

Online Appendix for
“On Moments of Folded and Truncated Multivariate
Normal Distributions”

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This online appendix contains some supplementary results that are referred to in the article. Sections 1 and 2 present the explicit expressions for some low order moments of folded and truncated multivariate normal distributions, respectively. We refer the reader to the article for the notation used here.

1 Folded Multivariate Normal: Explicit Expressions for Low Order Moments

The recurrence relation for $I_{\kappa}^n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ in the article can be used to obtain explicit expressions for the product moments of \mathbf{Y} . In the following, we provide explicit expressions for $E(\mathbf{Y}^{\kappa})$ up to the fourth order, i.e., $\sum_{i=1}^n k_i \leq 4$. In our expressions, we assume $\sigma_1 = \dots = \sigma_n = 1$. This implies that $\boldsymbol{\Sigma} = \mathbf{R}$, where $\mathbf{R} = (\rho_{ij})$ is the correlation matrix of \mathbf{X} , with $\rho_{ij} = \sigma_{ij}/(\sigma_i\sigma_j)$. For the general $\boldsymbol{\Sigma}$ case, we just need to replace μ_i in our expressions with μ_i/σ_i , and then multiply the result by $\sigma_1^{k_1} \dots \sigma_n^{k_n}$.

For univariate moments, Winkelbauer (2012) shows that

$$E(Y_i^k) = \frac{2^{\frac{k}{2}} \Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi}} {}_1F_1\left(-\frac{k}{2}; \frac{1}{2}; -\frac{\mu_i^2}{2}\right),$$

where ${}_1F_1(a; b; z)$ is the confluent hypergeometric function. It follows that the first four moments of Y_i are

$$\begin{aligned} E(Y_i) &= \mu_i \operatorname{erf}\left(\frac{\mu_i}{\sqrt{2}}\right) + 2\phi(\mu_i), \\ E(Y_i^2) &= 1 + \mu_i^2, \\ E(Y_i^3) &= \mu_i(3 + \mu_i^2) \operatorname{erf}\left(\frac{\mu_i}{\sqrt{2}}\right) + (4 + 2\mu_i^2)\phi(\mu_i), \\ E(Y_i^4) &= 3 + 6\mu_i^2 + \mu_i^4. \end{aligned}$$

Using the recurrence relation of confluent hypergeometric functions, higher order moments of Y_i can be obtained using¹

$$E(Y_i^k) = (\mu_i^2 + 2k - 3)E(Y_i^{k-2}) - (k-2)(k-3)E(Y_i^{k-4}) \quad (k \geq 4).$$

¹Elandt (1961) expresses the higher order moments of Y_i in terms of Fisher's I_k functions, and her expression (Eq. 8) is less efficient than ours.

For bivariate moments, we define $z_{i.j} = (\mu_i - \rho_{ij}\mu_j)/(1 - \rho_{ij}^2)^{1/2}$ and use (5) in the article repeatedly to obtain

$$\begin{aligned}
I_{(1,1)}^2((\mu_i, \mu_j)^T, \rho_{ij}) &= (\mu_i\mu_j + \rho_{ij})\Phi_2(\mu_i, \mu_j; \rho_{ij}) + \mu_i\phi(\mu_j)\Phi(z_{i.j}) \\
&\quad + \mu_j\phi(\mu_i)\Phi(z_{j.i}) + (1 - \rho_{ij}^2)\phi_2(\mu_i, \mu_j; \rho_{ij}), \\
I_{(2,1)}^2((\mu_i, \mu_j)^T, \rho_{ij}) &= \{(1 + \mu_i^2)\mu_j + 2\rho_{ij}\mu_i\}\Phi_2(\mu_i, \mu_j; \rho_{ij}) + (\mu_i\mu_j + 2\rho_{ij})\phi(\mu_i)\Phi(z_{j.i}) \\
&\quad + (1 + \mu_i^2 + \rho_{ij}^2)\phi(\mu_j)\Phi(z_{i.j}) + \mu_i(1 - \rho_{ij}^2)\phi_2(\mu_i, \mu_j; \rho_{ij}), \\
I_{(3,1)}^2((\mu_i, \mu_j)^T, \rho_{ij}) &= \{\mu_i\mu_j(3 + \mu_i^2) + 3\rho_{ij}(1 + \mu_i^2)\}\Phi_2(\mu_i, \mu_j; \rho_{ij}) \\
&\quad + \{(2 + \mu_i^2)\mu_j + 3\mu_i\rho_{ij}\}\phi(\mu_i)\Phi(z_{j.i}) \\
&\quad + \{\mu_i^3 + 3\mu_i(1 + \rho_{ij}^2) - \mu_j\rho_{ij}^3\}\phi(\mu_j)\Phi(z_{i.j}) \\
&\quad + (2 + \mu_i^2 + \rho_{ij}^2)(1 - \rho_{ij}^2)\phi_2(\mu_i, \mu_j; \rho_{ij})
\end{aligned}$$

for $i \neq j$. Summing up these terms for the four quadrants, i.e., with $(\mu_i, \mu_j, \rho_{ij})$ in the above expressions replaced by $(\mu_i, -\mu_j, -\rho_{ij})$, $(-\mu_i, \mu_j, -\rho_{ij})$, and $(-\mu_i, -\mu_j, \rho_{ij})$ in the other three quadrants, we obtain for $i \neq j$

$$\begin{aligned}
E(Y_i Y_j) &= (\mu_i\mu_j + \rho_{ij})p_2(\mu_i, \mu_j; \rho_{ij}) + 2\mu_i\phi(\mu_j)\operatorname{erf}\left(\frac{z_{i.j}}{\sqrt{2}}\right) \\
&\quad + 2\mu_j\phi(\mu_i)\operatorname{erf}\left(\frac{z_{j.i}}{\sqrt{2}}\right) + 4(1 - \rho_{ij}^2)\phi_2(\mu_i, \mu_j; \rho_{ij}), \\
E(Y_i^2 Y_j) &= \{(1 + \mu_i^2)\mu_j + 2\rho_{ij}\mu_i\}\operatorname{erf}\left(\frac{\mu_j}{\sqrt{2}}\right) + 2(1 + \mu_i^2 + \rho_{ij}^2)\phi(\mu_j) \\
E(Y_i^3 Y_j) &= \{\mu_i\mu_j(3 + \mu_i^2) + 3\rho_{ij}(1 + \mu_i^2)\}p_2(\mu_i, \mu_j; \rho_{ij}) \\
&\quad + 2\{(2 + \mu_i^2)\mu_j + 3\rho_{ij}\mu_i\}\phi(\mu_i)\operatorname{erf}\left(\frac{z_{j.i}}{\sqrt{2}}\right) \\
&\quad + 2\{\mu_i^3 + 3\mu_i(1 + \rho_{ij}^2) - \mu_j\rho_{ij}^3\}\phi(\mu_j)\operatorname{erf}\left(\frac{z_{i.j}}{\sqrt{2}}\right) \\
&\quad + 4(2 + \mu_i^2 + \rho_{ij}^2)(1 - \rho_{ij}^2)\phi_2(\mu_i, \mu_j; \rho_{ij}), \\
E(Y_i^2 Y_j^2) &= E(X_i^2 X_j^2) = (1 + \mu_i^2)(1 + \mu_j^2) + 4\mu_i\mu_j\rho_{ij} + 2\rho_{ij}^2,
\end{aligned}$$

where

$$p_2(\mu_i, \mu_j; \rho_{ij}) = 4\Phi_2(\mu_i, \mu_j; \rho_{ij}) - 2\Phi(\mu_i) - 2\Phi(\mu_j) + 1.$$

With the univariate and bivariate moments of \mathbf{Y} available, the expected value and covariance matrix of \mathbf{Y} for the general Σ case are

$$E(Y_i) = \mu_i \operatorname{erf}\left(\frac{\tilde{\mu}_i}{\sqrt{2}}\right) + 2\sigma_i\phi(\tilde{\mu}_i),$$

$$\begin{aligned}
\text{Var}(Y_i) &= \mu_i^2 + \sigma_i^2 - E(Y_i)^2, \\
\text{Cov}(Y_i, Y_j) &= (\mu_i \mu_j + \sigma_{ij}) \{4\Phi_2(\tilde{\mu}_i, \tilde{\mu}_j; \rho_{ij}) - 2\Phi(\tilde{\mu}_i) - 2\Phi(\tilde{\mu}_j) + 1\} \\
&\quad + 2\mu_i \sigma_j \phi(\tilde{\mu}_j) \text{erf}\left(\frac{\tilde{\mu}_i - \rho_{ij} \tilde{\mu}_j}{\sqrt{2}(1 - \rho_{ij}^2)^{\frac{1}{2}}}\right) + 2\mu_j \sigma_i \phi(\tilde{\mu}_i) \text{erf}\left(\frac{\tilde{\mu}_j - \rho_{ij} \tilde{\mu}_i}{\sqrt{2}(1 - \rho_{ij}^2)^{\frac{1}{2}}}\right) \\
&\quad + 4\sigma_i \sigma_j (1 - \rho_{ij}^2) \phi_2(\tilde{\mu}_i, \tilde{\mu}_j; \rho_{ij}) - E(Y_i)E(Y_j).
\end{aligned}$$

For trivariate moments, we define $z_{i \cdot jk} = (z_{i \cdot k} - \rho_{ij \cdot k} z_{j \cdot k}) / (1 - \rho_{ij \cdot k}^2)^{1/2}$, where $\rho_{ij \cdot k} = (\rho_{ij} - \rho_{ik} \rho_{jk}) / \{(1 - \rho_{ik}^2)(1 - \rho_{jk}^2)\}^{1/2}$. Let $\tilde{\boldsymbol{\mu}} = (\mu_i, \mu_j, \mu_k)^T$ and $\tilde{\mathbf{R}}$ be a 3 by 3 submatrix of \mathbf{R} that consists of the (i, j, k) th rows and columns of \mathbf{R} . Applying (5) in the article repeatedly, we obtain

$$\begin{aligned}
I_{(1,1,1)}^3(\tilde{\boldsymbol{\mu}}, \tilde{\mathbf{R}}) &= (\mu_i \mu_j \mu_k + \mu_i \rho_{jk} + \mu_j \rho_{ik} + \mu_k \rho_{ij}) \Phi_3(\tilde{\boldsymbol{\mu}}; \tilde{\mathbf{R}}) \\
&\quad + (\mu_j \mu_k + \rho_{ij} \rho_{ik} + \rho_{jk}) \phi(\mu_i) \Phi_2(z_{j \cdot i}, z_{k \cdot i}; \rho_{jk \cdot i}) \\
&\quad + (\mu_i \mu_k + \rho_{ij} \rho_{jk} + \rho_{ik}) \phi(\mu_j) \Phi_2(z_{i \cdot j}, z_{k \cdot j}; \rho_{ik \cdot j}) \\
&\quad + (\mu_i \mu_j + \rho_{ik} \rho_{jk} + \rho_{ij}) \phi(\mu_k) \Phi_2(z_{i \cdot k}, z_{j \cdot k}; \rho_{ij \cdot k}) \\
&\quad + \mu_i (1 - \rho_{jk}^2) \phi_2(\mu_j, \mu_k; \rho_{jk}) \Phi(z_{i \cdot jk}) \\
&\quad + \mu_j (1 - \rho_{ik}^2) \phi_2(\mu_i, \mu_k; \rho_{jk}) \Phi(z_{j \cdot ik}) \\
&\quad + \mu_k (1 - \rho_{ij}^2) \phi_2(\mu_i, \mu_j; \rho_{ij}) \Phi(z_{k \cdot ij}) + |\tilde{\mathbf{R}}| \phi_3(\tilde{\boldsymbol{\mu}}; \tilde{\mathbf{R}}), \\
I_{(2,1,1)}^3(\tilde{\boldsymbol{\mu}}, \tilde{\mathbf{R}}) &= \{(1 + \mu_i^2)(\mu_j \mu_k + \rho_{jk}) + 2\mu_i(\mu_j \rho_{ik} + 2\mu_k \rho_{ij}) + 2\rho_{ij} \rho_{ik}\} \Phi_3(\tilde{\boldsymbol{\mu}}; \tilde{\mathbf{R}}) \\
&\quad + \{\mu_i(\mu_j \mu_k + \rho_{jk}) + 2\mu_j \rho_{ik} + 2\mu_k \rho_{ij}\} \phi(\mu_i) \Phi_2(z_{j \cdot i}, z_{k \cdot i}; \rho_{jk \cdot i}) \\
&\quad + \{2\mu_i(\rho_{ik} + \rho_{ij} \rho_{jk}) + \mu_k(1 + \mu_i^2 + \rho_{ij}^2) - \mu_j \rho_{ij}^2 \rho_{jk}\} \phi(\mu_j) \Phi_2(z_{i \cdot j}, z_{k \cdot j}; \rho_{ik \cdot j}) \\
&\quad + \{2\mu_i(\rho_{ij} + \rho_{ik} \rho_{jk}) + \mu_j(1 + \mu_i^2 + \rho_{ik}^2) - \mu_k \rho_{ik}^2 \rho_{jk}\} \phi(\mu_k) \Phi_2(z_{i \cdot k}, z_{j \cdot k}; \rho_{ij \cdot k}) \\
&\quad + (\mu_i \mu_k + 2\rho_{ik} + \rho_{ij} \rho_{jk})(1 - \rho_{ij}^2) \phi_2(\mu_i, \mu_j; \rho_{ij}) \Phi(z_{k \cdot ij}) \\
&\quad + (\mu_i \mu_j + 2\rho_{ij} + \rho_{ik} \rho_{jk})(1 - \rho_{ik}^2) \phi_2(\mu_i, \mu_k; \rho_{ik}) \Phi(z_{j \cdot ik}) \\
&\quad + (1 + \mu_i^2 + \rho_{ij}^2 + \rho_{ik}^2)(1 - \rho_{jk}^2) \phi_2(\mu_j, \mu_k; \rho_{jk}) \Phi(z_{k \cdot ij}) \\
&\quad + \mu_i |\tilde{\mathbf{R}}| \phi_3(\tilde{\boldsymbol{\mu}}; \tilde{\mathbf{R}}),
\end{aligned}$$

where i, j, k are distinct positive integers. Summing up these terms over 8 different values of $(\boldsymbol{\mu}_s, \boldsymbol{\Sigma}_s)$ and after simplification, we obtain for distinct positive integers i, j, k ,

$$\begin{aligned}
E(Y_i Y_j Y_k) &= (\mu_i \mu_j \mu_k + \mu_i \rho_{jk} + \mu_j \rho_{ik} + \mu_k \rho_{ij}) p_3(\mu_i, \mu_j, \mu_k; \rho_{ij}, \rho_{ik}, \rho_{jk}) \\
&\quad + 2(\mu_j \mu_k + \rho_{ij} \rho_{ik} + \rho_{jk}) \phi(\mu_i) p_2(z_{j \cdot i}, z_{k \cdot i}; \rho_{jk \cdot i})
\end{aligned}$$

$$\begin{aligned}
& + 2(\mu_i\mu_k + \rho_{ij}\rho_{jk} + \rho_{ik})\phi(\mu_j)p_2(z_{i\cdot j}, z_{k\cdot j}; \rho_{ik\cdot j}) \\
& + 2(\mu_i\mu_j + \rho_{ik}\rho_{jk} + \rho_{ij})\phi(\mu_k)p_2(z_{i\cdot k}, z_{j\cdot k}; \rho_{ij\cdot k}) \\
& + 4\mu_i(1 - \rho_{jk}^2)\phi_2(\mu_j, \mu_k; \rho_{jk})\operatorname{erf}\left(\frac{z_{i\cdot jk}}{\sqrt{2}}\right) \\
& + 4\mu_j(1 - \rho_{ik}^2)\phi_2(\mu_i, \mu_k; \rho_{ik})\operatorname{erf}\left(\frac{z_{j\cdot ik}}{\sqrt{2}}\right) \\
& + 4\mu_k(1 - \rho_{ij}^2)\phi_2(\mu_i, \mu_j; \rho_{ij})\operatorname{erf}\left(\frac{z_{k\cdot ij}}{\sqrt{2}}\right) + 8|\tilde{\mathbf{R}}|\phi_3(\tilde{\boldsymbol{\mu}}; \tilde{\mathbf{R}}), \\
E(Y_i^2 Y_j Y_k) & = \{(1 + \mu_i^2)(\mu_j\mu_k + \rho_{jk}) \\
& + 2\mu_i(\mu_j\rho_{ik} + \mu_k\rho_{ij}) + 2\rho_{ij}\rho_{ik}\}p_2(\mu_j, \mu_k; \rho_{jk}) \\
& + 2\{2\mu_i(\rho_{ik} + \rho_{ij}\rho_{jk}) + \mu_k(1 + \mu_i^2 + \rho_{ij}^2) - \mu_j\rho_{ij}^2\rho_{jk}\}\phi(\mu_j)\operatorname{erf}\left(\frac{z_{k\cdot j}}{\sqrt{2}}\right) \\
& + 2\{2\mu_i(\rho_{ij} + \rho_{ik}\rho_{jk}) + \mu_j(1 + \mu_i^2 + \rho_{ik}^2) - \mu_k\rho_{ik}^2\rho_{jk}\}\phi(\mu_k)\operatorname{erf}\left(\frac{z_{j\cdot k}}{\sqrt{2}}\right) \\
& + 4(1 + \mu_i^2 + \rho_{ij}^2 + \rho_{ik}^2)(1 - \rho_{jk}^2)\phi_2(\mu_j, \mu_k; \rho_{jk}),
\end{aligned}$$

where

$$\begin{aligned}
& p_3(\mu_i, \mu_j, \mu_k; \rho_{ij}, \rho_{ik}, \rho_{jk}) \\
& = 8\Phi_3\left(\begin{bmatrix} \mu_i \\ \mu_j \\ \mu_k \end{bmatrix}; \begin{bmatrix} 1 & \rho_{ij} & \rho_{ik} \\ \rho_{ij} & 1 & \rho_{jk} \\ \rho_{ik} & \rho_{jk} & 1 \end{bmatrix}\right) - 4\Phi_2(\mu_i, \mu_j; \rho_{ij}) - 4\Phi_2(\mu_i, \mu_k; \rho_{ik}) \\
& - 4\Phi_2(\mu_j, \mu_k; \rho_{jk}) + 2\Phi(\mu_i) + 2\Phi(\mu_j) + 2\Phi(\mu_k) - 1.
\end{aligned}$$

For the fourth order product moment, we define $z_{i\cdot jkl} = (z_{i\cdot kl} - \rho_{ij\cdot kl}z_{j\cdot kl})/(1 - \rho_{ij\cdot kl}^2)^{1/2}$, where $\rho_{ij\cdot kl} = (\rho_{ij\cdot l} - \rho_{ik\cdot l}\rho_{jk\cdot l})/\{(1 - \rho_{ik\cdot l}^2)(1 - \rho_{jk\cdot l}^2)\}^{1/2}$. Let $\hat{\boldsymbol{\mu}} = (\mu_i, \mu_j, \mu_k, \mu_l)^\top$ and $\hat{\mathbf{R}}$ be a 4 by 4 submatrix of \mathbf{R} that consists of the (i, j, k, l) th rows and columns of \mathbf{R} . Applying (5) in the article repeatedly to obtain $I_{(1,1,1,1)}^4(\hat{\boldsymbol{\mu}}, \hat{\mathbf{R}})$ and then summing up the expression for 16 different values of $(\boldsymbol{\mu}_s, \boldsymbol{\Sigma}_s)$, we obtain for distinct positive integers i, j, k, l ,

$$\begin{aligned}
E(Y_i Y_j Y_k Y_l) & = (\mu_i\mu_j\mu_k\mu_l + \mu_i\mu_j\rho_{kl} + \mu_i\mu_k\rho_{jl} + \mu_i\mu_l\rho_{jk} + \mu_j\mu_k\rho_{il} + \mu_j\mu_l\rho_{ik} + \mu_k\mu_l\rho_{ij} \\
& + \rho_{ij}\rho_{kl} + \rho_{ik}\rho_{jl} + \rho_{il}\rho_{jk})p_4(\hat{\boldsymbol{\mu}}, \hat{\mathbf{R}}) \\
& + 2\{\mu_j\mu_k\mu_l + \mu_j(\rho_{ik}\rho_{il} + \rho_{kl}) + \mu_k(\rho_{ij}\rho_{il} + \rho_{jl}) + \mu_l(\rho_{ij}\rho_{ik} + \rho_{jk}) \\
& - \mu_i\rho_{ij}\rho_{ik}\rho_{il}\}\phi(\mu_i)p_3(z_{j\cdot i}, z_{k\cdot i}, z_{l\cdot i}; \rho_{jk\cdot i}, \rho_{jl\cdot i}, \rho_{kl\cdot i}) \\
& + 2\{\mu_i\mu_k\mu_l + \mu_i(\rho_{jk}\rho_{jl} + \rho_{kl}) + \mu_k(\rho_{ij}\rho_{jl} + \rho_{il}) + \mu_l(\rho_{ij}\rho_{jk} + \rho_{ik}) \\
& - \mu_j\rho_{ij}\rho_{jk}\rho_{jl}\}\phi(\mu_j)p_3(z_{i\cdot j}, z_{k\cdot j}, z_{l\cdot j}; \rho_{ik\cdot j}, \rho_{il\cdot j}, \rho_{kl\cdot j})
\end{aligned}$$

$$\begin{aligned}
& + 2\{\mu_i\mu_j\mu_l + \mu_i(\rho_{jk}\rho_{kl} + \rho_{jl}) + \mu_j(\rho_{ik}\rho_{kl} + \rho_{il}) + \mu_l(\rho_{ik}\rho_{jk} + \rho_{ij}) \\
& \quad - \mu_k\rho_{ik}\rho_{jk}\rho_{kl}\}\phi(\mu_k)p_3(z_{i\cdot k}, z_{j\cdot k}, z_{l\cdot k}; \rho_{ij\cdot k}, \rho_{il\cdot k}, \rho_{jl\cdot k}) \\
& + 2\{\mu_i\mu_j\mu_k + \mu_i(\rho_{jl}\rho_{kl} + \rho_{jk}) + \mu_j(\rho_{il}\rho_{kl} + \rho_{ik}) + \mu_k(\rho_{il}\rho_{jl} + \rho_{ij}) \\
& \quad - \mu_l\rho_{il}\rho_{jl}\rho_{kl}\}\phi(\mu_l)p_3(z_{i\cdot l}, z_{j\cdot l}, z_{k\cdot l}; \rho_{ij\cdot l}, \rho_{ik\cdot l}, \rho_{jk\cdot l}) \\
& + 4(1 - \rho_{ij}^2)(\mu_k\mu_l + \rho_{ik}\rho_{il} + \rho_{jk}\rho_{jl} + \rho_{kl})\phi_2(\mu_i, \mu_j; \rho_{ij})p_2(z_{k\cdot ij}, z_{l\cdot ij}; \rho_{kl\cdot ij}) \\
& + 4(1 - \rho_{ik}^2)(\mu_j\mu_l + \rho_{ij}\rho_{il} + \rho_{jk}\rho_{kl} + \rho_{jl})\phi_2(\mu_i, \mu_k; \rho_{ik})p_2(z_{j\cdot ik}, z_{l\cdot ik}; \rho_{jl\cdot ik}) \\
& + 4(1 - \rho_{il}^2)(\mu_j\mu_k + \rho_{ij}\rho_{ik} + \rho_{jl}\rho_{kl} + \rho_{jk})\phi_2(\mu_i, \mu_l; \rho_{il})p_2(z_{j\cdot il}, z_{k\cdot il}; \rho_{jk\cdot il}) \\
& + 4(1 - \rho_{jk}^2)(\mu_i\mu_l + \rho_{ij}\rho_{jl} + \rho_{ik}\rho_{kl} + \rho_{il})\phi_2(\mu_j, \mu_k; \rho_{jk})p_2(z_{i\cdot jk}, z_{l\cdot jk}; \rho_{il\cdot jk}) \\
& + 4(1 - \rho_{jl}^2)(\mu_i\mu_k + \rho_{ij}\rho_{jk} + \rho_{il}\rho_{kl} + \rho_{ik})\phi_2(\mu_j, \mu_l; \rho_{jl})p_2(z_{i\cdot jl}, z_{k\cdot jl}; \rho_{ik\cdot jl}) \\
& + 4(1 - \rho_{kl}^2)(\mu_i\mu_j + \rho_{ik}\rho_{jk} + \rho_{il}\rho_{jl} + \rho_{ij})\phi_2(\mu_k, \mu_l; \rho_{kl})p_2(z_{i\cdot kl}, z_{j\cdot kl}; \rho_{ij\cdot kl}) \\
& + 8\mu_i|\hat{\mathbf{R}}_{(1),(1)}|\phi_3(\hat{\boldsymbol{\mu}}_{(1)}; \hat{\mathbf{R}}_{(1),(1)})\operatorname{erf}\left(\frac{z_{i\cdot jkl}}{\sqrt{2}}\right) \\
& + 8\mu_j|\hat{\mathbf{R}}_{(2),(2)}|\phi_3(\hat{\boldsymbol{\mu}}_{(2)}; \hat{\mathbf{R}}_{(2),(2)})\operatorname{erf}\left(\frac{z_{j\cdot ikl}}{\sqrt{2}}\right) \\
& + 8\mu_k|\hat{\mathbf{R}}_{(3),(3)}|\phi_3(\hat{\boldsymbol{\mu}}_{(3)}; \hat{\mathbf{R}}_{(3),(3)})\operatorname{erf}\left(\frac{z_{k\cdot ijl}}{\sqrt{2}}\right) \\
& + 8\mu_l|\hat{\mathbf{R}}_{(4),(4)}|\phi_3(\hat{\boldsymbol{\mu}}_{(4)}; \hat{\mathbf{R}}_{(4),(4)})\operatorname{erf}\left(\frac{z_{l\cdot ijk}}{\sqrt{2}}\right) + 16|\hat{\mathbf{R}}|\phi_4(\hat{\boldsymbol{\mu}}; \hat{\mathbf{R}}),
\end{aligned}$$

where

$$\begin{aligned}
p_4(\hat{\boldsymbol{\mu}}, \hat{\mathbf{R}}) &= 16\Phi_4(\hat{\boldsymbol{\mu}}; \hat{\mathbf{R}}) - 8\Phi_3(\hat{\boldsymbol{\mu}}_{(1)}; \hat{\mathbf{R}}_{(1),(1)}) - 8\Phi_3(\hat{\boldsymbol{\mu}}_{(2)}; \hat{\mathbf{R}}_{(2),(2)}) \\
&\quad - 8\Phi_3(\hat{\boldsymbol{\mu}}_{(3)}; \hat{\mathbf{R}}_{(3),(3)}) - 8\Phi_3(\hat{\boldsymbol{\mu}}_{(4)}; \hat{\mathbf{R}}_{(4),(4)}) + 4\Phi_2(\mu_i, \mu_j; \rho_{ij}) + 4\Phi_2(\mu_i, \mu_k; \rho_{ik}) \\
&\quad + 4\Phi_2(\mu_i, \mu_l; \rho_{il}) + 4\Phi_2(\mu_j, \mu_k; \rho_{jk}) + 4\Phi_2(\mu_j, \mu_l; \rho_{jl}) + 4\Phi_2(\mu_k, \mu_l; \rho_{kl}) \\
&\quad - 2\Phi(\mu_i) - 2\Phi(\mu_j) - 2\Phi(\mu_k) - 2\Phi(\mu_l) + 1.
\end{aligned}$$

2 Truncated Multivariate Normal: Explicit Expressions for Low Order Moments

Using our recurrence relation for $F_{\boldsymbol{\kappa}}^n(\mathbf{a}, \mathbf{b}; \boldsymbol{\mu}, \boldsymbol{\Sigma})$, we present some low order product moments for the lower truncated multivariate normal. For the upper truncated multivariate normal,

$$E(\mathbf{Z}^{\boldsymbol{\kappa}}) = (-1)^{\sum_{i=1}^n k_i} E((- \mathbf{Z})^{\boldsymbol{\kappa}}) = (-1)^{\sum_{i=1}^n k_i} E((- \mathbf{X})^{\boldsymbol{\kappa}} \mid -\mathbf{X} > -\mathbf{b}).$$

Since $-\mathbf{X} \sim N(-\boldsymbol{\mu}, \boldsymbol{\Sigma})$, we just need to replace $\boldsymbol{\mu}$ with $-\boldsymbol{\mu}$ and \mathbf{a} with $-\mathbf{b}$ in the expression for the product moment of a lower truncated multivariate normal, and then multiply the result by $(-1)^{\sum_{i=1}^n k_i}$ to obtain the product moment of an upper truncated multivariate normal.² In our derivations, we assume $\sigma_1 = \dots = \sigma_n = 1$, i.e., $\boldsymbol{\Sigma} = \mathbf{R}$. The result for the general $\boldsymbol{\Sigma}$ case can be obtained by replacing a_i with a_i/σ_i , μ_i with μ_i/σ_i , and multiplying the result by $\sigma_1^{k_1} \dots \sigma_n^{k_n}$.

Let $\boldsymbol{\eta} = (\eta_1, \dots, \eta_n)^\top = \boldsymbol{\mu} - \mathbf{a}$. When $n = 1$, Cohen (1951a) expresses $E(Z^k)$ using Fisher's I_k functions, which is essentially equivalent to (6) in the article for the case of $n = 1$. However, we can also use the recursion for $F_k^1(a, \infty; \mu, 1)$ to obtain the more efficient recursion

$$E(Z^{k+1}) = \mu E(Z^k) + k E(Z^{k-1}) + \frac{a^k \phi(\eta)}{\Phi(\eta)} \quad (k \geq 1),$$

with the boundary condition $E(Z^0) = 1$. Using this recurrence relation, we obtain the first three moments of Z as

$$\begin{aligned} E(Z) &= \mu + \frac{\phi(\eta)}{\Phi(\eta)}, \\ E(Z^2) &= 1 + \mu^2 + \frac{(\mu + a)\phi(\eta)}{\Phi(\eta)}, \\ E(Z^3) &= 3\mu + \mu^3 + \frac{(\mu^2 + a\mu + a^2 + 2)\phi(\eta)}{\Phi(\eta)}. \end{aligned}$$

When $n = 2$, we use (5) and (6) in the article to obtain $E(Z_1^{k_1} Z_2^{k_2})$ for $1 \leq k_1 + k_2 \leq 3$. Specifically, we have

$$\begin{aligned} E(Z_1) &= \mu_1 + \frac{\phi(\eta_1)\Phi(w_{2.1}) + \rho_{12}\phi(\eta_2)\Phi(w_{1.2})}{\Phi_2(\eta_1, \eta_2; \rho_{12})}, \\ E(Z_1^2) &= 1 + \mu_1^2 + \frac{(\mu_1 + a_1)\phi(\eta_1)\Phi(w_{2.1})}{\Phi_2(\eta_1, \eta_2; \rho_{12})} \\ &\quad + \frac{\rho_{12}\{(2\mu_1 - \rho_{12}\eta_2)\phi(\eta_2)\Phi(w_{1.2}) + (1 - \rho_{12}^2)\phi_2(\eta_1, \eta_2; \rho_{12})\}}{\Phi_2(\eta_1, \eta_2; \rho_{12})}, \\ E(Z_1 Z_2) &= \mu_1 \mu_2 + \rho_{12} + \frac{(\mu_2 + \rho_{12}a_1)\phi(\eta_1)\Phi(w_{2.1}) + (\mu_1 + \rho_{12}a_2)\phi(\eta_2)\Phi(w_{1.2})}{\Phi_2(\eta_1, \eta_2; \rho_{12})} \\ &\quad + \frac{(1 - \rho_{12}^2)\phi_2(\eta_1, \eta_2; \rho_{12})}{\Phi_2(\eta_1, \eta_2; \rho_{12})}, \\ E(Z_1^3) &= 3\mu_1 + \mu_1^3 + \frac{(\eta_1^2 + 3a_1\mu_1 + 2)\phi(\eta_1)\Phi(w_{2.1})}{\Phi_2(\eta_1, \eta_2; \rho_{12})} \end{aligned}$$

²Although analytically attainable, we do not report the results for the doubly truncated multivariate normal distribution because the expressions of the product moments can be very lengthy.

$$\begin{aligned}
& + \frac{\rho_{12}\{3 + 3\mu_1^2 - 3\rho_{12}\mu_1\eta_2 + \rho_{12}^2(\eta_2^2 - 1)\}\phi(\eta_2)\Phi(w_{1.2})}{\Phi_2(\eta_1, \eta_2; \rho_{12})} \\
& + \frac{\rho_{12}(1 - \rho_{12}^2)(2\mu_1 + a_1 - \rho_{12}\eta_2)\phi_2(\eta_1, \eta_2; \rho_{12})}{\Phi_2(\eta_1, \eta_2; \rho_{12})}, \\
E(Z_1^2 Z_2) &= (1 + \mu_1^2)\mu_2 + 2\rho_{12}\mu_1 + \frac{\{(\mu_1 + a_1)\mu_2 + \rho_{12}(2 + a_1^2)\}\phi(\eta_1)\Phi(w_{2.1})}{\Phi_2(\eta_1, \eta_2; \rho_{12})} \\
& + \frac{\{1 + \mu_1^2 + 2\rho_{12}a_2\mu_1 + \rho_{12}^2(1 - a_2\eta_2)\}\phi(\eta_2)\Phi(w_{1.2})}{\Phi_2(\eta_1, \eta_2; \rho_{12})} \\
& + \frac{(1 - \rho_{12}^2)(\mu_1 + a_1 + \rho_{12}a_2)\phi_2(\eta_1, \eta_2; \rho_{12})}{\Phi_2(\eta_1, \eta_2; \rho_{12})},
\end{aligned}$$

where $w_{i.j} = (\eta_i - \rho_{ij}\eta_j)/(1 - \rho_{ij}^2)^{1/2}$.

When $n = 3$, we again use (5) and (6) in the article to obtain $E(Z_1^{k_1} Z_2^{k_2} Z_3^{k_3})$ for $1 \leq k_1 + k_2 + k_3 \leq$

3. Specifically, we have

$$\begin{aligned}
E(Z_1) &= \mu_1 + q_1 + \rho_{12}q_2 + \rho_{13}q_3, \\
E(Z_1^2) &= 1 + \mu_1^2 + (\mu_1 + a_1)q_1 + \rho_{12}(2\mu_1 - \rho_{12}\eta_2)q_2 + \rho_{13}(2\mu_1 - \rho_{13}\eta_3)q_3 \\
& + \rho_{12}(1 - \rho_{12}^2)h_1 + \rho_{13}(1 - \rho_{13}^2)h_2 + gh_3, \\
E(Z_1 Z_2) &= \mu_1\mu_2 + \rho_{12} + (\mu_2 + \rho_{12}a_1)q_1 + (\mu_1 + \rho_{12}a_2)q_2 \\
& + (\rho_{23}\mu_1 + \rho_{13}\mu_2 - \rho_{13}\rho_{23}\eta_3)q_3 + (1 - \rho_{12}^2)h_1 + \rho_{23}(1 - \rho_{13}^2)h_2 \\
& + \rho_{13}(1 - \rho_{23}^2)h_3, \\
E(Z_1^3) &= 3\mu_1 + \mu_1^3 + (\eta_1^2 + 3a_1\mu_1 + 2)q_1 + \rho_{12}\{3 + 3\mu_1^2 - 3\rho_{12}\mu_1\eta_2 + \rho_{12}^2(\eta_2^2 - 1)\}q_2 \\
& + \rho_{13}\{3 + 3\mu_1^2 - 3\rho_{13}\mu_1\eta_3 + \rho_{13}^2(\eta_3^2 - 1)\}q_3 \\
& + \rho_{12}(1 - \rho_{12}^2)(2\mu_1 + a_1 - \rho_{12}\eta_2)h_1 + \rho_{13}(1 - \rho_{13}^2)(2\mu_1 + a_1 - \rho_{13}\eta_3)h_2 \\
& + \left\{ 3g\mu_1 + \rho_{23}(\rho_{12}^3\eta_2 + \rho_{13}^3\eta_3) - \frac{\rho_{12}^2(3\rho_{13} - \rho_{12}\rho_{23})w_{2.3}}{(1 - \rho_{23}^2)^{\frac{1}{2}}} \right. \\
& \left. - \frac{\rho_{13}^2(3\rho_{12} - \rho_{13}\rho_{23})w_{3.2}}{(1 - \rho_{23}^2)^{\frac{1}{2}}} \right\} h_3 + \frac{g|\mathbf{R}|\phi_3(\boldsymbol{\eta}; \mathbf{R})}{(1 - \rho_{23}^2)\Phi_3(\boldsymbol{\eta}; \mathbf{R})}, \\
E(Z_1^2 Z_2) &= (1 + \mu_1^2)\mu_2 + 2\rho_{12}\mu_1 + \{(\mu_1 + a_1)\mu_2 + \rho_{12}(2 + a_1^2)\}q_1 \\
& + \{1 + \mu_1^2 + 2\rho_{12}a_2\mu_1 + \rho_{12}^2(1 - a_2\eta_2)\}q_2 \\
& + [\rho_{13}\{2\rho_{12} + \mu_2(2\mu_1 - \rho_{13}\eta_3)\} + \rho_{23}\{1 - \rho_{13}^2 + (\mu_1 - \rho_{13}\eta_3)^2\}]q_3 \\
& + (1 - \rho_{12}^2)(\mu_1 + a_1 + \rho_{12}a_2)h_1 + (1 - \rho_{13}^2)\{\mu_2\rho_{13} + \rho_{23}(\mu_1 + a_1 - \rho_{13}\eta_3)\}h_2 \\
& + \{ga_2 + \rho_{13}(1 - \rho_{23}^2)(2\mu_1 - \rho_{13}\eta_3)\}h_3 + \frac{\rho_{13}|\mathbf{R}|\phi_3(\boldsymbol{\eta}; \mathbf{R})}{\Phi_3(\boldsymbol{\eta}; \mathbf{R})},
\end{aligned}$$

$$\begin{aligned}
E(Z_1 Z_2 Z_3) &= \mu_1 \mu_2 \mu_3 + \rho_{23} \mu_1 + \rho_{13} \mu_2 + \rho_{12} \mu_3 \\
&+ \{\mu_2 \mu_3 + \rho_{12} \rho_{13} + \rho_{23} + a_1(\mu_2 \rho_{13} + \mu_3 \rho_{12} - \eta_1 \rho_{12} \rho_{13})\} q_1 \\
&+ \{\mu_1 \mu_3 + \rho_{12} \rho_{23} + \rho_{13} + a_2(\mu_1 \rho_{23} + \mu_3 \rho_{12} - \eta_2 \rho_{12} \rho_{23})\} q_2 \\
&+ \{\mu_1 \mu_2 + \rho_{13} \rho_{23} + \rho_{12} + a_3(\mu_1 \rho_{23} + \mu_2 \rho_{13} - \eta_3 \rho_{13} \rho_{23})\} q_3 \\
&+ (1 - \rho_{12}^2)(\mu_3 + a_1 \rho_{13} + a_2 \rho_{23}) h_1 + (1 - \rho_{13}^2)(\mu_2 + a_1 \rho_{12} + a_3 \rho_{23}) h_2 \\
&+ (1 - \rho_{23}^2)(\mu_1 + a_2 \rho_{12} + a_3 \rho_{13}) h_3 + \frac{|\mathbf{R}| \phi_3(\boldsymbol{\eta}; \mathbf{R})}{\Phi_3(\boldsymbol{\eta}; \mathbf{R})},
\end{aligned}$$

where

$$\begin{aligned}
g &= 2\rho_{12}\rho_{13} - \rho_{23}(\rho_{12}^2 + \rho_{13}^2), \\
q_1 &= \phi(\eta_1) \Phi_2(w_{2.1}, w_{3.1}; \rho_{23.1}) / \Phi_3(\boldsymbol{\eta}; \mathbf{R}), \\
q_2 &= \phi(\eta_2) \Phi_2(w_{1.2}, w_{3.2}; \rho_{13.2}) / \Phi_3(\boldsymbol{\eta}; \mathbf{R}), \\
q_3 &= \phi(\eta_3) \Phi_2(w_{1.3}, w_{2.3}; \rho_{12.3}) / \Phi_3(\boldsymbol{\eta}; \mathbf{R}), \\
h_1 &= \phi_2(\eta_1, \eta_2; \rho_{12}) \Phi(w_{3.12}) / \Phi_3(\boldsymbol{\eta}; \mathbf{R}), \\
h_2 &= \phi_2(\eta_1, \eta_3; \rho_{13}) \Phi(w_{2.13}) / \Phi_3(\boldsymbol{\eta}; \mathbf{R}), \\
h_3 &= \phi_2(\eta_2, \eta_3; \rho_{23}) \Phi(w_{1.23}) / \Phi_3(\boldsymbol{\eta}; \mathbf{R}),
\end{aligned}$$

and $w_{i.jk} = (w_{i.k} - \rho_{ij.k} w_{j.k}) / (1 - \rho_{ij.k}^2)^{1/2}$.

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