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Appendix A. Mathematical details of the continuous-time correlated random walk (CTCRW) model.

Appendix A: Mathematical Details of the CTCRW Model

Discretizing the continuous-time model

Here we present the details of discretizing the continuous-time CRW (CTCRW) model into a form which can be placed into a discrete-time state-space model, so that the fast Kalman filter and smoother can be used for estimation and smoothing. Durbin and Koopman (2001) present a similar approach for discretizing an integrated Brownian motion process for state-space analysis. Here, we modify their derivation for use on an integrated Ornstien-Uhlenbeck (OU) process.

First, we define the velocity process $\nu(t)$ and the position process $\mu(t)$ as a function of a Brownian motion process (note, we have left off the coordinate notation for simplicity),

$$\nu(t) = \gamma + \frac{\sigma e^{-\beta t}}{\sqrt{2\beta}} W\left(e^{2\beta t}\right) \tag{A.1}$$

and

$$\begin{aligned} \mu(t) &= \mu(0) + \int_0^t \nu(u)du \\ &= \mu(0) + \gamma t + \frac{\sigma}{\sqrt{2\beta}} \int_0^t e^{-\beta u} W\left(e^{2\beta u}\right) du, \end{aligned} \tag{A.2}$$

where γ is a constant over $(0, t]$, σ and β are parameters, and $W(\cdot)$ is a standard Brownian motion process with $E[W(t)] = 0$, variance $V[W(t)] = t$, and covariance $C[W(s), W(t)] = min(s, t)$.

The discretization of the CTCRW model begins be assuming that the process is sampled at times $t_1, \ldots, t_n$. Let $= \nu_i = \nu(t_i)$ for $i = 1, \ldots, n$. Then, assuming γ_i is a constant over $(t_i, t_{i+1}]$ and, for $t \in (t_i, t_{i+1}]$, $\nu(t)$ follows an OU process with mean γ_i, covariance parameters σ and β, and initial value ν_i, we obtain the transition equation for ν_{i+1}

$$\begin{aligned} \nu_{i+1} &= \gamma_i + \frac{\sigma_i e^{-\beta t_{i+1}}}{\sqrt{2\beta}} W\left(e^{2\beta t_{i+1}}\right) \\ &= \nu_i e^{-\beta\delta_i} + \gamma(1 - e^{-\beta\delta_i}) + \frac{\sigma e^{-\beta t_{i+1}}}{\sqrt{2\beta}} \left\{W(e^{2\beta t_{i+1}}) - W(e^{2\beta t_i})\right\} \\ &= \nu_i e^{-\beta\delta_i} + \gamma(1 - e^{-\beta\delta_i}) + \zeta_i, \end{aligned} \tag{A.3}$$

where

$$\zeta_i = \frac{\sigma e^{-\beta t_{i+1}}}{\sqrt{2\beta}} \left\{W(e^{2\beta t_{i+1}}) - W(e^{2\beta t_i})\right\} \tag{A.4}$$

is a Gaussian random variable with $E[\zeta_i] = 0$ and variance $V[\zeta_i] = \sigma^2\{1 - e^{-2\beta\delta_i}\}/2\beta$. The serial independence of the ζ_i result from the independent increments property of the Brownian process. Note, that the OU process parameters were not indexed, but, in general they might also vary by time intervals $(t_i, t_{i+1}]$. Even with shifts in mean or covariance, the process will remain smooth in continuous time. This can be seen by letting $\delta_i \to 0$ and noting that $\nu_{i+1} \to \nu_i$ for any β, σ, or γ_i as long as they are constant over $(t_i, t_{i+1}]$. This allows for inclusion of covariates which can affect the velocity process; as in the harbor seal example.

By using the velocity process formulation above, the position process at t_{i+1}, μ_{i+1}, can be formulated in terms of the position and velocity process at time t_i,

$$
\begin{aligned}
\mu_{i+1} = \mu_i + \int_{t_i}^{t_{i+1}} & \nu_i e^{-\beta(u-t_i)} + \gamma_i(1 - e^{-\beta(u-t_i)}) \\
& \qquad + \frac{\sigma e^{-\beta u}}{\sqrt{2\beta}} \left\{ W(e^{2\beta u}) - W(e^{2\beta t_i}) \right\} du \\
= \mu_i + \nu_i & \left(\frac{1 - e^{-\beta\delta_i}}{\beta} \right) + \gamma_i \left(\delta_i - \frac{1 - e^{-\beta\delta_i}}{\beta} \right) + \xi_i,
\end{aligned}
\tag{A.5}
$$

where

$$
\xi_i = \frac{\sigma}{\sqrt{2\beta}} \int_{t_i}^{t_{i+1}} e^{-\beta u} \left\{ W(e^{2\beta u}) - W(e^{2\beta t_i}) \right\} du. \tag{A.6}
$$

Again, due to the independent increments property of the Brownian process, the ξ_i are serially independent and $E[\xi_i] = 0$. Also, the fact that the Brownian process is continuous almost everywhere on $(t_i, t_{i+1}]$, permits exchange of integration order which allows evaluation of variance,

$$
\begin{aligned}
V[\xi_i] &= \frac{\sigma^2}{2\beta} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} e^{-\beta u} e^{-\beta v} \\
&\qquad \times E\left[\left\{ W(e^{2\beta u}) - W(e^{2\beta t_i}) \right\} \left\{ W(e^{2\beta v}) - W(e^{2\beta t_i}) \right\} \right] dv du \\
&= \frac{\sigma^2}{\beta^2} \left\{ \delta_i - \frac{2}{\beta}\left(1 - e^{-\beta\delta_i}\right) + \frac{1}{2\beta}\left(1 - e^{-2\beta\delta_i}\right) \right\}.
\end{aligned}
\tag{A.7}
$$

The covariance between ξ_i and ζ_i is also necessary for likelihood evaluation and can be calculated in a similar fashion to give,

$$
C[\xi_i, \zeta_i] = \frac{\sigma^2}{2\beta^2} \left\{ 1 - 2e^{-\beta\delta_i} + e^{-2\beta\delta_i} \right\}. \tag{A.8}
$$

Random drift model

The properties of the dual scale mean process model presented in the northern fur seal example can be readily derived given the results of the previous section. Before beginning, recall the definition of the mean process model

$$
\begin{aligned}
\nu(t) &= \gamma(t) + \vartheta(t) \\
&= \frac{\sigma_\gamma e^{-\beta t/\psi}}{\sqrt{2\beta/\psi}} W^{(\gamma)}(e^{2\beta/\psi}) + \frac{\sigma e^{-\beta t}}{\sqrt{2\beta}} W^{(\vartheta)}(e^{2\beta})
\end{aligned}
\tag{A.9}
$$

where $W^{(\gamma)}$ and $W^{(\vartheta)}$ are independent Brownian motions and $\psi > 1$ is a time scale multiplying factor. Note that we have, again, dropped the coordinate notation for simplicity.

Using the previous results, the transition equations for the state $\boldsymbol{\alpha}_{i+1} = [\mu_{i+1}, \vartheta_{i+1}, \gamma_{i+1}]'$, in reverse order, are

$$\begin{aligned}
\gamma_{i+1} &= e^{-\beta\delta_i/\psi}\gamma_i + \omega_i \\
\vartheta_{i+1} &= e^{-\beta\delta_i}\vartheta_i + \zeta_i \\
\mu_{i+1} &= \mu_i + \vartheta_i\left(\frac{1-e^{-\beta\delta_i}}{\beta}\right) + \gamma_i\psi\left(\frac{1-e^{-\beta\delta_i/\psi}}{\beta}\right) + \xi_i.
\end{aligned} \tag{A.10}$$

The state equation error variables are given by

$$\begin{aligned}
\omega_i &= \frac{\sigma_\gamma e^{-\beta t_{i+1}/\psi}}{\sqrt{2\beta/\psi}}\left\{W^{(\gamma)}(e^{2\beta t_{i+1}/\psi}) - W^{(\gamma)}(e^{2\beta t_i/\psi})\right\} \\
\zeta_i &= \frac{\sigma e^{-\beta t_{i+1}}}{\sqrt{2\beta}}\left\{W^{(\vartheta)}(e^{2\beta t_{i+1}}) - W^{(\vartheta)}(e^{2\beta t_i})\right\} \\
\xi_i &= \frac{\sigma}{\sqrt{2\beta}}\int_{t_i}^{t_{i+1}} e^{-\beta u}\left\{W^{(\gamma)}(e^{2\beta u}) - W^{(\gamma)}(e^{2\beta t_i})\right\} du \\
&\quad + \frac{\sigma_\gamma}{\sqrt{2\beta/\psi}}\int_{t_i}^{t_{i+1}} e^{-\beta u/\psi}\left\{W^{(\vartheta)}(e^{2\beta u/\psi}) - W^{(\vartheta)}(e^{2\beta t_i/\psi})\right\} du.
\end{aligned} \tag{A.11}$$

The variances and covariances given in the northern fur seal movement section are directly derived from results in the section above along with the independence of $W^{(\gamma)}$ and $W^{(\vartheta)}$.

Literature Cited

Durbin, J., and S. Koopman. 2001. Time Series Analysis by State Space Methods. Oxford University Press, Oxford, UK.