Online Supplement to "Pseudo-Ranks: The Better Way of Ranking?"

In this supplementary document, we provide detailed formal considerations regarding the counterintuitive behavior of pseudo-rank-based relative effects, thereby extending the brief discussion of the corresponding simulated data example that has been included in the main body of the manuscript.

1 Preliminaries

In the sequel we assume $i = 1, \ldots, a$ random variables Y_i with continuous distributions F_i and will consider for a vector $\alpha = (\alpha_1, \ldots, \alpha_a)$ of non-negative weights with $\sum_{i=1}^{a} \alpha_i = 1$ the reference distribution

$$H^\psi := \sum_{i=1}^{a} \alpha_i F_i.$$

Think of F_i representing the distribution of a specific target variable in a population stratum i. Based on the reference distribution H^ψ we define the relative effects

$$\psi_i := \int H^\psi dF_i = P(Z < Y_i),$$

where $Z \sim H^\psi$ is stochastically independent from all Y_i. Note that we use the superscript ψ in H^ψ to indicate that in fact, we will consider the unweighted reference distribution in the sequel.

For this discussion it is useful to express the relative effects in terms of the pair-wise and centered relative effects

$$\delta_{ij} := P(Y_i > Y_j) - 0.5, \quad i, j = 1, \ldots, a.$$

Note that $\delta_{ii} = 0$ and $\delta_{ij} = -\delta_{ji}$ for all i, j. The relative effect can then be written in terms of the δ_{ij} as

$$\psi_i := 0.5 + \sum_{\ell=1}^{a} \delta_{i\ell}\alpha_\ell = 0.5 + \sum_{\ell=1,\, \ell \neq i}^{a} \delta_{i\ell}\alpha_\ell.$$

2 Examples

We will focus here on changes in the relative effects when a specific population stratum is split into groups (that are then identical with respect to the distribution of Y). If we would consider the expectations μ_i of Y_i instead of relative effects, an artificial splitting of a specific stratum would not change the parameter values of the remaining groups. It would only reduce the efficiency of the group comparisons, since it would (unnecessarily) increase the number of parameters (by duplicates) and decrease the group sizes. The question we pose is whether we make a similar observation with pseudo-ranks and the underlying relative effects. The question is not purely academic. In applications it is quite common to join groups with similar distributions in order to gain power. Moreover, some multiple contrasts tests (*e.g.*, Williams' test) rely on the conjunction of groups.

If in the reference distribution H^ψ the distributions of the individual groups F_i are weighted equally (as for pseudo-ranks) then a split or junction of groups will change the reference distribution and thereby the relative effects also for those groups that remain unchanged. This may lead to a counter-intuitive behaviour of pseudo-ranks.

2.1 Splitting one of two groups

As a first simple example we start considering three groups where $F_1 \neq F_2 = F_3$ which implies $\delta_{23} = \delta_{32} = 0$. Obviously, the reference distribution with all three groups is

$$H^\psi = (1/3)F_1 + (2/3)F_2 \qquad \text{(i.e. } \alpha_1 = 1/3, \alpha_2 = 2/3, \alpha_3 = 0)$$

whereas it becomes

$$H^\psi = (1/2)F_1 + (1/2)F_2 \qquad (\alpha_1 = \alpha_2 = 1/2, \alpha_3 = 0)$$

when group 2 and 3 are rejoined. With the three groups, the relative effects are

$$\psi_1 = 0.5 + \delta_{12}\,2/3, \qquad \psi_2 = \psi_3 = 0.5 - \delta_{12}/3,$$

Note that we are slightly abusing notation here, because strictly speaking, all considerations are based on taking the unweighted average (*i.e.*, all α_i are equal). However, for sake of notational simplicity, we consider the equivalent two-group setting (which results from $F_2 = F_3$) and the corresponding unweighted averages.
When the first two groups are rejoined, the relative effects become

$$\psi_1 = 0.5 + \delta_{12}/2, \qquad \psi_2 = 0.5 - \delta_{12}/2.$$

Hence, the conjunction of the last two groups changes the relative effect, also for the first group. Both relative effects decrease when the last two groups are rejoined and $\delta_{12} > 0$.

Interestingly, the contrast $\psi_1 - \psi_2 = \delta_{12}$ is invariant with respect to the way we deal with the last two groups.

Similar considerations show that when joining k groups, i.e. starting with $F_1 \neq F_2 = \cdots = F_{k+1}$, we get a reference distribution with weights $\alpha_1 = 1/(k+1)$ and $\alpha_2 = k/(k+1)$ (and $\alpha_3 = \cdots = \alpha_{k+1} = 0$). Hence, the relative effects become

$$\psi_1 = 0.5 + \delta_{12}\,k/(k+1), \quad \psi_2 = \cdots = \psi_{k+1} = 0.5 - \delta_{12}/(k+1)\,.$$

Obviously, ψ_1 monotonically increases in k (when $\delta_{12} > 0$) with limit $0.5+\delta_{12}$. However, the contrast $\psi_1 - \psi_2 = \delta_{12}$ is invariant with respect to k (which is a remarkable property). This is due to the fact that actually, we have a two-group setting, with the different splittings only changing the weights. For two-group comparisons, the difference between the relative effects is invariant to the choice of those weights, which has been already mentioned in Section 3 of the manuscript. Note that for the invariance it is crucial to take the difference of the relative effects: One can show, for instance, that the ratio ψ_1/ψ_2 increases in k.

2.2 Splitting one of three groups

The following example shows that the invariance property of contrasts between pseudo-rank-based relative effects is not a general property and can fail in examples with more than two (statistically different) strata. Assume now four strata with $F_1 \neq F_2 = F_3 \neq F_4$. With all four strata ($\alpha_1 = \alpha_4 = 1/4$, $\alpha_2 = 1/2$, $\alpha_3 = 0$) we obtain the relative effects

$$\psi_1 = 0.5 + \delta_{12}/2 + \delta_{14}/4, \quad \psi_2 = 0.5 - (\delta_{12} + \delta_{42})/4, \quad \psi_4 = 0.5 - \delta_{14}/4 - \delta_{24}/2,$$

and consequently the contrast

$$\psi_1 - \psi_2 = \delta_{12}\, 3/4 + (\delta_{14} + \delta_{42})/4\ .$$

When joining the groups 2 and 3 ($\alpha_1 = \alpha_2 = \alpha_4 = 1/3$), we obtain the relative effects

$$\psi_1 = 0.5 + (\delta_{12} + \delta_{14})/3, \quad \psi_2 = 0.5 - (\delta_{12} + \delta_{42})/3, \quad \psi_4 = 0.5 - (\delta_{14} + \delta_{24})/3$$

and the contrast

$$\psi_1 - \psi_2 = \delta_{12}\, 2/3 + (\delta_{14} + \delta_{42})/3$$

This shows, that the contrast $\psi_1 - \psi_2$ is no longer invariant with respect to the way we handle the groups 2 and 3 (which would not be the case when comparing expectations). Moreover, we can construct examples (*e.g.*, from normally distributed Y_i with means μ_i satisfying $\mu_1 - \mu_4 = \mu_4 - \mu_2 = \delta > 0$ and variances σ_i^2 satisfying $\sigma_1^2 = \sigma_2^2 < \sigma_4^2$) where $\delta_{14} + \delta_{42} > 0$ is (much) smaller than $\delta_{12} > 0$. In this (and only this) case, the contrast $\psi_1 - \psi_2$ becomes smaller when rejoining the statistically identical second and third strata.

Similar calculations show that even $\psi_1 - \psi_4$ changes and may decrease when rejoining the F_2 and F_3 strata. With and without the split we get respectively:

$$\psi_1 - \psi_4 = \delta_{14}\,/2 + (\delta_{12} + \delta_{24})/2 \quad \text{and} \quad \psi_1 - \psi_4 = \delta_{14}\, 2/3 + (\delta_{12} + \delta_{24})/3.$$

The former is larger than the latter if $\delta_{12} + \delta_{24} > \delta_{14} > 0$ which is the case, for example, for normal Y_i with means $\mu_1 = 4$, $\mu_2 = \mu_3 = 2$, $\mu_4 = 0$ and variance $\sigma_i^2 = 0.25$ for $i \in \{1, 2, 3, 4\}$. A numerical illustration has been included in Section 3 of the manuscript.

Assuming a split of the F_2/F_3-strata into k groups leads to a reference distribution with $\alpha_1 = \alpha_4 = 1/(k+2)$ and $\alpha_2 = k/(k+2)$ and the relative effects

$$\begin{aligned}\psi_1 &= 0.5 + \delta_{12}\, k/(k+2) + \delta_{14}/(k+2),\\ \psi_2 &= 0.5 - (\delta_{12} + \delta_{42})/(k+2),\end{aligned}$$

$$\psi_4 = 0.5 - \delta_{14}/(k+2) - \delta_{24}k/(k+2)$$

The resulting contrasts are

$$\psi_1 - \psi_2 = \delta_{12}\,(k+1)/(k+2) + (\delta_{14} + \delta_{42})/(k+2)$$

and

$$\psi_1 - \psi_4 = \delta_{14}\,2/(k+2) + (\delta_{12} + \delta_{24})\,k/(k+2)$$

The first increases in k when $\delta_{12} > \delta_{14} + \delta_{42} > 0$, the second one if $\delta_{12} + \delta_{24} > \delta_{14} > 0$.

In summary, the contrasts $\psi_1 - \psi_2$ and $\psi_1 - \psi_4$ can become larger with an artificial split of the F_2/F_3-strata (not changing the data distribution) and even increase with the number of such splits. This is rather counter-intuitive from a statistical point of view as it favours artificial splits and the resulting over-parametrization. Moreover, asymptotically (*i.e.*, for $k \to \infty$), the pairwise relative effect δ_{14} vanishes in the contrast $\psi_1 - \psi_4$, which might be regarded as another unexpected result. In order to demonstrate the actual empirical consequences of our formal considerations, we provide some examples using simulated normally distributed data in the `R` code file `CExamples_psranks.R`.