Complex Number Addition and the Complex (Argand) Plane, Activity 4

Learning Goals:

1) Learn about the algebraic and geometric meanings of complex conjugation and the modulus of a complex number, as well as the relationships between these.

2) Gain algebraic and geometric understandings of the triangle inequality.

3) Use the *Mathematica* functions Dot (usually represented as a period infix operator "."), ReflectionMatrix, Manipulate, Locator, Abs.

Prerequisites:

1) Familiarity with vectors, linear algebra, and linear transformations

2) Comfort with geometric interpretations of absolute value

3) *Mathematica* content from Activities 1, 2, and 3 of this module

Introduction:

A fundamental part of geometry is the study of transformations that preserve various properties of geometric objects. An important class of transformations includes the so-called isometries, which preserve distances along lines and curves. Isometries include the familiar "rigid" translations and rotations, but they also include the slightly less familiar reflections. In the plane, reflections include transformations that map every point to its "mirror image" across some line. They also include transformations that map every point to its "mirror image" across from some point, although such a transformation can be though of more simply as a 180˚ rotation about that point. In the context of complex analysis, given a point $z \in \mathbb{C}$, its reflection across the horizontal (real) axis is given a special symbol $\overline{z}$ and is called the complex conugate of z. It plays a special role in complex analysis, and we will see that role get played out in this activity culminating in a proof of the fact that the shortest distance between two points is along the straight line segment between them.

Content:

The geometric concept of a reflection is useful in many mathematical contexts, including tests for symmetries in graphs of functions (which, in turn, can lead to simpler calculations in important applied math subjects like Fourier series) and applications of group theory to atoms, molecules, and solids (and to the sub-field of crystallography). In two-dimensional rectangular coordinate geometry, the simplest types of reflections are to reflect across the horizontal axis, to reflect across the vertical axis, and to reflect through the origin — though this last one is a different type than the first two. The reflection that maps $z = a + b\,i$ to its additive inverse $-z = -a - b\,i$ (denoted by $z \mapsto -z$ or $a + b\,i \mapsto -a - b\,i$) is a reflection through the origin. The reflection that maps z across the real (horizontal) axis is denoted by $z \mapsto \overline{z}$ or $a + b\,i \mapsto \overline{a + b\,i} = a - b\,i$ and is called ***complex conjugation*** — $\overline{z} = a - b\,i$ is called the complex conjugate of $z = a + b\,i$. The reflection that maps z across the imaginary (vertical) axis would take $a + b\,i$ and map it to $-a + b\,i$, which can be viewed as the composite transformation $z \mapsto \overline{z} \mapsto -\overline{z}$. Evidently this is the same thing as the composition in the reverse order $z \mapsto -z \mapsto \overline{-z}$, implying that $-\overline{z} = \overline{-z}$. In *Mathematica*, complex conjugation can be done for *particular* complex numbers with the function Conjugate and we can numerically confirm this last equality by entering the code in the following cell.

```
Manipulate[{a + b * I, -Conjugate[a + b * I], Conjugate[-(a + b * I)]},
 {{a, 2}, -4, 4}, {{b, 3}, -4, 4}, LabelStyle → Large]
```

The parallelogram law is *not the only reason* it is useful to view complex numbers as vectors, it can also be useful if we think of reflections and other kinds of linear transformations in terms of ideas from linear algebra. To be more specific, if we view the complex number $z = a + b\,i$ as a column vector $z = \begin{pmatrix} a \\ b \end{pmatrix}$, then reflections can be viewed as linear transformations that map $z = \begin{pmatrix} a \\ b \end{pmatrix}$ to a vector of the form $R\,z = R\begin{pmatrix} a \\ b \end{pmatrix}$, where R is some 2×2 matrix and $R\begin{pmatrix} a \\ b \end{pmatrix}$ represents the 2×2 matrix R times the 2×1 column vector $\begin{pmatrix} a \\ b \end{pmatrix}$, with the multiplication defined in the usual way.

For example, *reflection across the horizontal axis (i.e., complex conjugation)* can be viewed as $\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} a \\ -b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$, so that the reflection matrix in this situation is $R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. And *reflection across the vertical axis* can be viewed as $\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} -a \\ b \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$, so that the reflection matrix in this situation is $R = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. *Reflection through the origin* can be viewed as $\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} -a \\ -b \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$, so that the "reflection matrix" in this situation is $R = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ (actually, this is more commonly called a "rotation matrix" — see the introduction). The *Mathematica* function ReflectionMatrix can confirm the first two results. In general, the syntax **ReflectionMatrix[{a,b}]** produces a 2×2 matrix representing a reflection across a line perpendicular to the vector $\begin{pmatrix} a \\ b \end{pmatrix}$ (i.e., parallel to the vector $\begin{pmatrix} b \\ -a \end{pmatrix}$) though, as with Conjugate above, this seems to work best for *particular* (real) numbers a and b. MatrixForm is an output-formatting command that allows us to see the output as an actual 2×2 matrix (rather than a list of lists). Because reflection through the origin is not reflection across any particular line, we cannot use ReflectionMatrix to confirm our answer for that case (though the matrix used above for that case is still correct). We can, however, confirm our answers for the last two cases above. Reflection across the horizontal axis is reflection across a line perpendicular to the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$:

```
MatrixForm[ReflectionMatrix[{0, 1}]]
```

Reflection across the vertical axis is reflection across a line perpendicular to the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (here, MatrixForm is used "after the fact" using postfix notation with the Postfix operator //. This is often done when the application of a function in *Mathematica* is an "afterthought".)

```
ReflectionMatrix[{1, 0}] // MatrixForm
```

As already mentioned, reflection through the origin is really a rotation by 180° = π radians about the origin. We can use RotationMatrix to confirm this.

```
RotationMatrix[π] // MatrixForm
RotationMatrix[180 Degree] // MatrixForm
```

We can algebraically implement complex conjugation as a mapping $z = \begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ -b \end{pmatrix} = \bar{z}$ using the Dot infix operator (period) "." to do the matrix multiplication:

```
MatrixForm[(1 0; 0 -1).(a; b)]
```

$$\texttt{MatrixForm}\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \begin{pmatrix} a \\ b \end{pmatrix}\right]$$

Technically speaking the output of the code $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \begin{pmatrix} a \\ b \end{pmatrix}$ (without MatrixForm) is ***not*** the list {*a*, −*b*} whose entries are the numbers *a* and −*b*. Rather, it is the list {{*a*}, {−*b*}} whose entries are the lists {*a*} and {−*b*} — which is *Mathematica*'s way of representing the column matrix $\begin{pmatrix} a \\ -b \end{pmatrix}$ in terms of lists.

Again, note that MatrixForm is not being used below.

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \begin{pmatrix} a \\ b \end{pmatrix}$$

This might seem like a minor problem, but we do run into trouble with it if we want to plot points with ListPlot. There are two ways around this problem. One thing we can do is use Flatten to rid the list {{*a*}, {−*b*}} of its "inner lists", leaving just the numbers *a* and −*b* behind for the entries of the new list.

$$\texttt{Flatten}\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \begin{pmatrix} a \\ b \end{pmatrix}\right]$$

Alternatively, if we input the column matrix $\begin{pmatrix} a \\ b \end{pmatrix}$ as the *actual list* {*a*, *b*} and use the infix Dot operator (period) "." in the same kind of way as before, *Mathematica* still does the correct matrix/vector multiplication, but returns the (unformatted) answer as the list {*a*, −*b*}.

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \{a, b\}$$

Discussion 1: Describe other kinds of geometric transformations/motions of the plane — that map a given geometric object to another object that is either congruent to or similar to the original object. You can describe them verbally but you should also be more mathematically precise in your descriptions. For example, you could describe what happens to the complex numbers (vectors) $z = 1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $z = i = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ under these transformations. Is this last description sufficient to completely describe how the transformation acts on every point/vector? Is the transformation linear (does it have an associated matrix and map the zero vector to itself)? Is it affine (another type of transformation that has an associated matrix as part of its description)? Does it preserve distances? You may want to consider experimenting with the *Mathematica* function RotationMatrix to help you give a fuller description.

Response 1:

(You can type your thoughts and answers here formatted in text mode)

Grader/Instructor Response 1:

(The grader/instructor will give you feedback about your work here)

We can now use the *Mathematica* knowledge gained from the previous activities in this educational module on complex addition and the complex plane to produce a locator-enabled animation showing the geometric effect of conjugation on points in the complex plane (previously we called such a thing "cursor-enabled", but from now on we will call it "locator-enabled"). Study the code below, especially in light of the content related to Locator from Activity 3, to gain understanding and help you do the exercise further below. Note that the infix star operator * multiplies each element of a list by the same number (we've used this before).

```
R = (1  0
     0 -1);
Manipulate[Show[Graphics[{Thick, Red, Arrow[{{0, 0}, z}], Blue, Arrow[{{0, 0}, R.z}],
    Text[Style["z", Red, Large, Italic], 1.2 * z],
    Text[Style["z̅", Blue, Large, Italic], 1.2 * R.z]}],
  ListPlot[{z, R.z}, PlotStyle → {Black, PointSize[.02]}],
  Axes → True, AxesStyle → Large, TicksStyle → Black,
  PlotRange → 5, AspectRatio → Automatic, AxesLabel →
   {Text[Style["real", Large, Italic]], Text[Style["imaginary", Large, Italic]]},
  ImageSize → Large], {{z, {1, 2}}, Locator}]
```

If we want, we can even get fancy and include another animation parameter, call it λ, that allows us to actually visualize the reflection of z to $\bar{z}$ as λ varies from 0 to 1. This takes some creativity to construct. Do some research on how to use the conditional command If and the formatting command Dashing if you have the desire to understand it. We will have occasion to create similar animations in future modules, so it is worth your time to study. Either way, you can enjoy the dynamic output that occurs when you open up the λ-slider and play the animation.

```
R = (1  0
     0 -1); Manipulate[
 Show[Graphics[{Thick, Red, Arrow[{{0, 0}, z}], If[λ < .95, Dashing[.01], Dashing[1]],
    Blue, Arrow[{{0, 0}, (1 - λ) * z + λ * R.z}], Text[Style["z", Red, Large, Italic],
     1.2 * z], Text[Style["z̅", Blue, Large, Italic],
     If[λ < .95, 10^(6) * R.z, 1.2 * ((1 - λ) * z + λ * R.z)]]}],
  ListPlot[{z, (1 - λ) * z + λ * R.z}, PlotStyle → {Black, PointSize[.02]}],
  Axes → True, AxesStyle → Large, TicksStyle → Black,
  PlotRange → 5, AspectRatio → Automatic, AxesLabel →
   {Text[Style["real", Large, Italic]], Text[Style["imaginary", Large, Italic]]},
  ImageSize → Large], {{z, {1, 2}}, Locator}, {λ, 0, 1}, LabelStyle → Large]
```

***Mathematica* Exercise 1:** Make a locator-enabled animation showing the geometric effect of the mapping that sends z to its reflection $\overline{-z} = -\overline{z}$ with respect to the imaginary (vertical) axis. Label the points/vectors with z and $-\overline{z}$. As an optional part to this exercise, include an animation parameter that allows you to visualize the reflection as the parameter varies from 0 to 1.

Mathematica Work 1:

(Enter your code under this cell when *Mathematica* is in "Input mode" — make sure a horizontal line is showing before you start typing)

Grader/Instructor *Mathematica* Assessment 1:

(The grader/instructor will give you feedback about your work here)

Why is does the reflection across the real axis $z \mapsto \overline{z}$ play a more important role in complex analysis that reflection across the imaginary axis $z \mapsto \overline{-z} = -\overline{z}$? Evidently the first transformation must be more important since it gets its "own symbol" whereas the second transformation doesn't. Perhaps the most fundamental reason is algebraic: when z and $\overline{z}$ are multiplied, the result is a non-negative real number — moreover, it's a special non-negative real number, the square of the distance from the point z to the origin 0 (alternatively, the square of the length of z as a vector). It's true that $z + \overline{z}$ is always real as well, which is often useful to realize.

If we assume, as has already been mentioned, that we multiply complex numbers as if they are linear polynomials in the indeterminate i, replacing i^2 with -1, we get $z\,\overline{z} = (a + b\,i)\,(a - b\,i) = a^2 - a\,b\,i + a\,b\,i - b^2\,i^2 = a^2 + b^2 = \left(\sqrt{a^2 + b^2}\right)^2$. Since the absolute value of a real number measures its distance to the origin, it make sense to use the symbol $|z|$ to represent the distance between the point z and the origin in the complex plane — in other words, $|z| = |a + b\,i| = \sqrt{a^2 + b^2}$ by the Pythagorean Theorem. Hence, $z\,\overline{z} = |z|^2$. ***Don't ever forget this last equation!*** It's a basic fact that many students forget about, but it ends up being more useful than you might imagine — we will make use of it below. A couple more useful facts you will want to make sure you remember are $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ and $\overline{z_1\,z_2} = \overline{z_1}\,\overline{z_2}$, and you should take the time to verify these. The quantity $|z|$, which we could call the absolute value of the complex number z, is more commonly called the modulus of z — ***not*** to be confused with the modulo operation from number theory. Since *Mathematica* reserves Mod for the modulo operation from number theory, group theory, and ring theory, it uses Abs for the operation of finding the modulus of a complex number.

```
Abs[11 + 7 I]
Sqrt[11^2 + 7^2]
```

It is worth noting here that the double equal sign == can be thought of as an infix operator for the Equal function, which returns a Boolean value, either True or False, to test equality in many situations.

```
Equal[Abs[11 + 7 I], Sqrt[11^2 + 7^2]]
Abs[11 + 7 I] == Sqrt[11^2 + 7^2]
Sqrt[11^2 + 7^2] == 11 + 7
```

Since, when drawn in the standard position of the parallelogram rule, the complex numbers z_1, z_2, and $z_1 + z_2$ (as vectors) can be thought of as sides of a triangle, the moduli $|z_1|$, $|z_2|$, and $|z_1 + z_2|$ represent the lengths of the sides of that triangle.

Discussion 2: Suppose you are given two *arbitrary* complex numbers z_1 and z_2

and their sum $z_1 + z_2$, visualized in the way just described in the cell above. Is there a relationship between the moduli $|z_1|$, $|z_2|$, and $|z_1 + z_2|$ that is true no matter what z_1 and z_2 are? This relationship could be an equation, an inequality, or a set of inequalities. It also could involve some arithmetic operation(s). Draw pictures to help you form your hypothesis and then visually explain why it's true.

Response 2:

(You can type your thoughts and answers here formatted in text mode)

Grader/Instructor Response 2:

(The grader/instructor will give you feedback about your work here)

Since the shortest distance between any two points in a plane is the length of the straight line between them, *the length of any side of a triangle will be less than or equal to the sum of the lengths of the other two sides*, with equality only occurring when the triangle is "degenerate" — when it's really just a line segment because the three points all lie on the same line. Since the complex numbers (vectors) z_1, z_2, and $z_1 + z_2$ can be thought of as sides of a triangle, it follows that we can write $|z_1 + z_2| \leq |z_1| + |z_2|$. This inequality is aptly called the triangle inequality, and it is a very useful tool in complex analysis, both for estimation in various kinds of calculations and for proofs of theorems. Equality holds if and only if $z_2 = 0$ or $z_1 = c\, z_2$ for some $c \in \mathbb{R}$. For the mathematical analyst, the triangle inequality is like the American Express Card — they don't leave home without it — and they hope to use it well in order to avoid being called an infidel.

Are there symbolic ways of proving the triangle inequality? It's relatively easy if the z's are actually real numbers. Since $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$ for $x \in \mathbb{R}$, we can write $-|x_1| \leq x_1 \leq |x_1|$ and $-|x_2| \leq x_2 \leq |x_2|$ for all $x_1, x_2 \in \mathbb{R}$. Adding these inequalities leads to $-(|x_1| + |x_1|) = -|x_1| - |x_2| \leq x_1 + x_2 \leq |x_1| + |x_2|$, which means that $|x_1 + x_2| \leq |x_1| + |x_2|$ for all $x_1, x_2 \in \mathbb{R}$.

Now consider the complex case. It's definitely trickier. It's one of those proofs where you just have to keep pushing forward; doing various calculations and thinking carefully about what they mean. Suddenly, if you are persistent, the necessary flash of insight may finally hit you. This is a situation where it is handy to use the relations $|z|^2 = z\,\overline{z}$ and $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ mentioned above. Use the first of these on the quantity $|z_1 + z_2|^2$ to get, assuming that multiplication of complex number satisfies the distributive property, the equation $|z_1 + z_2|^2 = (z_1 + z_2)\left(\overline{z_1 + z_2}\right) = (z_1 + z_2)\left(\overline{z_1} + \overline{z_2}\right) = z_1\,\overline{z_1} + z_1\,\overline{z_2} + z_2\,\overline{z_1} + z_2\,\overline{z_2}$ $= |z_1|^2 + |z_2|^2 + z_1\,\overline{z_2} + z_2\,\overline{z_1}$. Evidently the expression $z_1\,\overline{z_2} + z_2\,\overline{z_1}$ is a real number since the various moduli in the last equation are real numbers (in fact, $z_2\,\overline{z_1} = \overline{z_1\,\overline{z_2}}$). If we can show that $z_1\,\overline{z_2} + z_2\,\overline{z_1} \leq 2\,|z_1|\,|z_2|$ we will be done since this fact will imply that $|z_1|^2 + |z_2|^2 + z_1\,\overline{z_2} + z_2\,\overline{z_1} \leq |z_1|^2 + 2\,|z_1|\,|z_2| + |z_2|^2 = (|z_1| + |z_2|)^2$ and we can take positive (real) square roots of both sides of the inequality $|z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2$ to finish the proof.

We can gain more insight by letting $z_1 = a + b\,i$ and $z_2 = c + d\,i$ to get
$z_1\,\overline{z_2} + z_2\,\overline{z_1} = (a + b\,i)(c - d\,i) + (c + d\,i)(a - b\,i)$
$= (a\,c + b\,d) + (-a\,d + b\,c)\,i + (a\,c + b\,d) + (-b\,c + a\,d)\,i = 2\,(a\,c + b\,d) = 2\,\mathrm{Re}(z_1\,\overline{z_2})$ — actually, we could have derived this from the fact that $z_2\,\overline{z_1} = \overline{z_1\,\overline{z_2}}$ without using the a, b, c, and d. Since $\mathrm{Re}(z) \leq |z|$ and $|\overline{z}| = |z|$ for any $z \in \mathbb{C}$ and since $|z_1\,z_2| = |z_1|\,|z_2|$ for any $z_1, z_2 \in \mathbb{C}$ (think about these statements), it follows that $z_1\,\overline{z_2} + z_2\,\overline{z_1} = 2\,\mathrm{Re}(z_1\,\overline{z_2}) \leq 2\,|z_1\,\overline{z_2}| = 2\,|z_1|\,|z_2|$. We are done. This proves

$|z_1 + z_2| \leq |z_1| + |z_2|$. Q.E.D.

***Mathematica* Exercise 2:** Make a locator-enabled animation showing the geometric meaning of the triangle inequality for various values of z_1 and z_2. Feel free to use your creativity to make this any way you want.

Mathematica Work 2:

(Enter your code under this cell when *Mathematica* is in “Input mode” — make sure a horizontal line is showing before you start typing)

Grader/Instructor *Mathematica* Assessment 2:

(The grader/instructor will give you feedback about your work here)

Since we know the difference of two complex numbers $z_1 - z_2$ can be visualized as a vector starting at the tip of z_2 and ending at the tip of z_1, when z_1 and z_2 are based at the same spot, we can also write what the triangle inequality means in this situation (enter the code below to remind yourself of the picture). One way it can be taken to mean is that $|z_1 - z_2| \leq |z_1| + |z_2|$. This inequality can also be thought about in a purely algebraic way using the form of the triangle inequality above as: $|z_1 - z_2| = |z_1 + (-z_2)| \leq |z_1| + |-z_2| = |z_1| + |z_2|$. The quantity $|z_1 - z_2|$ can be interpreted as the distance between z_1 and z_2 (as points). Hence, the distance between any two complex numbers, viewed as points, is less than or equal to the sum of their distances to the origin (the sum of the lengths of the complex numbers as vectors).

```
Manipulate[
 Show[Graphics[{Thick, Red, Arrow[{{0, 0}, z1}], LightRed, Arrow[{-z2, z1 - z2}],
    Blue, Arrow[{{0, 0}, z2}], Arrow[{z1 - z2, z1}], LightBlue,
    Arrow[{{0, 0}, -z2}], Green, Arrow[{z2, z1}], Arrow[{{0, 0}, z1 - z2}]}],
  Graphics[{Text[Style["z₂", Blue, Large, Italic], .5 * z2],
    Text[Style["z₂", Blue, Large, Italic], .5 z2 + (z1 - z2)],
    Text[Style["-z₂", LightBlue, Large, Italic], -.5 * z2],
    Text[Style["z₁", Red, Large], .5 * z1], Text[Style["z₁", LightRed, Large, Italic],
     .5 * z1 - z2], Text[Style["z₁-z₂", Green, Large, Italic], .5 * (z1 + z2)],
    Text[Style["z₁-z₂", Green, Large, Italic], .5 * (z1 - z2)]}],
  Axes → True, AxesLabel → {Text[Style["real", Large, Italic]],
    Text[Style["imaginary", Large, Italic]]}, TicksStyle → Large,
  PlotRange → {{-8.1, 8.1}, {-8.1, 8.1}}, ImageSize → Large],
 {{z1, {1, 3}}, Locator}, {{z2, {2, 2}}, Locator},
 LabelStyle → Large]
```

Alternative ways of viewing the triangle inequality in this situation that are sometimes useful are: $|z_1| = |(z_1 - z_2) + z_2| \leq |z_1 - z_2| + |z_2|$ and $|z_2| = |(z_2 - z_1) + z_1| \leq |z_1 - z_2| + |z_1|$.

■ Conclusion:

This ends the last activity of the learning module on complex addition and the complex plane. Hopefully you have gained insight into the natural geometric nature of complex numbers and the way that basic geometric facts can be expressed using complex numbers. These points of view and facts are fundamental to understanding complex analysis. Make sure they stay in your brain for the long haul by continually reminding yourself of them as you work more deeply into complex analysis.