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Appendix C. Calculations for the perturbation analysis.

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Demographic and dispersal matrices

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & F_1 e_s & F_2 e_s \\ s_j & s_j r_j & 0 & 0 \\ 0 & s_j(1-r_j) & 0 & 0 \\ 0 & 0 & s_a & s_a \end{bmatrix}$$

$$M_i\left(s\right) = \begin{bmatrix} 1 & 1 & mgf_1(s) & mgf_2(s) \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

where $mgf_i\ (s)$ is the moment-generating function of the WALD kernel, the subscript indicating the difference in parameter h_r used for the two adult-classes (noted thereafter h_{ri}), with i in {1,2}. The moment-generating function only exists for some finite interval around s = 0. It is assumed that there exists some value s_{max} such that all elements of $M_i\left(s\right)$ exist for all $0 \le s < s_{max}$ with s_{max} defined as:

$$s_{max} = \frac{V_t^2}{4 h_c \kappa \bar{U} \sigma_w}$$

For i = {1,2} we have:

$$mgf_i\left(s\right) = \exp[\frac{\gamma_i}{\mu_i}\left(1 - \sqrt{1 - \frac{2\mu_i^2 s}{\gamma_i}}\right)]$$

Perturbation analysis

The sensitivity of spread rate c* to a parameter x can be expressed in function of the sensitivity of c* to a matrix transition (from stage l to stage k) $a_{k,l}$:

$$\frac{\partial c^*}{\partial x} = \sum_{\mathrm{k,l}} \frac{\partial c^*}{\partial a_{k,l}} \frac{\partial a_{k,l}}{\partial x}$$

where $\dfrac{\partial c^*}{\partial a_{k,l}}$ is the conventional sensitivity of c* to a transition value (equations 26 of Neubert and Caswell 2000):

$$\frac{\partial c^*}{\partial a_{k,l}} = \frac{m_{k,l}}{s^* \rho_1} \frac{\partial \rho_1}{\partial h_{k,l}}$$

where s* is the value of s that minimizes c(s); $m_{k,l}$ is the (k,l) element of M(s*), ρ_1 is the dominant eigenvalue of $H(s) = A \circ M(s)$, $h_{k,l}$ is the (k,l) element of H(s*). See Neubert and Caswell 2000 for the calculation of $\frac{\partial \rho_1}{\partial h_{k,l}}$.

(Similarly the elasticity of c* to underlying parameters is given by:

$$\frac{x \partial c^*}{c^* \partial x} = \frac{x}{c^*} \sum_{k,l} \frac{\partial c^*}{\partial a_{k,l}} \frac{\partial a_{k,l}}{\partial x}$$

For the sake of clarity, we thereafter present the calculation of sensitivities only, as for any parameter *x*:

$$elasticity(x) = \frac{x}{c^*}.sensitivity(x)$$

In the case of dispersal parameters the sensitivity of c* to a given parameter *y* can then be expressed as:

$$\frac{\partial c^*}{\partial y} = \frac{a_{1,4}}{s^* \rho_1} \frac{\partial m_{1,4}}{\partial y} \frac{\partial \rho_1}{\partial h_{1,4}} + \frac{a_{1,5}}{s^* \rho_1} \frac{\partial m_{1,5}}{\partial y} \frac{\partial \rho_1}{\partial h_{1,5}}$$

as in our case, elements of M(s) describing dispersal have for coordinates (1,4) and (1,5), for values h_{r1} and h_{r2} respectively. Note that in the case of h_{r1} and h_{r2} the sensitivity is only half of the equation:

$$\frac{\partial c^*}{\partial h_{r1}} = \frac{a_{1,4}}{s^* \rho_1} \frac{\partial m_{1,4}}{\partial h_{r1}} \frac{\partial \rho_1}{\partial h_{1,4}}$$

$$\frac{\partial c^*}{\partial h_{r2}} = \frac{a_{1,5}}{s^* \rho_1} \frac{\partial m_{1,5}}{\partial h_{r1}} \frac{\partial \rho_1}{\partial h_{1,5}}$$

The calculation of sensitivities (and elasticities) requires finding the partial derivatives of the moment-generating function $\mathrm{mgf_i(s)}$ on the different parameters:

$$\frac{\partial m_{k,l}}{\partial y} = \frac{\partial mgf_i(s)}{\partial y} = \frac{\partial}{\partial y} \left(\exp\left[\frac{\gamma_i}{\mu_i} \left(1 - \sqrt{1 - \frac{2\mu_i{}^2 s}{\gamma_i}} \right) \right] \right)$$

where μ and γ are respectively the scale and shape parameters of the kernel, defined as:

$$\mu = \frac{\bar{U}h_{ri}}{v_t}$$

and

$$\gamma = \frac{\bar{U}{h_{ri}}^2}{2\kappa h_c \sigma_w}$$

By substituting the parameters μ and γ with the full set of parameters the formula can be calculated for each dispersal parameter:

Horizontal wind speed

$$\frac{\partial m_{k,l}}{\partial \bar{U}} = \frac{\partial mgf_i(s)}{\partial \bar{U}}$$

$$\frac{\partial m_{k,l}}{\partial \bar{U}} = \frac{\partial}{\partial \bar{U}} \left(\exp\left[\frac{h_{ri}v_t}{2\kappa h_c \sigma_w} \left(1 - \sqrt{1 - \frac{4\bar{U}h_c \kappa \sigma_w s}{{v_t}^2}} \right) \right] \right)$$

$$= \exp\left[\frac{h_{ri}v_t}{2\kappa h_c \sigma_w} \left(1 - \sqrt{1 - \frac{4\bar{U}h_c \kappa \sigma_w s}{{v_t}^2}} \right) \right] \frac{\partial}{\partial \bar{U}} \left(\frac{h_{ri}v_t}{2\kappa h_c \sigma_w} \left(1 - \sqrt{1 - \frac{4\bar{U}h_c \kappa \sigma_w s}{{v_t}^2}} \right) \right)$$

$$= mgf_i(s) \frac{\partial}{\partial \bar{U}} \left(\frac{h_{ri}v_t}{2\kappa h_c \sigma_w} \left(1 - \sqrt{1 - \frac{4\bar{U}h_c \kappa \sigma_w s}{{v_t}^2}} \right) \right)$$

$$= mgf_i(s) \frac{\partial}{\partial \bar{U}} \left(\frac{h_{ri}}{2\kappa h_c \sigma_w} \left(v_t - \sqrt{{v_t}^2 - 4\bar{U}h_c \kappa \sigma_w s} \right) \right)$$

$$= \frac{\partial}{\partial \bar{U}} \left(\frac{h_{ri}}{2\kappa h_c \sigma_w} (4\bar{U}h_c \kappa \sigma_w s - {v_t}^2) \right) . \frac{mgf_i\left(s\right)}{2\sqrt{{v_t}^2 - 4\bar{U}h_c \kappa \sigma_w s}}$$

$$\frac{\partial m_{k,l}}{\partial \bar{U}} = sh_{ri} \frac{mgf_i(s)}{R(s)}$$

with

$$R(s) = \sqrt{{v_t}^2 - 4\bar{U}h_c \kappa \sigma_w s}$$

Standard-deviation of vertical wind

$$\frac{\partial m_{k,l}}{\partial \sigma_w} = mgf_i(s)\frac{\partial}{\partial \sigma_w}\left(\frac{h_{ri}}{2\kappa h_c \sigma_w}\left(v_t - R(s)\right)\right)$$

$$= mgf_i(s)\left[\left(v_t - R(s)\right)\frac{\partial}{\partial \sigma_w}\left(\frac{h_{ri}}{2\kappa h_c \sigma_w}\right) + \frac{h_{ri}}{2\kappa h_c \sigma_w}\frac{\partial}{\partial \sigma_w}\left(v_t - R(s)\right)\right]$$

$$= \frac{-h_{ri}}{2\kappa h_c {\sigma_w}^2} mgf_i(s)\left[v_t - R(s) - \sigma_w\left(\frac{-4\bar{U}h_c\kappa s}{2R(s)}\right)\right]$$

$$= \frac{-h_{ri}}{2\kappa h_c {\sigma_w}^2} mgf_i(s)\left[v_t - \frac{R(s)^2 + 2\bar{U}h_c\kappa\sigma_w s}{R(s)}\right]$$

$$= -\frac{h_{ri}\, mgf_i(s)}{2\kappa h_c {\sigma_w}^2}\left[\frac{2\kappa h_c \bar{U}\sigma_w s - {v_t}^2}{R(s)} + v_t\right]$$

Canopy height and wind velocity variable (similar derivation to standard deviation of vertical wind)

$$\frac{\partial m_{k,l}}{\partial h_c} = -\frac{h_{ri} mgf_i(s)}{2\kappa \sigma_w {h_c}^2}\left[\frac{2\kappa h_c \bar{U}\sigma_w s - {v_t}^2}{R(s)} + v_t\right]$$

$$\frac{\partial m_{k,l}}{\partial \kappa} = -\frac{h_{ri} mgf_i(s)}{2\sigma_w h_c \kappa^2}\left[\frac{2h_c \kappa \bar{U}\sigma_w s - {v_t}^2}{R(s)} + v_t\right]$$

Height of release

$$\frac{\partial m_{k,l}}{\partial h_{ri}} = mgf_i(s)\frac{\partial}{\partial h_{ri}}\left(\frac{h_{ri}}{2\kappa h_c \sigma_w}\left(v_t - R(s)\right)\right)$$

$$= \frac{mgf_i(s)}{2\kappa h_c \sigma_w}\left(v_t - R(s)\right)$$

Seed terminal velocity

$$\frac{\partial m_{k,l}}{\partial v_t} = mgf_i(s)\frac{\partial}{\partial v_t}\left(\frac{h_{ri}}{2\kappa h_c \sigma_w}\left(v_t - \sqrt{{v_t}^2 - 4\bar{U}h_c \sigma_w s}\right)\right)$$

$$= mgf_i(s)\frac{h_{ri}}{2\kappa h_c \sigma_w} - \frac{\partial}{\partial v_t}\left(\frac{h_{ri}}{2\kappa h_c \sigma_w}\sqrt{{v_t}^2 - 4\bar{U}h_c \sigma_w s}\right)$$

$$= mgf_i\left(s\right)\frac{h_{ri}}{2\kappa h_c\sigma_w} - \left(\frac{h_{ri}}{2\kappa h_c\sigma_w}\frac{2v_t}{2\sqrt{v_t^{\,2} - 4\bar{U}h_c\sigma_w s}}\right)$$

$$= mgf_i\left(s\right)\frac{h_{ri}}{2\kappa h_c\sigma_w}\left(1 - \frac{v_t}{R(s)}\right)$$

Literature Cited

Neubert, M. G. and H. Caswell. 2000. Demography and dispersal: calculation and sensitivity analysis of invasion speed for structured populations. Ecology **81**:1613-1628.