

# Complex Number Addition and the Complex (Argand) Plane, Activity I

## ■ Learning Goals:

1) See that one “natural” view of complex numbers as linear “polynomials”, with real number coefficients, in the indeterminate  $i$  (forgetting, for the moment, that  $i^2 = -1$ ) leads to

(a) a definition of complex number addition that is reasonable algebraically and

(b) a reasonable geometric interpretation of complex numbers as points in a plane, analogous to the geometric interpretation of real numbers as points on a line.

2) Learn basic applications of the *Mathematica* functions ComplexExpand, ListPlot, Graphics, Table, and Show in this setting, as well as various *Mathematica* formatting options

## ■ Prerequisites:

1) Some comfort with abstract algebra, though not necessarily at the level of a course in abstract algebra

2) A basic familiarity with complex numbers in precalculus or calculus is helpful

3) Understanding of the way points are described in a rectangular (Cartesian) coordinate system

## ■ Introduction:

The arithmetic of complex numbers, which leads to deeper subject of complex analysis (calculus-related ideas and tools for understanding complex functions), can be thought about and worked through in purely symbolic rule-based (formal) ways. However, as with many areas of mathematics, the geometry of complex arithmetic gives the symbolic mathematical statements of fact richer meaning, leads to new insights, and informs possible real-world applications. This module, “Complex Number Addition and the

Complex (Argand) Plane”, provides an introduction to the marriage of the symbolic and the geometric within the study of complex arithmetic, providing a solid foundation for work in complex analysis. Activity 1 of this module focuses on the basic correspondence between a given complex number  $a + bi$  and the point  $z$  in the plane whose rectangular (Cartesian) coordinates are  $(a, b)$ . When this correspondence is specified and the axes are labeled with the words “real” (horizontal) and “imaginary” (vertical), the plane is then called “the” complex plane, or sometimes the Argand plane, in honor of the Swiss mathematician Jean-Robert Argand.

## ■ Content:

Complex numbers can be viewed in purely algebraic and symbolic ways, which is where we start in this first activity of a learning module on complex addition and the complex plane. However, our main goal in this learning module is to see that there are natural *geometric* ways of viewing complex numbers, and that this mode of thought is a powerful tool for understanding and problem-solving. This is most typically done with either rectangular or polar coordinates, and you will need to become very familiar with translating between these two coordinate systems if you want to be successful in understanding and using complex numbers and complex functions.

Algebraic expressions are combinations of symbols, such as  $x^2$ ,  $6x^2 + 5xy + y^3 + \frac{2}{3}x + z + 3z$ ,  $\frac{z}{5} - 11yx + 2x + x(4+x)$ , and  $\frac{x^2}{y} + \frac{x^3}{x}$ , that combine numbers and variables using operations from arithmetic, such as addition, subtraction, multiplication, and division — other operations that are sometimes allowed include things such as square roots, cube roots, and more exotic kinds of mathematical objects. The variables can be thought of as pure symbols with no meaning, but we usually think of them as representing numbers, and we usually are interested in determining what happens to the values of these expressions as the variables are allowed to change, or “vary”. In other words, we typically want to think of them as defining functions and then use our mathematical knowledge and tools to analyze the behavior of those functions. In mathematics classes, this is often a purely mental exercise, but it forms the foundation for nearly all advanced applications of mathematics, including in the physical sciences, the social sciences, computer science, medicine, engineering, finance, and statistics.

### Discussion 1: How should the expressions

$x^2$ ,  $6x^2 + 5xy + y^3 + \frac{2}{3}x - z + 3z$ ,  $\frac{z}{5} - 11yx + 2x + x(4+x)$ , and  $\frac{x^2}{y} + \frac{x^3}{x}$  be simplified, if at all? How should arithmetic (addition, subtraction, multiplication, and division) be performed and simplification be done? (**Hints:** recall what “like terms” and “common denominators” are. Also recall the distributive property and the rules for exponents). Write down some sample computations and simplifications using the given expressions. How does this work correspond to the properties of the arithmetic of the real numbers? For example, are you using the commutative property for addition or multiplication anywhere?

### Response 1:

(You can type your thoughts and answers here formatted in text mode)

## Grader/Instructor Response I:

**(The grader/instructor will give you feedback about your work here)**

The main point of Discussion 1 is that we do indeed often treat expressions as if the variables represent numbers; and indeed, we often use properties of real numbers to simplify these expressions and arithmetic combinations of these expressions. If you are at a point in your mathematical education where you can do such operations without much thought, that is a sign that you have internalized much of your pre-college mathematics well.

*Mathematica* can do purely symbolic arithmetic, based on these assumptions, though sometimes we must use the built-in symbolic manipulation functions [Expand](#), [Simplify](#), [FullSimplify](#), [Together](#), [Apart](#), or [Factor](#) to get the answer into a form that we want. Enter each of the following three lines of code to see examples of this.

$$\frac{x^2}{y} + \frac{x^3}{x}$$

$$\text{Together}\left[\frac{x^2}{y} + \frac{x^3}{x}\right]$$

$$\text{Simplify}\left[x^2 / (x^2 / y + x^3 / x)\right]$$

**Mathematica Exercise I:** Experiment with the arithmetic operations  $+$ ,  $-$ ,  $*$ ,  $/$  and the built-in symbolic manipulation functions mentioned in the preceding paragraph on the expressions in Discussion I to determine what these *Mathematica* functions do. Check *Mathematica*'s answers by hand. Are *Mathematica*'s answers always in the form you would leave your answers?

### Mathematica Work I:

**(Enter your code under this cell when *Mathematica* is in "Input mode" — make sure a horizontal line is showing before you start typing)**

**(You can type your thoughts and answers here formatted in text mode)**

## Grader/Instructor *Mathematica* Assessment I:

**(The grader/instructor will give you feedback about your work here)**

Initially, on a purely symbolic level, any **complex number**, such as  $4 + 3i$ , can be thought of as just an expression in the "variable"  $i$ , completely ignoring the fact that  $i^2 = -1$ . The meaning of the word "variable" here is a bit of a misnomer, however. We don't want to allow  $i$  to take on different values. It might be better to use a different word for the role that  $i$  plays here; mathematicians often use the word "indeterminate" instead.

**Discussion 2:** Based on the view of a complex number  $a + b\bar{i}$ , where  $a$  and  $b$  are **real numbers**, as just an expression in the indeterminate  $\bar{i}$ , describe how the addition of two or more complex numbers should be done. Give examples. Write down a general formula for the sum of two arbitrary complex numbers,

$a + b i$  and  $c + d i$ , where  $a, b, c,$  and  $d$  are arbitrary real numbers (often written  $a, b, c, d \in \mathbb{R}$  and read “ $a, b, c,$  and  $d$  are all elements of the set of real numbers  $\mathbb{R}$ ”). Does your definition of complex number addition satisfy the **associative property**? The **commutative property**? Why or why not? Base your answer to this last question on related facts about real number arithmetic. What role does the complex number  $0 + 0 i$  play? For a given complex number,  $a + b i$ , what can you say about the related complex number  $(-a) + (-b) i$ ? Is the set of real numbers  $\mathbb{R}$  a subset of the set of complex numbers  $\mathbb{C}$ ? If not, is there a “natural” way to “associate”  $\mathbb{R}$  with some subset of  $\mathbb{C}$ . Are there other “natural ways” to create such an association? Is one association superior to the others? Are these even good questions to ask in the first place?

### Response 2:

(You can type your thoughts and answers here formatted in text mode)

### Grader/Instructor Response 2:

(The grader/instructor will give you feedback about your work here)

A capital **I** is the most basic way to represent the imaginary unit  $i$  in input/output mode in *Mathematica*. [ComplexExpand](#) can be used to confirm or refute your ideas in Discussion 2 above.

```
ComplexExpand[(3 + 2 I) + (4 + 2 I)]
```

```
ComplexExpand[(a + b * I) + (c + d * I)]
```

Do you have any ideas about how the multiplication of two arbitrary complex numbers  $a + b i$  and  $c + d i$  should be done? Make a conjecture and then see what *Mathematica* gives for the answer.

```
ComplexExpand[(a + b * I) * (c + d * I)]
```

You can also check that the **distributive property** works in this setting. The following cell contains two lines of input code that will generate two outputs.

```
ComplexExpand[(a + b * I) * ((c + d * I) + (e + f * I))]
```

```
ComplexExpand[(a + b * I) * (c + d * I) + (a + b * I) * (e + f * I)]
```

Since a complex number  $a + b i$  is uniquely determined by the values of  $a$  and  $b$ , as well as their role in this expression — next to the  $i$  or not — it is natural to associate the complex number  $a + b i$  (which can also be represented as  $b i + a$ ) with the **ordered pair**  $(a, b)$  (which should **not** be written as  $(b, a)$ ). When we impose a **Cartesian (rectangular) coordinate system** on a plane, we can use these ordered pairs to visualize the complex numbers as points in this plane in a standard way. The “**real part**”  $a$ , of  $a + b i$  —the first coordinate of  $(a, b)$  — specifies the horizontal (right/left) perpendicular displacement, with respect to the vertical axis, of the number  $a + b i$  as a point in the plane; with positive displacement being to the right and negative displacement being to the left. The “**imaginary part**”  $b$ , of  $a + b i$  —the second coordinate of  $(a, b)$  — specifies the vertical (up/down) perpendicular displacement, with respect to the horizontal axis, of the number  $a + b i$  as a point in the plane; with positive displacement being upward and negative displacement being downward. When a complex number is plotted in the given plane in this way, we call this plane “the” **Complex (Argand) Plane**; we label the horizontal axis to be the **real axis** and we label the vertical axis to be the **imaginary axis**. These things can just be thought of as

labels, and no philosophical objections about the deeper nature of what we are doing need be sustained. Questions about applicability to [the real world](#), however, do need to eventually be addressed.

In *Mathematica*, the ordered pair  $(a, b)$  is typically represented as a “list” with two entries, or elements:  $\{a, b\}$  (see a summary page of many manipulations that can be done, in a very general setting, with all kinds of [lists in Mathematica](#)). These points can then be plotted with [ListPlot](#). For instance, we can, as done in [Figure \[1\] \(Ch. 1.1.1, p. 2\) of Tristan Needham's Visual Complex Analysis](#), plot the complex numbers  $4 = 4 + 0i$ ,  $4 + 3i$ ,  $3i = 0 + 3i$ ,  $-7 + i = (-7) + 1i$ ,  $-2 - 3i = (-2) + (-3)i$ , and  $2 - 2i = 2 + (-2)i$  using the code to follow. For efficiency's sake, we put these lists (of two elements each) in a “list of lists”  $\{\{4, 0\}, \{4, 3\}, \{0, 3\}, \{-7, 1\}, \{-2, -3\}, \{2, -2\}\}$  (with six total elements) in order to plot them all at once. Note also that the options [PlotStyle](#), [PlotRange](#), and [AxesLabel](#), can be used to, respectively, make the dots black and larger than normal, choose the plotting window, and label the axes. Note also the use of “**InitialFigure1** =” at the beginning of the line will store the resulting output graph in a variable we have named **InitialFigure1**.

```
InitialFigure1 = ListPlot[{{4, 0}, {4, 3}, {0, 3}, {-7, 1}, {-2, -3}, {2, -2}},
  PlotStyle -> {Black, PointSize[.02]},
  PlotRange -> {{-9, 9}, {-4, 4}}, AxesLabel -> {"real", "imaginary"}]
```

After the preceding line has been entered, the graph can be displayed again by just entering its name **InitialFigure1**.

```
InitialFigure1
```

The preceding line of code will not produce the desired output after we quit *Mathematica's kernel*, either through the **Evaluation menu** or by quitting the program.

Another option, [GridLines](#), can be used to create a standard rectangular grid in the background of the picture. The function [Show](#) combines the graphic outputs into one output and the option [AspectRatio](#) can be used to make the scales on the axes the same.

```
Show[InitialFigure1,
  GridLines -> {{-9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9},
  {-4, -3, -2, -1, 0, 1, 2, 3, 4}}, AspectRatio -> Automatic]
```

When there are many grid-lines, the locations of the grid-lines can be generated more efficiently with [Table](#), which is the basic list-generating command in *Mathematica* when the elements of a list have a “formula”. For example, **Table[ $i$ , { $i$ , -9, 9}]** generates the list  $\{-9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ .

```
Table[i, {i, -9, 9}]
```

As a related example, **Table[ $i^2$ , { $i$ , -9, 9}]** makes a list of the squares of the numbers in the preceding list.

```
Table[i^2, {i, -9, 9}]
```

Note that lists in *Mathematica* are not the same as sets in Mathematics. The preceding list is **not** equivalent to the *mathematical set*  $\{0, 1, 4, 9, 16, 25, 36, 49, 64, 81\}$ .

[Table](#) is very flexible. It can also generate, for example, lists of points,

```
Table[{i^3, i^4}, {i, -5, 5}]
```

or even lists of graphs.

```
Table[Plot[x^(i), {x, 0, 1}], {i, .5, 2, .5}]
```

We now have a way to create the graph of the complex numbers in the complex plane more efficiently (especially through the use of “Copy & Paste” to save typing time).

```
Show[InitialFigure1,
  GridLines → {Table[i, {i, -9, 9}], Table[i, {i, -4, 4}]}, AspectRatio → Automatic]
```

Finally, we may combine the built-in functions `Graphics` and `Text` to label these points in a way that will be useful, not just for producing a nice static output picture, but also for producing animations where the labeling of a point can move with the point while the animation is occurring. We have also included the option `AxesStyle` and the graphics directive `Thick` to make the axes thicker.

```
Show[InitialFigure1,
  Graphics[{Text[4, {4.5, .5}], Text[4 + 3 i, {4.5, 3.5}], Text[3 i, {.5, 3.5}],
    Text[-7 + i, {-7.3, 1.5}], Text[-2 - 3 i, {-2.5, -2.5}], Text[2 - 2 i, {2.5, -1.5}]}],
  GridLines → {Table[i, {i, -9, 9}], Table[i, {i, -4, 4}]},
  AxesStyle → Thick, AspectRatio → Automatic]
```

**Mathematica Exercise 2:** Use *Mathematica* to make a plot, in the complex plane, of the points  $2 - 3i$ ,  $-5 + 8i$ ,  $1 + 2i$ ,  $-4i$ , and  $10$ . Make sure your axes are labeled appropriately, your window is chosen well, your points are larger than the default (and make them colored `Red`), the grid-lines are included, the scales on the axes are the same, the axes are thick, and the points are labeled. Experiment with your code to strive to make it as efficient as possible (that is, as short as possible — and possibly use just one cell of code rather than multiple cells). Note that a semicolon “;” will suppress output in *Mathematica* but can also be used to separate distinct lines of code within the same cell if you wish.

### Mathematica Work 2:

(Enter your code under this cell when *Mathematica* is in “Input mode” — make sure a horizontal line is showing before you start typing)

### Grader/Instructor Mathematica Assessment 2:

(The grader/instructor will give you feedback about your work here)

In Activity 2 of this learning module on complex number addition and the complex plane, we will consider the geometric importance of this view for complex number addition.