Establishing the Basis for Rational Finite Prodsums

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Abstract

This paper investigates non-empty sets of rational numbers, denoted by ps={a1,a2,, an}, where the sum of the elements is equal to the product of the elements, i.e., i=1nai =i=1nai . Our investigation will primarily focus on understanding the properties of finite sets, initially examining solutions within the more constrained domain of positive integers using algebraic and combinatorial techniques. This foundation will then allow us to explore the broader landscape of rational solutions and develop a more abstract characterization of such sum-product sets.

Introduction

The interplay between the fundamental arithmetic operations of addition and multiplication gives rise to numerous fascinating problems in mathematics. Consider the seemingly simple question: Can a set of numbers possess the unique property where the sum of its elements is equal to their product? This paper explores the characteristics of such sets, where the additive and multiplicative identities surprisingly coincide We will primarily focus on investigating finite sets of rational numbers, beginning with an analysis of solutions within the more constrained domain of positive integers before expanding to the rational numbers and deriving analogous rules, concluding with a comparison of the properties observed in both integer and rational families.

Keywords: Prodsums, Recursive Sets, Sum Equals Product

Introducing Common Notation

A prodsum is a set of numbers whose sum equals their product.
By their definition, all prodsum sets must satisfy this equation referred to as the "Fundamental Law of Prodsums" , $\sum_{i=1}^{n} a_i = \prod_{i=1}^{n} a_i$. In short, it means that the product and sum of every element must approach some common value, which we will refer to as the o-value, denoted with o. While this is not necessarily essential, a common notation denoting prodsums will be of great use. I propose, we denote prodsums as the following: $pds(2, 2) = o$. Where the arguments of pds represent the elements of the set of the underlying product sum, and o represents the product-sum's o-value. With this notation, you can instinctually recognize that this is a product sum whose product and sum will, by definition, equal o

The Integer Family Of Prodsums

As noted in the introduction, there are some sets of purely integer numbers for which the sum of their elements equals the product of their elements..

The Smallest Integer Prodsum Set

It is worth acknowledging that some definitions might consider a set containing a single element, {a}, as a Prodsum set, as the sum of the elements (a) is trivially equal to the product of the elements (a). In this light, any single number could be seen as forming such a set. However, the focus of most discussions and research on Prodsums lies in the scenarios where the interplay between at least two numbers leads to this equality, revealing deeper mathematical relationships rather than the self-evident property of a single element. Therefore, while a single-element set technically fits the definition, we will assume a set with two elements to be the smallest possible Prodsum set. Of these, there are two, $\{0,0\}$ and $\{2,2\}$

Let us consider $pds(2, 2)$

It must satisfy the base rule because

$2 + 2 = 4$, addition

$2 \times 2 = 4$, multiplication

$2 \times 2 = 2 + 2$, substitution

$4 = 4$, simplification, verification of the base rule

For the sake of completeness, $pds(0, 0)$ also fulfills this rule.

$0 + 0 = 0$, addition

$0 \times 0 = 0$, multiplication

$0 \times 0 = 0 + 0$, substitution

$0 = 0$, simplification, verification of the base rule

Base Rules For Integer Prodsums.

For integer Prodsums, the following rules apply

Rule: Element Equality Rule: For all Prodsums with more than two elements, not all elements in the set may be equal to each other

Explanation: Consider the behavior of the number 1 under the operations of addition and multiplication.

- Addition: For any number a, adding 1 to it increases its value: $a+1>a$ (assuming a is such that this holds, e.g., real numbers).
- Multiplication: For any number a, multiplying it by 1 leaves its value unchanged: $a\times1=a$

This fundamental difference in how addition and multiplication interact with the number 1 is the key reason why such sets can exist. The '1' acts as a bridge, allowing the sum and product of a set with other unequal, non-zero elements to become equal.

One exception to this rule is if all elements in the product sum set equal zero, in which case, the product will equal the sum because $0 \times 0 = 0 + 0$.

Rule: O-Value Sign Rule: For a Prodsum set of integers that does not contain zero, the sign of the o-value is determined by the number of negative elements in the set.

Explanation: Consider $pds(a_1, a_2, \cdots, a_n)$ where none of the elements are zero. The sign of the product of the elements is determined by the number of negative elements. Each negative element contributes a factor of -1 to the product.

1. If there is an odd number of negative elements, the product will have a negative sign. Since the set is a Prodsum set, the sum must equal the product, and therefore the o-value (the common value) will be negative.
2. If there is an even number of negative elements, the product will have a positive sign (since pairs of negative numbers multiply to a positive number). Consequently, the sum must also be positive, and the o-value will be positive.

Rule: Zero Rule: If a Prodsum set contains zero, then the o-value of the set must equal zero.

Explanation: Consider $pds(a_1, a_2, \cdots, a_n)$ that includes the element zero. By the definition of a Prodsum set, the sum of its elements must be equal to the product of its elements, and this common value is the o-value. When zero is an element of the set, the product of all elements in S will always be zero, as any number multiplied by zero results in zero. Since the product of the set is zero and the set is a Prodsum set, the sum of the elements must also be zero. Therefore, if a Prodsum set contains zero, both its sum and its product are zero, and consequently, the o-value of the set must equal zero.

Rule: For a Prodsum set consisting of positive integers with more than two elements, the number 1 must be an element of the set.
Explanation: Assume, for the sake of contradiction, that there exists a Prodsum set of positive integers with more than two elements where none of the elements is equal to 1. This implies that all elements in the set are integers greater than or equal to 2. Consider a set of three such integers, {a,b,c}, where $a \geq 2, b \geq 2, c \geq 2$. The product is abc $2 \times 2 \times 2 = 8$, while the sum is i $a + b + c \leq 3 \times max(a, b, c)$. Even in the smallest case {2,2,2}, the product (8) is greater than the sum (6). As the number of elements increases (say to $n \geq 3$) and each element remains at least 2, the product, being at least $2n$, grows exponentially, while the sum grows at most linearly with the magnitude of the elements. This fundamental difference in growth rates ensures that for any set of three or more positive integers greater than or equal to 2, the product will always be strictly greater than the sum. Therefore, our initial assumption must be false, and any Prodsum set of positive integers with more than two elements must necessarily include the number 1 to balance the sum and the product.

Rule: If a Prodsum set contains zero, then the o-value of the set must equal zero
Explanation: Consider a $pds(0, a_1, a_2, \cdots, a_n)$ By the definition of a Prodsum set, the sum of its elements must be equal to the product of its elements, and this common value is the o-value. When zero is an element of the set, the product of all elements in S will always be zero, as any number multiplied by zero results in zero. Since the product of the set is zero and the set is a Prodsum set, the sum of the elements must also be zero. Therefore, if a Prodsum set contains zero, both its sum and its product are zero, and consequently, the o-value of the set must equal zero.

The Rational Family Of Prodsums

This subset concerns expanding the domain of available Prodsums to rational numbers, significantly broadening our initial focus on integers. We will not yet delve into a formal, abstract definition in this subsection; instead, this exploration will serve as a crucial stepping stone and foundational investigation. By moving to the dense domain of rational numbers, we aim to uncover a potentially richer variety of Prodsums, which may exhibit different properties and structures compared to their integer counterparts. This initial investigation will allow us to build intuition through concrete examples and establish a solid groundwork that will be essential for understanding and formulating a more abstract definition of Prodsums over the rational numbers in later sections. The insights gained here will illuminate the fundamental relationships between sums and products when fractional values are permitted, paving the way for a more comprehensive theoretical framework.
Like in sub-section 1, for example $pds(a_1, a_2, \cdots, a_n)$, is a Prodsum that fulfills the *Fundamental Law of Prodsums.*

Smallest Rational Prodsum

Like the integers, some may consider the smallest rational Prodsum set, in terms of the number of elements, to be a set with a single element. If the set is {q}, where q is any rational number, then the sum of the elements is q, and the product of the elements is also q. Since the sum equals the product, any single rational number forms a Prodsum set.
While it is technically true that a set containing a single rational number satisfies the condition of the sum equaling the product, the notion of a "set" often implies a collection of multiple elements. Furthermore, the concept of a Prodsum set becomes more meaningful and mathematically interesting when considering the non-trivial balance between addition and multiplication involving at least two numbers. The equality

of sum and product in a single-element set is self-evident and holds for any rational number, making it somewhat of a degenerate case. Therefore, when seeking the smallest rational Prodsum set in a more significant and structural sense, we should consider sets with at least two elements to observe the interplay between different numbers under these operations. Thus, the smallest rational Prodsum set must be at least two elements, one example of which is $pds(3, \frac{3}{2})$.

The sum of the elements is $3 + \frac{3}{2} = \frac{9}{2}$ $(note\ that\ 3 = \frac{6}{2})$

The product of the elements is $3 \times \frac{3}{2} = \frac{9}{2}$ $(note\ that\ 3 = \frac{6}{2})$

Since the sum equals the product (29=29), the set {3,23} is an example of a rational Prodsum set with two elements. This demonstrates a non-trivial case where two distinct rational numbers satisfy the Prodsum property.

Rules For Rational Prodsums

These rules are derived directly from the fundamental definition of a Prodsum set applied to rational numbers and the specific properties of 0, 1, and -1 under addition and multiplication.

<u>*Rule*</u>: If a rational Prodsum set contains the element zero, then the sum of the elements must be zero, and the product of the elements must be zero. Consequently, the o-value of the set is zero.

<u>*Explanation*</u>: This rule has been proven already in the 1.2.3 ZERO RULE section; it has been repeated here for cases $a_i = \frac{0}{p}$ where p is an integer. Because $\frac{0}{p} = 0$, this rule still holds true.

<u>*Rule*</u>: A set containing exactly two rational numbers $\{a, b\}$ is a Prodsum set if and only if the relationship $(a - 1)(b - 1) = 1$ holds. This implies that if $a \neq 1$, then $b = 1 + \frac{1}{a-1}$. Note that neither a nor b can be equal to 1 in a two-element rational Prodsum set.

<u>*Explanation*</u>: Consider a set S containing exactly two rational numbers, a and b, so S={a,b}. For S to be a Prodsum set, the sum of its elements must equal their product, meaning a+b=ab. Rearranging this equation, we get $ab - a = 0$. Adding 1 to both sides yields $ab - a - b + 1 = 1$, which can be factored as $(a - 1)(b - 1) = 1$. If $a = 1$, we can divide by $(a - 1)$ to get $(b - 1) = \frac{1}{a-1}$, and thus $b = 1 + \frac{1}{a-1}$. Conversely, if $b \neq 1$, then $a = 1$. If either a or b were equal to 1, say a=1, then the equation $(a - 1)(b - 1) = 1$ would become $(1 - 1)(b - 1) = 0 \times (b - 1) = 0$, which contradicts itself because $0 \neq 1$. Therefore, in a two-element rational Prodsum set, neither element can be equal to 1.

<u>*Rule*</u>: If a rational Prodsum set contains the number 1 and other elements (denoted here as $a_1, a_2, \cdots a_i, \cdots, a_n$), then the product of the remaining elements must be one greater than their sum. In symbols, this is $o = 1 + \prod_{i=2}^{n} a_i$

<u>*Explanation*</u>: Let $S = \{a_1, a_2, \cdots, a_n\}$ be a rational Prodsum set containing the number 1, where q2,q3,...,qn are other rational numbers. The sum of the elements in S is 1+q2+q3+...+qn. The product of

the elements in S is $1 \times q2 \times q3 \times ... \times qn = q2 \times q3 \times ... \times qn$. Since the set is a Prodsum set, the sum must equal the product. $1 + q2 + q3 + ... + qn = q2 \times q3 \times ... \times qn$. This can be rewritten as $1 + \prod_{i=2}^{n} a_i$

Rule: If a rational Prodsum set contains the number -1 and other elements (denoted here as $a_1, a_2, \cdots a_i, \cdots, a_n$), the o-value of the set must equal positive one. In symbols, this is $1 = o = \sum_{i=1}^{n} a_i + \prod_{i=2}^{n} a_i$

Explanation: Let $S = \{-1, a_1, a_2, \cdots, a_n\}$ be a rational Prodsum set where $a_1, a_2, \cdots, a_n$ are other rational numbers. The sum of the elements in S is $-1 + a_2 + a_3 + ... + a_n$. The product of the elements in S is $(-1) \times a_2 \times a_3 \times ... \times a_n = -(a_2 \times a_3 \times ... \times a_n)$. Since the set is a Prodsum set, the sum must equal the product $-1 + a_2 + a_3 + ... + a_n = -(a_2 \times a_3 \times ... \times a_n)$. If you allow $T = \sum_{i=2}^{n} a_i, P = \prod_{i=2}^{n} a_i$ then the equation becomes $-1 + T = -P$, which can be rearranged to $P + T = 1$. Which is identical, though substitution, to $1 = \sum_{i=1}^{n} a_i + \prod_{i=2}^{n} a_i$

Rule: For all rational Prodsums with more than two elements, not all elements in the set may be equal to each other, except for a set where all elements are equal to zero.
Explanation: Consider a rational Prodsum set where all n elements are equal to some rational number q. The sum of the elements is nq, and the product of the elements is qn. For the set to be a Prodsum set, we must have nq=qn.
If q=0, then n×0=0n, which holds for all n≥1. Thus, a set consisting entirely of zeros is a Prodsum set, and this is an exception to the rule about unequal elements.
If q=0, we can divide by q to get n=qn−1.

- If n=1, then 1=q0=1, which is true for any non-zero rational q.

- If n=2, then 2=q1, so q=2. The set {2,2} has a sum of 4 and a product of 4, so it is a Prodsum set.

- If n>2, consider the equation n=qn−1, where q is a non-zero rational number. If q=p/r where p and r are integers with no common factors other than 1, and r>0, then n=(p/r)n−1, which means n⋅rn−1=pn−1. For n>2, this equation generally does not hold for rational q unless ∣q∣=1.

    - If q=1, then n=1n−1=1, which contradicts n>2.
    - If q=−1, then n=(−1)n−1. This requires n=1 (if n−1 is even) or n=−1 (if n−1 is odd), neither of which satisfies n>2.

Therefore, for rational Prodsums with more than two elements, the condition n=qn−1 with q=0 is generally not satisfied by a single rational number q, implying that not all elements in such sets can be equal, except for the case where all elements are zero.

Exploring The Similarities And Differences Between Rational And Integer Prodsums

Having established the properties and rules governing Prodsums within both the integer and rational domains, this section will directly compare these two families of sets. By examining the rules, examples,

and constraints discussed previously, we aim to illuminate the key similarities and differences that arise when considering solutions within these distinct number systems.
There are several notable similarities between the rules governing integer and rational Prodsums:
Firstly, the Zero Rule is consistent across both domains: if a Prodsum set, whether composed of integers or rational numbers, contains zero, then the o-value (the common value of the sum and the product) must be zero. This stems directly from the property that any number multiplied by zero equals zero.
Secondly, the number one plays a crucial role in both types of sets. For positive integer Prodsums with more than two elements, the Rule of One states that 1 must be an element to help balance the slower growth of the sum compared to the product. Similarly, for rational Prodsums containing one, there is a specific relationship that the other elements must satisfy, indicating that the presence of 1 imposes constraints in both contexts.
Thirdly, the number negative one also influences both integer and rational Prodsums. While the Integer Rules have specific conditions based on the parity of negative elements affecting the sign of the o-value, the Rational Rules include a condition that must be met when -1 is an element of the set, highlighting its impact on the balance between the sum and the product in both number systems.
Finally, the fundamental definition of a Prodsum set – where the sum of the elements equals their product – is the underlying principle for all rules in both integer and rational domains. The rules we derive are essentially specific applications or consequences of this definition under various conditions related to the elements of the set.

Two Element Sets: For integers greater than 1, the only two-element Prodsum set is $\{2,2\}$. For rational numbers, there are infinitely many two-element Prodsums, characterized by the relationship $(a-1)(b-1)=1$, where $a,b \in Q$ and neither is equal to 1.

Rule of One: For positive integer Prodsums with more than two elements, the number 1 must be an element. While there is a rule for rational Prodsums involving the number 1 (Rule Involving One), it describes a condition on the other elements and does not state that 1 is strictly necessary for larger sets of positive rationals greater than 1.

Rules Involving Negatives: For integer sets, we have distinct rules based on the parity of the number of negative elements (Odd Negatives Rule and Even Negatives Rule), determining the sign of the o-value. For rational sets, we have a specific rule focusing on the case where -1 is an element, providing a relationship that the other elements must satisfy. A general rule about the sign based on the number of negative rational elements was not explicitly stated as a separate rule, but would follow the same principle as with integers.

Element Equality Rule: The rule stating that not all elements in a Prodsum set with more than two integers may be equal to each other was specific to the integer domain. We haven't discussed a direct counterpart for rational numbers.