Anatomy of the Pythagoras' tree

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The grand architecture of nature can be seen at play in a tree: no two are alike. Nevertheless, there is an inescapable similarity that makes us identify a tree anywhere in the world. Just saying "tree" recalls words like green, root, leaves, still, strong, branches. The tree of primitive Pythagorean triples is no different (Figure 1). It has a root, or a beginning. It is rooted not on earth, but on the soil of our mind. It has branches that spring from that root as it grows with the action of nature and time. In this case, it is not the proverbial Mother Nature, but the human nature—a nature formed by the human interpretation of reality. The Pythagoras' tree presented by Berggren in 1934 has stood still and strong for almost a century, but probably it is even older. Its leaves are triples, and they grow throughout its branches. Ultimately, when one looks at the Pythagoras' tree, one looks at a 'tree'. The root is the triple (3, 4, 5). All branches and leafs emerge from, and are dependent, of this root. Like any tree, all it requires is a seed and soil, and all develops automatically. The architecture that defines the tree is present throughout the tree and is a reflection of the beginning—the root. In other words, the root (3, 4, 5) plus the same movement repeated over and over again creates the tree. In this paper, we will look at how this basic geometrical and mathematical movement governs the birth and growth of the Pythagoras' tree. Pythagoras is included in secondary education around the world including in Australian Curriculum (ACARA, n.d.), and hence this paper will be of interest to all.

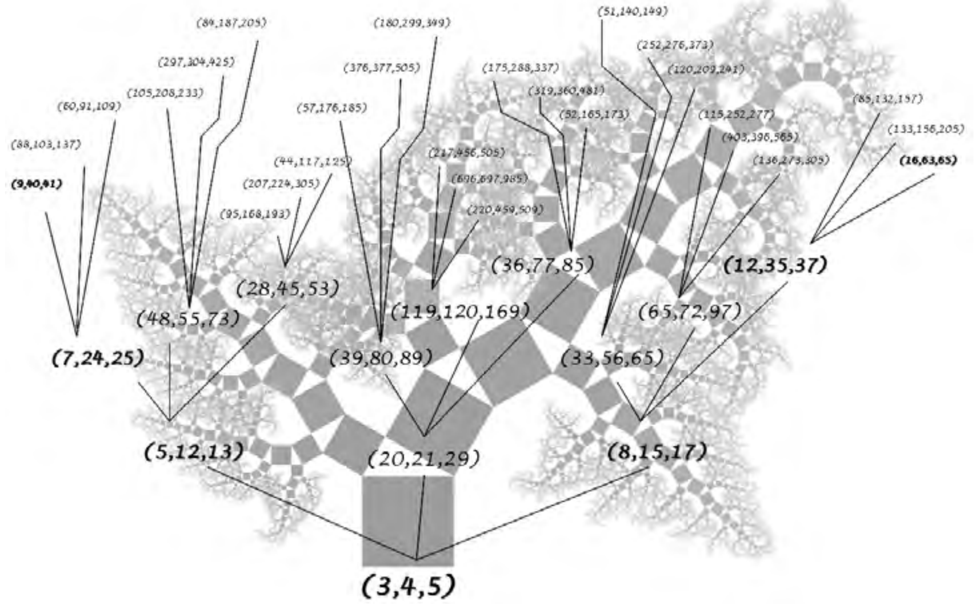

Figure 1. The tree of primitive Pythagorean triples. (Berggren, 1934; Vieth, 2016)

The structure of this tree can be better appreciated when drawn from left to right, as shown in Figure 2.

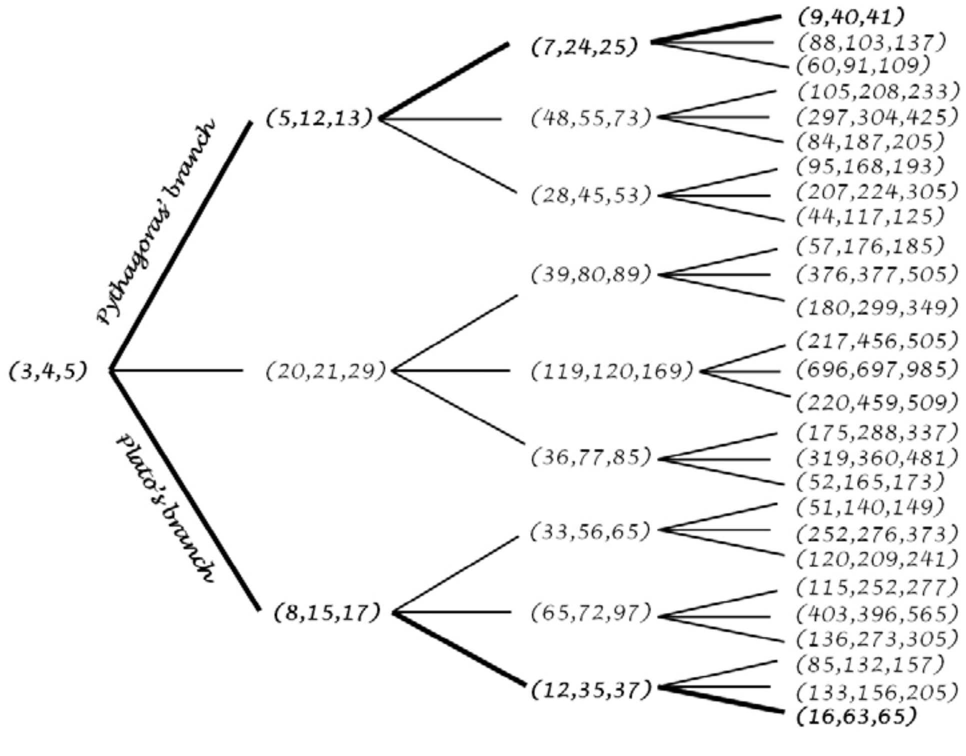

Figure 2. The tree of primitive Pythagorean triples (Figure 1) redrawn from left to right.

History tells us that Pythagoras in 300 BC discovered the branch that grows with odd sides 3, 5, 7, 9, etc. (Heath, 1956). This forms the top branch in Figure 2. A century later, Plato discovered the lower branch that grows with even sides 4, 8, 12, 16, etc. A tree, however, is not made only of two branches.

There is a variety of branches that also grow with their specific numbers. What governs all this? An explanation begins with the central square theory (Teia, 2015). It showed how both Pythagoras' and Plato's families of triples are governed solely by discrete increments in x. Or, in other words, the values y and z are only dependent on the combination of the value of x plus a geometric pattern. What is this geometric pattern? The central square theory states that all triangles of triples relate to each other via intermediate squares. Figure 3 shows how the central square theory interconnects parent-child triples. The right side of the equation z^2 is composed geometrically of four right-angled triangles rotating around a central square $(y - x)^2$. When these triangles are enclosed, it forms a new central square (side 7 for (3, 4, 5)) about which other Pythagorean triples revolve (like the child triple (5, 12, 13)). Enclosing this gives a new central square [17]. Iterative interconnections between triples using the central square theory form branches. This iterative process sounds very much like a plant growing. Ultimately, the branches together form the tree.

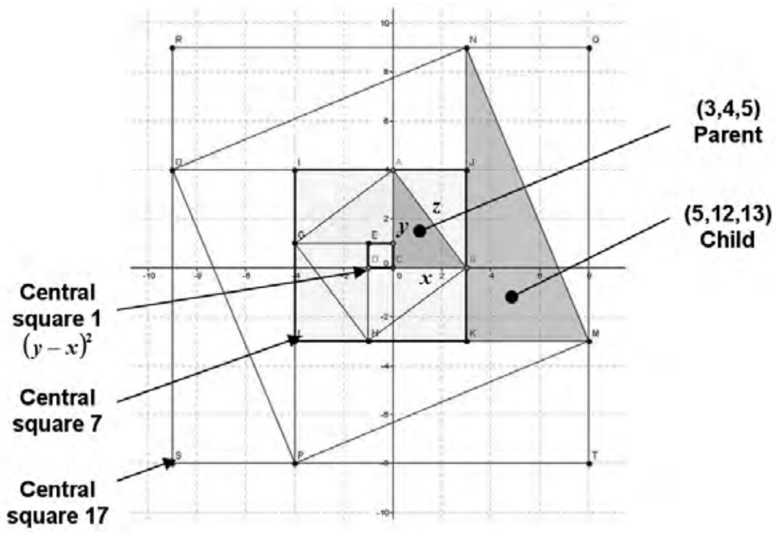

Figure 3. Geometrical interpretation of triples using the central square theory (Teia, 2015).

Hypothesis

How do branches appear, and grow? Having observed that both Pythagoras' and Plato's branches grow in a fixed incremental pattern, the hypothesis is that all branches behave in this manner. That is, they all grow from a specific triple, increasing by a fixed predetermined amount towards infinity. In the process of growth, each branch generates new triples that, in turn, give new branches. If triples generate branches, and vice versa, there must be a pattern, a basic movement that governs the growth of the tree. In this way, all triples are geometrically interconnected all the way to the first triple (3, 4, 5).

Basic movement

The movement that generates the three child triples (5, 12, 13), (8, 15, 17) and (20, 21, 29) from the parent triple (3, 4, 5) is shown mathematically in Figure 4(a), and geometrically in Figure 4(b). The movement is now explained:

1. Start with the first triangle rectangle (3, 4, 5) rotating around the unit side square (Figure 4(b)). Enclosing it gives square side [7]. Pythagoras' branch grows in steps of 2, hence the next triple starts with 3 + 2 = 5 (Figure 4(a)).
2. Summing the short side with the side of the square gives the longer side 5 + [7] = 12.
3. Connecting the ends gives the child triple (5, 12, 13) (Figure 4(b)). This triple revolves around the square side [7]. Plato's branch grows in steps of 4, hence the next child triple, after (3, 4, 5), starts with 4 + 4 = 8 (Figure 4(a)).
4. Summing the short side with the side of the square gives the longer side 8 + [7] = 15.
5. Connecting the ends gives the triple (8, 15, 17), which revolves around the square side [7] (Figure 4(b)). The middle triple (20, 21, 29) comes from the interaction of (3, 4, 5) with the child triples on either side, that is (5, 12, 13) and (8, 15, 17). This is now described. This movement generated two new branches highlighted by thick lines in Figure 4(a). From the side of the Pythagoras' branch, a new branch 12 + 4 = +{16} is formed.
6. Similarly from the side of the Plato's branch, a new branch 15 + 3 = +{18} is formed.
7. Adding the root (3, 4, 5) with the specific growth of the new branches +{16} and +{18} gives 4 + {16} = 20 and 3 + {18} = 21, giving the middle triple (20, 21, 29). Like the parent triple (3, 4, 5), the middle triple (20, 21, 29) revolves around the square [1], as shown in Figure 4(b). This completes the basic movement.

The side of the square, about which a triple revolves, is found by subtracting the long side with the short, or $y - x$. For example, (5, 12, 13) and (8, 15, 17) revolve around the square side 12 – 5 = 15 – 8 = [7]. Similarly, (3, 4, 5) and (20, 21, 29) revolve around the square side 4 – 3 = 21 – 20 = [1].

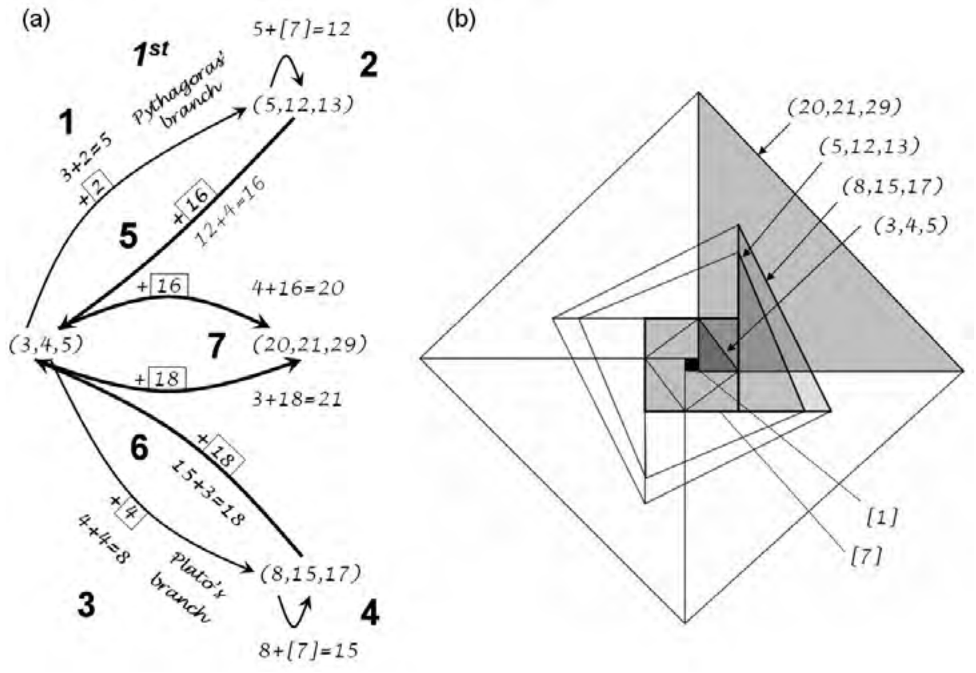

Figure 4. Basic movement explained by (a) mathematics and (b) geometry.

Building the tree

The movement in Figure 4(a) created two new branches +{16} and +{18}. These values correspond to how much x increases along each branch. Let us return to the hypothesis, which says that all branches grow from a triple and x increases by a fixed predetermined amount towards infinity. If this is the case, extending the branches is straight forward. One only needs to add the increment to x to create the subsequent triple in the branch. For example, branch +{16} grows from triple (20, 21, 29) with the small side increasing as x = 20, 36, 52, etc. (Figure 5). Similarly, branch +{18} grows as x = 21, 39, 57, etc. The Pythagoras' and Plato's branches behave exactly in this manner, or 5, 7, 9, 11, etc. and 4, 8, 12, 16, etc.

Applying this basic movement (step 1–5) to the second level of branches starting in triples (5, 12, 13), (20, 21, 29) and (8, 15, 17) gives the child triples (7, 24, 25), (28, 45, 53), (39, 80, 89), (36, 77, 85), (33, 56, 65) and (12, 35, 37) (Figure 6(a)). Following, the new middle triples are obtained by applying steps 5–7 described before (Figure 6(b)).

Another step-by-step example of the basic movement, applied to (5, 12, 13) in the second level (Figure 6(a) top right corner), is described below (the steps are equivalent to the previous description for Figure 4):

1–2. Continuing the Pythagoras' branch +{2} gives 5 + {2} = 7. Adding this to the respective central square 7 + [17] = 24 gives (7, 24, 25) (Figure 6(a)). Note that the side of the square results from the enclosure of the revolving triples, giving 5 + 12 = [17].

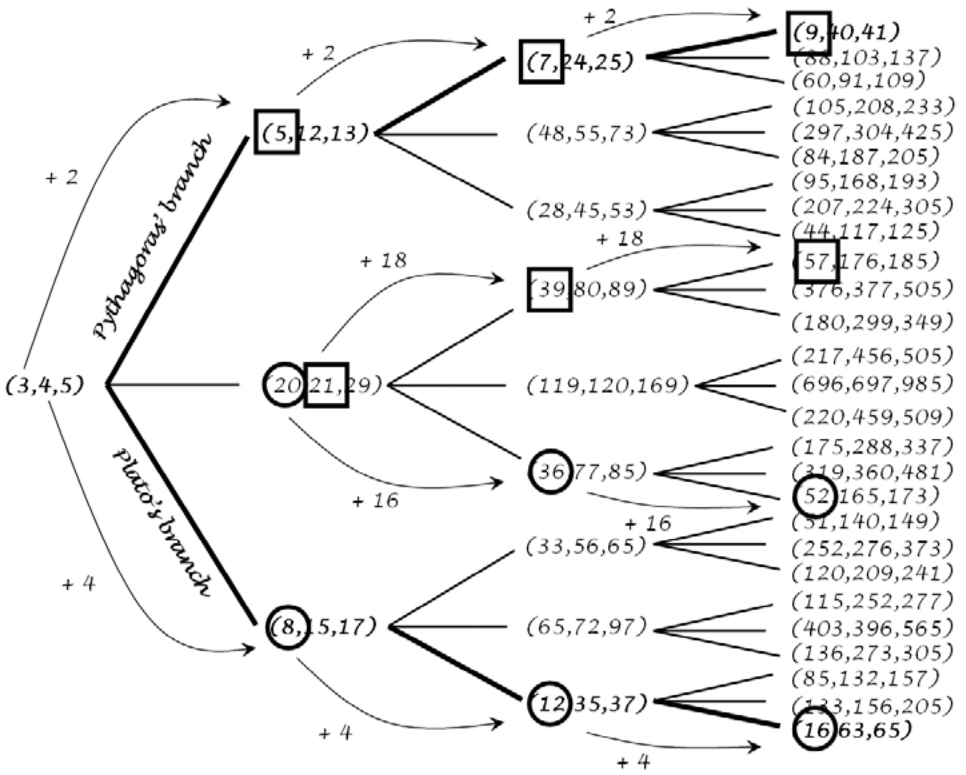

Figure 5. The Pythagoras' tree showing branches +{16} and +{18}.

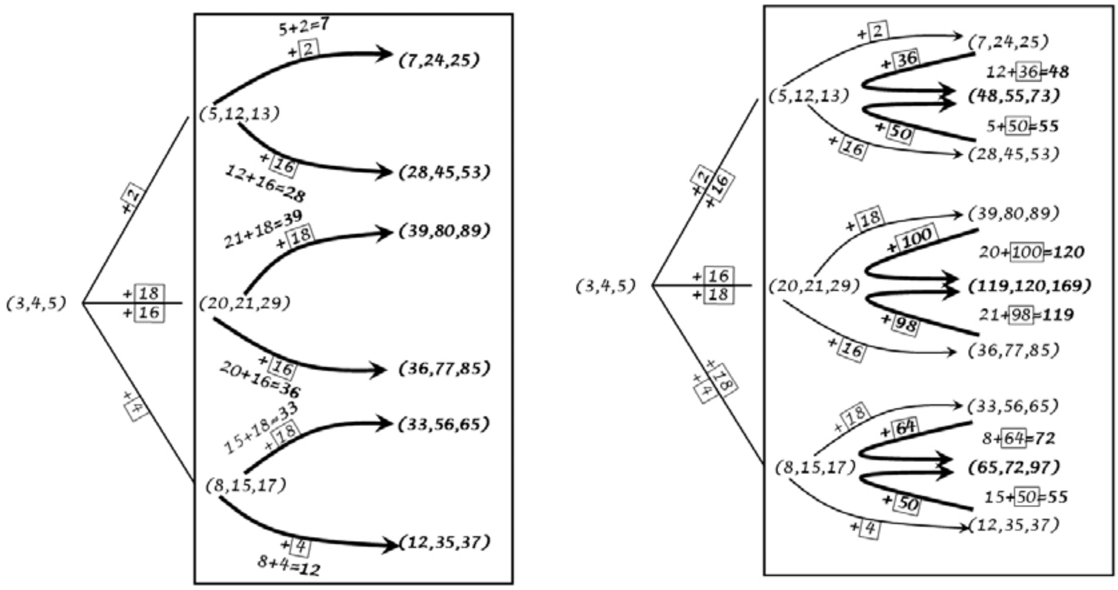

Figure 6. Basic movement applied to the 2nd level, forming (a) outer triples and (b) middle triple.

3–4. Continuing the +{16} branch gives 12 + {16} = 28. Adding this to the respective central square 28 + [17] = 45 gives (28, 45, 53).

5–6. New branches 12 + 24 = +{36} and 12 + 28 = +{50} are formed from (5, 12, 13) (Figure 6(b)).

7. Adding the new branches to (5, 12, 13), or 5 + {50} = 55 and 12 + {36} = 48, gives the middle triple (48, 55, 73).

The tree continues to grow as parent triples generate child triples by means of this basic movement (Figure 7). This movement is applicable anywhere in the tree. It is how the tree grows.

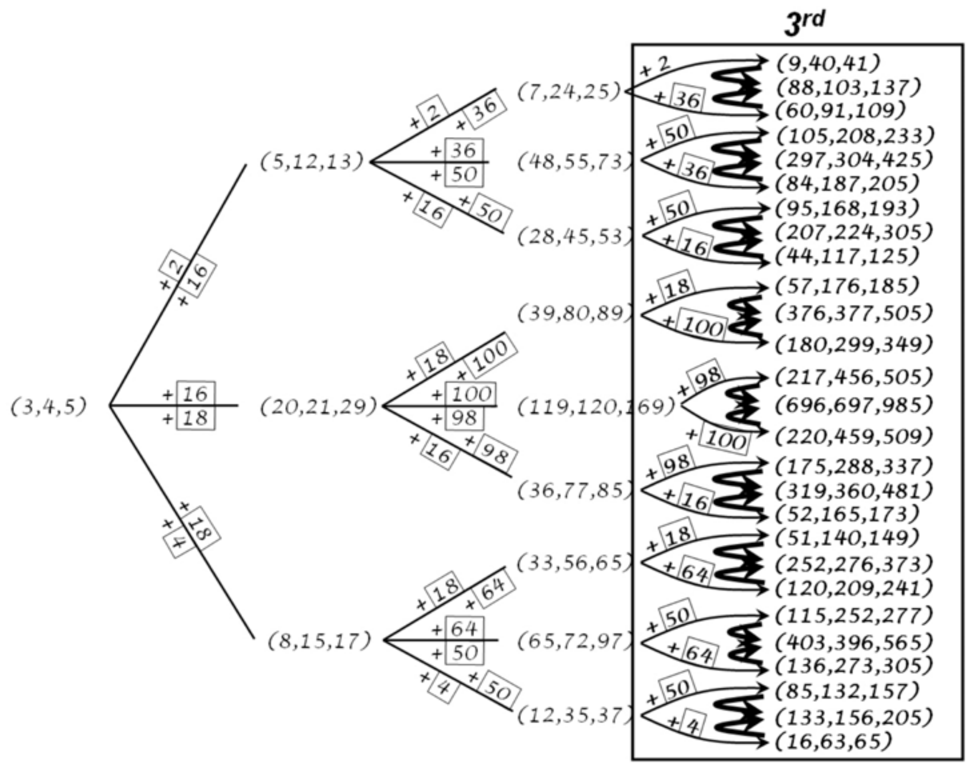

Figure 7. Basic movement applied to the 3rd level, forming outer and inner triples.

Branches

Figure 7 shows that two new branches are born from each triple. To the exception of the Pythagoras' branch +{2} and Plato's branch +{4}, all other branches come in pairs, and grow steadily along the tree (as illustrated in Figure 8(a)). For example, two branches +{18} are born at the root (3, 4, 5) and grow separately along the highlighted thick lines. Another example is two branches +{50} born in triple (5, 12, 13) at the upper side of the tree, that also grow steadily along the tree. One could map the branches in a similar manner as done for the triples in the Pythagoras' tree. All branches in the Pythagoras' tree obey the central square theory. Figure 8(b) shows, as an example, the geometrical representation of branch +{18} (identified in Figure 8(a)) created with the central square theory.

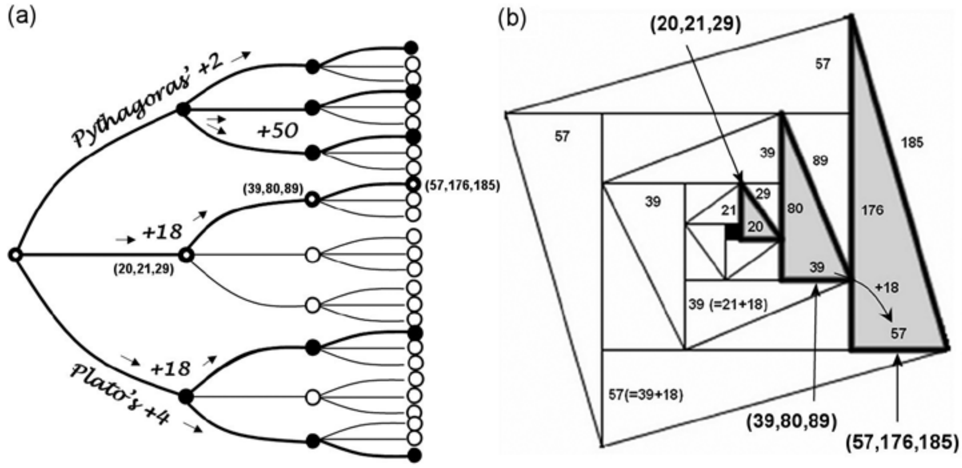

Figure 8. (a) Some branches of the Pythagoras' tree and (b) the geometry of the branch +{18}.

Tree of squares

Triples are not expressed only by triangles, but also by the underlying squares about which they revolve. Since each triple revolves around a specific square, a new tree of squares underlying the tree of triples appears. The side of each square is always the difference in sides of the corresponding triple, that is, $y - x$. Applying this transformation to the tree of triples (in Figure 2) results in the tree of squares (in Figure 9).

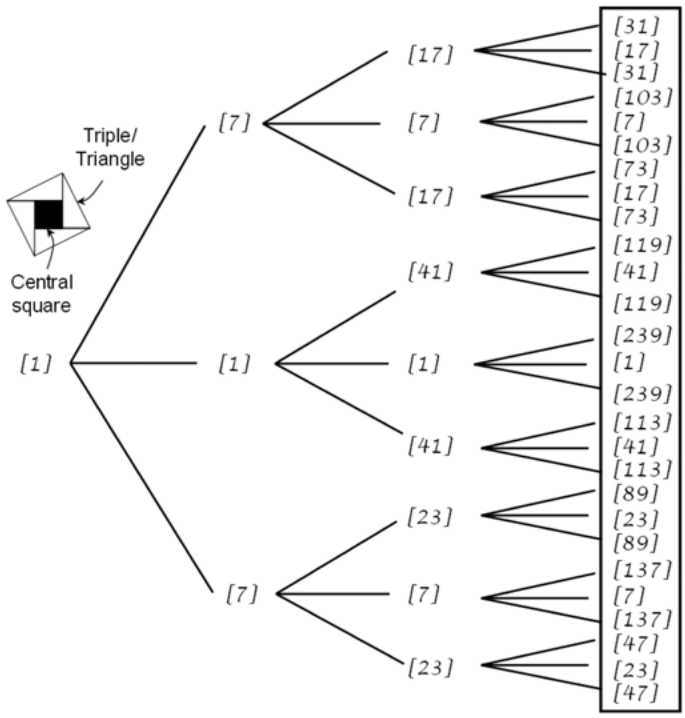

Figure 9. Tree of squares

It is interesting to note that back in 1934, Berggren mathematically identified the numbers in the tree of squares without knowing that it belonged to a family of squares (Figure 10). In reality, these two trees are numerical images or interpretations of the geometrical Pythagoras' tree that grows with triangles and squares. It is also interesting to note that a fractal Pythagoras' tree is built with both triangles and squares (Figure 11).

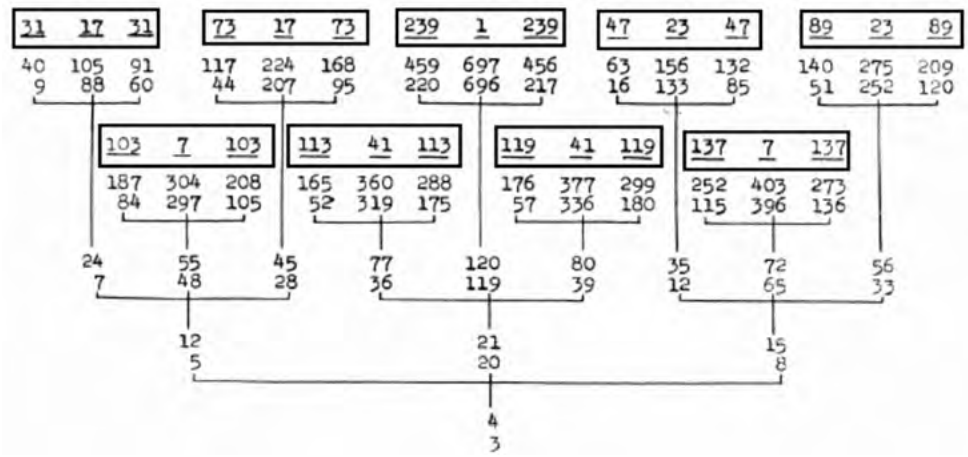

Figure 10. Pythagoras' tree identified by Berggren in 1934.

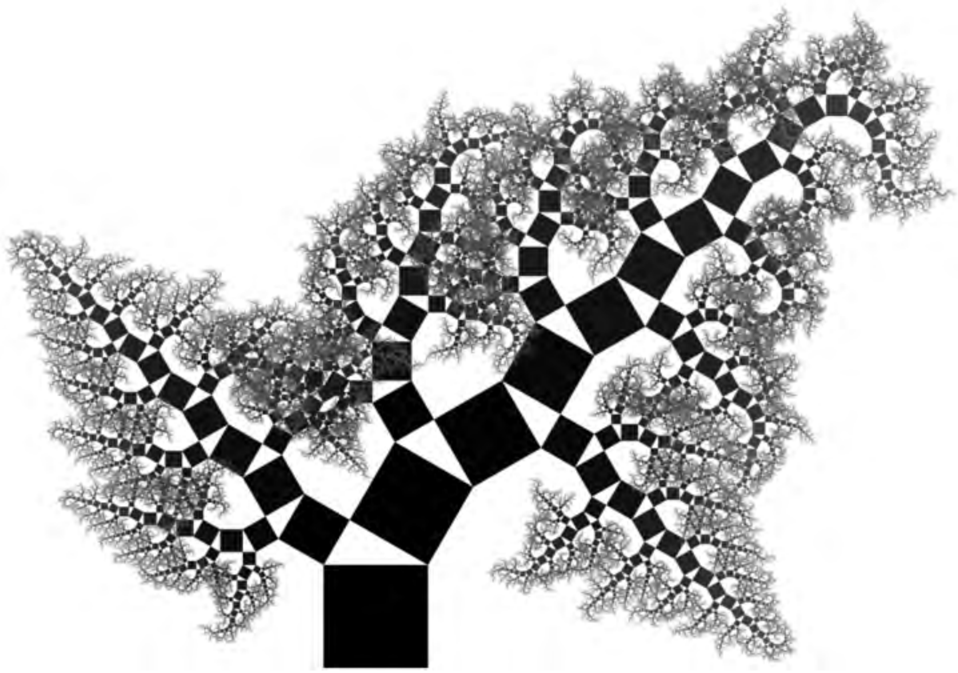

Figure 11. Fractal Pythagoras' tree built using squares and triangles (Vieth, 2016).

Conclusion

The Pythagoras' tree behaves just as a 'tree' in that the root (3,4,5) plus the same movement repeated over and over again grows from a seed, to a plant, to a tree. In human life, this movement is termed cell division. With triples, this movement is a geometrical and mathematical propagation explained by the central square theory. This movement explains how triples split into other triples via branches. It is a chain in space and time, where all branches grow from a specific triple, and vice versa. Examples applied at different locations in the tree have shown the same thing – the basic movement is the law that governs the birth and growth of the Pythagoras' tree. All branches come in pairs, and grow steadily along the tree. And finally, it was found that the geometric composition of the tree is heterogeneous. That is, the Pythagoras' tree is not only a tree made of triples/triangles, but also of squares. This led to the discovery of the tree of squares. These two trees are numerical images or interpretations of the same geometrical Pythagoras' tree that grows not only with triangles, but also with their central squares.

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