Universal Gear and the Pythagorean Theorem

Luis Teia
Von Karman Institute for Fluid Dynamics
Belgium
<luis.teia@vki.ac.be> or < luisteia@sapo.pt>

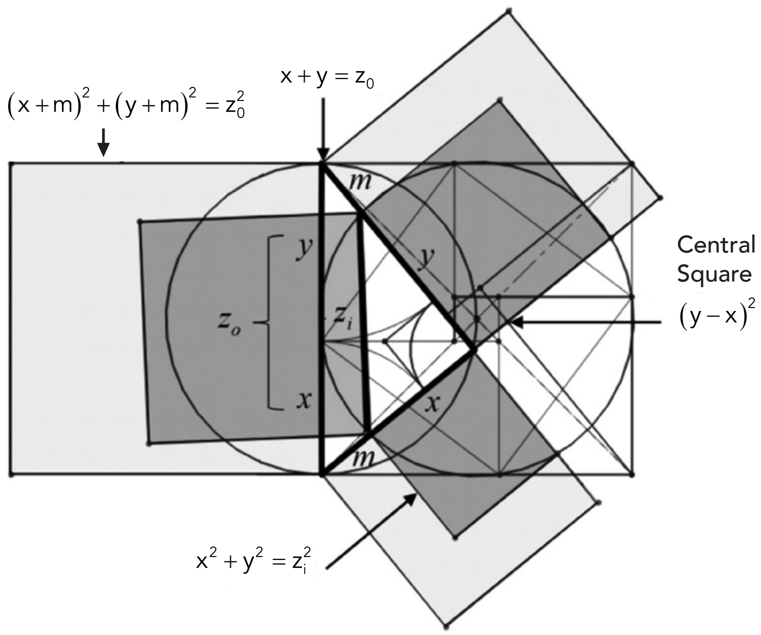
This article discusses the advanced characteristics of a mathematical and geometrical system explained and published in the course of several preceding journal articles (Teia, 2015, 2016, 2018a and 2018b). The reading of the preceding articles is encouraged for a clearer understanding of the discussion that follows.

Introduction

Revolutionary leaps in knowledge, like the formulation of Pythagoras' Theorem during the times of the Babylonians, are always accompanied by a leap in the capacity of human's thought processes. This paper follows this tradition by proposing a leap in knowledge, that comprises the explanation of the three-dimensional integration of Pythagoras' Theorem, requiring from the reader the leap in capacity of their own thought process (like three dimensional visualisation) to a level beyond the current average. A unique geometrical connection is formed from the integration of the three equations in Figure 1. This connection crystalises as two overlapping right-triangles governed by Pythagoras' Theorem, where x and y are the sides of the smaller right triangle of hypotenuse z_i, m is the increment added to the legs to transform the smaller right triangle to the larger having as hypotenuse z_o. The overall geometrical drawing, governing the relation between these three equations and whose construction is explained in Teia (2018b), was named the Pythagorean Gear.

The Pythagorean Gear reaches an interesting alignment for the particular case x=y, when the right-angled triangle becomes isosceles (Figure 2a). Indeed, the clay tablet in Figure 2b shows that this particular alignment has been an object of study since the times of the Babylonians.

Reality is naturally three-dimensional, thus this two-dimensional theorem must have a three dimensional expression. The particular case of this expression in Teia (2018b) showed that just as an isosceles triangle forms by two perpendicular lines, as shown in Figure 3a, so does a truncated tetrahedron form in between three mutually



$$\begin{cases} x+y=z_o \\ x^2+y^2=z_i^2 \\ (x+m)^2+(y+m)^2=z_o^2 \end{cases} \tag{1}$$

Figure 1. Two-dimensional geometric construct connecting the three equations (Teia, 2018a).

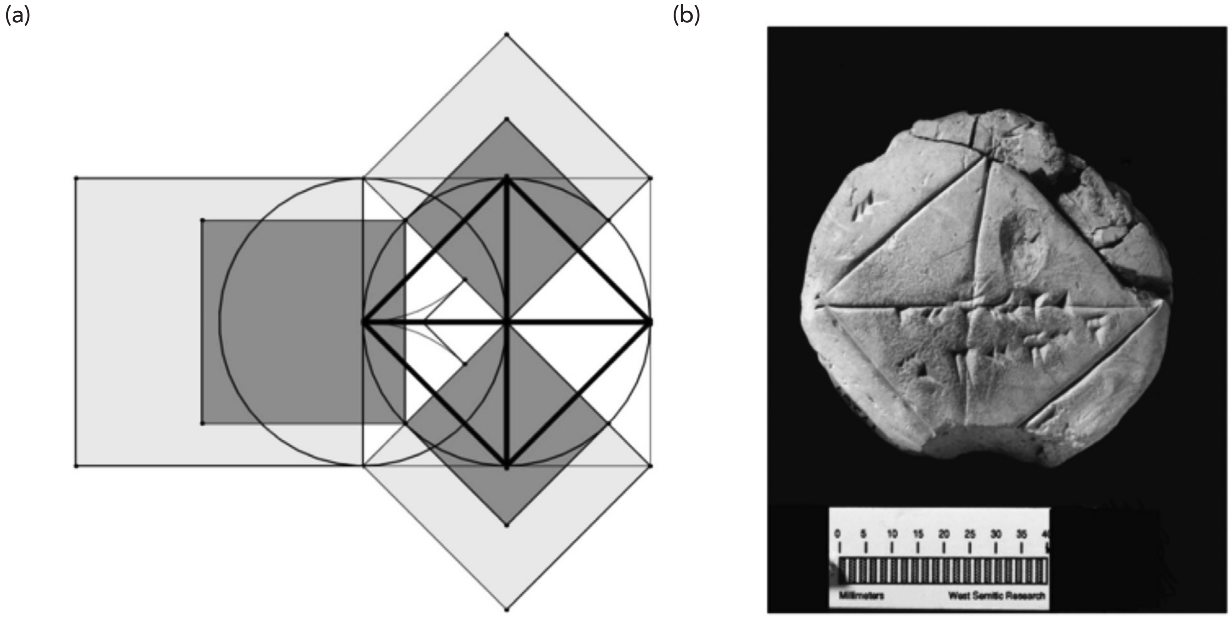


Figure 2. Particular case $x=y$: (a) the Pythagorean gear and (b) Babylonian clay tablet YBC 7289 (Courtesy of the Yale Babylonian Collection).

orthogonal triangles built up as shown in Figures 3b, 3c and 3d. From this insight it emerges that as the hypotenuse is the diagonal of a square, so analogously the hexagon is the diagonal of the cube. Generally speaking, this means that just as in two dimensions Pythagoras' Theorem governs the shape of a right-angled triangle, in three dimensions the Universal Gear governs the connection between orthogonal Pythagorean Theorems. The details of the theorem transformation from two to three dimensions will be discussed in the remainder of this paper. Both Pythagoras' Theorem and geometry in general form part of most secondary education curricula around the world, including in the *Australian Curriculum* (ACARA, n.d.), hence this paper will be of general interest.

Hypothesis

Pythagoras' Theorem, normally applied to either a right-angled isosceles or scalene triangle, forms the basis for a much larger three-dimensional orthogonally

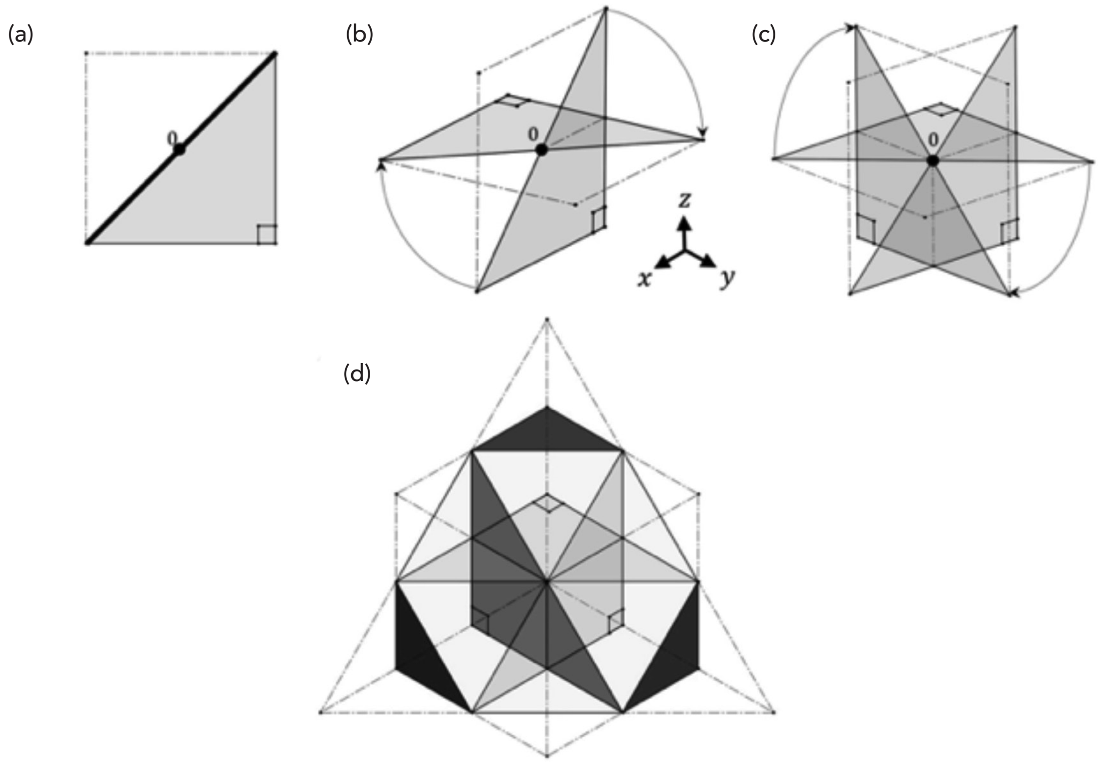


Figure 3 (a–d). Evolution of the Pythagorean expression from two- to three-dimensions [Teia, 2018b].

aligned and synchronous concept, hereafter named the Universal Gear.

Theory

The Universal Gear

Understanding the meaning of the Universal Gear requires a step-by-step approach, starting from Pythagoras' Theorem and evolving towards the complexity that is its three-dimensional expression.

Pythagorean Theorem and the truncated tetrahedron

The transformation from two- to three-dimensions shown in Figure 3 is now further explained. Let us start with the two-dimensional case. Consider the particular case $x=y$ (i.e., Figure 3a is now shown in Figure 4a with more detail) composed of a group of isosceles triangles of side x with diagonal $\sqrt{2}x$. The four inner congruent triangles A together form an inner square of side $\sqrt{2}x$, while another four outer triangles B complete it, forming a larger outer square of side 2x. All eight triangles are congruent. Because the eight triangles are right-angled isosceles triangles, the Pythagorean identity $z_i^2 = x^2 + y^2$ trivially shows that $z_i^2 = x^2 + x^2$ and thus $z_i = \sqrt{2}x$ with the side of the outer square as $z_o = x + x$. This means that in two dimensions, the balance within the Universal Gear (defined by Pythagoras' Theorem along a given plane) is that of the sum of areas, where for the particular case $x=y$ the addition of two areas size x^2 give another area size z_i^2, and the area of the outer square is:

$$\left(\sqrt{2}x\right)^2 + \left(\sqrt{2}x\right)^2 = (2x)^2 \equiv z_o^2 \tag{2}$$

This means that the transition from a sum of lines $z_o = x + x$ to a sum of squares (2) is a squaring process: $\sqrt{2}\sqrt{2} = 2$. For the general case $x \neq y$, the sum of the areas of the four inner triangles $4 \times \frac{xy}{2}$, plus the four outer triangles $4 \times \frac{xy}{2}$ and the central square $(y-x)^2$, result in the outer square z_o^2, or:

$$4 \times \frac{xy}{2} + (y-x)^2 + 4 \times \frac{xy}{2} = z_o^2 = (x+y)^2 \tag{3}$$

Knowing that the inner square z_i^2 is composed of the central square and four triangles $z_i^2 = (y-x)^2 + 4 \times \frac{xy}{2}$

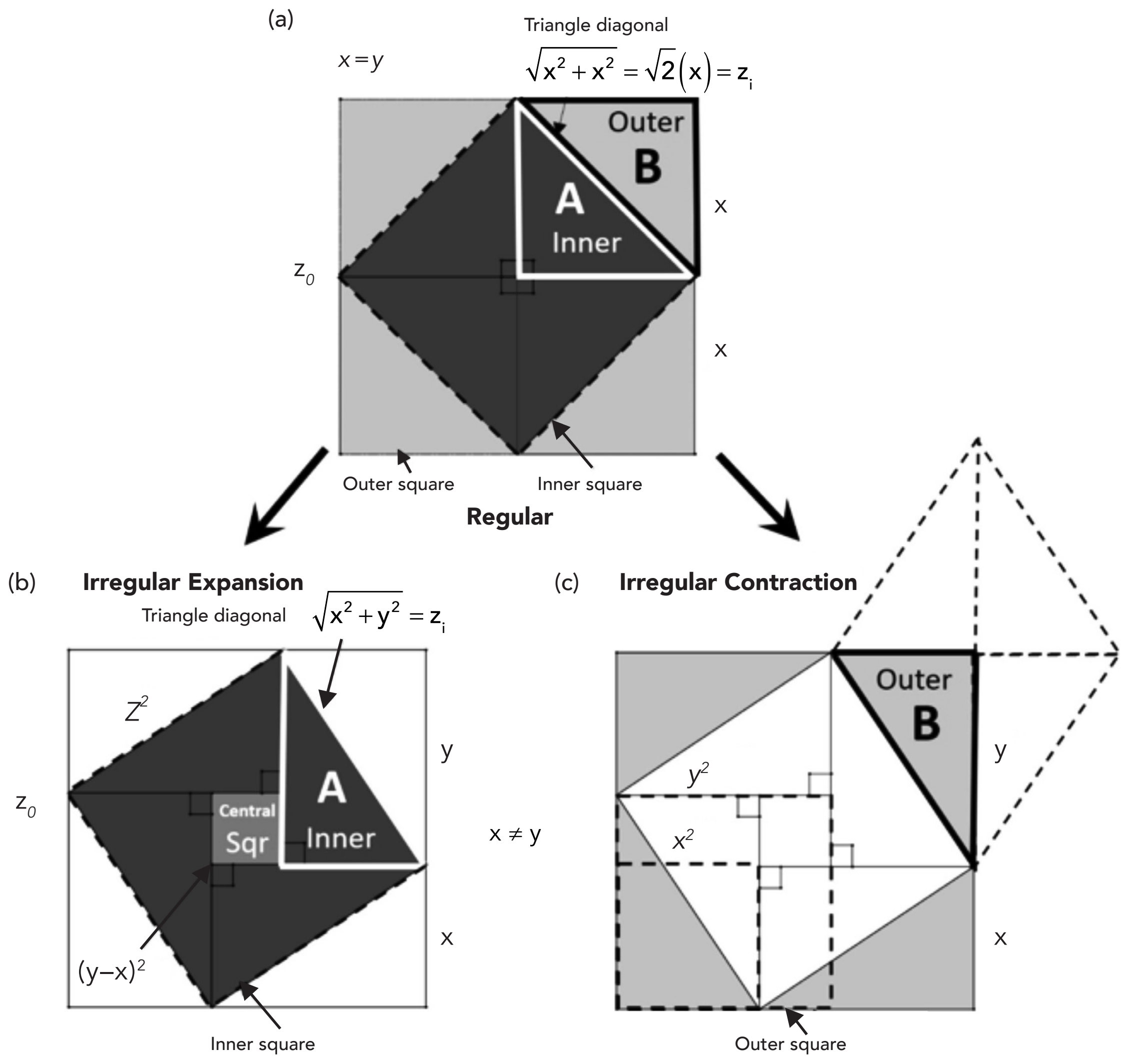


Figure 4. Pythagorean theorem: (a) regular for $x=y$, (b) expanded irregular for $x \neq y$ and (c) contracted irregular for $x \neq y$.

with the Pythagorean Theorem as $z_i^2 = x^2 + y^2$ simplifies (6) into:

$$4 \times \frac{xy}{2} + z_o^2 = z_i^2 = x^2 + y^2 \tag{4}$$

A connection is established between the orthogonal $z_o = y + x$ and the diagonal $z_i = \sqrt{y^2 + x^2}$. The total area of the outer square is $(x+y)^2$ and the inequality of x and *y* results in the appearance of a central square of size $(y-x)^2$. Further insight into the importance of this square is given by Teia (2015). That is, the area of the central square appears as the difference between the isosceles and scalene triangles. The area of the parallelogram shown in Figure 4c, constructible from the four outer triangles, will reduce as the difference between y and x increases.

Similarly, just as in two-dimensions two perpendicular lines form in between them a triangle, in three dimensions three perpendicular triangles form in between them the three dimensional equivalent—the truncated tetrahedron. This is true for both the regular particular case $x = y$ in Figure 5a, and for the irregular general case $x \neq y$ in Figures 5b and 5c, respectively. The volume of the tetrahedron V_T is readily shown to be half the volume of a cube side x:

$$V_T = \frac{x^3}{2} \tag{5}$$

Note that the volume of the tetrahedron is composed of several smaller volumes partitioned naturally by the inner orthogonal triangles. The partition results in three smaller tetrahedrons, a complete smaller central square, and three smaller cubes missing a section. Consequentially, it is interesting to note that the missing section of each smaller cube is in fact the smaller tetrahedron. That is, placing the smaller tetrahedrons in the missing corner of the smaller cubes gives three complete smaller cubes.

For the general case *x≠y*, the same disposition of three perpendicular triangles is still present, but a necessary offset appears in their alignment. This offset applies to the inner "expanded" tetrahedron (Figure 5b), but not to the outer "contracted" tetrahedron (Figure 5c). Note that for the particular case $x = y$ the inner and outer tetrahedron are the same. The inner right-angled triangle A (previously in Figure 4b) is found inside the inner expanded tetrahedron (Figure 5b), while the outer right-angled triangle B (from Figure 4c) is found inside the outer expanded tetrahedron (Figure 5c). Where the central square of size $(y-x)^2$ arises in the two dimensional irregular case, the central cube of size $(y-x)^3$ and parallelepiped extensions $x(y-x)^2$ arises in the three dimensional irregular case. In the three-dimensional general irregular case $x \neq y$, the definition of the tetrahedron volume becomes less trivial than the regular case, as the smaller tetrahedrons skew and the smaller cube becomes a parallelepiped. It can be shown that the volume of the inner expanded tetrahedron V_{Ti} is:

$$V_{Ti} = \frac{1}{(x+y)^3}\left\{-\frac{1}{2}x^6 + \frac{7}{4}x^4y^2 + \frac{5}{2}x^3y^3 + \frac{1}{4}x^2y^4\right\} \tag{6}$$

While the volume of the outer contracted tetrahedron V_{To} is:

$$V_{To} = \frac{1}{(x+y)^3}\left\{\frac{3}{2}x^3y^3 + \frac{5}{4}\left[x^4y^2 + x^2y^4\right]\right\} \tag{7}$$

Derivation of these two equations (6) and (7) is too extensive to be show in the main text. Therefore, the proofs are enclosed in the appendix. A way to readily partially confirm the validity of both equations is by replacing $x = y$, and reducing both of them back to equation (5). This is a good classroom exercise in algebra. Note that all two-dimensional hypotenuse lines always cross at the same point 0, which displaces in space as x varies. Even though the hexagon distorts from Figure 5a to Figures 5b and 5c, it is interesting to note that its surface is still a composition of six identical right-triangles, isosceles for $x = y$ and scalene for $x \neq y$. Note that Figure 5b and 5c offer a whole new perspective to the classical Pythagorean theorem "windmill" analogy.

Truncated octahedron

For the particular case $x = y$, just as rotating four isosceles triangles around an axis gives an inner square (Figure 4a), revolving eight regular truncated tetrahedra around all three axes gives a regular truncated octahedron (Figure 6a). A square is the dual of itself (Kepler, 1619), and in the context of the particular case of the Universal Gear, it is the particular version of the octahedron—the truncated octahedron—that is the dual of the cube. Similarly, for the general case $x \neq y$, all the orthogonal planes defined by the two dimensional Pythagorean Theorem move in perfect synchronicity to achieve the new balance. In the process, two types of octahedra are formed corresponding to the two types of truncated tetrahedra that compose them. The inner truncated octahedron expands, where the central cube plus the parallelepiped extensions common to all tetrahedrons compose the core (Figure 6b). The outer truncated octahedron contracts, taking the form of Figure 6c. The triangle rectangles A and B (shown before in Figure 5b and 5c), composing each tetrahedron found within both the inner and outer octahedrons, are the same.

Cube

Just as in two-dimensions four isosceles triangles surrounding an inner square form an outer square (Figure 4a), so does in three-dimensions eight truncated tetrahedrons surrounding an inner truncated octahedrons form

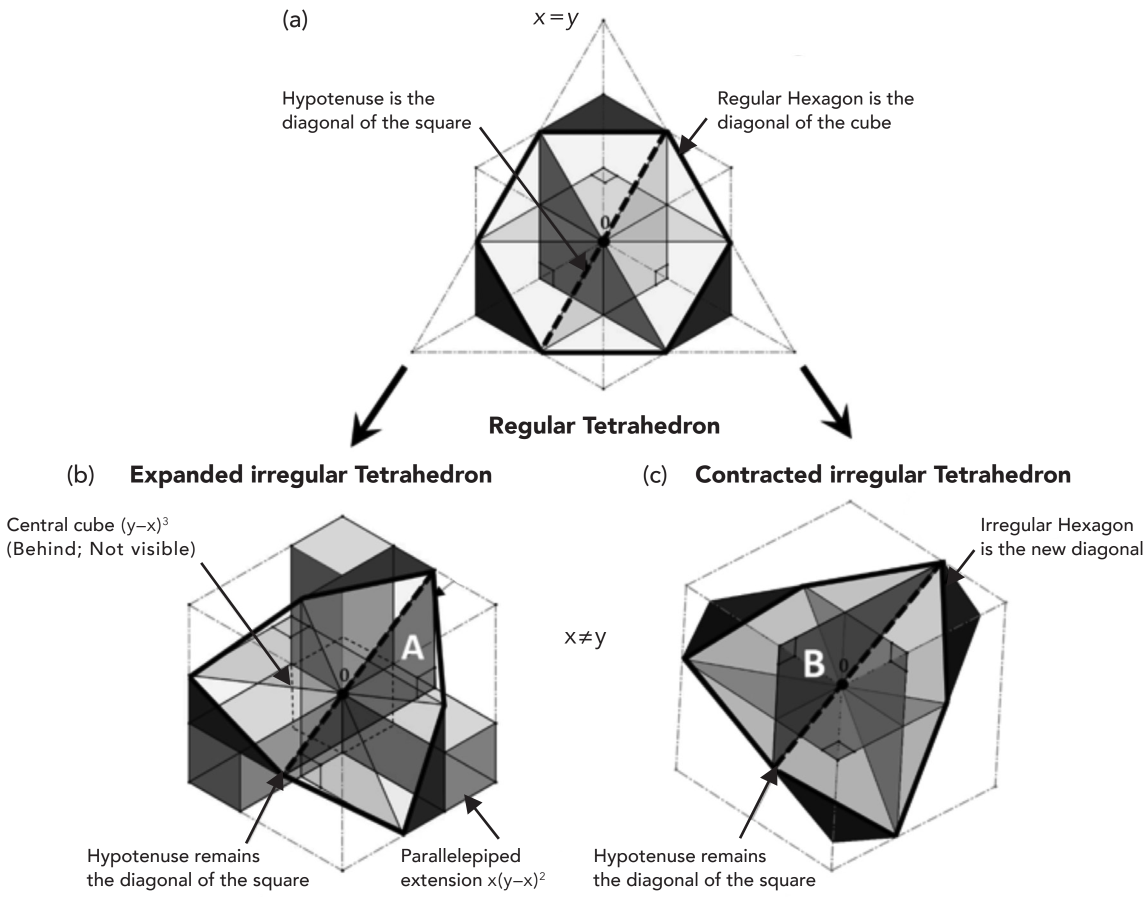


Figure 5. Truncated tetrahedron: (a) regular for $x=y$, (b) expanded irregular for $x \neq y$ and (c) contracted irregular for $x \neq y$.

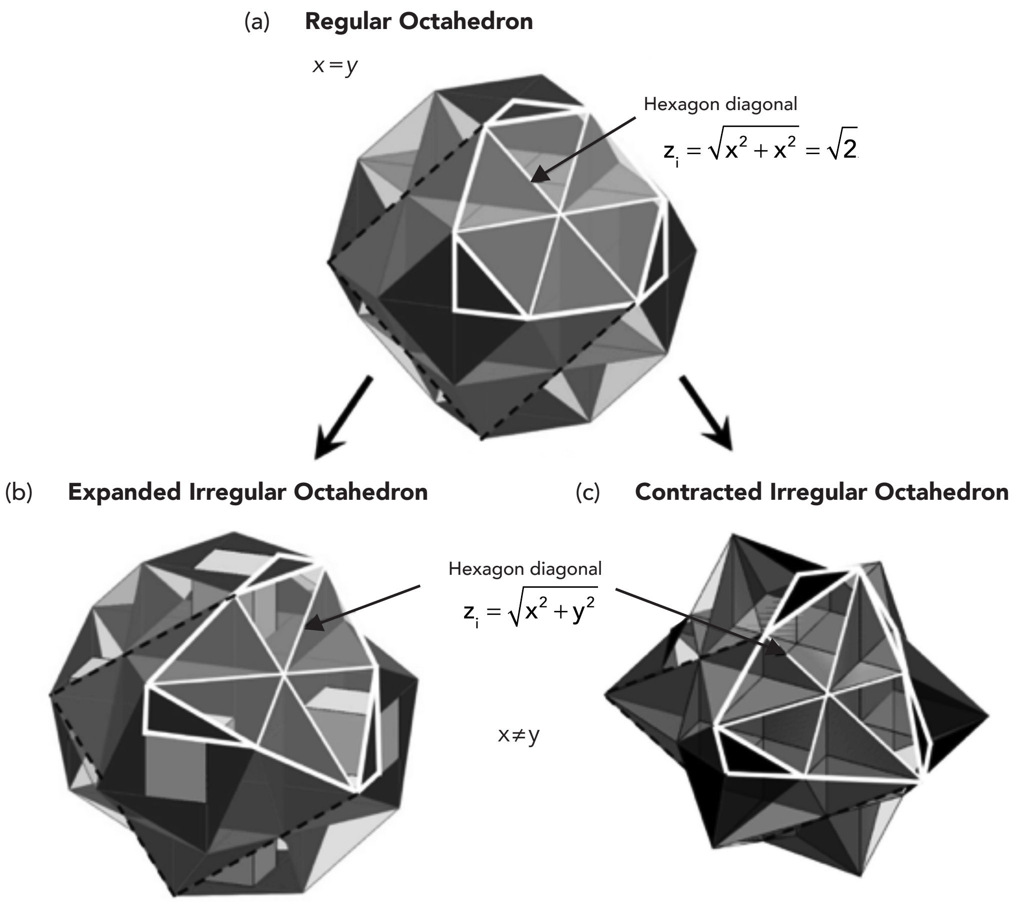


Figure 6. Truncated octahedron: (a) regular for $x=y$, (b) expanded irregular for $x \neq y$ and (c) contracted irregular for $x \neq y$.

a cube (Figure 7a). This is true for both particular and general cases. For the particular case $x=y$, the eight component regular tetrahedra (Figure 5a) around the inner regular octahedron (Figure 6a) are shown in the exploded view of Figure 7a. These eight outer tetrahedra form the outer octahedron, and are identical to those that form the inner octahedron. Together they form the outer cube of side $z_0=x+x$, shown on the right of the same Figure 7a. The transition to the general case $x \neq y$ is shown in Figure 7b. The expansion of the inner octahedron (seen before in Figure 6b) and contraction of the outer octahedron (shown in Figure 6c that surround the inner octahedron) integrate perfectly resulting in the outer cube side $z_o=y+x$, shown on the right of the same Figure 7b. Note that the assembly is reciprocal, that is, the contracted octahedron can in turn be placed at the center, with the expanded octahedron surrounding it. Even though the outer cubes of side z_o in Figure 7a and 7b can be seen as identical, the partition of volumes, areas and lengths within the cube changed when x becomes different to y.

In three dimensions, the balance within the gear is that of the sum of volumes. For the particular case $x=y$, this means that the addition of eight truncated tetrahedrons (that form two truncated octahedrons) volumes size x^3 give a cube size z_o^3. The hexagon of the truncated tetrahedron has diagonal length $\sqrt{2}(x)$ which is the cornerstone to express the sum of volumes as:

$$\sqrt{2}\left(\sqrt{2}(x)\right)^3+\sqrt{2}\left(\sqrt{2}(x)\right)^3=(2x)^3\equiv z_o^3 \tag{8}$$

This means that the transition from a sum of areas (2) to a sum of volumes (8) is again a squaring process: $\sqrt{2}\sqrt{2}=2$.

For the general case $x \neq y$, the volume of a cube side z_o is given as the summed volume of eight inner expanded truncated tetrahedron and eight outer contracted truncated tetrahedron, plus the central cube $(y-x)^3$ and respective six parallelepiped extensions $x(y-x)^2$, resulting in:

$$8V_{Ti}+(y-x)^3+6x(y-x)^2+8V_{To}=z_o^3 \tag{9}$$

Equation (9) is the general version of equation (8). The expressions for V_{Ti} and V_{To} were already given in equations (6) and (7), and replacing in equation (9), with some algebraic development results in (10). (Next page.)

This is the three-dimensional equivalent to (3). Indeed, an evolution of the mathematical equation expressing the one-dimensional sum of lengths, two-dimensional sum of areas and three-dimensional sum of volumes is seen when comparing $z_o=y+x$, (3) and (10).

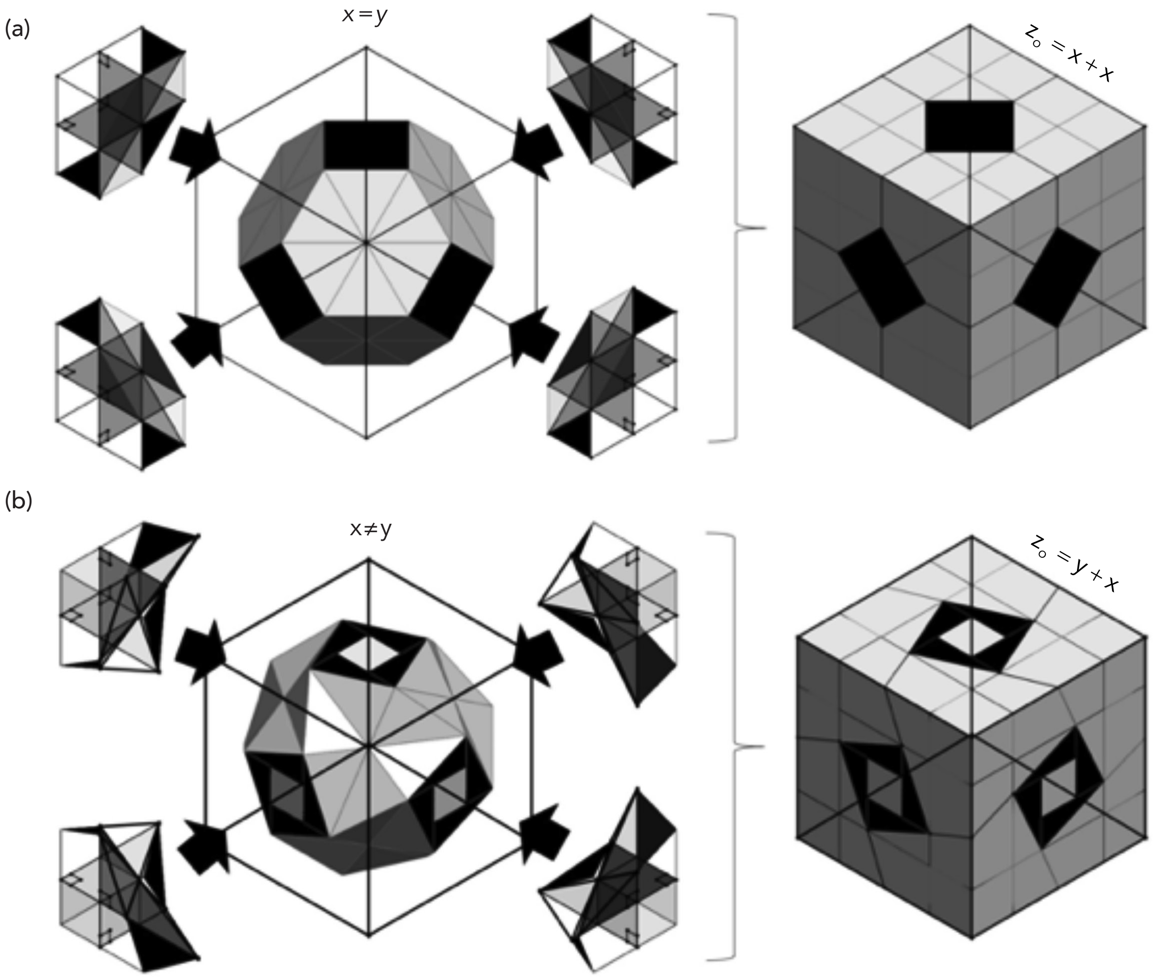


Figure 7. Exploded view of cube side z_o, tetrahedrons and inner octahedrons for: (a) case x=y and (b) case x≠y.

$$8\frac{1}{(x+y)^3}\left\{-\frac{1}{2}x^6+3x^4y^2+4x^3y^3+\frac{3}{2}x^2y^4\right\}+(y-x)^3+6x(y-x)^2=z_o^3=(y+x)^3 \quad (10)$$

3.1.4 Pythagorean Theorem orthogonal integration

Figure 8 shows how the Universal Gear is constructed using the same Pythagorean Theorem aligned in all three orthogonal directions. Figure 8a,c,e and g shows the construction of the particular case x=y while Figure 8b,d, f and h shows the construction of the general case x≠y.

As mentioned, both the particular case x=y and general case x≠y need to be satisfied in all three orthogonal planes in order for the assembly to be volumetrically correct. The result is shown in Figure 8g for the particular case x=y, and in Figure 8h for the general case x≠y.

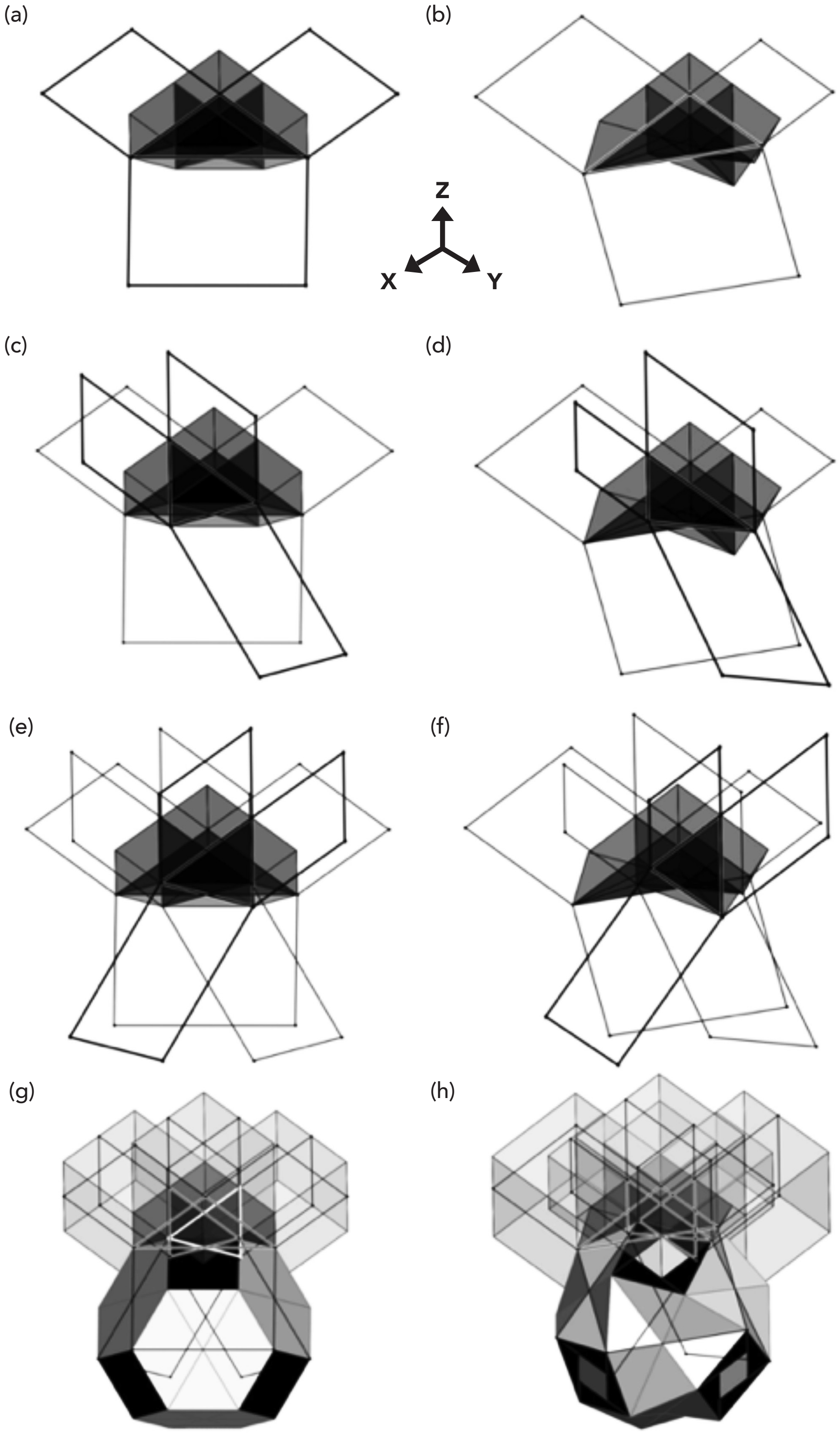


Figure 8. Three-dimensional integration of the Pythagorean Theorem: (a,b) along XY plane, (c,d) along YZ plane, (e,f) along XZ plane and (g,h) the resulting volume.

Looking more specifically to one half sections of Figures 8g and 8h shown in Figure 9a and 9b, the one-dimensional sum of lengths, the two-dimensional sum of areas and the three-dimensional sum of volumes are observable. It is also noticeable that just as in two-dimensions two squares are related to a larger square via an intermediate right triangle, in three-dimensions two cubes are related to half a truncated octahedron via an intermediate truncated tetrahedron. Note that the entire three-dimensional structure both in Figures 9a and 9b is solely established by the orthogonally interlaced Pythagoras' Theorem. Overall, the Universal Gear shows that at least for the particular case $x = y$, two cubes relate to half a regular truncated octahedron (the three dimensional dual within this context) via a regular truncated tetrahedron (Figure 9a). This is different from saying that two cubes relate to a third, as suggested by $x^3 + y^3 = z^3$. That is, the above construction is not isomorphic to the general case of two cubes adding to a third cube. Conversely, two truncated octahedrons are related to a cube via the truncated tetrahedron, as shown previously in Teia (2018b).

3.2 Fractal nature

The online Cambridge dictionary defines the word 'fractal' as "a complicated pattern in mathematics built from simple repeated shapes that are reduced in size every time they are repeated" (Cambridge, n.d.). For the particular case $x = y$, this pattern is visible in Figure 10a, where enclosed within the outer square $z_o = x + x = 2x$ there is the inner square $z_i = \sqrt{x^2 + x^2} = \sqrt{2}x$, whose difference are the four isosceles triangles A, B, C and D that are ultimately governed by the Pythagorean Theorem. This relation between squares repeats itself by assuming the inner square to become the outer square, and both are now related by intermediate isosceles triangles A', B', C' and D' that are also governed by Pythagoras' Theorem. These steps can be repeated indefinitely in both directions, either inwardly or outwardly. Therefore, it follows from the initial definition that the nature of Pythagoras' Theorem is fractal, in that it is a complicated pattern repeated from simple shapes, like squares and triangles, reducing in size with every repetition. Since the Universal Gear is a construct of orthogonally interlaced Pythagorean Theorems, the Universal Gear also has a fractal nature (Figure 10b). The outer cube side $z_o = 2x$ encloses an inner truncated octahedron with diagonal side $\frac{z_i}{2} = \frac{\sqrt{2}x}{2}$ via eight truncated tetrahedrons. The square sides of the octahedron are in the same plane as those of the cube. In the same way, the truncated octahedron side $\frac{\sqrt{2}x}{2}$ encloses an inner cube whose vertices are the centre of the hexagonal faces of the octahedron, which in turn encloses a smaller truncated octahedron. The process repeats itself inwardly and outwardly towards infinity.

For the general case $x \neq y$, the same fractal nature is present in all dimensions. In two dimensions, the outer square side $z_o = x + y$ now connects to the inner square

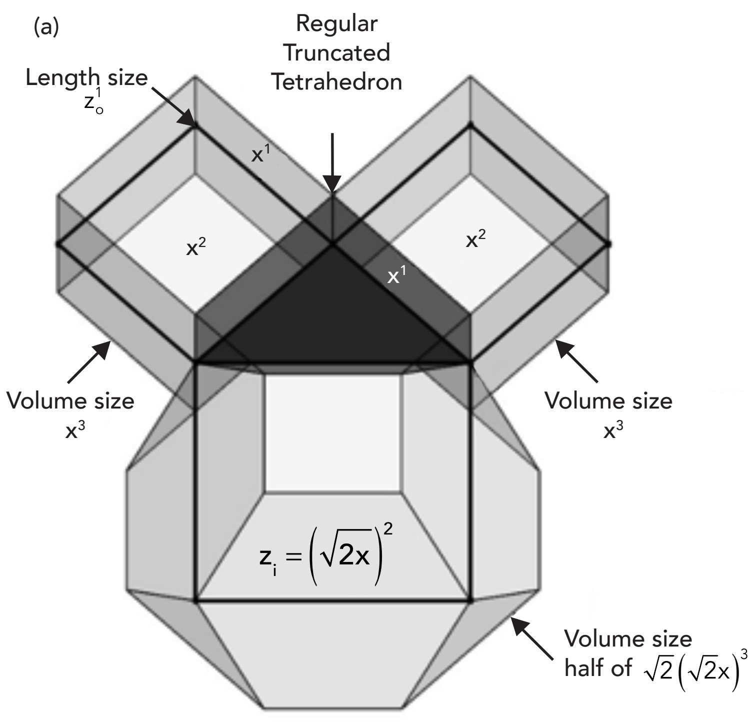


Figure 9. The geometry of the one-, two-, and three-dimensional gear: (a) regular for x=y and (b) irregular for x≠y.

$z_i = \sqrt{x^2+y^2}$ via four scalene triangles A'', B'', C'' and D'' of sides x and y (Figure 10c). Taking the latter to be the new outer square, the process repeats itself to find the new inner square, that is now surrounded by four new triangles A''', B''', C''' and D'''. As before, this evolves towards an inward and outward infinity. With all dimensions together as a Universal Gear, the general case x≠y also behaves in the same fractal manner (Figure 10d). The outer cube side $z_o = x+y$ encloses an inner irregular expanded truncated octahedron side $\frac{z_i}{2} = \frac{\sqrt{x^2+y^2}}{2}$ via eight irregular contracted truncated tetrahedrons. The square sides of the octahedron are in the same plane as those of the cube. In the same way, the truncated octahedron encloses an inner cube whose vertices are the centre of the hexagonal faces of the octahedron, which in turn encloses a smaller truncated octahedron. The fractal process extends towards infinity, inwardly and outwardly.

3.3 One-, two- and three-dimensional grids

For the particular case x = y, all the triangles are isosceles, equal and aligned with each other, forming two dimensional grids enclosed within infinite planes orthogonal to each other, and in each plane the one-dimensional grid (i.e., an infinite connection of equal finite line segments) governs the length formed by the outline of the triangles, while the two-dimensional grid governs the area information formed by the interior of the triangles (Figure 11a). Looking at a level higher, one can see that all the squares (for example, A, B, C, D and E formed by four isosceles triangles each) are identical and perfectly connected with each other. In turn, these infinite number of planes orthogonal to each other form a three-dimensional grid governing the volume information in between the triangles. The space in between the triangles was shown previously in Figure 3, where the three orthogonal triangles form a truncated tetrahedron. As a consequence, in three dimensions the volume defined by the orthogonal intersections of all the grids define equal and perfectly connecting tetrahedrons (that in turn together form octahedrons) and occupy the entirety of the three dimensional space (Figure 11b).

For the general case x≠y, all triangles are again equal but are now scalene and out of alignment with each other, forming in turn distorted one- and two-dimensional grids that are out of alignment (Figure 11c). A new balance between lengths and areas is formed where a new

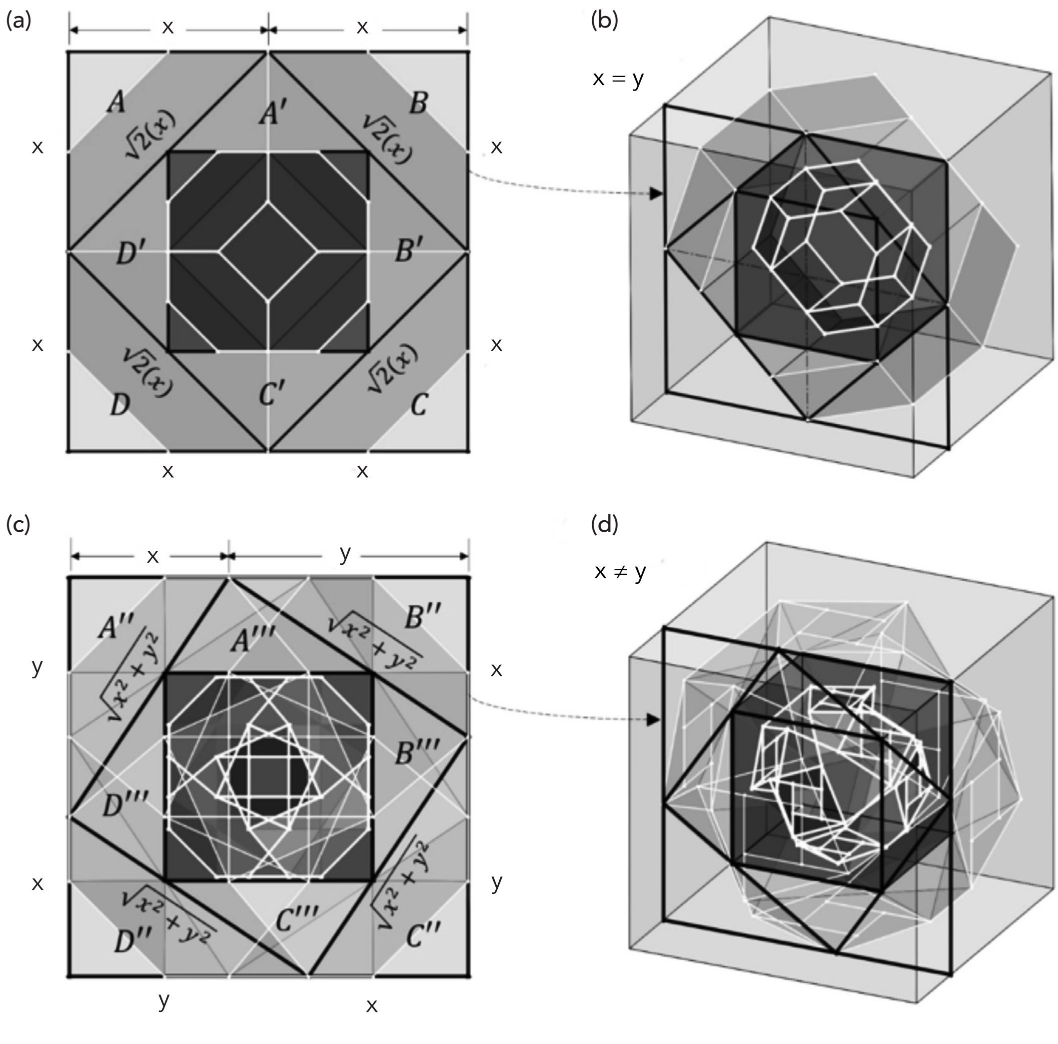


Figure 10. Fractal nature of the Universal Gear: (a) particular case x=y and (b) general case x≠y.

geometrical element appears naturally to allow for the re-adjustment of length and area information within the grids [i.e., line (y–x) in 1D and square $(y-x)^2$ in 2D]. The previously exemplified squares A, B, C and D in Figure 11a (formed each now by four scalene triangle) have expanded, creating a smaller square at their center. In turn, square E at the center contracted. These scalene triangles and the new central squares represent in fact a change in partition of length and area within the one- and two-dimensional unaligned grids. Consequently, the volume between the triangles and the squares also affects the shape of the tetrahedrons (and thus also of the contracted and expanded octahedrons that they compose), forming the three-dimensional grid (Figure 11d). The appearance of the smaller central squares in two dimensions gives rise to new volumetric central elements to appear [i.e., cube $(y-x)^3$ and its parallelepiped extensions $x(y-x)^2$], as a natural re-adjustment from an aligned state $x=y$ to an unaligned state $x\neq y$.

Each truncated octahedron, contracted or expanded, can be seen as an individual gear, while many of them interconnected can be seen as a network gear of many gears. For the particular case $x=y$, many of the one- and two-dimensional gears in Figure 11a aggregate to form a larger network gear of many gears in Figure 12a. Within this network gear, information flows from the orthogonal grid (i.e. composed of vertical and horizontal lines) to the diagonal grid (composed of the oblique lines) via the intermediate nodes, and back again. Similarly, in three-dimensions, the individual gears formed by the octahedrons in Figure 11b aggregate to form larger network gears of many octahedrons in Figure 12b. The particularity of this truncated octahedron is that its square and hexagonal surfaces are a perfect match to other neighbouring truncated octahedrons, thus forming the special equilibrium where the infinite multidimensional grid occupies all space.

For the general case $x\neq y$, the difference between *x* and *y* causes a distortion in the two dimensional orthogonal grid, and Figure 11c transforms into Figure 12c. In a dynamic situation like in a network of pendulums, this non-alignment migrates from the orthogonal grid to

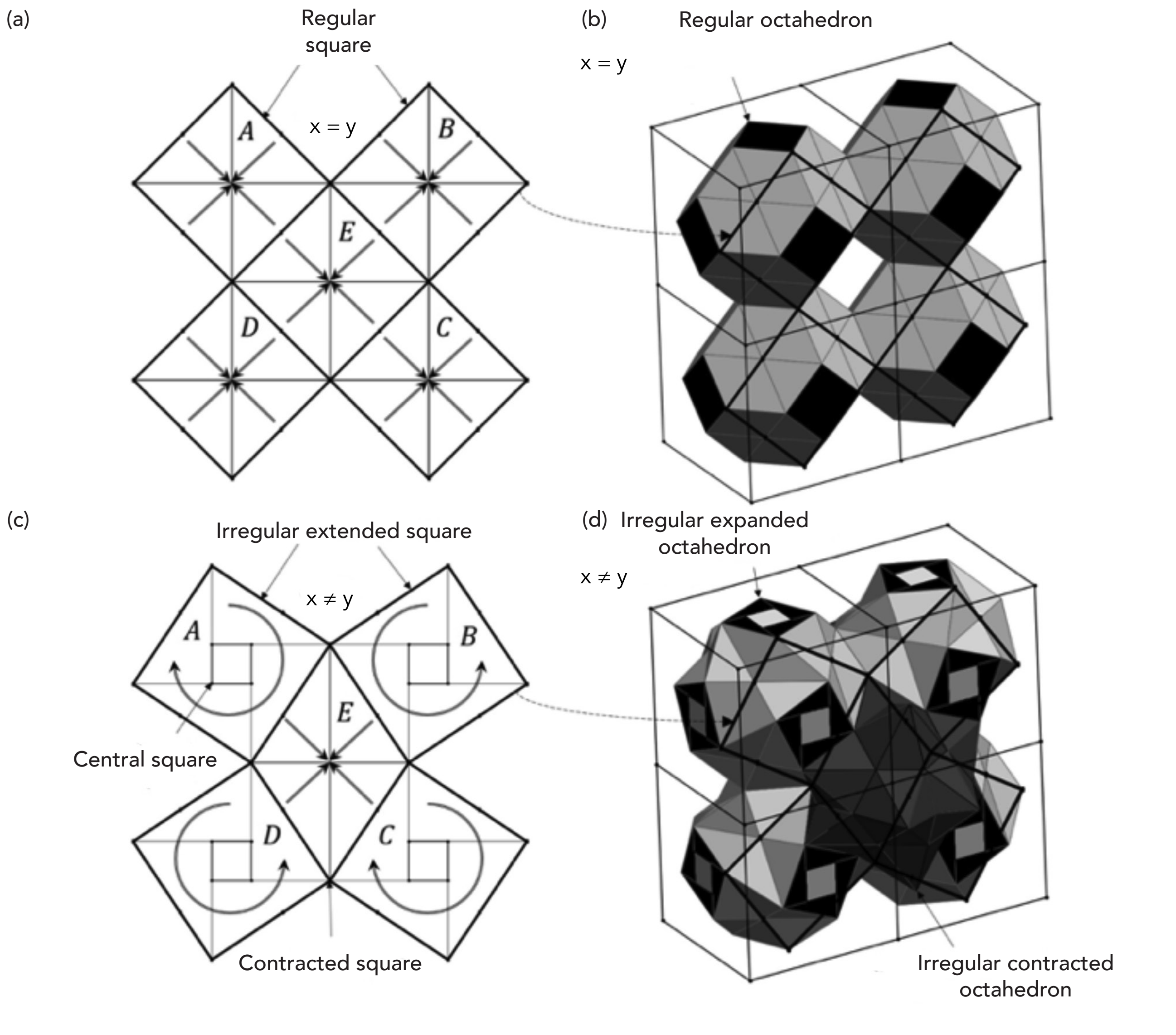


Figure 11. Grid integration of the Universal Gear: (a) particular case x=y and (b) general case x≠y.

the diagonal and back again, very much like a perpetual motion machine. And just as in a pendulum, if the regular position is considered to be the particular case x = y, then the general case x≠y oscillates around this particular case. This infinite network gear shown in Figure 12d composed of one-, two- and three-dimensional grids, is the Universal Gear. Within the Universal Gear, the motion of the expanding octahedrons is matched with that of the contracting octahedrons, such that all distorted hexagonal and square faces of neighbouring octahedrons match each other, thus forming the general state of the infinite multidimensional grid occupying all space. Ultimately, the Universal Gear is like a clock governed by the jewel movement of the internal orthogonal and diagonal grids composed of right-angled triangles, changing state in a dynamic motion between the regular case x = y (Figure 12b) and irregular case x≠y (Figure 12d).

3.4 Application to computational fluid dynamics

In computational fluid dynamics, a network of geometrical elements (tetrahedral and prismatic) are used to study the behaviour (e.g., pressure, temperature, air speed, etc) of a fluid within or around a body (Anderson Jr., 1995). Figure 13 shows how the volume around a CAD (Computer Aided Design) YF–17 fighter jet geometry was split into a variable size network of tetrahedral and prismatic elements (often called "cells" in the aeronautical world) [Tomac and Eller, 2014]. The Pythagorean structure of the Universal Gear could be used as an alternative means of mapping the cells. In such case, different truncated octahedral cells could have different sizes depending on the Pythagorean relation between x and y, and groups of these octahedrons could be merged to form larger cells, often filling large spaces presenting small aerodynamic gradients to reduce cell count. Hence, the Universal Gear offers a grid construction method to interrelate tetrahedral (already used in unstructured meshes) using the order of the Pythagorean Theorem as the means to map them volumetrically.

3.5 Classroom exercise—construction of the truncated octahedron

The base of the Universal Gear is the truncated octahedron, both for the particular and general case, as it was shown

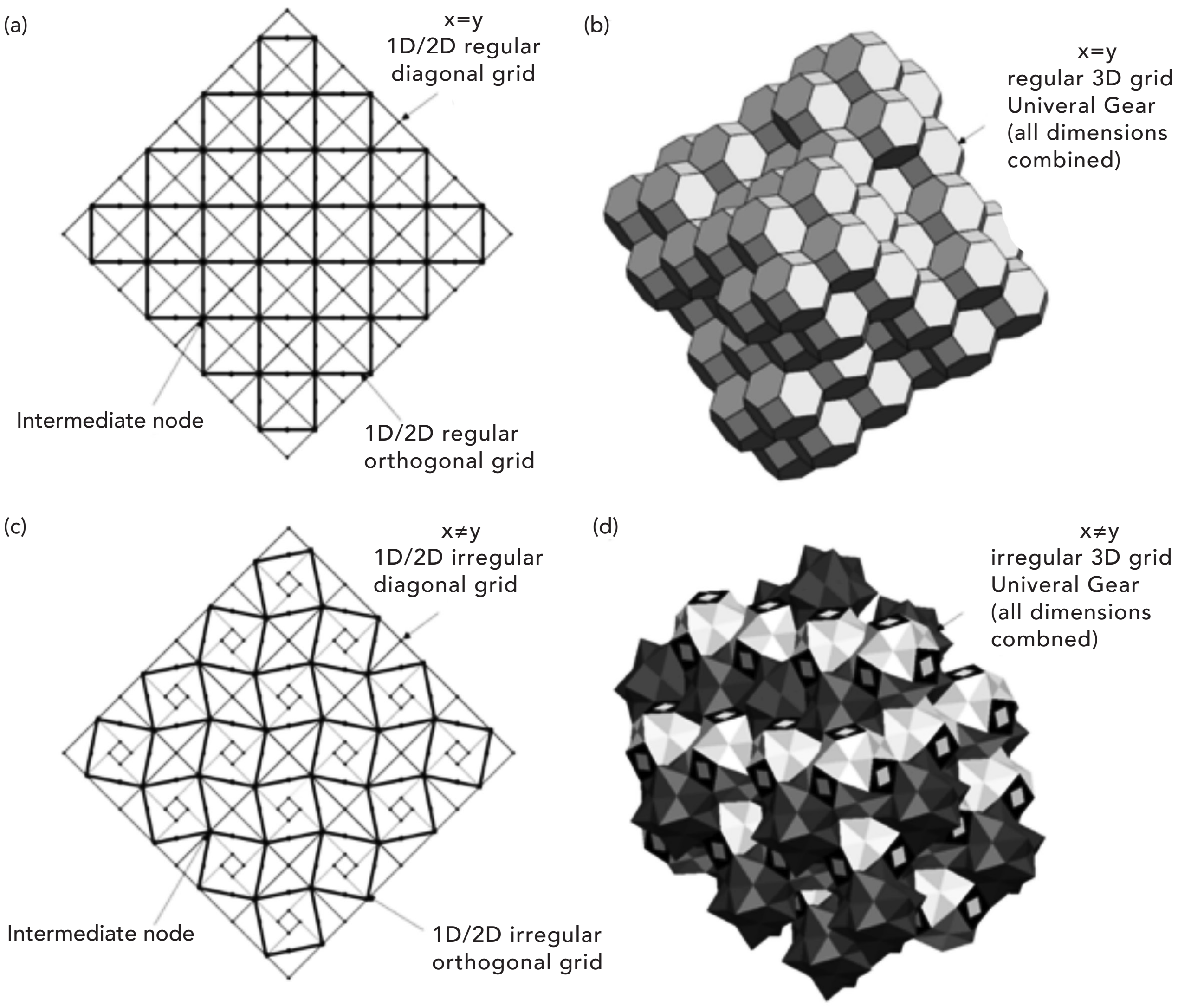


Figure 12 (a–d). Two- and three-dimensional perspectives of the Universal Gear for both x=y and x≠y.

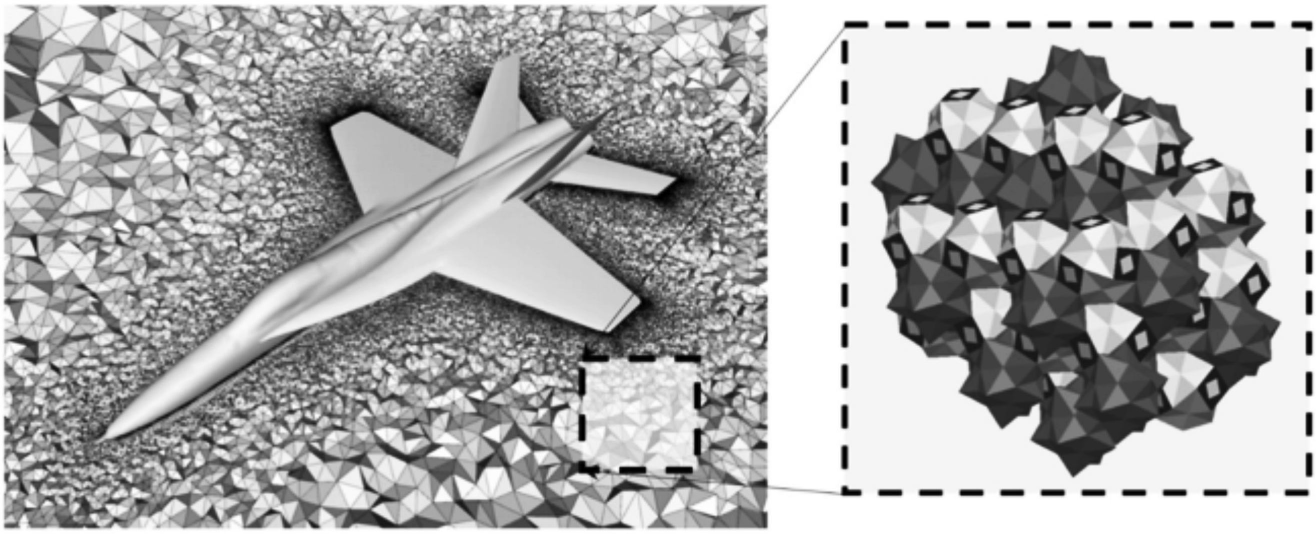


Figure 13. Potential application of the Universal Gear in computational fluid dynamics.

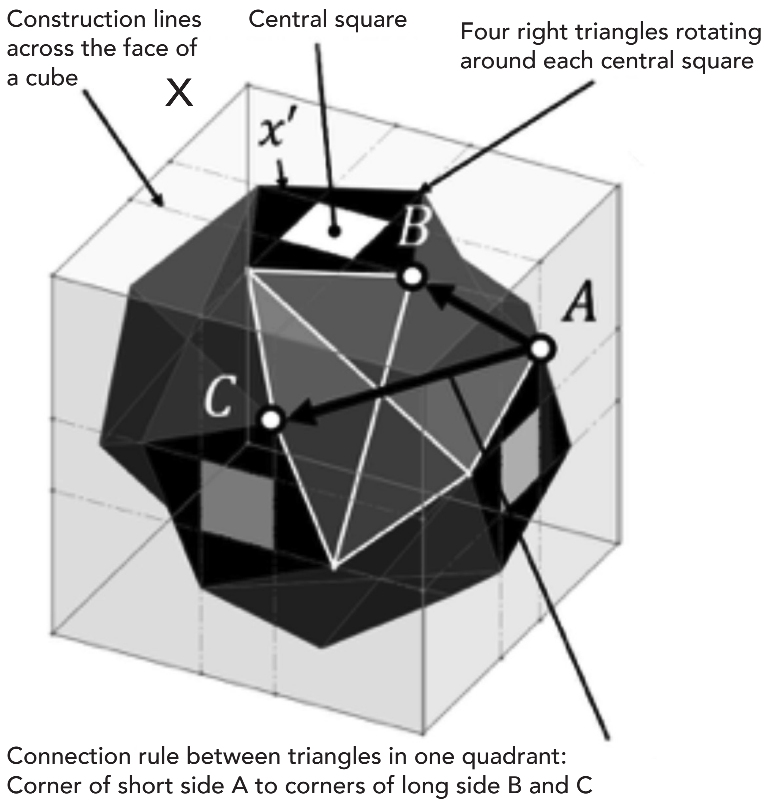


Figure 14. Applied exercise for the construction of the truncated octahedron—the building block of the Universal Gear.

to contain all other one-, two- and three-dimensional relations and form the infinite grid in Figure 12 and 13. Therefore, it would be of interest that students are able to construct it geometrically using a Computer-Aided Design (CAD) software, of which an example is the open-source Free-CAD tool (that is free to install and use in a classroom environment). The layout of CAD programs in general may be different, but they all have the same functions (e.g., sketches, extrusion, planes, etc). The solution to this exercise (Figure 14) is described in the following three main steps:

1. Create a cube side z_o within a sketch. An efficient way to do this is to create a cube from a two dimensional sketch, and then use a 3D sketch (that is a sketch that is not bounded by a plane), and draw lines snapping to the existing cube edges. Then, create a point at distance x from all corners along all edges. If the two distances starting from each corner match, then by defini-tion it is the regular case $z_o = x + x$. If they don't match, then it is the irregular case $z_o = y + x$. From these new points at distance x, draw construction lines (bounded to the face and perpendicular to the edge) across the face of the cube. In so doing, central squares are formed automatically by the intersection of these lines.
2. Using regular (continuous) lines, draw on each side of the cube four right-triangles rotating around each central square. These are smaller than right-triangles discussed before where the short leg of each triangle is given as $x' = \frac{xy}{x+y} < x$.
3. Again using regular lines, connect the three opposing small triangles (one in each of the three sides of the cube surrounding one corner), using as a rule short-to-long side connection. That is, the end of each short side of one triangle (e.g., point A in triangle I with white line as diagonal) connects to the end of the long sides of the other two triangles (e.g., point B in triangle II and point C in triangle III also with white lines as diagonals).

Repeating steps 2–3 for all other cube corners results in an expanded truncated octahedron defined by the regular lines (as shown in Figure 14b) enclosed in a cube defined by construction lines. To obtain the regular octahedron, simply make x equal to y in the drawing. Note that the sense of rotation of the small triangles defines the symmetry of the octahedron, that is one octahedron with the triangles rotating in a clockwise (CW) sense matches its faces with a symmetrical octahedron with the triangles rotating in a counter-clockwise (CCW) sense.

A proposed further improvement is to add faces to all triangles and squares with different colour coding to make it more visible (like in Figure 14). For those who are adventurous, an assembly of eight expanded octahedrons (alternating type of neighboured octahedron between CW and CCW to match the sides) will form the shape of the collapsed octahedron (as shown in Figure 6c alone and integrated in Figure 11d). Ultimately, these eight expanded octahedrons formed by collapsed octahedrons inside and around them, are the building block for the larger grid assembly that is the Universal Gear for the particular case $x = y$ in Figure 12b and for the general case $x \neq y$ in Figure 12d.

Conclusion

A leap in knowledge often is accompanied by a necessary leap in the human thought process. This was true at the time when the Babylonians learnt Pythagoras' Theorem, and it is true now. The present leap in knowledge challenges one's ability for three-dimensional visualization. It insights the Pythagorean Theorem as being part of a larger three-dimensional concept, here named the Universal Gear. In a nutshell, the Universal Gear comprises of infinite interlaced Pythagorean Theorems embedded in planes along all three orthogonal directions that are constantly connected and in perfect balance. In so doing, this concept governs information exchange in one- (lines), two- (areas) and three-dimensions (volumes) throughout the grid. From this enhanced perspective, the Universal Gear is bigger than the Pythagorean Theorem in that not only it satisfies the later individually—by interrelating the area of three squares via an intermediate triangle—but it also establishes a volumetric equivalent (to the Pythagorean Theorem) by relating two cubes to a half truncated octahedron via a truncated tetrahedron,and vice versa, which is different from saying that the volumetric sum of two cubes relate to a third. Noting that all in all, it is the Pythagorean Theorem that governs this three-dimensional structure. Thus, the Universal Gear establishes that the truncated tetrahedron is the three-dimensional equivalent to the right-triangle—and a truncated tetrahedron is indeed formed by three orthogonally alignment right-triangles—and that the diagonal of the cube, that is the hexagon, is the three-dimensional equivalent to the diagonal of a square (the hypotenuse of a triangle). Ultimately, the Universal Gear is governed by the Pythagorean Theorem's dynamics expressed synchronously in a three-dimensional network. Like reality, the Universal Gear presents a fractal nature in that its structure repeats itself infinitely both inwardly and outwardly. One of its practical applications has been identified in the study of fluids, and more particularly in computation fluid dynamics.

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Appendix

The purpose of this appendix is to prove equations (6) and (7) that define the volume of the inner expanded and outer contracted tetrahedrons, respectively.

$$V_{Ti} = \frac{1}{(x+y)^3}\left\{-\frac{1}{2}x^6 + \frac{7}{4}x^4y^2 + \frac{5}{2}x^3y^3 + \frac{1}{4}x^2y^4\right\} \tag{11}$$

$$V_{To} = \frac{1}{(x+y)^3}\left\{\frac{3}{2}x^3y^3 + \frac{5}{4}\left[x^4y^2 + x^2y^4\right]\right\} \tag{12}$$

The inner tetrahedron is composed of a central cube V_{Ti1} and three sets of a parallelepiped with two missing corners V_{Ti2}, a tetrahedron V_{Ti3} and half of a smaller parallelepiped V_{Ti4}, as shown in Figure 15a

$$V_{Ti} = V_{Ti1} + 3\left\{V_{Ti2} + V_{Ti3} + V_{Ti4}\right\} \tag{13}$$

where each is given as

$$V_{Ti1} = \left(\frac{x^2}{x+y}\right)^3 = \frac{x^6}{(x+y)^3} \tag{14}$$

$$V_{Ti2} = \frac{5}{6}\left(\frac{x^2}{x+y}\right)\left(y - x + \frac{x^2}{x+y}\right)\left(x - \frac{x^2}{x+y}\right) = \frac{5}{6}\frac{x^3y^3}{(x+y)^3} \tag{15}$$

$$V_{Ti3} = \frac{1}{6}\left(x - \frac{x^2}{x+y}\right)^2\left(\frac{y-x}{2} + \frac{x^2}{x+y}\right) = \frac{1}{6}\left(\frac{x^2y^4 + x^4y^2}{2(x+y)^3}\right) \tag{16}$$

$$V_{Ti4} = \frac{1}{2}(y-x)\left(\frac{x^2}{x+y}\right)^2 = \frac{1}{2}\frac{\left(-x^2+y^2\right)x^4}{(x+y)^3} \tag{17}$$

Applying equations (14)–(17) to (13) gives

$$V_{Ti} = \frac{1}{(x+y)^3}\left\{x^6 + 3\left[\frac{1}{6}\left(\frac{x^2y^4 + x^4y^2}{2}\right) + \frac{5}{6}x^3y^3 + \frac{1}{2}\left(-x^2+y^2\right)x^4\right]\right\}$$

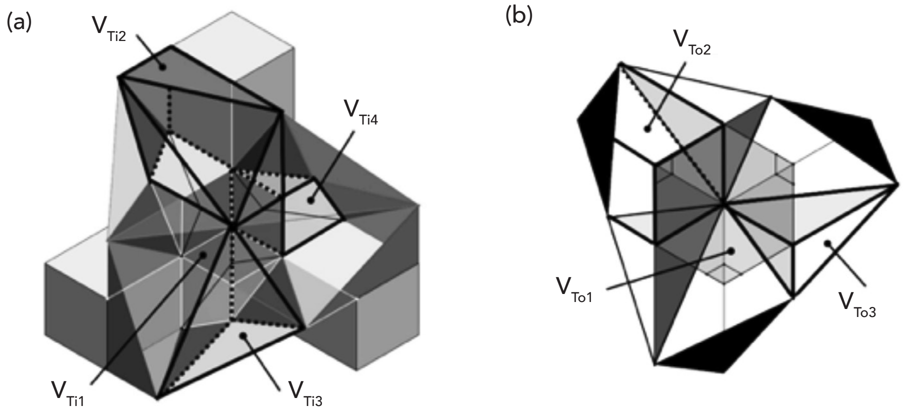

Figure 15. Volumetric partitioning of the: (a) inner expanded tetrahedron and (b) outer contracted tetrahedron.

Simplifying gives as required equation (11)

$$V_{Ti} = \frac{1}{(x+y)^3}\left\{-\frac{1}{2}x^6 + \frac{7}{4}x^4y^2 + \frac{5}{2}x^3y^3 + \frac{1}{4}x^2y^4\right\}$$

The outer tetrahedron is composed of a cube V_{To1} and three sets of a parallelepiped with a missing corner V_{To2} and a tetrahedron V_{To3}, as shown in Figure 15b.

$$V_{To} = V_{To1} + 3\left\{V_{To2} + V_{To3}\right\} \tag{18}$$

where each is given as

$$V_{To1} = \left(x - \frac{x^2}{x+y}\right)^3 = \frac{(xy)^3}{(x+y)^3} \tag{19}$$

$$V_{To1} = \frac{1}{2}\left(y - x + 2\frac{x^2}{x+y}\right)\left(x - \frac{x^2}{x+y}\right) - \frac{1}{6}\left(y - x + 2\frac{x^2}{x+y}\right)\frac{1}{2}\left(x - \frac{x^2}{x+y}\right)^2 = \tag{20}$$

$$= \frac{1}{2}\left\{\frac{5}{6}\left(\frac{x^2+y^2}{x+y}\right)\left(\frac{xy}{x+y}\right)^2\right\}$$

$$V_{To3} = \frac{1}{6}\left(\frac{x^2}{x+y}\right)\left(y - x + \frac{x^2}{x+y}\right)\left(x - \frac{x^2}{x+y}\right) = \frac{1}{6}\left(\frac{x^3y^3}{(x+y)^3}\right) \tag{21}$$

Applying equations (19)–(21) to (18) gives

$$V_{To} = \frac{(xy)^3}{(x+y)^3} + \left\{\frac{1}{6}\left(\frac{x^3y^3}{(x+y)^3}\right) + \frac{1}{2}\left\{\frac{5}{6}\left(\frac{x^2+y^2}{x+y}\right)\left(\frac{xy}{x+y}\right)^2\right\}\right\}$$

Simplifying gives equation (12) as required $V_{To} = \frac{1}{(x+y)^3}\left\{\frac{3}{2}x^3y^3 + \frac{5}{4}\left[x^4y^2 + x^2y^4\right]\right\}$

This completes the proof.