Construction of the Transnatural Numbers In Transset Theory

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Abstract

We assert that the universe of all classes is partitioned by the universe of all sets and the universe of all antinomies. The usual set operations are applied to all classes, giving a total transset-theory that contains a set theory and an antinomy theory. Transset theory is just naive set-theory with antinomies so it is pedagogically simple.

We define that a transset that contains only itself is an atom and that all other venues, including the empty transset, are molecules. Transset theory can model all set-theories, including those with atoms, and the whole of category theory, using atoms as category objects and molecules as category relations.

We construct the transnatural numbers such that: the natural numbers are von Neumann ordinals; transnatural infinity, being the greatest ordinal, is built from the universal set; and transnatural nullity, being the only unordered, transnatural number, is built from the universal antinomy.

We extend transnatural arithmetic to transordinal arithmetic and show that the classical paradoxes of set theory are dissolved in transset theory, with counting that is consistent with transordinal arithmetic.

We use the fact that all antinomies have subsets to provide a foundation for paraconsistent logics and to explain how scientific theories can be useful, despite having both internal contradictions in explanations and external contradictions with observations.

1 Introduction

Theories of sets and numbers have had a profound influence on each other during the historical development of mathematics. We continue this tradition by considering how set theory can respond to the recent introduction of the transreal numbers.

After some years of development, the consistency of transreal arithmetic was established by machine proof [7] but, despite this proof, the transreals gave rise to some controversy [24]. Subsequently human proofs established the consistency of transreal [11] and transcomplex [15] numbers by constructing them from the real and complex numbers. In the

of extensionality does not admit sets that are unequal to themselves so we introduce a new class, antinomies, that are defined by the property that they are unequal to themselves. Henceforth we may refer to R as the Russell Antinomy, which we say exists as an antinomy with the property $R \neq R$.

We call classes, transsets and, in a bold move, we assert that the universe of all transsets, U, called the universal transset, is partitioned by the universe of all sets, V, called the universal set, and the universe of all antinomies, W, called the universal antinomy. We then obtain a total transset-theory by applying the usual set operations to all transsets and sorting them into sets and antinomies. This is to say that we adopt naive set-theory with antinomies. The sorting may be syntactic, as in the set theories ZFC and NFU or, more generally, it may be semantic, as in our discussion of the Russell Antinomy, where a proof is given to perform the sort. Our assertion of the partition is bold because it implies that the sorting can always be done, despite, so called, undecidable classes and logical gaps.

Having equipped ourselves with a transset-membership predicate, we provide specialisations of this to set membership and antinomy membership. This gives us three theories: transset theory exists in itself and contains both a set theory and an antinomy theory.

There is a technical difficulty with antinomies that we cannot assign them to a variable using equality. For example we cannot assign the Russell Antinomy, R, to the variable x, by writing $x = R$, because we would then have $x = R \neq R \neq x$, but $x \neq x$ is satisfied by all antinomies, not just R. We handle this by introducing an interchangeability predicate, $x \doteq y$, which we read as: *x is interchangeable with y*. Now we can assign R to x by writing $x \doteq R$. The interchangeability predicate is an equivalence relation over all transsets, whereas equality is an equivalence relation only over sets. We arrange that two objects are interchangeable if they have the same subsets and subantinomies, and are equal if they have the same subsets and no subantinomies. Thus equality is a restricted version of interchangeability. The distinction between interchangeability and equality bears on philosophical discussions of identity and justifies the view that unequal scientific theories can be interchangeable in so far as they describe those predictions and observations that have actually been made – though further predictions or observations might rule out some interchanges. Hence the business of practicing scientist is, at least, to reduce the range of satisfactory interchanges. As antinomies are self-contradictory this implies that practicing scientists have access to a paraconsistent logic that allows effective reasoning over inconsistency.

Russell's Paradox is just one of the classical paradoxes of set theory. We set out to show that the usual paradoxes are dissolved by transset theory. Paradoxes may be allowed, harmlessly, or they may be blocked when comprehension is limited to the universal set and counting is consistent with transordinal arithmetic. One of our manoeuvres is to construct some otherwise paradoxical sets, in a top-down way, by subtracting members from a universe, rather than constructing them, in a bottom-up way, from the empty set. The experience of Computer Science is that there is a synergy between bottom-up and top-down methods so that solutions can be obtained by mixed methods that, in practice, are not accessible to either reasoning method alone. We wonder if this synergy runs deeper: we already know that admitting a universal set produces a set theory that is more expressive than ZFC but does admitting comprehension limited to the universal set admit some true theorems that are blocked by stratification? In other words can a mixture of bottom-up and top-down mathematics deliver more than is logically accessible to either form of mathematics alone?

NFU and some other set theories have urelements or atoms. Atoms are usually admitted as objects that have no members but this means that atoms cannot be distinguished by extensionality. We define that atoms are transsets that have themselves as their only member. Conversely we define that molecules are transsets that do not have themselves as their only member. Hence the empty transset is a molecule! (This is more sensible than it appears at first blush. Atoms are defined by a singular property – they contain only themselves – whereas the empty transset is defined by a dual property: it has no subsets and it has no subantinomies.)

We arrange that we may have arbitrarily many of our atoms by correlating them with arbitrarily many sentence letters. Our atoms have different properties from the usual atoms but we can still use our atoms to model the atoms in any set theory. We can also model the whole of category theory by taking category objects as atoms and category relations as molecules. Category theory is notoriously difficult to teach because of its abstractness. We wonder if a more effective syllabus is to start with naive set theory and then introducing antinomies, before introducing category theory?

Our main business is to construct the transnatural numbers in a total set-theory. We employ the usual construction of the natural numbers, as von Neumann ordinals, so that they are ordered by membership. Stratification blocks this construction in NFU, where ordinals are described by other means. We construct transnatural infinity in a top-down fashion. We set transnatural infinity equal to the universal set, excluding the singleton set whose member is the universal set. Hence transnatural infinity is not a member of itself so it is not less than itself; this blocks the Burali-Forti paradox and ensures that transnatural infinity is equal to itself, which agrees with transreal arithmetic. We show that transnatural infinity has the same cardinality as the universal antinomy and the universal transset. Hence we can count all classes by forming bijections with sets. In other words, sets provide all of the numbers we need to count all transsets. Hence, by construction, transnatural infinity is both the greatest cardinal and the greatest ordinal – regardless of what sets are admitted as transfinite numbers. Thus transnatural infinity is installed as the greatest transnumber, as required by transreal arithmetic. This top-down construction of infinity, from the universal set, is not available in ZFC. Similarly we define that transnatural nullity is the universal antinomy, excluding the singleton antinomy whose member is the universal antinomy. Antinomies are not available as objects in any set theory. Hence nullity is not a member of any set, giving it the required property of being unordered with respect to all transreal numbers.

Some of the classical paradoxes of set theory rely on counting, including counting the members of transfinite sets whose cardinality is less than the universal set. We extend transnatural arithmetic to transordinal arithmetic so that we can dissolve these paradoxes.

In the Discussion we consider the validity of partitioning the class of all classes, the universal transset, into the universal set and the universal antinomy. We consider what the impact might be of reading all of the usual non-existence and diagonalisation proofs as establishing the existence of antinomies. We use the fact that all antinomies have subsets to provide a foundation for paraconsistent logics, that is logics that allow reasoning over inconsistency, and to explain how scientific theories can be useful, despite having both internal contradictions in explanations and external contradictions with observations. We also discuss the pedagogical advantages of transset theory.

We conclude with a statement of the main, original contributions of the paper and leave open the question of whether stratification is too conservative.

2 Transset Theory

We seek to establish a total set-theory whose objects are sets and antinomies. The defining characteristic of an antinomy is that it is unequal to itself, which raises some delicate issues.

We express our theory in first-order logic with equality. As usual we assume that this base language is abstracted from our theory, so that we may use the language as is, but that this language can be encoded within the theory so that nothing is lost to the theory. We assume that our base language has arbitrarily many sentence letters so that we have the freedom to use some of them as atoms. We will see, later, that the cardinality of atoms and hence of sentence letters is very high so the letters cannot be discrete glyphs collated in an alphabetical sequence, instead we think of letters as being drawn from a continuous alphabet of glyphs. (It may help to imagine a candy stick of rock with the glyph "1" at one end that varies continuously into the glyph "2" at the other end. A cross-section of the rock shows a glyph and taking cross-sections at an arbitrarily high cardinality delivers a continuum of glyphs. Taking the cross-sections with the cardinality of the universal set provides one glyph, or letter, for each element of the universal set.)

We begin by introducing interchangeability as an equivalence relation over all transsets.

Axiom 1 (Reflexivity of Interchangeability). *The interchangeability predicate, $x \doteq y$, is read as: x is interchangeable with y. Its base case is defined by the reflexive relation: $x \doteq x$.*

Axiom 2 (Commutativity of Interchangeability). $x \doteq y \iff y \doteq x$.

Axiom 3 (Transitivity of Interchangeability). $x \doteq y \,\&\, y \doteq z \iff x \doteq z$.

The interchangeability operator gets its name from the fact that the eponymous interchangeability of any transsets x and y, in our theory, can be effected by substitutions that actually interchange the sentences x' and y', in our base language, where x' defines x and y' defines y.

We link equality to interchangeability with

Axiom 4 (Equality Implies Interchangeability)**.** $x = y \implies x \doteq y$.

Just as equality generalises to interchangeability so every compound operator, involving equality, generalises to an operator involving interchangeability. Where the compound equality-operator has a glyph, we denote the generalised interchangeability-operator by the same glyph, with a dot set over it. For example, less than or equal, $\leq$, generalises to less than or interchangeable, $\dot{\leq}$, and greater than or equal, $\geq$, generalises to greater than or interchangeable, $\dot{\geq}$.

We introduce transset membership, which we later specialise to set membership and antinomy membership.

Axiom 5 (Transset Membership)**.** *The transset membership predicate, $x \in y$, is read as: x is a transset member of y. It is defined as: $x \in y \doteq \phi_y(x)$. Here x and y are anyranssets and $\phi_y(x)$ is a formula, quantified over arbitrarily many terms, that defines the transset, y, by the unification $y = \phi_y(x)$. The formula $\phi_y(x)$ may be Curried or may be applied to a tuple, x, so that the formula may apply to zero, one, or many arguments.*

As a convenient shorthand we say that x is a member of y whenever x is a transset member of y. We then speak explicitly of set-membership and antinomy-membership as specialisations of the more general (transset) membership. We may treat other transset operations similarly.

The method of specifying a potentially infinite collection, y, by unification with a formula, $\phi_y(x)$, that has a parameter, x, is known, in Computer Science, as lazy evaluation. Universal quantification is implied by an uninstantiated parameter, the variable x, and existential quantification is implied by an instantiated parameter, $x = k$, for any constant k. We may revert to the usual universal, $\forall$, and existential quantifiers, $\exists$, wherever the programmatic method might cause confusion.

We define that atoms are transsets whose only member is equally themselves. We define molecules conversely.

Definition 6 (Atoms)**.** *We say that α_i is an atom if and only if its only member is equal to α_i. Thus: $y \in \alpha_i \iff y = \alpha_i$.*

Each atom, α_i, in our theory, is an identical sentence letter, α_i, in our base language.

Definition 7 (Molecules)**.** *We say that μ_i is a molecule if and only if μ_i is not an atom.*

We now use transset membership to define an analogue of the usual set-builder notation and to name the universal transset, U, the universal set, V, and the universal antinomy, W. Henceforth the latin braces, $\{$ and $\}$, bracket transsets. We later partition transsets into sets and antinomies.

Definition 8 (Comprehension Limited to the Universal Transset)**.**
$\{x \mid \phi_y(x)\} \doteq \phi_y(x)$.

Limiting comprehension to the universal transset is no limitation at all! This form of comprehension is usually called universal comprehension but we prefer our, more systematic, name. Later we will discuss comprehension limited to the universal set and comprehension limited to the universal antinomy.

Naive set-theory employs universal comprehension. Transset theory differs from naive set-theory only in that it provides the class of antinomies – though this leads to the introduction of the transordinals, in place of the ordinals of naive set-theory.

We define a narrative transset whose main purpose is to allow transsets to be specified in natural language. It relies on the specification being interpreted by a competent speaker of the language!

Definition 9 (Narrative Transset). $\{x \mid s\} \doteq \{x \mid \Phi_s(x)\}$. *Here s is a sentence in any language, such as our base language, our theory or a natural language such as English or Portuguese. This sentence is unified with a transset, $\Phi_s(x)$, in our base language.*

We now define some more formal notations. Notice that the empty set and the universal set are defined by equality, whereas the other transsets are defined by interchangeability. This anticipates our partition of the universal transset into the universal set and the universal antinomy. The partition could be established, using more formal language, before these shorthand notations are introduced but we prefer our simpler presentation.

Definition 10 (Empty Set). $\{\} = \{x \mid F\}$.

Definition 11 (Enumerated Transset). $\{x_1, x_2, ..., x_i\} \doteq \{x \mid (x \doteq x_1) \vee (x \doteq x_2) \ ... \vee (x \doteq x_i)\}$.

Definition 12 (Universal Transset). *The universal transset, U, is given by $U \doteq \{x \mid T\}$.*

Definition 13 (Universal Set). *The universal set, V, is given by $V = \{x \mid x = x\}$.*

Definition 14 (Universal Antinomy). *The universal antinomy, W, is given by $W \doteq \{x \mid x \neq x\}$.*

We define set extensionality as usual, though totallity means we can drop the usual guarding clause that the arguments to extensionality are sets. For us if two objects are equal then they are sets so we adopt

Axiom 15 (Set Extensionality). $x = y \implies (z \in x \implies z \in y)$.

We define analogues of the usual operations of set theory. Our definitions are lexically identical to the usual definitions, except that we allow arguments to be any transsets and do not require them to be sets. We may generalise any set theory similarly.

Definition 16 (Transset Complement). *The transset complement, x^c, of any transset, x, is given by: $x^c \doteq \{y \mid y \notin x\}$.*

Definition 17 (Transset Union). *The transset union, $x \cup y$, of any transsets x and y, is given by $x \cup y \doteq \{z \mid z \in x \vee z \in y\}$.*

All of the unary and binary connectives of the usual set-theories can be constructed from transset complement and transset union so these two operations give a second method for generalising set theories.

We now assert the partitioning of the universal transset into the universal set and the universal antinomy. This axiom asserts that either an object is equal to itself or else it is not equal to itself. It is a transset version, in our theory, of the Axiom of the Excluded Middle, in our base language.

Axiom 18 (Universal Transset Partitioned by the Universal Set and the Universal Antinomy)**.** *$U \doteq V \cup W$, with $V \cap W = \{\}$.*

We accept the usual definition of set cardinality but allow it to apply to any transsets.

Definition 19 (Transset Cardinality)**.** *Given any transset, x, a transset cardinal, y, of x, is given by $|x|$, such that $|x| \doteq y$ if and only if there is a bijection from the members of x to the members of y.*

We give a top-down definition of the powertransset.

Definition 20 (Powertransset)**.** *The powertransset, $\mathcal{P}(x)$, of any transset, x, is the transset of all subtranssets of x, given by $\mathcal{P}(x) \doteq \{z \mid z \doteq x \setminus y\}$. Here y ranges over all members of the universal transset.*

Theorem 21 (Powertransset of Some Universes)**.** *$\mathcal{P}(U) = U$, $\mathcal{P}(V) = V$, $\mathcal{P}(W) \doteq W$.*

Proof. Let the y^c be arbitrary members of U, then $\mathcal{P}(U) \doteq \{z \mid z \doteq U \setminus y\} \doteq \{z \mid z \doteq y^c\}$ so $y^c \in \mathcal{P}(U)$. Now $y^c \in U \iff y^c \in \mathcal{P}(U)$, therefore $P(U) \doteq U$. Similarly $\mathcal{P}(V) = V$, $\mathcal{P}(W) \doteq W$. □

We wish to dissolve the classical paradoxes of set theory but some of these involve counting so we first arm ourselves with the transnatural and transordinal numbers. We take this opportunity to prove some theorems concerning the cardinality of certain transsets.

3 Construction of the Transnaturals

We define transnatural nullity, Φ, and infinity, ∞, as follows. Notice that nullity is defined by interchangeability and infinity is defined, more tightly, by equality.

Definition 22 (Nullity)**.** $\Phi \doteq W \setminus \{W\}$.

Definition 23 (Infinity)**.** $\infty = V \setminus \{V\}$.

We accept the usual definition of the von Neumann ordinals as sets but with two changes. Firstly where von Neumann would write "if and only if" we write "if" so that we have the freedom, if needed, to add ∞ as an ordinal. We need this freedom if V is not strictly well ordered but if it can be proved, as a theorem, that V is strictly well ordered then we can revert to von Neumann's use of "if and only if." Secondly we take well ordering with respect to transset membership, not the stricter set membership. This gives us the necessary freedom to obtain the ordering of Φ. As usual we say that a finite ordinal is a natural number.

Definition 24 (Von Neumann Ordinal)**.** *A set, S, is an ordinal if S is strictly well ordered, with respect to transset membership, and every member of S is also a subset of S.*

Axiom 25 (Infinity is the Ordinal Type of the Von Neumann Ordinals)**.** *Transnatural infinity, ∞, is the ordinal type of the von Neumann Ordinals, which is to say that it is the least ordinal that is greater than all von Neumann Ordinals.*

Theorem 26 (Transnatural Ordering Holds)**.** *The usual ordering of the transnatural numbers holds for the finite, von Neumann ordinals. Furthermore transnatural infinity is equal to itself and is greater than every von Neumann ordinal. Finally nullity is equal to itself and is unordered with respect to every other transnatural number.*

Proof. The finite, von Neumann ordinals, or natural numbers, are ordered as usual. Every von Neumann ordinal, including every natural number, is less than ∞ because the von Neumann ordinals are sets and ∞ is the universal set, excluding one set which is not a von Neumann ordinal. Nullity, Φ, is unordered with respect to every von Neumann ordinal and infinity, ∞, because these numbers are sets and Φ is an antinomy. $\Phi \doteq \Phi$ because $\Phi \doteq W \setminus \{W\} \doteq \Phi$ and interchangeability, $\doteq$, is an equivalence relation. Similarly $\infty = \infty$ or, more loosely, $\infty \doteq \infty$. □

Henceforth we shall write all 'equations' of transnatural arithmetic as interchanges. That is we shall use interchange, $\doteq$, wherever transnatural arithmetic would ordinarily write equality, $=$. In other circumstances it might be convenient to write equality for interchangeability in an abuse of notation but we refrain from this abuse here.

We now establish that the universal transset, U, the universal set, V, and the universal antinomy, W, all have the same cardinality. This justifies taking only sets as ordinals because we can obtain a bijection between any transsets and sets.

Definition 27 (Kuratowski Pair)**.** *The Kuratowski Pair, $\langle x, y\rangle$, is ordered so that x is the first member of the pair and y is the second member of the pair. It is given by $\langle x, y\rangle \doteq \{\{x\}, \{x, y\}\}$.*

Axiom 28 (Atomic Transset Names)**.** *Every distinct transset, T_i, is named by a distinct atom, α_j, with T_i bijective to $\langle \alpha_j, T_i\rangle$.*

We note, in passing, that this Axiom 28 forces the universal transset to have an extremely high cardinality.

Definition 29 (Universal Set of Atoms)**.** *The universal set of atoms, A, is given by $A = \{\alpha \mid \alpha \text{ is an atom}\}$.*

Theorem 30 (Cardinality of Some Universes)**.** $|U| \doteq |V| \doteq |W| \doteq |A|$.

Proof. By the Axiom of Atomic Transset Names, $|U| \doteq |A|$. V, W are transsets so $|V|, |W| \dot{\leq} |U|$. For every atom, α_i, it is the case that $\{\alpha_i\} \in V$ and $\langle \alpha_i, x \neq x\rangle \in W$ so $|V|, |W| \dot{\geq} |U|$. Therefore $|V|, |W| \doteq |U|$. □

It follows, from Theorem 30, that there are pairwise bijections between U, V, W, A. Hence it is sufficient to take the ordinals and canonical cardinals as sets.

Theorem 31 (Transnatural Infinity is the Greatest Cardinal and Ordinal)**.**

Proof. U is bijective with V so no cardinal or ordinal, however defined, is greater than V. In Definition 23, transnatural infinity, ∞, is V, excluding one set so, by the Hotel Paradox, $|V| = \infty$. Therefore ∞ is the greatest cardinal. By Axiom 25, ∞ is an ordinal type and is therefore ordinal. Hence ∞ is the greatest ordinal. $\square$

We have now established the ordering of all of the transordinal numbers, which includes the ordering of the transnatural numbers. Transnatural arithmetic may be developed by extending, in transset theory, any of the usual, set-theoretical developments of the natural numbers. For example we may generalise the existing developments of the transreal [11] and transcomplex numbers [15] given in ZFC. Similarly a large part of the usual mathematics generalises to a total form in transset theory.

Various ordinal arithmetics can be defined with exponentiation and non-associative addition and multiplication. These can be extended to total, transordinal arithmetics as follows. Firstly assert that, as usual, Φ is absorptive over addition, subtraction, multiplication, division, and exponentiation. Secondly assert that $0 \times \infty \doteq \Phi$. Thirdly assert that for a given ordinal, Ω, it is the case that the left and right subtractions have $\Omega - \Omega \doteq \Phi$ and, similarly, left and right divisions have $\Omega/\Omega \doteq \Phi$. Fourthly assert that, in all cases not already treated, left subtraction is the inverse of left addition, left division is the inverse of left multiplication, right subtraction is the inverse of right addition, and right division is the inverse of right multiplication. Fifthly assert $\Omega^\infty = \infty$. Sixthly, in an extension to the usual ordinal arithmetics, assert that the logarithm is the inverse of the exponential.

4 Dissolving Paradoxes

We set out to show that transset theory is immune to the paradoxes of naive set-theory and to show how this immunity arrises. This latter is an exercise in exploiting the additional structure of transset theory to reason coherently over antinomies: it is an example of paraconsistent reasoning.

4.1 Russell

In the Introduction we showed that the Russell Antinomy is not equal to itself. Here we want more. We seek a total characterisation of all of the members of the universal transset with respect to membership of the Russell Antinomy. That is we seek to characterise all of the members, non-members and gaps of the Russell Antinomy.

We begin by defining the Russell Transset, R_U, using interchangeability, not equality. Later on we shall obtain the Russell Antinomy, R_W, and the Russell Set, R_V.

Definition 32 (Russell Transset). *The Russell Transset, R_U, is given by $R_U \doteq \{x \mid x \notin x\}$.*

Now the Russell Transset, R_U, exists as a sentence in our base language. The transset has the non-paradoxical property $R_U \doteq R_U$. Thus the *existence* of the Russell Transset is non-paradoxical but we may still construct paradoxical *membership* of the Russell Transset. In order to facilitate discussion we add subscripts to the definition: $R_U \doteq \{x_1 \mid x_2 \notin x_3\}$, where $x_1 \doteq x_2 \doteq x_3$. Suppose $x_1 \doteq R_U$, then we are supposing $R_U \in R_U$, whence $R_U \notin R_U$, by $x_2 \notin x_3$, with $x_2 \doteq x_3 \doteq R_U$. That is: $R_U \in R_U \implies R_U \notin R_U$. Conversely suppose that $x_2 \doteq x_3 \doteq R_U$, then we are supposing $R_U \notin R_U$, whence $R_U \in R_U$, by $x_1 \doteq R_U$. That is: $R_U \notin R_U \implies R_U \in R_U$. Combining these two implications we have the usual bi-implication: $R_U \in R_U \iff R_U \notin R_U$.

What are we to make of the membership paradox: $R_U \in R_U \iff R_U \notin R_U$? We cannot assert that R_U does not exist because it does exist, in our terms, as a transset. We cannot dissolve the paradox by asserting one of $R_U \in R_U$ or else $R_U \notin R_U$ because, in either case, the paradox would then be a contradiction. The Axiom of the Excluded Middle, in our base language, prevents us from asserting the dialethia: $R_U \in R_U \,\&\, R_U \notin R_U$. What remains? A gap remains: $R_U \in R_U$ has no degree of truth or falsehood. It would usually be said that $R_U \in R_U$ is undecidable or incomputable but we prefer to recognise this situation as a gap.

Theorem 33 (The Russell Transset is an Antinomy)**.**

Proof. The Axiom of Extensionality gives: $x = y \implies (z \in x \implies z \in y)$. Taking $x = y = z = R_U$ gives $R_U = R_U \implies (R_U \in R_U \implies R_U \in R_U)$ but we have $R_U \in R_U \iff R_U \notin R_U$. Therefore $R_U \neq R_U$, which is to say that R_U is an antinomy: $R_U \in W$. Therefore the Russell Transset is interchangeable with the Russell Antinomy: $R_U \doteq R_W$. $\square$

Antinomies may have subsets and subantinomies. We now show that the Russell Antinomy and its complement have infinitely many subsets and infinitely many subantinomies – putting their existence beyond doubt!

Theorem 34 (The Russell Antinomy has Infinitely Many Subsets and Infinitely Many Subantinomies)**.**

Proof. For every atom, α_i: firstly $\langle \alpha_i, \{\}\rangle = \langle \alpha_i, \{\}\rangle$ and $\langle \alpha_i, \{\}\rangle \notin \langle \alpha_i, \{\}\rangle$ so $|R_W| \doteq |R_W \cap V| \doteq \infty$; secondly $\langle \alpha_i, x \neq x\rangle \neq \langle \alpha_i, x \neq x\rangle$, because, by hypothesis, $x \neq x$, and it is the case that $\langle \alpha_i, x \neq x\rangle \notin \langle \alpha_i, x \neq x\rangle$, so $|R_W \cap W| \doteq \infty$. $\square$

Theorem 35 (The Complement of the Russell Antinomy has Infinitely Many Subsets and Infinitely Many Subantinomies)**.**

Proof. Firstly, by the Axiom of Atomic Transset Names, every atom is a member of itself so every atom is a member of the complement, R_W^{c}, of the Russell Antinomy. Therefore $|R_W^{\mathrm{c}}| \doteq |R_W^{\mathrm{c}} \cap V| \doteq \infty$. Secondly for each antinomy, w, it is the case that $W \setminus w \in W \setminus w$ so $|R_W^{\mathrm{c}} \cap W| \doteq \infty$. $\square$

Thus we see that the Russell Transset is not a set but is an antinomy. This justifies calling the Russell Transset the Russell Antinomy. There is nothing paradoxical about the existence of the Russell Antinomy but the Russell Antinomy's membership of itself is paradoxical in the sense that it is a logical gap. One could follow the usual practice of regarding gaps as undecidable or incomputable values or one might prefer to represent gaps explicitly. Whatever one's preferences are, one might consider encoding undecidable/incomputable/gap values in a three-valued, trans-Boolean logic [11] with the values False ($-\infty$), True (∞), and Gap (Φ).

At this stage we have dissolved the Russell Paradox and have shown how transset theory is immune to this paradox – but totality demands more. We disposed of the paradox by introducing the transset, R_U, and identifying R_U as an antinomy, R_W. This leaves open the question of whether, in our terms, a Russell Set, R_V, exists and what its properties are.

Definition 36 (Russell Set)**.** *The Russell Set, R_V, is given by $R_V \doteq R_W \cap V$.*

Theorem 37 (The Russell Antinomy is not a Member of the Russell Set)**.**

Proof. By definition R_V is a set so, by extensionality, all of its members are sets and none of its members are antinomies, but R_W is an antinomy, therefore $R_W \notin R_V$. □

Thus the Russell Set is the set of all sets that do not contain themselves. This statement is non-paradoxical by virtue of excluding all antinomies and the Russell Antinomy in particular. In our terms the sentence, "The set of all sets that do not contain themselves." has comprehension limited to our universal set and is therefore non-paradoxical; whereas, in Russell's terms, the sentence has unlimited comprehension and is therefore paradoxical.

Taking all of this together, we may summarise the salient issues by saying: the Russell Antinomy is a member of the Universal Antinomy, the Russell Antinomy's membership of itself is a gap, the Russell Set is a member of the Russell Antinomy, and neither the Russell Antinomy nor the Russell Set is a member of the Russell Set.

We may now define a non-paradoxical set-theory by defining set membership as a specialisation of transset membership, whence totality demands that we also define antinomy membership.

Definition 38 (Set Membership)**.** *The set membership predicate, $x \underset{\mathrm{V}}{\in} y$, is read as: x is a set member of y. It is defined as: $x \underset{\mathrm{V}}{\in} y = x \in V \,\&\, x \in y = (x = x) \,\&\, x \in y$.*

Definition 39 (Antinomy Membership)**.** *The antinomy membership predicate, $x \underset{\mathrm{W}}{\in} y$, is read as: x is an antinomy member of y. It is defined as: $x \underset{\mathrm{W}}{\in} y = x \in W \,\&\, x \in y = (x \neq x) \,\&\, x \in y$.*

Now we have three theories: a transset theory, a set theory and an antinomy theory. The transset theory contains both the set theory and the antinomy theory. We conjecture that each of these theories dissolve all of the usual paradoxes of set theory, some examples of which are illustrated next.

4.2 Cantor – Diagonalisation

In classical logic a contradiction explodes, making all theorems true. Paraconsistent logics, by contrast, control the effects of a contradiction so that only a limited number of theorems are true; such logics support coherent reasoning over contradictions, which is what we set out to achieve in transset theory.

In the usual mathematics, contradictions are closely associated with existence and non-existence. When an hypothesis leads to a contradiction, the contradiction is taken to indicate the non-existence of whatever was hypothesised. For example the hypothesis of the Russell Set usually leads to a contradiction so the Russell Set is said not to exist. We have just used transset theory to overturn this conclusion but we do not want to overturn all of the conclusions of the usual mathematics! In the usual mathematics an hypothesis can be negated, a contradiction then proves the existence of whatever was hypothesised. We can retain any of the usual results by using contradictions to demonstrate the existence and non-existence of sets, while creating harmless antinomies. As a useful example of this we now show, firstly, that the transnatural numbers, $\mathbb{N}^T = \mathbb{N} \cup \{-\infty, \infty, \Phi\}$, have the same cardinality as the natural numbers, $\mathbb{N}$, secondly the transreals, $\mathbb{R}^T = \mathbb{R} \cup \{-\infty, \infty, \Phi\}$, have the same cardinality as the reals, $\mathbb{R}$, and, thirdly, the cardinality of the transreals is greater than the cardinality of the transnaturals. This involves a critical use of Cantor's diagonalisation argument, with careful handling of contradiction and antinomy. When we have worked through this example we will be in a position to state what distinguishes contradictions from antinomies.

We begin by establishing the cardinality of the transreals and the transnaturals.

Theorem 40 (The Transnatural Numbers Have the Same Cardinality as the Natural Numbers)**.**

Proof. We enumerate the transnatural numbers: $0 \to \Phi, 1 \to -\infty, 2 \to \infty$, then $n \to n-3$. □

Theorem 41 (The Transreal Numbers Have the Same Cardinality as the Real Numbers)**.**

Proof. We establish a bijection between the transreals and the reals as follows. Firstly the transnatural numbers are taken bijectively with the natural numbers as in the above Theorem 40. Secondly the non-transnatural transreals are taken bijectively with the non-natural reals by identity. □

In order to show that $|\mathbb{R}^T| > |\mathbb{N}^T|$, it is sufficient to show that $|\mathbb{R}| > |\mathbb{N}|$ but we give a slightly wider consideration of the decimal form of the transreals. Our treatment of Cantor's diagonalisation argument is based on the presentation in [26].

As usual the transintegers are $\mathbb{Z}^T = \mathbb{Z} \cup \{-\infty, \infty, \Phi\}$. Now let a_0 be a transinteger, $a_0 \in \mathbb{Z}^T$, and let $a_1, a_2, ..., a_n$ be an indefinitely long sequence of digits, indexed over the natural numbers, $n > 0$, such that $a_n \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. We may take the digits as von Neumann ordinals and the sequence of digits as a nested Kuratowski Pair. Then the series

$$x \doteq a_0 + \sum_{n \in \mathbb{N} \setminus \{0\}} \frac{a_n}{10^n}$$

converges. As usual we require that a proper, decimal expansion does not have an indefinitely large number of terms $a_n = 9$. Even so, when a_0 is a strict transinteger, $a_0 \in \{-\infty, \infty, \Phi\}$, then $x \doteq a_0$, regardless of the sum indexed over the natural numbers $n > 0$, so the expansion is not unique. Hence we define that the proper, decimal expansion of the strictly transreal numbers is given by the transinteger, a_0, without the sum. If the sum were taken exactly to transnatural infinity then the term $a_\infty/(10^\infty) = a_\infty/\infty = 0$, regardless of the digit a_∞, so the expansion would not be unique. Finally if the sum were taken to transnatural nullity, then the term $a_\Phi/(10^\Phi) = a_\Phi/\Phi = \Phi$, regardless of a_0, a_Φ and any a_n, with $n > 0$. Hence we take the sum over the naturals $n > 0$. Taking all of this together, the proper, decimal expansion of a transreal number is either a strict transinteger or else it is the usual, proper, decimal expansion of a real number.

We may now follow through, very briefly, with Cantor's proof that $|\mathbb{N}| \neq |\mathbb{R}|$. Cantor enumerates the decimal expansions so that the n'th expansion is given as: $a^{(n)} = a_0^{(n)}.a_1^{(n)}a_2^{(n)}...a_n^{(n)}...a_{n+i}^{(n+i)}$. Cantor then constructs a proper, decimal expansion, $b = b_0.b_1b_2...b_n...b_{n+i}$, so that diagonal digits $b_j \neq a_j^{(j)}$. If $|\mathbb{N}| = |\mathbb{R}|$ then it must be possible to insert b into the enumeration, at some position n, so that $b = a^{(n)}$ but then the digit $b_n \neq a_n^{(n)}$. For Cantor and for us, this is a contradiction: by construction $b \in \mathbb{R}$ but the expansion of b cannot have a set, which is the digit b_n, such that $b_n = a_n^{(n)}$. Therefore $b \notin \mathbb{N}$. For us, alone, $(b_n = a_n^{(n)})\,\&(b_n \neq a_n^{(n)})$ is an antinomy. We can satisfy the diagonalisation with an antinomy but no antinomy is in $\mathbb{N}$ so Cantor's result holds, despite the antinomy.

Now we can see what the difference between an antinomy and a contradiction is. A contradiction, which is a sentence in our base language, exists as an antinomy, in out theory, and does not exist as a set, in our theory. Put another way: in our theory an antinomy, which exists, is the specification of a contradictory set, which does not exist.

4.3 Cantor – Powerset

Cantor proves that for any set, A, its powerset, $\mathcal{P}(A)$, has greater cardinality: $|\mathcal{P}(A)| > |A|$. But we have $V = \mathcal{P}(V)$ so $|V| = |\mathcal{P}(V)|$. This is a dilemma. In order to resolve this dilemma we need to find how Cantor's powerset proof fails in our set theory.

Our treatment of Cantor's powerset proof is based on the presentation in [26]. Cantor starts by showing that $|A| \leq |\mathcal{P}(A)|$ because, for every $x \in A$, there is a singleton set $\{x\} \in \mathcal{P}(A)$. We agree with this.

Cantor then shows that $|A| \neq |\mathcal{P}(A)|$. His method is to assume, to the contrary, that $|A| = |\mathcal{P}(A)|$, and then show a contradiction. We will resolve the dilemma by showing that Cantor makes, in our terms, a category error, which undercuts his contradiction.

Cantor assumes there is a bijection, $f : A \to \mathcal{P}(A)$, such that $f(x) = B_x$. Then B_x is a subset of A because every member of $\mathcal{P}(A)$ is a subset of A. Consider any element, $x \in A$, and its image, B_x. Then either $x \in B_x$ or $x \notin B_x$. The set of all x with the latter property is denoted by C, so that $C = \{x \mid (x \in A) \,\&(x \notin B_x)\}$. This is false, in our terms, because C is an antinomy, not a set, so the proof by contradiction fails at this point. The fact that C is an antinomy is established in the last line of Cantor's proof so we follow it through.

Cantor states that C is a subset of A and, since f is surjective, C must be an image – that is, there exists an $a \in A$ such that $C = B_a$. Now $a \in C \iff a \notin B_a$, since C is the set of all x satisfying $x \notin B_x$. And, as just noted, $B_a = C$, whence $a \notin B_a \iff a \notin C$. Therefore $a \in C \iff a \notin B_a$ and $a \notin B_a \iff a \notin C$, whence $a \in C \iff a \notin C$. This is a contradiction. Cantor takes it as given that C is a set so, for him, this contradiction shows that $|A| \neq |\mathcal{P}(A)|$ but we take it to show that C is an antinomy, whence $C \notin A, \mathcal{P}(A)$ and the proof that $|A| \neq |\mathcal{P}(A)|$ fails. In our terms, Cantor mistakes an antinomy for a set, which destroys his proof that $|A| < |\mathcal{P}(A)|$.

At this point we have established that Cantor's powerset proof does not hold in our set theory. Totality demands more. Cantor considered the antinomy, C, of all $x \in A$, such that $x \notin B_x$, but we must also consider the set, D, of all $x \in A$ such that $x \in B_x$. That is, we must consider the set $D = \{x \mid (x \in A) \,\&(x \in B_x)\}$. Now D obtains when $f(x) = \{x\}$. Note that f maps V to V, so $A = \mathcal{P}(A)$, at least when $A = V$. Hence Cantor's powerset proof establishes $|V| = |\mathcal{P}(V)|$, as required by transset theory.

Of course many transsets, including many sets, have $|\mathcal{P}(A)| = 2^{|A|} > |A|$ but the universal set, V, has $|\mathcal{P}(V)| = 2^{|V|} = 2^{\infty} = \infty = |V|$, in agreement with transreal arithmetic. Similarly $|\mathcal{P}(U)| = |\mathcal{P}(V)| = |\mathcal{P}(W)| = |U| = |V| = |W| = \infty$.

Thus we have resolved the dilemma in transset theory but we have identified a potentially serious issue for the usual set theories. If a set theory has no objects which are antinomies then it may be reasonable to assume that C is a set, in which case Cantor's powerset argument proves, as usual, that $|S| < |\mathcal{P}(S)|$, for some small sets S. But this ignores the D set which established the existence of some large sets T, such that $|T| = |\mathcal{P}(T)|$. We find it astonishing that this has been missed!

The consequences of this apparent omission might be harmless, set theories might be consistent and unnecessarily limited to small sets, S, but admitting large sets, T, as a fuller treatment of Cantor's powerset argument appears to do, may establish that the usual set theories are inconsistent.

Large sets are further allowed by our dissolution of the next two paradoxes.

4.4 Burali-Forti

The Burali-Forti paradox, reported in [22], establishes that there is no greatest ordinal. We have a greatest transordinal, ∞. This is a dilemma which must be resolved.

The Burali-Forti paradox establishes that the set of all ordinals is not the greatest ordinal. We agree with this. Our greatest transordinal, ∞, is not the set of all transordinals because it does not contain itself. Thus we avoid the Burali-Forti paradox by constructing infinity as $\infty = V \setminus \{V\}$ in Definition 23. Nullity is given, similarly, in Definition 22, so it avoids analogous paradoxes.

4.5 Specker

Specker's Theorem, reported in [22], proves that $|\mathcal{P}(V)| < |V|$. We have $|\mathcal{P}(V)| = |V|$. This is a dilemma which must be resolved.

Specker's theorem involves a proof by contradiction where it is shown, in effect, that the type of the universal set is one greater than the cardinality of the universal set: $\mathcal{T}\{|V|\} = |V| + 1 > |V|$. This is taken to be a contradiction but we have $|V| = \infty$ and $\infty + 1 = \infty$ so there is no contradiction and the proof of Specker's theorem fails in our transset theory.

5 Discussion

The ambition of transmathematics is to arrange that every system is a total system, for example that all functions are total functions and all operators are closed. It might be thought that this ambition is bound to fail on the simplest questions of arithmetic. For example: which number is less-than and greater-than zero? There is no such number so the question cannot be answered; it is a partial question. We can totalise it by asking a more general question: what is the set of numbers that are less-than and greater-than zero? This set is the empty set so we have an answer. But we cannot use set theory to answer every question. We cannot answer the question: what is the set of all sets that are not members of themselves? For that we need a more general theory, such as category theory or our transset theory.

Our motivation to find total, mathematical theories is a practical one. We want to use such theories in the design and programming of digital computers that have no logical error states, so that any program which compiles cannot suffer an abnormal end (crash) for any but physcial reasons. We want to apply total, mathematical theories and programs in physics so that we can analyse all physical systems, including singular ones. And we want mathematics to be expressed in computer-proof systems so that: consistency can be checked, in detail; revised systems can be checked quickly; and mathematics can be applied by computer users, not just by mathematicians.

We also have pedagogic ambitions. We want both traditional and computerised mathematics to be easy to teach. At present naive set-theory is taught in primary and secondary schools. This theory is overturned in tertiary education where the paradoxes of set theory are presented and more restricted set theories such as ZFC and NFU are taught. Overturning theories is a waste of earlier learning but we can avoid it with transset theory. Transset theory is naive set-theory with antinomies so if we add antinomies and classes as objects, nothing need be withdrawn.

This brings us to a terminological issue. It is the practice in transmathematics to add the prefix, *trans* to the ordinary descriptions of mathematics. This is meant as a service to the reader, to warn that the usual objects of mathematics are being extended. But such terminological variety might prove an obstacle to the learner. Why, for example, are there *trans*sets but no *trams*antinomies? A transset is nothing other than a class so, if transmathematics is to be taught in primary and secondary schools, more widely than has already been done, then perhaps one should prefer to say that one is dealing with class theory, set theory and antinomy theory.

In tertiary education it is notoriously difficult to teach category theory – the subject is so abstract that students find it difficult to obtain a foot hold. We wonder if transset theory can be used to ease the passage to category theory by introducing categories as classes that have transset atoms as category objects and transset molecules as category relations?

We are neutral on points of vocabulary and leave it to professional educators, among whom we number, to decide if and when it is appropriate to teach transmathematics.

Our decision to assert that sets and antinomies partition all classes or transsets is bold. The partition requires that every transset can be sorted into exactly one of a set or an antinomy. This sorting is total, which is what we want, but it excludes the possibility of there being absolutely undecidable cases. We can still have undecidability in limited systems, such as the undecidability of first order logic, Turing undecidability, and so on; but we cannot have undecidability in the universal transset because every function appears there. Hence everything is decidable in top-down mathematics that starts from the universal transset, even though some things may be inaccessible to bottom-up mathematics that starts from the empty set. We wonder if it can be proved that top-down mathematics has some content? In other words, can it be proved that there are top-down theorems that are inaccessible to bottom-up methods? This would amount to showing that stratification is too tight. This question is of both mathematical and philosophical interest. If stratification is too tight then a truly creative system, such as a human, computer or robot, must have access to untyped reasoning. In the nomenclature of debate in Artificial Intelligence, it would require that the most general intelligences are *scruffy*, not *neat*.

We introduced *interchangeability* as an equivalence class over all transsets, extending the notion of *equality* which is defined only for sets. This gives us the technical ability to identify antinomies and prove facts about them. We may now read our earlier results in Computer Science in this new, mathematical, light: the Not-a-Number objects of floating-point arithmetic [1][2] are not members of the universal set because $NaN_i \neq NaN_j$ for all i, j – including $i = j$! This complicates the writing of numerical programs, as discussed in [6], with corrections in [17]. We suggest that interchangeability is a coherent operator that could be used with $NaNs$ as a replacement for the IEEE, floating-point operator, *unordered*, which we proved is incoherent [6]. Alternatively, as shown in [6], if $NaNs$

are replaced by transreal nullity then the usual relational operators – less-than, equal-to and greater-than – are sufficient. Hence there is no need for an unordered or interchangeability operator in trans-floating-point arithmetic.

Having an interchangeability predicate, that is distinct from equality, bears on philosophical issues of identity. It is usual to take identity over equality but many of the philosophical paradoxes of identity can be read as attempts to assert identity over physical interchanges that alter some parts of an object, thereby calling into question its identity. We wonder if the separate notions of interchangeability and equality can be used to clarify these philosophical questions?

We introduced atoms so that they can be distinguished by extensionality. This is not original but it is significant. It obliges us to have a continuum of sentence letters in our base language so that we have enough atoms to establish the cardinality results we want. We may still have all of the usual results of mathematics that are built up from a finite vocabulary but we can have more. It might be thought that practical computing systems are necessarily finite but this is not, quite, certain. If the many universes hypothesis is true then quantal computers might operate in an infinitude of universes, before returning a result to our universe. Regardless of whether or not such computers can be constructed, we want to adopt a mathematics that does not block the possibility of constructing such powerful machines – so we have a practical reason for adopting continuous sentence letters, as well as exercising our mathematical freedom to have them.

There are mathematical consequences to adopting our atoms. We have the freedom to encode anything whatsoever – any mathematics and any physical atoms or particles – as our atoms. Hence we may trivially model any given mathematics in transset theory, including the whole of category theory. In particular we may choose our atoms to be the atoms of other set theories, we may choose our atoms to be category theory objects and we may choose our molecules to be category theory relations. There is no limit to what could be encoded in transset theory but, of course, our choices have mathematical consequences. Specker's theorem proves that in set theories like NFU, the powerset of the universal set has lower cardinality than the universal set! This arises because the powerset contains only sets and no atoms of the sort that are not distinguishable by extensionality, let us call them urelements to avoid confusion with our atoms. Thus Specker proves that there are more urelements than there are sets. We adopt the Axiom of Atomic Transset Names so that, in particular, there are as many atoms as there are sets. We do this so that the cardinalities of transset theory support transnatural and transordinal arithmetic. Having adopted a transarithmetic, Specker's Theorem and the Burali-Forti Paradox no longer hold so we are free to have different cardinalities than the usual set theories. As one would expect, the properties of urelements and atoms depend on the other axioms of the theories they are embedded in. In particular the choice of arithmetic has a profound influence on cardinality.

We chose to build transset theory using the base language of first-order logic. This is an eminently practical choice and is sufficient to our needs. It does not preclude us from porting transset theory to different languages, such as a multivalued logic, category theory, or whatever. We might have mathematical or practical reasons to change the base language but first-order logic is a sufficient bootstrap.

The Russell Paradox has had a profound influence on the historical development of set theories. This influence continues in transset theory. We show that the Russell Antinomy is distinct from the Russell Set. Both objects exist. The Russell Antinomy's membership of itself is paradoxical in that it is a logical gap. Various other memberships are non-paradoxical: the Russell Antinomy is a member of the Universal Antinomy, the Russell Set is a member of the Universal Set, the Russell Set is a member of the Russell Antinomy, the Russell Set is not a member of itself, the Russell Antinomy is not a member of the Russell Set, and so on. Thus we arrive at a wider understanding of the Russell Paradox.

Cantor's Diagonalisation Proof is upheld in transset theory. Cantor goes to considerable trouble to show that the diagonalised expansion exists so that its absence from the enumerated expansions proves that there are more real than natural numbers. The case of Cantor's Powerset Theorem is more delicate. Cantor has no doubt that he may identify an object, $C = \{x \mid x \notin f(x)\}$, with a set, whence he proves that every set has a smaller cardinality than its powerset. For us $C' \doteq \{x \mid x \notin f(x)\}$ is an antinomy. We cannot simply assume it is a set unless we explicitly require this. We may easily re-write Cantor's Powerset Theorem to make this assumption explicit. Hence Cantor's Powerset Theorem holds for some sets. But our concern with totality drives us to notice the set $D = \{x \mid x \in f(x)\}$, whence certain sets, such as our universal set, do have the same cardinality as their powersets. We find it astonishing that this appears to have been missed in the usual treatments of Cantor's Powerset Theorem. The consequences of missing it might be harmless. NFU is immune to Cantor's Powerset Theorem: the universal set is larger than its powerset but all other sets are smaller than their powersets. ZFC adopts small sets so the existence of large sets whose cardinality is equal to their powerset does not arise. This might be an unnecessary but harmless restriction or it might be a case of monster barring that blocks inconsistency in ZFC. We would like to know if ZFC can support our Universal Set and Universal Antinomy? If so, all of the usual results of ZFC may be totalised. Other totalisations are available. For example our set membership predicate could totalise set theories over all sets that exists, taking our antinomies as non-existent sets. In the computer age we are desperately concerned to establish totality in all mathematics so these questions are worth exploring.

Most generally the usual non-existence proofs establish, in our terms, the existence of an antinomy. An antinomy may have many, even infinitely many, members so discarding, so called, non-existent objects may discard a very great deal of mathematical structure. We recommend examining all non-existence proofs to see if anything of value has been discarded.

Having both sets and antinomies gives us access to every method of paraconsistent reasoning. This might be of use in the formal development of paraconsistent logics and, more generally, in the conduct of science.

In many cases scientific theories embrace inconsistent explanations and are inconsistent with empirical observations, yet they provide a useful basis for scientific reasoning and experiment. This cannot be explained in classical logic where any inconsistency blows up to make all statements logically true. This is not the case in paraconsistent logics or in transset theory – both preserve only a limited number of statements as true theorems or as sets. But does our set theory provide any insights into how science should be conducted?

Karl Popper [25] advanced the one-sided method of setting up scientific theories so that they can be refuted. He noted, with some disparagement, that practicing scientists often set out to confirm their theories, not to refute them but, for us, it is rational to do both things. The complement of everything that is true is everything that is false and everything that is an antinomy. Similarly the complement of everything that is false is everything that is true and everything that is an antinomy. Antinomies occur in both complements. Between truth and falsehood there is a no-man's land of antinomy. The boundaries of knowledge can be advanced, on both sides, by establishing what is true and by establishing what is false. Both confirmation and refutation have a role to play in minimising antinomy. Perhaps this should be the goal of science: to maximise truth and to minimise both falsehood and antinomy.

How far might we go in criticising Popper? Are his views on science a mistake, founded in the old paradigm of classical logic? Can we do better?

6 Conclusion

We extend naive set theory by introducing antinomies as objects. At a stroke this dissolves many of the usual paradoxes of set theory and gives us access to transnatural and transordinal arithmetics in place of the usual, but more limited, natural and ordinal arithmetics. We are astonished to find that the usual treatments of Cantor's Powerset Theorem appear to miss the case where a large set has the same cardinality as its powerset. For us this raises questions about the consistency of Zermelo-Fraenkel set theory and whether it would benefit from introducing a universal set and a universal antinomy?

We have dealt, very briefly, with some philosophical issues, including questioning Popper's views on science. We show that both confirmation and refutation have distinct, but synergistic, roles to play in establishing scientific truth, dismissing falsehood, and minimising logical gaps.

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