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Scaling from trees to forests: tractable macroscopic equations for forest dynamics.

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Appendix B: Stability of the stationary state given by equations 16-18

In this Appendix, we derive equations that govern the local stability of the stationary state given by equations (16-18). Let $\eta(s,t_0)$ and $\varepsilon(t_0)$ be:

$$s^*(t) = \hat{s}^* + \varepsilon(t) \tag{B-1}$$

$$N(s,t) = \widehat{N}(s) + \eta(s,t) \tag{B-2}$$

Substituting (B-1) and (B-2) into the von Forester equation (11) we get

$$\frac{\partial \eta(s,t)}{\partial t} = -G(s,(\hat{s}^*+\varepsilon(t)))\frac{\partial(\widehat{N}(s)+\eta(s,t))}{\partial s} - (\widehat{N}(s)+\eta(s,t))W(s,(\hat{s}^*+\varepsilon(t))). \tag{B-3}$$

We now linearize equation (B-3) with respect to the small perturbations by Taylor expanding and retaining only linear terms.

$$\frac{\partial \eta(s,t)}{\partial t} = -\varepsilon(t)Q_1(s) - Q_2(s)\frac{\partial \eta(s,t)}{\partial s} - Q_3(s)\eta(s,t) \tag{B-4}$$

where,

$Q_1(s) = \left(\frac{\partial G(s,\hat{s}^*)}{\partial s^*}\frac{d\widehat{N}(s)}{ds} + \frac{\partial W(s,\hat{s}^*)}{\partial s^*}\widehat{N}(s)\right),$

$Q_2(s) = G(s,\hat{s}^*),$

$Q_3(s) = W(s,\hat{s}^*).$

Turning now to the PPA equation (10), we introduce the perturbations (B-1) and (B-2) and get:

$$1 = \int_{\hat{s}^*+\varepsilon(t)}^{\infty} (\widehat{N}(s)+\eta(s,t))A(\hat{s}^*+\varepsilon(t),s)ds \tag{B-5}$$

After linearization of (B-5) and using (10) we have

$$0 = \int_{\hat{s}^*}^{\infty} \eta(s,t)A(\hat{s}^*,s)ds + \varepsilon(t)\int_{\hat{s}^*}^{\infty} \widehat{N}(s)\frac{\partial A(\hat{s}^*,s)}{\partial s^*}ds \tag{B-6}$$

The second integral is a constant, which we can denote as $C = \int_{\hat{s}^*}^{\infty} \widehat{N}(s)\frac{\partial A(\hat{s}^*,s)}{\partial \hat{s}^*}ds$. Then equation (B-6) gives us $\varepsilon(t)$ as an explicit function of $\eta(s,t)$:

$$\varepsilon(t) = -\frac{\int_{\hat{s}^*}^{\infty} \eta(s,t)A(\hat{s}^*,s)ds}{C}. \tag{B-7}$$

Equation (B-4) after substituting (B-7) becomes

$$\frac{\partial \eta(s,t)}{\partial t} = \frac{Q_1(s)}{C}\int_{\hat{s}^*}^{\infty} \eta(s,t)A(\hat{s}^*,s)ds - Q_2(s)\frac{\partial \eta(s,t)}{\partial s} - Q_3(s)\eta(s,t). \tag{B-8}$$

Finally, we introduce perturbations (B-1), (B-2) into the boundary condition equation (12)

$$\widehat{N}(s_0,t) + \eta(s_0,t) = \int_{s_0}^{\infty} (\widehat{N}(s)+\eta(s,t))F(\hat{s}^*+\varepsilon(t),s)ds/G(s_0,s^*,t). \tag{B-9}$$

Equation, (B-9) after linearization and substitution of (12), becomes

$$\eta(s_0,t) = \int_{s_0}^{\infty} \eta(s,t)F(\hat{s}^*,s)ds + \int_{s_0}^{\infty} \widehat{N}(s)\frac{\partial F(\hat{s}^*,s)}{\partial s^*}\varepsilon(t)ds. \tag{B-10}$$

Now we substitute (B-7) into (B-10) to get

$$\eta(s_0,t) = \int_{s_0}^{\infty} \eta(s,t)F(\hat{s}^*,s)ds - \int_{s_0}^{\infty} \widehat{N}(s)\frac{\partial F(\hat{s}^*,s)}{\partial s^*}\left[\frac{\int_{\hat{s}^*}^{\infty} \eta(s',t)A(\hat{s}^*,s')ds'}{C}\right]ds. \tag{B-11}$$

Therefore, the stationary state for a tree monoculture described by the boundary value problem (10), (11), (12), and $N(s,0)$ is given by equations (16), (17), and (18). The linear stability analysis of this stationary state can be performed by introduction of small perturbations (B-1), (B-2). The formula (B-7) gives $\varepsilon(t)$ as an explicit function of $\eta(s,t)$. Then the stability of the stationary state is determined by the boundary value problem of the partial-differential equation (B-8) with the boundary (B-11) and the initial $\eta(s,t_0)$ conditions.