Transnatural Numbers as Elementary Objects In a Slight (?) Modification of Set Theory NFU

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Abstract

The consistency of the transreals, relative to the reals, was previously proved by constructing the transreals as tuples of reals. We now give a more fundamental construction of the transnaturals as elementary objects in the set theory New Foundations with Urelements (NFU), which set theory we extend by making certain operations total over atoms as well as sets. Hence the transnumbers are established in the (new) foundations of mathematics.

1 Introduction

Transreal arithmetic augments real arithmetic with three definite, non-finite numbers. Critically it delivers a total and consistent arithmetic. Totallity is essential to the use of transnumbers in computer science as a means of avoiding logical exceptions. Totality ensures that every syntactically correct program is semantically correct in the sense that it does not crash for any logical reason when executed, though it may crash for a physical reason. Totality also enables very strong mathematical reasoning by exclusion. During the present development of the transnatural numbers, we shall totalise certain operations of the set theory New Foundations with Urelements (NFU). Thus we also provide a, possibly slight, extension of set theory.

There is a machine proof [2] of the consistency of transreal arithmetic and a human, constructive proof [4] of the consistency of transreal and transcomplex arithmetic relative to real arithmetic. The constructive proof is expressed in Zermelo-Fraenkel set theory with the axiom of Choice (ZFC), using tuples of a real numerator and denominator such that, for all positive k: positive infinity is defined by $\infty = 1/0 \equiv k/0$; negative infinity is defined by $-\infty = -1/0 \equiv -k/0$; and nullity is defined by $\Phi = 0/0$.

The arrangement of the transreal numbers is shown in Figure 1. The real numbers are shown on a finitely long line. All of the non-finite, transreal numbers – negative infinity, nullity, and positive infinity – are separated from the real-number line by gaps. (This arrangement may be deduced by calculating ϵ-neighbourhoods in transreal arithmetic.) Nullity

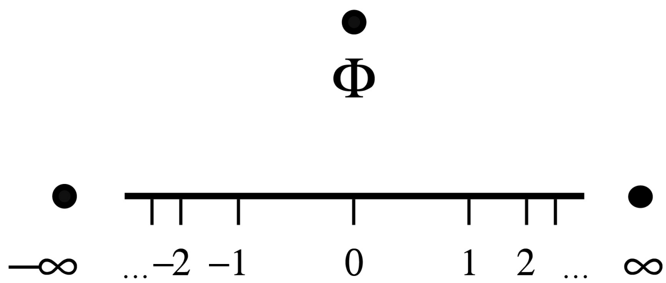

Figure 1: Transreal-Number Line.

is unordered and so may be shown anywhere off the extended-real-number line from negative infinity to positive infinity; it is shown, here, above zero. Some simple theorems of transreal arithmetic establish that nullity is the only unordered, transreal number; that infinity is greater than any other ordered, transreal number and that negative infinity is less than any other ordered, transreal number. So much for proof. We then rely on an intuition that positive infinity is the greatest ordinal. This intuition gains credence when we recall that non-Archimedean number systems have infinitesimal numbers whose magnitude is greater than zero but less than any real number [8] [3]. The reciprocals of these infinitesimal numbers are transfinite numbers, infinities, arranged such that the closer an infinitesimal is to zero, the greater the magnitude of its reciprocal infinity. We suppose that this relationship continues to zero so that the reciprocal of zero has the greatest magnitude. Hence transreal, positive infinity is the unique reciprocal of zero, $\infty = 1/0$, and transreal, negative infinity is the unique, negative of the reciprocal of zero, $-\infty = -1/0$. The non-zero infinitesimals form, negative and positive, open intervals about zero and their reciprocals form, negative and positive, open intervals far from zero. We say that these infinities are non-terminal. Each of the non-terminal infinities has a magnitude which is strictly less than the magnitude of transreal, positive and negative, infinity, which we call terminal infinities. As transreal, positive infinity is terminal, it has an ordinal position: just as zero is the first ordinal, so its reciprocal – transreal, positive infinity – is the last ordinal. One of the goals of the present paper is to put this intuition on a firmer footing by establishing the transnatural numbers as elementary objects in a suitable set theory.

ZFC is the most popular set theory. It is usual, in ZFC, to represent natural numbers, and all non-terminal, ordinal numbers, with the von Neumann Ordinals: $0 = \{\}$; $1 = \{0\} = \{\ \{\}\ \}$; $2 = \{0, 1\} = \{\ \{\}, \{\{\}\}\ \}$; $3 = \{0, 1, 2\} = \{\ \{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}\ \}$; and so on. The von Neumann ordinals, n_i, have the advantage that cardinality and ordering emerge in the elementary way: $n_i = |n_i|$ and $n_1 < n_2$ iff $n_1 \in n_2$. But the set theory ZFC is incompatible with a greatest ordinal so we cannot use it to represent transreal infinity in an elementary way. We must use some other set theory.

By contrast the set theory NFU provides a universal set [7] [6]. Having a universal set ensures that set complement exists as part of a Boolean algebra. Boolean algebras enable reasoning by exclusion so NFU is very well placed to exploit the totality delivered by transarithmetics.

NFU has a stratified comprehension rule which exploits type theory to block Russell's paradox, making the set theory consistent [7]. However, stratified comprehension has a number of profound influences. Firstly the member relation, $\in$, does not exist as an elementary relation; instead it is defined via the subset relation so that $x \in y$ is identical to $\{x\} \subseteq y$. Secondly stratified comprehension complicates the construction of some sets. In particular the construction of the von Neumann ordinals cannot proceed in the usual way. Instead, in NFU, it is usual to construct each of the non-terminal ordinals – $0, 1, 2$, and so on – as the set of all sets with n elements [6]. This is adequate for developing ordinal arithmetics but it blocks elementary cardinality and ordering as instantiated by the von Neumann ordinals. This structure also means that ordinal arithmetic, in NFU, is implemented, not on sets, but on an arbitrarily chosen element of a set, all of which elements must have the same cardinality. But this means that NFU's ordinal arithmetic is not defined on any set which lacks this structure, making the arithmetic partial, not total, over all sets. This loss of totality would be grievous to the transarithmetics.

We address these problems by exploiting two further properties of NFU. In addition to providing sets, NFU also provides urelements, that is it provides atoms, and it adopts the Axiom of Choice so all sets are well ordered. In particular the set of all atoms is well ordered. Given ordered atoms, α_i, we define the canonical form of the transordinal numbers: nullity, Φ, is the atom $\Phi = \alpha_0$; zero, 0, is the empty set of atoms, $0 = \{\}$; every non-zero, non-terminal, transordinal, n, has $0 < n < \infty$ and is the set $n = \{\alpha_1, \alpha_2, ..., \alpha_n\}$; infinity, ∞, is the set of all atoms $\infty = \{\alpha_0, \alpha_1, \alpha_2, ..., \alpha_\infty\}$. This restores an elementary instantiation of cardinality and ordering, in NFU, as $n_i = |n_i|$ and $n_1 < n_2$ iff $n_1 \subset n_2$.

We then take transordinal nullity, Φ, equivalent to each free atom and take each transordinal set of atoms equivalent to any set with the same cardinality. We reduce any atom or set to its canonical form, before performing arithmetic, and reduce the result of each arithmetical operation to canonical form. Hence transordinal arithmetic is total over all NFU objects and, in particular, transnatural arithmetic is total over all NFU objects equivalent to a transnatural number.

We suppose that transnatural arithmetic may be extended, in the usual way, to other arithmetics, such as transinteger, transrational, transreal, and transcomplex arithmetic; but we do not undertake that work here.

It will be helpful to recall that transreal arithmetic is totally associative and commutative and that it has some absorptivities. Nullity is totally absorptive over the arithmetical operations of addition, subtraction, multiplication and division but is not generally absorptive; for example nullity may appear as an argument of functions whose value is not nullity. Infinity is partially absorptive. Also we shall want some of the non-finite powers of two for discussion of the powerset.

Observation 1. *For all transreal x: $\Phi \ \circ \ x = \Phi$, where $\circ$ is any one of the operations of addition, subtraction, multiplication, division.*

Observation 2. *For all transreal x:*

$$\infty + x = \begin{cases} \Phi & , \textit{ if } x \in \{\Phi, -\infty\} \\ \infty & , \textit{ otherwise} \end{cases} .$$

Observation 3. *For all transreal x:*

$$\infty \times x = \begin{cases} \Phi & , \textit{ if } x \in \{\Phi, 0\} \\ \infty & , \textit{ if } x > 0 \\ -\infty & , \textit{ if } x < 0 \end{cases} .$$

Observation 4. *For all transreal x: $\Phi \not< x$ and $x \not< \Phi$.*

Observation 5. *For all transreal $x \notin \{\Phi, -\infty, \infty\}$: $-\infty < x < \infty$.*

Observation 6. *It is a theorem of transreal arithmetic that $2^{\Phi} = \Phi$, $2^{\infty} = \infty$.*

2 Construction

Our goal is to construct transnatural arithmetic [2] using elementary objects in the set theory NFU. We begin by recalling two axioms of NFU, given here as Axiom 7 and Axiom 9. Compare with [6].

Axiom 7. *The universal set $V := \{x \mid x = x\}$ exists.*

Observation 8. *Observe, in passing, that the Not-a-Number objects of floating-point arithmetic are not members of the universal set because $NaN_i \neq NaN_j$ for all i, j – including $i = j$! This complicates the writing of numerical programs, as discussed in [1], with corrections in [5].*

Axiom 9. *If x is an atom then for all y: $y \not\subseteq x$.*

Observation 10. *The empty set, $x = \{\}$, is not an atom because there is a $y = \{\}$ such that $y \subseteq x$.*

The binary operations of ZFC are total because all of the objects of ZFC are pure sets but the corresponding binary operations of NFU are partial because NFU also provides atoms. The question of how each binary operation should be totalised is delicate so we provide an axiom scheme, to handle many operations, and say which particular operations we totalise here. The scheme employs nullity, Φ, which is defined as an NFU object in Axiom 12 below. The totalisations are disjoint from the usual operations and so are trivially consistent with them.

Axiom Scheme 11. *The binary operation $A \circ B = \Phi$ when A or B is an atom and $\circ$ is a specified operation. Here we specify: set union, $\cup$; subset, $\subset$; disjoint sum, $\oplus$; Cartesian product, usually written $\times$ but written here as $\otimes$; set difference, $\setminus$.*

In ZFC two sets have the same cardinality iff there is a bijection between the elements of the sets. This is a total definition because all of the objects of ZFC are pure sets but the corresponding definition in NFU is partial because NFU also provides atoms. We define the cardinality of free atoms in Axiom 13 below. The totalisation of cardinality is disjoint from the usual cardinality and so is trivially consistent with it.

NFU adopts the Axiom of Choice so all sets are well ordered. In particular there is an ordering of the atoms $\alpha_0, \alpha_1, \alpha_2$, and so on. This ordering defines the predecessor relationship, $\prec$, for all atoms. We obtain each transordinal number as a unique atom or set of atoms, relative to a given ordering; but there are many orderings so it is more helpful to observe that our definitions are unique up to an isomorphism of atoms and sets. The invariants of all such descriptions are the cardinalities and subset relations developed below.

Axiom 12. *Transordinal nullity, Φ, is the free atom $\Phi := \alpha_0$.*

Axiom 13. *Each free atom, α_i, has cardinality nullity, $|\alpha_i| := \Phi$.*

Axiom 14. *Transordinal zero, 0, is the empty set of atoms $0 := \{\}$.*

Axiom 15. *Each non-terminal, transordinal, n, is the set of atoms $n := \{\alpha \mid \alpha_0 \prec \alpha \prec \alpha_n\} \cup \{\alpha_n\}$.*

Axiom 16. *The last atom, α_∞, such that $\alpha_i \prec \alpha_\infty$ for all $\alpha_i \neq \alpha_\infty$, exists.*

Axiom 17. *Terminal, transordinal infinity, ∞, is the set of all atoms $\infty := \{\alpha \mid \alpha \text{ is an atom}\}$.*

Observation 18. *The atom α_0 is a marker for the strictly transordinal numbers, Φ and ∞, in the sense that α_0 is equal only to Φ and is an element only of ∞. The atom α_0 plays a critical role in establishing the topology shown in Figure 1. Firstly, as a free atom, α_0 is disjoint from all of the transordinal numbers described by sets so it describes the point at nullity, Φ, which is disjoint from all other transreal numbers. Secondly every transordinal $\{\alpha_1, \alpha_2, ..., \alpha_n\}$, including zero, $\{\}$, that is less than infinity, has an immediate successor, $\{\alpha_1, \alpha_2, ..., \alpha_n, \alpha_{n+1}\}$, but infinity is not the immediate successor to any of these, non-terminal, ordinals because infinity contains α_0. Thus α_0 is intimately related to the gaps in Figure 1. This inclines us to the view that α_0 is the vacuum or empty atom. We shall hold this thought in mind in case we find any further parallels between the empty set and the empty atom that provide deeper insights into our modified set theory.*

Definition 19. *For all x, the canonical form of x as a transordinal number, $\mathcal{C}(x)$, is given by:*

$$\mathcal{C}(x) = \begin{cases} \Phi & , \textit{ if } x \textit{ is an atom} \\ \{\} & , \textit{ if } x = \{\} \\ \infty & , \textit{ if } x \textit{ is bijective with } \infty \\ \{\alpha_1, ..., \alpha_n\} & , \textit{ otherwise if } x \textit{ is bijective with } \{\alpha_1, ..., \alpha_n\} \end{cases} .$$

Axiom 20. *Transordinal ordering is provided by the less-than relation, $<$, such that $x < y$ iff $\mathcal{C}(x) \subset \mathcal{C}(y)$.*

Theorem 21. *Nullity is unordered with respect to every other transordinal. As required by Observation 4.*

Proof. In other words, nullity is equal to itself but is not less than or greater than any transordinal, including itself. It is sufficient to show that $\Phi \not< x$ where x is any transordinal. Thus $\Phi < x \iff \mathcal{C}(\Phi) \subset \mathcal{C}(x) \iff \Phi \subset \mathcal{C}(x)$ but Φ is an atom so, by the inclusion of the subset relation in Axiom Scheme 11, it is the case that $\Phi \not\subset \mathcal{C}(x)$. Therefore $\Phi \not< x$. $\square$

Theorem 22. *The universal set has cardinality infinity.*

Proof. Every atom is an element of V so $|V| \geq |\infty|$. Name each element of V with a distinct atom then $|\infty| \geq |V|$. Hence $|V| = |\infty|$. Naming may be done as follows. For each set in V there is a distinct atom in V which we take to be the name of the set. If there are any additional atoms in V then we take it that each atom names itself. $\square$

Observation 23. *Infinity is the greatest transordinal because it is ordered and infinity is the greatest cardinal. To see this cardinality consider any object x. Regardless of whether x is an object of NFU, x itself and each distinct subcomponent of x, if any, can be named as a distinct atom in V. Thus the cardinality of any collection of objects in x is no greater than $|V|$ but $|V| = |\infty|$. It follows that every component of the class of all classes and every object and arrow in the category of all categories can be named by atoms in V. Compare with Observation 5.*

NFU supports a powerset function, which we write as $\mathcal{P}(X) = Y$, where X is an arbitrary set and Y is the set of all subsets of X. In general $|\mathcal{P}(X)| = 2^{|X|}$. We now totalises the power function and consider it further in the Discussion.

Axiom 24. *For each atom, α, $|\mathcal{P}(\alpha)| = \Phi$.*

The cardinality of the powerset of a largest set is considered in various set theories. We present a theorem here and consider it further in the Discussion.

Theorem 25. *If $|X| = |\infty|$ then $|\mathcal{P}(X)| = |\infty|$.*

Proof. For each element, $e \in X$, there is a distinct singleton set, $\{e\} \in |\mathcal{P}(X)|$, so $|\mathcal{P}(X)| \geq |X|$ or, identically, $|\mathcal{P}(X)| \geq |\infty|$. Name each element of $|\mathcal{P}(X)|$ with a distinct atom then $|\infty| \geq |\mathcal{P}(X)|$. Hence $|\mathcal{P}(X)| = |\infty|$. $\square$

Observation 26. *Taking the general case $|\mathcal{P}(X)| = 2^{|X|}$ of NFU and other set theories, together with Axiom 24 and Theorem 25, we have $|\mathcal{P}(X)| = 2^{|X|}$ for all X, where $2^{|X|}$ is calculated in transreal arithmetic. See Observation 6.*

We have now instantiated infinity and nullity in NFU. The arithmetic of the natural numbers, along with many other number systems, has already been given in NFU [6]. It remains only to totalise NFU's operations of addition, $+$, and multiplication, $\times$, so that they may involve the strictly transnatural numbers infinity and nullity. We begin by rehearsing a usual

Definition 27. *The disjoint sum of sets S and T, written $S \oplus T$, is the set $\{(x,y) \mid (x \in S$ and $y=0)$ or $(x \in T$ and $y=1)\}$.*

In ZFC this definition is total because all objects are sets but in NFU the definition is partial because the disjoint sum, as given so far, is not defined for a sum of atoms. We totalise the disjoint sum by specifying it in Axiom Scheme 11.

Axiom 28. *The transordinal sum of atoms or sets x, y is $x+y := \mathcal{C}(x \oplus y)$.*

Theorem 29. *For all transordinal x: $x + \Phi = \Phi$. As required by Observation 1.*

Proof. $x + \Phi = \mathcal{C}(x \oplus \Phi) = \mathcal{C}(\Phi) = \Phi$. □

Theorem 30. *For all transordinal $x \neq \Phi$: $x + \infty = \infty$. As required by Observation 2.*

Proof. $x + \infty = \mathcal{C}(x \oplus \infty) = \mathcal{C}(y)$ for some $y \in V$ so $|V| = |\infty| \geq |y|$ but $(\alpha_i, 1) \in y$ for all atoms α_i so $|y| \geq |\infty|$. Hence $|y| = |\infty|$, therefore $\mathcal{C}(y) = \infty$. □

Axiom 31. *The transordinal product, $\times$, of atoms or sets x, y is as follows. Firstly $\infty \times 0 = \Phi$, as required by Observation 3. Otherwise $x \times y := \mathcal{C}(x \otimes y)$.*

Theorem 32. *For all transordinal x: $x \times \Phi = \Phi$. As required by Observation 1.*

Proof. $x \times \Phi = \mathcal{C}(x \otimes \Phi) = \mathcal{C}(\Phi) = \Phi$. □

Theorem 33. *For all transordinal $x \notin \{0, \Phi\}$: $x \times \infty = \infty$. As required by Observation 3.*

Proof. $x \times \infty = \mathcal{C}(x \otimes \infty) = \mathcal{C}(y)$ for some $y \in V$ so $|V| = |\infty| \geq |y|$ but $(x_i, \alpha_j) \in y$ for all elements $x_i \in x$ and for all atoms α_j so $|y| \geq |\infty|$. Hence $|y| = |\infty|$, therefore $\mathcal{C}(y) = \infty$. □

We have now obtained transordinal arithmetic.

NFU has a type system in which atoms are of type 0 and the empty set is of type 1. It follows from stratified comprehension that, in particular, if all of the elements of a set are of type t then the set is of type $t+1$. Our transordinal arithmetic is adequate for implementing this type system over all non-terminal sets. In addition we have the terminal type ∞ and we have the type Φ which is the type of all objects that are not in V. We consider types further in the Discussion.

When the transordinals are restricted to the transnaturals we obtain transnatural arithmetic.

3 Discussion

The motivation for the present paper comes from computer science where total functions are extremely desirable because, self evidently, there is no need to write code or have hardware to process exceptional cases, because there are none! During the course of the paper we totalise various operations of set theory.

We set out to describe the transnatural numbers, which have a single unordered number, Φ, called *nullity*, and a greatest number, ∞, called *infinity*. The usual set theory ZFC is not suitable because it is inconsistent with a greatest cardinal or ordinal number so we cannot have ∞ as an elementary object. We turn, instead, to the set theory NFU which has a universal set, V, and which has many atoms. We define infinity as the set of all atoms and prove $|\infty| = |V|$ in Theorem 22. Thus infinity is the greatest cardinal. It takes a little more work to establish the ordering of the transnatural numbers.

We find it convenient to work with the larger set of transordinal numbers. Our Axiom 16 asserts that there is a last atom, α_∞. Hence a well ordering of the atoms exists which has a first, α_0, and a last, α_∞, element: $\alpha_0, \alpha_1, \alpha_2, ..., \alpha_\infty$. We define that nullity is the first atom, $\Phi := \alpha_0$; that zero is the empty set, $0 := \{\}$; that each non-terminal, transordinal, $n \neq 0$, is the set of atoms with first element α_1, thus $n := \{\alpha_1, \alpha_2, ..., \alpha_n\}$; and that, as has been said, infinity is the set of all atoms: $\infty = \{\alpha_0, \alpha_1, \alpha_2, ..., \alpha_\infty\}$. We then totalise the subset relation, $\subset$, using Axiom Scheme 11, so that $x \subset y = \Phi$ iff x or y is an atom. This leads to the result that: Φ is unordered, as established in Theorem 21; that all of the non-terminal numbers, including zero, are ordered, by construction; and that ∞ is the greatest, therefore terminal, transordinal, as noted in Observation 23.

There is a subtlety in the construction of ∞. Let us express a fine distinction: there is a last, non-terminal, transordinal, n, such that $n = \{\alpha_1, \alpha_2, ..., \alpha_{\infty-1}\}$ where $\alpha_{\infty-1}$ is the predecessor to α_∞ but this is not the last transordinal, because $n < \infty$, nor is it the immediate predecessor to ∞ because ∞ contains α_0 and n does not. Thus nullity, $\Phi = \alpha_0$, introduces a gap in the sequence of transordinal numbers, reflecting the gap in Figure 1 between the real-number line and infinity.

There is a subtlety, too, in the construction of nullity. The atom α_0 occurs, as a free atom, only in the number nullity, $\Phi = \alpha_0$, and is unordered with respect to every other transordinal. This is reflected by the fact that the point at nullity, Φ, is the only isolated point in Figure 1.

Taking these two subtleties together, we see that the atom α_0 is a marker for the strictly transordinal numbers, Φ and ∞, and is related to the gaps in Figure 1. For this reason we consider that the atom α_0 is the *vacuum* or *empty atom*. We shall bear this thought in mind in case we come across further examples of how nullity behaves as an empty atom with respect to atoms, paralleling the behaviour of the empty set with respect to sets.

We then establish transordinal arithmetic but observe that when the transordinals are restricted to transnaturals, we obtain transnatural arithmetic as elementary operations on elementary objects of a set theory. This is as much as we set out to do. That our use of set theory is consistent is established by construction: we have constructed a model of transnatural arithmetic and that arithmetic is known to be consistent.

We might, however, have had a wider ambition to show that the whole of our extension of the set theory NFU is consistent. Let us now address some remarks to this extended set theory, in the hope of demonstrating that it is sufficiently valuable to justify the labour of proving it consistent.

In Observation 26 we generalise the power function, $|\mathcal{P}(x)| = 2^{|x|}$, for all atoms and sets, x, where $2^{|x|}$ is calculated in transreal arithmetic. Thus the power function is established as a total function with no exceptions. It does, however, stand in contrast to results obtained in other domains of discourse. Specker's theorem, reported in [6], establishes that the cardinality of the powerset is $|\mathcal{P}(V)| < |V|$; we establish $|\mathcal{P}(V)| = |V|$; and Cantor, reported in [6], establishes $|\mathcal{P}(V)| > |V|$. This is a trilemma. It might have been worse. One can conceive of exploiting the quadrachotomy of transreal arithmetic to obtain the unordered result $|\mathcal{P}(V)| = \Phi$! The trilemma is easily resolved. We note that Specker's theorem involves a proof by contradiction where it is shown, in effect, that the type of the universal set is one greater than the cardinality of the universal set, $\mathcal{T}\{|V|\} > |V|$. This is taken to be a contradiction but we have $|V| = \infty$ and $\infty + 1 = \infty$ so there is no contradiction and the proof of Specker's theorem fails in our domain of discourse. The proof of Cantor's theorem involves a non-terminating iteration and is taken to prove that the universal set, V, does not exist in any domain of discourse where Cantor's theorem holds. However, we establish our result statically, without iteration, so Cantor's theorem does not hold and, secondly, we require terminating iterations to get to the terminal value ∞ so, again, Cantor's theorem does not hold in our domain of discourse. Thus we obtain the total and intuitive result, $|\mathcal{P}(x)| = 2^{|x|}$ for all atoms and sets, x.

We have established that ∞ is the greatest ordinal, hence there is no cardinal or ordinal number too large to be expressed in our extension of NFU. In particular there is no cardinal objection to having proper classes in our extension of NFU. We are free to label each component of a proper class with a distinct atom and then use tuples to record the class relations. Thus our extension of NFU has considerable expressivity.

Let us see how far we can get in instantiating Russel's set of all sets that are not elements of themselves. We begin with the universal set, V. This is the set of everything, atoms and sets, that are well formed in our set theory. This is not the set of everything, for example the Not-a-Number objects of floating-point arithmetic are not elements of V. See Observation 8. The universal set is, precisely, the set of all elements that are well formed in our set theory. We now form the set, $S = V \setminus \infty$, which is the set of all sets that are well formed in our set theory. Next we exclude from S the elements which are elements of themselves, leaving the set R as the well formed set of all well formed sets that are not elements of themselves. This might produce some cognitive dissonance in the reader so let us perform the exclusion using the methods of computer science before expressing the exclusion in mathematical notation.

We take the transreal-number line, in Figure 1, as an abstract data structure representing a set, X. We suppose that there is a bijection, obtained by well ordering, between the points on the line and the elements of V. We associate one bit with each point on the line. If the bit is set, the corresponding element of V is in X, otherwise the bit is clear and the corresponding element is not in X. We provide a fragment of a programming language in which variables are declared on first use. We provide a function: *retract(x, X)* which clears the bit corresponding to element x in the variable X. The loop *foreach x in X do <body> endforeach* associates a machine with every element x in X which executes the *<body>*. We allow ourselves conditionals and mathematical notation, including the name, V, of the universal set, and ∞ of the set of atoms. So as to avoid confusion between program assignment and mathematical equality, we take it that $x \to y$ assigns the value of x to the variable y. Let us now construct R.

$V \to R$
foreach x in R do
        if $x \in \infty$ then retract(x, R) endif
endforeach
foreach x in R do
        if $x \in x$ then retract(x, R) endif
endforeach

The first *foreach* loop sets R equal to the set of all sets, which is the set S, above. The second *foreach* loop removes, from R, all of the sets that are elements of themselves. This leaves R as the well formed set of all well formed sets that are not elements of themselves. The set R is not empty; it contains, at least, the empty set because the empty set is not an element of itself. The program uses an unstratified predicate, $x \in x$, but this is harmless.

Now let us carry out the same exclusion, taking up from where S is the well formed set of all well formed sets.

$$R = S \setminus \{x \mid x \in S \ \& \ x \in x\}$$

Here $\{x \mid x \in S \ \& \ x \in x\}$ is unstratified but requiring that x is an element of the previously defined S blocks Russel's paradox. If we want more, we may have it by describing R in category theory, expressed in our set theory, using our atoms for the objects of category theory and our tuples for the arrows of category theory.

An important property of our abstract machine is that we may have distinct machines associated with every point in a data structure. For the machines, there are no inaccessible elements, though there may be elements that are inaccessible to any particular program, such as one written in the usual mathematical notation. Seen as a specification for our machine, our transordinal arithmetic applies to infinity, nullity and to all ordinal numbers; a restriction of it applies to the transordinals.

4 Conclusion

We extend the set theory New Foundations with Urelements (NFU) by making its operations, including the powerset function, total over free atoms and by introducing a last atom to terminate well orderings. This leads to new cardinality results. Trivially we have sufficient cardinality, among the atoms, to describe the proper class of all classes and to describe the category of all categories in category theory. We find that the powerset of the universal set has the same cardinality as the universal set, thereby disposing of the paradoxes of Cantor and Specker. By removing elements from the universal set, we give a non-paradoxical construction of Russel's set of all sets that do not contain themselves. We describe abstract machines which can access every set in the universal set, though, as usual, particular programs, for the machines, may be unable to access some sets. We construct transreal nullity as the empty atom and transreal infinity as the set of all atoms, thereby describing these numbers in an elementary way in our set theory. We describe transordinal arithmetic, a restriction of which delivers transnatural arithmetic.

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Thanks are due to Walter Gomide for reporting that philosophers would find the transreals more accessible if there were an account of the transreals in elementary set theory, for asking how nullity differs from the empty set, and for suggesting that a set theory with urelements might be adequate to the task.

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