Ecological Archives E083-007-A4

Sebastian Diehl. 2002. Phytoplankton, light, and nutrients in a gradient of mixing depths: theory. *Ecology* 83:386-398.

Appendix D: Effects of background turbidity on equilibrium conditions in the closed system model with mixing depth-dependent algal loss rate.

For a given incident light intensity and mixing depth, light availability depends on background turbidity and algal biomass. The equilibrium equations can thus be written as

$$\frac{dW}{dt} = P(R^*, W^*, K_{bg}) - l(v,z)W^* = 0 \tag{D.1}$$

$$\frac{dR_s}{dt} = cl(v,z)W^* - rR_s^* = 0 \tag{D.2}$$

$$zR_{tot} = cW^* + R_s^* + zR^* \tag{D.3}$$

where $P(R,W,K_{bg})$ is total algal production in the mixed layer and corresponds to the integral in Eq. 1a. Implicit differentiation of Eqs. D.1-D.3 with respect to K_{bg} and rearrangement yields

$$\frac{dW^*}{dK_{bg}} = \frac{1}{b}\left(\frac{\partial P}{\partial R}\frac{dR^*}{dK_{bg}} + \frac{\partial P}{\partial K_{bg}}\right) \tag{D.4}$$

$$\frac{dR_s^*}{dK_{bg}} = \frac{cl(v,z)}{r}\frac{dW^*}{dK_{bg}} \tag{D.5}$$

$$z\frac{dR^*}{dK_{bg}} + c\frac{dW^*}{dK_{bg}} + \frac{dR_s^*}{dK_{bg}} = 0 \tag{D.6}$$

with $b = l(v,z) - \partial P/\partial W$. Substitution of Eqs. D.4 and D.5 into Eq. D.6 yields

$$\frac{dR^*}{dK_{bg}} = \frac{-\dfrac{\partial P}{\partial K_{bg}}\left[r + l(v,z)\right]}{\dfrac{\partial P}{\partial R}\left[r + l(v,z)\right] + \dfrac{rbz}{c}}. \tag{D.7}$$

In Appendices A and B, I have shown that $\partial P/\partial R > 0$ and that, at equilibrium, $b > 0$. Furthermore, $\partial P/\partial K_{bg} < 0$, because, at constant W and R, an increase in background turbidity means that less light is available for algal growth. Thus the numerator and the denominator of Eq. D.7 are both positive and $dR^*/dK_{bg} > 0$. In other words, nutrient availability at an interior equilibrium (i.e., $W^*>0$) is positively related to background turbidity .

Substitution of Eq. D.7 into Eq. D.4 and rearrangement yields

$$\frac{dW^*}{dK_{bg}} = \frac{1}{b}\frac{\partial P}{\partial K_{bg}}\left[1 - \frac{r + l(v,z)}{r + l(v,z) + \dfrac{rbz}{c\,\partial P/\partial R}}\right] \tag{D.8}$$

which is negative. Subsequent substitution of Eq. D.8 into Eq. D.5 reveals that $dR_s{*}/dK_{bg} < 0$. Finally, substitution of $\omega^* z = W^*$ into Eq. D.8 reveals that $d\omega^*/dK_{bg} < 0$. In other words, at an interior equilibrium, algal biomass concentration as well as the standing stocks of algal biomass and of sedimented nutrients are all negatively related to background turbidity.

The concentration of total nutrients in the mixed layer, R_m, can be calculated using the mass balance constraint . Substitution of $R_m{*} = cW^*/z + R^*$ into Eq. D.3 and subsequent implicit differentiation with respect to K_{bg} yields

$$\frac{dR_m{*}}{dK_{bg}} = -\frac{1}{z}\frac{dR_s{*}}{dK_{bg}}. \tag{D.9}$$

Because $dR_s{*}/dK_{bg} < 0$, this expression is positive. In other words, at an interior equilibrium, the concentration of total nutrients in the mixed layer is positively related to background turbidity.

Finally, $dR^*/dK_{bg} > 0$ and $dW^*/dK_{bg} < 0$ can only be simultaneously fulfilled, if, with increasing background turbidity, algal production becomes increasingly limited by light (i.e., if $dI_{out}{*}/dK_{bg} < 0$). Hence, at an internal equilibrium, I_{out}* must be negatively related to background turbidity.