Appendices for *Carbon Transit Through Degradation Networks*, Forney DC, Rothman DH.

A Detailed solution to the two-state example problem

The solution to (4) is readily found by assuming solutions exist of the form

$$\mathbf{x}(t) = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \mathrm{e}^{-\lambda \mathrm{t}}. \tag{A.1}$$

Substituting (A.1) for $\mathbf{x}$ in equation (4) results in the eigenvalue problem

$$-\lambda \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -k_1 & 0 \\ fk_1 & -k_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{A.2}$$

which, after subtracting the left hand side from both sides of (A.2) is

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -k_1 + \lambda & 0 \\ fk_1 & -k_2 + \lambda \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{A.3}$$

However, there are only a few possible values of the eigenvalues λ and eigenvectors $\mathbf{u}$ which can satisfy (A.3). This occurs when the matrix is singular and its determinant is zero, providing the characteristic equation

$$(-k_1 + \lambda)(-k_2 + \lambda) = 0, \tag{A.4}$$

which has the two solutions

$$\lambda_1 = k_1 \tag{A.5}$$

$$\lambda_2 = k_2. \tag{A.6}$$

The directions of the eigenvectors $\mathbf{u_1}$ and $\mathbf{u_2}$ are found by plugging λ_1 and λ_2 into (A.3).

The solution $\mathbf{x}$ is a superposition of both exponential decays,

$$\mathbf{x} = \alpha_1 \mathbf{u_1} \mathrm{e}^{-\lambda_1 \mathrm{t}} + \alpha_2 \mathbf{u_2} \mathrm{e}^{-\lambda_2 \mathrm{t}} \tag{A.7}$$

which is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \alpha_1 \begin{bmatrix} k_2 - k_1 \\ fk_1 \end{bmatrix} \mathrm{e}^{-\lambda_1 \mathrm{t}} + \alpha_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathrm{e}^{-\lambda_2 \mathrm{t}} \tag{A.8}$$

The weights α for each eigenvector are determined by substituting the initial condition $\mathbf{x}(0)$ into (A.7) at $t = 0$, resulting in the system of equations

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} \mathbf{u_1} & \mathbf{u_2} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}. \tag{A.9}$$

The solution to (A.9) is

$$\alpha_1 = \frac{1}{k_2 - k_1} x_1(0) \tag{A.10}$$

$$\alpha_2 = x_2(0) - \frac{fk_1}{k_2 - k_1} x_1(0) \tag{A.11}$$

The total mass of the system, $G(t) = x_1(t) + x_2(t)$ therefore decays as

$$G = r_1 \mathrm{e}^{-\lambda_1 \mathrm{t}} + \mathrm{r}_2 \mathrm{e}^{-\lambda_2 \mathrm{t}} \tag{A.12}$$

where, as seen from (A.8)

$$\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} \sum_i u_{1,i} & 0 \\ 0 & \sum_i u_{2,i} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}. \tag{A.13}$$

Solving (A.13) provides the estimates of the mass fractions r_1 and r_2 in eigenspace

$$r_1 = x_1(0)\left(1 + \frac{f k_1}{k_2 - k_1}\right), \tag{A.14}$$

$$r_2 = x_2(0) - x_1(0)\frac{f k_1}{k_2 - k_1}. \tag{A.15}$$