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APPENDIX B: Mathematical Details

The Lagrange Multipliers.

The partition function (Eq. A.4) for $R(n, \varepsilon)$ is given by solving the integral and sum in the denominator of text Eq.3. Solving the integral first, we obtain:

$$Z(\lambda_1, \lambda_2) = \sum_{n=1}^{N_0} \int_0^{E_0} e^{-\lambda_1 n} e^{-\lambda_2 n \varepsilon} d\varepsilon = \sum_{n=1}^{N_0} \frac{e^{-\lambda_1 n}}{\lambda_2 n} (1 - e^{-\lambda_2 n E_0}) \quad (\text{B.1})$$

Now, we apply Eq. A.5 to the determination of λ_2 using

$$\frac{\partial \ln(Z)}{\partial \lambda_k} = \frac{1}{Z} \frac{\partial Z}{\partial \lambda_k} \quad (\text{B.2})$$

Since the average energy per species is E_0/S_0 , Eq. A.5 becomes

$$\frac{E_0}{S_0} = \frac{1}{Z} \frac{\partial Z(\lambda_1, \lambda_2)}{\partial \lambda_2} = \frac{1}{Z} \sum_{n=1}^{N_0} \frac{e^{-\lambda_1 n} (1 - (1 + \lambda_2 E_0) e^{-\lambda_2 n E_0})}{\lambda_2^2 n} \quad (\text{B.3})$$

Comparing Eqs. B.1, B.3, we see that if we neglect the term $e^{-\lambda_2 n E_0}$ in each of the equations, we obtain $\lambda_2 = S_0/E_0$ as in Eq. 7a in the text. Because $n \geq 1$, the exponential term is less than or equal to $(1+S_0) e^{-S_0} \ll 1$, and thus neglect of the exponential term is self-consistently justified.

Eq. A.5 applied to λ_1 and neglecting terms of order $e^{-\lambda_1 N_0}$ gives

$$\frac{1}{\ln\left(\frac{1}{1 - e^{-\lambda_1}}\right)} \left(\frac{e^{-\lambda_1}}{1 - e^{-\lambda_1}}\right) = \frac{N_0}{S_0} \quad (\text{B.4})$$

If we now approximate $e^{-\lambda_1} \approx 1 - \lambda_1$, we obtain text eq. 7b. Again, self-consistency justifies our approximation because for our data sets $\lambda_1 \ll 1$ and $e^{-\lambda_1 N_0} \ll 1$. Although the analytical

simplifications are extremely accurate, we used numerical solutions to the exact equations for the Lagrange multipliers for purposes of testing theory.

Again using Eq. A.5 and text Eqs. 3 and 9, the Lagrange multiplier λ_p is obtained from

$$\frac{n_0 A}{A_0} = \frac{\sum_{n=1}^{n_0} n x^n}{\sum_{n=1}^{n_0} x^n} = \frac{1}{1 - x^{n_0+1}} \left(\frac{x}{1-x} - x^{n_0+1} \left(n_0 + \frac{1}{1-x} \right) \right) \quad (\text{B.5})$$

where $x \equiv e^{-\lambda_p}$. No closed-form solution exists for all scales, and so again numerical solutions were used.