



# Making Sense of Algebra with Realistic Mathematics Education

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# Expectations?

- How can RME help teachers and students to understand and meet the common core goals of Algebra?
- How can students develop strategies for solving systems-of-equations?
- What can I do with my 6<sup>th</sup> (7<sup>th</sup>, 8<sup>th</sup>) grade students?
- How may a learning trajectory for (systems of) equations look like?
- Sources?

# Tug-of-war

How would a sixth grade student solve this problem?

Four oxen are as strong as five horses.



An elephant is as strong as one ox and two horses.



5. Which animals will win the tug-of-war below? Give a reason for your prediction.

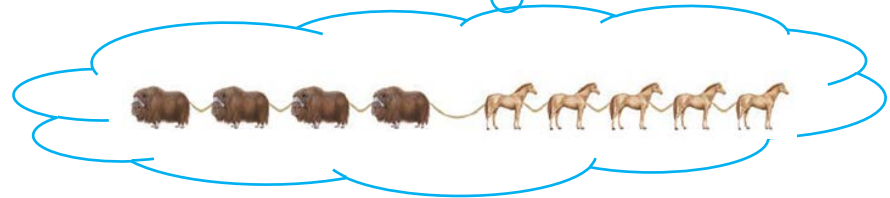
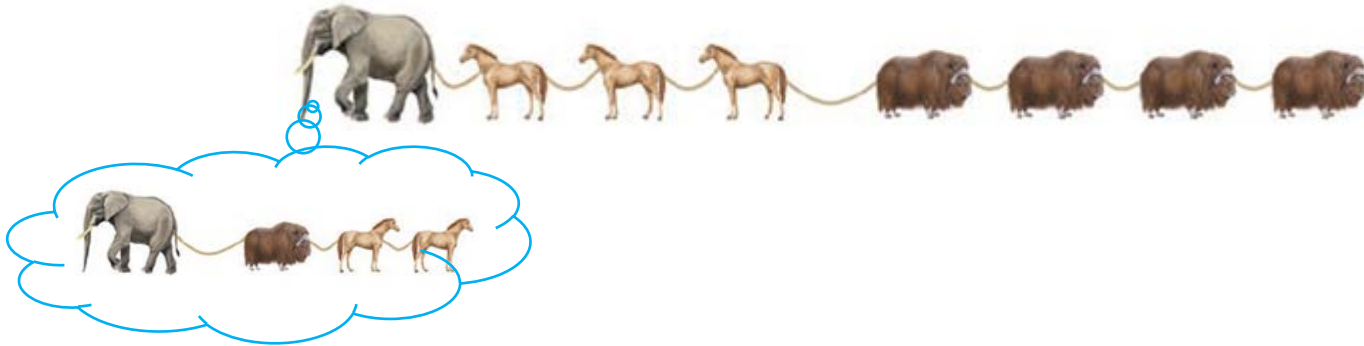


# Substitute and compare

Four oxen are as strong as five horses.



An elephant is as strong as one ox and two horses.



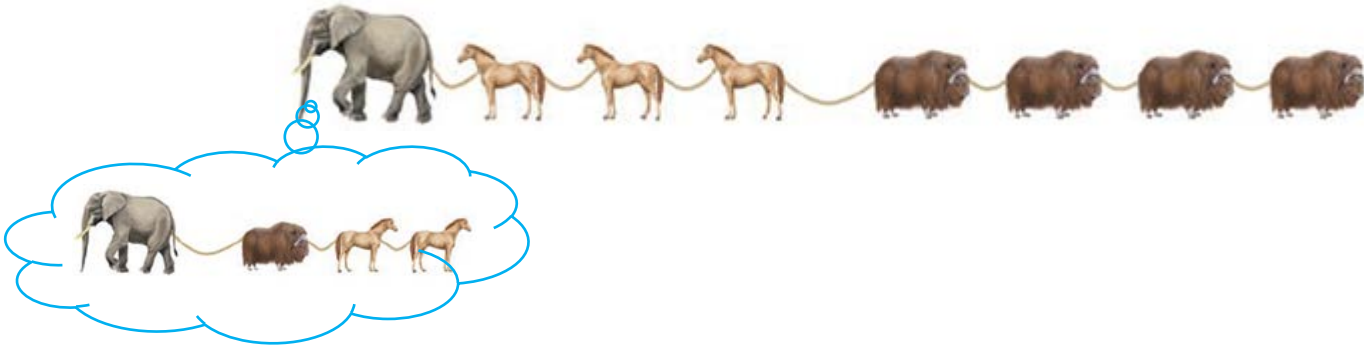
This side wins

# Substitute and compare

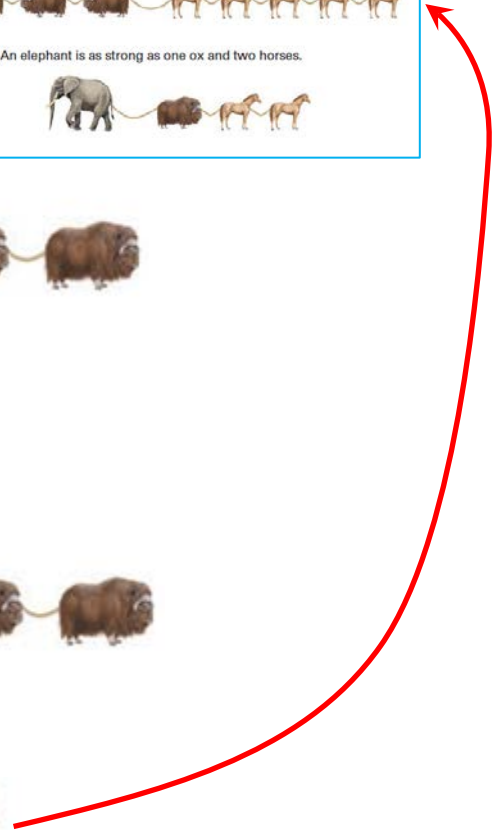
Four oxen are as strong as five horses.



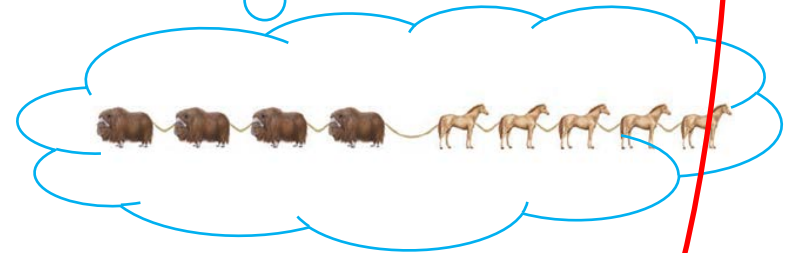
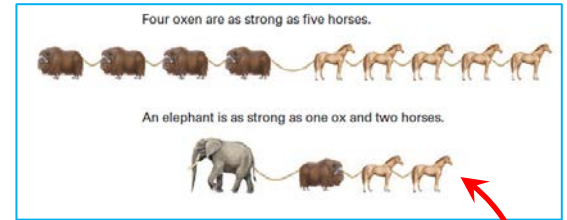
An elephant is as strong as one ox and two horses.



This side wins



# Substitute and compare



This side wins

# ICEBERG METAPHOR

FORMAL

TIP OF THE ICEBERG

$$\begin{aligned} L &= 30y + 320 \\ L &= 35y + 295 \end{aligned}$$

$$15 + 8x = 37 - 3x$$

INFO

# Three frogs

How would a sixth grade student solve this problem? Visualizations?



MT decides to hold a jumping contest. The three contestants are Sunny, Cal, and Legs. In this contest, all of a frog's steps are the same size. Also, when a frog jumps, he always travels the same distance (the referee's frog-jumping rule applies). For the contest, each frog must complete two jump sequences and then take steps to end up in the same place. Your problem is to find out which frog has the longest jump.

## ■ Sunny's Results:

When Sunny jumps 4 times and takes 11 steps **forward**, he lands in the same place as when he jumps 5 times and takes 4 steps **forward**.

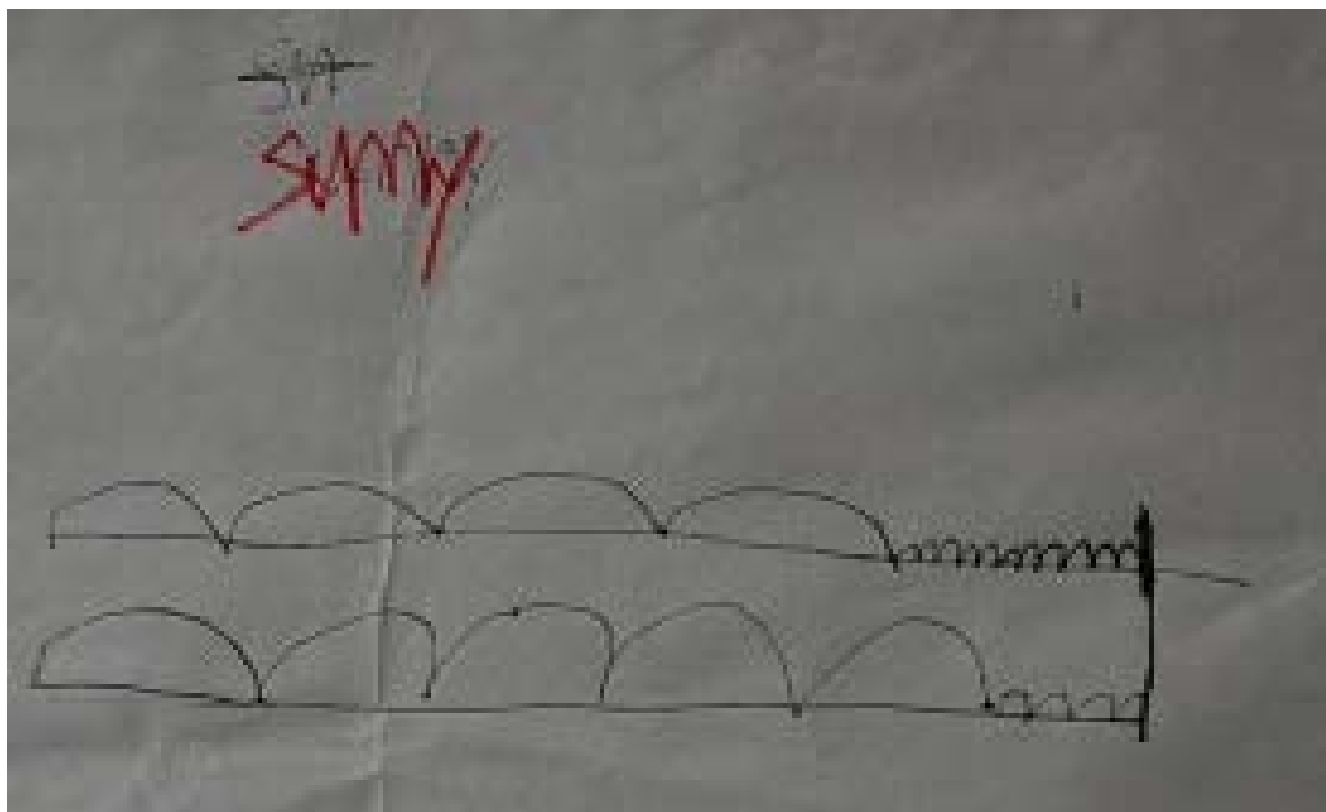
## ■ Cal's Results:

When Cal jumps 3 times and takes 6 steps **forward**, he lands in the same place as when he jumps 4 times and takes 2 steps **backward**.

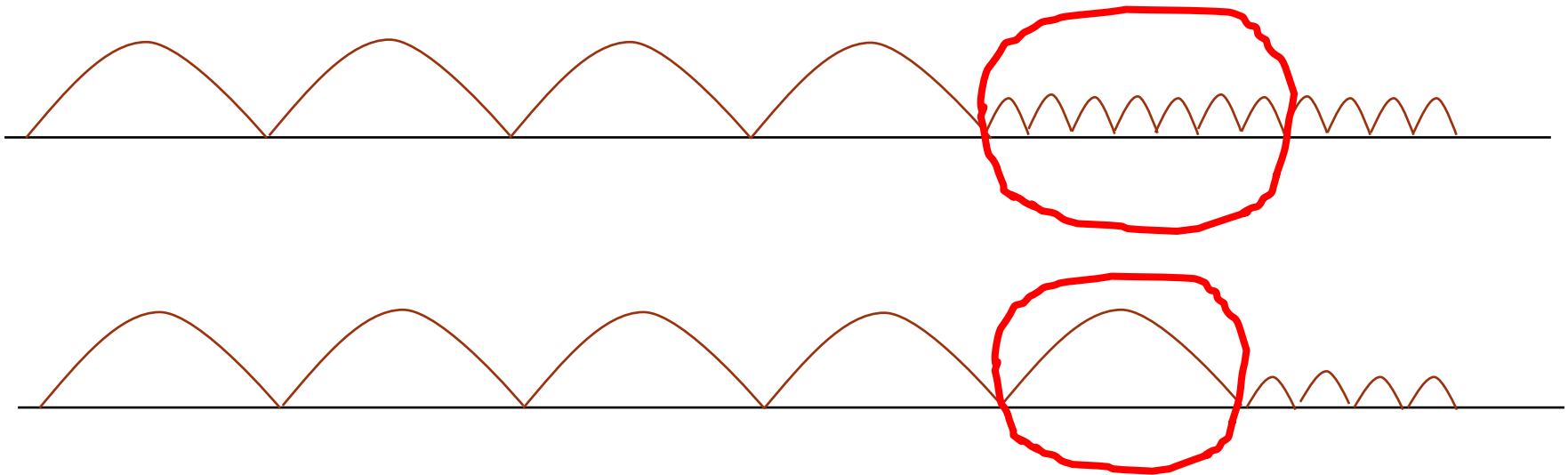
## ■ Legs' Results:

When Legs jumps 2 times and takes 13 steps **forward**, he lands in the same place as when he jumps 4 times and takes 5 steps **backward**.



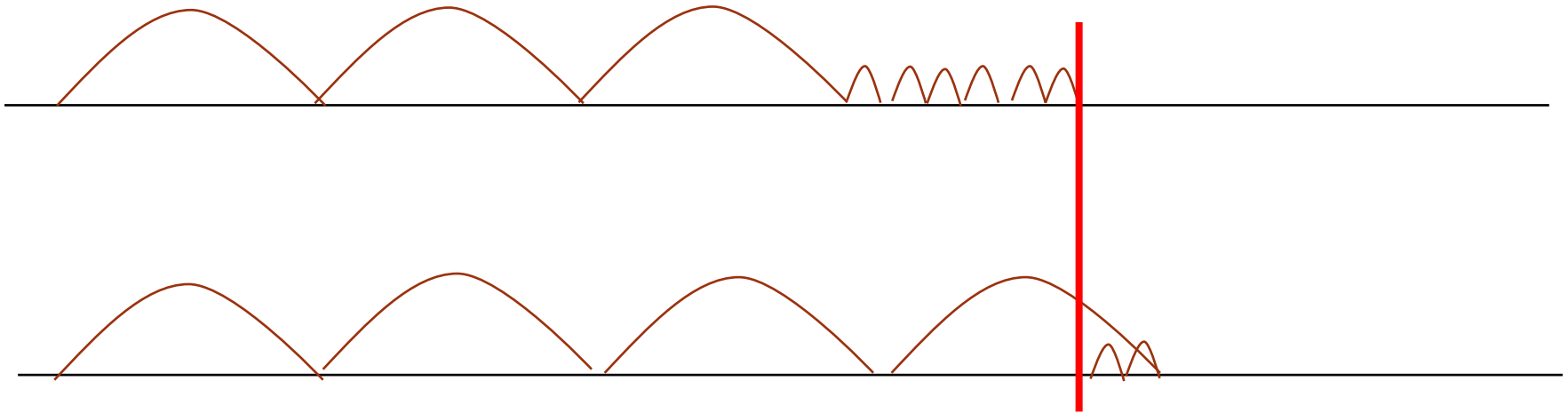


# Sunny



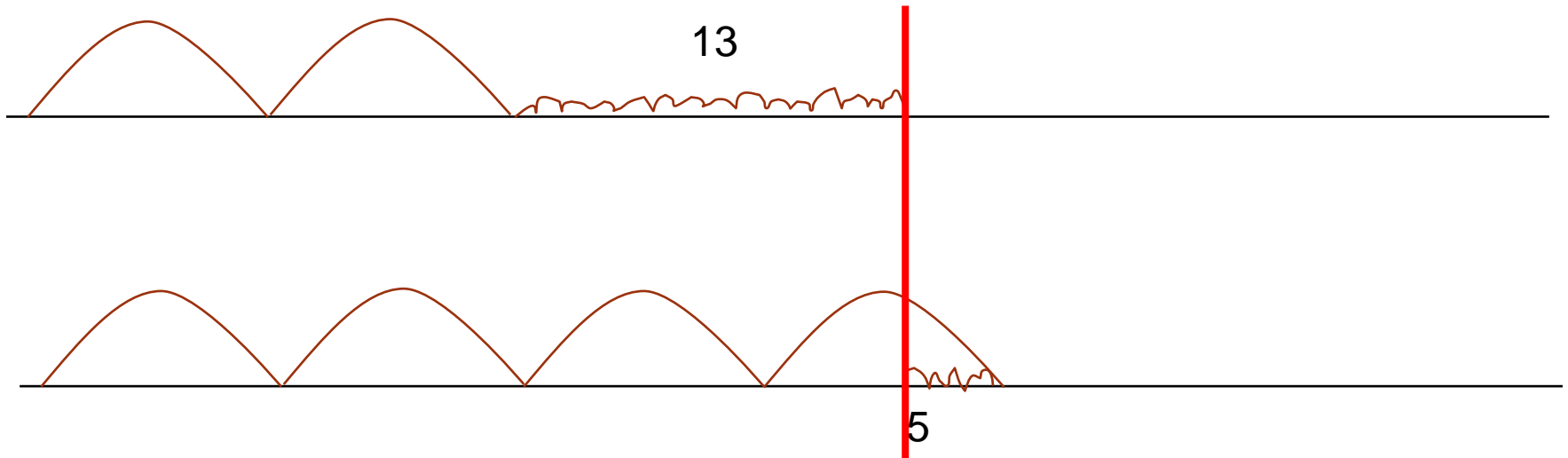
1 jump = 7 steps

# Cal



1 jump = 8 steps

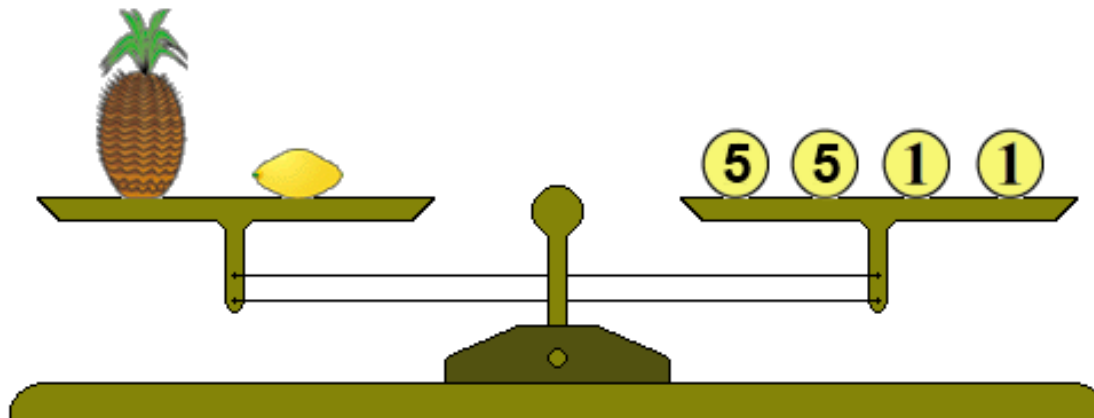
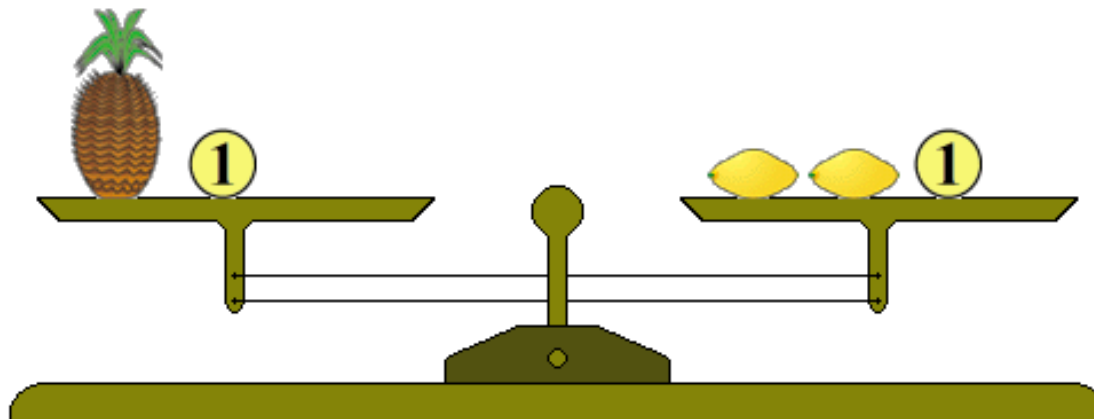
# Leg



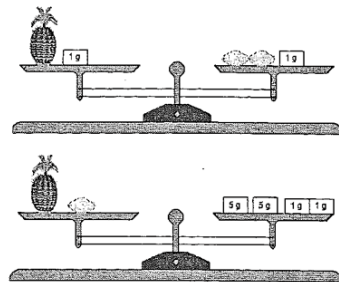
2 jump = 18 steps  
1 jump = 9 steps

# Balance scales

How can this problem be solved using informal strategies?

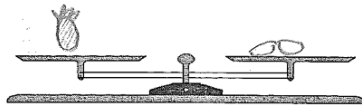


# 1



Compare the student work.  
What are the differences,  
what is the same?  
What concepts are developed?

Step 1:

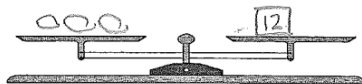


What did you do? subtract 1g  
from each side.

Why will the scale remain balanced?

I took the same amount  
away from both sides.

Step 2:

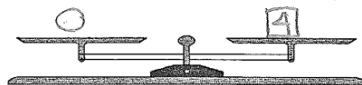


What did you do? substituted  
the pineapple for 2  
lemons.

Why will the scale remain balanced?

The substitution weight  
was equivalent.

Step 3:

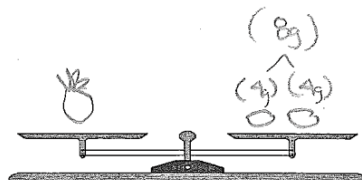


What did you do? Divided both  
sides by 3 on each  
side.

Why will the scale remain balanced?

I did the exact same  
thing to both sides.

Step 4:



What did you do? substituted  
the lemons for 4g each  
to show that the pine-  
apple was = to 8 grams

Why will the scale remain balanced?

4 grams per lemon

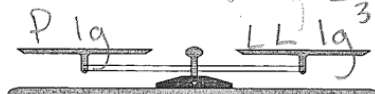
8 grams per pineapple

# 2

P = pineapple  
L = lemon

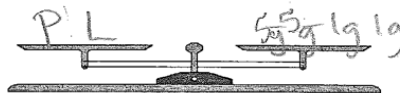
$$1P + 1L = 12g$$

Step 1:



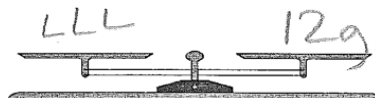
$$\begin{array}{r} P \ 1g \\ - 1g \\ \hline P = LL \end{array}$$

Step 2:



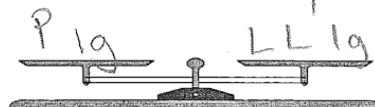
$$P + L = 5g$$

Step 3:



$$\begin{array}{r} LLL \\ \hline 3 \\ 12g \\ \hline 3 \end{array}$$

Step 4:



$$\begin{array}{r} P \ 1g \\ - 1g \\ \hline 4g + 4g = 8g + 1g = 9g \end{array}$$

4 grams per lemon

Consider the balance scales on the left.  
Find the weight of one lemon and one pineapple using a *substitution strategy*.

For each step, draw the balance scales and explain your method.

What did you do?

I subtracted 1g from both side of the top scale  
Why will the scale remain balanced?  
1 pineapple = 2 lemons

What did you do?

I substituted the pineapple on the bottom scale for 2 lemons  
Why will the scale remain balanced?  
 $3L = 5g$

What did you do?

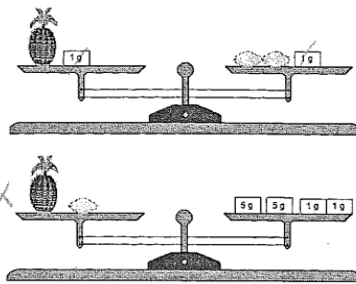
I divided both sides of the scale by 3.  
Why will the scale remain balanced?  
 $\frac{3}{3} = 1$   $\frac{12}{3} = 4$

What did you do?

I substituted the lemons for 4g on the right side then subtracted 1g from the  
Why will the scale remain balanced?  
 $1L = 4g$   $1P = 8g$  left.

8 grams per pineapple

I put  
my work  
on the back



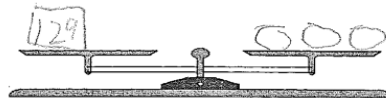
5

Consider the balance scales on the left.

Find the weight of one lemon and one pineapple using a *substitution strategy*.

For each step, draw the balance scales and explain your method.

Step 1:



What did you do? I took away one gram from both sides and substituted the pineapple for 12 grams.  
Why will the scale remain balanced? because 12 g is equal to one pineapple and one lemon.

Step 2:



What did you do?

$$1P + 1L = 2L + 12$$

$$-1L$$

$$1P = 1L + 12$$

1L is half of a P

balanced?

$$1P = 1L + 12$$

$$12 = 3L$$

$$\frac{12}{3} = 1L$$

balanced?

$$1P + 1L = 12$$

$$2P + 2L = 24$$

$$1P = 6$$

4

$$1P + 4 = 12$$

balanced?

4

grams per lemon

6

grams per pineapple



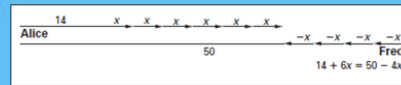
# ICEBERG METAPHOR

FORMAL

TIP OF THE ICEBERG

$$L=30y+320$$
$$L=35y+295$$

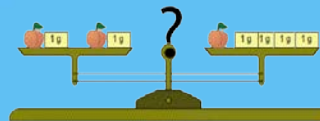
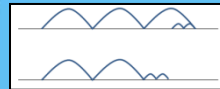
$$15 + 8x = 37 - 3x$$



PRE-FORMAL

FLOATING  
CAPACITY

## MODELS



Order	Sandwich	Total cost
1	3	5
2	3	1
3	3	1
4	3	1
5	3	1
6	3	1

Use a combination strategy to find the price of one apple.



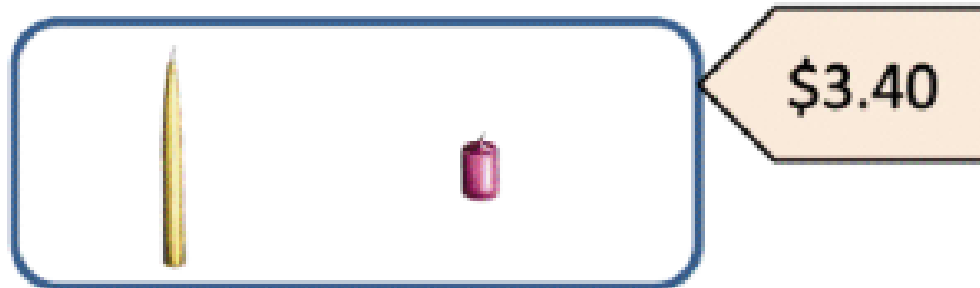
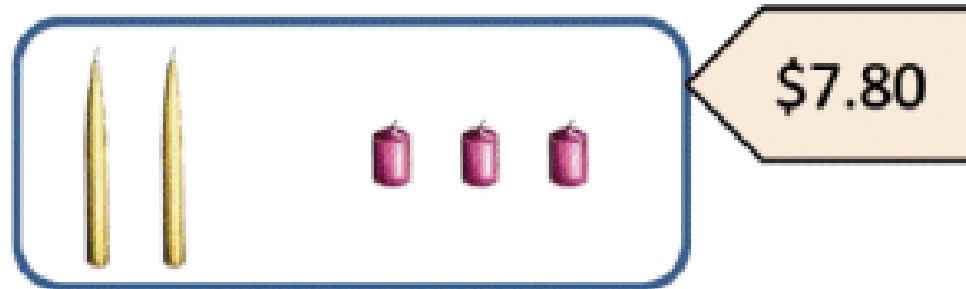
Find the price of one apple.



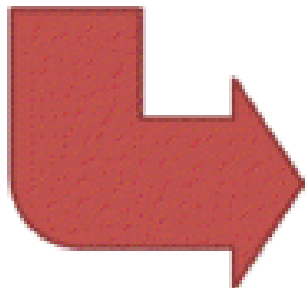
INFORMAL



# Generate new equations



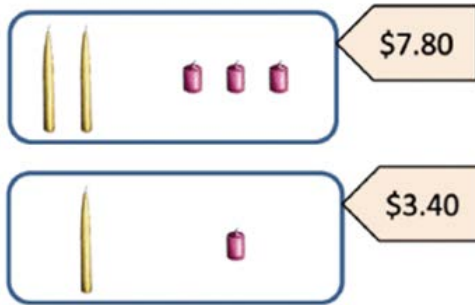
Make your  
own  
combination,  
and find the  
price.



# Generate new equations



# Combination chart



Find the difference

[illegible]

# A Learning trajectory

FORMAL

FLOATING CAPACITY

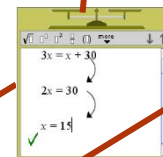
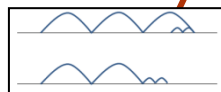
Equations

$$8 + 4x = 10 - 2x$$

Sets of Equations

$$\begin{aligned} y &= 2x + 5 \\ 2y &= 4x - 3 \end{aligned}$$

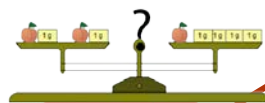
$$3j - 2 = 2j + 2$$



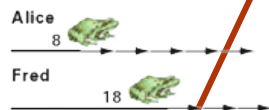
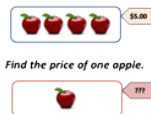
Find the difference

Tall	Short	Total
2	3	\$7.80
1	1	\$3.40
2	2	\$6.80
0	1	\$1.00

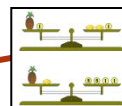
X 2



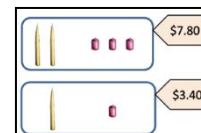
Balance strategy



Compare jumps and steps



$$\begin{aligned} L &= 4j + 5 \\ L &= 2j + 15 \end{aligned}$$



Generate new equations

Notebook strategy

Substitute and compare



- **MIC (Mathematics in Context)**

<http://mathincontext.eb.com>

Units:

Comparing Quantities

Graphing Equations

- **MiTC (Mathematics in the City)**

<http://mitccny.org>

<http://www.contextsforlearning.com>

Unit:

The California Frog-Jumping Contest: Algebra

- **DME (Digital Mathematics Environment)**

<http://dme.colorado.edu/dwo/colorado.html>, or

<http://www.fi.uu.nl/dwo/en> (login as guest)

Math Secondary Education> Algebra> Lesson-Systems of equations

More information:

<http://www.fisme.uu.nl/wisweb/en>

- **RME conference in Boulder.**

4th International RME Conference, 2013 sept 27-29, University of Colorado, Boulder

See for more information: [www.fius.org](http://www.fius.org)

- **Utrecht Summer School on Mathematics and Science Education**

August 19 - 30, 2013 in Utrecht, the Netherlands.

<http://www.utrechtsummerschool.nl/index.php?type=courses&code=H7>

<http://www.utrechtsummerschool.nl>

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- [google.com/+RaymondJohnson](https://www.google.com/+RaymondJohnson)