Detailed Explanation of Extended Navier-Stokes Equation Calculations with Validation

Introduction

The extended Navier-Stokes equations describe the motion of fluids, such as liquids and gases, and are fundamental in fluid dynamics. These calculations are essential for modeling flows, such as those in an inline oil-water separator, where oil must be separated from water within a pipeline. This document provides a detailed, step-by-step explanation of symbolic and numerical calculations, including validation steps such as smoothness checks and continuity.

Overview of the Calculations

The calculations include:

- **Symbolic Equations:** These equations represent the fluid motion in the x-, y-, and z-directions symbolically, allowing a general study of fluid behavior.
- **Numerical Evaluation:** Specific values are substituted to evaluate the fluid behavior at a particular point, yielding practical, real-world solutions.
- **Validation Tests:** Smoothness checks and the continuity equation are used to ensure the accuracy and physical consistency of the solution.

The parameters include density (ρ), viscosity (ν), and gravitational acceleration (g), with values typical for oil. The velocity components u, v, w, and pressure p are defined as functions of position and time.

Detailed Step-by-Step Explanation

1. Symbolic Extended Navier-Stokes Equation in x-Direction

The extended Navier-Stokes equation in the x-direction is:

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

where:

- $\rho = 850\,\mathrm{kg/m}^3$ is the density of oil.
- $\nu = 0.1\,\mathrm{Pa \cdot s}$ is the viscosity.
- u, v, w are the velocity components in the x-, y-, and z-directions.
- p is the pressure.

We substitute the following example functions for u, v, w, and p:

$$u = 5\sin(y), \quad v = 3\cos(x), \quad w = 1, \quad p = 101325 - 20z + 300\sin(t)$$

Expanding the equation symbolically with these substitutions provides the full extended Navier-Stokes equation in the x-direction.

2. Symbolic Extended Navier-Stokes Equation in y-Direction

Similarly, the equation for the y-direction is given by:

$$\rho \left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \right) = -\frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right)$$

Using the example values for u, v, w, and p gives the symbolic form for the y-direction.

3. Symbolic Extended Navier-Stokes Equation in z-Direction

The equation in the z-direction is:

$$\rho \left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} + \nu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) - \rho g$$

where $g = 9.81\,\mathrm{m/s}^2$ represents gravitational acceleration. Substituting the values results in the symbolic z-direction equation.

4. Continuity Equation

The continuity equation ensures mass conservation within the fluid:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

By substituting the expressions for u, v, and w, we obtain the symbolic form of the continuity equation, verifying that mass is conserved.

5. Smoothness Check

A smoothness check is conducted on each velocity component to confirm that the functions for u, v, and w are differentiable and have no discontinuities:

$$\text{Smoothness of } u: \text{True}, \quad \text{Smoothness of } v: \text{True}, \quad \text{Smoothness of } w: \text{True}$$

This validation ensures that the velocity fields are continuous and physically plausible.

Numerical Evaluation of the Extended Navier-Stokes Equations

To analyze a specific location and time in the fluid, we substitute values $x = 1$, $y = 1$, $z = 1$, and $t = 1$ into the symbolic equations. These substitutions yield practical numerical solutions.

Example Calculation and Final Solution in the x-Direction

Using these values, we find:

$$5276.25\frac{\partial u}{\partial t} + 1377.77\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + 0.2\frac{\partial^2 u}{\partial x^2} + 0.1\frac{\partial^2 u}{\partial y^2}$$

This numerical solution represents the rates of change and forces acting in the x-direction at this specific point.

Final Numerical Solution for Each Direction

The calculations for the remaining directions and the continuity equation yield the following final numerical results:

- **Numerical Solution in y-Direction:**

$$3077.77\frac{\partial v}{\partial t} + 3576.25\frac{\partial v}{\partial x} = -\frac{\partial p}{\partial y} + 0.2\frac{\partial^2 v}{\partial y^2} + 0.1\frac{\partial^2 v}{\partial x^2}$$

- **Numerical Solution in z-Direction:**

$$6654.02\frac{\partial w}{\partial t} = 0.3\frac{\partial^2 w}{\partial x^2} - \frac{\partial p}{\partial z} - 8338.5$$

- **Continuity Check (numerical):**

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Conclusion

Through symbolic and numerical solutions, along with validation via smoothness checks and the continuity equation, we achieve a comprehensive understanding of fluid flow in a given system. This approach is foundational for applications like inline separators, where accurate modeling and validation of fluid dynamics are essential for optimal oil-water separation.