B]}}]}], Graphics[{Green, Polygon[{2 b, 2 c, 2 c + {Re[2 Ce * I],
         Im[2 Ce * I]}, 2 c + {Re[2 Ce * I], Im[2 Ce * I]} - {Re[2 Ce], Im[2 Ce]}}]}],
  Graphics[{Magenta, Polygon[{2 c, {0, 0}, {Re[2 De * I], Im[2 De * I]},
      {Re[2 De * I], Im[2 De * I]} - {Re[2 De], Im[2 De]}}]}], Graphics[{Thick, Black,
    Line[{{0, 0}, 2 a}]}], Graphics[{Thick, Black, Line[{2 a, 2 b}]}], Graphics[
   {Thick, Black, Line[{2 b, 2 c}]}], Graphics[{Thick, Black, Line[{2 c, {0, 0}}]}],
  Graphics[{Thickness[.02], Brown, Line[{p, r}]}], Graphics[{Thickness[.02], Brown,
    Line[{q, s}]}], ListPlot[{p, q, r, s}, PlotStyle → {Black, PointSize[.04]}],
  Axes → True, AxesLabel → {Text[Style["real", Large, Italic]],
    Text[Style["imaginary", Large, Italic]]}, ImageSize → Large],
 {{a, {2, 1}}, Locator}, {{b, {3, -1}}, Locator}, {{c, {1.5, -2.5}}, Locator}]
```

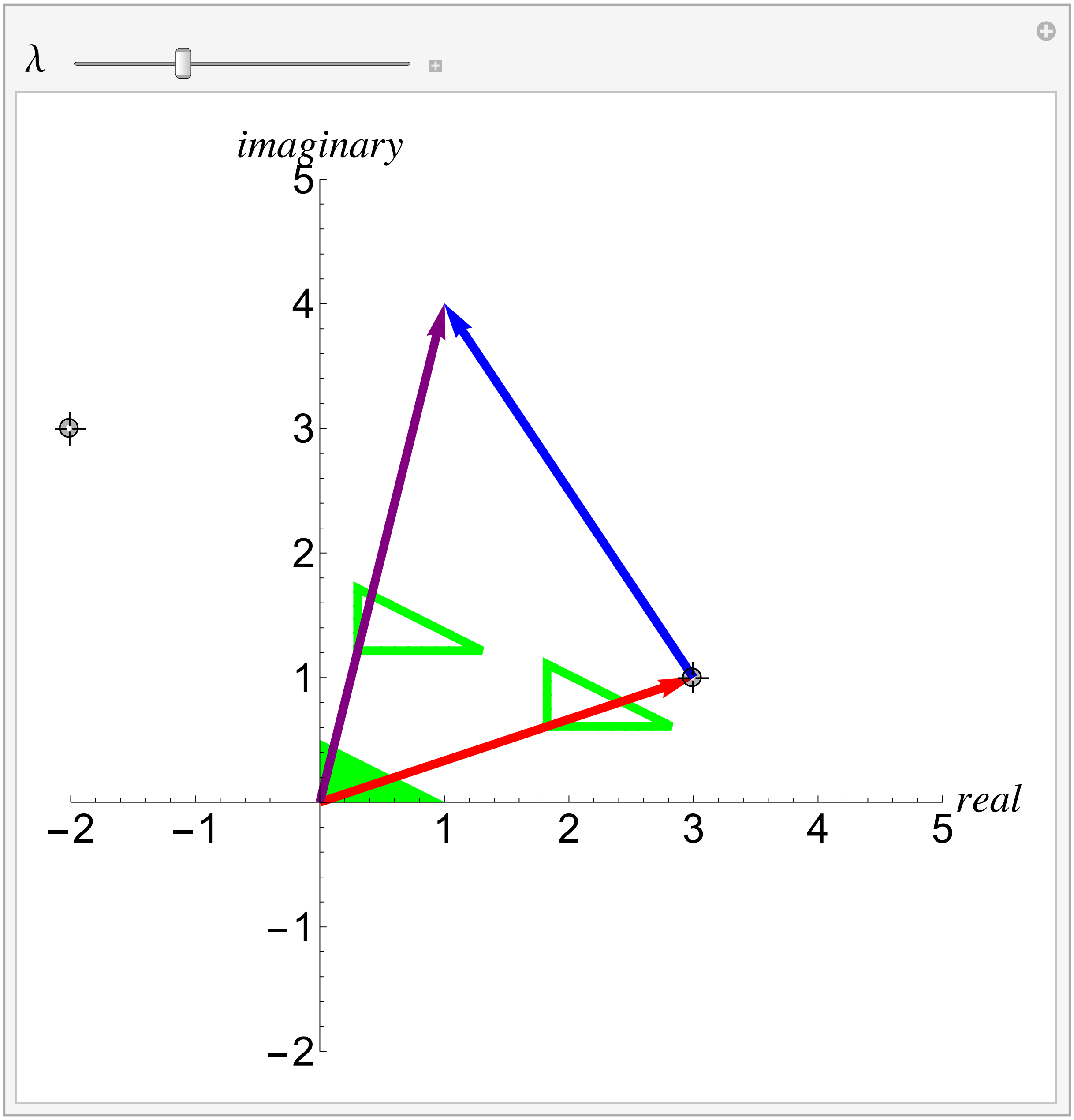
Out[4]=



Commutativity of Translation Compositions (Chapter 1, Figure 13a (page 18))

In[5]:=
```
FilledTriangle[{a_, b_, c_}] := Graphics[{Green, Polygon[{a, b, c}]}];
UnFilledTriangle[{a_, b_, c_}] :=
 Graphics[{Thickness[.01], Dashed, Green, Line[{a, b, c, a}]}];
BaseTriangle = {{0, 0}, {1, 0}, {0, .5}};
Manipulate[Show[FilledTriangle[BaseTriangle],
  FilledTriangle[If[λ < .5, {{1000, 0}, {1001, 0}, {1000, .5}},
    Table[BaseTriangle[[i]] + v, {i, 1, 3}]]], FilledTriangle[If[λ < .99,
    {{1000, 0}, {1001, 0}, {1000, .5}}, Table[BaseTriangle[[i]] + v + w, {i, 1, 3}]]],
  UnFilledTriangle[Table[BaseTriangle[[i]] + (v + w) * λ, {i, 1, 3}]],
  If[λ < .5, UnFilledTriangle[Table[BaseTriangle[[i]] + v * 2 λ, {i, 1, 3}]],
   UnFilledTriangle[Table[BaseTriangle[[i]] + v + w * (2 λ - 1), {i, 1, 3}]]], Graphics[
   {Thickness[.01], Red, Arrow[{{0, 0}, v}], Thickness[.01], Blue, Arrow[{v, v + w}],
    Thickness[.01], Purple, Arrow[{{0, 0}, v + w}]}], Axes → True, AxesLabel →
   {Text[Style["real", Large, Italic]], Text[Style["imaginary", Large, Italic]]},
  PlotRange → {{-2, 5}, {-2, 5}}, TicksStyle → Large, ImageSize → Large],
 {{v, {3, 1}}, Locator}, {{w, {-2, 3}}, Locator}, {λ, 0, 1}, LabelStyle → Large]
```

Out[5]=

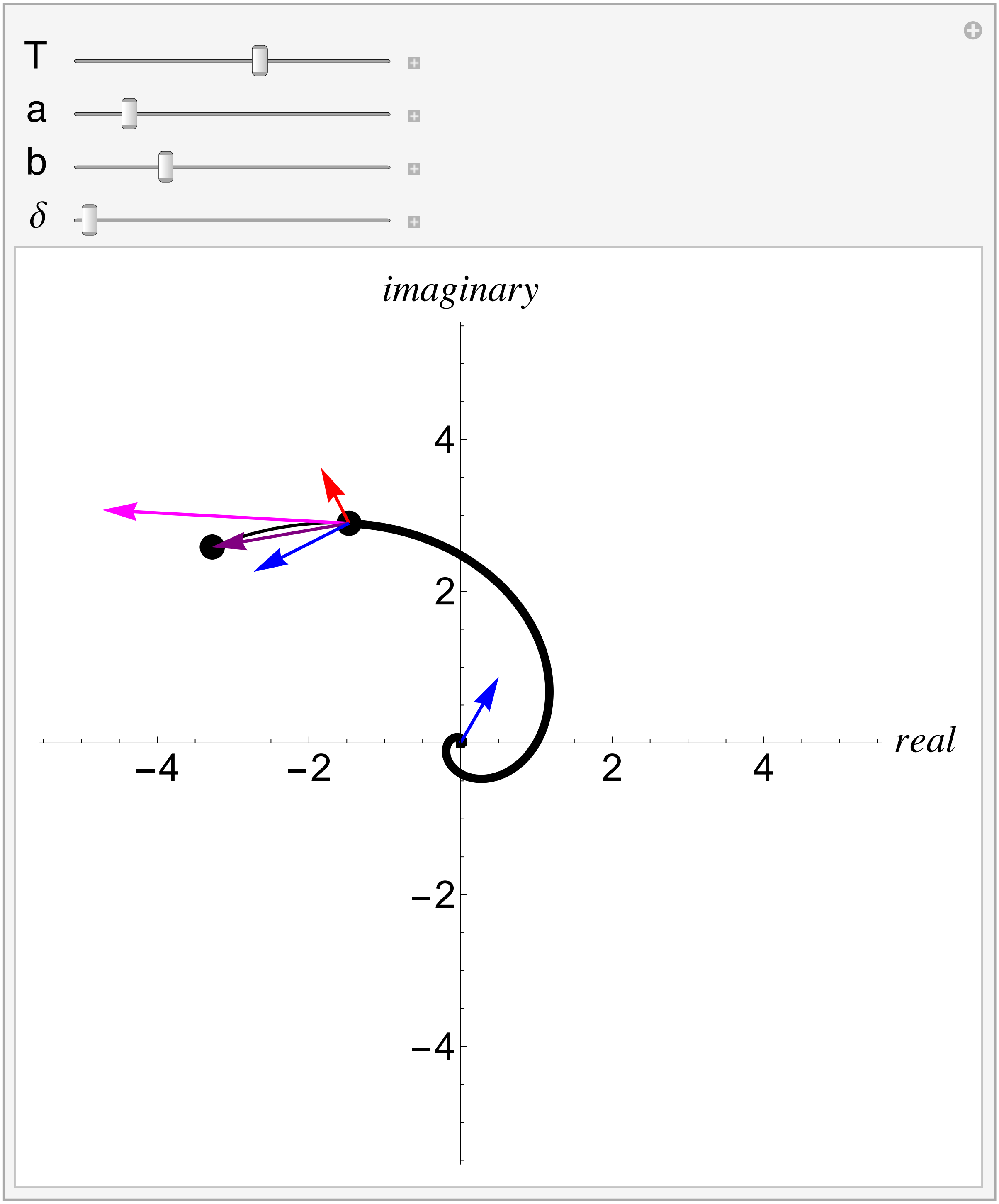


How the Trajectory of

$Z(t) = e^{(a+b\,i)\,t} = e^{a\,t}\cos(b\,t) + i\,e^{a\,t}\sin(b\,t)$ changes as $a + b\,i$ changes and how the velocity vector $V(t)$ is the limiting value of $\frac{Z(t+\delta)-Z(t)}{\delta}$ as $\delta \longrightarrow 0$ (Chapter 1, Figure 16 (page 21))

In[6]:=
```
Za_,b_[t_] := E^(a*t)*E^(I*b*t); Va_,b_[t_] := Za,b'[t];
Manipulate[Show[ParametricPlot[{Re[Za,b[t]], Im[Za,b[t]]},
    {t, -4 π, T}, PlotStyle → {Thickness[.01], Black}],
  ParametricPlot[{Re[Za,b[t]], Im[Za,b[t]]}, {t, T, T + δ}, PlotStyle → {Thick, Black}],
  ListPlot[{{Re[Za,b[T]], Im[Za,b[T]]}, {Re[Za,b[T + δ]], Im[Za,b[T + δ]]}},
   PlotStyle → {Black, PointSize[.03]}],
  Graphics[{Thick, Blue, Arrow[{{0, 0}, {a, b}}], Thick, Magenta, Arrow[{{Re[Za,b[T]],
        Im[Za,b[T]]}, {Re[Za,b[T]], Im[Za,b[T]]} + {Re[Va,b[T]], Im[Va,b[T]]}}], Thick,
    Purple, Arrow[{{Re[Za,b[T]], Im[Za,b[T]]}, {Re[Za,b[T + δ]], Im[Za,b[T + δ]]}}],
    Thick, Red, Arrow[{{Re[Za,b[T]], Im[Za,b[T]]},
      {Re[Za,b[T]], Im[Za,b[T]]} + a * δ * {Re[Za,b[T]], Im[Za,b[T]]}}],
    Thick, Blue, Arrow[{{Re[Za,b[T]], Im[Za,b[T]]},
      {Re[Za,b[T]], Im[Za,b[T]]} + b * δ * {-Im[Za,b[T]], Re[Za,b[T]]}}]}],
  PlotRange → 10 * Abs[a], AxesOrigin → {0, 0}, TicksStyle → Large, AxesLabel →
   {Text[Style["real", Large, Italic]], Text[Style["imaginary", Large, Italic]]},
  ImageSize → Large], {{T, 3 π / 4}, -4 π + .001, 4 π},
 {{a, 1 / 2}, .1, 3}, {{b, √3 / 2}, .1, 3},
 {δ, .5, .0001}, LabelStyle → Large]
```

Out[6]=



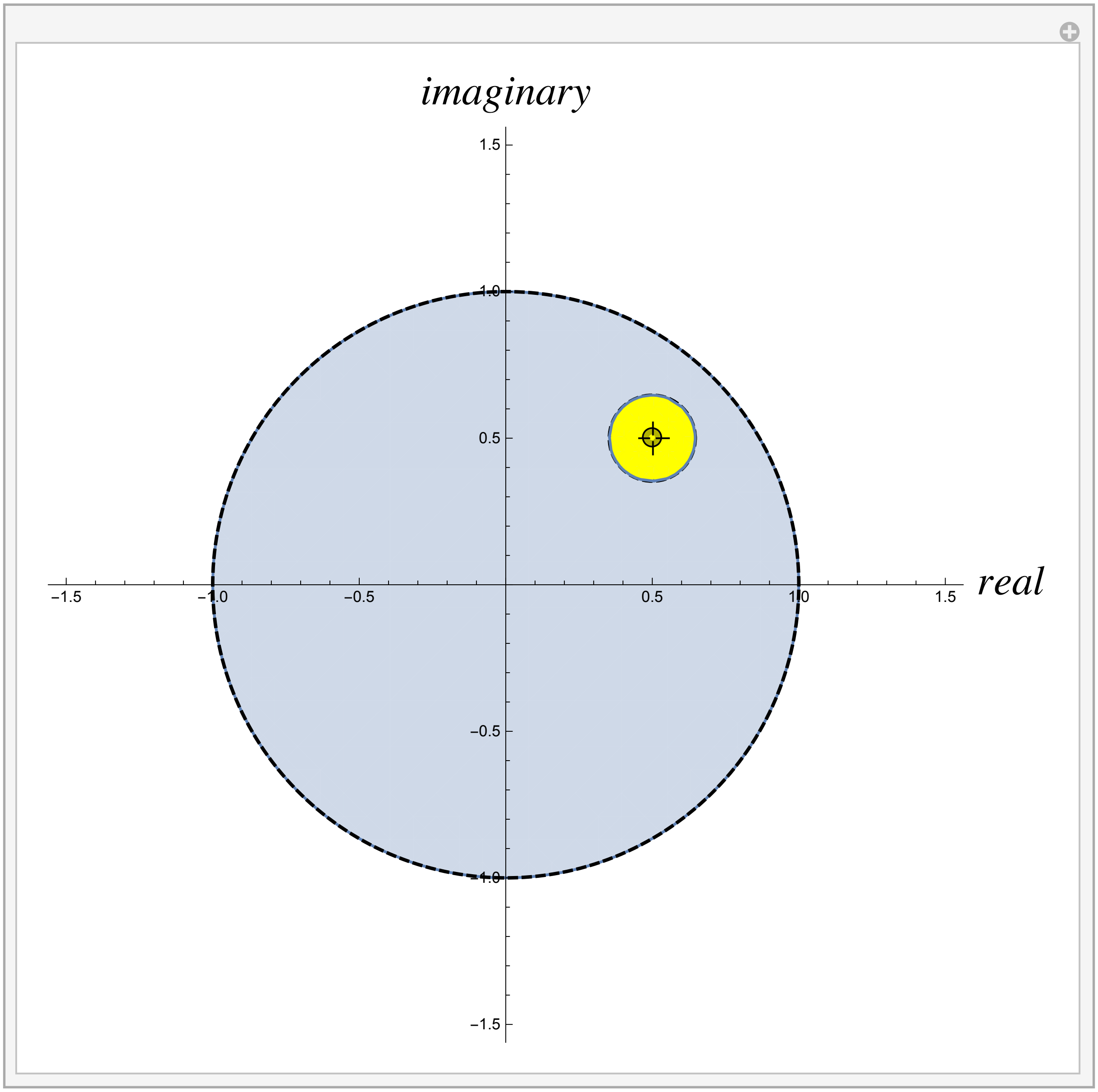
Miscellaneous

other Demonstrations

Open Sets in the Complex Plane

In[7]:=
```
Manipulate[Show[RegionPlot[x^2 + y^2 < 1, {x, -1.5, 1.5},
    {y, -1.5, 1.5}, Frame → False, Axes → True, PerformanceGoal → "Quality"],
   Graphics[{Thick, Dashed, Circle[{0, 0}]}],
   Graphics[{Thick, Dashed, Circle[z0, (1 - Abs[z0[[1]] + I * z0[[2]]])/2]}],
   RegionPlot[(x - z0[[1]])^2 + (y - z0[[2]])^2 < ((1 - Abs[z0[[1]] + I * z0[[2]]])/2)^2,
    {x, -5, 5}, {y, -5, 5}, PlotStyle → Yellow,
    PlotPoints → 100, PerformanceGoal → "Quality"], AxesLabel →
    {Text[Style["real", Large, Italic]], Text[Style["imaginary", Large, Italic]]},
   ImageSize → Large], {{z0, {.5, .5}}, Locator}]
```

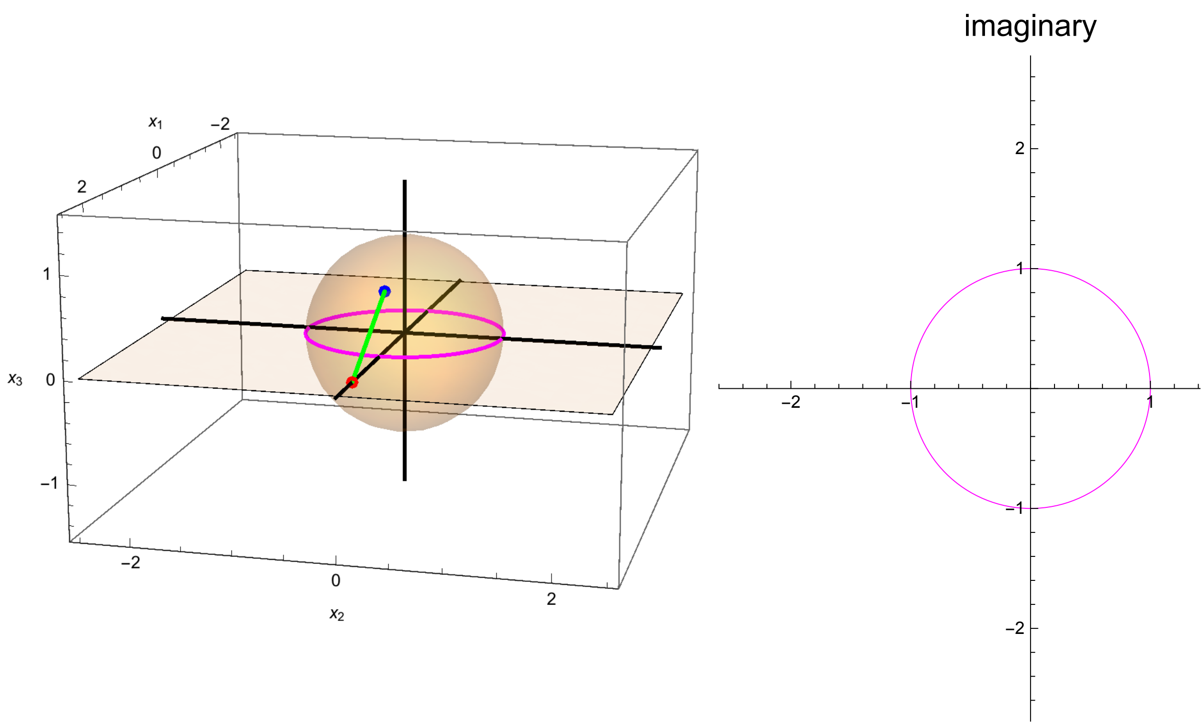
Out[7]=



Stereographic Projection onto Riemann Sphere

In[8]:=
```
Π[{p_, q_, r_}] := {2 p/(p^2 + q^2 + 1), 2 q/(p^2 + q^2 + 1), (p^2 + q^2 - 1)/(p^2 + q^2 + 1)};
axes = ParametricPlot3D[{{t, 0, 0}, {0, t, 0}, {0, 0, t}},
   {t, -100, 100}, PlotStyle → {{Thick, Black}}]; equator =
  ParametricPlot3D[{Cos[t], Sin[t], 0}, {t, 0, 2 Pi}, PlotStyle → {{Thick, Magenta}}];
Manipulate[Grid[{{Show[axes, equator, ContourPlot3D[x^2 + y^2 + z^2 == 1,
       {x, -2, 2}, {y, -2, 2}, {z, -1, 1}, BoxRatios → {2, 2, 1},
       ContourStyle → Opacity[.2], Mesh → None, PerformanceGoal → "Quality"],
      ContourPlot3D[z == 0, {x, -2.5, 2.5}, {y, -2.5, 2.5}, {z, -1, 1},
       BoxRatios → {2, 2, 1}, ContourStyle → Opacity[.1], Mesh → None],
      ListPointPlot3D[{Append[pt, 0]}, PlotStyle → {{Red, PointSize[.02]}}],
      ListPointPlot3D[{Π[Append[pt, 0]]}, PlotStyle → {{Blue, PointSize[.02]}}],
      ParametricPlot3D[Append[pt, 0] * (1 - t) + t * Π[Append[pt, 0]],
       {t, 0, 1}, PlotStyle → {{Thickness[.007], Green}}],
      PlotRange → {{-2.5, 2.5}, {-2.5, 2.5}, {-1.5, 1.5}},
      AxesLabel → {x1, x2, x3}, ViewPoint → {4, 1, 1}, ImageSize → Medium],
     Show[ListPlot[{pt}, PlotStyle → {Red, PointSize[.03]}],
      Graphics[{Magenta, Circle[]}], PlotRange → {{-2.5, 2.5}, {-2.5, 2.5}},
      AxesLabel → {"real", "imaginary"}, AspectRatio → Automatic,
      ImageSize → Medium]}}], {{pt, {2, 0}}, Locator}]
```

Out[8]=

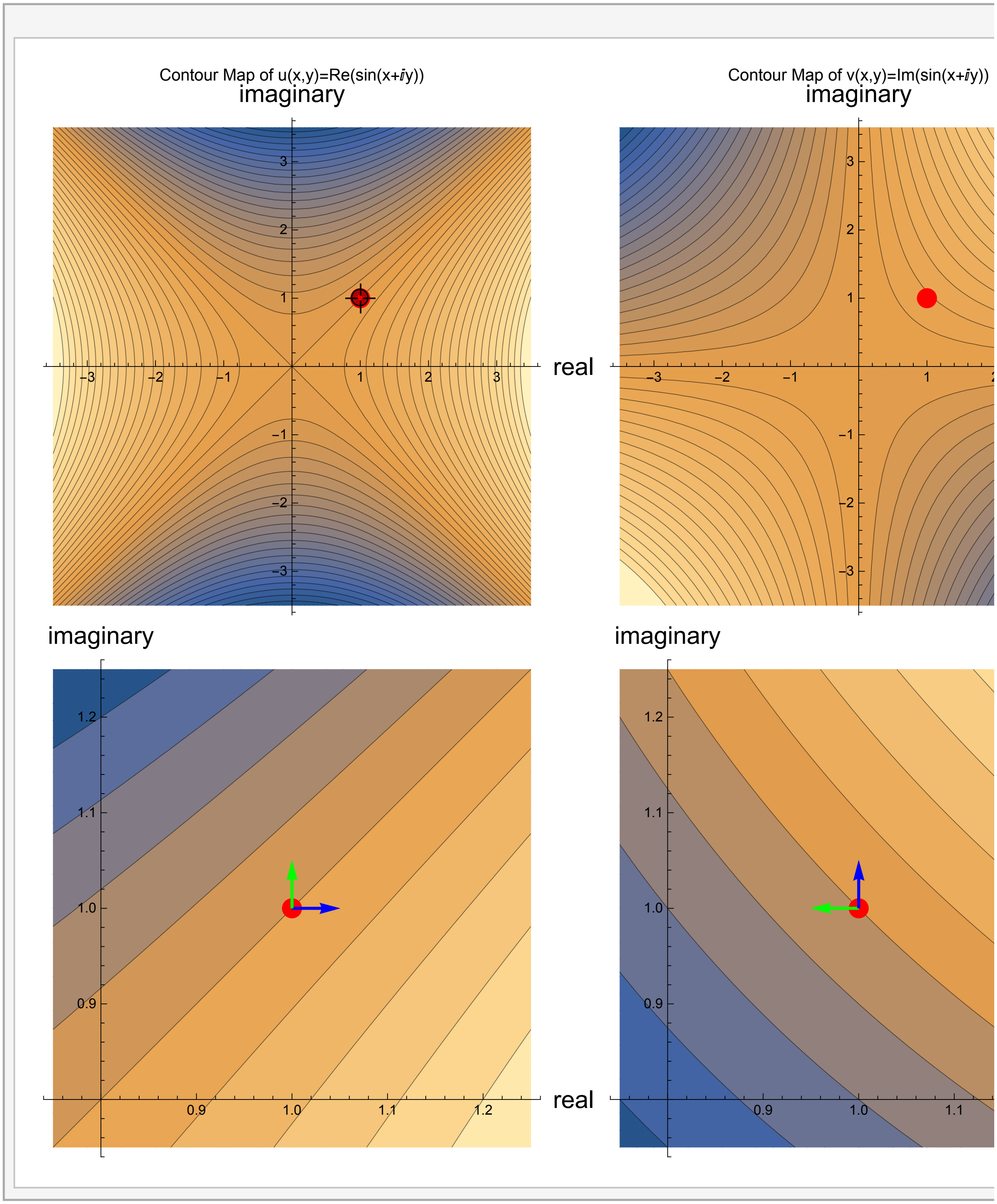


Cauchy-Riemann Equations

$f(z) = \sin(z)$

In[9]:=
```
u[x_, y_] := Cosh[y] Sin[x]; v[x_, y_] := Cos[x] Sinh[y];
Manipulate[Grid[{{Show[ContourPlot[u[x, y], {x, -3.5, 3.5},
      {y, -3.5, 3.5}, Contours → 40, PerformanceGoal → "Quality"],
     ListPlot[{pt}, PlotStyle → {Red, PointSize[.04]}], Frame → False,
     Axes → True, PlotLabel → "Contour Map of u(x,y)=Re(sin(x+iy))",
     AxesLabel → {"real", "imaginary"}, ImageSize → Medium],
    Show[ContourPlot[v[x, y], {x, -3.5, 3.5}, {y, -3.5, 3.5},
      Contours → 40, PerformanceGoal → "Quality"],
     ListPlot[{pt}, PlotStyle → {Red, PointSize[.04]}], Frame → False,
     Axes → True, PlotLabel → "Contour Map of v(x,y)=Im(sin(x+iy))",
     AxesLabel → {"real", "imaginary"}, ImageSize → Medium]},
   {Show[ContourPlot[u[x, y], {x, pt[[1]] - .25, pt[[1]] + .25},
      {y, pt[[2]] - .25, pt[[2]] + .25}, PerformanceGoal → "Quality"],
     ListPlot[{pt}, PlotStyle → {Red, PointSize[.04]}],
     Graphics[{Thick, Blue, Arrow[{pt, pt + {.05, 0}}]}],
     Graphics[{Thick, Green, Arrow[{pt, pt + {0, .05}}]}], Frame → False,
     Axes → True, AxesLabel → {"real", "imaginary"}, ImageSize → Medium],
    Show[ContourPlot[v[x, y], {x, pt[[1]] - .25, pt[[1]] + .25},
      {y, pt[[2]] - .25, pt[[2]] + .25}, PerformanceGoal → "Quality"],
     ListPlot[{pt}, PlotStyle → {Red, PointSize[.04]}],
     Graphics[{Thick, Green, Arrow[{pt, pt + {-.05, 0}}]}],
     Graphics[{Thick, Blue, Arrow[{pt, pt + {0, .05}}]}],
     Frame → False, Axes → True, AxesLabel → {"real", "imaginary"},
     ImageSize → Medium]}}], {{pt, {1, 1}}, Locator}]
```

Out[9]=



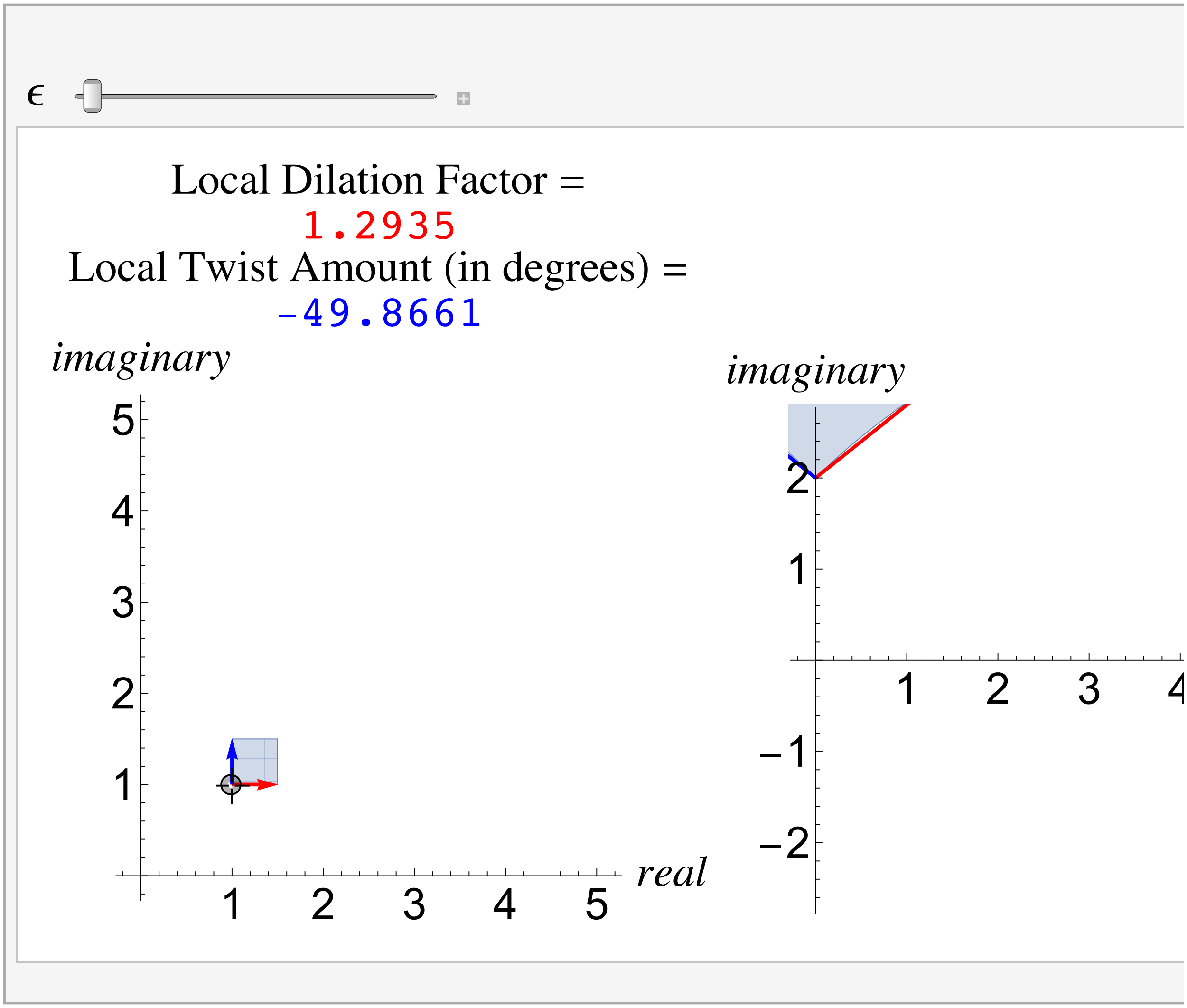
The Derivative as a Local "Amplitwist"...terminology credit

to Tristan Needham (from his book "Visual Complex Analysis")

$f(z) = \sin(z)$

```
In[10]:= u[x_, y_] := Cosh[y] * Sin[x]; v[x_, y_] := Cos[x] * Sinh[y];
Φ[{x_, y_}] := {u[x, y], v[x, y]};
ΦPrime[{x_, y_}] := {Cos[x] Cosh[y], -Sin[x] Sinh[y]};
Manipulate[Grid[{{Style[Text["Local Dilation Factor ="], Large],},
   {Item[NumberForm[Style[N[Norm[ΦPrime[z0]]], Large, Red], {6, 4}]],},
   {Style[Text["Local Twist Amount (in degrees) ="], Large],},
   {Item[NumberForm[Style[N[(180 / π) * ArcTan[ΦPrime[z0][[1]], ΦPrime[z0][[2]]]],
       Large, Blue], {6, 4}]],}, {Show[ParametricPlot[z0 + {x, y}, {x, 0, ϵ},
      {y, 0, ϵ}, PerformanceGoal → "Quality"], Graphics[{Thick, Red,
       Arrow[{z0, z0 + {ϵ, 0}}]}], Graphics[{Thick, Blue, Arrow[{z0, z0 + {0, ϵ}}]}],
     PlotRange → {{0, 5}, {0, 5}}, ImageSize → Medium, Frame → False, Axes → True,
     TicksStyle → Large, AxesLabel → {Text[Style["real", Large, Italic]],
       Text[Style["imaginary", Large, Italic]]}], Show[
     ParametricPlot[Φ[z0 + {x, y}], {x, 0, ϵ}, {y, 0, ϵ}, PerformanceGoal → "Quality"],
     Graphics[{Thick, Red, Arrow[{Φ[z0], Φ[z0 + {ϵ, 0}]}]}],
     Graphics[{Thick, Blue, Arrow[{Φ[z0], Φ[z0 + {0, ϵ}]}]}],
     PlotRange → {{0, 5}, {-2.5, 2.5}}, ImageSize → Medium, Frame → False, Axes → True,
     TicksStyle → Large, AxesLabel → {Text[Style["real", Large, Italic]],
       Text[Style["imaginary", Large, Italic]]}]}}],
  {{z0, {1, 1}}, Locator}, {ϵ, .5, .001}, LabelStyle → Large]
```

Out[10]=

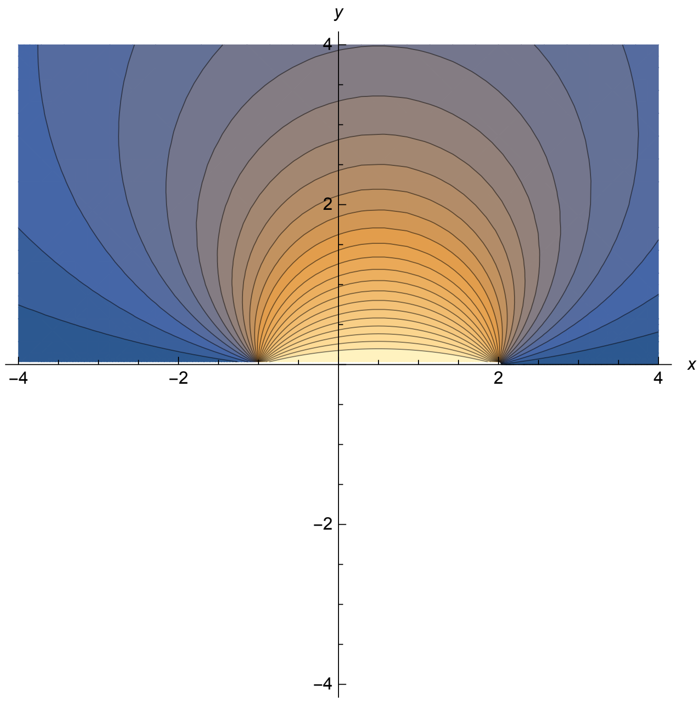


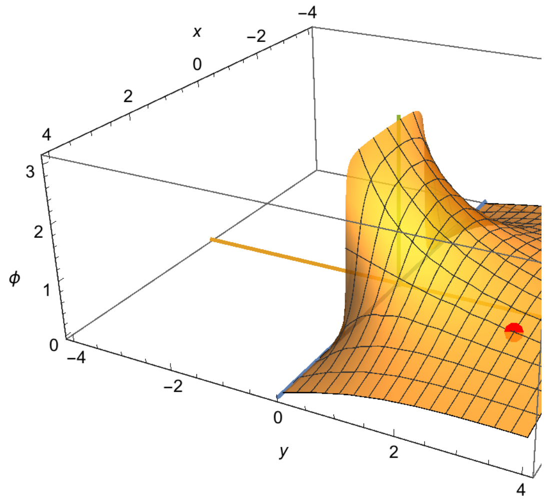
Solving Laplace's Equation

In[11]:=

```
ϕ[x_, y_] := -Arg[(x + I * y) + 1] + Arg[(x + I * y) - 2];
Grid[{{ContourPlot[ϕ[x, y], {x, -4, 4}, {y, -4, 4}, Contours → 50,
    Frame → False, Axes → True, RegionFunction → Function[{x, y, z}, y ≥ 0],
    AxesLabel → {x, y}, ImageSize → Medium],
   Show[Plot3D[ϕ[x, y], {x, -4, 4}, {y, -4, 4}, PlotStyle → Opacity[.8],
     RegionFunction → Function[{x, y, z}, y ≥ 0]], ParametricPlot3D[
     {{t, 0, 0}, {0, t, 0}, {0, 0, t}}, {t, -20, 20}, PlotStyle → Thick],
    ListPointPlot3D[{{2, 3, Pi / 4}}, PlotStyle → {{Red, PointSize[.03]}}],
    ViewPoint → {2, 1, 1}, Axes → True, AxesLabel → {x, y, ϕ}, ImageSize → Medium]}}]
```

Out[11]=



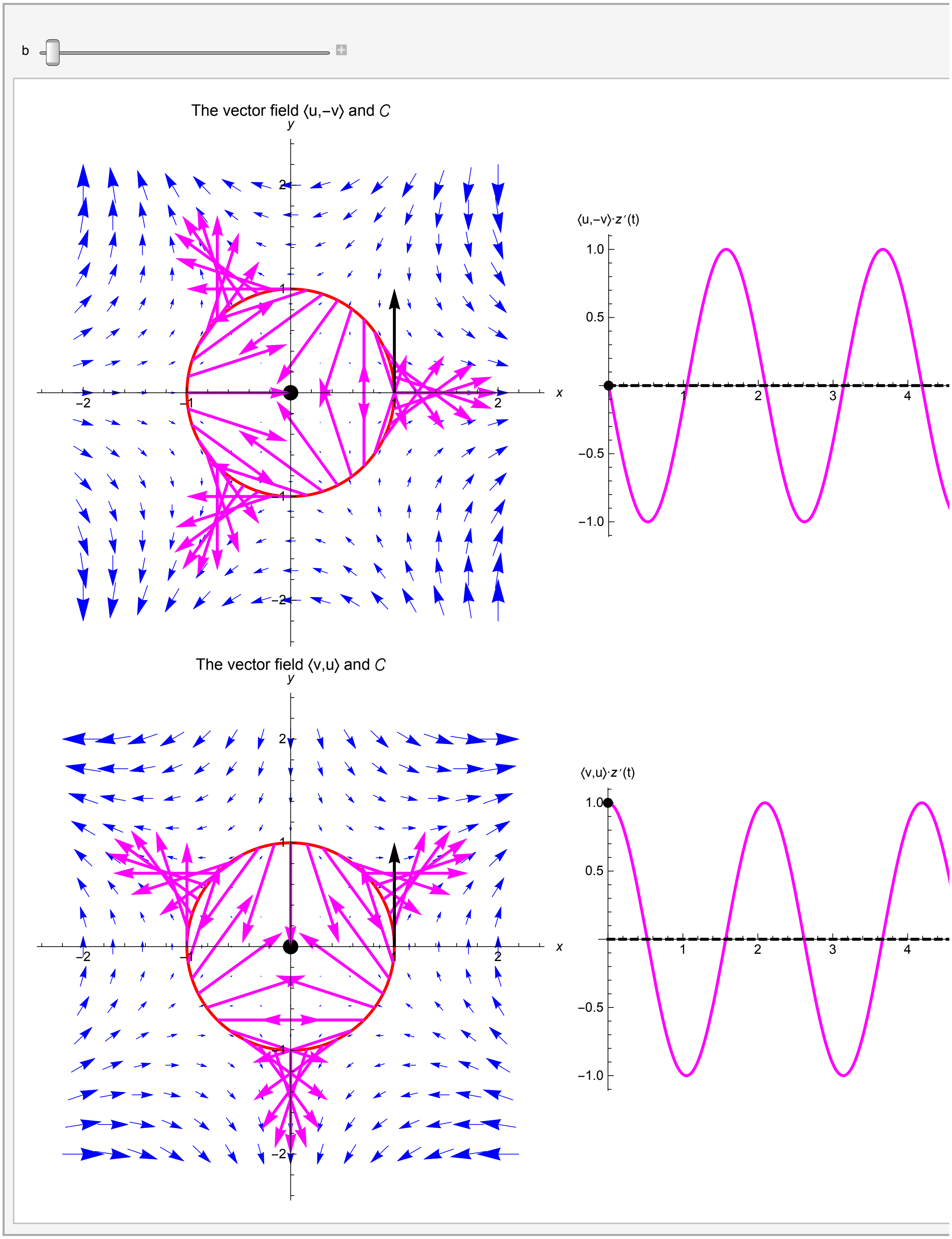


Complex Integration, viewed as $\int_C f(z)\, dz = \int_C \langle u, -v\rangle \cdot ds + i \int_C \langle v, u\rangle \cdot ds$

$f(z) = z^2$

```
In[12]:= u[x_, y_] := x^2 - y^2; v[x_, y_] := 2 x * y; x[t_] := Cos[t];
  y[t_] := Sin[t]; z[t_] := {x[t], y[t]}; uNegvBackGround =
   Show[VectorPlot[{u[x, y], -v[x, y]}, {x, -2, 2}, {y, -2, 2}, VectorStyle → Blue],
    ParametricPlot[z[t], {t, 0, 2 π}, PlotStyle → {Thick, Red}],
    ListPlot[{{0, 0}}, PlotStyle → {Black, PointSize[.03]}], Flatten[Table[Graphics[
       {Thick, Magenta, Arrow[{z[θ], z[θ] + {u[x[θ], y[θ]], -v[x[θ], y[θ]]}}]}],
      {θ, 0, 2 π, 2 π / 40}]], Frame → False, Axes → True, AxesLabel → {x, y},
    PlotLabel → "The vector field ⟨u,-v⟩ and C"]; vuBackGround =
   Show[VectorPlot[{v[x, y], u[x, y]}, {x, -2, 2}, {y, -2, 2}, VectorStyle → Blue],
    ParametricPlot[z[t], {t, 0, 2 π}, PlotStyle → {Thick, Red}],
    ListPlot[{{0, 0}}, PlotStyle → {Black, PointSize[.03]}], Flatten[Table[
      Graphics[{Thick, Magenta, Arrow[{z[θ], z[θ] + {v[x[θ], y[θ]], u[x[θ], y[θ]]}}]}],
      {θ, 0, 2 π, 2 π / 40}]], Frame → False, Axes → True, AxesLabel → {x, y},
    PlotLabel → "The vector field ⟨v,u⟩ and C"]; Manipulate[
   Grid[{{Show[uNegvBackGround, Graphics[{Thick, Black, Arrow[{z[b], z[b] + z'[b]}]}],
        ImageSize → Medium], Show[Plot[{u[x[t], y[t]], -v[x[t], y[t]]}.z'[t],
         {t, 0, 2 π}, PlotStyle → {Thick, Magenta}],
        ListPlot[{{b, {u[x[b], y[b]], -v[x[b], y[b]]}.z'[b]}}, PlotStyle → {Black,
           PointSize[.02]}], Graphics[{Thick, Black, Dashed, Line[{{0, 0}, {2 π, 0}}]}],
        ImageSize → Medium, AxesLabel → {"t", "⟨u,-v⟩·z′(t)"}]},
     {Show[vuBackGround, Graphics[{Thick, Black, Arrow[{z[b], z[b] + z'[b]}]}],
        ImageSize → Medium], Show[Plot[{v[x[t], y[t]], u[x[t], y[t]]}.z'[t],
         {t, 0, 2 π}, PlotStyle → {Thick, Magenta}],
        ListPlot[{{b, {v[x[b], y[b]], u[x[b], y[b]]}.z'[b]}}, PlotStyle → {Black,
           PointSize[.02]}], Graphics[{Thick, Black, Dashed, Line[{{0, 0}, {2 π, 0}}]}],
        ImageSize → Medium, AxesLabel → {"t", "⟨v,u⟩·z′(t)"}]}}], {b, 0, 2 π}]
```

Out[12]=



Complex Integration, viewed as

$\int_C f(z)\, dz =$, $\int_C (u + i\, v)\, dz = \Delta F = \Delta U + i\, \Delta V$ including the calculation of these changes as limits near branch cuts

$f(z) = \frac{1}{(z-i)(z-2)}$

```
In[13]:= f[z_] := 1/((z - I) * (z - 2)); ∫f[z] ⅆz
```

Out[13]= $\left(\frac{1}{5} - \frac{2\,i}{5}\right) \text{ArcTan}\left[\frac{-2+z}{1+2\,z}\right] + \left(\frac{1}{5} + \frac{i}{10}\right) \text{Log}\left[(2-z)^2\right] - \left(\frac{1}{5} + \frac{i}{10}\right) \text{Log}\left[1+z^2\right]$

In[14]:=
```
z1[t_] := I + E^(I * t); z2[t_] := 2 + E^(I * t); z3[t_] := 3 E^(I * t);

Residue[f[z], {z, I}]

2 π * ⅈ * Residue[f[z], {z, I}]

Residue[f[z], {z, 2}]

2 π * ⅈ * Residue[f[z], {z, 2}]

N[2 π * ⅈ * Residue[f[z], {z, I}]]
NIntegrate[f[z1[t]] * z1'[t], {t, 0, 2 π}]

N[2 π * ⅈ * Residue[f[z], {z, 2}]]
NIntegrate[f[z2[t]] * z2'[t], {t, 0, 2 π}]

NIntegrate[f[z3[t]] * z3'[t], {t, 0, 2 π}]
```

In[15]:=
```
z1[t_] := I + E^(I * t); z2[t_] := 2 + E^(I * t);
z3[t_] := 3 E^(I * t); f[z_] := 1/((z - I) * (z - 2));
F[z_] := (1/5 - 2 ⅈ/5) ArcTan[(-2 + z)/(1 + 2 z)] + (1/5 + ⅈ/10) Log[(2 - z)^2] - (1/5 + ⅈ/10) Log[1 + z^2];
Grid[{{Show[ParametricPlot3D[{{t, 0, 0}, {0, t, 0}, {0, 0, t}}, {t, -5, 5},
      PlotStyle → Thick], Plot3D[Re[F[x + I * y]], {x, -4, 4}, {y, -4, 4}],
    ParametricPlot3D[{{Re[z1[t]], Im[z1[t]], Re[F[z1[t]]]},
       {Re[z2[t]], Im[z2[t]], Re[F[z2[t]]]}, {Re[z3[t]], Im[z3[t]], Re[F[z3[t]]]}},
      {t, 0, 2 π}, PlotStyle → {{Thickness[.02], Red}, {Thickness[.02], Blue},
        {Thickness[.02], Darker[Green]}}], ImageSize → Medium,
    PlotRange → {{-4, 4}, {-4, 4}, {-1, 1}}, ViewPoint → {2, 1, 1},
    AxesLabel → {Text[Style["Re(z)", Large]], Text[Style["Im(z)", Large]],
      Text[Style["Re(F(z))", Large]]}, BoxRatios → {1, 1, 1}],
   Show[ParametricPlot3D[{{t, 0, 0}, {0, t, 0}, {0, 0, t}}, {t, -5, 5},
      PlotStyle → Thick], Plot3D[Im[F[x + I * y]], {x, -4, 4}, {y, -4, 4}],
    ParametricPlot3D[{{Re[z1[t]], Im[z1[t]], Im[F[z1[t]]]},
       {Re[z2[t]], Im[z2[t]], Im[F[z2[t]]]}, {Re[z3[t]], Im[z3[t]], Im[F[z3[t]]]}},
      {t, 0, 2 π}, PlotStyle -> {{Thickness[.02], Red}, {Thickness[.02], Blue},
        {Thickness[.02], Darker[Green]}}], ImageSize → Medium,
    PlotRange → {{-4, 4}, {-4, 4}, {-1.5, 1.5}}, ViewPoint → {2, 1, 1},
    AxesLabel → {Text[Style["Re(z)", Large]], Text[Style["Im(z)", Large]],
      Text[Style["Im(F(z))", Large]]}, BoxRatios → {1, 1, 1}]}}]
```

Out[15]=