Geometric Loci Arising from a Complex Rational Function Parameterized by Recursive Prodsum Properties

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1 Introduction: Recursive Prodsum Sets and Complex Numbers

This paper explores the intersection of complex analysis and number theory by examining the geometric loci generated by a specific complex rational function and its connection to recursive prodsum sets. We begin by defining these sets, as introduced in "Expanding Prodsum sets with Recursion" by Lozier-Davis (Academia.edu), and then investigate how properties of these sets can parameterize a complex function, leading to interesting geometric interpretations.

The paper "Expanding Prodsum sets with Recursion" investigates non-empty sets of rational numbers, denoted by $S = \{a_1, a_2, ..., a_n\}$, where the sum of the elements is equal to the product of the elements, i.e., $\sum_{i=1}^{n} a_i = \prod_{i=1}^{n} a_i$. These sets are called recursive because new numbers can be generated recursively, allowing the complete set to fulfill this base rule.

It's crucial to observe that the initial element, a_1, can be chosen arbitrarily (with some restrictions, as we'll discuss). This choice then dictates the subsequent elements in the set, according to the recursive formula.

Definition 1 (Recursive Prodsum Sets)**.** *Given an initial element a_1, the recursive prodsum set R is generated by the following rule: For $n > 1$, the nth element, a_n, is defined as:*

$$a_n = \frac{\sum_{i=1}^{n-1} a_i}{\prod_{i=1}^{n-1} a_i - 1}$$

The set R is then the set of all a_i generated by this process.

For example, if we choose $a_1 = 2$, then $a_2 = \frac{2}{2-1} = 2$, and $a_3 = \frac{2+2}{2*2-1} = \frac{4}{3}$. The set would start as $\{2, 2, 4/3, ...\}$.

Definition 2 (Standard Prodsum Numbers (x_1 Recursive Prodsums))**.** *The x_1 recursive prodsum set, denoted as R_1, is the set of positive integers n that are equal to the product of their proper divisors.*

The sequence of standard prodsum numbers begins with $R_1 = \{6, 120, 672, 8064, 32640, ...\}$. Note that the standard prodsum numbers do not, in general, follow the recursive prodsum set generation rule from Definition 2. They are a separate concept, but as shown in the paper, can be used as the base set for generating higher order recursive prodsum sets.

Definition 3 (x_2 Recursive Prodsum Sets)**.** *The x_2 recursive prodsum set, denoted as R_2, is a set of integers m such that m is equal to the product of its proper divisors, and all of these proper divisors are elements of R_1.*

For example, if a number m has proper divisors $\{d_1, d_2, ..., d_k\}$ and $m = d_1 * d_2 * ... * d_k$, and all d_i belong to R_1, then m belongs to R_2.

2 Complex Rational Function and its Geometric Locus

Let us consider the complex rational function:

$$f(a) = \frac{a + bi}{a - 1 + bi}$$

where $a \in \mathbb{R}$ is a variable and $b \in \mathbb{R}, b \neq 0$ is a constant. We aim to determine the geometric locus of the output of this function in the complex plane as a varies.

Let $z = x+yi = f(a)$. We proceed by multiplying the numerator and denominator by the conjugate of the denominator to separate the real and imaginary parts:

$$z = \frac{(a + bi)(a - 1 - bi)}{(a - 1 + bi)(a - 1 - bi)}$$

Expanding the numerator and denominator:

$$z = \frac{a(a - 1) - abi + bai - b^2 i^2}{(a - 1)^2 - (bi)^2}$$

Since $i^2 = -1$, we have:

$$z = \frac{a^2 - a - abi + abi + b^2}{(a - 1)^2 + b^2}$$

$$z = \frac{(a^2 - a + b^2) - bi}{a^2 - 2a + 1 + b^2}$$

Separating the real and imaginary parts, we obtain:

$$x = \frac{a^2 - a + b^2}{a^2 - 2a + 1 + b^2}$$

$$y = \frac{-b}{a^2 - 2a + 1 + b^2}$$

Now, we manipulate these equations to eliminate a and find a relationship between x and y. From the equation for y, we have:

$$y((a-1)^2 + b^2) = -b$$

$$(a-1)^2 + b^2 = -\frac{b}{y}$$

$$(a-1)^2 = -\frac{b}{y} - b^2$$

Next, consider $x - 1$:

$$x-1 = \frac{a^2 - a + b^2}{a^2 - 2a + 1 + b^2} - 1 = \frac{a^2 - a + b^2 - (a^2 - 2a + 1 + b^2)}{a^2 - 2a + 1 + b^2} = \frac{a-1}{(a-1)^2 + b^2}$$

Substituting $(a-1)^2 + b^2 = -\frac{b}{y}$:

$$x - 1 = \frac{a-1}{-\frac{b}{y}} = -\frac{y(a-1)}{b}$$

$$b(x-1) = -y(a-1)$$

$$(a-1) = -\frac{b(x-1)}{y}$$

Squaring both sides:

$$(a-1)^2 = \frac{b^2(x-1)^2}{y^2}$$

Equating the two expressions for $(a-1)^2$:

$$\frac{b^2(x-1)^2}{y^2} = -\frac{b}{y} - b^2$$

Multiplying both sides by y^2:

$$b^2(x-1)^2 = -by - b^2y^2$$

Dividing by b^2 (since $b \neq 0$):

$$(x-1)^2 = -\frac{y}{b} - y^2$$

Rearranging:

$$(x-1)^2 + y^2 + \frac{y}{b} = 0$$

Completing the square for the y terms:

$$(x-1)^2 + \left(y^2 + \frac{y}{b} + \frac{1}{4b^2}\right) = \frac{1}{4b^2}$$

$$(x-1)^2 + \left(y + \frac{1}{2b}\right)^2 = \left(\frac{1}{2|b|}\right)^2$$

This is the equation of a circle in the complex plane.

Theorem 1. *The geometric locus of the complex function $f(a) = \frac{a+bi}{a-1+bi}$, where $a \in \mathbb{R}$ varies and $b \in \mathbb{R}, b \neq 0$ is a constant, is a circle with center $C = 1 - \frac{1}{2b}i$ and radius $r = \frac{1}{2|b|}$.*

Proof. The derivation of the circle equation follows from the algebraic manipulations shown above. The equation $(x-1)^2 + (y + \frac{1}{2b})^2 = (\frac{1}{2|b|})^2$ is in the standard form of a circle's equation, $(x-h)^2 + (y-k)^2 = r^2$, where (h,k) is the center and r is the radius. By comparing the derived equation with the standard form, we can directly identify the center as $1 - \frac{1}{2b}i$ and the radius as $\frac{1}{2|b|}$. □

3 Parameterization with Recursive Prodsum Sets

We now investigate how the properties of recursive prodsum sets, particularly the relationship between R_1 and R_2, can be used to parameterize the constant b in the circle equation. We focus on using the properties of the x_2 set to define the parameter b in our circle equation.

3.1 Parameterization using Properties of x_2 Sets Based on x_1 Divisors

Given an x_2 recursive prodsum $m \in R_2$, its proper divisors $D(m)$ are all elements of R_1. We can parameterize b in the circle equation based on properties of these x_1 divisors.

- Number of x_1 proper divisors: Set $b = |D(m)|$. The resulting circle has center $1 - \frac{1}{2|D(m)|}i$ and radius $\frac{1}{2|D(m)|}$.
- Sum of x_1 proper divisors: Set $b = \sum_{d \in D(m)} d$. The circle has center $1 - \frac{1}{2(\sum_{d \in D(m)} d)}i$ and radius $\frac{1}{2|\sum_{d \in D(m)} d|}$.
- The x_2 prodsum itself: Set $b = m$. The circle has center $1 - \frac{1}{2m}i$ and radius $\frac{1}{2m}$.

Proposition 1. *If b is parameterized using a property of an x_2 prodsum m, the resulting circle's center will always lie on the line $x = 1$ in the complex plane.*

Proof. This follows directly from Theorem 1. The center of the circle is given by $1 - \frac{1}{2b}i$. Regardless of how b is defined (as long as it's a real number, which is implied by its connection to the divisors of an integer), the real part of the center will always be 1. □

4 Rules and Observations

Here are some observations and rules derived from the previous sections:

1. The complex function $f(a) = \frac{a+bi}{a-1+bi}$ maps the real line (parameterized by a) to a circle in the complex plane for $b \neq 0$.
2. When $b = 0$, the function $f(a) = \frac{a}{a-1}$ maps the real line (excluding $a = 1$) to the real line (excluding $f(a) = 1$).
3. The center of the circle generated by $f(a)$ is always on the line $x = 1$ in the complex plane.
4. The radius of the circle is inversely proportional to $|b|$. As $|b|$ increases, the radius decreases, and the circle becomes smaller.
5. Parameterizing b using properties of x_2 recursive prodsums establishes a relationship between the circle's geometry and the structure of these higher-order number sets.

5 Conclusion

This paper has explored the geometric locus of a complex rational function and its connection to recursive prodsum sets. We have shown that the function generates a circle in the complex plane, and we have discussed how properties of x_2 recursive prodsums, which are based on the x_1 (standard) prodsum numbers, can be used to parameterize the circle's center and radius. This parameterization provides a potential link between complex geometry and the recursive structure of prodsum sets, offering a novel way to visualize and analyze these number-theoretic objects. Future research could focus on explicitly identifying x_2 prodsum sets and exploring deeper geometric implications of this parameterization.