Appendix to "Is There a Jump in the Transition?"

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Abstract

This appendix contains all the proofs of the theorems in the paper.

A Proof of Theorems

A.1 Proof of Theorem 1

The consistency of the least squares estimator is standard and does not matter whether or not the parameter value lies at a boundary value of the parameter space. Thus, we begin with deriving the convergence rate of the estimators. Then we construct a sequence of localized criterion functions and show their weak convergence by deriving the finite dimensional convergence and stochastic equicontinuity. The asymptotic distribution of the estimator then follows by the argmax continuous mapping theorem.

Notation: with a slight abuse of notation we let $\mathbb{S}_n(\theta)$ stand for the sum of squared residuals for a given parameter value $\theta = (\alpha', h, \gamma)'$, while $\mathbb{S}_n(h, \gamma)$ indicates the profiled sum of squared residuals for a given (h, γ). And we write $\mathcal{K}_i(h, \gamma)$ for $K(q_i, h, \gamma)$, $\mathcal{I}_i(\gamma)$ for $1\{q_i > \gamma\}$, and $\mathcal{X}_i(h, \gamma) = x_i \mathcal{K}_i(h, \gamma)$ and $\mathcal{X}_i(\gamma) = x_i \mathcal{I}_i(\gamma)$. The weak convergence is signified by $\Rightarrow$.

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Convergence Rate

The following two lemmas are crucial to our proof.

Lemma 1. *Let*

$$\mathbb{G}_n(h,\gamma) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \varepsilon_i x_i \mathcal{K}_i(h,\gamma).$$

Then, for some $C < \infty$ and for any γ_0 and $\eta > 0$,

$$E \sup_{h<\eta, |\gamma-\gamma_0|<\eta} |\mathbb{G}_n(h,\gamma) - \mathbb{G}_n(0,\gamma_0)| \leq C\eta^{1/2}.$$

Proof of Lemma 1 This can be verified by means of van der Vaart and Wellner's (1996) Theorem 2.14.2. Specifically, we need to establish the finiteness of the bracketing integral and the order of L_s-norm of an envelope function. For the finiteness of the entropy integral $J(1,\mathcal{F})$, note that this is an empirical process indexed by the product of $\varepsilon_i x_i$ and bounded monotone functions $k(\cdot)$. In view of Andrews' (1994) Theorem 6 and the polynomial bounds on the bracketing entropy of the class of monotone functions (Theorem 2.7.5 of van der Vaart and Wellner (1996)), the entropy integral condition is satisfied. It remains to show that the envelope has L_2-norm of magnitude $\eta^{1/2}$. Specifically, for $0 \leq h \leq \eta$, $|\gamma - \gamma_0| < \eta$, we note that

$$|K(q;h,\gamma) - 1(q > \gamma_0)| \leq |K(q;h,\gamma) - 1(q > \gamma)| + 1(|q - \gamma_0| \leq \eta)$$

and

$$\begin{aligned} |K(q;h,\gamma) - 1(q > \gamma)| &\leq 1(|q-\gamma_0| \leq \eta) + K(q;\eta,\gamma_0 - \eta)\, 1(q - \gamma_0 \leq -\eta) \\ &+ (1 - K(q;\eta,\gamma_0+\eta))\, 1(q - \gamma_0 > \eta). \end{aligned}$$

Then, a natural envelope function for the centered empirical process $\mathbb{G}_n(h,\gamma) - \mathbb{G}_n(0,\gamma_0)$ is the multiple of $2|\varepsilon_i x_i|$ to the right side terms in the preceding inequality. Furthermore, since $E\left(|\varepsilon_i x_i|^2 |q_i\right)$ is bounded by C,

$$E|\varepsilon_i x_i|^2 1(|q-\gamma_0| \leq \eta)^2 \leq C\eta$$

and using the change-of-variables formula

$$\begin{aligned} E\left|\varepsilon_i x_i\right|^2 K\left(q;\eta,\gamma_0-\eta\right)^2 1\left(q-\gamma_0\le-\eta\right) &\le C \quad \int_{-\infty}^{\gamma_0-\eta} K\left(q;\eta,\gamma_0-\eta\right)^2 p\left(q\right) dq \\ &= C \quad \eta\int_{-\infty}^{0} k\left(x\right) p\left(\eta x+\gamma_0-\eta\right) dx, \end{aligned}$$

where the integral is bounded since $k\left(\cdot\right)$ is bounded. And, the same argument applies for the last term $\left(1-K\left(q;\eta,\gamma_0+\eta\right)\right)1\left(q-\gamma_0>\eta\right)$. Thus, this establishes the order of magnitude of the envelope as $\eta^{1/2}$ in terms of L_2-norm. ■

Lemma 2. *For each $\eta>0$, there exist random variables $\{R_n\}$ of order $O_p\left(1\right)$ and a positive constant C such that*

$$\left|\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i c'x_i\left(\mathcal{K}_i\left(h,\gamma\right)-\mathcal{I}_i\left(\gamma_0\right)\right)\right|\le\eta n^{-\psi}\left(\left|\gamma-\gamma_0\right|+h\right)+n^{-1+\psi}R_n^2,$$

for any $|\gamma-\gamma_0|,h\le C$. Here, γ_0 can be replaced by any sequence $\gamma_n\to\gamma_0$.

Proof of Lemma 2 Let $g=|\gamma-\gamma_0|+h$ and define $A_{n,j}=\{g:(j-1)^3n^{-1+2\psi}\le g<j^3n^{-1+2\psi}\}$ and

$$R_n^2=n^{1-\psi}\sup_{g\le C}\left\{\left|\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i c'x_i\left(\mathcal{K}_i\left(h,\gamma\right)-\mathcal{I}_i\left(\gamma_0\right)\right)\right|-\eta g n^{-\psi}\right\}.$$

There exists a positive constant C such that

$$\begin{aligned} &P\left\{R_n>m\right\} \\ &=P\left\{\left|n^{-\psi}\sum_{i=1}^{n}\varepsilon_i c'x_i\left(\mathcal{K}_i\left(h,\gamma\right)-\mathcal{I}_i\left(\gamma_0\right)\right)\right|>\eta g n^{1-2\psi}+m^2\text{ for some }g\le C\right\} \\ &\le\sum_{j=1}^{\infty}P\left\{n^{-\psi}\left|\sum_{i=1}^{n}\varepsilon_i c'x_i\left(\mathcal{K}_i\left(h,\gamma\right)-\mathcal{I}_i\left(\gamma_0\right)\right)\right|>\eta(j-1)^3+m^2\quad\text{for some }g\in A_{n,j}\right\} \\ &\le\sum_{j=1}^{\infty}\frac{Cj^{3/2}}{\left(\eta\left(j-1\right)^3+m^2\right)}, \end{aligned}$$

for all $m>0$, where the last equality is due to Lemma 1. Since the above sum is finite for all $m>0$, the conclusion follows. ■

We are now ready to derive the convergence rates of our estimators. To begin with, decompose the centered sample criterion function $\mathbb{S}_n$ as follows

$$
\begin{aligned}
&\mathbb{S}_n(\theta) - \mathbb{S}_n(\theta_0) \\
&= \underbrace{\frac{-2}{n}\sum_{t=1}^{n} \varepsilon_i x_i'\delta_0 \left(\mathcal{K}_i\left(h,\gamma\right) - \mathcal{I}_i\left(\gamma_0\right)\right) + \frac{1}{n}\sum_{t=1}^{n} \left(x_i'\delta_0\right)^2 \left|\mathcal{K}_i\left(h,\gamma\right) - \mathcal{I}_i\left(\gamma_0\right)\right|}_{=A_{1n}(h,\gamma)} \\
&+ \underbrace{\frac{-2}{n}\sum_{t=1}^{n} \varepsilon_i \mathcal{X}_i\left(h,\gamma\right)'\left(\alpha-\alpha_0\right)}_{=A_{2n}(\alpha,h,\gamma)} + \underbrace{\frac{2}{n}\sum_{t=1t}^{n} x_i'\delta_0 \left(\mathcal{K}_i\left(h,\gamma\right) - \mathcal{I}_i\left(\gamma_0\right)\right)\mathcal{X}_i\left(h,\gamma\right)'\left(\alpha-\alpha_0\right)}_{=A_{3n}(\alpha,h,\gamma)} \qquad (1) \\
&+ \underbrace{\left(\alpha-\alpha_0\right)'\frac{1}{n}\sum_{t=1}^{n} \mathcal{X}_i\left(h,\gamma\right)\mathcal{X}_i\left(h,\gamma\right)'\left(\alpha-\alpha_0\right)}_{=A_{4n}(\alpha,h,\gamma)},
\end{aligned}
$$

and consider each term in the η-neighborhood of $(\alpha_0, 0, \gamma_0)$.

First, note that for sufficiently small η there exists some $c_1 > 0$ such that

$$
A_{1n}\left(h,\gamma\right) \geq c_1\left(\left|\gamma-\gamma_0\right| + h\right)n^{-2\psi} + O_p\left(n^{-1}\right),
$$

due to Lemma 2 and the uniform law of large numbers together with the following expansion: for any $\eta > 0$ and for $\eta/2 < h, |\gamma - \gamma_0| < \eta$,

$$
\begin{aligned}
&E\left(x_i'\delta_0\right)^2\left(\mathcal{K}_i\left(h,\gamma\right) - \mathcal{I}_i\left(\gamma_0\right)\right)^2 \\
&\geq Cn^{-2\psi}\int_{-\infty}^{\gamma_0} k\left(\frac{q-\gamma}{h}\right)^2 p\left(q\right)dq + \int_{\gamma_0}^{\infty}\left(1-\left(\frac{q-\gamma}{h}\right)\right)^2 p\left(q\right)dq \\
&\geq C'^{-2\psi}h\left(\int_{-C'''}^{0} k\left(x-\frac{\gamma-\gamma_0}{h}\right)^2 dx + \int_{0}^{C'''}\left(1-k\left(x-\frac{\gamma-\gamma_0}{h}\right)\right)^2 dx\right) \\
&\geq C''^{-2\psi}\left(h + \left|\gamma-\gamma_0\right|\right)
\end{aligned}
$$

for some $C'' > 0$, $C''' > 2$, where we applied the change-of-variables and the fact that $p\left(x\right)$ is bounded away from zero in a neighborhood of γ_0 for the second inequality and we used the following fact for the last inequality that

$$\int_{-C'''}^{0} k\left(x-\frac{\gamma-\gamma_0}{h}\right)^2 dx+\int_{0}^{C'''}\left(1-k\left(x-\frac{\gamma-\gamma_0}{h}\right)\right)^2 dx$$
$$=\int_{-C'''}^{-\frac{\gamma-\gamma_0}{h}} k\left(x\right)^2 dx+\int_{-\frac{\gamma-\gamma_0}{h}}^{C'''}\left(1-k\left(x\right)\right)^2 dx$$
$$=2\int_{-C'''}^{-\left|\frac{\gamma-\gamma_0}{h}\right|} k\left(x\right)^2 dx+\left|\frac{\gamma-\gamma_0}{h}\right|$$

since $1/2<\left|\frac{\gamma-\gamma_0}{h}\right|<2$. By the same reasoning,

$$\begin{aligned} & E\left|\left(x_i'\delta_0\right)\left(\mathcal{K}_i\left(h,\gamma\right)-\mathcal{I}_i\left(\gamma_0\right)\right)\left(\mathcal{X}_i\left(h,\gamma\right)'\left(\alpha-\alpha_0\right)\right)\right| \\ \leq\ & Cn^{-2\psi}\left(h+\left|\gamma-\gamma_0\right|\right)\left|\alpha-\alpha_0\right| \\ \leq\ & c_1 n^{-2\psi}\left(h+\left|\gamma-\gamma_0\right|\right)/4 \end{aligned}$$

by choosing η small enough. This leads us to conclude that

$$\left|A_{1n}\right|+\left|A_{3n}\right|\geq c_1 n^{-2\psi}\left(h+\left|\gamma-\gamma_0\right|\right)/2+O_p\left(n^{-1}\right)$$

for any α,h,γ in the η-neighborhood. Turning to A_{2n}, note that it follows from Lemma 2 and the central limit theorem

$$\left|A_{2n}\right|\leq\left(n^{-\psi}\left[h+\left|\gamma-\gamma_0\right|\right]+O_p\left(n^{-1+\psi}\right)+O_p\left(n^{-1/2}\right)\right)\left|\alpha-\alpha_0\right|$$

Finally, for the A_{4n} note that $E\left(\mathcal{X}_i\left(h,\gamma\right)'\left(\alpha-\alpha_0\right)\right)^2\geq C\left|\alpha-\alpha_0\right|^2$ and thus $\left|A_{4n}\right|\geq C\left|\alpha-\alpha_0\right|^2$ with probability arbitrarily close to one, due to the uniform law of large numbers.

Then, put all the bounds together with the fact that

$$\mathbb{S}_n(\theta)-\mathbb{S}_n(\theta_0)\leq 0$$

at $\theta = \widehat{\theta}$. Namely,

$$\begin{aligned} & O_p\left(n^{-1}\right) + \left(n^{-\psi}\left[h + |\gamma - \gamma_0|\right] + O_p\left(n^{-1/2}\right)\right)|\alpha - \alpha_0| \\ \geq \quad & c_2 n^{-2\psi}\left(h + |\gamma - \gamma_0|\right) + C\left|\alpha - \alpha_0\right|^2 . \end{aligned} \tag{2}$$

Suppose $O_p\left(n^{-1}\right) \leq \left(n^{-\psi}\left[h + |\gamma - \gamma_0|\right] + O_p\left(n^{-1/2}\right)\right)|\alpha - \alpha_0|$. Otherwise, $|\alpha - \alpha_0| = O_p\left(n^{-1/2}\right)$ and it in turn implies that $(h + |\gamma - \gamma_0|) = O_p\left(n^{-1+2\psi}\right)$ as we wanted.[1] Then,

$$|\alpha - \alpha_0| \leq C'\left(n^{-\psi}\left[h + |\gamma - \gamma_0|\right] + O_p\left(n^{-1/2}\right)\right).$$

Again, if $\left(n^{-\psi}\left[h + |\gamma - \gamma_0|\right] \leq O_p\left(n^{-1/2}\right)\right)$, we get the same conclusion. Otherwise, putting it back to (2) yields

$$\left(c_2 - c_3\right) n^{-2\psi}\left(h + |\gamma - \gamma_0|\right) \leq O_p\left(n^{-1}\right)$$

for some $c_3 < c_2$ by choosing η small enough. This in turn implies $|\alpha - \alpha_0| = O_p\left(n^{-1/2}\right)$. Thus, we considered all the possible cases, completing the proof.

Weak Convergence of localized processes

Given the convergence rate we obtained in the previous section, we may consider the weak convergence of the localized criterion function

$$n\left[\mathbb{S}_n\left(\alpha_0 + an^{-1/2}, hn^{-1+2\psi}, \gamma_0 + gn^{-1+2\psi}\right) - \mathbb{S}_n\left(\theta_0\right)\right], \tag{3}$$

where $(a', h, g)'$ belongs to an arbitrary compact set and $a = (b', d')'$ accordingly to β and δ. Thus, let $|a|, h, |g| \leq C$. Then, the asymptotic distribution of our estimator follows as a consequence due to the argmax continuous mapping theorem.

It is convenient to apply the same decomposition in the previous section to the localized process (3) and derive the weak convergence of each term. Recall the definitions in (1) and let

$$\widetilde{A}_{1n}\left(h, g\right) = nA_{1n}\left(hn^{-1+2\psi}, \gamma_0 + gn^{-1+2\psi}\right)$$

[1] $a + b \leq c$ implies that $a \leq c$ when a and b are non-negative.

and

$$\widetilde{A}_{jn}\left(a,h,g\right)=nA_{jn}\left(\alpha_0+an^{-1/2},hn^{-1+2\psi},\gamma_0+gn^{-1+2\psi}\right),$$

for $j=2,3,4$. We begin by showing that

$$\sup_{a,h,g}\left|\widetilde{A}_{jn}\left(a,h,g\right)-\widetilde{A}_{jn}\left(a,0,0\right)\right|=o_p\left(1\right),$$

for $j=2,3,4$. The case of $j=4$ is a simple consequence of the uniform law of large numbers for triangular arrays. For the case of $j=2$, note that due to Lemma 1,

$$\begin{aligned}
& E\sup_{h,g}\left|\widetilde{A}_{2n}\left(a,h,g\right)-\widetilde{A}_{2n}\left(a,0,0\right)\right| \\
= & E\sup_{h,g}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\varepsilon_i\left(\mathcal{K}_i\left(hn^{-1+2\psi},\gamma_0+gn^{-1+2\psi}\right)-\mathcal{I}_i\left(\gamma_0\right)\right)x_i'd\right| \\
= & 3Cn^{-1/2+\psi}.
\end{aligned}$$

The case of $j=3$ is similar and thus details are omitted. Also note that $\widetilde{A}_{3n}\left(a,0,0\right)=0$ for any a.

In the meantime, the result so far yields an oracle property, namely, the asymptotic distribution of $\left(\widehat{h},\widehat{\gamma}\right)$ is solely determined by $\widetilde{A}_{1n}\left(h,g\right)$ and that of $\widehat{\alpha}$ solely by $\widetilde{A}_{2n}\left(a,0,0\right)+\widetilde{A}_{4n}\left(a,0,0\right)$ since the functions of $\left(h,g\right)$ and those of a are separable.

By the central limit theorem and law of large numbers,

$$\widetilde{A}_{2n}\left(a,0,0\right)+\widetilde{A}_{4n}\left(a,0,0\right)\Rightarrow-2Z'a+a'\left[E\mathcal{X}_i\left(\gamma_0\right)\mathcal{X}_i\left(\gamma_0\right)'\right]a,$$

where $Z\sim\mathcal{N}\left(0,E\varepsilon_i^2\mathcal{X}_i\left(\gamma_0\right)\mathcal{X}_i\left(\gamma_0\right)'\right)$.

Turning to $\widetilde{A}_{1n}\left(h,g\right)$, let

$$\begin{aligned}
G_{1n}\left(h,g\right) &= \sum_{i=1}^{n}\varepsilon_i x_i'\delta_0\left(\mathcal{K}_i\left(hr_n^{-1},\gamma_0+gr_n^{-1}\right)-\mathcal{I}_i\left(\gamma_0\right)\right) \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\varepsilon_i x_i'c_0r_n^{1/2}\left(\mathcal{K}_i\left(hr_n^{-1},\gamma_0+gr_n^{-1}\right)-\mathcal{I}_i\left(\gamma_0\right)\right),
\end{aligned}$$

which is the first term of the decomposition of $\widetilde{A}_{1n}\left(h,g\right)$. The tightness of the empirical process G_{1n} can be verified by Theorem 2.11.23 of van der Vaart and Wellner (1996). As we argued in the proof of Lemma 1, the entropy integral is uniformly bounded for all n. Thus, it is sufficient to verify the L_2-continuity condition and two conditions for the envelope function in (2.11.21) of van der Vaart and Wellner. For the latter, first construct an envelope as in the proof of Lemma 1 by replacing η with Cr_n^{-1}. Then, the L_2-norm of this envelope is bounded proceeding as in the proof of Lemma 1. The tail condition in (2.11.21) is trivial as K and the indicator is bounded. Finally, the L_2-continuity follows from the monotonicity of k and the square integrability of $k(x) - 1(x > 0)$ and from the dominated convergence theorem.

Turning to the finite dimensional convergence of G_{1n}, apply the standard CLT for an independent triangular array and the Cramer-Rao device, and get the asymptotic normality for $(G_{1n}(h_1, g_1), G_{1n}(h_2, g_2))'$ for arbitrary h_1, h_2, g_1, g_2. More specifically, a straightforward algebra using the change-of-variables formula yields that the asymptotic covariance of $(G_{1n}(h_1, g_1), G_{1n}(h_2, g_2))'$ is given by

$$E\left(G_{1n}(h_1, g_1) G_{1n}(h_2, g_2)\right) \to m_1 p_0 EW(h_1, g_1) W(h_2, g_2),$$

where $m_1 = E\left(\varepsilon_i^2 (x_i' c_0)^2 | q_i = \gamma_0\right) = c_0' V c_0$, $p_0 = p(\gamma_0)$ and W is defined in Section 3.2. Note that the same algebra shows that the covariance between $G_{1n}(h, g)$ and $\widetilde{A}_{2n}(a, 0, 0)$ goes to zero. This implies that they are asymptotically independent due to Gausianity. Also note that its covariance kernel satisfies the following property:

$$\begin{aligned}
\Omega(ah_1, ag_1, ah_2, ag_2) &= \int_{-\infty}^{0} k\left(\frac{x - ag_1}{ah_1}\right) k\left(\frac{x - ag_2}{ah_2}\right) dx \\
&= \int_{-\infty}^{0} k\left(\frac{xa^{-1} - g_1}{h_1}\right) k\left(\frac{xa^{-1} - g_2}{h_2}\right) dx \\
&= a\Omega(h_1, g_1, h_2, g_2).
\end{aligned}$$

Therefore, we can write

$$\sqrt{a} W(h, g) \overset{d}{=} W(ah, ag).$$

On the other hand, a straightforward algebra and the uniform law of large numbers yield that the second term in the decomposition of $\widetilde{A}_{1n}(h, g)$ converges in probability to

$p_0 m_2 B(h,g)$, where $m_2 = E\left((x_i' c_0)^2 | q_i = \gamma_0\right) = c_0' D c_0$. Note that $B(ah, ag) = aB(h,g)$. Therefore,

$$\frac{m_2}{m_1}\left(2\sqrt{p_0 m_1} W(h,g) + p_0 m_2 B(h,g)\right) \overset{d}{=} 2W(\xi h, \xi g) + B(\xi h, \xi g),$$

where $\xi = \frac{m_2^2 p_0}{m_1}$. The minimum of the process on the right hand side on $\mathbb{R}^2$ is equivalent to that of $2W(h,g) + B(h,g)$ on $\mathbb{R}^2$ and its minimizer is equivalent to that of $2W(h,g) + B(h,g)$ times ξ^{-1}. We conclude

$$\xi \arg\min_{h,g} \widetilde{A}_{1n}(h,g) \Rightarrow \arg\min_{h,g}\left(2W(h,g) + B(h,g)\right)$$

due to the argmax continuous mapping theorem. ■

A.2 Proof of Theorem 2

This is a straightforward corollary of the proof of Theorem 1 and the same argument as in Hansen's (2000) proof of Theorem 2. That is, the estimation error in the $\widehat{\alpha}$ becomes negligible in the QLR statistics. Then, since the minimum and the constrained minimum and their difference is a continuous functional the result follows from the continuous mapping theorem and the change-of-variables as in the proof of Theorem 1. ■

A.3 Asymptotic Distribution of $\mathbb{Q}_{3n}$

The asymptotic distribution of $\mathbb{Q}_{3n}$ under the null of $\delta_0 = 0$ is given by

$$\mathbb{Q}_{3n} \Rightarrow -\sigma^{-2} \min_{\theta: c=0}\left(2Z(\theta) + V(\theta)\right) + \sigma^{-2} \min_{\theta: b=0 \text{ and } d=0}\left(2Z(\theta) + V(\theta)\right),$$

where $Z(\theta)$ be a mean zero Gaussian process with covariance kernel

$$\begin{aligned}
\varkappa(\theta_1, \theta_2) &= E d_1' x_i d_2' x_i \varepsilon_i^2 k\left(\frac{q_i - g_1}{h_1}\right) k\left(\frac{q_i - g_2}{h_2}\right) + E(b_1 - c_1)' x_i x_i' (b_2 - c_2) \varepsilon_i^2 \\
&\quad + E d_1' x_i x_i' (b_2 - c_2) \varepsilon_i^2 k\left(\frac{q_i - g_1}{h_1}\right) + E(b_1 - c_1)' x_i d_2' x_i \varepsilon_i^2 k\left(\frac{q_i - g_2}{h_2}\right)
\end{aligned}$$

where $\theta = (b', c', d', g, h)'$, and

$$V(\theta) = E\left(d'x_i k\left(\frac{q_i - g_1}{h_1}\right) + b'x_i\right)^2 - E\left(c'x_i\right)^2.$$

The derivation is analogous to Lee et al. (2011) in view of the tightness proof in Theorem 1. That is, due to Lee et al. (2011) we can consider the functions defined on the expanded parameter space θ while the tightness of the process is essentially the same as that in Theorem 1. In particular, for the finiteness of the entropy integral, recall the discussion in the proof of Lemma 1. Here we need to consider the whole parameter space for h and γ, not just neighborhoods of the true values of h_0 and γ_0. However, the class of functions is still a class of monotone functions and thus the same argument yields the finiteness of the entropy integral. Furthermore, the boundedness of the function K and the moment conditions for ε_i and x_i guarantees the envelope's L_2 norm to be bounded. The derivation of the finite dimensional convergence is standard. ■

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