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Appendix A. Analytical maximum-likelihood estimators of θ from Eqs. 7 and 8.

Proof 1: Analytical maximum-likelihood estimator for θ assuming known ψ where the likelihood for all count observations assumes the stationary Poisson distribution for observations on the first sampling occasion (k = 1) and the transition density for all other observations (k > 1). Let:

J = the number of sampling units

k = sampling occasion 1,...,K

$\alpha_{jk} = \frac{1}{\psi}\left(1 - e^{-\psi t_{jk}}\right)$

t_{jk} = length of time since previous sampling event at sample unit j, or length of time since sampling event k-1 at sample unit j

Then the likelihood for all count observations is:

$$L\left(x_{11}, \dots, x_{jk} \mid \alpha_{12}, \dots, \alpha_{JK}, \theta\right) = \prod_{j=1}^{J} \left[\frac{\left(\frac{\theta}{\psi}\right)^{x_{j1}} e^{-\frac{\theta}{\psi}}}{x_{j1}!} \prod_{k=2}^{K} \frac{(\theta\alpha_{jk})^{x_{jk}} e^{-\theta\alpha_{jk}}}{x_{jk}!} \right] \tag{A.1}$$

Simplifying the log-likelihood and solving for the MLE:

$$L\left(x_{11}, \dots, x_{jk} \mid \alpha_{12}, \dots, \alpha_{JK}, \theta\right) = \prod_{j=1}^{J} \frac{\left(\frac{\theta}{\psi}\right)^{x_{j1}} e^{-\frac{\theta}{\psi}}}{x_{j1}!} \prod_{j=1}^{J} \prod_{k=2}^{K} \frac{(\theta\alpha_{jk})^{x_{jk}} e^{-\theta\alpha_{jk}}}{x_{jk}!} \tag{A.2}$$

Factoring the likelihood:

$$= \left(\frac{\theta}{\psi}\right)^{\sum_{j=1}^{J} x_{j1}} e^{-J\left(\frac{\theta}{\psi}\right)} \theta^{\sum_{j=1}^{J}\sum_{k=2}^{K} x_{jk}} e^{-\theta \sum_{j=1}^{J}\sum_{k=2}^{K} \alpha_{jk}} \left[\prod_{j=1}^{J} \frac{1}{x_{j1}!}\right] \left[\prod_{j=1}^{J} \prod_{k=2}^{K} \frac{\alpha_{jk}{}^{x_{jk}}}{x_{jk}!}\right] \tag{A.3}$$

Taking the natural logarithm:

$$ln\left[L\left(x_{11},\dots,x_{jk}\,|\,\alpha_{12},\dots,\alpha_{JK},\theta\right)\right] =$$

$$\left(\sum_{j=1}^{J}x_{j1}\right)ln\theta \;-\; \left(\sum_{j=1}^{J}x_{j1}\right)ln\psi \;-\; J\left(\frac{\theta}{\psi}\right) \;+ \left(\sum_{j=1}^{J}\sum_{k=2}^{K}x_{jk}\right)ln\theta - \theta\sum_{j=1}^{J}\sum_{k=2}^{K}\alpha_{jk} \;+$$

$$\sum_{j=1}^{J}ln\left(\frac{1}{x_{j1}!}\right) + \sum_{j=1}^{J}\sum_{k=2}^{K}ln\left(\frac{\alpha_{jk}{}^{x_{jk}}}{x_{jk}!}\right) \tag{A.4}$$

Taking the derivative with respect to theta:

$$\frac{d}{d\theta}\left[ln\left[L\left(x_{11},\dots,x_{jk}\,|\,\alpha_{12},\dots,\alpha_{JK},\theta\right)\right]\right] = \frac{\sum_{j=1}^{J}x_{j1}}{\theta} - \frac{J}{\psi} + \frac{\sum_{j=1}^{J}\sum_{k=2}^{K}x_{jk}}{\theta} - \sum_{j=1}^{J}\sum_{k=2}^{K}\alpha_{jk} \tag{A.5}$$

Setting the derivative equal to zero and solving for the maximum-likelihood estimate of theta:

$$0 = \frac{\sum_{j=1}^{J}\sum_{k=1}^{K}x_{jk}}{\hat{\theta}_{mle}} - \frac{J}{\psi} - \sum_{j=1}^{J}\sum_{k=2}^{K}\alpha_{jk} \tag{A.6}$$

$$\frac{J}{\psi} + \sum_{j=1}^{J}\sum_{k=2}^{K}\alpha_{jk} = \frac{\sum_{j=1}^{J}\sum_{k=1}^{K}x_{jk}}{\hat{\theta}_{mle}} \tag{A.7}$$

$$\hat{\theta}_{mle}\left(\frac{J}{\psi} + \sum_{j=1}^{J}\sum_{k=2}^{K}\alpha_{jk}\right) = \sum_{j=1}^{J}\sum_{k=1}^{K}x_{jk} \tag{A.8}$$

$$\hat{\theta}_{mle} = \frac{\sum_{j=1}^{J}\sum_{k=1}^{K}x_{jk}}{\frac{J}{\psi} + \sum_{j=1}^{J}\sum_{k=2}^{K}\alpha_{jk}} \tag{A.9}$$

Proof 2: Analytical maximum-likelihood estimator for θ assuming known ψ and the stationary Poisson distribution for all count observations.

Let:

$$X^{*} \sim Poisson\left(\lambda = \frac{\theta}{\psi}\right)$$

n = total number of count observations

Then the likelihood for all count observations is:

$$L(x_{11}, \dots, x_{JK} \mid \theta, \psi) = \prod_{j=1}^{J} \prod_{k=1}^{K} \frac{\left(\frac{\theta}{\psi}\right)^{x_{jk}} e^{-\frac{\theta}{\psi}}}{x_{jk}!} \tag{A.10}$$

Factoring the likelihood:

$$\left(\frac{\theta}{\psi}\right)^{\sum_{j=1}^{J} \sum_{k=1}^{K} x_{jk}} e^{-n\left(\frac{\theta}{\psi}\right)} \left[\prod_{j=1}^{J} \prod_{k=1}^{K} \frac{1}{x_{jk}!}\right] \tag{A.11}$$

Taking the natural logarithm:

$$ln\left[L(x_{11}, \dots, x_{JK} \mid \theta, \psi)\right] = \left(\sum_{j=1}^{J} \sum_{k=1}^{K} x_{jk}\right) ln\theta \; - \; \left(\sum_{j=1}^{J} \sum_{k=1}^{K} x_{jk}\right) ln\psi \; - \; n\left(\frac{\theta}{\psi}\right) + \sum_{j=1}^{J} \sum_{k=1}^{K} ln\left(\frac{1}{x_{jk}!}\right) \tag{A.12}$$

Taking the derivative with respect to theta:

$$\frac{d}{d\theta}\left[ln\left[L(x_{11}, \dots, x_{JK} \mid \theta, \psi)\right]\right] = \frac{\sum_{j=1}^{J} \sum_{k=1}^{K} x_{jk}}{\theta} - \frac{n}{\psi} \tag{A.13}$$

Setting the derivative equal to zero and solving for the maximum-likelihood estimate of theta:

$$0 = \frac{\sum_{j=1}^{J} \sum_{k=1}^{K} x_{jk}}{\hat{\theta}_{mle}} - \frac{n}{\psi} \tag{A.14}$$

$$\frac{n}{\psi} = \frac{\sum_{j=1}^{J} \sum_{k=1}^{K} x_{jk}}{\hat{\theta}_{mle}} \tag{A.15}$$

$$\hat{\theta}_{mle} \frac{n}{\psi} = \sum_{j=1}^{J} \sum_{k=1}^{K} x_{jk} \tag{A.16}$$

$$\hat{\theta}_{mle} = \frac{\psi}{n} \sum_{j=1}^{J} \sum_{k=1}^{K} x_{jk} \tag{A.17}$$